

Assessment, Mitigation, and Backtesting of Extreme Risks in Insurance and Finance

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

In this thesis, we focus on the quantitative assessment, mitigation, and backtesting of extreme risks in insurance and finance. We adopt tools from extreme value theory (EVT), dependence modeling, and statistical testing to address fundamental challenges in managing low-probability but high-impact events. The research contributes to three key aspects of extreme risk management: systemic risk assessment, catastrophe risk mitigation, and risk model evaluation.

We begin with the assessment of systemic risk under extreme scenarios. Systemic events are characterized by strong dependence among individual entities and heavy-tailed risk behavior. In Chapter 3, we develop an asymptotic framework for systemic risk measures, covering both Value-at-Risk-based and expectile-based risk measures. Second-order asymptotic approximations are derived to improve the accuracy of risk quantification beyond conventional first-order results. Special attention is given to expectile-based systemic risk measures, which provide a more conservative assessment of systemic risk.

The limitation of diversification for heavy-tailed risks has been well documented in the literature, especially for extremely heavy-tailed risks with infinite first moment. However, in practical insurance markets, catastrophic risks often exhibit extremely heavy tails but are also subject to truncation due to limited liability or policy design. In Chapter 4, we investigate the effectiveness of catastrophe risk pooling under these realistic constraints. Building on EVT, we characterize conditions under which diversification benefits remain achievable. The analysis incorporates tail heaviness, loss scaling, liability structure, and risk-sharing rules, providing theoretical foundations and practical guidance for catastrophe risk management.

In Chapter 5, we turn to the problem of risk model evaluation through backtesting. Conventional backtesting methods often rely on strict model assumptions and may fail under model misspecification, structural change or dependence uncertainty. Building on existing model-free testing approaches using e-values and e-processes, we extend these ideas to develop a backtesting framework for identifiable and elicitable risk measures, including Value-at-Risk, Expected Shortfall, and expectiles. The proposed framework delivers valid statistical inference for both standard and comparative backtests, and supports robust risk assessment across a wide range of risk levels, not limited to extreme events.

Throughout the thesis, theoretical results are complemented by extensive simulation studies and real-world applications. These findings contribute to advancing the theoretical foundations and practical methodologies for extreme risk management in modern insurance and financial systems.

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Dedication

To my family

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Chapter 1

Introduction

The growing occurrence and global impact of “Black Swan” events have underscored the importance of continuously developing risk management methodologies in financial and insurance markets. Extreme events such as the COVID-19 pandemic, the 2021 US stock market crash, and increasingly frequent large-scale natural disasters have illustrated the complexity and magnitude of risks that can arise in such environments. In particular, these events reveal the challenges associated with accurately capturing heavy-tailed distributions, complex dependence structures, and structural instability that characterize extreme risk scenarios. These characteristics present significant considerations for the design and implementation of risk assessment and mitigation tools, especially when managing low-probability but high-impact events.

These developments underscore the necessity for systematic, theoretically sound, and data-driven methodologies that can more effectively address the challenges posed by extreme risks in insurance and financial markets. Accurate risk measurement techniques capable of capturing the tail behavior and dependence among risks, effective risk-sharing mechanisms to mitigate the financial burden of catastrophic events, and reliable tools to monitor and validate risk models under realistic, potentially unstable conditions are essential to enhance the stability and risk-bearing capacity of these systems. Advancing these areas is key to improving the resilience of financial and insurance markets and ensuring their ability to function effectively under conditions of heightened uncertainty.

This thesis contributes to the quantitative management of extreme risks through three fundamental and interconnected components: (1) assessment of extreme risks, (2) mitigation of extreme risks, and (3) monitoring the performance of risk forecasts. Specifically, this thesis focuses on: (1) developing a more accurate asymptotic framework for systemic risk measurement under general dependence structures and heavy-tailed risk distributions, (2) proposing effective diversification mechanisms for extremely heavy-tailed natural catastrophe risks, and (3) developing a model-free backtesting framework for identifiable and elicitable risk measures. While the first two components address the unique challenges posed by systemic financial events and catastrophic losses, the third component provides a general statistical framework for evaluating risk models, applicable not only to extreme risk scenarios but also to risk assessment under more moderate conditions. Together, these research directions align with the key stages of the actuarial control cycle and form a systematic foundation for enhancing the resilience of insurance and financial systems.

The structure of this thesis is organized as follows. In Chapter 2, we prepare some mathematical concepts and tools regarding heavy-tailed distributions and risk measures that will be widely used in the rest of the thesis.

In Chapter 3, we advance the theoretical foundations of systemic risk measurement in the context of financial systemic events. We provide an asymptotic framework for systemic risk measures, classifying them into two families: Value-at-Risk (VaR)-based and expectile-based measures. In particular, we conduct a comparative study of two expectile based systemic risk measures, the Individual Conditional Expectile (ICE) and the Systemic Individual Conditional Expectile (SICE). These measures serve as expectile based counterparts to the widely used Marginal Expected Shortfall and Systemic Expected Shortfall. They extend existing concepts and offer a systematic approach to evaluating systemic risk under mutually dependent and heavy tailed risks. To capture the complex dependence structures commonly observed in practice, we adopt a flexible multivariate Sarmanov distribution framework. Building on this, we derive second order asymptotic approximations for both VaR based and expectile based systemic risk measures, which improve the approximation accuracy compared to conventional first order results. We further investigate the associated diversification benefits and show that expectile based measures tend to produce more conservative risk assessments. This property makes them appealing from a prudential

regulatory perspective. Our results contribute to the development of theoretically sound and practically relevant tools for systemic risk quantification in modern risk management.

In Chapter 4, we investigate effective mechanisms for catastrophe risk pooling to enhance diversification benefits for participants facing extremely heavy-tailed natural catastrophe risks. While risk pools have been widely advocated as a practical solution to improve financial resilience against catastrophic events, their effectiveness is fundamentally challenged by the extremely heavy-tailed nature of such losses. Building on extreme value theory, we develop a theoretical framework that characterizes conditions under which risk pools can be successfully formed and deliver meaningful diversification benefits. Specifically, we examine different pool structures, including uniform and proportional liability schemes, as well as uniform and proportional risk-sharing rules, and identify scenarios where fair and effective risk pooling is achievable. These conditions are derived through an asymptotic analysis of Value-at-Risk-based diversification ratios, providing practical guidance for designing sustainable catastrophe risk pools. Our findings underscore the critical influence of factors such as tail heaviness, liability design, and the number of participants on pooling outcomes. The theoretical results are further supported by extensive simulations and real data analysis. These results provide practical guidance for insurers, policymakers, and stakeholders to enhance catastrophe risk management amid rising natural disaster exposures.

In Chapter 5, we extend recent model-free testing approaches based on e-values and e-processes to both standard and comparative backtesting of identifiable and elicitable risk measures, contributing to more reliable risk assessment in both regulatory and risk management contexts. Standard backtests focus on detecting risk underestimation for individual models, while comparative backtests assess the relative performance of competing forecasting methods. We construct rigorous e-tests tailored to each of these tasks, without requiring distributional assumptions. Our approach leverages the identifiability and elicibility properties of risk measures to design sequential tests applicable to a broad class of risk measures, including the mean, variance, Value-at-Risk, Expected Shortfall, and expectile. The proposed methodology ensures valid Type-I error and false discovery rate control, even in the presence of structural changes. Extensive numerical simulations and empirical analysis using real financial data demonstrate the effectiveness and practical

relevance of our framework for improving risk forecast evaluation in practice.

Several potential topics for further research are presented in [Chapter 6](#).

Chapter 2

Preliminaries

In this chapter, we prepare some mathematical concepts and tools necessary for the rest of the thesis.

2.1 Univariate Risk Measures

Value-at-Risk (VaR) is probably the most widely used risk measure in insurance institutions. It has a prominent role in the Basel regulatory framework and has also been influential in Solvency II (see [Embrechts et al. \(2011\)](#)). We define VaR as follows.

Definition 2.1.1 *The **Value-at-Risk (VaR)** at level $p \in (0, 1)$ of a loss variable X with distribution function $F_X(x) = \mathbb{P}(X \leq x), x \in \mathbb{R}$ is defined as the p -quantile of the loss distribution:*

$$\text{VaR}_p(X) = F_X^{\leftarrow}(p) = \inf\{x \in \mathbb{R} : F_X(x) \geq p\},$$

where $F_X^{\leftarrow}(p)$ is the inverse of the distribution function.

VaR is sometimes criticized for a number of different reasons. Most important are its lack of subadditivity property and the fact that it completely ignores the severity of losses

in the far tail of the loss distribution. The coherent risk measure Expected Shortfall (ES) was introduced to solve these issues.

Definition 2.1.2 *The **Expected Shortfall (ES)** also called Tail Value-at-Risk (TVaR) or Conditional Tailed Expectation (CTE) at level $p \in (0, 1)$ of a loss variable X with $\mathbb{E}(|X|) < \infty$ is defined as*

$$\text{ES}_p(X) = \frac{1}{1-p} \int_p^1 \text{VaR}_t(X) dt.$$

ES can be intuitively understood as the expected loss that is incurred in the event that VaR is exceeded, and thus, it looks further into the tail of the loss distribution. However, ES is not elicitable (see [Gneiting \(2011\)](#)). Elicitability can be interpreted as the property that the functional of probability distributions can be estimated by generalized regression, which is a helpful criterion for the determination of optimal point forecasts. Another property, which makes elicibility an important concept, is that it can be used for comparing the performance of different forecast methods (see [Gneiting \(2011\)](#) for a detailed discussion). It is interesting to look for risk measures that are coherent and elicitable.

Since VaR is not coherent and ES lacks direct elicibility, Expectiles have been suggested as coherent and elicitable alternatives to both VaR and ES.

Definition 2.1.3 *For a confidence level $p \in (0, 1)$ and square integrable X , the **Expectile** $\text{ex}_p(X)$ is defined as*

$$\text{ex}_p(X) = \arg \min_{\theta \in \mathbb{R}} \left\{ p \mathbb{E} \left[((X - \theta)_+)^2 \right] + (1 - p) \mathbb{E} \left[((X - \theta)_-)^2 \right] \right\},$$

where $x_+ := \max(x, 0)$ and $x_- := \min(x, 0)$.

The presence of terms X_+^2 and X_-^2 makes this definition well defined indeed as soon as $X \in L^2$ (i.e. $\mathbb{E}|X|^2 < \infty$). A more general but less intuitive definition is suggested by the following observation:

Lemma 2.1.1 (*Newey and Powell (1987) or Bellini and Bignozzi (2015)*)

If X is an integrable random variable, $e_p(X)$ is the unique solution θ of the equation

$$\theta - \mathbb{E}[X] = \frac{2p-1}{1-p} \mathbb{E}[(X - \theta)_+].$$

Thus, expectiles of a distribution function F with a finite absolute first-order moment are well defined. Expectiles summarize the distribution function in much the same way as the quantiles; see [Gneiting \(2011\)](#). Actually, the expectile ex_p with $p \geq \frac{1}{2}$ is the only risk measure which is elicitable, law-invariant and coherent; see [Bellini and Bignozzi \(2015\)](#). The study on expectiles also becomes increasingly popular in finance and econometric literature; see, for example, [De Rossi and Harvey \(2009\)](#), [Kuan et al. \(2009\)](#).

2.2 Marginal Risk Contributions

For risk management purposes, especially in extreme cases, it is useful to decompose the portfolio-wide risk into components (risk contributions) associated with the sub-portfolios or assets the portfolio comprises. There are quite a few approaches to this problem. In this thesis, to measure the risk contribution from an individual risk, we follow the Euler principle, whose definition is given below.

Definition 2.2.1 Let $X_1, X_2, \dots, X_n$ be loss variables such that $S_n = \sum_{i=1}^n X_i$ and let ρ be a risk measure. If the derivative $\frac{d\rho(S_n + hX_i)}{dh}$ exists at $h = 0$ then the risk contribution of X_i to $\rho(S_n)$ is defined by

$$\rho(X_i|S_n) = \left. \frac{d\rho(S_n + hX_i)}{dh} \right|_{h=0} \quad (2.1)$$

Applying the above definition to the Expected Shortfall yields the following theorem.

Theorem 2.2.1 The **Marginal Expected Shortfall (MES)** of X_m with respect to the portfolio S_n is defined as the risk contribution of X_m to the portfolio's Expected Shortfall. If

the partial derivative in (2.1) exists for ρ taken as the Expected Shortfall, for a confidence level $p \in (0, 1)$,

$$\text{MES}_{p,m}(S_n) = \text{ES}_p(X_m \mid S_n) = \mathbb{E}[X_m \mid S_n > \text{VaR}_p(S_n)].$$

Definition 2.2.2 For a confidence level $p \in (0, 1)$, the **Systemic Expected Shortfall (SES)** of a risk X_m is defined as

$$\text{SES}_{p,m}(S_n) = \mathbb{E}[(X_m - \text{VaR}_p(X_m))_+ \mid S_n > \text{VaR}_p(S_n)].$$

Analogous to MES, SES can be intuitively understood as the expected amount a financial institution is undercapitalized in a future catastrophic scenario in which the overall portfolio is undercapitalized.

2.3 Regular Variation

Regularly varying functions are those functions which behave asymptotically like power functions, which is an essential analytical tool for dealing with heavy tails, long-range dependence and domains of attraction.

Definition 2.3.1 A measurable function $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is said to be regular varying at $t_0 \in [-\infty, \infty]$ with index $\alpha \in \mathbb{R}$, denoted by $f \in \text{RV}_\alpha^{t_0}$, if for all $x > 0$

$$\lim_{t \rightarrow t_0} \frac{f(tx)}{f(t)} = x^\alpha.$$

When $t_0 = \infty$, we write $\text{RV}_\alpha^{t_0} = \text{RV}_\alpha$. In addition, if $\alpha = 0$, then f is said to be slowly varying at infinity.

The definition above implies that a smaller value of the tail index α corresponds to a heavier tail of the distribution function F . When considering two loss variables X and Y with unequal tail indices, it's widely acknowledged that the contribution of the lighter tail

to the aggregated loss $X + Y$ becomes asymptotically negligible compared to that of the heavier tail.

In the following lemma, we list some important properties of regular variation that this thesis will use for proofs.

