

Aspects of Quantum Gravity: From Black Hole Thermodynamics to Holography

by

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Author's Declaration

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of Contributions

This thesis is based on the following published papers.

- Chapter 3 is based on

Jiayue Yang and Robert B. Mann. Dynamic behaviours of black hole phase transitions near quadruple points. JHEP, 08:028, 2023. [1]

- Chapter 4 is based on

Jiayue Yang and Andrew R. Frey. Complexity, scaling, and a phase transition. JHEP, 09:029, 2023. [2]

Abstract

Understanding the nature of quantum gravity remains one of the cutting-edge challenges in modern theoretical physics. Despite widespread interest and extensive research, the formulation of a complete theory of quantum gravity remains an unsolved problem. This thesis aims to explore quantum gravity from two primary perspectives: black hole thermodynamics and holography.

Black hole chemistry extends black hole thermodynamics by incorporating the cosmological constant, treating the black hole mass as enthalpy rather than energy, and interpreting the cosmological constant and its conjugate variable as pressure and volume, respectively. Holographic complexity investigates the duality between gravitational quantities in the bulk and quantum complexity on the boundary. This includes various proposals such as complexity=volume (CV), complexity=spacetime volume (CV2.0), complexity=action (CA), and complexity=anything proposals.

In our exploration of black hole thermodynamics, we investigate phase transitions near quadruple points, where four distinct black hole phases coexist. Utilizing the free-energy landscape technique within the framework of four-dimensional Einstein gravity coupled with nonlinear electrodynamics (NLE), we undertake the first investigation into the dynamics of recently discovered multicriticality in black holes. By treating off-shell Gibbs free energy as a potential and solving the Smoluchowski equation numerically, we capture the evolution of the state probability distributions during black hole phase transitions. Our study reveals how off-shell Gibbs free energy, shaped by ensemble temperatures, influences these transitions and highlights intricate behaviours such as weak and strong oscillations.

In our investigation of holography, we analyze the complexity of conformal field theories (CFTs) compactified on a circle with a Wilson line, corresponding to magnetized solitons in four-dimensional and five-dimensional Anti-de Sitter space (AdS). We explore three distinct proposals for holographic complexity. Our study reveals that these proposals exhibit a confinement-deconfinement phase transition driven by variations in the Wilson line, with the complexity of formation acting as the order parameter for this transition. We find that the proposed volume and action complexity functionals obey a scaling relation with the radius of the circle. We show that this scaling law is applicable to a broad family of potential complexity functionals and conjecture that the scaling law applies to the complexity of conformal field theories on a circle in more general circumstances.

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Dedication

This thesis is dedicated to my parents, who have inspired and supported me as I pursue my passion in science.

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Chapter 1

Introduction

1.1 Motivation

In the early 20th century, quantum mechanics (QM) [3] emerged as a fundamental theory in physics, describing the behaviour of objects at microscopic scales. There are two major formulations of quantum mechanics: Erwin Schrödinger's wave mechanics, which is based on the Schrödinger equation, and Werner Heisenberg's matrix mechanics formulation, with key contributions from Max Born and Pascual Jordan. Quantum mechanics provides a comprehensive framework for understanding a wide range of phenomena and plays an essential role in numerous technological advancements.

In 1905, Albert Einstein proposed the theory of special relativity (SR) [4]. In 1915, he established his groundbreaking theory of general relativity (GR) [5, 6], which revolutionized our understanding of the intricate relationship between space, time, and matter. Since then, general relativity has become an essential and powerful tool for the study of astrophysics and cosmology. Numerous subsequent experiments and observations have verified the correctness and validity of general relativity, including the perihelion precession of Mercury [7], the deflection of light by the Sun [8], gravitational redshift [9], detection of gravitational waves [10], and the existence of black holes [11].

Quantum mechanics and relativity stand as two of the landmark discoveries of 20th-century physics. Despite their importance, the two theories are fundamentally irreconcilable: quantum mechanics is a probabilistic theory that generally applies to microscopic scales, whereas general relativity is a deterministic theory that describes the gravitational effects of massive objects. Physicists have undertaken extensive efforts to reconcile quantum mechanics with relativity. Among these efforts, quantum field theory (QFT) [12] is

particularly notable as it successfully combines quantum mechanics with special relativity. There are several candidate quantum gravity theories; for example, string theory [13, 14, 15] and loop quantum gravity [16, 17] have been proposed to unify quantum mechanics with general relativity and establish an elegant theory of quantum gravity. However, a complete theory of quantum gravity has not yet been found.

One of the central goals in modern theoretical physics is to unravel the mysteries of quantum gravity. Despite widespread interest and research, the exact essence of quantum gravity remains elusive. To address this challenge, this thesis aims to explore quantum gravity through various approaches and gain insights from different perspectives. In particular, this thesis focuses primarily on black hole thermodynamics [18, 19, 20, 21, 22, 23] and holography [24, 25, 26]. Black hole thermodynamics provides a framework for understanding the thermodynamic properties of black holes, while holography offers a framework for connecting quantum field theories to gravitational theories through dualities. By employing these two approaches, this thesis aims to contribute to the ongoing efforts to better understand quantum gravity. Through a detailed analysis of the dynamic behaviours of black hole phase transitions [1] in Chapter 3 and the holographic complexity of magnetized AdS solitons [2] in Chapter 4, we hope to shed light on the fundamental nature of spacetime and the intersection between quantum and relativistic theories.

1.2 Overview

Since the 1970s, studies in black hole thermodynamics [18, 19, 20, 21, 22, 23] have uncovered potential deep connections between gravitational theory, quantum theory, and thermodynamics. In classical general relativity, black holes are solutions to Einstein's field equations, and within this framework, black holes absorb all matter, with no possibility of matter escaping. As a result, a classical black hole has neither temperature nor entropy. However, in the context of quantum field theory in curved spacetime, black holes can emit radiation [22, 23], where surface gravity corresponds to temperature, and the event horizon area is related to entropy. This led to the identification of an analogous relationship between the laws of black hole mechanics and the laws of thermodynamics, which in turn led to the formation of the field of black hole thermodynamics.

Recent research has begun to consider the influence of the cosmological constant on black hole thermodynamics to further refine the correspondence between the laws of black hole mechanics and thermodynamics [27, 28]. This has led to the introduction of thermodynamic pressure and thermodynamic volume. Within this theoretical framework, black holes

have been found to exhibit behaviors similar to substances in chemical systems. For instance, black holes have been found to resemble Van der Waals fluids [29, 30, 31, 32, 33, 34], a practical model used to describe the critical behaviors of real fluids. They can exhibit a triple point [35, 36, 37], similar to that of water, where solid, liquid, and gas phases coexist. They can also have multicritical points [38, 39, 40] like those have been observed in chemical systems such as polymers and colloids [41, 42]. Additionally, black holes can undergo phase transitions between different phases [29, 35, 43], which are distinguished by the sizes of the black holes. These newly discovered properties have led to the emergence of a new field known as black hole chemistry [27, 28]. Black hole chemistry extends the ideas of real-world chemistry into the realm of strong gravity, offering new ways to understand and explore the universe by treating black holes as complex thermodynamic systems.

Real-world holography is a technique that uses two-dimensional interference patterns of light waves to reproduce three-dimensional light field images (holograms) through diffraction. Holography is used in various practical applications, including artistic displays, security holograms, and data storage. The main idea of holography is to encode information about a higher-dimensional structure in a lower-dimensional format. In theoretical physics, the holographic principle [24, 25, 26] suggests that all of the information contained within a bulk can be represented as a theory living on a lower-dimensional boundary of that bulk spacetime. The holographic principle emerged from studies of black hole thermodynamics and the realization that the entropy of a black hole scales with the surface area of its event horizon rather than its volume [23].

Studies of the holographic principle have revealed profound connections between gravitational theory, quantum theory, and information theory. A famous subfield of the holographic principle is the Anti-de Sitter space/Conformal Field Theory correspondence (AdS/CFT duality) [25], also known as the gauge/gravity duality. AdS is a maximally symmetric spacetime with constant negative curvature, while CFT is a type of quantum field theory that is invariant under conformal transformations. The AdS/CFT correspondence states that a theory of quantum gravity in $d + 1$ dimensions that is asymptotically AdS is dual to a CFT in d dimensions without gravity. AdS/CFT is holographic because the gravitational theory exists in one dimension higher than the quantum field theory. It is a duality because any physical quantity that can be calculated in one theory can also be calculated in the dual theory. There is an AdS/CFT dictionary that connects these two theories, providing a precise correspondence between them. For example, concepts like entanglement entropy and quantum complexity, which originate from information theory and are usually defined in the quantum field theory living on the boundary, have their gravitational duals in the bulk. These are referred to as holographic entanglement entropy [44, 45, 46, 47, 48] and holographic complexity [49, 50, 51, 52, 53, 54, 55, 56].

Black hole thermodynamics and holography are indeed two different perspectives on quantum gravity. Black hole thermodynamics emphasizes the thermodynamic and statistical mechanics aspects of black holes, while holography provides a bridge connecting gravity and quantum theory. Historically, holography was motivated by research on black hole thermodynamics. The connection between black hole thermodynamics and holography lies in their shared emphasis on the relationship between gravity, quantum mechanics, entropy, and information, as well as the quantum aspects of black holes. Both perspectives help us better understand the properties of black holes and probe the quantum structure of spacetime.

In the framework of black hole chemistry, we will investigate the dynamics of multicriticality in black holes, focusing on the behavior of phase transitions near quadruple points [1]. Within the AdS/CFT correspondence framework, we will use the holographic complexity of formation density as an order parameter to explore phase transitions between AdS solitons and periodic AdS [2]. The common theme across these studies is the investigation of phase transitions in different physical contexts and through different methodologies. Both approaches seek to unravel deeper aspects of quantum gravity by studying how these phase transitions occur. Ultimately, the goal is to enhance our understanding of quantum gravity by integrating these insights and advancing our exploration of the fundamental connections between these diverse phenomena.

1.3 Outline

This thesis is organized as follows.

Firstly, in Chapter 2, we provide the background information and theoretical framework for black hole thermodynamics [18, 19] and holography [24, 25, 26]. We first review black hole thermodynamics and then move to black hole chemistry [27, 28], which extends black hole thermodynamics to include a cosmological constant. Next, we cover holography, discussing topics from holographic entanglement entropy [44] to holographic complexity [49, 50]. This chapter reviews key concepts and previous work in these areas.

In Chapter 3, we present a concrete investigation into the dynamic behaviours of black hole phase transitions near quadruple points [1]. This study is conducted within the framework of Einstein gravity coupled with nonlinear electrodynamics (NLE) in four-dimensional spacetime, using the free-energy landscape technique [57, 58, 59, 60]. The dynamic process of black hole phase transitions is described on the basis of the Smoluchowski equation. We analyze the dynamic evolution of the black hole state distribution and find the rela-

tionship between the state distribution evolution, off-shell Gibbs free energy, and ensemble temperature.

In Chapter 4, we provide a detailed analysis of the holographic complexity of magnetized solitons in four-dimensional and five-dimensional Anti-de Sitter space (AdS) using three different proposals [2]. This study is conducted within the Anti-de Sitter/conformal field theory (AdS/CFT) correspondence framework [25]. We examine the confinement-deconfinement phase transition using the complexity of formation as the order parameter and uncover a general scaling relation that offers new insights into the nature of holographic complexity.

Finally, in Chapter 5, we summarize our findings from both projects—on black hole thermodynamics and holography. We also discuss their broader implications for the field of quantum gravity, and propose potential directions for future research.

Chapter 2

Background

2.1 Black hole thermodynamics with Lambda

Given the considerable efforts by physicists to bridge quantum mechanics and relativity, one promising approach to understanding quantum gravity is to investigate black hole thermodynamics [18, 19].

Intuitively, black holes are incredibly dense objects, where massive amounts of matter are compressed into an extremely tiny space. In classical general relativity, black holes are regions of spacetime with an event horizon beyond which nothing, not even light, can escape due to strong gravity [61]. Quantum black holes arise from attempts to reconcile general relativity with quantum mechanics. Unlike classical black holes, which are completely black, quantum black holes exhibit a phenomenon known as Hawking radiation [20, 22]. This radiation results from quantum effects near the event horizon, causing the black hole to emit thermal radiation with a specific temperature called the Hawking temperature. The black holes considered in the field of black hole thermodynamics are quantum black holes, which have a temperature. What makes quantum black holes interesting is that they happen to be where gravity and quantum theory come together, neither gravitational nor quantum effects can be ignored.

Now let us briefly review black hole mechanics and black hole thermodynamics. In 1971, Stephen Hawking proved the area theorem, which states that the area of a black hole's event horizon does not decrease over time [62]. This law is analogous to the second law of thermodynamics. In 1972, Jacob Bekenstein proposed that a black hole possesses a thermodynamic entropy that is proportional to the area of the black hole's event horizon,

now known as black hole entropy or Bekenstein-Hawking entropy [21]. In 1973, Brandon Carter, James Bardeen, Jacob Bekenstein, and Stephen Hawking published the pioneering paper “The Four Laws of Black Hole Mechanics” [18], which considered the event horizon as a null hypersurface generated by a Killing vector field. Here is the formulation of the four laws of black hole mechanics [18].

Zeroth law: The surface gravity κ , defined by $\xi^a \nabla_a \xi^b = \kappa \xi^b$ where ξ^a is a normalized Killing vector generating the event horizon, is constant over the event horizon of a stationary black hole.

$$\kappa = \text{const.} \quad (2.1)$$

First law: For perturbations of black holes, the variation of the black hole mass M is related to the variation of the area A of the event horizon, the variation of angular momentum J , and the variation of charge Q by

$$\delta M = \frac{\kappa}{8\pi G} \delta A + \Omega \delta J + \Phi \delta Q, \quad (2.2)$$

where Ω is its angular velocity, and Φ is the conjugate potential for the U(1) charge, G is the gravitational constant.

Second law: The area A of the event horizon of a black hole is non-decreasing,

$$\delta A \geq 0, \quad (2.3)$$

this is also known as Hawking’s area theorem.

Third law: The surface gravity κ of a black hole cannot be reduced to zero through a finite process [63].

In 1975, Stephen Hawking made a groundbreaking discovery about black holes, showing that they are not completely black [20, 22]. By applying quantum field theory in curved spacetime, Hawking found that black holes could emit particles due to quantum effects. This emitted radiation, now known as Hawking radiation, has a spectrum similar to that of black-body thermal radiation. The corresponding characteristic temperature, known as the Hawking temperature, is given by

$$T = \frac{\hbar \kappa}{2\pi}, \quad (2.4)$$

where $\hbar$ is the reduced Planck constant, and κ is the surface gravity of the black hole. Note that we have set the Boltzmann constant $k_B = 1$, and the speed of light $c = 1$ in the above

formula. We also emphasize $\hbar$ appearing in the formula, indicating the quantum nature of black hole temperature. In the classic limit when $\hbar \rightarrow 0$, the classical black hole has a temperature of absolute zero. This is how the subject of the thermodynamics of black holes was born, which has since profoundly influenced theoretical physics and fascinated researchers.

Notably, there is an amazing analogy between the mechanics of black holes and thermodynamics. By comparing equation (2.2) with the first law of thermodynamics

$$\delta U = T\delta S - P\delta V + \Phi\delta Q + \sum_i \mu_i \delta N_i, \quad (2.5)$$

one can find the important relation that connects black hole entropy S and the area A of the black hole event horizon,

$$S = \frac{A}{4\hbar G}. \quad (2.6)$$

Therefore, the first law of black hole thermodynamics is

$$\delta M = T\delta S + \Phi\delta Q + \Omega\delta J. \quad (2.7)$$

The black hole mass M is identified with the energy U , the rotational and electromagnetic work terms $\Phi\delta Q + \Omega\delta J$ are analogous to the chemical potential and electromagnetic work terms $\Phi\delta Q + \sum_i \mu_i \delta N_i$ in equation (2.5). These thermodynamic quantities are connected by a useful Smarr–Gibbs–Duhem relation, which, in four dimensions, is expressed as

$$M = 2(TS + \Omega J) + \Phi Q. \quad (2.8)$$

One would notice that pressure and volume do not appear in the standard form of the first law of black hole thermodynamics, as given by equation (2.7). In recent years, a new field called black hole chemistry [27, 28] has emerged, incorporating these concepts into black hole thermodynamics. Black hole chemistry, also known as black hole thermodynamics with Lambda, is based on the idea of considering black holes within an environment characterized by a negative cosmological constant Lambda (Λ). In this context, the negative cosmological constant is regarded as a positive pressure,

$$P = -\frac{\Lambda}{8\pi}. \quad (2.9)$$

The corresponding thermodynamic volume is defined as the conjugate variable of the thermodynamic pressure

$$V \equiv \left(\frac{\partial M}{\partial P} \right)_{S,Q,J}, \quad (2.10)$$

where the black hole mass M is interpreted as the chemical enthalpy [64], which includes both the internal energy U and the energy PV needed to place the black hole in an environment with a negative cosmological constant,

$$M = U + PV. \quad (2.11)$$

Therefore, in the framework of black hole chemistry, the generalized first law of black hole thermodynamics is given by

$$\delta M = T\delta S + V\delta P + \Phi\delta Q + \Omega\delta J. \quad (2.12)$$

The generalized Smarr–Gibbs–Duhem relation in four dimensions is given by

$$M = 2(TS + \Omega J) + \Phi Q - 2PV. \quad (2.13)$$

With thermodynamic pressure and volume defined above, the thermodynamic phase space can be extended. In this theoretical framework, a variety of rich thermodynamic and chemical phenomena, such as black hole phase transitions [43], triple points [35, 36, 37], and Van der Waals-like behaviour [29, 30, 31, 32, 33, 34], have been discovered. Exploring the thermodynamic properties and thermodynamic phenomena of quantum black holes provides valuable insights into the fundamental principles of quantum gravity. These efforts are crucial in paving the way toward a unified theory that reconciles quantum theory and general relativity.

We will discuss a specific example for black hole phase transitions in the context of black hole chemistry in detail in Chapter 3.

2.2 Holographic complexity

In addition to black hole thermodynamics, holography [24, 25, 26] is also a widely recognized approach to understanding quantum gravity.

Inspired by black hole thermodynamics, in the 1990s, physicists such as Leonard Susskind, Juan Maldacena, and Edward Witten expanded upon Gerard 't Hooft's insights to develop the holographic principle [24, 25, 26]. This principle is grounded in the notion of a duality between two theories. The basic idea is that the bulk gravitational theory is equivalent to the boundary quantum field theory that is one dimension lower than the bulk theory. A notable example of this principle is the Anti-de Sitter space/Conformal field theory correspondence (AdS/CFT duality) [25] proposed by Maldacena in 1997. The holographic

principle significantly advanced our comprehension of quantum gravity and paved the way for novel directions in theoretical physics. One such direction is the study of quantum information in quantum gravity, which aims to understand the connections between quantum information theory and gravitational theory, and how they bring new insights to each other. A key development in this area is the understanding of holographic entanglement entropy based on the seminal work of Shinsei Ryu and Tadashi Takayanagi [44]. In 2006, Ryu and Takayanagi proposed the first formula for holographic entanglement entropy [44], known as the Ryu-Takayanagi (RT) formula. The RT formula states that the entanglement entropy of a subregion in the boundary CFT is proportional to the area of the minimal extremal surface in the bulk AdS spacetime that is dual to the CFT,

$$S(X) = \frac{Area(S_{RT})}{4G}. \quad (2.14)$$

Here, G is the gravitational constant, $S(X)$ is the holographic entanglement entropy of the boundary subregion X , and S_{RT} is the codimension-2 spacelike minimal extremal surface anchored to the AdS boundary and homologous to region X , also known as the Ryu-Takayanagi (RT) surface. The minimal extremal condition should also be emphasized in determining the RT surface in the formula (2.14). The significance of the RT formula lies in its ability to connect gravitational theory and quantum information theory in a holographic and profound way.