Lemma 2.3.1 (*Bingham et al. (1989)*)

Let $f \in \text{RV}_\alpha$ for some $0 < \alpha < \infty$. We have the following:

1. (Potter's bounds) For every $\epsilon > 0$, there exists some $x_0 > 0$ such that for $x, y \geq x_0$,

$$(1 - \epsilon) \left(\left(\frac{y}{x} \right)^{\alpha + \epsilon} \wedge \left(\frac{y}{x} \right)^{\alpha - \epsilon} \right) \leq \frac{f(y)}{f(x)} \leq (1 + \epsilon) \left(\left(\frac{y}{x} \right)^{\alpha + \epsilon} \vee \left(\frac{y}{x} \right)^{\alpha - \epsilon} \right)$$

2. Let $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a non-decreasing function with $f(\infty) = \infty$. Then, $f \in \text{RV}_\alpha$ if and only if $f^\leftarrow \in \text{RV}_{1/\alpha}$.
3. (Karamata's theorem) Suppose there exists $x_0 > 0$ such that $f(x)$ is positive and locally bounded for $x \geq x_0$. If $\alpha \geq -1$ then

$$\lim_{x \rightarrow \infty} \frac{xf(x)}{\int_{x_0}^x f(s)ds} = \alpha + 1.$$

If $\alpha < -1$, or $\alpha = -1$ and $\int_0^\infty f(s)ds \leq \infty$, then

$$\lim_{x \rightarrow \infty} \frac{xf(x)}{\int_x^\infty f(s)ds} = -\alpha - 1.$$

The second-order regular variation (2RV) is a concept that is a generalization of regular variation (RV). The idea of 2RV was initially proposed to investigate the rate of convergence of the extreme order statistics in EVT; see [De Haan and Stadtmüller \(1996\)](#) and [De Haan and Ferreira \(2006\)](#).

Definition 2.3.2 A measurable function $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is said to be second-order regular varying with the first-order index $\alpha \in \mathbb{R}$ and second-order index $\beta \leq 0$, denoted by $f \in$

$2\mathcal{RV}_{\alpha,\beta}^{t_0}$, if there exists some eventually positive or negative measurable function $A(\cdot)$ with $A(t) \rightarrow 0$ as $t \rightarrow t_0$ such that

$$\lim_{t \rightarrow t_0} \frac{\frac{f(tx)}{f(t)} - x^\alpha}{A(t)} = x^\alpha \frac{x^\beta - 1}{\beta} =: H_{\alpha,\beta}(x), \quad \text{for all } x > 0.$$

Here $H_{\alpha,\beta}(x)$ is $x^\alpha \log x$ if $\beta = 0$, and $A(\cdot)$ is called an auxiliary function of f . If $t_0 = \infty$, we write $2\mathcal{RV}_{\alpha,\beta}^{t_0} = 2\mathcal{RV}_{\alpha,\beta}$.

It is worth noting that the auxiliary function $A(\cdot) \in \mathcal{RV}_\beta$; see for example Theorem 2.3.3 of [De Haan and Ferreira \(2006\)](#). Moreover, let X denote a loss variable with distribution function F . Here $\bar{F}(\cdot)$ means the survival function $\bar{F}(x) = 1 - F(x)$. The tail quantile function associated with the distribution function F is denoted by $U_F(\cdot) = (1/\bar{F})^\leftarrow(\cdot) = F^\leftarrow(1 - 1/\cdot)$. Note that $\bar{F}(\cdot) \in \mathcal{RV}_{-\alpha}$ for all $\alpha \in \mathbb{R}$ is equivalent to $U_F(\cdot) \in \mathcal{RV}_{1/\alpha}$ (see Corollary 1.2.10 of [De Haan and Ferreira \(2006\)](#)). Furthermore, if $\bar{F}(\cdot) \in 2\mathcal{RV}_{-\alpha,\beta}$ with $\alpha > 0, \beta \leq 0$ and auxiliary function $A(\cdot)$, by Theorem 2.3.9 of [De Haan and Ferreira \(2006\)](#), one can easily check that $U_F(\cdot) \in 2\mathcal{RV}_{1/\alpha,\beta/\alpha}$ with auxiliary function $\alpha^{-2}A \circ U_F(\cdot)$. Generally, the equality $F(F^\leftarrow(p)) = p$ does not hold true. It can be shown that if $\bar{F}(\cdot) \in \mathcal{RV}_{-\alpha}$ with $\alpha > 0$, then $F(F^\leftarrow(p)) \sim p$ as $p \uparrow 1$. If we further assume that $\bar{F}(\cdot) \in 2\mathcal{RV}_{-\alpha,\beta}$ with auxiliary function $A(\cdot)$, then

$$U_F(1/\bar{F}(t)) = t(1 + o(A(t))) \quad \text{as } t \rightarrow \infty, \quad F(F^\leftarrow(p)) = p(1 + o(F^\leftarrow(p))) \quad \text{as } p \uparrow 1; \quad (3.2)$$

see [Mao and Yang \(2015\)](#) and Exercise 2.11 of [De Haan and Ferreira \(2006\)](#).

Chapter 3

Value-at-Risk- and Expectile-based Systemic Risk Measures and Second-order Asymptotics: With Applications to Diversification ¹

3.1 Introduction

Insurers and governments establish risk-sharing pools to deal with catastrophe risks, including natural disasters, to mitigate risk and achieve enhanced financial stability. However, when faced with correlated risks and a deteriorating external environment, the effectiveness of the risk-sharing pool is compromised. It becomes crucial to accurately approximate the risk contribution of each loss within the pool, particularly in catastrophic scenarios when the risk pool is more likely to be undercapitalized. In this chapter, we provide an asymptotic treatment to approximate the systemic risk contributions from risks. The treatment has three steps. The first step is that the measuring methods are classified

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into two representative families: Value-at-Risk- (VaR-) and expectile-based ones. From the definitions, VaR_p and e_p exhibit distinct mathematical properties. As we can see below, both of them lead to many methods to approximating systemic risk contributions, which also known as systemic risk measures; e.g., ES_p (or CTE_p), $\text{MES}_{p,m}$ and $\text{SES}_{p,m}$ which are categorized within the family of VaR-based risk measures. As a result, VaR_p and e_p can serve as building blocks in approximating systemic risk contributions and form essential foundations for further study. Let $X_1, \dots, X_n$ denote the loss variables within the risk pool and the aggregate $S_n = \sum_{i=1}^n X_i$. Taylor (2008) introduced an expectile-based alternative to ES, known as the Conditional Expectile (CE_p):

$$\text{CE}_p(X) := \mathbb{E}[X | X > \text{ex}_p(X)],$$

where CE_p represents the expectation of exceedances beyond the p -th expectile ex_p of the distribution of X .

To evaluate each risk contribution to the portfolio, we further propose two new expectile-based systemic risk measuring methods. We call them the *Individual Conditional Expectile* (ICE) and the *Systemic Individual Conditional Expectile* (SICE):

$$\text{ICE}_{p,m}(S_n) := \mathbb{E}[X_m | S_n > \text{ex}_p(S_n)],$$

and

$$\text{SICE}_{p,m}(S_n) := \mathbb{E}[(X_m - \text{ex}_p(X_m))_+ | S_n > \text{ex}_p(S_n)].$$

ICE and SICE stand from a view of expectile to capture an individual agent's risk profile conditional on a system-wide catastrophe. For comparison, the VaR estimation knows only whether an observation is below or above the predictor. It would be inaccurate to measure an extreme risk based on only the frequency of tail losses and not on their values. This problem is typical in extreme cases where there are often few observations available, corresponding to very high confidence levels. The expectile makes more efficient use of the available data since it optimizes the discrepancy between the observations and the predictor. Particularly, ICE represents the potential losses an individual would contribute conditional on the tail of the portfolio's loss distribution. SICE is an improved version of ICE and reveals the individual's excess loss to his/her expectile $\text{ex}_p(X_m)$ conditional on

the systemic catastrophe.

In the second step, to characterize a general dependence and heavy-tailed risks, we assume that the risks $X_1, \dots, X_n$ are dependent on each other through a multivariate Sarmanov distribution. This characterizes a very general dependence structure, e.g., compared to the commonly-used Farlie-Gumbel-Morgenstern (FGM) copula; see [Yang and Hashorva \(2013\)](#). In particular, the Sarmanov distribution is flexible in joining different types of marginals, making it suitable for modeling various risks. The advantage of using the Sarmanov distribution lies in its ability to capture dependencies and helps the evaluation of joint probabilities.

Name	First-order asymptotics	Second-order asymptotics
$\text{VaR}_p(S_n)$	Bingham et al. (1989) ; Barbe et al. (2006) ; Embrechts et al. (2009) and so on.	Degen et al. (2010) ; Mao and Yang (2015) ; Theorem 3.3.1 of this chapter
$\text{CTE}_p(S_n)$	Alink et al. (2005) ; Chen et al. (2012) ; Kley et al. (2020) and so on.	Mao and Hu (2013) ; Lv et al. (2013) ; Theorem 3.3.1 of this chapter
$\text{MES}_{p,m}(S_n)$	Asimit et al. (2011) ; Joe and Li (2011) ; Jaunė and Šiaulys (2022) and so on.	Hua and Joe (2011) ; Theorem 3.3.2 of this chapter
$\text{SES}_{p,m}(S_n)$	Chen and Liu (2022)	Theorem 3.3.2 of this chapter
$\text{ex}_p(S_n)$	Bellini et al. (2014) ; Bellini and Di Bernardino (2017)	Mao et al. (2015) ; Mao and Yang (2015) ; Theorem 3.4.1 of this chapter
$\text{CE}_p(S_n)$	Dhaene et al. (2022)	Theorem 3.4.1 of this chapter
$\text{ICE}_{p,m}(S_n)$	Emmer et al. (2015) ; Tadese and Drapeau (2020)	Theorem 3.4.2 of this chapter
$\text{SICE}_{p,m}(S_n)$	Theorem 3.4.2 of this chapter	Theorem 3.4.2 of this chapter

Table 3.1: Contribution of this chapter compared to the literature. Here VaR-based systemic risk measures include VaR, CTE, MES and SES, while expectile-based systemic risk measures include ex, CE, ICE and SICE.

In the third step, we obtain the second-order asymptotics of the two families of systemic risk measures. Here we make our most theoretical contributions on asymptotic approximations. First, we investigate the second-order expansions of tail probability of

S_n under multivariate Sarmanov distribution (Proposition 3.3.1 below), which generalizes the results of Mao and Hu (2013) and Theorem 4.4 of Mao and Yang (2015). Second, we study second-order asymptotic formulas of $\text{VaR}_p(S_n)$, $\text{CTE}_p(S_n)$ (Theorem 3.3.1 below) and $\text{MES}_{p,m}(S_n)$, $\text{SES}_{p,m}(S_n)$ (Theorem 3.3.2 below). Third, we use different methods to obtain the second-order asymptotic estimation of expectile, which extends theorem 3.1 of Mao and Yang (2015) (Proposition 3.4.1 below). Fourth, we consider the second-order asymptotic formulas of $\text{ex}_p(S_n)$, $\text{CE}_p(S_n)$ (Theorem 3.4.1 below) and $\text{ICE}_{p,m}(S_n)$, $\text{SICE}_{p,m}(S_n)$ (Theorem 3.4.2). Lastly, we apply two examples to explain different risk measures based on VaR and expectile. Numerical and analytical examples illustrate that our second-order asymptotics provide an accurate estimate and behave much better than the first-order asymptotics. Further, we compare these two families of systemic risk measures and find that expectile-based systemic risk measures produce a larger risk evaluation sensitivity than that of VaR-based systemic risk measures.

Moreover, we explore economic insights on diversification with our asymptotic treatment. In risk management, evaluating diversification benefits properly is a key step, since risk diversification is one of the main purpose to establish the risk pool that may reduce a company's need for risk-based capital. The purposes of establishing the portfolio diversification dates back to the celebrated Markowitz mean-variance model, revealing the importance of mitigating risks in the investment. Diversification hence becomes a crucial topic in banking and insurance for risk management, integral to regulatory frameworks like Basel II and Solvency II. A lot of works propose quantitative ways to measure the advantage of benefits. E.g., Chen et al. (2022) delved into comparing diversification advantages under the worst-case VaR and ES in the context of dependence uncertainty; see Chen et al. (2024) for more results. Among them, the *diversification benefit*, proposed by Bürgi et al. (2008), signifies the preserved capital achieved through collectively considering all risks in a portfolio versus addressing each risk in isolation; see Section 3.6 for a detailed definition. Based on our asymptotic treatment, we obtain the second-order asymptotics for the diversification benefits based on different risk measures, including VaR, CTE, ex and MES. We find that CE, an expectile-based systemic risk measure, provides the most accurate approximation with an error range of less than 5%. Further, the expectile-based diversification benefits consistently deduce an underestimation and suggest a conservative approximation, while the VaR-based diversification benefits consistently deduce an over-

estimation and suggest an optimistic approximation.

Finally, we discuss our results compared with the literature. In the field of risk management and capital allocation, it is necessary to determine how to allocate the acquired economic capital among different risks. In this case, the solvency capital has already been calculated using risk aggregation techniques; see [Blanchet et al. \(2020\)](#) for a recent treatment of risk aggregation. In recent years, the focus is on selection of appropriate models for multivariate risk factors, such as the choice of dependence structure model and the distributions of the marginals. Some recent contributions include the use of the FGM distribution ([Yang and Hashorva \(2013\)](#), [Chen and Yang \(2014\)](#), etc) and the Sarmanov distribution ([Qu and Chen \(2013\)](#), [Yang and Hashorva \(2013\)](#), etc), and multivariate regular variation (MRV) ([Embrechts et al. \(2009\)](#), [Asimit et al. \(2011\)](#), etc). They usually aim to obtain the first-order asymptotics of some risk measures. On the contrary, our results provide a series of second-order asymptotics, which is much more accurate than the first-order asymptotics; this will be shown in the tables and figures later. This fact is also studied in [Degen et al. \(2010\)](#), [Mao et al. \(2012\)](#), [Mao and Hu \(2013\)](#), but they attain the second-order asymptotics for independent and identically distributed (iid) random variables (rvs) with second-order regularly varying tails. This independence assumption is too restrictive for practical problems. [Mao and Yang \(2015\)](#) studied the case that $X_1, \dots, X_n$ are dependent on each other through a multivariate FGM distribution, and derived second-order approximations of the risk concentrations of Value-at-Risk and expectile. However, our study contributes to the advancement of systemic risk measurement by introducing novel measures, developing a modeling framework, and providing enhanced asymptotic tools for risk approximation. Technically, the proposed two wide families of systemic risk measures can include the result in [Mao and Yang \(2015\)](#). In particular, we provide rigorous and necessary lemmas for asymptotic treatment (Lemmas [A.1-A.3](#)) and offer a simpler proof for the key results.

The rest of the chapter is organized as follows. In Section [3.2](#), we briefly discuss the n -dimensional Sarmanov distribution. In Sections [3.3](#) and [3.4](#), we present several second-order asymptotics of risk measures based on VaR and expectile and examples to explain the main results. In Section [3.5](#), we give two examples to illustrate some measures under VaR and expectile. Also, according to the second-order asymptotics of VaR, CTE, expectile

and CE, we further extend results on measuring diversification benefit. In Section 3.6, we apply our results to discuss the diversification benefit. Lastly, Section 3.7 concludes the chapter. Appendix A provides details for some proofs.

3.2 Multivariate Sarmanov Distribution

The Sarmanov distribution was originally introduced by Sarmanov (1966) in the bivariate case. Its applications in many insurance contexts show its flexible structure when modeling the dependence between multivariate risks given the distribution of the marginals; see Qu and Chen (2013), Abdallah et al. (2016), Ratovomirija (2016) and so on. It was then extended by Ting Lee (1996) and Kotz et al. (2004) in the multivariate case:

$$\mathbb{P}(X_1 \in dx_1, \dots, X_n \in dx_n) = \left(1 + \sum_{1 \leq i < j \leq n} a_{ij} \phi_i(x_i) \phi_j(x_j)\right) \prod_{k=1}^n dF(x_k), \quad (2.1)$$

where F is the corresponding marginal distribution of $X_1, X_2, \dots, X_n$. Particularly, the parameters a_{ij} are real numbers and the kernels ϕ_i are functions satisfying

$$\mathbb{E}[\phi_i(X_i)] = 0, \quad i = 1, \dots, n,$$

and

$$1 + \sum_{1 \leq i < j \leq n} a_{ij} \phi_i(x_i) \phi_j(x_j) > 0, \quad \text{for all } x_i \in D_{X_i}, i = 1, \dots, n,$$

where $D_{X_i} = \{x \in \mathbb{R} : \mathbb{P}(X_i \in (x - \delta, x + \delta)) > 0 \text{ for all } \delta > 0\}, i = 1, \dots, n$.

Similarly to those pointed out in Yang and Wang (2013), three common choices for the kernels $\phi_i, i = 1, \dots, n$ are listed below:

- (i) $\phi_i(x) = 1 - 2F(x)$ for all $x \in D_{X_i}$, leading to the well-known standard FGM distribution;
- (ii) $\phi_i(x) = (e^{-x} - c_i) \mathbf{1}_{\{x \geq 0\}}$ with $c_i = \mathbb{E}[e^{-X_i} \mathbf{1}_{\{X_i \geq 0\}}] / \mathbb{P}(X_i \geq 0)$ for all $x \in D_{X_i}$;
- (iii) $\phi_i(x) = x^p - \mathbb{E}[X_i^p]$ for all $x \in D_{X_i}$ and $p \in \mathbb{R}$ such that $\mathbb{E}[X_i^p] < \infty$.

We remark that for any proper bivariate Sarmanov distribution, the kernels are bounded. Thus, in case (iii), $D_{X_i}, i = 1, \dots, n$ are bounded sets. We further discuss the dependence

structure of two rvs (X_1, X_2) following a Sarmanov distribution with different kernel functions. As before, the kernel functions are written in the form $\phi(x) = f(x) - E[f(X)]$ with f properly chosen. To model the dependence between the two rvs X_1 and X_2 , we shall use Pearson's correlation coefficient, which is defined as

$$\rho_{12} = \frac{\mathbb{E}[X_1 X_2] - \mathbb{E}[X_1]\mathbb{E}[X_2]}{\sqrt{\text{Var}(X_1)}\sqrt{\text{Var}(X_2)}}.$$

In the case of the Sarmanov distribution, ρ_{12} can be rewritten as

$$\rho_{12} = \frac{a_{12}\mathbb{E}[X_1\phi_1(X_1)]\mathbb{E}[X_2\phi_2(X_2)]}{\sqrt{\text{Var}(X_1)}\sqrt{\text{Var}(X_2)}}. \quad (2.2)$$

Based on (2.2), we hereafter present Pearson's correlation coefficient for different kernel functions along with its maximal and minimal value.

Case 1: Let the kernel functions be $\phi_i(x) = 1 - 2F(x)$, which corresponds to the FGM distribution. It is well known that Pearson correlation coefficients ρ_{12} of the FGM distribution lie between $-\frac{1}{3}$ and $\frac{1}{3}$ (see [Schucany et al. \(1978\)](#)), which is an important drawback of the FGM distribution. [Huang and Kotz \(1984\)](#) show that, by considering the iterated generalization of FGM distribution proposed by [Johnson and Kotz \(1977\)](#), the range of correlation coefficients can be enlarged.

Case 2: Let the kernel functions be $\phi_i(x) = x^p - \mathbb{E}[X_i^p]$. A usual choice is $p = 1$, which leads to the Pearson correlation coefficients $\rho_{12} = a_{12}\sigma_1\sigma_2$. In this situation, according to [Ting Lee \(1996\)](#), we assume that the correlation coefficient of X_1 and X_2 exists. Let T_1 and T_2 be the corresponding upper truncation points, and suppose the marginal probability density functions are defined only for non-negative values. Under these conditions, a_{12} satisfies

$$\max \left\{ \frac{-1}{\mu_1\mu_2}, \frac{-1}{(T_1 - \mu_1)(T_2 - \mu_2)} \right\} \leq a_{12} \leq \min \left\{ \frac{1}{\mu_1(T_2 - \mu_2)}, \frac{1}{\mu_2(T_1 - \mu_1)} \right\},$$

where $\mu_i = \mathbb{E}[X_i]$. Then, the maximal and the minimal values of the correlation coefficient

are, respectively, given by

$$\rho_{12}^{\max} = \frac{\sigma_1 \sigma_2}{\max(\mu_1(T_2 - \mu_2), \mu_2(T_1 - \mu_1))}, \quad \rho_{12}^{\min} = \frac{-\sigma_1 \sigma_2}{\max\{\mu_1 \mu_2, (T_1 - \mu_1)(T_2 - \mu_2)\}}.$$

Case 3: Let the kernel functions be $\phi_i(x) = e^{-x} - g$ with $g = \mathbb{E}[e^{-X_i}]$ for all $x \in D_{X_i}$. In this case, if the correlation coefficient of X_1 and X_2 exists, the range of a_{12} is (see [Ting Lee \(1996\)](#))

$$\frac{-1}{\max(\mathcal{L}_1(1)\mathcal{L}_2(1), (1 - \mathcal{L}_1(1))(1 - \mathcal{L}_2(1)))} \leq a_{12} \leq \frac{1}{\max(\mathcal{L}_1(1)(1 - \mathcal{L}_2(1)), \mathcal{L}_2(1)(1 - \mathcal{L}_1(1)))},$$

where $\mathcal{L}_i(t) = \int_0^\infty \exp(-tx_i) dF_i(x_i)$. Then, according to (2.2), the maximal value of Pearson's correlation coefficient ρ_{12} can be written as follows

$$\rho_{12}^{\max} = \frac{[-\mathcal{L}'_1(1) - \mathcal{L}_1(1)\mu_1][-\mathcal{L}'_2(1) - \mathcal{L}_2(1)\mu_2]}{\max(\mathcal{L}_1(1)(1 - \mathcal{L}_2(1)), \mathcal{L}_2(1)(1 - \mathcal{L}_1(1)))\sigma_1\sigma_2}$$

and the minimal value can be expressed as

$$\rho_{12}^{\min} = -\frac{[-\mathcal{L}'_1(1) - \mathcal{L}_1(1)\mu_1][-\mathcal{L}'_2(1) - \mathcal{L}_2(1)\mu_2]}{\max(\mathcal{L}_1(1)\mathcal{L}_2(1), (1 - \mathcal{L}_1(1))(1 - \mathcal{L}_2(1)))\sigma_1\sigma_2},$$

where $\mu_i = \mathbb{E}[X_i]$.

3.3 Second-order Asymptotics of VaR-based Systemic Risk Measures

In this section, we study the second-order asymptotics of VaR-based systemic risk measures with multivariate Sarmanov distributions. Let $X_1, X_2, \dots, X_n$ denote the financial losses, which are identically distributed random variables with a distribution function F . Denote the distribution function of the aggregate risk $S_n = \sum_{i=1}^n X_i$ by $G(t) := P(S_n \leq t)$. Before stating some results, we use the following notation:

- (i) $\eta_\alpha := \alpha \int_0^{1/2} ((1-x)^{-\alpha} - 1) x^{-\alpha-1} dx + 2^{2\alpha-1} - 2^\alpha$;
- (ii) $\mu := \mathbb{E}[X]$;
- (iii) $\mu(t) := \int_0^t x dF(x)$;
- (iv) $\mu_i(t) := \int_0^t x \phi_i(x) dF(x), i = 1, \dots, n$;

First, we establish the second-order asymptotics of the random sum under multivariate Sarmanov distributions.

Proposition 3.3.1 *Let $X_1, \dots, X_n$ be nonnegative random variables with common marginal distribution F satisfying that $\bar{F}(\cdot) \in 2\text{RV}_{-\alpha, \beta}$ with $\alpha > 0, \beta \leq 0$ and an auxiliary function $A(\cdot)$. Suppose that $(X_1, \dots, X_n)$ follows an n -dimensional Sarmanov distribution given by (2.1) and $\lim_{t \rightarrow \infty} \phi_i(t) = d_i \in \mathbb{R}, \phi_i(\cdot) - d_i \in \text{RV}_{\rho_i}$ with $\rho_i \leq 0$ for each $i = 1, \dots, n$. Then as $t \rightarrow \infty$, we get that*

$$\frac{\bar{G}(t)}{\bar{F}(t)} = n \left(1 + \widetilde{A}_n(t) (1 + o(1)) \right),$$

where

$$\widetilde{A}_n(t) = \begin{cases} \alpha t^{-1} \mu_n^*(t) + o \left(A(t) + \sum_{i=1}^n (\phi_i(t) - d_i) \right), & \alpha \geq 1, \\ \eta_\alpha \kappa_n \bar{F}(t) + o \left(A(t) + \sum_{i=1}^n (\phi_i(t) - d_i) \right), & 0 < \alpha < 1, \end{cases} \quad (3.1)$$

and

$$\mu_n^*(t) := (n-1)\mu(t) + \sum_{1 \leq i < j \leq n} \frac{a_{ij}(d_i\mu_j(t) + d_j\mu_i(t))}{n},$$

$$\kappa_n := n-1 + \sum_{1 \leq i < j \leq n} \frac{2a_{ij}d_id_j}{n}.$$

Proof. For $t > 0$, denote the region $\Omega_t = \{(x_1, \dots, x_n) \in \mathbb{R}_+^n : \sum_{i=1}^n x_i > t\}$. In addition, let $X_1^*, \dots, X_n^*$ be iid with a distribution function F . According to Proposition 1.1 of [Yang and Wang \(2013\)](#), there exist n constants $c_i > 1, i = 1, \dots, n$, such that $|\phi_i(x_i)| \leq c_i - 1$ for all $x_i \in D_{X_i}$. Let $\widetilde{X}_1^*, \dots, \widetilde{X}_n^*$ be mutually independent rvs with marginal distributions $\widetilde{F}_1, \dots, \widetilde{F}_n$, which are also independent of $X_1^*, \dots, X_n^*$. Particularly, $\widetilde{F}_1, \dots, \widetilde{F}_n$ are defined by

$$d\widetilde{F}_i(x_i) := \left(1 - \frac{\phi_i(x_i)}{c_i}\right) dF(x_i), \quad i = 1, \dots, n.$$

By Lemma [A.1](#) in Appendix, here we have $\phi(t) = 1 - \frac{\phi_i(x_i)}{c_i}$. Thus, $\lim_{t \rightarrow \infty} \phi(t) = 1 - d_i/c_i$ and $\phi_i(\cdot) - 1 + d_i/c_i \in \text{RV}_{\rho_i}$. It follows that $\widetilde{F}_i(\cdot) \in 2\text{RV}_{-\alpha, \gamma_i}$ with $\gamma_i = \max\{\beta, \rho_i\}$ and auxiliary function $\widetilde{A}^i(\cdot) = A(\cdot) + \frac{\rho_i\alpha}{(c_i-d_i)(\alpha-\rho_i)}(\phi_i(\cdot) - d_i)$. In addition, as $t \rightarrow \infty$, we obtain

$$\frac{\widetilde{F}_i(t)}{\widetilde{F}(t)} = \left(1 - \frac{d_i}{c_i}\right) \left(1 - \frac{\alpha}{(c_i - d_i)(\alpha - \rho_i)}(\phi_i(t) - d_i)(1 + o(1))\right),$$

for all $i = 1, \dots, n$. Write $\mu_i(t) := \int_0^t x\phi_i(x)dF(x), i = 1, \dots, n$. We have that

$$\int_0^t x d\widetilde{F}_i(x) = \int_0^t x \left(1 - \frac{\phi_i(x)}{c_i}\right) dF(x) = \mu(t) - \frac{\mu_i(t)}{c_i}.$$

Next, we can split $\overline{G}(t)$ as

$$\begin{aligned}
\overline{G}(t) &= \int_{\Omega_t} \left(1 + \sum_{1 \leq i < j \leq n} a_{ij} \phi_i(x_i) \phi_j(x_j) \right) \prod_{k=1}^n dF(x_k) \\
&= \int_{\Omega_t} \left(1 + \sum_{1 \leq i < j \leq n} a_{ij} c_i c_j \left(1 - \left(1 - \frac{\phi_i(x_i)}{c_i} \right) - \left(1 - \frac{\phi_j(x_j)}{c_j} \right) \right. \right. \\
&\quad \left. \left. + \left(1 - \frac{\phi_i(x_i)}{c_i} \right) \left(1 - \frac{\phi_j(x_j)}{c_j} \right) \right) \right) \prod_{k=1}^n dF(x_k) \\
&= \int_{\Omega_t} \prod_{k=1}^n dF(x_k) + \sum_{1 \leq i < j \leq n} a_{ij} c_i c_j \left(\int_{\Omega_t} \prod_{k=1}^n dF(x_k) - \int_{\Omega_t} \prod_{k=1, k \neq i}^n dF(x_k) d\widetilde{F}_i(x_i) \right. \\
&\quad \left. - \int_{\Omega_t} \prod_{k=1, k \neq j}^n dF(x_k) d\widetilde{F}_j(x_j) + \int_{\Omega_t} \prod_{k=1, k \neq i, j}^n dF(x_k) d\widetilde{F}_i(x_i) d\widetilde{F}_j(x_j) \right) \\
&:= I(t) + \sum_{1 \leq i < j \leq n} a_{ij} c_i c_j (I(t) - I_i(t) - I_j(t) + I_{i,j}(t)). \tag{3.2}
\end{aligned}$$

To deal with $I(t)$, according to Propositions 3.6, 3.7 and Remark 3.1 of [Mao and Ng \(2015\)](#) with common distribution $\overline{F}(\cdot) \in 2\text{RV}_{-\alpha, \beta}$, it follows that

$$\frac{I(t)}{\overline{F}(t)} = \begin{cases} n(1 + (n-1)\alpha t^{-1}\mu(t)(1 + o(1))) + o(A(t)), & \alpha \geq 1, \\ n(1 + (n-1)\eta_\alpha \overline{F}(t)(1 + o(1))) + o(A(t)), & 0 < \alpha < 1. \end{cases}$$