However, bulk geometry cannot be fully reconstructed solely from the entanglement entropy [65, 66, 67]. In black hole physics, the growth of the spatial volume behind the horizon of the black hole continues long after the black hole has reached thermal equilibrium, where entropy is maximized. At this stage, something other than entropy keeps increasing. Thus entanglement entropy is not enough, a new notion called complexity is needed. The concept of complexity originates from computer science, where it measures the difficulty of implementing a specific algorithm. In quantum circuit theory, complexity is defined as the minimum number of quantum gates needed to transform a reference state into a target state, representing the optimized number of elementary quantum operations required to achieve this transformation. If the quantum state $|\psi\rangle$ can be obtained from the reference state $|\psi_0\rangle$ through the following transformation, which can be expressed as an optimized sequence of n (minimal number) quantum gates

$$|\psi\rangle = U |\psi_0\rangle = g_n g_{n-1} \cdots g_1 |\psi_0\rangle, \quad (2.15)$$

then the relative complexity of the two quantum states is

$$C(|\psi\rangle, |\psi_0\rangle) = n. \quad (2.16)$$

One notable advantage of complexity is its ability to capture the growing geometry of black hole interiors even after thermal equilibrium is reached. On one hand, the quantum complexity of the boundary grows linearly with time, with entropy being the growth rate.

$$\frac{dC(t)}{dt} \propto ST \quad (2.17)$$

On the other hand, the volume in the interior region of a two-sided black hole spacetime grows linearly with time. The growth rate is the area A of the horizon, which corresponds to the black hole's entropy.

$$\frac{dV(t)}{dt} \propto lAT \quad (2.18)$$

where A is the horizon area, T is the black hole temperature, and l can be taken as the AdS length scale.

Thus, the first holographic complexity proposal, known as “complexity=volume” (CV) [49, 50], was introduced. This proposal states that the complexity is proportional to the spatial volume, and depends on the AdS length scale

$$C \propto \frac{V}{Gl}, \quad (2.19)$$

where l is the AdS length scale, and G is the Newton's gravitational constant introduced through dimension analysis.

Motivated by this, various holographic complexity proposals were formulated, including “complexity=spacetime volume” (CV 2.0) [51], “complexity=action” (CA) [52, 53, 54], and “complexity=anything” program [55, 56]. To be more precise, here are the formulations for these proposals on holographic complexity.

Complexity=Volume: the complexity of the boundary CFT state is proportional to the volume V of the maximal volume spacelike hypersurface anchored at a fixed boundary time.

$$C_V = \frac{(d-1)V}{2\pi^2 Gl}, \quad (2.20)$$

where d is the dimension of boundary CFT, and l is the length scale. We use this normalization of complexity in the CV proposal to ensure clarity in our subsequent calculations.

Complexity=Volume 2.0: the complexity of the boundary CFT state is proportional to the spacetime volume of the Wheeler-DeWitt patch (WDW patch) V_{WDW} .

$$C_2 = \frac{V_{WDW}}{Gl^2}, \quad (2.21)$$

where l is the length scale, and G is the gravitational constant. The WDW patch is the region formed by the union of all the possible spacelike hypersurfaces anchored at a fixed boundary time as introduced in the original CV proposal. So the WDW patch is a spacetime region bounded by null surfaces.

Complexity=Action: the complexity of the boundary CFT state is proportional to the action of the Wheeler-DeWitt patch S_{WDW} .

$$C_A = \frac{S_{WDW}}{\pi}. \quad (2.22)$$

Holographic complexity is a field of study that delves into the intriguing relationship between gravitational quantities in the bulk spacetime and quantum complexity on the boundary of that spacetime. This duality represents a specific manifestation of the broader AdS/CFT correspondence.

We will discuss these holographic complexity proposals in detail with a specific example in Chapter [4](#).

Chapter 3

Dynamic behaviours of black hole phase transitions near quadruple points

In this chapter, we explore quantum gravity through black hole thermodynamics. We conduct the first investigation into the dynamics of the recently discovered multicriticality in black holes. Specifically, we examine black hole quadruple points within the framework of four-dimensional Einstein gravity coupled with nonlinear electrodynamics. By numerically solving the Smoluchowski equation, which governs the evolution of the black hole probability distribution function at various ensemble temperatures, we observe thermodynamic phase transitions among the four black hole phases, as well as weak and strong oscillatory phenomena. Additionally, we find that the evolution of the black hole probability distribution is closely linked to the structure of the off-shell Gibbs free energy.

3.1 Introduction

Following the establishment of the four laws of black hole thermodynamics [18, 19, 20, 21, 22, 23], significant strides have been achieved over the past fifty years in the field of black hole thermodynamics. One particularly intriguing subfield within this area is the study of phase transitions and critical phenomena in the context of black hole thermodynamics. A pioneering work in this direction was the Hawking-Page phase transition [43], which characterizes the transition between thermal radiation and large black holes, akin

to the solid-liquid phase transition. The study of the Hawking-Page phase transition not only provided profound insight into the thermodynamic behaviour of black holes but also inspired numerous interesting works in this field.

In recent times, the understanding of black hole critical phenomena has been significantly enhanced within the framework of black hole chemistry [27, 28], where the core idea is to view the black hole mass as enthalpy instead of energy [64]. This approach involves interpreting the cosmological constant and its conjugate variable as the thermodynamic pressure and volume of the black hole system, respectively. From this perspective, there exists a phase transition between large and small charged anti-de Sitter (AdS) black holes, analogous to the gas-liquid phase transition in the Van der Waals model [29, 30, 31, 32, 33, 34]. This analogy has opened up new avenues for exploring the microstructure of black holes and understanding their thermodynamic properties in greater detail.

The underlying kinetics of black hole phase transitions have been explored using the free energy landscape model quite recently [57, 58, 59, 60]. In this framework, these phase transitions are driven by thermal fluctuations, with the black hole horizon radius serving as the order parameter of the phase transition. The dynamics of black hole phase transitions are analyzed by solving the Smoluchowski equation [68, 69, 70, 71], a probabilistic Fokker-Planck equation that models the diffusion process of a system in some potential barriers. In the context of black hole phase transitions, the effective potential barrier is represented by the off-shell Gibbs free energy, a continuous function of the black hole horizon radius at a given ensemble temperature, indicating the stability of various black hole states. Peaks correspond to unstable states, while valleys correspond to stable states. This approach provides a comprehensive way to visualize how black hole phase transitions occur.

The finding that black holes can exhibit triple points has further supported the argument for treating black holes as chemical systems [35, 36, 37]. At black hole triple points, three stable phases – small, intermediate, and large black hole phases – meet together, analogous to the triple point of water where solid, liquid, and gas coexist at specific pressure and temperature conditions. The dynamic behaviour of black hole triple points has been recently investigated [72]. In this study, it was discovered that black holes, initially in small, intermediate, or large phases, can transition to the other two coexistent phases at the triple point. This finding shows that black hole thermodynamic phase transitions can indeed occur dynamically. Both weak and strong oscillatory behaviours were observed. In the weak oscillatory behaviour scenario, the probability of a non-initial state reaches a maximum before decaying to a stationary state, with its probability never surpassing that of the initial state. In the strong oscillatory behaviour scenario, the probability of a non-initial state surpasses that of the initial state. This work highlights the complex nature of black hole phase transitions and provides deeper insight into the stability and

transition dynamics of black hole phases.

Beyond the triple point, multicritical points have been observed in chemical systems such as polymers and colloids [41, 42]. Multicritical points emerge when multiple distinct phases in a system coexist at a specific choice of values of thermodynamic parameters, extending the idea of a triple point. Black hole multicriticality is a recent breakthrough in black hole chemistry. This phenomenon has been observed in various contexts, including 4-dimensional Einstein gravity coupled to nonlinear electrodynamics [38], multiply rotating Kerr-AdS black holes [39], and spherically symmetric black holes in Lovelock gravity [40]. In these scenarios, black hole solutions exhibiting multicritical behaviour have been demonstrated. Multicritical behaviour in black hole physics is significantly less understood compared to black hole triple points and other phase behaviours. This indicates a need for further in-depth research to fully comprehend the underlying dynamics of multicritical phenomena in black holes.

Inspired by the above discussion, we explore the dynamic behaviour of black hole quadruple points within the framework of 4-dimensional Einstein gravitational theory coupled to nonlinear electrodynamics, aiming to gain insights into the nature of black hole phase transitions. Our investigation reveals that the early evolution of the probability distribution of black hole phases near the quadruple point depends on the first passage time, which correlates with the size of the Gibbs potential barrier between phases. Moreover, we observe that the distribution of the final stationary phases is determined by the relative heights of the potential wells in the off-shell Gibbs free energy.

3.2 Quadruple points in black hole phase transitions

We investigate 4-dimensional Einstein gravity coupling nonlinear electromagnetic fields, where the action is given by [73]

$$S = \int d^4x [\sqrt{-g}(R - 2\Lambda - \sum_{i=1}^N \alpha_i (F^2)^i)], \quad (3.1)$$

where g is the determinant of the metric tensor, R is the Ricci scalar, Λ is the cosmological constant, $F^2 = F_{\mu\nu}F^{\mu\nu}$, with the electromagnetic field tensor $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$, A_μ is the electromagnetic four-potential, and α_i are dimensional coupling constants. The sum over i may truncate at some finite value of N , but it is also possible to consider $N \rightarrow \infty$, for example, as in Born-Infeld electrodynamics. The coupling constants α_i can be treated as independent thermodynamic variables, each possessing its own conjugate variable. The

occurrence of multicritical points relies on how the α_i values are chosen relative to the other thermodynamic variables.

We consider the following ansatz that describes spherically symmetric static black holes

$$ds^2 = -U(r)dt^2 + \frac{1}{U(r)}dr^2 + r^2 d\Omega^2, \quad (3.2)$$

$$A_\mu = [\Phi(r), 0, 0, 0]. \quad (3.3)$$

If the spacetime is asymptotically AdS, and Φ asymptotically vanishes, Φ and U can be assumed to have the following series forms,

$$\Phi = \sum_{r=1}^K b_i r^{-i}, \quad U = 1 + \sum_{i=1}^K c_i r^{-i} + \frac{r^2}{l^2}. \quad (3.4)$$

The field equations can then be solved for the coefficients b_i and c_i in terms of the mass M , charge Q , and the coupling constants α_i . Any specific solution depends on a finite number of parameters, which can be indexed either by N (the number of α_i couplings in the action) or by K (the number of parameters in the solution), in addition to M and Q . In the latter case the expressions for the gauge field and metric function are explicitly given by (3.4) whereas in the former case these expressions can be determined implicitly. In either case, black hole multicritical thermodynamic phenomena can be observed by selecting appropriate values for a finite number of the coupling constants α_i . And the values of K or N determining the maximal degree of multicriticality [38].

To admit quadruple points [38], let us concentrate on a scenario where α_i are nonzero and independent for $i \leq 7$. For $i \geq 8$ all remaining α_i can be expressed in terms of these quantities. This case is the simplest non-trivial example to observe multicritical dynamical behaviour in 4-dimensional Einstein gravity coupled to nonlinear electrodynamics. The solution is given by

$$\Phi = \frac{Q}{r} + \frac{b_5}{r^5} + \frac{b_9}{r^9} + \frac{b_{13}}{r^{13}} + \frac{b_{17}}{r^{17}} + \frac{b_{21}}{r^{21}} + \frac{b_{25}}{r^{25}}, \quad (3.5)$$

$$U = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} + \frac{b_5 Q}{2r^6} + \frac{b_9 Q}{3r^{10}} + \frac{b_{13} Q}{4r^{14}} + \frac{b_{17} Q}{5r^{18}} + \frac{b_{21} Q}{6r^{22}} + \frac{b_{25} Q}{7r^{26}} + \frac{r^2}{l^2}, \quad (3.6)$$

where the expression of b_i are known in terms of the coupling α_i and the charge Q , as given in this work [38], and b_i vanishes for $i > 25$.

The black hole thermodynamic quantities, temperature T , entropy S , pressure P , and

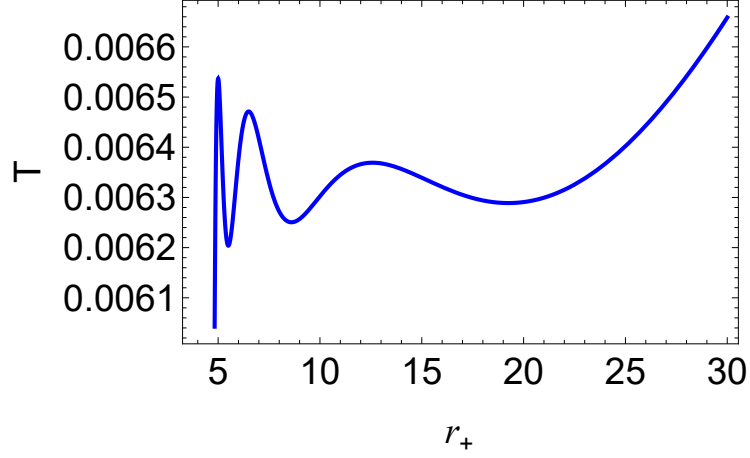


Figure 3.1: Temperature is plotted as a function of the horizon radius. Black hole quadruple point parameters with pressure $P = 0.0000689999421$, and charge $Q = 6.751117513$ are used in plotting the T - r_+ diagram.

volume V are given by

$$T = \frac{1}{4\pi r_+} \left(1 + \frac{3r_+^2}{l^2} - \frac{Q^2}{r_+^2} - 5\frac{b_5 Q}{2r_+^6} - 3\frac{b_9 Q}{r_+^{10}} - 13\frac{b_{13} Q}{4r_+^{14}} - 17\frac{b_{17} Q}{5r_+^{18}} - 7\frac{b_{21} Q}{2r_+^{22}} - 25\frac{b_{25} Q}{7r_+^{26}} \right), \quad (3.7)$$

$$S = \pi r_+^2, \quad P = \frac{3}{8\pi l^2}, \quad V = \frac{4\pi r_+^3}{3}, \quad (3.8)$$

and the equation of state is

$$P = \frac{T}{2r_+} - \frac{1}{8\pi r_+^2} + \frac{Q^2}{8\pi r_+^4} + \frac{5b_5 Q}{16\pi r_+^8} + \frac{3b_9 Q}{8\pi r_+^{12}} + \frac{13b_{13} Q}{32\pi r_+^{16}} + \frac{17b_{17} Q}{40\pi r_+^{20}} + \frac{7b_{21} Q}{16\pi r_+^{24}} + \frac{25b_{25} Q}{56\pi r_+^{28}}, \quad (3.9)$$

where all formulae are expressed in Planck units, Q is the black hole charge, and r_+ denotes the horizon radius, the size of the black hole.

The Gibbs free energy $G(T) = M - TS$, whose global minimum determines the thermodynamically stable state, is a powerful tool for finding black hole multicritical behaviours [38, 39]. The presence of $(N - 1)$ swallowtails in the $G - T$ diagram indicates N distinct black hole phases. The self-intersection points of the swallowtails signify first-order phase transitions between two different phases. Each cusp in a swallowtail corresponds

to an extremum of temperature T as a function of horizon radius r_+ , N distinct phases require $2N - 2$ distinct extrema in $T(r_+)$. If two adjacent inflections of $T(r_+)$ occur at the same temperature, then the intersection points of corresponding swallowtails will merge, resulting in a tricritical point. If this happens for $j \leq N$ different inflections, then the j swallowtails will have a common self-intersection point, which corresponds to a j -th order multicritical point. These critical points can be found by finely tuning the thermodynamic parameters. We can obtain a quadruple point, where the $T(r_+)$ curve exhibits three pairs of local maxima and minima, as illustrated in figure 3.1, by specifying that the thermodynamic parameters have the following values,

$$\left\{ \begin{array}{l} P = 0.0000689999421 \\ Q = 6.751117513 \\ b_5 = -5078.980603 \\ b_9 = 6.054804813 \times 10^6 \\ b_{13} = -4.131510152 \times 10^9 \\ b_{17} = 1.399821234 \times 10^{12} \\ b_{21} = -2.014970449 \times 10^{14} \\ b_{25} = 1.016449472 \times 10^{16}. \end{array} \right. \quad (3.10)$$

To understand the dynamics of black hole phase transition near quadruple points, we consider a canonical ensemble at a fixed temperature T_E , which does not need to be the Hawking temperature T . For the parameter values specified in (3.10), the temperature range where quadruple points can exist is $T_E \in (0.00628883, 0.00636918)$, as shown in figure 3.1. By treating the horizon radius as the order parameter, we extend the on-shell Gibbs free energy to include black hole spacetimes with all possible radii. Thus, the off-shell Gibbs free energy G_L is then defined as a function of the horizon radius with an input ensemble temperature T_E instead of the Hawking temperature T . This yields

$$\begin{aligned} G_L &= M - T_E S \\ &= \frac{1}{840r_+^{25}} (60b_{25}Q + 70b_{21}Qr_+^4 + 84b_{17}Qr_+^8 + 105b_{13}Qr_+^{12} + 140b_9Qr_+^{16} + 210b_5Qr_+^{20} \\ &\quad + 420Q^2r_+^{24} + 420r_+^{26} + 1120P\pi r_+^{28}) - T_E\pi r_+^2, \end{aligned} \quad (3.11)$$

where M represents the black hole mass (the conserved charge associated with the timelike Killing vector $\xi = \partial_t$ [38]), and S denotes the entropy of black hole. From equation (3.11), we observe that the off-shell Gibbs free energy, as a function of r_+ , also depends on the ensemble temperature T_E , the thermodynamic pressure P , and the black hole charge Q . A

union of black hole spacetimes with a wide range of arbitrary horizon radii forms what is known as the black hole landscape. Within this landscape, the off-shell free energy serves as the effective potential during the black hole phase transition process, where the horizon radius acts as the order parameter.

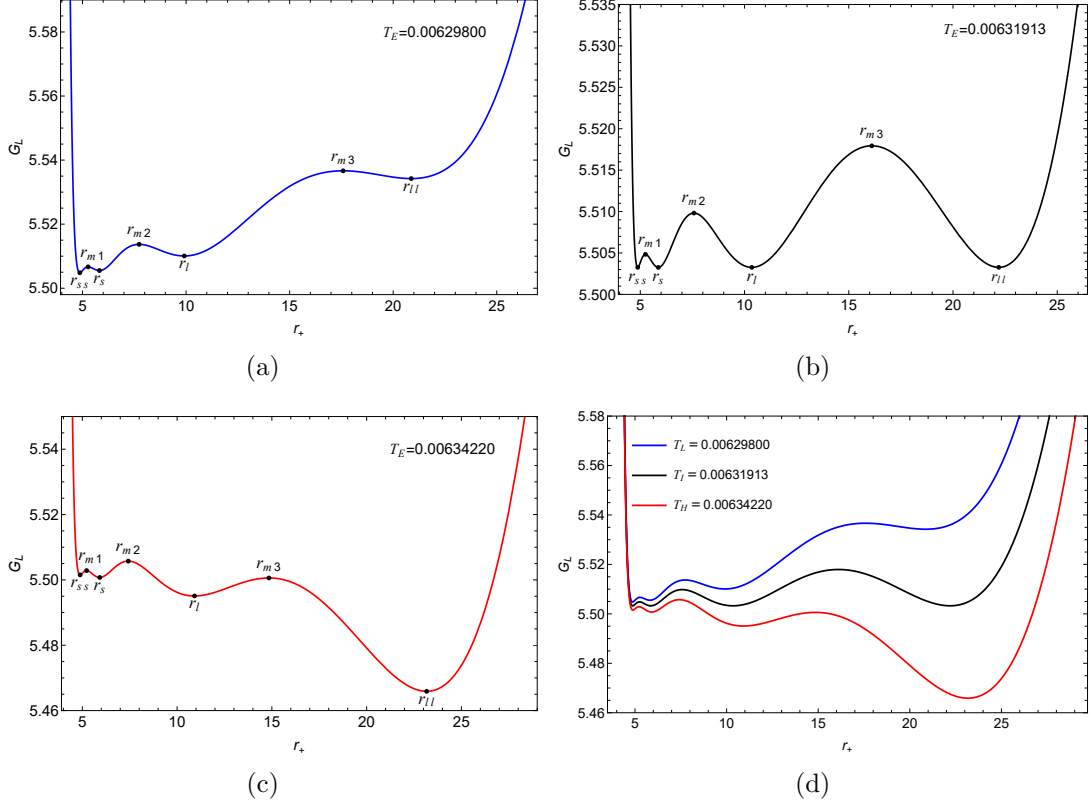


Figure 3.2: Off-shell Gibbs free energy diagrams at various ensemble temperatures. (a) $T_E = 0.006298$, (b) $T_E = 0.00631913$, (c) $T_E = 0.0063422$. (d) Off-shell Gibbs free energy at three different ensemble temperatures are presented in one panel with low temperature $T_L = 0.006298$, intermediate (quadruple point) temperature $T_I = 0.00631913$, and high temperature $T_H = 0.0063422$.