By the similar analysis, $\widetilde{F}_i(\cdot) \in 2\text{RV}_{-\alpha, \gamma_i}$ and (A.1) in Appendix. We have

$$\frac{I_i(t)}{\overline{F}(t)} = \begin{cases} \left(n - \frac{d_i}{c_i} \right) (1 + \alpha(n-1)t^{-1}\mu(t)(1 + o(1))) - \left(\frac{\alpha(n-1)\mu_i(t)}{c_i t} + \frac{\alpha(\phi_i(t) - d_i)}{c_i(\alpha - \rho_i)} \right) (1 + o(1)) \\ \quad + o(A(t)), & \alpha \geq 1, \\ \left(n - \frac{d_i}{c_i} \right) + \eta_\alpha(n-1) \left(n - \frac{2d_i}{c_i} \right) \overline{F}(t)(1 + o(1)) - \frac{\alpha(\phi_i(t) - d_i)}{c_i(\alpha - \rho_i)} (1 + o(1)) \\ \quad + o(A(t)), & 0 < \alpha < 1, \end{cases}$$

and

$$\frac{I_{i,j}(t)}{\bar{F}(t)} = \begin{cases} \left(n - \frac{d_i}{c_i} - \frac{d_j}{c_j} \right) (1 + \alpha t^{-1} (n-1) \mu(t) (1 + o(1))) - \alpha \left(\left(n - 1 - \frac{d_j}{c_j} \right) \frac{\mu_i(t)}{c_i t} \right. \\ \quad \left. + \left(n - 1 - \frac{d_i}{c_i} \right) \frac{\mu_j(t)}{c_j t} + \frac{(\phi_i(t) - d_i)}{c_i(\alpha - \rho_i)} + \frac{(\phi_j(t) - d_j)}{c_j(\alpha - \rho_j)} \right) (1 + o(1)) + o(A(t)), & \alpha \geq 1, \\ \left(n - \frac{d_i}{c_i} - \frac{d_j}{c_j} \right) + \eta_\alpha \left((n-1) \left(n - 2 \left(\frac{d_i}{c_i} + \frac{d_j}{c_j} \right) \right) + \frac{2d_i d_j}{c_i c_j} \right) \bar{F}(t) (1 + o(1)) \\ \quad - \left(\frac{\alpha(\phi_i(t) - d_i)}{c_i(\alpha - \rho_i)} + \frac{\alpha(\phi_j(t) - d_j)}{c_j(\alpha - \rho_j)} \right) (1 + o(1)) + o(A(t)), & 0 < \alpha < 1. \end{cases}$$

Pulling all the asymptotics for $I(t)$, $I_i(t)$, and $I_{i,j}(t)$ into (3.2) yields that

$$\frac{\bar{G}(t)}{n\bar{F}(t)} = \begin{cases} 1 + \alpha t^{-1} \left((n-1) \mu(t) + \sum_{1 \leq i < j \leq n} \frac{a_{ij}(d_i \mu_j(t) + d_j \mu_i(t))}{n} \right) (1 + o(1)) \\ \quad + o \left(A(t) + \sum_{i=1}^n (\phi_i(t) - d_i) \right), & \alpha \geq 1, \\ 1 + \eta_\alpha \left(n - 1 + \sum_{1 \leq i < j \leq n} \frac{2a_{ij} d_i d_j}{n} \right) \bar{F}(t) (1 + o(1)) \\ \quad + o \left(A(t) + \sum_{i=1}^n (\phi_i(t) - d_i) \right), & 0 < \alpha < 1, \end{cases}$$

Thus, this ends the proof. ■

Remark 3.3.1 For different choices of the kernel function $\phi_i(x)$, the corresponding constants d_i and ρ_i are summarized as follows:

- If $\phi_i(x) = 1 - 2F(x)$, then $d_i = -1$ and $\rho_i = -\alpha$.
- If $\phi_i(x) = x^p - \mathbb{E}[X_i^p]$ for $p \leq 0$, then $d_i = -\mathbb{E}[X_i^p]$ and $\rho_i = p$.
- If $\phi_i(x) = e^{-x} - g$, then $d_i = -g$ and $\rho_i = -\infty$.

In addition, let $\rho_0 = \max_{1 \leq i \leq n} \rho_i$, associated with the kernel function $\phi_0(\cdot)$ and constant $d_0 \in \mathbb{R}$.

In view of Proposition 3.3.1, we can easily obtain the 2RV property of $\bar{G}(\cdot)$.

Corollary 3.3.1 *Under the conditions of Proposition 3.3.1, we have $\overline{G}(\cdot) \in 2\text{RV}_{-\alpha, \lambda}$ with $\lambda = \max\{-1, -\alpha, \beta\}$ and an auxiliary function $A_G^{(n)}(\cdot)$ given by*

$$A_G^{(n)}(t) = \begin{cases} A(t) - \alpha t^{-1} \mu_n^*(t), & \alpha \geq 1, \\ A(t) - \alpha \eta_\alpha \kappa_n \overline{F}(t), & 0 < \alpha < 1. \end{cases} \quad (3.3)$$

Proof. According to the definition of $\widetilde{A}_n(t)$, it is easy to check that $\widetilde{A}_n(t) \in \text{RV}_{-\tilde{\lambda}}$, where $\tilde{\lambda} = \min\{1, \alpha\}$. Note that $\overline{F}(t) \in 2\text{RV}_{-\alpha, \beta}$ with an auxiliary function $A(t)$, for any $x > 0$, as $t \rightarrow \infty$, we have that

$$\begin{aligned} \frac{\overline{G}(tx)}{\overline{G}(t)} &= \frac{\overline{G}(tx)}{\overline{F}(tx)} \frac{\overline{F}(tx)}{\overline{F}(t)} \frac{\overline{F}(t)}{\overline{G}(t)} \\ &= \frac{n \left(1 + \widetilde{A}_n(tx) (1 + o(1)) \right)}{n \left(1 + \widetilde{A}_n(t) (1 + o(1)) \right)} \left(x^{-\alpha} + H_{-\alpha, \beta}(x) A(t) (1 + o(1)) \right) \\ &= x^{-\alpha} + H_{-\alpha, \beta}(x) A(t) (1 + o(1)) + x^{-\alpha} (x^{\tilde{\lambda}} - 1) \widetilde{A}_n(t) (1 + o(1)) \\ &= \begin{cases} x^{-\alpha} + (H_{-\alpha, \beta}(x) A(t) - H_{-\alpha, -1}(x) \alpha t^{-1} \mu_n^*(t)) (1 + o(1)) + o \left(\sum_{i=1}^n (\phi_i(t) - d_i) \right), & \alpha \geq 1, \\ x^{-\alpha} + (H_{-\alpha, \beta}(x) A(t) - H_{-\alpha, -\alpha}(x) \alpha \eta_\alpha \kappa_n \overline{F}(t)) (1 + o(1)) + o \left(\sum_{i=1}^n (\phi_i(t) - d_i) \right), & 0 < \alpha < 1. \end{cases} \end{aligned}$$

Thus, we complete this proof. ■

Remark 3.3.2 (1) When $\phi_i(x) = 1 - 2F(x)$ for all $x \in D_{X_i}, i = 1, \dots, n$, Proposition 3.3.1 and Corollary 3.3.1 reduce to Theorem 4.4 and Corollary 4.5 of [Mao and Yang \(2015\)](#).

(2) If $\alpha \geq 1$ and $\beta < -1$, then $A(t) = o(\mu_n^*(t))$. If $\alpha < 1$ and $\beta < -\alpha$, then $A(t) = o(\overline{F}(t))$. If $\beta > -(1 \wedge \alpha)$, then $\mu_n^*(t) = o(A(t))$ and $\overline{F}(t) = o(A(t))$.

Second, we are ready to show the second-order asymptotics of $\text{VaR}_p(S_n)$ and $\text{CTE}_p(S_n)$.

Theorem 3.3.1 *Under the conditions of Proposition 3.3.1, we have, as $p \uparrow 1$,*

$$\frac{\text{VaR}_p(S_n)}{F^{\leftarrow}(p)} = \begin{cases} n^{1/\alpha} \left(1 + \left(\frac{\mu_n^*(F^{\leftarrow}(p))}{n^{1/\alpha} F^{\leftarrow}(p)} + \frac{n^{\beta/\alpha}-1}{\alpha\beta} A(F^{\leftarrow}(p)) \right) (1 + o(1)) \right) \\ \quad + o \left(\sum_{i=1}^n (\phi_i(t) - d_i) \right), & \alpha \geq 1, \\ n^{1/\alpha} \left(1 + \left(\frac{\eta_{\alpha, \beta} \kappa_n \bar{F}(F^{\leftarrow}(p))}{\alpha n} + \frac{n^{\beta/\alpha}-1}{\alpha\beta} A(F^{\leftarrow}(p)) \right) (1 + o(1)) \right) \\ \quad + o \left(\sum_{i=1}^n (\phi_i(t) - d_i) \right), & 0 < \alpha < 1. \end{cases}$$

For $\alpha > 1$, as $p \uparrow 1$,

$$\frac{\text{CTE}_p(S_n)}{F^{\leftarrow}(p)} = \frac{\alpha n^{1/\alpha}}{\alpha - 1} (1 + \zeta_{\alpha, \beta}^n A(F^{\leftarrow}(p)) (1 + o(1))) + \frac{\mu_n^*(F^{\leftarrow}(p))}{F^{\leftarrow}(p)} (1 + o(1)) + o \left(\sum_{i=1}^n (\phi_i(t) - d_i) \right),$$

and

$$\zeta_{\alpha, \beta}^n = \frac{1}{\alpha\beta} \left(\frac{n^{\beta/\alpha}(\alpha - 1)}{\alpha - \beta - 1} - 1 \right). \quad (3.4)$$

Here, by convention, $\frac{n^{\beta/\alpha}-1}{\alpha\beta} := \alpha^{-2} \log n$ and $\zeta_{\alpha, \beta}^n := \alpha^{-2} \log n$ if $\beta = 0$. Clearly, the first-order asymptotics of $\text{VaR}_p(S_n)$ and $\text{CTE}_p(S_n)$ are $n^{1/\alpha} F^{\leftarrow}(p)$ and $\frac{\alpha n^{1/\alpha}}{\alpha-1} F^{\leftarrow}(p)$.

Proof. Since $\bar{F}(\cdot) \in 2\text{RV}_{-\alpha, \beta}$ with an auxiliary function $A(\cdot)$ and Theorem 2.3.9 of De Haan and Ferreira (2006), one can easily check that $U_F(\cdot) \in 2\text{RV}_{1/\alpha, \beta/\alpha}$ with an auxiliary function $\alpha^{-2} A \circ U_F(\cdot)$. Let $t = G^{\leftarrow}(p)$. If $p \uparrow 1$, then $t \rightarrow \infty$. By the relation (3.2)

and Proposition 3.3.1, we get that

$$\begin{aligned}
\frac{\text{VaR}_p(S_n)}{F^{\leftarrow}(p)} &= \frac{G^{\leftarrow}(p)}{F^{\leftarrow}(p)} = \frac{U_F(1/\overline{F}(t))}{U_F(1/\overline{G}(t))(1+o(A(t)))} \\
&= \left(\frac{\overline{G}(t)}{\overline{F}(t)}\right)^{1/\alpha} \left(1 + \frac{\left(\frac{\overline{G}(t)}{\overline{F}(t)}\right)^{\beta/\alpha} - 1}{\beta/\alpha} \alpha^{-2} A \circ U_F\left(\frac{1}{\overline{G}(t)}\right)\right) \\
&= \left(n \left(1 + \widetilde{A}_n(t)(1+o(1))\right)\right)^{1/\alpha} \left(1 + \frac{n^{\beta/\alpha} - 1}{\alpha\beta} A \circ U_F\left(\frac{1}{\overline{G}(t)}\right)(1+o(1))\right),
\end{aligned}$$

where we use the transformation $t \mapsto G^{\leftarrow}(p)$ and the first-order Taylor expansion. Note that $\widetilde{A}_n(G^{\leftarrow}(p)) \sim \widetilde{A}_n(n^{1/\alpha}F^{\leftarrow}(p)) \sim n^{\tilde{\lambda}/\alpha}\widetilde{A}_n(F^{\leftarrow}(p))$ with $\tilde{\lambda} = -\min\{1, \alpha\}$. As $p \uparrow 1$, we have that

$$\begin{aligned}
\frac{\text{VaR}_p(S_n)}{F^{\leftarrow}(p)} &= n^{1/\alpha} \left(1 + \frac{1}{\alpha} \widetilde{A}_n(G^{\leftarrow}(p))(1+o(1)) + \frac{n^{\beta/\alpha} - 1}{\alpha\beta} A(F^{\leftarrow}(p))(1+o(1))\right) \\
&= n^{1/\alpha} \left(1 + \frac{1}{\alpha} n^{\tilde{\lambda}/\alpha} \widetilde{A}_n(F^{\leftarrow}(p))(1+o(1)) + \frac{n^{\beta/\alpha} - 1}{\alpha\beta} A(F^{\leftarrow}(p))(1+o(1))\right) \\
&= \begin{cases} n^{1/\alpha} \left(1 + \left(\frac{\mu_n^*(F^{\leftarrow}(p))}{n^{1/\alpha}F^{\leftarrow}(p)} + \frac{n^{\beta/\alpha}-1}{\alpha\beta} A(F^{\leftarrow}(p))\right)(1+o(1))\right) \\ \quad + o\left(\sum_{i=1}^n (\phi_i(t) - d_i)\right), & \alpha \geq 1 \\ \\ n^{1/\alpha} \left(1 + \left(\frac{\eta_{\alpha\kappa n}}{\alpha n} \overline{F}(F^{\leftarrow}(p)) + \frac{n^{\beta/\alpha}-1}{\alpha\beta} A(F^{\leftarrow}(p))\right)(1+o(1))\right) \\ \quad + o\left(\sum_{i=1}^n (\phi_i(t) - d_i)\right). & 0 < \alpha < 1. \end{cases}
\end{aligned}$$

By Proposition 3.3.1 and the definition of $\text{CTE}_p(S_n)$, we have

$$\begin{aligned}
\frac{\text{CTE}_p(S_n)}{F^{\leftarrow}(p)} &= \frac{\mathbb{E}[S_n | S_n > \text{VaR}_p(S_n)]}{F^{\leftarrow}(p)} \\
&= \frac{\alpha}{\alpha-1} \left(1 + \frac{1}{\alpha(\alpha-\lambda-1)} A_G^{(n)}(G^{\leftarrow}(p)) (1 + o(1)) \right) \frac{\text{VaR}_p(S_n)}{F^{\leftarrow}(p)} \\
&= \frac{\alpha n^{1/\alpha}}{\alpha-1} \left(1 + \frac{n^{\lambda/\alpha}}{\alpha(\alpha-\lambda-1)} A_G^{(n)}(F^{\leftarrow}(p)) (1 + o(1)) \right) \\
&\quad \cdot \left(1 + \left(\frac{\mu_n^*(F^{\leftarrow}(p))}{n^{1/\alpha} F^{\leftarrow}(p)} + \frac{n^{\beta/\alpha} - 1}{\alpha\beta} A(F^{\leftarrow}(p)) \right) (1 + o(1)) \right) \\
&= \frac{\alpha n^{1/\alpha}}{\alpha-1} \left(1 + \frac{1}{\alpha\beta} \left(\frac{n^{\beta/\alpha}(\alpha-1)}{\alpha-\beta-1} - 1 \right) A(F^{\leftarrow}(p)) (1 + o(1)) \right) + \frac{\mu_n^*(F^{\leftarrow}(p))}{F^{\leftarrow}(p)} (1 + o(1)) \\
&\quad + o \left(\sum_{i=1}^n (\phi_i(t) - d_i) \right).
\end{aligned}$$

This ends the proof. ■

Remark 3.3.3 According to $\text{VaR}_p(S_n)$ in Theorem 3.3.1, we include Theorem 4.6 of [Mao and Yang \(2015\)](#) and our result provides a simpler proof. If $a_{ij} = 0$, for all $1 \leq i \neq j \leq n$, the n -dimensional Sarmanov distribution reduces to the independent r.v.s. In this case, $\text{CTE}_p(S_n)$ of Theorem 3.3.1 is consistent with Theorem 3.1 of [Mao et al. \(2012\)](#).

The following example is used to illustrate Theorem 3.3.1 under the Pareto distribution with different parameters α . In addition, we consider different a_{12} under a Sarmanov distribution with $\phi_i(\cdot) = 1 - 2F(\cdot)$, $i = 1, 2$.

Example 3.3.1 (The Pareto distribution) A Pareto distribution function F satisfies that

$$F(x) = 1 - \left(\frac{k}{x+k} \right)^\alpha, \quad x, k, \alpha > 0.$$

It can be described that $\overline{F}(\cdot) \in 2\text{RV}_{-\alpha, -1}$ with an auxiliary function $A(t) = \alpha k/t$ and $F^{\leftarrow}(p) = k((1-p)^{-1/\alpha} - 1)$. If $\alpha > 1$, we have $\mu = k/(\alpha-1)$ and $\mu(t) = k/(\alpha-1) - (\alpha t +$

$k)/(\alpha - 1)\bar{F}(t)$. Let X_1 and X_2 have an identical Pareto distribution F . Suppose that the random vector (X_1, X_2) follows a Sarmanov distribution in (2.1) with $\phi_i(\cdot) = 1 - 2F(\cdot)$, $i = 1, 2$. Clearly, $d_i = -1$, $\rho_i = -\alpha$ and $\mu_i(t) = k/(2\alpha - 1) - (2\alpha t + k)/(2\alpha - 1) (\bar{F}(t))^2 - \mu(t)$, $i = 1, 2$. Then

$$\text{VaR}_p(S_2) = \begin{cases} 2^{1/\alpha} F^{\leftarrow}(p) + \mu(F^{\leftarrow}(p)) - a_{12}\mu_1(F^{\leftarrow}(p)) + k(2^{1/\alpha} - 1), & \alpha \geq 1, \\ 2^{1/\alpha} F^{\leftarrow}(p) \left(1 + \frac{\eta_\alpha(1+a_{12})}{2\alpha}(1-p)\right) + k(2^{1/\alpha} - 1), & 0 < \alpha < 1, \end{cases}$$

and for $\alpha > 1$,

$$\text{CTE}_p(S_2) = \frac{2^{1/\alpha}\alpha}{\alpha - 1} (F^{\leftarrow}(p) + k) - k + \mu(F^{\leftarrow}(p)) - a_{12}\mu_1(F^{\leftarrow}(p)).$$

Table 3.2: Simulated values (MC) versus first-order (1st) and second-order (2nd) asymptotics values $\text{VaR}_p(S_2)$ and $\text{CTE}_p(S_2)$ with various values of α . The Pareto distribution with $k = 1$ and $p = 0.99$, $a_{12} = 0.5$. The simulated values are the averages over 100 repetitions with a sample size of 10^6 each.

α	$\text{VaR}_p(S_2)_{MC}$	$\text{VaR}_p(S_2)_{1st}$	$\text{VaR}_p(S_2)_{2nd}$	$\text{CTE}_p(S_2)_{MC}$	$\text{CTE}_p(S_2)_{1st}$	$\text{CTE}_p(S_2)_{2nd}$	$\frac{\text{VaR}_p(S_2)_{1st}}{\text{VaR}_p(S_2)_{MC}}$	$\frac{\text{VaR}_p(S_2)_{2nd}}{\text{VaR}_p(S_2)_{MC}}$	$\frac{\text{CTE}_p(S_2)_{1st}}{\text{CTE}_p(S_2)_{MC}}$	$\frac{\text{CTE}_p(S_2)_{2nd}}{\text{CTE}_p(S_2)_{MC}}$
1.1	126.8065	121.5690	126.1943	1561.6015	1337.2588	1360.6628	0.9587	0.9952	0.8563	0.8713
1.5	35.3132	32.6000	34.9843	102.9805	97.8000	103.3591	0.9232	0.9907	0.9497	1.0037
2.0	14.4215	12.7262	14.1893	28.5141	25.4524	28.3298	0.8824	0.9839	0.8926	0.9935
2.5	8.2435	7.0060	8.0581	13.8310	11.6767	13.6085	0.8499	0.9775	0.8442	0.9839
3.0	5.5535	4.5847	5.4053	8.5145	6.8771	8.3277	0.8255	0.9733	0.8077	0.9781
4.0	3.2479	2.5708	3.1404	4.5244	3.4277	4.3938	0.7915	0.9669	0.7576	0.9711
5.0	2.2591	1.7378	2.1739	2.9994	2.1722	2.8956	0.7692	0.9623	0.7242	0.9654

In Table 3.2 we find that the values of VaR and CTE decrease as α rises and the second-order asymptotics of VaR and CTE are closer to the empirical value as $p \uparrow 1$. However, the asymptotic values of CTE are not accurate as α is close to 1, because CTE may not exist as $\alpha \uparrow 1$.

The following theorem obtains the second-order asymptotics of MES and SES under an n -dimensional Sarmanov distribution. These results are important in a wide range of marginal risk contributions, capital allocation and systemic risk.

Theorem 3.3.2 Let $X_1, \dots, X_n$ be nonnegative random variables with a common marginal distribution F satisfying that $\bar{F}(\cdot) \in 2\text{RV}_{-\alpha, \beta}$ with $\alpha > 1$, $\beta \leq 0$ and auxiliary function $A(\cdot)$. Suppose that $(X_1, \dots, X_n)$ follows an n -dimensional Sarmanov distribution given by (2.1) and $\lim_{x_i \rightarrow \infty} \phi_i(x_i) = d_i \in \mathbb{R}$, $\phi_i(\cdot) - d_i \in \text{RV}_{\rho_i}$ with $\rho_i \leq 0$ for each $i = 1, \dots, n$. Then as $t \rightarrow \infty$, we get that

$$\frac{\text{MES}_{p,m}(S_n)}{F^{\leftarrow}(p)} = \frac{\alpha n^{1/\alpha}}{(\alpha - 1)n} \left(1 + \zeta_{\alpha, \beta}^n A(F^{\leftarrow}(p)) (1 + o(1)) \right) + o \left(t^{-1} + o \left(\sum_{i=1}^n (\phi_i(t) - d_i) \right) \right),$$

where $\zeta_{\alpha, \beta}^n$ is defined in (3.4) and

$$\frac{\text{SES}_{p,m}(S_n)}{F^{\leftarrow}(p)} = \frac{\text{MES}_{p,m}(S_n)}{F^{\leftarrow}(p)} - \frac{1}{n}.$$

Obviously, the first-order asymptotics of $\text{MES}_p(S_n)$ and $\text{SES}_p(S_n)$ are $\frac{\alpha n^{1/\alpha} F^{\leftarrow}(p)}{(\alpha - 1)n}$ and $\frac{\alpha n^{1/\alpha} F^{\leftarrow}(p)}{(\alpha - 1)n} - \frac{F^{\leftarrow}(p)}{n}$.

Proof. Here we require Lemma A.4 in Appendix A and Theorem 3.3.1. Define $\tilde{B}(t) = \frac{1}{\alpha(\alpha - \beta - 1)} A(t) - \frac{\mu_n^*(t)}{t}$. We have that $\tilde{B} \in \text{RV}_{\rho}$ with $\rho = \min\{-1, \beta\}$. It follows that, as $p \uparrow 1$,

$$\begin{aligned} \frac{\text{MES}_{p,m}(S_n)}{F^{\leftarrow}(p)} &= \frac{\mathbb{E}[X_m | S_n > \text{VaR}_p(S_n)]}{F^{\leftarrow}(p)} \\ &= \frac{\alpha}{(\alpha - 1)n} \left(1 + \tilde{B}(\text{VaR}_p(S_n)) (1 + o(1)) \right) \frac{\text{VaR}_p(S_n)}{F^{\leftarrow}(p)} \\ &= \frac{\alpha n^{1/\alpha}}{(\alpha - 1)n} \left(1 + n^{\rho/\alpha} \tilde{B}(F^{\leftarrow}(p)) (1 + o(1)) \right) \\ &\quad \cdot \left(1 + \left(\frac{\mu_n^*(F^{\leftarrow}(p))}{n^{1/\alpha} F^{\leftarrow}(p)} + \frac{n^{\beta/\alpha} - 1}{\alpha \beta} A(F^{\leftarrow}(p)) \right) (1 + o(1)) \right) \\ &= \frac{\alpha n^{1/\alpha}}{(\alpha - 1)n} \left(1 + \frac{1}{\alpha \beta} \left(\frac{n^{\beta/\alpha}(\alpha - 1)}{\alpha - \beta - 1} - 1 \right) A(F^{\leftarrow}(p)) \right) (1 + o(1)) + o(t^{-1} + o(\phi_0(t) - d_0)). \end{aligned}$$

Here we require Lemma A.5 in Appendix A and Theorem 3.3.1, we conclude that

$$\begin{aligned}\frac{\text{SES}_{p,m}(S_n)}{F^{\leftarrow}(p)} &= \frac{\mathbb{E}[(X_m - \text{VaR}_p(X_m))_+ | S_n > \text{VaR}_p(S_n)]}{F^{\leftarrow}(p)} \\ &= \frac{\text{MES}_{p,m}(S_n)}{F^{\leftarrow}(p)} - \frac{1}{n}.\end{aligned}$$

This completes the proof for $\text{MES}_{p,m}(S_n)$ and $\text{SES}_{p,m}(S_n)$. ■

Lastly, Example 3.3.2 is used to explain Theorem 3.3.2 with the Burr distribution. We use $B(u, v)$ to represent the Beta distribution.

Example 3.3.2 (*Burr distribution*) A Burr distribution function F satisfies that

$$F(x) = 1 - (1 + x^{-\beta})^{\alpha/\beta}, \quad x, \alpha > 1, \beta < 0.$$

It is easy to check that $\bar{F}(\cdot) \in 2\text{RV}_{-\alpha, \beta}$ with auxiliary function $A(t) = \alpha t^\beta$, $\mu = -\alpha/\beta B\left(\frac{1-\alpha}{\beta}, \frac{\beta-1}{\beta}\right)$ and $F^{\leftarrow}(p) = ((1-p)^{\beta/\alpha} - 1)^{-1/\beta}$. Let X_1 and X_2 have an identical Burr distribution F . Suppose that the random vector (X_1, X_2) follows a Sarmanov distribution in (2.1) with $\phi_i(\cdot) = 1 - 2F(\cdot)$. Clearly, $d_i = -1, i = 1, 2$. Then,

$$\text{MES}_p(S_2) = \frac{2^{1/\alpha} \alpha F^{\leftarrow}(p)}{2(\alpha - 1)} \left(1 + \frac{1}{\beta} \left(\frac{2^{\beta/\alpha} (\alpha - 1)}{\alpha - \beta - 1} - 1 \right) (F^{\leftarrow}(p))^\beta \right),$$

and

$$\text{SES}_p(S_2) = \frac{2^{1/\alpha} \alpha F^{\leftarrow}(p)}{2(\alpha - 1)} \left(1 + \frac{1}{\beta} \left(\frac{2^{\beta/\alpha} (\alpha - 1)}{\alpha - \beta - 1} - 1 \right) (F^{\leftarrow}(p))^\beta \right) - \frac{F^{\leftarrow}(p)}{2}.$$

It is easy to see that the first-order asymptotics of $\text{MES}_p(S_2)$ and $\text{SES}_p(S_2)$ are $\frac{2^{1/\alpha} \alpha F^{\leftarrow}(p)}{2(\alpha - 1)}$ and $\frac{2^{1/\alpha} \alpha F^{\leftarrow}(p)}{2(\alpha - 1)} - \frac{F^{\leftarrow}(p)}{2}$. By Figure 3.1, the second-order asymptotics of $\text{MES}_p(S_2)$ and $\text{SES}_p(S_2)$ are much closer to the simulation value than the first-order asymptotics for $p \in [0.95, 1)$.

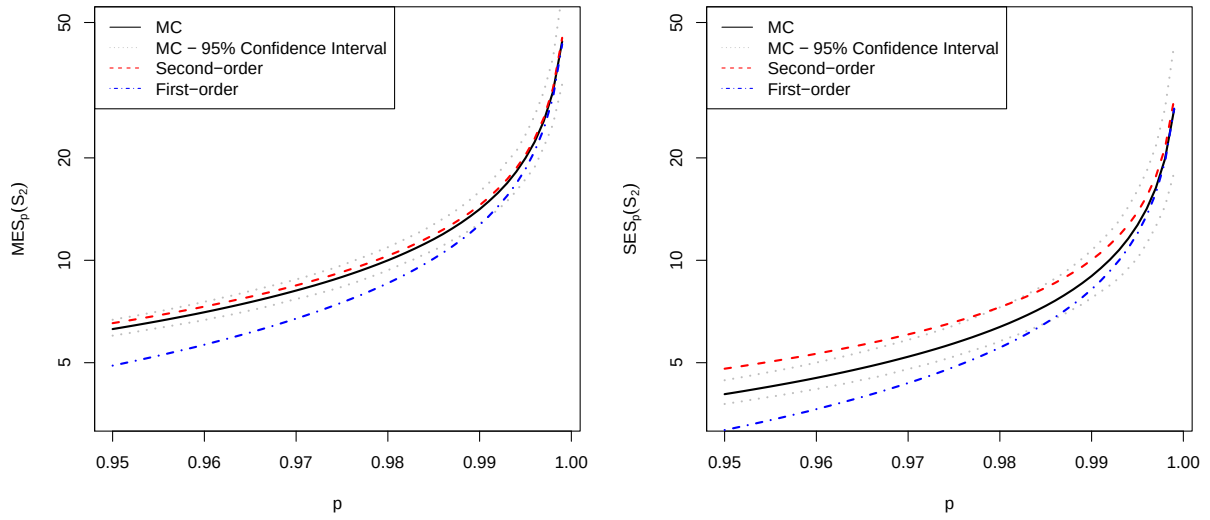


Figure 3.1: Simulated values (MC) versus first-order and second-order asymptotic values of $MES_p(S_2)$ (left) and $SES_p(S_2)$ (right) with $\alpha = 2, \beta = -0.5$ and $a_{12} = 0.5$. The simulated values are the averages over 100 repetitions with a sample size of 10^6 each.