As shown in figure 3.2, we depict the off-shell Gibbs free energy G_L v.s. the horizon radius r_+ at a quadruple point with pressure $P = 0.0000689999421$, charge $Q = 6.751117513$ (as given by (3.10)) at cold, intermediate and high ensemble temperatures. Only the local

extrema on the $G_L(r_+)$ curve correspond to physical black hole states, on-shell solutions to the field equations that derived from the action given in (3.1). For each curve, there are four local minima, i.e. $r_k=r_{ss}$, r_s , r_l , and r_{ll} , representing four thermodynamic local stable black hole phases in order of increasing horizon size. There are also three local maxima, i.e. r_{m1} , r_{m2} and r_{m3} , representing three unstable black hole phases or what we call intermediate transition states. All the other points on the curve are transient black hole states during the phase transition process, which are off-shell solutions to the field equations.

3.3 Dynamic processes near the quadruple point

In this section, we investigate the dynamics of black hole phase transitions at the quadruple point ensemble temperatures, as well as at slightly lower and higher ensemble temperatures. From now on, we denote the black hole horizon radius r_+ by r for simplicity. We introduce the probability distribution function $\rho(t, r)$ to describe the probability that the system stays in a specific black hole state with horizon radius r at time t . According to the free energy landscape, black hole systems can experience thermodynamic phase transitions and transition to different black hole states via a thermal fluctuation of the order parameter r . This process can be described by a particular case of the Fokker-Planck equation, known as the Smoluchowski equation [74],

$$\frac{\partial \rho(t, r)}{\partial t} = D \frac{\partial}{\partial r} \left(e^{-\beta G_L(r)} \frac{\partial}{\partial r} (e^{\beta G_L(r)} \rho(t, r)) \right), \quad (3.12)$$

where $G_L(r)$ represents the off-shell Gibbs free energy acting as the potential, β is the inverse ensemble temperature, $D = k_B T_E / \zeta$ is the diffusion coefficient, with k_B being Boltzmann's constant and ζ the dissipation coefficient. To investigate the dynamic evolution of various black hole phases near the quadruple point, we numerically solve the Smoluchowski equation to find the probability distribution function $\rho(t, r)$. In the solving procedure we set $k_B = \zeta = 1$ without loss of generality. Solving this partial differential equation requires both initial and boundary conditions. As initial conditions, we assume the system starts in a Gaussian wave packet peaked at one of the four distinct stable black hole phases with the horizon radius r_i

$$\rho(0, r) = \frac{1}{\sigma \sqrt{\pi}} e^{-\frac{(r-r_i)^2}{\sigma^2}}, \quad (3.13)$$

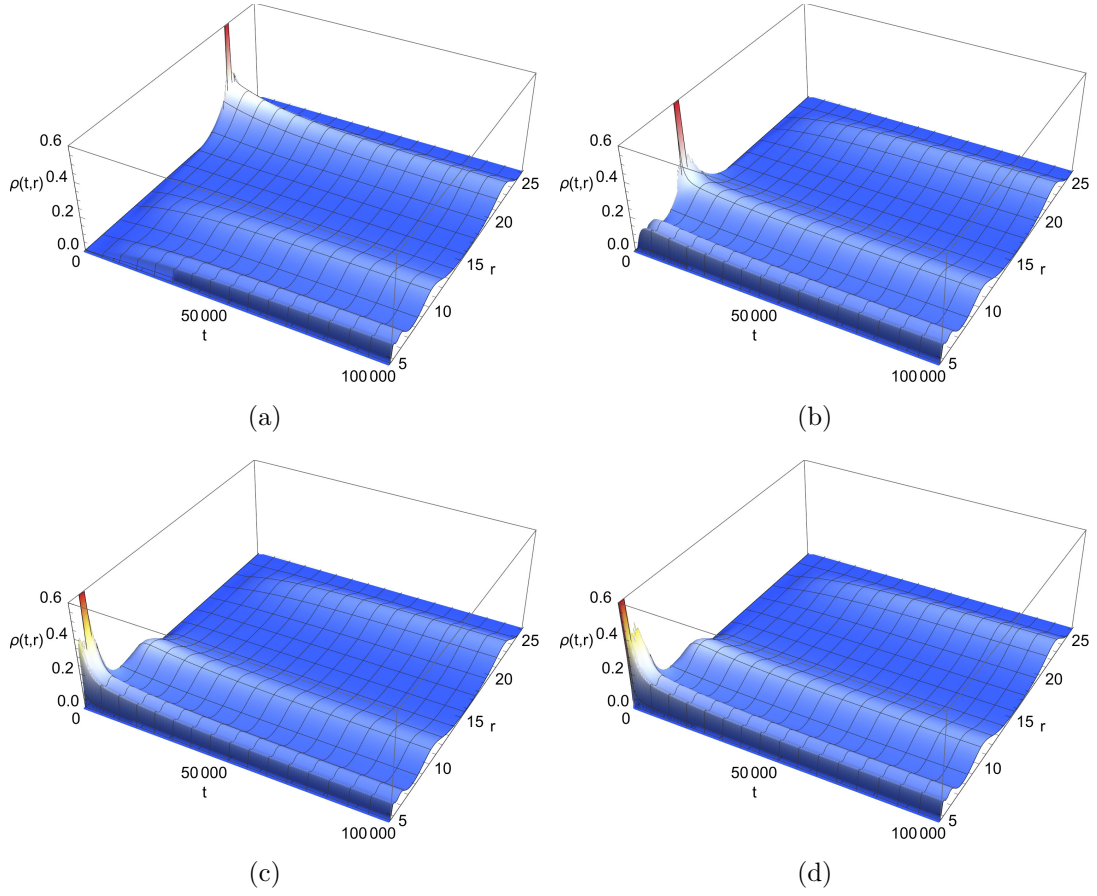


Figure 3.3: Time evolution of the probability distribution function $\rho(t, r)$ at quadruple point ensemble temperature $T_E = 0.00631913$. The initial Gaussian wave packet is set to be peaked at the coexistent (a) largest (b) large (c) small and (d) smallest black hole states.

where σ is related to the standard deviation of the distribution. In practice, we numerically set $\sigma = 0.2$ and initialize the system to be peaked at one of the four black hole states: either with the smallest horizon radius r_{ss} , small horizon radius r_s , large horizon radius r_l , or largest horizon radius r_{ll} . Reflective boundary conditions are imposed at $r = 0$ and $r = \infty$, where extremely high potential barriers are present, to maintain the conservation of total probability over time.

$$e^{-\beta G_L(r)} \frac{\partial}{\partial r} (e^{\beta G_L(r)} \rho(t, r)) \Big|_{r=r_{bdy}} = 0. \quad (3.14)$$

To solve the Smoluchowski equation numerically, we select the left and right boundaries at $r = 1$ and $r = 30$. The final stationary probability distributions, $\rho(t, r)$ should not vary with time, ensuring the probability current vanishes. Therefore, the final probability distributions $\rho(r) \propto e^{-\beta G_L(r)}$ must depend on the off-shell Gibbs free energy structure, which is determined by the ensemble temperature. Structure of the off-shell Gibbs free energy G_L for three distinct values of the ensemble temperature T_E is illustrated in figure 3.2. As shown in figure 3.2(b), at the intermediate ensemble temperature which corresponds to the quadruple point, the four local minima representing four stable black hole phases share the same Gibbs free energy value $G_L = 5.50326$. For a slightly smaller value of T_E , shown in figure 3.2(a), the four minima Gibbs free energy values are no longer degenerate, but instead increase from the smallest black hole phase with horizon radius r_{ss} to the largest black hole phase with r_{ll} . From figure 3.2(c), we observe that for a slightly larger value of T_E the minima value decrease from horizon radius r_{ss} to r_{ll} .

3.3.1 Quadruple point ensemble temperature

Let us first investigate the dynamics of black hole phase transitions at the quadruple point temperature $T_E = 0.00631913$. In figure 3.3, we depict the behaviour of the probability distribution function ρ as a function of possible horizon radii r and time t by numerically solving the Smoluchowski equation for initial states peaked at each of the horizon radii $r_i = r_{ll}, r_l, r_s$, and r_{ss} . In figure 3.4 we plot for each of these successive cases the probability evolution of $\rho(t)$ by fixing the horizon radius r at each of the values $r_i = r_{ss}, r_s, r_l$, and r_{ll} .

Consider the first scenario depicted in figure 3.3(a) and figure 3.4(a). Initially, the probability is highly concentrated in the black hole state with the largest horizon radius,

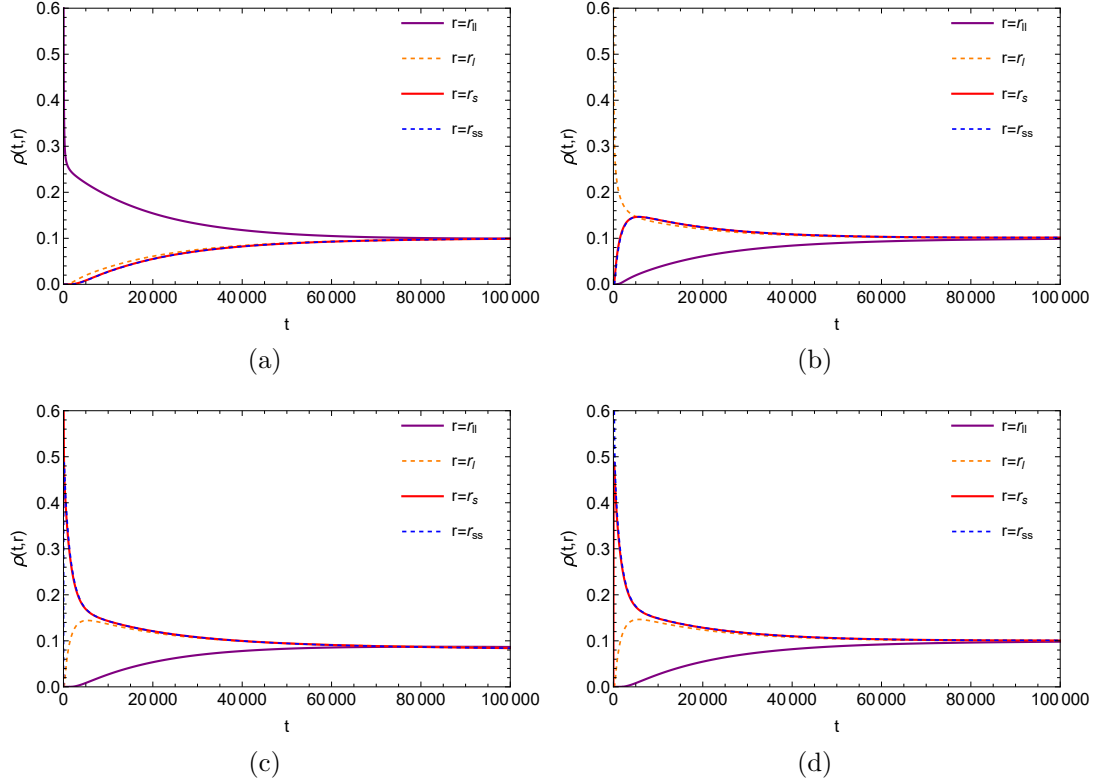


Figure 3.4: Behaviours of $\rho(t, r)$ for four stable black hole phases at quadruple point ensemble temperature $T_E = 0.00631913$. The solid purple curve represents the largest black hole phase, the dashed orange curve corresponds to the large phase, the solid red curve represents the small phase, and the dashed blue curve corresponds to the smallest phase. The initial Gaussian wave packet is set to be peaked at the coexistent (a) largest (b) large (c) small and (d) smallest black hole states.

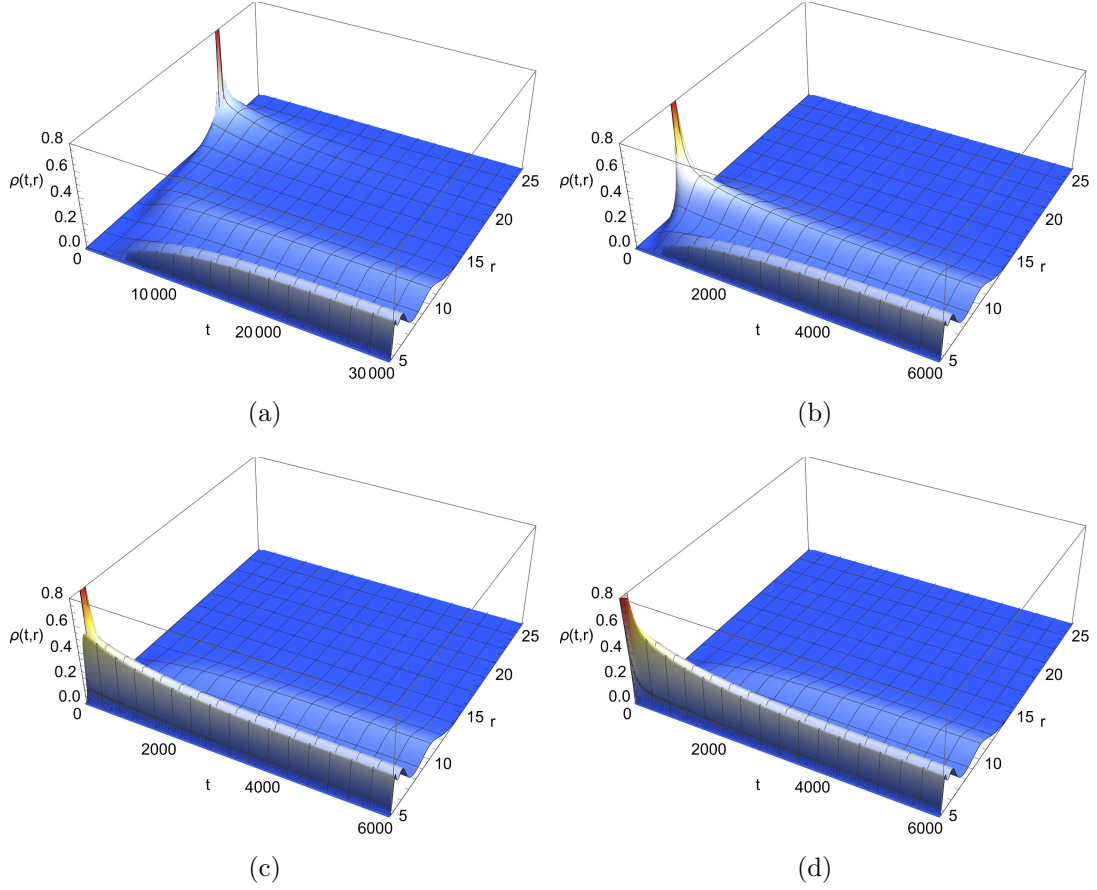


Figure 3.5: Time evolution of the probability distribution function $\rho(t, r)$ at low ensemble temperature $T_E = 0.006298$. The initial Gaussian wave packet is set to be peaked at the coexistent (a) largest (b) large (c) small and (d) smallest black hole states.

meaning $\rho(0, r_l) \gg \rho(0, r_l) \approx \rho(0, r_s) \approx \rho(0, r_{ss}) \approx 0$. As time increases, the probability associated with the largest phase with radius r_l decreases dramatically, whereas the probabilities for the other three phases steadily rise. We also observe that during this transition, the probabilities follow the order $\rho(t, r_l) > \rho(t, r_s) > \rho(t, r_{ss})$ with $\rho(t, r_s) \approx \rho(t, r_{ss})$. This pattern arises because the off-Shell Gibbs free energy, acting as a potential, presents more barriers between the largest black hole phase and the two smallest black hole phases than between the largest phase and the large phase. The phenomenon that $\rho(t, r_s)$ and $\rho(t, r_{ss})$ have close probability arises from the relatively small barrier width (defined as the distance between a minimum and the adjacent maximum) and height between these two black hole

states. Eventually, after a sufficiently long time, say $t > 100000$, the four states reach equilibrium, forming a multicritical state where each state shares the same probability, $\rho(r_k, t) \approx 0.1$ for $k = ll, l, s$, and ss , due to the degenerate local minima value in the Gibbs free energy diagram, as shown in figure 3.2(b).

Consider the next scenario shown in figure 3.3(b) and 3.4(b), where the initial wave packet is located at the large black hole state with horizon radius r_l . In the early stages, we have $\rho(0, r_l) \gg \rho(0, r_{ll}) \approx \rho(0, r_s) \approx \rho(0, r_{ss}) \approx 0$. At early times, the probability of the phase with large radius declines rapidly, while the probabilities $\rho(t, r_{ll}), \rho(t, r_s)$, and $\rho(t, r_{ss})$ increase. Notably, the probability leakage to the small and smallest black hole phases grows much faster than that to the largest phase. This is attributed to the relatively low potential barrier heights and narrow widths between the large state and the two smallest states. The probabilities $\rho(t, r_s)$, and $\rho(t, r_{ss})$ quickly reach their peak values, even slightly exceeding $\rho(t, r_l)$, the probability of the given initial state. This indicates a shift in the dominant state, in this case from the large black hole phase to the small and smallest phases. This phenomenon, known as strong oscillatory behaviour, is also observed in the black hole phase transition at the triple point [72]. Ultimately, the four probabilities stabilize and equilibrate to nearly the same value, as expected due to the shape of off-shell Gibbs free energy at quadruple point temperature.

Let us consider the next two scenarios. If the initial wave packet peaks at $r = r_s$, as shown in figure 3.3(c) and figure 3.4(c), the probability diffuses from $\rho(t, r_s)$ to $\rho(t, r_{ss})$ very quickly at early times due to the extremely small potential barrier between the small and smallest black hole states. And the probability functions for the small and smallest black hole phases almost overlap after a short time. For the evolution of the large black hole, the probability $\rho(t, r_l)$ initially increases to a maximum value smaller than those of $\rho(t, r_s)$ and $\rho(t, r_{ss})$, and subsequently decreases to a stationary value. This phenomenon is referred to as weak oscillatory behaviour [72]. During the phase transition process, $\rho(t, r_{ll})$ for the largest black hole state has the lowest value and grows the slowest. This is due to the largest phase being farthest from the initial state. Finally, in figure 3.4(d), when the distribution initially peaks at the smallest phase, the behaviour of all four probability curves resembles those of figure 3.4(c), the only distinction being that the roles $\rho(t, r_s)$ and $\rho(t, r_{ss})$ play are swapped.

3.3.2 Low and high ensemble temperatures

At an ensemble temperature $T_E = 0.006298$ slightly below the quadruple point temperature, the Gibbs free energy for the four stable black hole phases shows the ordering

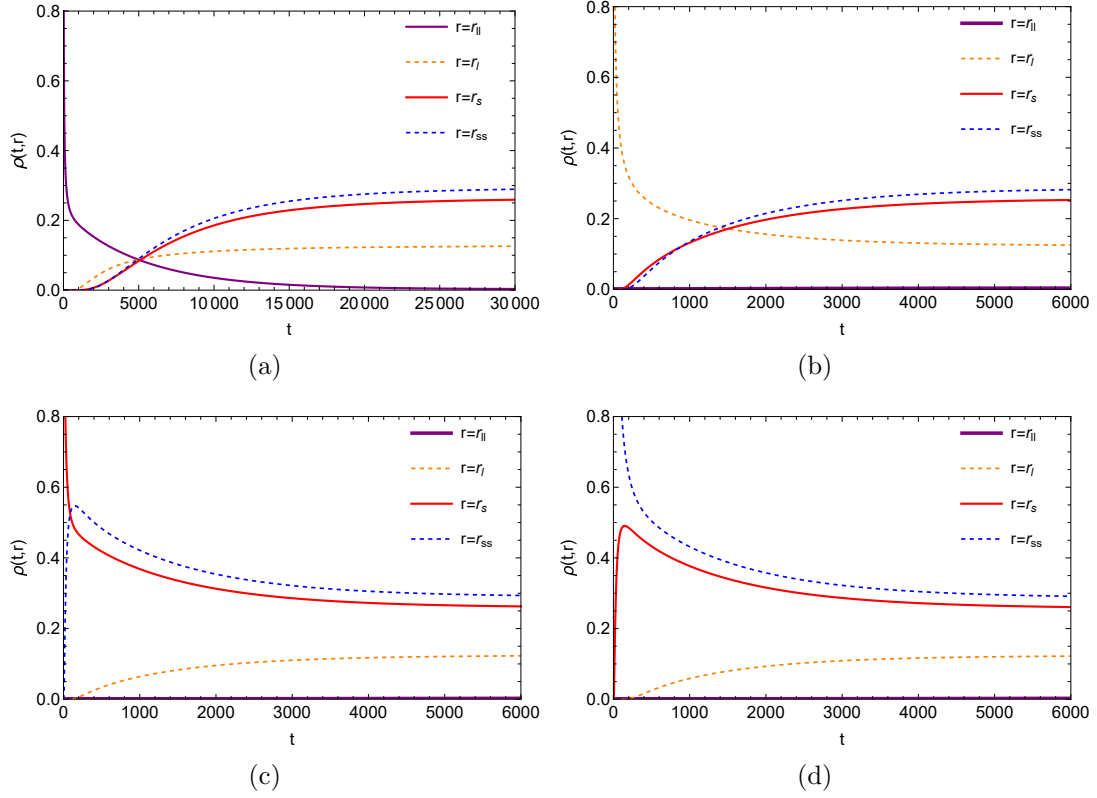


Figure 3.6: Behaviours of $\rho(t, r)$ for four stable black hole phases at low ensemble temperature $T_E = 0.006298$. The solid purple curve represents the largest black hole phase, the dashed orange curve corresponds to the large phase, the solid red curve represents the small phase, and the dashed blue curve corresponds to the smallest phase. The initial Gaussian wave packet is initialized to peak at the coexistent (a) largest (b) large (c) small and (d) smallest black hole states.

$G_L(r_{ss}) < G_L(r_s) < G_L(r_l) < G_L(r_u)$, as shown in figure 3.2(a). Therefore, as depicted in figure 3.5 and figure 3.6, the smallest black hole phase is expected to be the ultimate dominant state regardless of the given initial conditions, with the final stationary state probability distributions being $\rho(t, r_{ss}) > \rho(t, r_s) > \rho(t, r_l) > \rho(t, r_u)$. When the smallest phase is the initial state, as seen in figure 3.6(d), weak oscillatory behaviour is exhibited. In contrast, strong oscillatory phenomena are observed when the initial state is peaked at one of the three radii r_u , r_l , and r_s , as respectively shown in figure 3.6(a), figure 3.6(b), and figure 3.6(c). The final probability for the largest black hole state is extremely small

compared to the other three states because its highest Gibbs free energy value makes it less thermodynamically favorable. The final probability $\rho(t, r_{ss})$ and $\rho(t, r_s)$ are relatively similar due to the smallest barrier size between these two phases. More interesting structures can be viewed with more discernment. For instance, as depicted in figure 3.6(a), the probability of the large black hole phase increases more rapidly than that of the small and smallest black hole phases initially. This is because the large black hole phase is closer to the initial state peaked at r_{ll} . However, the lower Gibbs free energy of phases at $r = r_s$ and $r = r_{ss}$ reverses this trend after $t \approx 5000$.

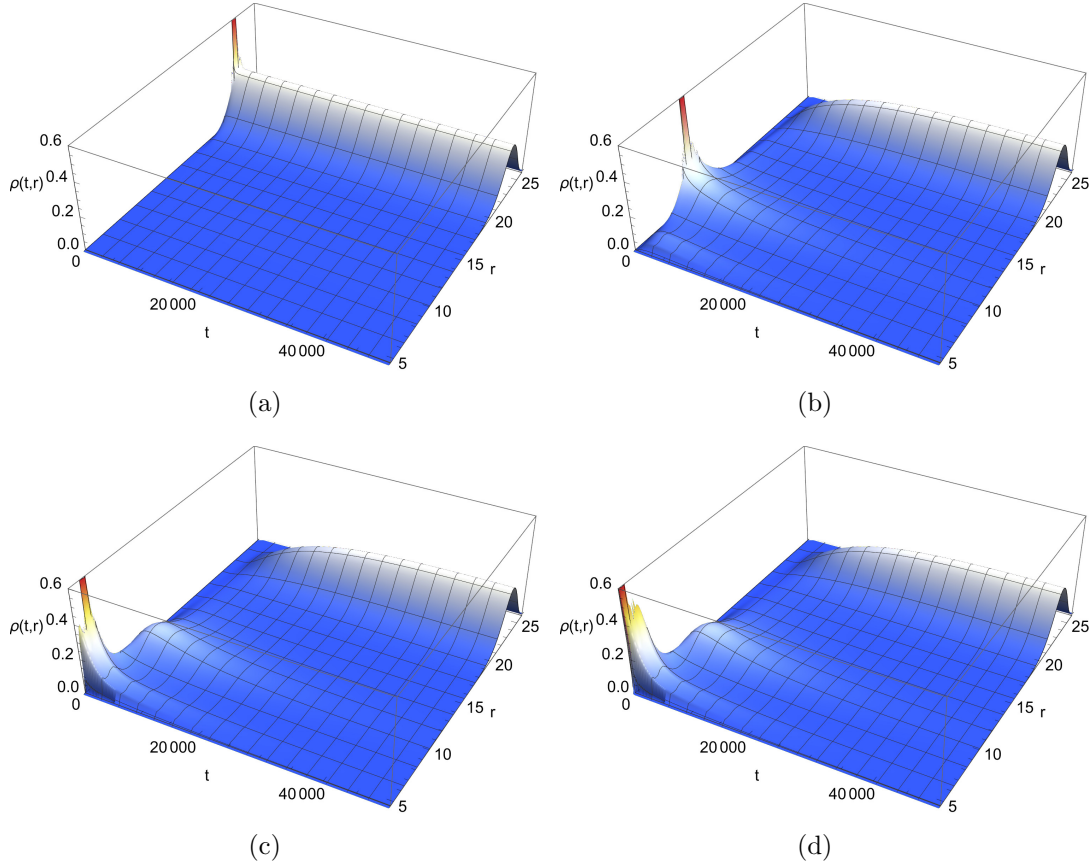


Figure 3.7: Time evolution of the probability distribution function $\rho(t, r)$ at high ensemble temperature $T_E = 0.0063422$. The initial Gaussian wave packet is initialized to peak at the coexistent (a) largest (b) large (c) small and (d) smallest black hole states.

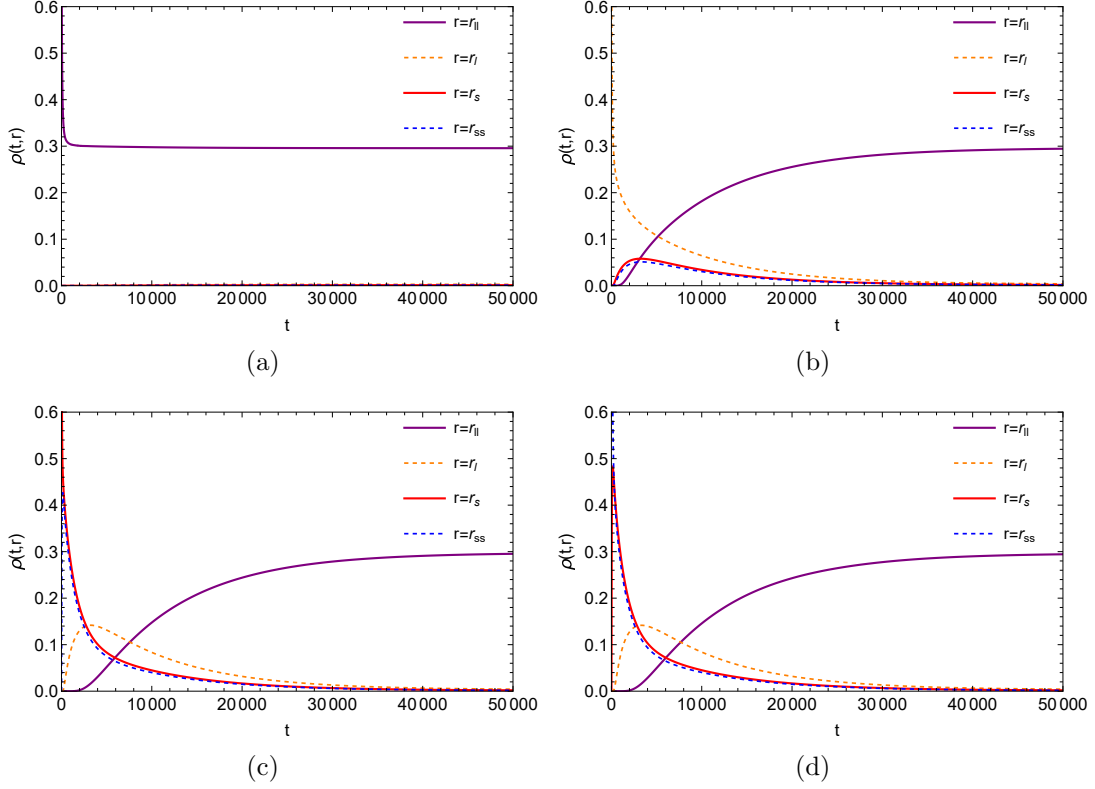


Figure 3.8: Behaviours of $\rho(t, r)$ for four stable black hole phases at the high ensemble temperature $T_E = 0.0063422$. The solid purple curve represents the largest black hole phase, the dashed orange curve corresponds to the large phase, the solid red curve represents the small phase, and the dashed blue curve corresponds to the smallest phase. The initial Gaussian wave packet is set to be peaked at the coexistent (a) largest (b) large (c) small and (d) smallest black hole states.

For a higher ensemble temperature, say $T_E = 0.0063422$, shown in figure 3.2(c), note that in the off-shell Gibbs free energy diagram $G_L(r_{ss}) > G_L(r_s) > G_L(r_l) > G_L(r_{ll})$, and the barrier height and width to transit from the largest black hole phase with radius r_{ll} to the other three phases are significantly large. Thus, the largest black hole phase ultimately becomes dominant, as illustrated in figure 3.7, regardless of the initial state. The final probability of the largest phase is much greater than that of the other three phases, as indicated by the final stationary state probability distribution $\rho(t, r_{ll}) \gg \rho(t, r_l) > \rho(t, r_s) > \rho(t, r_{ss})$ in figure 3.8. That is why strong oscillatory phenomena exist in figure 3.8(b), figure 3.8(c), and figure 3.8(d). Upon closer examination of figure 3.8(b), initially the probability of the

large black hole $\rho(t, r_l)$ diffuses to the other three states and $\rho(t, r_s) \gtrsim \rho(t, r_{ss}) > \rho(t, r_{ll})$ because the barrier widths amongst r_s, r_{ss} and r_l are much smaller than that between r_{ll} and r_l . Afterward $\rho(t, r_{ss}) \approx \rho(t, r_s) < \rho(t, r_{ll})$ due to the lowest Gibbs energy value at $r = r_{ll}$. Similar phenomena can be observed in figure 3.8(c), and figure 3.8(d). Therefore, in the initial stages, barrier widths and heights significantly influence the evolution process, while in the later stages, the value of the Gibbs free energy becomes pivotal.

3.4 The first passage events at different ensemble temperatures

In this section, we explore the underlying physics of black hole phase transition behaviours near quadruple points by examining first passage events. On the free energy landscape, the first passage time for a given initial phase refers to the time required to first ascend to the peaks of neighboring Gibbs free energy barriers corresponding to unstable phases. Reflecting boundary conditions are enforced at $r = 0$ and $r = \infty$. In order to model the first passage event, absorbing boundary conditions are applied at r_{m1}, r_{m2} , and r_{m3} for each unstable phase,

$$\rho(t, r_{m1}) = \rho(t, r_{m2}) = \rho(t, r_{m3}) = 0. \quad (3.15)$$

Let us focus on the intermediate ensemble temperature shown in figure 3.3, the first passage time distributions as functions of time are denoted by $F_{p1}(t)$, $F_{p2}(t)$, $F_{p3}(t)$ and $F_{p4}(t)$ for the four initial cases $r_i = r_{ss}, r_s, r_l$, and r_{ll} , respectively. Applying the Smoluchowski equation with boundary conditions, the first passage time distributions are given by

$$F_{p1}(t) = -\frac{\partial \rho(t, r_{m1})}{\partial r}, \quad (3.16)$$

$$F_{p2}(t) = \frac{\partial \rho(t, r_{m1})}{\partial r} - \frac{\partial \rho(t, r_{m2})}{\partial r}, \quad (3.17)$$

$$F_{p3}(t) = \frac{\partial \rho(t, r_{m2})}{\partial r} - \frac{\partial \rho(t, r_{m3})}{\partial r}, \quad (3.18)$$

$$F_{p4}(t) = \frac{\partial \rho(t, r_{m3})}{\partial r}. \quad (3.19)$$

The first passage time distributions at three different ensemble temperatures are displayed in figure 3.9, figure 3.10, and figure 3.11. Each first passage time distribution curve,

$F_p(t)$, exhibits a peak shortly after the start, followed by a rapid decay. This suggests a high probability that the first passage event happens within a very short time, providing insight into the early-time behaviour of $\rho(t, r)$.

In Table 3.1, the height and width of each Gibbs free energy barrier are listed at the three different ensemble temperatures. For each temperature, the extreme point t of the first passage time distribution is determined for each initial case. These values are presented in Table 3.2. The differences among the various barrier widths are larger than the differences among the barrier heights, indicating that barrier widths are more crucial in determining the black hole phase transition behaviour. Therefore, we use the width of the barrier W to label the size of the barrier in the following context. From Table 3.1 and Table 3.2, we conclude that the time corresponding to the peak of the first passage time distribution t is highly correlated with the Gibbs free energy potential barrier size. For instance, let us first investigate the cases at intermediate temperature T_I and high temperature T_H . From t_{ll3} to t_{ss1} , the time t that corresponds to first passage time peak decreases, i.e., $t_{ll3} > t_{l3} > t_{l2} > t_{s2} > t_{s1} > t_{ss1}$, and we have the barrier widths ordering $W_{ll3} > W_{l3} > W_{l2} > W_{s2} > W_{s1} > W_{ss1}$. This is consistent with the fact that it is easier to surmount a potential barrier with smaller width. Likewise at low temperature T_L , we have $t_{l3} > t_{ll3} > t_{l2} > t_{s2} > t_{s1} > t_{ss1}$, with corresponding barrier widths $W_{l3} > W_{ll3} > W_{l2} > W_{s2} > W_{s1} > W_{ss1}$. The interchange of the longest first passage time is accompanied by an interchange of the widest barrier.

To explain the dynamic behaviour of multicritical black hole phase transitions at early times, we assume that the time needed for the probability to diffuse from the barrier top to the adjacent local minimum is significantly shorter than the first passage time to climb to the barrier top from the initial local stable state [58]. Thus we obtain the first passage time scales from each initial black hole state to other stable states, indicating the difficulty of transitioning between stable phases, as demonstrated in Table 3.3. Let us focus on the quadruple point ensemble temperature $T_E = 0.00631913$. First, for the initial largest black hole state with horizon radius r_{ll} , we have $t(ll \rightarrow l) < t(ll \rightarrow s) \lesssim t(ll \rightarrow ss)$, i.e., $1593 < 1817 \lesssim 1827$. This indicates that the probability more readily transitions to the state with radius r_l . This is why in figure 3.4(a), we observe the phenomenon that $\rho(t, r_l) > \rho(t, r_s) \gtrsim \rho(t, r_{ss})$ during the early stages. For a large initial black hole state with radius r_l , we have the early time behaviour $\rho(t, r_s) \gtrsim \rho(t, r_{ss}) > \rho(t, r_{ll})$ shown in figure 3.4(b), which we can understand from the phase transition time scale relations $t(l \rightarrow s) \lesssim t(l \rightarrow ss) < t(l \rightarrow ll)$. For a small initial black hole state, shown in figure 3.4(c), we observe that $t(s \rightarrow ss) < t(s \rightarrow l) < t(s \rightarrow ll)$, explaining why $\rho(t, r_{ss})$ initially grows the fastest, and $\rho(t, r_{ll})$ the slowest, with $\rho(t, r_{ss}) > \rho(t, r_l) > \rho(t, r_{ll})$. Similarly, for the smallest initial black hole state, shown in figure 3.4(d), the reason why $\rho(t, r_s) > \rho(t, r_l) > \rho(t, r_{ll})$ at early

times is that the transition times follow the order $t(ss \rightarrow s) < t(ss \rightarrow l) < t(ss \rightarrow ll)$.

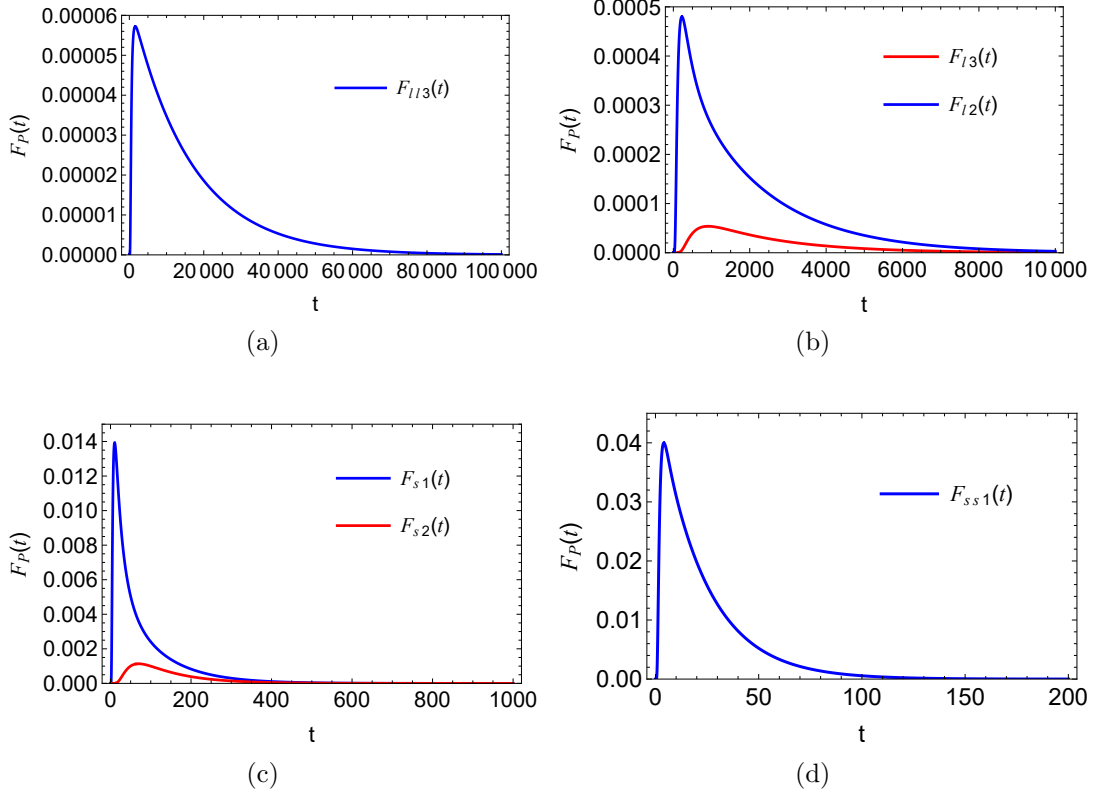


Figure 3.9: First passage time distribution at intermediate (quadruple point) ensemble temperature $T_E = 0.00631913$ for initial (a)largest, (b) large, (c) small, (d) smallest black hole phases respectively.