3.4 Second-order Asymptotics of Expectile-based Systemic Risk Measures

In this section, we consider the second-order asymptotics of expectile-based systemic risk measures under multivariate Sarmanov distributions. Firstly, we establish the second-order asymptotics of the expectile under a different method with the literature.

Proposition 3.4.1 *Let $\overline{F}(\cdot) \in 2\text{RV}_{-\alpha, \beta}$ with $\alpha > 1, \beta \leq 0$ and from auxiliary function $A(\cdot)$. Then we have that as $p \uparrow 1$,*

$$\frac{\text{ex}_p(X)}{F^{\leftarrow}(p)} = (\alpha - 1)^{-1/\alpha} (1 + \xi_{\alpha, \beta} A(F^{\leftarrow}(p)) (1 + o(1))) + \frac{\mu}{\alpha F^{\leftarrow}(p)} (1 + o(1)),$$

where

$$\xi_{\alpha, \beta} = \frac{1}{\alpha\beta} \left(\frac{(\alpha - 1)^{1-\beta/\alpha}}{\alpha - \beta - 1} - 1 \right). \quad (4.1)$$

Here $\xi_{\alpha, \beta} := -\alpha^{-2} \log(\alpha - 1)$ if $\beta = 0$.

Proof. See Appendix. ■

Proposition 3.4.1 is derived by a different method with Corollary 1 of [Daouia et al. \(2018\)](#). In addition, due to $1 - p \sim \overline{F}(F^{\leftarrow}(p))$ as $p \uparrow 1$ and $\alpha > 1$, we derive that $1 - p = o(1/F^{\leftarrow}(p))$. Thus, Proposition 3.4.1 is consistent with Proposition 3.1 of [Mao et al. \(2015\)](#) or Theorem 3.1 of [Mao and Yang \(2015\)](#).

Secondly, we obtain the second-order asymptotics of $\text{ex}_p(S_n)$ and $\text{CE}_p(S_n)$ with an n -dimensional Sarmanov distribution.

Theorem 3.4.1 *Let $X_1, \dots, X_n$ be nonnegative random variables with a common marginal distribution F satisfying that $\overline{F}(\cdot) \in 2\text{RV}_{-\alpha, \beta}$ with $\alpha > 1, \beta \leq 0$ and auxiliary function $A(\cdot)$. Suppose that $(X_1, \dots, X_n)$ follows an n -dimensional Sarmanov distribution given by*

(2.1) and $\lim_{x_i \rightarrow \infty} \phi_i(x_i) = d_i \in \mathbb{R}$ for each $i = 1, \dots, n$. Then as $p \uparrow 1$, we get that

$$\begin{aligned} \frac{\exp_p(S_n)}{F^{\leftarrow}(p)} &= \left(\frac{n}{\alpha-1} \right)^{1/\alpha} \left(1 + \frac{1}{\alpha\beta} \left(\frac{n^{\beta/\alpha}(\alpha-1)^{1-\beta/\alpha}}{\alpha-\beta-1} - 1 \right) A(F^{\leftarrow}(p)) (1 + o(1)) \right) \\ &\quad + \frac{(\alpha-1)\mu_n^*(F^{\leftarrow}(p)) + n\mu}{\alpha F^{\leftarrow}(p)} (1 + o(1)) + o\left(\sum_{i=1}^n (\phi_i(t) - d_i) \right), \end{aligned}$$

and

$$\begin{aligned} \frac{\text{CE}_p(S_n)}{F^{\leftarrow}(p)} &= \frac{\alpha n^{1/\alpha}}{(\alpha-1)^{1/\alpha+1}} (1 + \chi_{\alpha,\beta}^n A(F^{\leftarrow}(p)) (1 + o(1))) \\ &\quad + \frac{(\alpha-2)\mu_n^*(F^{\leftarrow}(p)) + n\mu}{(\alpha-1)F^{\leftarrow}(p)} (1 + o(1)) + o\left(\sum_{i=1}^n (\phi_i(t) - d_i) \right), \end{aligned}$$

where

$$\chi_{\alpha,\beta}^n = \frac{1}{\alpha\beta} \left(\left(\frac{n}{\alpha-1} \right)^{\beta/\alpha} \frac{\alpha+\beta-1}{\alpha-\beta-1} - 1 \right). \quad (4.2)$$

Here $\chi_{\alpha,\beta}^n := \alpha^{-2}(\log n - \log(\alpha-1))$ if $\beta = 0$. In addition, the first-order asymptotics of $\exp_p(S_n)$ and $\text{CE}_p(S_n)$ are $\frac{n^{1/\alpha} F^{\leftarrow}(p)}{(\alpha-1)^{1/\alpha}}$ and $\frac{\alpha n^{1/\alpha} F^{\leftarrow}(p)}{(\alpha-1)^{1/\alpha+1}}$.

Proof. Firstly, to deal with $\mathbb{E}[S_n]$, owing to $(X_1, \dots, X_n)$ follows an n -dimensional Sarmanov distribution in (2.1), we have

$$\begin{aligned} \mathbb{E}[S_n] &= \int_{[0,\infty]^n} \sum_{i=1}^n x_i \left(1 + \sum_{1 \leq i < j \leq n} a_{ij} \phi_i(x_i) \phi_j(x_j) \right) \prod_{k=1}^n dF(x_k) \\ &= \sum_{i=1}^n \int_{[0,\infty]^n} x_i \left(1 + \sum_{1 \leq j \leq n, j \neq i} a_{ij} \phi_i(x_i) \phi_j(x_j) \right) \prod_{k=1}^n dF(x_k) \\ &= n\mu. \end{aligned}$$

According to Proposition 3.4.1 and Corollary 3.3.1, we have

$$\begin{aligned}
\text{ex}_p(S_n) &= (\alpha - 1)^{-1/\alpha} G^\leftarrow(p) \left(1 + \xi_{\alpha, \lambda} A_G^{(n)}(G^\leftarrow(p)) (1 + o(1)) \right) + \frac{\mathbb{E}(S_n)}{\alpha} (1 + o(1)) \\
&= (\alpha - 1)^{-1/\alpha} G^\leftarrow(p) \left(1 + n^{\lambda/\alpha} \xi_{\alpha, \lambda} A_G^{(n)}(F^\leftarrow(p)) (1 + o(1)) \right) + \frac{n\mu}{\alpha} (1 + o(1)) \\
&= \left(\frac{n}{\alpha - 1} \right)^{1/\alpha} F^\leftarrow(p) \left(1 + \left(\frac{\mu_n^*(F^\leftarrow(p))}{n^{1/\alpha} F^\leftarrow(p)} + \frac{n^{\beta/\alpha} - 1}{\alpha\beta} A(F^\leftarrow(p)) \right) (1 + o(1)) \right) \\
&\quad \cdot \left(1 + n^{\lambda/\alpha} \xi_{\alpha, \lambda} A_G^{(n)}(F^\leftarrow(p)) (1 + o(1)) \right) + \frac{n\mu}{\alpha} (1 + o(1)) \\
&= \left(\frac{n}{\alpha - 1} \right)^{1/\alpha} F^\leftarrow(p) \left(1 + \frac{1}{\alpha\beta} \left(\frac{n^{\beta/\alpha}(\alpha - 1)^{1-\beta/\alpha}}{\alpha - \beta - 1} - 1 \right) A(F^\leftarrow(p)) (1 + o(1)) \right) \\
&\quad + \alpha^{-1} ((\alpha - 1)\mu_n^*(F^\leftarrow(p)) + n\mu) (1 + o(1)) + o\left(\sum_{i=1}^n (\phi_i(t) - d_i) \right).
\end{aligned}$$

According to the definition of $\text{CE}_p(S_n)$ and Theorem 3.4.1, we obtain, as $p \uparrow 1$,

$$\begin{aligned}
\frac{\text{CE}_p(S_n)}{F^\leftarrow(p)} &= \frac{\mathbb{E}[S_n | S_n > \text{ex}_p(S_n)]}{F^\leftarrow(p)} \\
&= \frac{\alpha}{\alpha - 1} \left(1 + \frac{1}{\alpha(\alpha - \lambda - 1)} A_G^{(n)}(\text{ex}_p(S_n)) (1 + o(1)) \right) \frac{\text{ex}_p(S_n)}{F^\leftarrow(p)} \\
&= \frac{\alpha}{\alpha - 1} \left(1 + \frac{1}{\alpha(\alpha - \lambda - 1)} \left(\frac{n}{\alpha - 1} \right)^{\lambda/\alpha} A_G^{(n)}(F^\leftarrow(p)) \right) \\
&\quad \cdot \left(\left(\frac{n}{\alpha - 1} \right)^{1/\alpha} \left(1 + \frac{1}{\alpha\beta} \left(\frac{n^{\beta/\alpha}(\alpha - 1)^{1-\beta/\alpha}}{\alpha - \beta - 1} - 1 \right) A(F^\leftarrow(p)) (1 + o(1)) \right) \right. \\
&\quad \left. + \frac{(\alpha - 1)\mu_n^*(F^\leftarrow(p)) + n\mu}{\alpha F^\leftarrow(p)} (1 + o(1)) \right) \\
&= \frac{\alpha n^{1/\alpha}}{(\alpha - 1)^{1/\alpha+1}} \left(1 + \frac{1}{\alpha\beta} \left(\left(\frac{n}{\alpha - 1} \right)^{\beta/\alpha} \frac{\alpha + \beta - 1}{\alpha - \beta - 1} - 1 \right) A(F^\leftarrow(p)) (1 + o(1)) \right) \\
&\quad + \frac{(\alpha - 2)\mu_n^*(F^\leftarrow(p)) + n\mu}{(\alpha - 1)F^\leftarrow(p)} (1 + o(1)) + o\left(\sum_{i=1}^n (\phi_i(t) - d_i) \right).
\end{aligned}$$

Thus, this completes the proof of the theorem. ■

Table 3.3: Simulated values (MC) versus first-order (1st) and second-order (2nd) asymptotic values $\text{exp}_p(S_2)$ and $\text{CE}_p(S_2)$ with different p . The Student t_α with $\alpha = 2.5$ and the Sarmanov with parameter $a_{12} = -0.5$. The simulated values are the averages over 100 repetitions with a sample size of 10^6 each.

p	$\text{exp}_p(S_2)_{MC}$	$\text{exp}_p(S_2)_{1st}$	$\text{exp}_p(S_2)_{2nd}$	$\text{CE}_p(S_2)_{MC}$	$\text{CE}_p(S_2)_{1st}$	$\text{CE}_p(S_2)_{2nd}$	$\frac{\text{exp}_p(S_2)_{1st}}{\text{exp}_p(S_2)_{MC}}$	$\frac{\text{exp}_p(S_2)_{2nd}}{\text{exp}_p(S_2)_{MC}}$	$\frac{\text{CE}_p(S_2)_{1st}}{\text{CE}_p(S_2)_{MC}}$	$\frac{\text{CE}_p(S_2)_{2nd}}{\text{CE}_p(S_2)_{MC}}$
0.9500	5.7991	4.0097	5.5194	9.2238	6.6828	9.0031	0.6914	0.9518	0.7245	0.9761
0.9600	6.2368	4.4366	5.9522	9.9000	7.3943	9.7178	0.7114	0.9544	0.7469	0.9816
0.9700	6.8510	5.0387	6.5609	10.8592	8.3978	10.7250	0.7355	0.9577	0.7733	0.9876
0.9800	7.8270	6.0016	7.5313	12.4033	10.0027	12.3340	0.7668	0.9622	0.8065	0.9944
0.9900	9.8652	8.0309	9.5699	15.6877	13.3849	15.7213	0.8141	0.9701	0.8532	1.0021
0.9950	12.5120	10.6764	12.2216	20.0218	17.7940	20.1339	0.8533	0.9768	0.8887	1.0056
0.9990	22.2665	20.4956	22.0482	36.1840	34.1594	36.5034	0.9205	0.9902	0.9440	1.0088
0.9999	53.1965	51.3901	52.9461	88.2526	85.6502	87.9961	0.9660	0.9953	0.9705	0.9971

The following example is applied to illustrate the interpretation of Theorem 3.4.1 with the absolute Student t_α distribution.

Example 3.4.1 (*Absolute Student t_α Distribution*) A standard Student t_α distribution with density function satisfies that

$$f(x) = \frac{\Gamma((\alpha + 1)/2)}{\sqrt{\alpha\pi}\Gamma(\alpha/2)} \left(1 + \frac{x^2}{\alpha}\right)^{-(\alpha+1)/2}, \quad x \in \mathbb{R}, \quad \alpha > 1.$$

Denote by F the distribution function of $|X|$, according to Example 4.2 of Mao et al. (2012) and Example 4.4 of Hua and Joe (2011), we know that $\bar{F} \in 2\text{RV}_{-\alpha, -2}$ with auxiliary function $A(t) = \frac{\alpha^2}{\alpha+2}t^{-2}$, $\mu = E|X| = \alpha/(\alpha - 1)$ and $F^\leftarrow(p) = t_\alpha^\leftarrow((1+p)/2)$. Let X_1 and X_2 have an identical Student t_α distribution F . Suppose that the random vector (X_1, X_2) follows a Sarmanov distribution in (2.1) with $\phi_i(\cdot) = 1 - 2F(\cdot)$, $i = 1, 2$. Clearly, $d_i = -1$. Then

$$\text{exp}_p(S_2) = \frac{2^{1/\alpha}F^\leftarrow(p)}{(\alpha - 1)^{1/\alpha}} + \frac{\alpha - 1}{\alpha} (\mu(F^\leftarrow(p)) - a_{12}\mu_1(F^\leftarrow(p))) + \frac{2}{\alpha - 1},$$

and

$$\text{CE}_p(S_2) = \frac{2^{1/\alpha}\alpha F^\leftarrow(p)}{(\alpha - 1)^{1/\alpha+1}} + \frac{\alpha - 2}{\alpha - 1} (\mu(F^\leftarrow(p)) - a_{12}\mu_1(F^\leftarrow(p))) + \frac{2\alpha}{(\alpha - 1)^2}.$$

According to Table 3.3, the second-order asymptotics of $\text{ex}_p(S_2)$ and $\text{CE}_p(S_2)$ are much closer to the simulation value as $p \in [0.95, 1)$.

Next, we get the second-order asymptotics of $\text{ICE}_{p,m}(S_n)$ and $\text{SICE}_{p,m}(S_n)$ with an n -dimensional Sarmanov distribution.