Let us explore the case of the low ensemble temperature $T_E = 0.006298$, with its Gibbs free energy structure illustrated in figure 3.2(a). When starting with the initial black hole phase peaked at the largest horizon radius r_{ll} , as shown in figure 3.6(a), the probability is more likely to leak to the black hole phase with radius r_l at early times. This is evidenced by the fact that $\rho(t, r_l) > \rho(t, r_s) \gtrsim \rho(t, r_{ss})$, which is consistent with the relation $t(ll \rightarrow l) < t(ll \rightarrow s) \lesssim t(ll \rightarrow ss)$. However, the situation reverses at later times: as observed in figure 3.6(a), $\rho(t, r_{ll}) < \rho(t, r_l) < \rho(t, r_s) < \rho(t, r_{ss})$. This is because the Gibbs free energy for the four stable states are ordered as $G_L(r_{ss}) < G_L(r_s) < G_L(r_l) < G_L(r_{ll})$.

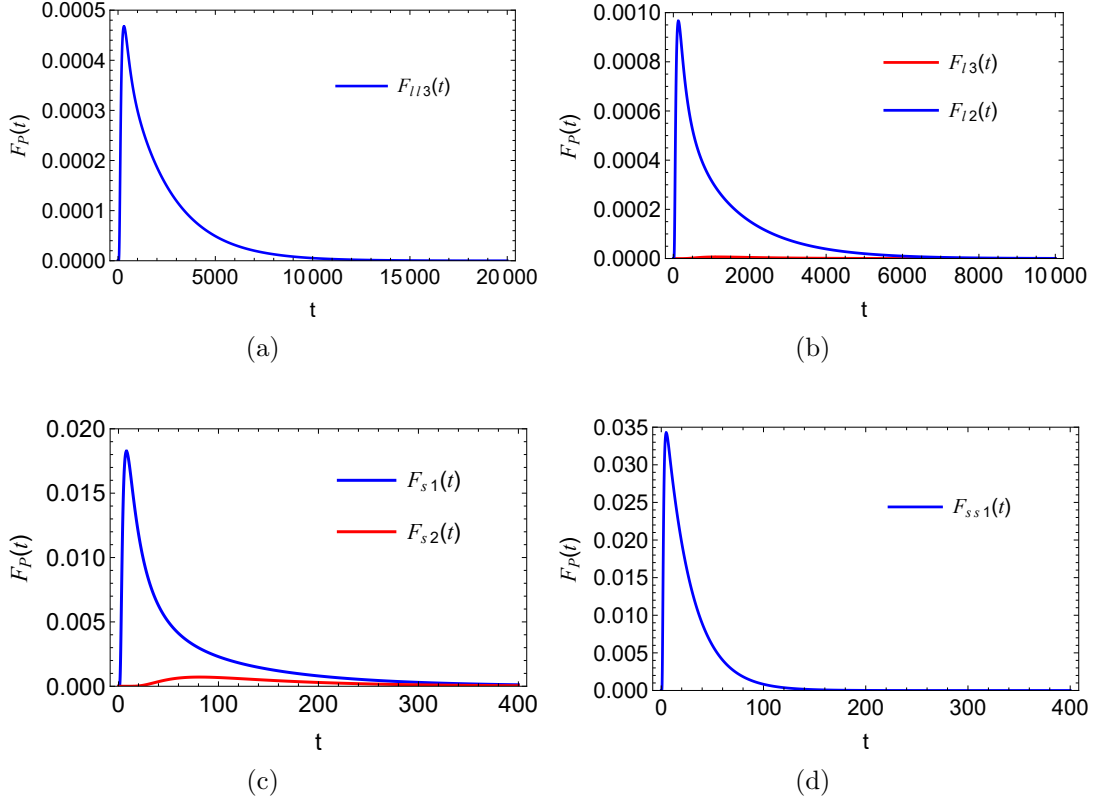


Figure 3.10: First passage time distribution at low ensemble temperature $T_E = 0.006298$ for initial (a)largest, (b) large, (c) small, (d) smallest black hole phases respectively.

at low ensemble temperature, and the final state probability distributions $\rho(r) \propto e^{-\beta G_L(r)}$, meaning states with lower Gibbs free energy have a higher probability of being occupied. The different orderings of the probability for the four stable states in the early and late stages result in the intersection points of these curves in figure 3.6(a), demonstrating what we call strong oscillatory behaviour. For the large initial black hole phase with radius r_l , figure 3.6(b) shows that $\rho(t, r_s) \gtrsim \rho(t, r_{ss}) \gg \rho(t, r_u)$ at early times, due to the phase transition time scale relationship $t(l \rightarrow s) \lesssim t(l \rightarrow ss) < t(l \rightarrow ll)$. Figure 3.6(c) illustrates the probability distribution function evolution for the initial small black hole case. The function $\rho(t, r)$ exhibits the most rapid growth at r_{ss} and the slowest growth rate at r_u during the early stages of evolution. Specifically, we observe $\rho(t, r_{ss}) > \rho(t, r_l) > \rho(t, r_u)$, which can be attributed to the transition time scales ordering $t(s \rightarrow ss) < t(s \rightarrow l) < t(s \rightarrow ll)$. Likewise, for the smallest initial black hole state depicted in figure 3.6(d), we find that

	H_{ll3}	H_{l3}	H_{l2}	H_{s2}	H_{s1}	H_{ss1}
T_L	0.0024	0.027	0.0036	0.0082	0.0011	0.0018
T_I	0.015	0.015	0.0066	0.0066	0.0016	0.0016
T_H	0.035	0.0055	0.011	0.0050	0.0021	0.0013
	W_{ll3}	W_{l3}	W_{l2}	W_{s2}	W_{s1}	W_{ss1}
T_L	3.3	7.7	2.2	1.9	0.54	0.40
T_I	6.1	5.8	2.8	1.7	0.62	0.37
T_H	8.3	3.9	3.5	1.5	0.69	0.34

Table 3.1: The sizes of the potential barriers in the off-shell Gibbs free energy under different ensemble temperatures. Barrier height H and barrier width W from initial largest ($r = r_{ll}$), large ($r = r_l$), small ($r = r_s$) and smallest ($r = r_{ss}$) black hole states to the adjacent top of the barrier $r = r_{m1}$, $r = r_{m2}$, and $r = r_{m3}$.

	t_{ll3}	t_{l3}	t_{l2}	t_{s2}	t_{s1}	t_{ss1}
T_L	307	1160	132	81	8	5
T_I	1593	912	224	70	10	4
T_H	3309	466	363	58	13	3

Table 3.2: Time corresponds to the peak in the first passage time distribution under different ensemble temperatures, starting from the initial largest ($r = r_{ll}$), large ($r = r_l$), small ($r = r_s$) and smallest ($r = r_{ss}$) black hole states to the adjacent tops of the barriers $r = r_{m1}$, $r = r_{m2}$, and $r = r_{m3}$.

$\rho(t, r_s)$ grows the fastest at early times, followed by $\rho(t, r_l)$ and $\rho(t, r_{ll})$, consistent with $t(ss \rightarrow s) < t(ss \rightarrow l) < t(ss \rightarrow ll)$. And for the final states in figure 3.6(b), figure 3.6(c), and figure 3.6(d), we observe the same phenomenon as figure 3.6(a), $\rho(t, r_{ll}) < \rho(t, r_l) < \rho(t, r_s) < \rho(t, r_{ss})$, which is determined only by the structure of the off-shell Gibbs free energy in figure 3.2(a) and is independent of the initial probability distribution. Similarly, at a higher ensemble temperature $T_E = 0.0063422$, the early time dynamics shown in figure 3.8 can also be well explained utilizing the first passage time presented in Table 3.3, while the late time behaviour is governed by the off-shell Gibbs free energy, as shown in figure 3.2(c).

We now investigate how ensemble temperatures influence first passage events. According to our findings in Table 3.1 and Table 3.2, as the ensemble temperature goes up, the barrier widths W_{ll3}, W_{l2}, W_{s1} become wider, resulting in an increase for the respective first passage times t_{ll3}, t_{l2}, t_{s1} . Conversely, an increase in temperature results in

	$t(ll \rightarrow l)$	$t(ll \rightarrow s)$	$t(ll \rightarrow ss)$
T_L	307	439	447
T_I	1593	1817	1827
T_H	3309	3672	3685
	$t(l \rightarrow s)$	$t(l \rightarrow ss)$	$t(l \rightarrow ll)$
T_L	132	140	1160
T_I	224	234	912
T_H	363	376	466
	$t(s \rightarrow ss)$	$t(s \rightarrow l)$	$t(s \rightarrow ll)$
T_L	8	81	1241
T_I	10	70	982
T_H	13	58	524
	$t(ss \rightarrow s)$	$t(ss \rightarrow l)$	$t(ss \rightarrow ll)$
T_L	5	86	1246
T_I	4	74	986
T_H	3	61	527

Table 3.3: The first passage time scale from the initial largest ($r = r_u$), large ($r = r_l$), small ($r = r_s$) and smallest ($r = r_{ss}$) black hole phases to other stable phases under different ensemble temperatures.

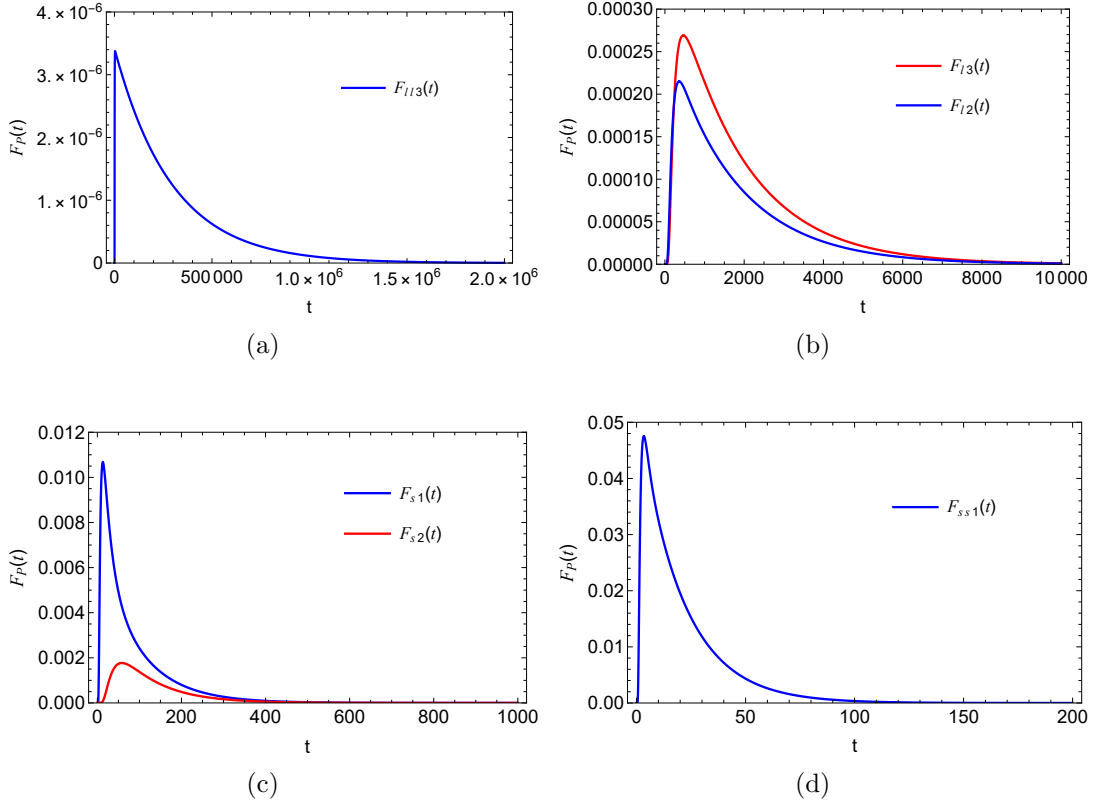


Figure 3.11: First passage time distribution at high ensemble temperature $T_E = 0.0063422$ for initial (a)largest, (b) large, (c) small, (d) smallest black hole phases respectively.

smaller W_{l3}, W_{s2}, W_{ss1} , causing t_{l3}, t_{s2}, t_{ss1} to decrease. This further supports the existence of a positive correlation between the peak time of the first passage time distribution and barrier width. Additionally, figure 3.2 shows that a change in temperature does not significantly alter the size of the barrier between the smallest and small phases, whereas the barrier between the large and largest phases undergoes a substantial change. Now let us consider the cases where the initial state is centered on the largest black hole states at various ensemble temperatures. The first passage times to reach the state with radius r_l from r_u is represented by F_{ll3} . As illustrated in figure 3.2, with rising ensemble temperature, both the barrier height and width for the initial largest black hole increase. By comparing figure 3.10(a), figure 3.9(a), and figure 3.11(a), we conclude that with increasing temperature, the peak of the first passage time distributions moves rightward and becomes lower, signifying the increased difficulty in surmounting higher and wider barriers. The

first passage times, calculated as $t_{l3} = 307, 1593$, and 3309 for low, intermediate, and high ensemble temperatures T_E , respectively, indicate that a significantly longer time is required to transition from the largest black hole state to other states at high temperatures. The extremely low value of F_{l3} and long first passage time at high temperature, as depicted in figure 3.11(a), indicate that it is almost impossible to transition to other states from largest black hole, explaining the final predominance of the largest black hole phase at high temperatures shown in figure 3.8.

3.5 Summary

Exploring black hole phase transitions provides valuable insights into the thermodynamic behaviour and microscopic structure of black holes. In this study, we conducted the first investigation into the dynamic behaviours of black hole multicritical phase transitions, specifically focusing on black hole quadruple points in Einstein gravity coupled with non-linear electrodynamics. A quadruple point in the context of black hole chemistry is a point in the phase diagram where four distinct black hole phases coexist. The quadruple point is characterized by three pairs of local maxima and minima on a T - r_+ diagram, with all inflection points occurring at the same temperature T , the temperature of multicriticality. On the free energy landscape, with the off-shell Gibbs free energy G_L as the potential and the horizon radius r_+ as the order parameter, there are four potential wells at the quadruple points, representing four coexisting black hole phases.

To investigate the dynamics of black hole multicritical phase transitions, we initialize the system in one of the four stable black hole phases and use the Smoluchowski equation to determine the evolution of the probability distribution function. Our results show that the probability density of the initial state diffuses to the other three states, with the diffusion rate affected by the first passage time distribution. This distribution is, in turn, controlled by the size of the potential barriers between states in the off-shell Gibbs free energy diagram. At late times, the probability distribution stabilizes, and the probabilities of the four stable black hole phases are determined by the values of the off-shell Gibbs free energy. The first passage time distribution exhibits a characteristic pattern: it quickly rises to a peak within a short time and then sharply declines. This indicates a high likelihood of the first passage event occurring within a short time period. The shorter the corresponding first passage time, the faster the probability density of the initial state diffusing to an adjacent phase, reflecting a greater tendency for the system to rapidly transition between phases.

The different orderings of $\rho(t, r)$ for the four stable black hole phases at early and late

times result in strong oscillatory phenomena: for example, as shown in figure 3.6(a), at early times we have $\rho(t, r_u) > \rho(t, r_l) > \rho(t, r_s) \gtrsim \rho(t, r_{ss})$, whereas at late times the situation reverses and we obtain $\rho(t, r_u) < \rho(t, r_l) < \rho(t, r_s) < \rho(t, r_{ss})$. The different orderings of the probability for the four stable states in the early and late stages lead to the intersection points of these curves in figure 3.6(a), which is what we call strong oscillatory behaviour. Moreover, even with the same orderings, weak oscillatory phenomena may exist. And the orderings of the probability distribution function $\rho(t, r)$ are governed by the off-shell Gibbs free energy G_L . Thus, the evolution of black hole states near the quadruple point is actually determined by the structure of the off-shell Gibbs free energy diagram. Our findings indicate that temperature plays a significant role in modifying the structure of this diagram, thereby impacting black hole phase transitions.

Our results build on previous work carried out on black hole triple point phase transitions [72], extending our understanding to the more complex scenario of multicritical points. We have focused on investigating the dynamic behaviours of black hole phase transitions near the quadruple point under various ensemble temperatures, demonstrating how temperature influences the off-shell Gibbs free energy and thus affects probability distributions in a multiphase scenario. By analyzing the initial diffusion process and the final stationary behaviours of the four stable black hole states, we also explain the oscillatory behaviours observed during the evolution process. This comprehensive analysis highlights the critical role of temperature in shaping the structure of the off-shell Gibbs free energy, which in turn governs the phase transitions of black holes.

In conclusion, this chapter investigates the intriguing phenomena of black hole phase transitions from the perspective of stochastic dynamics. This study may enhance our understanding of the underlying physics of black hole phase transitions and the microscopic structure of black holes. Furthermore, the free energy landscape model can be applied to examine the dynamic behaviours of quintuple points or higher multicritical points [38, 39, 40], where even more complex behaviours may become manifest. Our findings may pave the way for future research into the thermodynamic properties and phase transitions of black holes in increasingly complex scenarios, thereby contributing to a deeper comprehension of these enigmatic astrophysical objects.

Chapter 4

Holographic complexity of Magnetized AdS solitons

In this chapter, we explore holographic complexity, an alternative approach to understanding quantum gravity. Specifically, we investigate the holographic complexity of conformal field theories (CFTs) compactified on a circle with a Wilson line, which are dual to magnetized solitons in Anti-de Sitter space (AdS). These theories exhibit a confinement-deconfinement phase transition driven by changes in the Wilson line, with the complexity of formation serving as an order parameter. We demonstrate a scaling relation that connects the proposed volume and action complexity to the radius of the circle. Furthermore, we show that a wide range of potential complexity functionals exhibit this scaling behaviour. Therefore, we conjecture that this scaling law broadly applies to the complexity of CFTs compactified on a circle.

4.1 Introduction

In the context of the AdS/CFT correspondence, holographic complexity is the gravity dual of the complexity of a gauge theory. This complexity is a measure of the distance in the Hilbert space between the gauge theory's state and a specified reference state. The topic of holographic complexity has been extensively studied, with foundational work including complexity=volume (CV), complexity=spacetime volume (CV 2.0), and complexity=action (CA) proposals [49, 50, 51, 52, 53, 54]. For comprehensive reviews and additional references, see [75, 76, 77, 78, 79]. Given the variety of distance measures and

potential reference states, multiple formulations of holographic complexity are expected. Studies such as [55, 56] have shown that a broad family of functionals in asymptotically AdS spacetimes exhibit the anticipated complexity behaviour on thermal states. This is called the complexity=anything program.

An important feature of complexity is that its physical significance comes from changes in complexity within the parameter space, rather than its absolute value. In black hole spacetimes, the time derivative of complexity is a key quantity of interest. It is also crucial to understand how complexity varies with other parameters, even in static situations. To achieve this goal, we analyze the holographic complexity of a conformal field theory (CFT) on a circle, considering its dependence on both the radius of the circle and the Wilson line associated with a $U(1)$ gauge field around the circle. The gravitational dual of a compactified CFT with a Wilson line is a magnetized generalization of the AdS soliton, as initially discussed in [80, 81]. Similar to the standard AdS soliton, the periodic direction in these magnetized solitons shrinks at a finite AdS radius [82, 83, 84, 85], which means that, unlike black hole spacetimes, the spacetime does not possess a horizon. The holographic complexity of the standard AdS soliton has been studied by Reynolds and Ross [86]. In our work, we extend their analysis by evaluating the complexity of formation of the magnetized solitons in both AdS_4 and AdS_5 , the difference in complexity between the magnetized soliton and AdS with a periodic boundary [52, 53, 87].

Furthermore, if the Wilson line is below a critical value, the energy of the soliton spacetime is negative compared to AdS spacetime, making it the ground state for these boundary conditions and leading to confinement in the gauge theory. Once the Wilson line surpasses this critical value, the energy of the soliton spacetime becomes positive, resulting in a zero-temperature phase transition from confinement to deconfinement. In this regime, the ground state is described by empty AdS with a periodic boundary direction and a Wilson line. As highlighted in [85], the state with a critical Wilson line value is also supersymmetric. Therefore, tuning the Wilson line also enables us to explore how supersymmetry breaking affects complexity. Since the ground state with a large Wilson line corresponds to periodic AdS, the complexity of formation becomes zero in this deconfined phase. This means that the complexity of formation acts as an order parameter for the confinement-deconfinement phase transition. To our knowledge, this is the first investigation to use complexity as an order parameter in a holographic confining scenario.¹ Employing complexity as an order parameter is significant because it provides a direct and quantitative measure of the phase transition. This approach is novel as it applies complexity in a first-principles holographic framework to analyze confinement and deconfinement phases.

¹though previous studies [88, 89] have explored complexity in phenomenological models of confinement

In section 4.2, we review the magnetized AdS soliton backgrounds, discuss the associated energy density, and examine the confining-deconfining phase transition as the Wilson line Φ is varied. Then we calculate the complexity of formation in section 4.3 using volume complexity proposal, which includes both the complexity of a maximal spatial slice and that of a spacetime region. We also consider complexity of formation as a spacetime volume in section 4.4 and as an action in section 4.5. In section 4.6, we explore how complexity of formation depends on the Wilson line and its role in phase transitions. Additionally, we demonstrate that the density of complexity of formation (per unit volume of the boundary CFT) scales as the inverse $(d - 1)$ st power of the circumference of the boundary circle.² This scaling behaviour is observed across a broad range of formulations for holographic complexity. The scaling for the standard soliton was first found by Reynolds and Ross [86], who observed a consistent decrease in complexity with increasing radius in a lattice fermion model. Our key findings build on this by showing that the scaling factors from the dependence on the Wilson line. Additionally, we provide a proof that this scaling behaviour is universal for any holographic complexity functional that obeys a set of specific assumptions. We summarize our results and propose several conjectures in section 4.7.

4.2 Magnetized AdS solitons and phase transition

In this section, we review the geometry of magnetized AdS_{d+1} solitons and the corresponding physics of dual gauge theory, based largely on the work [85]. We consider a gauge theory on a flat spacetime with one spatial coordinate, ϕ , periodically identified with period $\Delta\phi$. In this setup, the gauge theory can have a $U(1)$ Wilson line around the periodic ϕ direction. There are three regular solutions to the holographic dual gravity theory involving a $U(1)$ gauge field. This is a bosonic subsector of gauged supergravity with 8 supercharges in either 4 or 5 dimensions, so our calculations below are limited to $d = 3, 4$, where d denotes the boundary CFT dimensions.

The first solution is AdS with a periodic direction and a Wilson line in the bulk. For reference, the metric and gauge field are given by

$$ds^2 = \frac{r^2}{l^2} (-dt^2 + d\vec{x}^2 + d\phi^2) + \frac{l^2}{r^2} dr^2, \quad A = -\frac{\Phi}{\Delta\phi} d\phi, \quad (4.1)$$

where l is the AdS length scale, ϕ is the periodic direction, and Φ is the holonomy of the boundary gauge field, which is also related to the net magnetic flux. The other two

²Due to the UV divergences, we calculate the complexity with a UV cutoff. Since the divergence structure is the same for both the AdS soliton and AdS, the complexity of formation remains finite. This scaling behaviour emerges as the cutoff is taken to infinity.

solutions are generalizations of the AdS soliton, and both have the following metric (for $d = 3, 4$)

$$ds^2 = \frac{r^2}{l^2} (-dt^2 + d\vec{x}^2 + f(r)d\phi^2) + \frac{l^2}{r^2 f(r)} dr^2, \quad f(r) \equiv 1 - \frac{\mu l^2}{r^d} - \frac{Q^2 l^2}{r^{2d-2}}, \quad (4.2)$$

and corresponding gauge field

$$A = \sqrt{7-d} Q \left(\frac{1}{r^{d-2}} - \frac{1}{r_0^{d-2}} \right) d\phi, \quad Q \equiv \frac{1}{\sqrt{7-d}} \frac{r_0^{d-2} \Phi}{\Delta\phi}. \quad (4.3)$$

Like the AdS soliton, the bulk spacetimes terminate at r_0 , the largest root of $f(r)$. To make the generalized AdS soliton geometries regular at r_0 , the ϕ coordinate must have the period

$$\Delta\phi = \frac{4\pi l^2}{r_0^2 f'(r_0)}, \quad (4.4)$$

note that ϕ has units of length, so $\Delta\phi$ represents a kind of circumference. For both values of d we consider, there are two solutions to $f(r_0) = 0$ and equation (4.4)

$$\mu = \frac{r_0^{2d-2} - Q^2 l^2}{l^2 r_0^{d-2}}, \quad r_0 = \frac{2\pi l^2}{d\Delta\phi} \left(1 \pm \sqrt{1 - \frac{\Phi^2}{\Phi_{max}^2}} \right), \quad (4.5)$$

where $\Phi_{max} = g(d)\pi l$, and $g(d) \equiv \sqrt{(46-10d)/d}$. The shorter soliton, which is defined as the solution with the larger r_0 , recovers the AdS soliton solution when setting $Q = 0$, whereas the longer soliton with the smaller r_0 goes to the periodic AdS solution in the vanishing Q limit. For completeness, we note that the two magnetic solitons have field strength

$$F = -\sqrt{7-d}(d-2) \frac{Q}{r^{d-1}} dr \wedge d\phi. \quad (4.6)$$

With the usual boundary conditions of fixed geometry and gauge field, the variables of the CFT are the periodicity $\Delta\phi$ and the Wilson line Φ .

There are two approaches to determine the energy density of each solution and thereby identify the ground state of the CFT for a given periodicity and Wilson line. The first approach is holographic renormalization [90, 91, 92], and the second approach is the extrinsic curvature method [93]. We will first discuss the holographic renormalization method. Here, we perform a Fefferman-Graham expansion [90] of the asymptotic form of the bulk metric as follows:

$$ds^2 = l^2 \frac{dz^2}{z^2} + \frac{l^2}{z^2} \left(\gamma_{ij}^{(0)} + \sum_{n=d}^{\infty} z^n \gamma_{ij}^{(n)} \right) dx^i dx^j; \quad (4.7)$$

with $x^i \equiv [t, \vec{x}, \phi]$ in our case, and the conformal boundary is located at $z = 0$. In the expansion, $\gamma_{ij}^{(0)}$ is related to the boundary CFT background metric, and $\gamma_{ij}^{(d)}$ is used to compute the expectation value of the CFT energy-momentum tensor. In general, the sum can include terms with $1 \leq n < d$, as well as terms proportional to $z^n \ln(z)$ for $n \geq d$ when d is even. However, these latter terms vanish for the solitons we consider. Apparently

$$\int \frac{dz}{z} = - \int \frac{dr}{r \sqrt{f(r)}} \quad \Rightarrow \quad \ln(z/l) = -\ln(r/l) + \frac{\mu l^2}{2dr^d} + \cdots \quad (4.8)$$

using a large radius expansion. Solving iteratively, we obtain

$$r = \frac{l^2}{z} \left(1 + \frac{\mu l^2}{2d} \frac{z^d}{l^{2d}} + \cdots \right). \quad (4.9)$$

In holographic renormalization, as discussed in references such as [91, 92], the expectation value of boundary stress-energy tensor is

$$\langle T_{ij} \rangle = \frac{dl^{d-1} \gamma_{ij}^{(d)}}{16\pi G} + \cdots, \quad (4.10)$$

where G denotes the $(d+1)$ -dimensional Newton constant in the AdS spacetime and the dots are terms that vanish on the soliton solutions. Therefore, the energy density is

$$\langle T_{tt} \rangle = -\frac{\mu}{16\pi G l^{d-1}}. \quad (4.11)$$

An alternative approach to computing the energy density of the asymptotically AdS spacetime is to compare the extrinsic curvature of a boundary surface at a cutoff radius r_m with that of a surface with the same geometry in pure AdS, as described in the paper by Hawking and Horowitz [93]. Specifically, we choose a surface at fixed time t and large r . The energy of spacetime is then given by

$$E = -\frac{1}{8\pi G} \int N(K - K_0), \quad (4.12)$$

with the integral over the surface. Here, N denotes the lapse, K is the extrinsic curvature of the surface in the spacetime being considered (the generalized AdS solitons in our case), and K_0 the extrinsic curvature of the surface in empty AdS. The extrinsic curvature is given by $\int K = n^\mu \partial_\mu A$, where n^μ is the radial unit normal vector and A is the area of the surface at fixed t, r . By comparing the extrinsic curvatures K and K_0 , we can determine the energy density of the spacetime of interest relative to AdS.

For the soliton backgrounds with a surface at $r = r_m$, the lapse function is $N = r_m/l$, the radial unit vector has only one nontrivial component $n^r = r_m \sqrt{f(r_m)}/l$, and the area of the surface in the limit $r_m \rightarrow \infty$ is given by

$$A = V_{\vec{x}} \Delta\phi \left(\frac{r_m}{l} \right)^{d-1} \sqrt{f(r_m)}, \quad (4.13)$$

with $V_{\vec{x}}$ denotes the volume along the $\vec{x}$ directions. Thus, we obtain

$$\int NK = \frac{V_{\vec{x}} \Delta\phi}{l^{d+1}} \left[(d-1)r_m^d + \left(1 - \frac{d}{2} \right) \mu l^2 \right]. \quad (4.14)$$

And NK_0 has the same form with $\mu = 0$, except that we must change the asymptotic periodicity by taking $\Delta\phi \rightarrow \sqrt{f(r_m)} \Delta\phi$, ensuring that the two surfaces have the same proper size at the cutoff radius r_m , as emphasized in [80]. Finally, we obtain

$$E = -\frac{V_{\vec{x}} \Delta\phi \mu}{16\pi G l^{d-1}}, \quad (4.15)$$

which is consistent with the holographic stress-energy tensor. Instead of matching the proper size of the cutoff surfaces in the soliton and periodic AdS spacetimes, we could alternatively take cutoff surfaces at the same Fefferman-Graham coordinate z , as will be demonstrated in the later discussion on the complexity of formation. Amusingly, this approach yields the same result for the energy of the magnetized AdS solitons. However, this is not a general feature of this formalism for determining the energy of asymptotically AdS spacetimes.

The long soliton (with small r_0) always has negative μ and positive energy density, while the short soliton (with large r_0) has negative energy density at small Φ and positive energy when $\Phi \geq \Phi_S \equiv 2\pi l/[\sqrt{3}]$ for $d = 3$ [$d = 4$]. We also set the energy density of periodic AdS to zero as a reference. Therefore, the system undergoes a phase transition from the short AdS soliton with negative energy density at small Φ to periodic AdS with zero energy density when $\Phi > \Phi_S$. Given that the ground state of the theory is either periodic AdS or the short magnetized AdS soliton, we will refer to the short magnetized AdS soliton as the soliton solution from now on. Since the soliton solutions cap off smoothly in the infrared rather than having a horizon at r_0 , they exhibit confinement, similar to the standard AdS soliton with $Q = 0$ [80, 81]. Therefore, tuning the Wilson line Φ through Φ_S induces a confining-deconfining phase transition, where the system transitions from a confined phase at small Φ to a deconfined phase for $\Phi > \Phi_S$. Furthermore, the short soliton is supersymmetric at $\Phi = \Phi_S$, as emphasized in [85].

As noted above, the usual Dirichlet boundary condition on the gauge field $\delta A_\mu = 0$ indicates that the boundary CFT is defined in terms of the fixed Wilson line Φ . In thermodynamics, this setup corresponds to the grand canonical ensemble. By adding a term to the action corresponding to the Maxwell field on the conformal boundary of AdS, or more precisely, on a cutoff surface at a fixed large radius r , the natural variable of the boundary CFT becomes Q [85, 94]. With the boundary term included in the action, the variational problem for the gauge field has Neumann boundary conditions $\delta F_{\mu\nu} = 0$. Multiplication of the boundary action term by an arbitrary coefficient results in Robin, or mixed boundary conditions. In the Euclidean theory, adding the boundary term corresponds to the theory in the canonical ensemble.

4.3 Volume complexity of formation

The initial proposal for holographic complexity is “complexity=volume” [49, 50], known as CV. This is the simplest proposal for holographic complexity, which states that the complexity of the boundary state is dual to a gravitational quantity that is proportional to volume of extremal/maximal time slice anchored at the boundary.

For an asymptotically AdS spacetime with appropriate normalization, the volume complexity is given by [86]

$$C_V = \frac{(d-1)V}{2\pi^2 G l}, \quad (4.16)$$

where V is the volume of a maximal volume slice anchored at a fixed time on the boundary. The overall normalization is not important, but to compute the complexity of magnetized AdS solitons with a varying Wilson line, we should use a fixed length scale, so we choose the AdS length scale for simplicity. The remainder of the normalization constant is chosen so that the AdS-Schwarzschild black hole saturates Lloyd’s bound for the time derivative of complexity at late times [86]. Although we have written the normalization of complexity with the AdS length scale l , the choice of length scale is, in fact, ambiguous and could be an arbitrary length scale.

All the backgrounds we consider possess both time translation and time reversal symmetries, indicating that the maximal volume surface is a constant time slice. We will always choose to measure the complexity at the boundary time $t = 0$. Since the volume diverges near the conformal boundary, we can only integrate up to a finite radius, and we denote this radial cutoff by r_m . For both the magnetized AdS solitons and periodic AdS,

this volume has the following form

$$V = V_{\vec{x}} \Delta\phi \int_{r_0}^{r_m} dr \left(\frac{r}{l}\right)^{d-2} = \frac{V_{\vec{x}} \Delta\phi}{d-1} \frac{r_m^{d-1} - r_0^{d-1}}{l^{d-2}}, \quad (4.17)$$

with $r_0 \rightarrow 0$ for periodic AdS, where $V_{\vec{x}}$ is the volume along the $\vec{x}$ directions, $\Delta\phi$ is the period of ϕ direction. To determine the complexity of formation for the magnetized AdS soliton, we need to subtract the corresponding volume of periodic AdS from the volume given in (4.17) [87]. As a result, periodic AdS, by definition, has zero complexity of formation, whereas the complexity of formation for solitons is observed to be negative. However, there is a subtlety to consider. Instead of comparing volumes at the same cutoff radius r_m , we need to compare them using the same Fefferman-Graham coordinate z as defined in (4.8). If r'_m is the radial cutoff in periodic AdS and r_m is the cutoff in the magnetized AdS soliton, then (4.9) implies that

$$r_m^{d-1} = r'^{d-1}_m \left(1 + \frac{\mu l^2 (d-1)}{2dr'^d_m} + \dots\right). \quad (4.18)$$

Therefore, as r_m goes to infinity, the divergent terms in the maximal volumes for the magnetized AdS soliton and the periodic AdS cancel each other out. This results in a negative density of complexity of formation for the magnetized AdS soliton

$$\mathcal{C}_V = -\frac{r_0^{d-1}}{2\pi^2 G l^{d-1}}. \quad (4.19)$$

Similarly, the difference in r_m between solitons with different values of Φ also becomes negligible as we take $r_m \rightarrow \infty$.

Two notable features of $\mathcal{C}_V$ are immediately evident. First, $\mathcal{C}_V$ scales as an inverse power of the circumference, following the relation $\mathcal{C}_V \propto \Delta\phi^{-(d-1)}$, and $\mathcal{C}_V$ exhibits a discontinuity at the phase transition point, where $\Phi = \Phi_S$, due to the fact that r_0 is non-zero at³ $\Phi = \Phi_S$. The dimensionless quantity $c_V \equiv G\mathcal{C}_V(\Delta\phi/l)^{d-1}$ as a function of Φ/Φ_{max} is shown in figure 4.1. Note that c_V jumps from a negative value to zero at $\Phi = \Phi_S$, the supersymmetric value, due to the confinement-deconfinement phase transition that occurs at this Wilson line value.

The negative value of the density of complexity of formation $\mathcal{C}_V$ is noteworthy. Engelhardt and Folkestad [95] showed that maximal volume slices in a wide variety of asymptotically AdS spacetimes are always larger than maximal volume slices in AdS. However, their

³Refer to Section 4.5 for the case where the natural parameter for the boundary CFT is Q rather than Φ . Additionally, note that the total complexity of formation scales as $C_V \propto \Delta\phi^{-(d-2)}$.

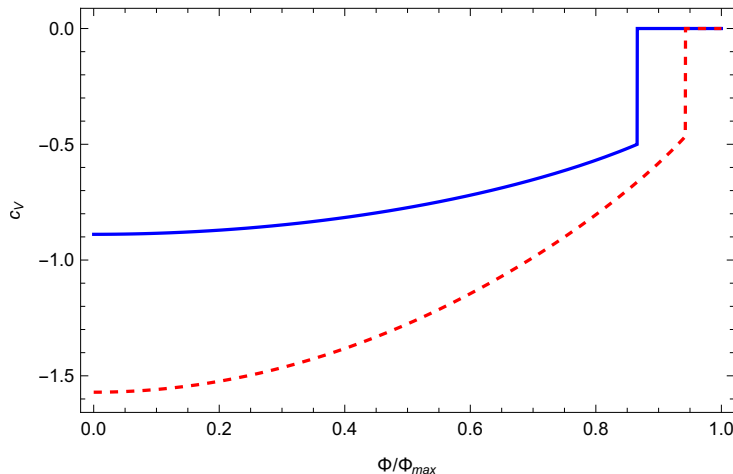


Figure 4.1: CV complexity of formation per unit volume for $d = 3$ (solid blue) and $d = 4$ (dashed red). Plotted curves are $c_V \equiv G\mathcal{C}_V(\Delta\phi/l)^{d-1}$, so complexity scales with an inverse power of $\Delta\phi$.

results rely on the assumption of maximal symmetry in the spatial slices of the conformal boundary, an assumption that does not apply to AdS solitons or periodic AdS. As a result, their theorems do not extend to these cases and thus we observe a negative $\mathcal{C}_V$.

4.4 Spacetime Volume complexity of formation

Another proposal for holographic complexity is “complexity= spacetime volume” [51], known as CV2.0. It states that complexity should be determined not just by the volume of the maximal slice but by the spacetime volume of the entire domain of these spatial surfaces with null boundaries. In this framework, the complexity is given by

$$C_2 = \frac{V_{WDW}}{Gl^2}, \quad (4.20)$$

where V_{WDW} is the spacetime volume of the Wheeler–DeWitt (WDW) patch. The WDW patch is the spacetime region bounded by future-directed and past-directed lightsheets emitted from the boundary time slice where the complexity is measured. Once again, we use the AdS length l as the length scale in the complexity formula for simplicity, and emphasize that the overall normalization of complexity is not physically important. In

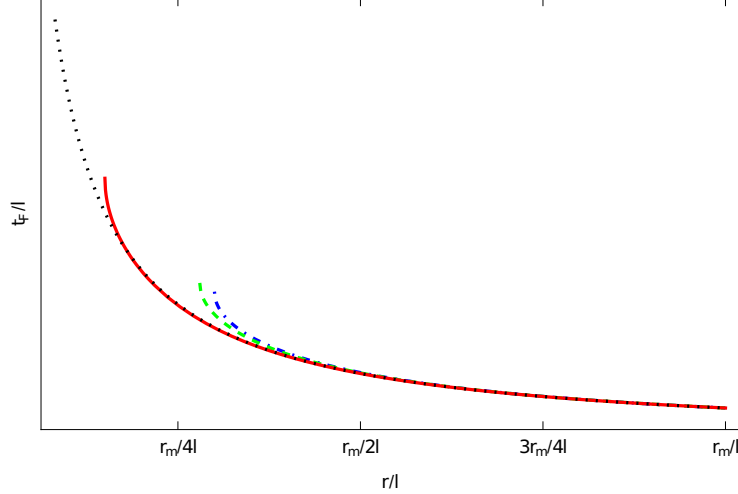


Figure 4.2: The future lightsheets $t_F(r)$ for $\Phi/\Phi_{max} = 0$ (dot-dashed blue), 0.5 (dashed green), 1 (solid red). In this instance, $\Delta\phi r_m/\ell^2 = 10\pi/3$ and $d = 4$. For reference, $t_F(r)$ for periodic AdS is dotted black.

defining the WDW patch, we set the boundary conditions so that both light sheets meet at $t = 0$ at the radial cutoff $r = r_m$ as a regulator, instead of taking $t = 0$ as $r \rightarrow \infty$ and then cutting off the patch at $r = r_m$. A demonstration that these regulators are equivalent when appropriately defined in the context of action complexity can be found in [96, 97].