Theorem 3.4.2 *Under the conditions of Theorem 3.4.1, as $p \uparrow 1$, we get that*

$$\begin{aligned} \frac{\text{ICE}_{p,m}(S_n)}{F^{\leftarrow}(p)} &= \frac{\alpha}{(\alpha-1)n} \left(\left(\frac{n}{\alpha-1} \right)^{1/\alpha} (1 + \chi_{\alpha,\beta}^n A(F^{\leftarrow}(p)) (1 + o(1))) \right. \\ &\quad \left. + \frac{n\mu - \mu_n^*(F^{\leftarrow}(p))}{\alpha F^{\leftarrow}(p)} (1 + o(1)) \right) + o \left(\sum_{i=1}^n (\phi_i(t) - d_i) \right), \end{aligned}$$

with $\chi_{\alpha,\beta}^n$ defined in (4.2) and

$$\frac{\text{SICE}_{p,m}(S_n)}{F^{\leftarrow}(p)} = \frac{\text{ICE}_{p,m}(S_n)}{F^{\leftarrow}(p)} - \frac{\text{ex}_p(X_m)}{nF^{\leftarrow}(p)}.$$

Moreover, the first-order asymptotics of $\text{ICE}_{p,m}(S_n)$ and $\text{SICE}_{p,m}(S_n)$ are $\frac{n^{1/\alpha} \alpha F^{\leftarrow}(p)}{n(\alpha-1)^{1+1/\alpha}}$ and $\frac{n^{1/\alpha} \alpha F^{\leftarrow}(p)}{n(\alpha-1)^{1+1/\alpha}} - \frac{F^{\leftarrow}(p)}{n(\alpha-1)^{1/\alpha}}$.

Proof. By the definition of $\text{ICE}_{p,m}(S_n)$, Theorem 3.4.1 and Lemma A.4 (here we require

Lemma A.4 in Appendix A), as $p \uparrow 1$, we have that

$$\begin{aligned}
\frac{\text{ICE}_{p,m}(S_n)}{F^{\leftarrow}(p)} &= \frac{\mathbb{E}[X_m | S_n > \text{ex}_p(S_n)]}{F^{\leftarrow}(p)} \\
&= \frac{\alpha}{(\alpha-1)n} \left(1 + \tilde{B}(\text{ex}(S_n))(1+o(1)) \right) \frac{\text{ex}(S_n)}{F^{\leftarrow}(p)} \\
&= \frac{\alpha}{(\alpha-1)n} \left(\left(\frac{n}{\alpha-1} \right)^{1/\alpha} \left(1 + \frac{1}{\alpha\beta} \left(\frac{n^{\beta/\alpha}(\alpha-1)^{1-\beta/\alpha}}{\alpha-\beta-1} - 1 \right) A(F^{\leftarrow}(p))(1+o(1)) \right) \right. \\
&\quad \left. + \frac{(\alpha-1)\mu_n^*(F^{\leftarrow}(p)) + n\mu}{\alpha F^{\leftarrow}(p)} (1+o(1)) \right) \left(1 + \left(\frac{n}{\alpha-1} \right)^{\rho/\alpha} \tilde{B}(F^{\leftarrow}(p))(1+o(1)) \right) \\
&= \frac{\alpha}{(\alpha-1)n} \left(\left(\frac{n}{\alpha-1} \right)^{1/\alpha} (1 + \chi_{\alpha,\beta}^n A(F^{\leftarrow}(p))(1+o(1))) \right. \\
&\quad \left. + \frac{n\mu - \mu_n^*(F^{\leftarrow}(p))}{\alpha F^{\leftarrow}(p)} (1+o(1)) \right) + o\left(\sum_{i=1}^n (\phi_i(t) - d_i) \right).
\end{aligned}$$

Applying Theorem 3.4.1 and Lemma A.5 (here we require Lemma A.5 in Appendix A), we conclude that

$$\begin{aligned}
\frac{\text{SICE}_{p,m}(S_n)}{F^{\leftarrow}(p)} &= \frac{\mathbb{E}[(X_m - \text{ex}_p(X_m))_+ | S_n > \text{ex}_p(S_n)]}{F^{\leftarrow}(p)} \\
&= \frac{\text{ICE}_{p,m}(S_n)}{F^{\leftarrow}(p)} - \frac{\text{ex}_p(X_m)}{n F^{\leftarrow}(p)}.
\end{aligned}$$

This completes the proof of $\text{ICE}_{p,m}(S_n)$ and $\text{SICE}_{p,m}(S_n)$. ■

Lastly, the following example is applied to this result of Theorem 3.4.2 under the Fréchet distribution.

Example 3.4.2 (*Fréchet distribution*) *A Fréchet distribution function F satisfies that*

$$F(x) = 1 - \exp(-x^{-\alpha}), \quad \alpha > 1.$$

Obviously, $\bar{F}(\cdot) \in 2\text{RV}_{-\alpha, -\alpha}$ with auxiliary function $A(t) = \alpha t^{-\alpha}/2$. Let X_1 and X_2 have an identical Fréchet distribution F . Suppose that the random vector (X_1, X_2) follows a

Sarmanov distribution in (2.1) with $\phi_i(\cdot) = (\cdot)^{-1} - E[X_i^{-1}]$. Then, $d_i = -E[X_i^{-1}]$ and $\rho_i = -1, i = 1, 2$. Thus

$$\text{ICE}_{p,1}(S_2) = \frac{2^{1/\alpha-1}\alpha F^{\leftarrow}(p)}{(\alpha-1)^{1/\alpha+1}} + \frac{2\mu - \mu(F^{\leftarrow}(p)) - a_{12}d_1\mu_1(F^{\leftarrow}(p))}{2(\alpha-1)},$$

and

$$\text{SICE}_{p,1}(S_2) = \text{ICE}_{p,1}(S_2) - \left(\frac{F^{\leftarrow}(p)}{2(\alpha-1)^{1/\alpha}} + \frac{\mu}{2\alpha} \right).$$

It is clear that Tables 3.4 and 3.5 reveal that the second-order asymptotics of $\text{ICE}_p(S_2)$ and $\text{CE}_p(S_2)$ tend to the simulation values for $p \in [0.95, 1)$ and provide much better estimates than the first-order asymptotics.

3.5 Numerical Examples

In this section, we numerically illustrate our asymptotic results.

Example 3.5.1 Under Example 3.3.1 with $\alpha = 2$, $k = 1$ and $a_{12} = -1, 0, 1$, by Theorems 3.3.1, 3.3.2, 3.4.1 and 3.4.2, we have that the first-order asymptotics (as $p \uparrow 1$) of VaR, Expectile, MES and ICE are equivalent (the ratio to $F^{\leftarrow}(p)$ is $2^{1/2}$). Those of CTE and CE are equivalent (the ratio to $F^{\leftarrow}(p)$ is $2^{3/2}$). Those of SES and SICE are equivalent (the ratio to $F^{\leftarrow}(p)$ is $2^{1/2} - 1/2$). We have the following observations:

- The second-order asymptotics of systemic risk measures are closer to the simulation values than the first-order asymptotics as $p \uparrow 1$. The expectile-based systemic risk measures are larger than VaR-based ones. One should consider a lower confidence level for expectile-based risk measures.

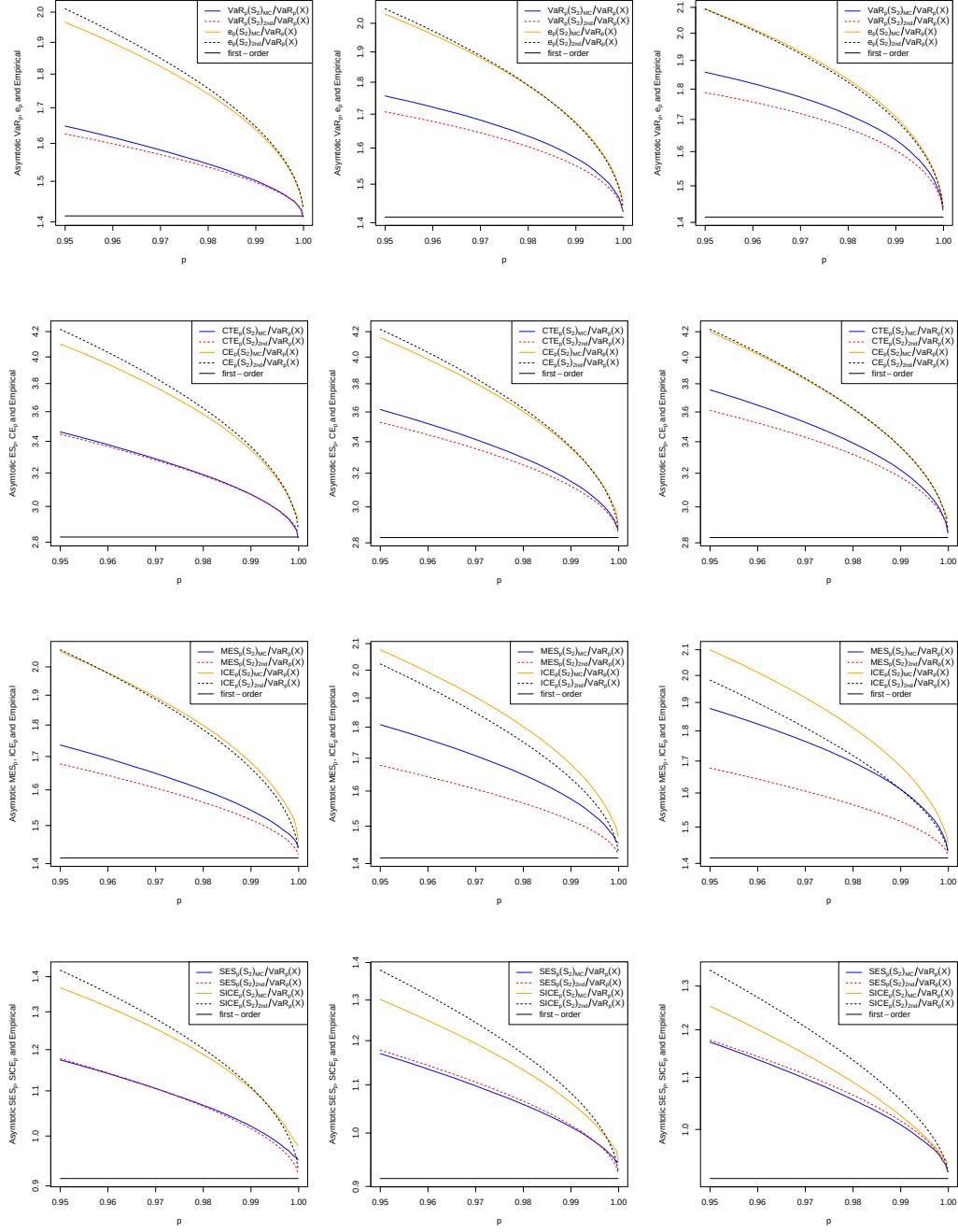


Figure 3.2: Comparison of MC-simulated, first-order, and second-order asymptotic values of various risk measure ratios for S_2 relative to X . Results are based on Pareto margins with Sarmanov dependence ($a_{12} = -1, 0, 1$ for left, middle, and right panels). The simulated values are the averages over 100 repetitions with a sample size of 10^6 each.

Table 3.4: Simulated values $\text{ICE}_p(S_2)_{MC}$ versus the first-order asymptotic values $\text{ICE}_p(S_2)_{1st}$ and the second-order asymptotic values $\text{ICE}_p(S_2)_{2nd}$ with $\alpha = 2, a_{12} = 0.5$. The simulated values are the averages over 100 repetitions with a sample size of 10^6 each.

p	$\text{ICE}_{p,1}(S_2)_{MC}$	$\text{ICE}_{p,1}(S_2)_{1st}$	$\text{ICE}_{p,1}(S_2)_{2nd}$	$\frac{\text{ICE}_{p,1}(S_2)_{1st}}{\text{ICE}_{p,1}(S_2)_{MC}}$	$\frac{\text{ICE}_{p,1}(S_2)_{2nd}}{\text{ICE}_{p,1}(S_2)_{MC}}$
0.9500	8.0518	6.2481	7.2566	0.7760	0.9012
0.9600	8.8178	7.0051	7.9763	0.7944	0.9046
0.9700	9.9471	8.1096	9.0379	0.8153	0.9086
0.9800	11.8197	9.9749	10.8508	0.8439	0.9180
0.9900	16.0617	14.1623	14.9689	0.8817	0.9320
0.9990	47.6943	45.0164	45.7053	0.9439	0.9583
0.9999	146.0483	144.7467	145.3974	0.9911	0.9955

- The second-order asymptotics of VaR, CTE and MES follow a_{12} (i.e. dependence coefficient) decreases from bottom simulation values to top. The second-order asymptotics of expectile and CE are closer to the simulation values as a_{12} increases.
- The second-order asymptotics of ICE is closer to the empirical value as a_{12} decreases and that of ICE is closer to the simulation value than that of MES.
- The second-order asymptotic of SICE is closer to the simulation values as a_{12} decreases and that of SES is closer to the simulation values than that of SICE.

Example 3.5.2 Consider the setting of Example 3.3.2 with $\alpha = 1.5$, $\beta = -0.5$ or -2 , under a Sarmanov distribution as in (2.1) with $\phi_i(\cdot) = 1 - 2F(\cdot)$, $i = 1, 2$ and $a_{12} = -0.5$. Then, by Theorem 3.3.1, 3.3.2, 3.4.1 and 3.4.2, as $p \uparrow 1$:

- Figures 3.3-3.6 compare the second-order asymptotics with Monte Carlo simulation results of Theorems 3.3.1, 3.3.2, 3.4.1 and 3.4.2. It is clear that the second-order asymptotics can capture the trend as $p \uparrow 1$ with the first-order asymptotics.
- Figures 3.3 and 3.5 reveal that the second-order asymptotics of VaR and MES with $\beta = -0.5$ are more accurate than $\beta = -2$, which the expectile and ICE are opposite to VaR with different β .