The lightsheets on the WDW patch boundary can be described by functions $t_F(r)$ and $t_P(r) = -t_F(r)$, where F, P represent the future and past lightsheets respectively. The lightsheets are given by

$$\frac{dt_F}{dr} = -\frac{l^2}{r^2\sqrt{f(r)}}, \quad (4.21)$$

and extended along the $\vec{x}$ and ϕ directions. For a fixed value of $\Delta\phi r_m$, increasing Φ decreases r_0 for the short soliton, thus lengthening the soliton. Figure 4.2 compares t_F with different values of Φ/Φ_{max} in $d = 4$. A small value of $\Delta\phi r_m$ was chosen to highlight the differences in the lightsheets near the soliton cap. Note that the end of each curve on the left corresponds to the end of the spacetime at $r = r_0$. Additionally, one feature of $t_F(r)$ is that it is finite but forms a cusp with an infinite derivative at $r = r_0$. For reference, the curve $t_F(r) = l^2/r - l^2/r_m$ for periodic AdS spacetime is also shown.

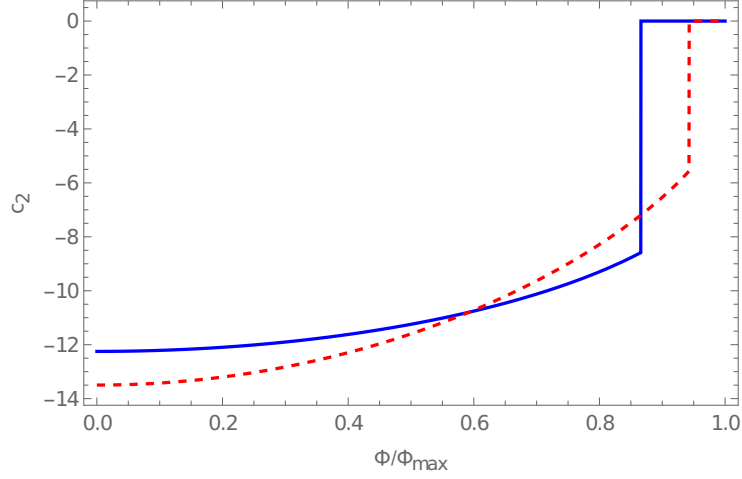


Figure 4.3: CV2.0 complexity of formation per boundary volume for $d = 3$ (solid blue) and $d = 4$ (dashed red). Plotted curves are $c_2 \equiv G\mathcal{C}_2(\Delta\phi/l)^{d-1}$, so complexity scales with an inverse power of $\Delta\phi$.

The spacetime volume of the WDW patch is given by

$$V_{WDW} = \int_{WDW} d^{d+1}x \sqrt{-g} = 2V_{\vec{x}} \Delta\phi \int_{r_0}^{r_m} dr \left(\frac{r}{l}\right)^{d-1} t_F(r), \quad (4.22)$$

we expect the complexity of formation to be negative — the increase in t_F near r_0 is insufficient to compensate for the shortening of spacetime. In the following, we provide the numerical calculation of $\mathcal{C}_2$, the density of complexity of formation, while keeping the Wilson line Φ (rather than Q) fixed.

Let us first determine $t_F(r) = -t_P(r)$ by numerical integration with $t = 0$ at $r = r_m$. It turns out that in the simplest $Q = 0$ case, it can be expressed in terms of hypergeometric functions; however, this does not extend to more general cases. It is straightforward to observe that the $r_0 t_F/l$ vs r/r_0 curve is influenced by two parameters $\Delta\phi r_m$ and Φ/Φ_{max} . The key point to note is that $f(r)$ does not depend on $\Delta\phi$ when expressed using the dimensionless radial variable $\tilde{r} = r/r_0$, with Φ held constant. Then, we obtain

$$t_F(r) = \frac{l^2}{r_0} \int_{\tilde{r}}^{r_m/r_0} \frac{d\tilde{r}'}{\tilde{r}'^2 \sqrt{f(\tilde{r}')}} \equiv \frac{l}{r_0} \tilde{t}_F(\tilde{r}), \quad (4.23)$$

where $\tilde{t}_F$ depends on the circumference only through the boundary condition that $\tilde{t}_F = 0$ at $\tilde{r} = r_m/r_0 \propto \Delta\phi r_m$. We compute $t_F(r)$ numerically.

To calculate the complexity of formation, we also need the spacetime volume of the WDW patch for periodic AdS, which is given by

$$V_{AdS} = 2V_{\bar{x}}\Delta\phi \int_0^{r'_m} dr \left(\frac{r}{l}\right)^{d-1} \left(\frac{l^2}{r} - \frac{l^2}{r_m}\right) = \frac{2V_{\bar{x}}\Delta\phi l^2}{d(d-1)} \left(\frac{r'_m}{l}\right)^{d-1}. \quad (4.24)$$

As previously mentioned, the difference between r'_m and r_m does not affect V_{AdS} as r_m approaches infinity, according to equation (4.18).

Since V_{WDW} must be evaluated numerically, we express V_{AdS} as

$$V_{AdS} = \frac{2V_{\bar{x}}\Delta\phi l}{d} \left\{ \frac{l}{d-1} \left(\frac{r_0}{l}\right)^{d-1} + \int_{r_0}^{r_m} dr \left(\frac{r}{l}\right)^{d-2} \right\}, \quad (4.25)$$

and subtract integrands. Thus we obtain the boundary density of the complexity of formation

$$\begin{aligned} \mathcal{C}_2 &= \frac{2}{G} \left\{ -\frac{1}{d(d-1)} \left(\frac{r_0}{l}\right)^{d-1} + \int_{r_0}^{r_m} dr \left(\frac{r}{l}\right)^{d-1} \left[\frac{t_F(r)}{l^2} - \frac{1}{d} \frac{1}{r} \right] \right\} \\ &= \frac{2}{G} \left(\frac{r_0}{l}\right)^{d-1} \left\{ -\frac{1}{d(d-1)} + \int_1^{r_m/r_0} d\tilde{r} \tilde{r}^{d-1} \left[\frac{\tilde{t}_F(\tilde{r})}{l} - \frac{1}{d} \frac{1}{\tilde{r}} \right] \right\}. \end{aligned} \quad (4.26)$$

We note that the factor in braces depends on the period $\Delta\phi$ only through the upper limit on the integral and the boundary condition on $\tilde{t}_F$. Moreover, $\Delta\phi$ only appears in the combination $r_m\Delta\phi$. Therefore, in the limit as the UV cutoff r_m goes to infinity, the quantity in braces no longer depends on $\Delta\phi$. As a result, $\mathcal{C}_2$ is proportional to $\Delta\phi^{-(d-1)}$ due to the overall prefactors of r_0 . To evaluate the numerical integral, we found that the result varies by less than 1% between values $r_m\Delta\phi = 400l^2$ and $r_m\Delta\phi = 500l^2$ for $\Phi = 0$, which has the smallest ratio r_m/r_0 at any $\Delta\phi$. Therefore, we chose $r_m\Delta\phi = 500l^2$.

The above calculation reveals a scaling relation $\mathcal{C}_2 \propto \Delta\phi^{-(d-1)}$ as the cutoff is removed by letting $r_m \rightarrow \infty$. Notably, this scaling is the same as that of $\mathcal{C}_V$. We therefore plot the CV2.0 density of complexity of formation $c_2 \equiv G\mathcal{C}_2(\Delta\phi/l)^{d-1}$ in figure 4.3. Like the CV density of complexity of formation shown in figure 4.1, the magnitude of c_2 decreases as the Wilson line Φ increases, and c_2 jumps to zero at the deconfining phase transition when $\Phi > \Phi_s$.

4.5 Action complexity of formation

In the “complexity=action” or CA proposal [52, 53, 54], the action complexity is given by

$$C_A = \frac{S_{WDW}}{\pi}, \quad (4.27)$$

where S_{WDW} is the action evaluated on the WDW patch, including appropriate boundary terms [52, 53, 54]. Altogether, this action is

$$S_{WDW} = S_{bulk} + S_{bdy} + S_{joint}, \quad (4.28)$$

where the bulk action, the boundary term, and the joint term are given by

$$S_{bulk} = \frac{1}{16\pi G} \int_{WDW} d^{d+1}x \sqrt{-g} \left(R + \frac{d(d-1)}{l^2} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right), \quad (4.29)$$

$$S_{bdy} = -\frac{1}{8\pi G} \int_F d\lambda d^{d-2}\vec{x} d\phi \sqrt{\gamma} \left(\kappa_F + \Theta_F \ln |\beta l \Theta_F| \right) \\ + \frac{1}{8\pi G} \int_P d\lambda d^{d-2}\vec{x} d\phi \sqrt{\gamma} \left(\kappa_P + \Theta_P \ln |\beta l \Theta_P| \right), \quad (4.30)$$

$$S_{joint} = -\frac{1}{8\pi G} \int_{F \cap P} d^{d-2}\vec{x} d\phi \sqrt{\gamma} \ln |k_F \cdot k_P / 2|. \quad (4.31)$$

Here, λ is the lightlike parameter along each lightsheet on the future and past null boundaries, with λ increasing towards the future in each case. The vectors $k_F, k_P \equiv dx/d\lambda$ are the future-directed null vectors along the respective lightsheets, and $\kappa_{F,P}$ is defined by $k^\mu \nabla_\mu k^\nu = \kappa k^\nu$. γ_{ij} represents the induced metric on the spatial slices at fixed λ along the lightsheet. Θ_F, Θ_P are the expansions of the lightsheets, defined as the logarithmic derivative of $\sqrt{\gamma}$ with respect to λ . The $\Theta_{F,P}$ terms are counterterms necessary to maintain reparameterization invariance of S_{bdy} , and β is an arbitrary parameter that sets the length scale of the counterterm. Note that different choice of β could affect the complexity in principle. However, if we choose the same β for both lightsheets and for all backgrounds, we will find that β dependence cancels in the complexity of formation when subtracting the periodic AdS action in the $r_m \rightarrow \infty$ limit. It is worth mentioning that in $d = 4$, the bulk action could also include a Chern-Simons term for the gauge field. However, this term vanishes for the soliton solutions. The action diverges, so we regulate it by introducing the cutoff r_m and restricting integration to $r \leq r_m$ only. To obtain the complexity of formation, we also subtract the corresponding action of periodic AdS integrated up to cutoff radius r'_m , which is related to r_m as specified by (4.18). It is worth noting that the Wilson line does not contribute to the action of periodic AdS due to vanishing field strength.

Similar to the boundary terms for gravitational degrees of freedom (4.30), the WDW patch action also includes a boundary term for the Maxwell field

$$\Delta S_{bdy} = \frac{\nu}{16\pi G} \int_F d\lambda d^{d-2}\vec{x} d\phi \sqrt{\gamma} k_F^\mu A^\nu F_{\mu\nu} + \frac{\nu}{16\pi G} \int_P d\lambda d^{d-2}\vec{x} d\phi \sqrt{\gamma} k_F^\mu A^\nu F_{\mu\nu}, \quad (4.32)$$

where ν is an arbitrary constant [98]. This serves as the lightlike analog of the term on the AdS conformal boundary that modifies the natural variable of the CFT. However, as particularly highlighted by the “complexity=anything” program [55, 56], there is no logical connection between ν and the boundary conditions at $r = r_m$. This indicates that ν can vary independently of whether Φ or Q is chosen as the natural variable of the CFT. Moreover, since the complexity is given by the on shell action, ΔS_{bdy} is equivalent to a multiple of the bulk Maxwell action in the absence of sources [98]. Consequently, we can adjust the prefactor of $F_{\mu\nu}F^{\mu\nu}$ in (4.29) to $(2\nu - 1)/4$ rather than using $-1/4$, thereby avoiding the need for an additional boundary term and simplifying the total action.

Here we provide a detailed calculation of the action in equations (4.29, 4.30, 4.31). Again, a sample of the lightsheets function $t_F(r)$ at fixed $\Delta\phi r_m$ and various values of Φ/Φ_{max} is shown in figure 4.2. The main difference in the shape of the lightsheets is due to the reduction in r_0 as Φ/Φ_{max} increases. At large radius, t_F approaches $(1/r - 1/r_m)$ as in periodic AdS. We also note that $\kappa_{F,P} = 0$ if we choose λ such that $dr/d\lambda = \mp \alpha_{F,P} \sqrt{f(r)}$, with $\alpha_{F,P}$ being arbitrary positive dimensionless constants. These derivatives give the $\mu = t$ and r components of $k_{F,P}^\mu$. The induced metric for spatial slices of either lightsheet gives $\sqrt{\gamma} = \sqrt{f(r)}(r/l)^{d-1}$.

Since $dr/d\lambda$ has opposite sign on the past and future lightsheets, their contributions to S_{bdy} are identical when written as an integral over radius. For notational convenience, we define

$$\Theta_{F,P}(r) = \frac{dr}{d\lambda} \left(\frac{d-1}{r} + \frac{1}{2f(r)} \frac{df}{dr} \right) \equiv \frac{dr}{d\lambda} \Theta(r). \quad (4.33)$$

The identity $\sqrt{\gamma}\Theta = d\sqrt{\gamma}/dr$ allows us to convert several boundary terms to joint terms. We then perform an additional integration by parts after adding and subtracting $\ln(r/l)$ to cancel a logarithmic divergence between the joint and boundary terms. All terms involving $\alpha_{F,P}$ also cancel out between S_{bdy} and S_{joint} .

To simplify the contribution of bulk action, we compute the trace of the Einstein equation, which yields the well-known identity

$$R = -\frac{d(d+1)}{l^2} + \frac{1}{4} \frac{d-3}{d-1} F_{\mu\nu}F^{\mu\nu}. \quad (4.34)$$

The Maxwell term without the additional boundary term (i.e., $\nu = 0$) is more negative, leading to an overall negative contribution from the Maxwell field strength to the bulk action. However, with the Maxwell boundary term, the terms proportional to $(F_{\mu\nu})^2$ contribute positively when $\nu > 1/(d-1)$.

The total action of AdS soliton can be further simplified, we obtain

$$\begin{aligned}
& S_{bulk} + S_{bdy} + S_{joint} \\
= & \frac{V_{\vec{x}}\Delta\phi}{8\pi G} \int_{r_0}^{r_m} dr \left(\frac{r}{l}\right)^{d-1} \left\{ 2\sqrt{f(r)} \left[\frac{1}{r} + \Theta(r) \left(\ln(r\Theta(r)) + \frac{1}{2} \ln(f(r)) \right) \right] \right. \\
& - t_F(r) \left[\frac{2d}{l^2} + \left(1 - \nu(d-1) \right) \frac{(d-2)^2 r_0^{2d-4} \Phi^2}{\Delta\phi^2} \frac{1}{r^{2d-2}} \right] \Big\} \\
& + \frac{V_{\vec{x}}\Delta\phi}{4\pi G} \left(\frac{r_m}{l}\right)^{d-1} \sqrt{f(r_m)} \ln(\beta). \tag{4.35}
\end{aligned}$$

It should be noted that the arbitrary constants $\alpha_{F,P}$ cancel between boundary and joint terms. Since $f(r) \rightarrow 1$ and $r\Theta \rightarrow d-1$ at large r , the divergences of the soliton action are the same as those of the empty periodic AdS action. Therefore, by subtracting the action of periodic AdS with a cutoff radius r'_m , as described in (4.18), we eliminate the divergences and obtain the complexity of formation. Since $r_m = r'_m(1 + \mathcal{O}(1/r_m^d))$, we take $r'_m = r_m$. Further discussion is provided below. To handle the numerical cancellation of divergences more easily, we rewrite the action of periodic AdS as

$$S_{AdS} = \frac{V_{\vec{x}}\Delta\phi}{4\pi G} \left\{ \left(\frac{r_m}{l}\right)^{d-1} \ln(\beta) + \ln(d-1) \left[\left(\frac{r_0}{l}\right)^{d-1} + \int_{r_0}^{r_m} dr \left(\frac{d-1}{l}\right) \left(\frac{r}{l}\right)^{d-2} \right] \right\}. \tag{4.36}$$

The terms depending on β cancel out as $r_m \rightarrow \infty$ because $f-1 \sim \mathcal{O}(r_m^{-d})$. Therefore, we can safely omit these terms from now on.

A specific reparameterization of the integration variable, as described in [86], is useful for both numerically computing the action and demonstrating scaling properties of the complexity. When expressed in terms of the variable $\tilde{r} = r/r_0$, the function f remains independent of r_0 and, therefore, independent of $\Delta\phi$, once μ and Q are determined in relation to $\Delta\phi$ and Φ . Similarly, $\tilde{t}_F(\tilde{r}) \equiv r_0 t_F/l$ depends on $\Delta\phi$ only via the boundary condition at $r_m/r_0 \propto r_m \Delta\phi$. Then the total action becomes

$$S = S_{bulk} + S_{bdy} + S_{joint} - S_{AdS} = \frac{V_{\vec{x}}\Delta\phi}{8\pi G} \left(\frac{r_0}{l}\right)^{d-1} \left[-\ln(d-1) + I(r_m/r_0, \Phi) \right], \tag{4.37}$$

where $I(r_m/r_0, \Phi)$ is given by the integral

$$\begin{aligned}
I(r_m/r_0, \Phi) = & \int_1^{r_m/r_0} d\tilde{r} \tilde{r}^{d-1} \left\{ 2\sqrt{f(\tilde{r})} \left[\frac{1}{\tilde{r}} + \tilde{\Theta}(\tilde{r}) \left(\ln(\tilde{r}\tilde{\Theta}(\tilde{r})) + \frac{1}{2} \ln(f(\tilde{r})) \right) \right] \right. \\
& - \frac{\tilde{t}_F(\tilde{r})}{l} \left[2d + (1 - \nu(d-1)) \frac{d^2(d-2)^2}{d-1} \frac{g(d)^2}{4\tilde{r}^{2d-2}} \frac{\Phi^2/\Phi_{\max}^2}{\left(1 + \sqrt{1 - \Phi^2/\Phi_{\max}^2}\right)^2} \right] \\
& \left. - 2\ln(d-1) \frac{d-1}{\tilde{r}} \right\}, \tag{4.38}
\end{aligned}$$

and $\tilde{\Theta}(\tilde{r}) = r_0\Theta = (d-1)/\tilde{r} - (df/d\tilde{r})/2f$ is the dimensionless expansion. An important point to note is that, when expressed in terms of the Wilson line Φ , $I(r_m/r_0, \Phi)$ also depends on $\Delta\phi$ only through the integration limit r_m/r_0 , which is proportional to $r_m\Delta\phi$. However, subtracting S_{AdS} ensures that I converges as the upper limit goes to infinity, resulting in I having a value independent of $\Delta\phi$. Therefore, upon removing the cutoff, the density of complexity of formation for standard boundary conditions is $\mathcal{C}_A \propto (\Delta\phi/l)^{-(d-1)}$, where the scaling is determined by the prefactor r_0^{d-1} .

Since the complexity of formation remains finite even as $r_m \rightarrow \infty$, we have evaluated (4.38) for $\Phi = 0$, where r_0 is largest, and checked where the integral has converged sufficiently. In both $d = 3$ and $d = 4$ cases, the value of $I(r_m/r_0, 0)$ changes by less than 1% between $r_m\Delta\phi = 400l^2$ and $r_m\Delta\phi = 500l^2$ for $\Phi = 0$ (which is the shortest soliton). Therefore, we use $r_m\Delta\phi = 500l^2$ for all numerical calculations of the complexity for convenience.

The Wilson line Φ in the CFT affects the action complexity evaluated at fixed periodicity $\Delta\phi$ in several ways. As previously mentioned, increasing Φ extends the soliton (decreasing r_0), which tends to increase the magnitudes of S_{bulk} and S_{bdy} . The change in the shape of function $t_F(r)$ influences the expansion $\Theta_{F,P}$ of the lightsheets at the WDW patch boundary, especially in the interior region. Additionally, the field strength itself contributes to the Lagrangian density (including a positive semidefinite contribution to the curvature for $d \geq 3$). This contribution is negative when ν is small but turns positive for $\nu > 1/(d-1)$.

In the grand canonical ensemble with variables $\Delta\phi$ and Φ appropriate to standard boundary conditions on AdS, the density of formation complexity $\mathcal{C}_A$ scales as $\Delta\phi^{-(d-1)}$, akin to the scaling observed in volume complexities $\mathcal{C}_V$ and $\mathcal{C}_2$, as shown previously. Considering that $\mathcal{C}_A$ is significantly more intricate to calculate than volume complexity, it is

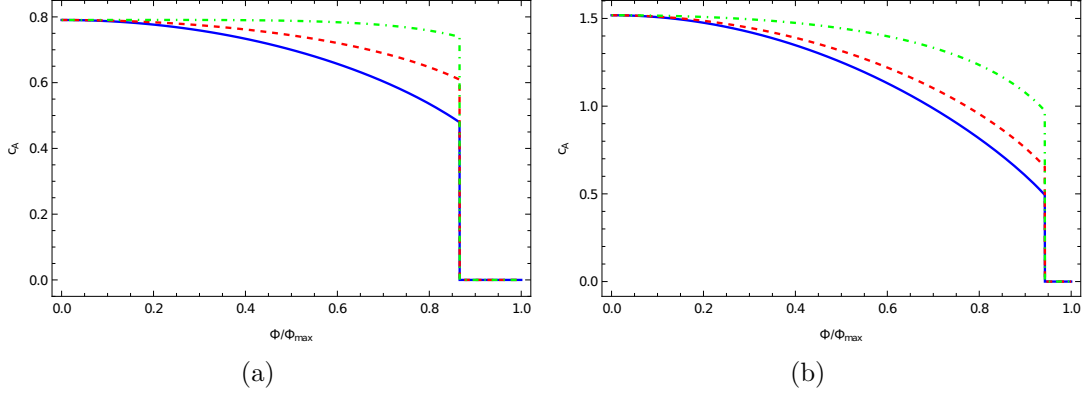


Figure 4.4: CA complexity of formation per unit boundary volume for (a) $d = 3$ and (b) $d = 4$ with $\nu = 0$ (solid blue), $\nu = 1/(d - 1)$ (dashed red), and $\nu = 1$ (dot-dashed green). Plotted curves are $c_A = G\mathcal{C}_A(\Delta\phi/l)^{d-1}$ as a function of the Wilson line Φ .

surprising that it adheres to the same simple scaling relation. Note that this scaling does not hold in the canonical ensemble, where complexity is evaluated as a function of Q . In that case, the scaling property of the action indicates that $\mathcal{C}_A \propto \Delta\phi^{-(d-1)}$ along lines where $Q\Delta\phi^{d-1}$ remains constant (see equations (4.3,4.5)).

We show the quantity $c_A \equiv G\mathcal{C}_A(\Delta\phi/l)^{d-1}$ as a function of Wilson line Φ in figure 4.4. In contrast to $\mathcal{C}_V$ and $\mathcal{C}_2$, the density of action complexity of formation $\mathcal{C}_A$ is positive semi-definite in the confining phase (and equals zero in the deconfined phase). Without the lightsheet boundary term for the Maxwell field (i.e., $\nu = 0$), the complexity is maximized with no Wilson line and decreases as the Wilson line increases towards $\Phi = \Phi_S$ (solid blue curve). When $\nu = 1/(d - 1)$, the $F_{\mu\nu}F^{\mu\nu}$ term is canceled out. As a result, the corresponding dashed red curve illustrates only the geometric effects of the Wilson line (including increased soliton length and altered WDW patch shape) on the complexity. Finally, for $\nu = 1$ (dot-dashed green curve), the $\mathcal{C}_A$ curve for $d = 3$ increases at $\Phi = 0$ and reaches a maximum before $\Phi = \Phi_S$. In this case, the positive contribution of the $F_{\mu\nu}F^{\mu\nu}$ term increases with Φ , but the overall scaling with r_0^{d-1} , which decreases with Φ , eventually dominates. For $\nu = 1$ and $d = 4$, the quantity c_A decreases as Φ increases.

4.6 Scaling with circumference in general complexity

In the previous two sections, we have shown that the complexity of formation density $\mathcal{C}$ for magnetized AdS solitons scales as $1/\Delta\phi^{d-1}$ when expressed as a function of the grand canonical variables $\Delta\phi, \Phi$, with the UV cutoff going to infinity in the CV, CV2.0, and CA holographic complexity proposals. The notable differences in the details and properties of these complexity measures (such as their positivity or negativity on the soliton) suggest that there may be a general principle at work, rather than mere coincidence.

Specifically, “Complexity equals anything” program [56] argued that a large family of functionals of the form

$$C = \frac{1}{Gl^2} \int_{\mathcal{M}} d^{d+1}x \sqrt{-g} \mathcal{G} + \frac{1}{Gl} \int_{\mathcal{M}^+} d^d\sigma \sqrt{h} \mathcal{F}^+ + \frac{1}{Gl} \int_{\mathcal{M}^-} d^d\sigma \sqrt{h} \mathcal{F}^- \quad (4.39)$$

all satisfy basic properties expected of holographic complexity. Here, $\mathcal{M}$ denotes a bulk region in asymptotically AdS spacetime, with $\mathcal{M}^\pm$ correspond to spacelike surfaces at the future and past boundaries of $\mathcal{M}$. The coordinates σ represent worldvolume coordinates on $\mathcal{M}^\pm$, while $\mathcal{G}$ and $\mathcal{F}^\pm$ are dimensionless scalar functions of the metric and curvatures. Since the magnetized solitons contain matter, we generalize to allow $\mathcal{G}, \mathcal{F}^\pm$ to be functions not only of the metric and curvatures but also of the gauge potential and field strength. The bulk region $\mathcal{M}$ itself optimizes a possibly different functional of the same form. The boundaries of $\mathcal{M}^\pm$ form a joint at the time slice on the AdS boundary (more precisely, at the cutoff radius) where the CFT complexity is to be evaluated. The paper [56] implies that additional contributions to C on this joint are included, potentially in a way that cancels boundary terms from the $\mathcal{M}^\pm$ integrals. Moreover, there are limits of the optimization procedure choosing $\mathcal{M}$ that leads to lightlike $\mathcal{M}^\pm$, so the WDW patch is a possible bulk region $\mathcal{M}$.

Here, we will argue that a general complexity functional, as expressed in (4.39), leads to a density for complexity of formation $\mathcal{C} \propto r_0^{d-1} \propto \Delta\phi^{-(d-1)}$ for the magnetized AdS soliton backgrounds described in section 4.2 when considering the soliton as a function of the grand canonical variables $\Delta\phi$ and Φ .

To support this argument, we assume that the complexity of an empty periodic AdS is given by $C = (aV_{\vec{x}}\Delta\phi/G)(r_m/l)^{d-1}$ when evaluated with a cutoff radius r_m , where a is a numerical constant, $V_{\vec{x}}$ is the volume in the $\vec{x}$ direction, and $\Delta\phi$ is the period in the ϕ direction. This is the behaviour observed in the CV, CV2.0, and CA proposals and is the strongest divergence possible for the bulk term in AdS since curvature invariants are constant and the embedding $t_\pm(r)$ of $\mathcal{M}^\pm$ satisfies $|\partial_r t_\pm(r)| \leq l^2/r^2$ for spacelike and

lightlike surfaces. The absence of subleading divergences (or a leading divergence of a lower power) allows us to express this term as

$$C = \frac{aV_{\vec{x}}\Delta\phi}{G} \left[\left(\frac{r_0}{l} \right)^{d-1} + (d-1) \int_{r_0}^{r_m} dr r^{d-2} \right], \quad (4.40)$$

which is suitable for subtraction from the AdS soliton's complexity.

To proceed, we express the metric (4.2) and gauge field (4.3,4.6) in terms of rescaled coordinates $\tilde{r} = r/r_0$ and $\tilde{t} = r_0 t/l$, which [86] introduced:

$$\begin{aligned} ds^2 &= \tilde{r}^2 \left(-d\tilde{t}^2 + \frac{r_0^2}{l^2} d\vec{x}^2 + \frac{r_0^2}{l^2} f(\tilde{r}) d\phi^2 \right) + \frac{l^2}{\tilde{r}^2 f(\tilde{r})} d\tilde{r}^2, \quad f(\tilde{r}) = 1 - \frac{1 - \tilde{Q}^2}{\tilde{r}^d} - \frac{\tilde{Q}^2}{\tilde{r}^{2d-2}}, \\ A &= \sqrt{7-d} r_0 \tilde{Q} \left(\frac{1}{\tilde{r}^{d-2}} - 1 \right) d\phi, \quad F = -\sqrt{7-d} (d-2) \frac{r_0 \tilde{Q}}{\tilde{r}^{d-1}} d\tilde{r} \wedge d\phi, \\ \tilde{Q}^2 &\equiv \frac{d^2/4}{7-d} g(d)^2 \left(\frac{\Phi}{\Phi_{max}} \right)^2 \left(1 + \sqrt{1 - \Phi^2/\Phi_{max}^2} \right)^{-2}. \end{aligned} \quad (4.41)$$

Since we are comparing the density of complexity of formation (ie, complexity per unit volume $V_{\vec{x}}\Delta\phi$) across different values of $\Delta\phi$ and Φ , it is necessary to keep the $\vec{x}$ and ϕ coordinates unrescaled. Similarly, the UV cutoff is set at a fixed value of the unrescaled radial coordinate $r = r_m$, which corresponds to a fixed Fefferman-Graham coordinate.

In these coordinates, the tensor components $g_{\mu\nu}, A_\mu, F_{\mu\nu}$ contain exactly one factor of r_0 for each leg along the $\vec{x}$ or ϕ directions. It is straightforward to verify that the same holds true for the Riemann tensor with all indices lowered. The Christoffel symbols are also such that covariant derivatives in the $\vec{x}, \phi$ directions introduce a factor of r_0 , while those in the $\tilde{t}, \tilde{r}$ directions do not. This argument also depends on the translational invariance in the $\vec{x}, \phi$ directions. Consequently, any bulk scalar function $\mathcal{G}$ constructed from the metric, the Riemann tensor, the gauge field, or their covariant derivatives remains independent of r_0 — the inverse metric components needed for contractions cancel them. Additionally, on either spatial surface $\mathcal{M}^\pm$, the normal vector n_μ can only have components in the $\tilde{t}, \tilde{r}$ directions due to translational invariance, and its only nonzero covariant derivatives have only those legs also. As a result, the extrinsic curvature has no factors of r_0 . Similar arguments also apply to the curvature κ and expansion Θ if $\mathcal{M}^\pm$ is lightlike. Therefore, if we assume $\mathcal{F}^\pm$ are constructed only from the metric, gauge field, and those curvatures, they contain no explicit factors of r_0 either. Altogether, any functional C of the form (4.39) is proportional to r_0^{d-1} , with all those factors arising from determinants of the metric, except for a dependence on the region of integration.

Under these assumptions, the density of complexity of formation takes the form

$$\mathcal{C} = \frac{1}{G} \left(\frac{r_0}{l} \right)^{d-1} \left\{ -a + \int_1^{r_m/r_0} d\tilde{r} \left[(\tilde{t}_+(\tilde{r}) - \tilde{t}_-(\tilde{r})) \mathfrak{g}(\tilde{r}) + \mathfrak{f}^+(\tilde{r}; \tilde{t}_+(\tilde{r})) + \mathfrak{f}^-(\tilde{r}; \tilde{t}_-(\tilde{r})) \right] \right\}, \quad (4.42)$$

where $\mathfrak{g}, \mathfrak{f}^\pm$ are obtained by evaluating $\mathcal{G}, \mathcal{F}^\pm$ on the AdS soliton with the AdS complexity subtracted, and $\tilde{t}_\pm(\tilde{r})$ describe the surfaces $\mathcal{M}^\pm$. The dependence of $\mathfrak{f}^\pm$ on $\tilde{t}_\pm(\tilde{r})$ includes dependence on the derivatives of that embedding function. Here we consider the case where $\mathcal{M}^\pm$ are spacelike, but the timelike case is similar. From the above discussion, $\mathfrak{g}, \mathfrak{f}^\pm$ must be independent of r_0 . Consequently, any dependence on $\Delta\phi$ can only arise from upper limit of integration or implicitly in $\tilde{t}_\pm$. However, $\tilde{t}_\pm$ are determined by optimization of a functional of the form (4.39), where the dependence on r_0 scales out of the integrand. Thus, $\tilde{t}_\pm$ depends on r_0 only through the boundary condition $\tilde{t}_\pm(\tilde{r} = r_m/r_0) = 0$. But given that $\mathcal{C}$ is by design convergent in the $r_m \rightarrow \infty$ limit, so the integral cannot depend on r_0 in that limit. The overall scaling is therefore given by

$$\mathcal{C} \propto r_0^{d-1} \propto \Delta\phi^{-(d-1)}. \quad (4.43)$$

4.7 Discussion and conjecture

In summary, we have evaluated different proposals for holographic complexity for a 3-dimensional or 4-dimensional CFT on a circle (times Minkowski spacetime) with a U(1) gauge field, which corresponds to magnetized AdS soliton backgrounds through gauge/gravity duality. One obvious observation is that “complexity=volume” and “complexity=spacetime volume” proposals have negative complexity of formation, while the complexity of formation for the “complexity=action” proposal is positive. As a result, we speculate that the reference states for the CV and CV2.0 proposals are located in a different region of the CFT’s Hilbert space compared to the reference states for CA complexity proposals (which may differ with varying boundary conditions for the gauge field). Another interpretation is that the volume and action complexity proposals have similar reference states but very different distance measures, though this seems less likely to explain the observed sign change.

As emphasized by [85], the compactified CFT exhibits two behaviours near the critical Wilson line value $\Phi = \Phi_S$: supersymmetry breaking for $\Phi \neq \Phi_S$ and a phase transition from the magnetized AdS soliton (confining) phase to the periodic AdS (deconfined) phase. The latter effect appears to be more significant for the complexity; the complexity of formation functions as an order parameter, vanishing in the deconfined phase and exhibiting

a discontinuous change at the phase transition point. This is perhaps unsurprising given the nature of the phase transition, with periodic AdS serving as the ground state in the deconfined phase. On the other hand, the degree of supersymmetry breaking does not seem to affect the complexity; $\mathcal{C}$ remains constant for $\Phi > \Phi_S$ (by definition), but below Φ_S , $|\mathcal{C}|$ grows generically as Φ decreases. (With further decreases in Φ , $|\mathcal{C}|$ decreases again in some cases.)

A notable finding is that the dependence of the complexity of formation on the grand canonical variables $\Delta\phi, \Phi$ factorizes, with the density behaving as $\mathcal{C} = h(\Phi)/\Delta\phi^{d-1}$. This extends the original observation of [86] for the soliton without a magnetic field to magnetized solitons, though only in $d = 3, 4$. This scaling law applies to the three specific forms of holographic complexity that we computed. Moreover, we showed that the general holographic complexity functional, as proposed by the “complexity=anything” program [55, 56], exhibits the same scaling relation under certain mild assumptions. Therefore, we propose a series of progressively stronger conjectures. The first conjecture is simply that this scaling extends to all magnetic AdS solitons of any dimensionality d . Next, we conjecture that the density of complexity of formation for any d -dimensional holographic field theory on a circle scales as the inverse $(d-1)$ st power of the circumference $\Delta\phi$, when the theory is expressed in terms of grand canonical variables. A stronger conjecture is that this scaling holds for the complexity of formation for any conformal field theory on a circle, or potentially even any quantum field theory.

A key point in our arguments is the finiteness of the complexity of formation, which enables us to remove the UV divergence when taking $r_m \rightarrow \infty$. This is evident since the soliton is asymptotically AdS, so the complexity of the soliton has the same divergence structure as periodic AdS, and the divergence cancels out in the complexity of formation subtraction. However, for the complexity of formation to scale with the circumference as conjectured, we assumed that the complexity of AdS itself must exhibit a pure divergence proportional to r_m^{d-1} , the strongest possible divergence. If we presuppose that the density of complexity of formation should scale as $\Delta\phi^{-(d-1)}$, this imposes instead an additional requirement for a functional to describe holographic complexity (beyond the current criteria, such as switchback effect and linear growth at late time in black hole backgrounds). Given that this appears to be an unnatural condition for theories with a scale, it may imply that complexity might not exhibit a straightforward scaling behaviour in non-conformal field theories.

Next, we note that the Casimir energy density of a d -dimensional field theory on one finite dimension scales as d powers of the inverse length of that dimension. In fact, we can see this behaviour in the holographic energy density as reviewed in section 4.2. It is intriguing to explore whether there is a deeper connection between the scaling of the

Casimir energy and of complexity of formation. On the other hand, both scaling laws may simply arise because both are given by integration over field modes in the noncompact directions of the field theory.

Finally, it is worth exploring the relationship between our findings and those of [99] regarding the complexity of another type of soliton in AdS_5 . There are two major distinctions from our work: first, the solitons studied in [99] have positive energy and are thus never the ground states; second, they are asymptotic to AdS_5 in global coordinates, so the compact boundary dimension corresponds to one of the angular directions of the boundary S^3 and cannot have arbitrary circumference. Therefore, it is challenging to make a direct comparison with our work. Nonetheless, [99] observe that the complexity of formation for those solitons also follows a scaling law, in their case with the thermodynamic volume of the soliton. Given that [100] reports a similar scaling for black holes, it would be intriguing to investigate what kind of relationship there may be between that scaling law and the one we have discussed here.

Chapter 5

Conclusion

In this thesis, we have explored two significant approaches in the pursuit of understanding quantum gravity: black hole thermodynamics and holography. In particular, we first provided the necessary background and theoretical framework for our investigations, and then discussed the dynamic behaviours of black hole phase transitions [1] and the holographic complexity of magnetized AdS solitons [2].

In the framework of black hole chemistry [27, 28] and the free energy landscape model [57, 58, 59, 60], we conducted the first investigation into the dynamics of multicriticality in black holes, focusing on black hole quadruple points in four-dimensional Einstein gravity coupled with nonlinear electrodynamics [1]. Our numerical solutions to the Smoluchowski equation uncovered the intricate dynamics of black hole phase transitions under varying ensemble temperatures. We observed how the probability distributions of black hole states evolve over time, influenced by potential barriers in the off-shell Gibbs free energy. In addition, we observed oscillatory behaviours in the probability distributions in various cases, reflecting the complex dynamics near the quadruple point.

The significance of this work lies in its contribution to understanding the dynamics of black hole phase transitions and the role of the off-shell Gibbs free energy in these processes. This study offers new insights into the behaviour of black holes at multicritical points and highlights the need for further exploration of these phenomena. Future research could expand on these findings by investigating higher-order multicritical points, such as quintuple points, and exploring their dynamics. Additionally, studying phase transitions in more complex gravity models, including multiply rotating Kerr-AdS black holes and Lovelock gravity, could further enhance our understanding of black hole behaviour under diverse conditions.

In the framework of AdS/CFT correspondence, we explored various holographic complexity proposals for conformal field theories compactified on a circle with a U(1) gauge field, corresponding to magnetized AdS soliton gravitational spacetime [2]. Complexity=volume (CV [49, 50] and CV2.0 [51]) exhibited negative complexity of formation, while complexity=action (CA [52, 53, 54]) showed positive complexity of formation. Despite these differences, common features emerged: all proposals demonstrated a confinement-deconfinement phase transition controlled by the Wilson line, with the complexity of formation consistently serving as an order parameter for this transition. Furthermore, both volume-based and action-based complexity, along with a broader family of complexity functionals [55, 56], were observed to adhere to the same scaling relation.

The significance of this work lies in its thorough analysis of the differences and similarities among various proposals of holographic complexity. This analysis provides valuable insights for future research on the relationship between holographic complexity, phase transitions, and scaling laws. Future research could extend these findings by exploring additional complexity functionals and their behaviour in diverse holographic settings. Investigating higher-dimensional spacetimes, more complex gauge fields, or alternative boundary conditions may provide more insights into the general applicability of the observed scaling relations. Additionally, further examination of how these complexity functionals relate to different phase transitions could enhance our understanding of quantum gravity.

Whether a complete theory of quantum gravity exists and what such a theory would look like are still open questions. Despite black hole thermodynamics and the holographic complexity approach, there are numerous approaches to quantum gravity, such as string theory [13, 14, 15], loop quantum gravity [16, 17], twistor theory [101, 102], double copy theory [103, 104] and causal set theory [105], each differing in which aspects of general relativity and quantum theory they retain or modify. More efforts are needed to advance the construction of a complete theory of quantum gravity. Hopefully, this thesis will contribute to a deeper understanding of the quantum nature of spacetime and offer valuable insights for the field of quantum gravity.

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