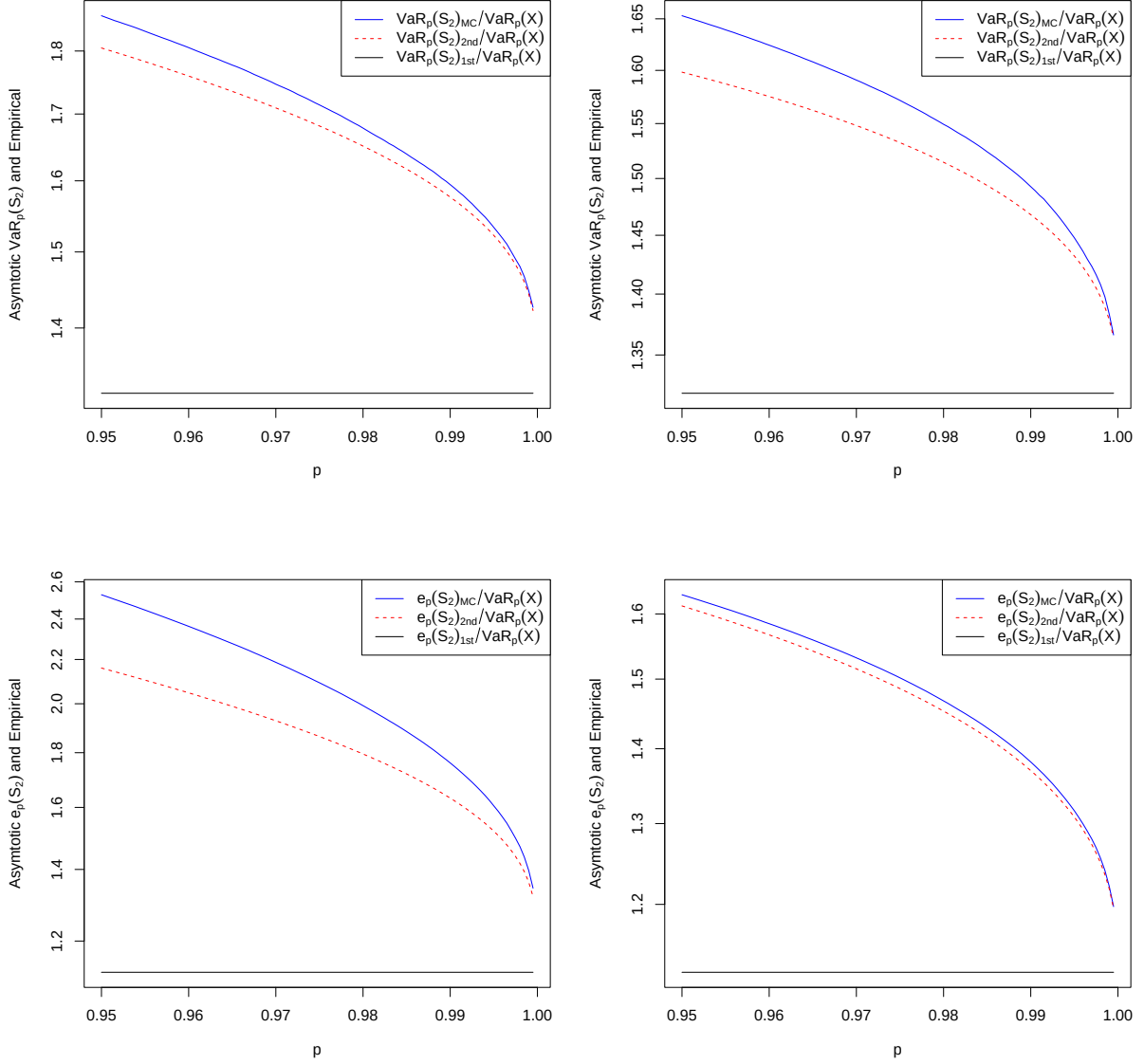


Figure 3.3: Comparison of the simulation values (MC), first-order and second-order asymptotic values of $\text{VaR}_p(S_2)/\text{VaR}_p(X)$ and $\exp_p(S_2)/\text{VaR}_p(X)$. We use the Burr distribution with $\beta = -0.5$ for the left panel and $\beta = -2$ for the right panel.

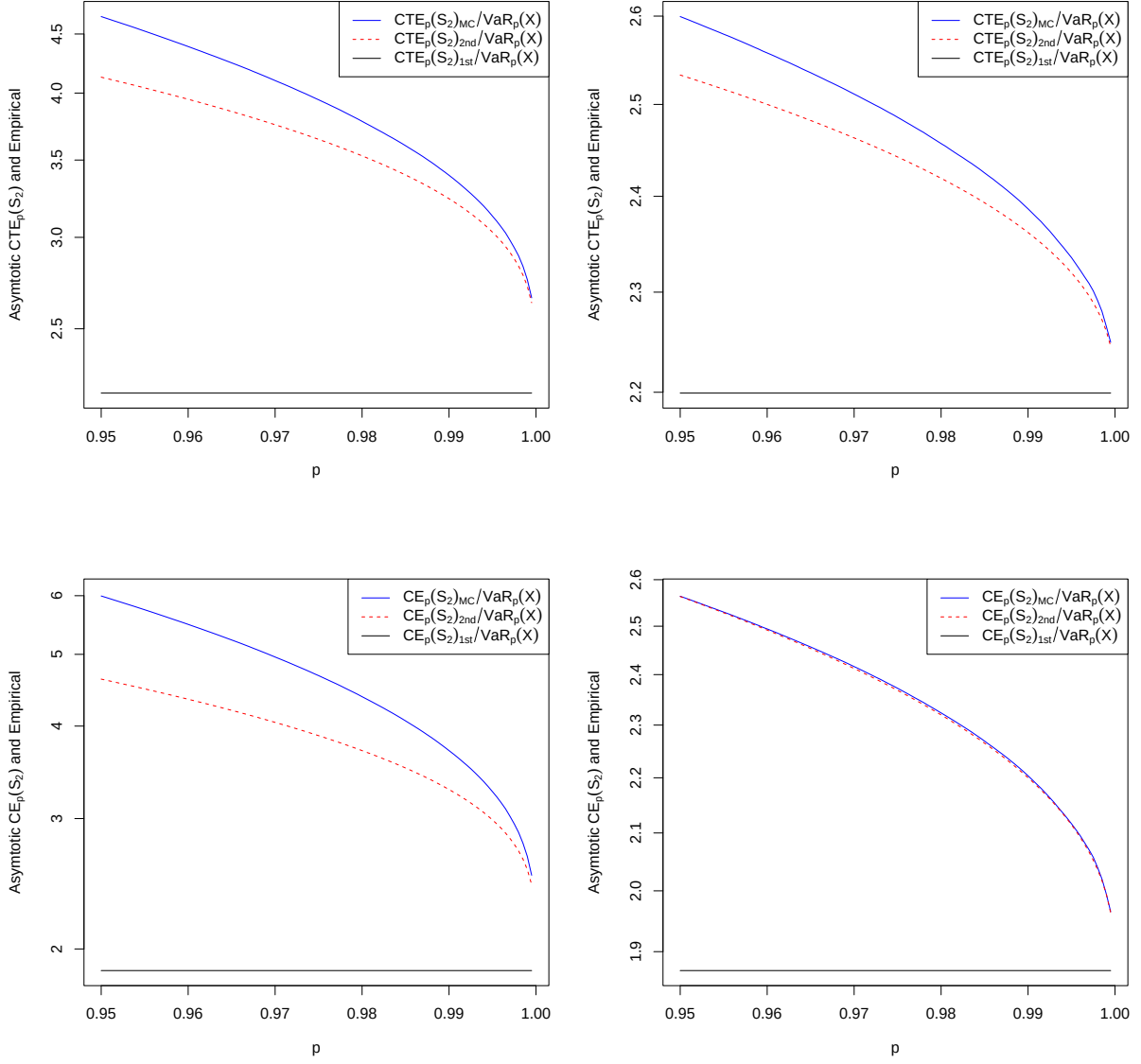


Figure 3.4: Comparison of the simulation values (MC), first-order and second-order asymptotic values of $\text{CTE}_p(S_2)/\text{VaR}_p(X)$ and $\text{CE}_p(S_2)/\text{VaR}_p(X)$, Burr distribution with $\beta = -0.5$ for the left panel and $\beta = -2$ for the right panel. The simulated values are the averages over 100 repetitions with a sample size of 10^6 each.

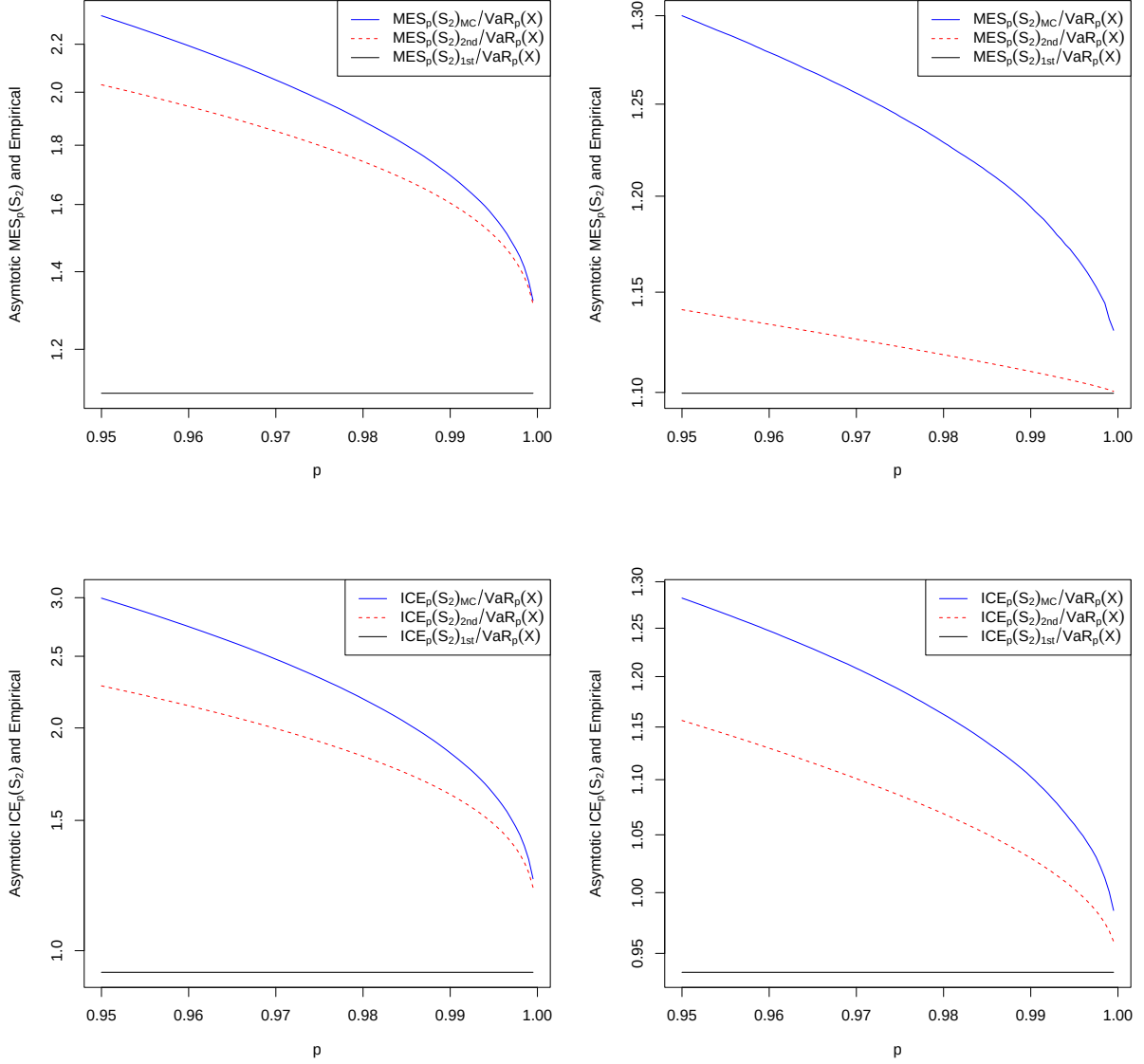


Figure 3.5: Comparison of the simulation values (MC), first-order and second-order asymptotic values of $MES_p(S_2)/VaR_p(X)$ and $ICE_p(S_2)/VaR_p(X)$, Burr distribution with $\beta = -0.5$ for the left panel and $\beta = -2$ for the right panel.

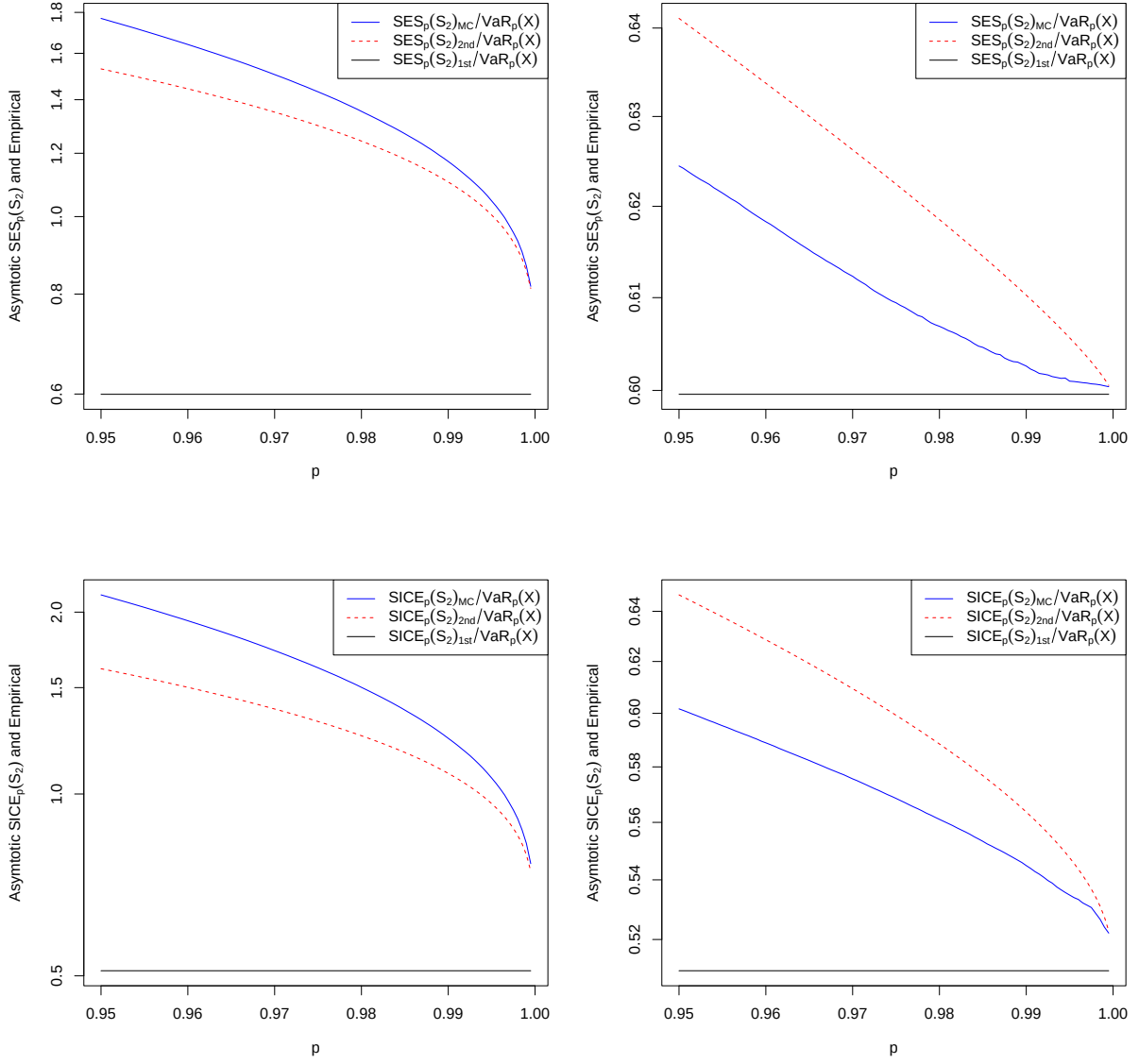


Figure 3.6: Comparison of the simulation values (MC), first-order and second-order asymptotic values of $\text{SES}_p(S_2)/\text{VaR}_p(X)$ and $\text{SICE}_p(S_2)/\text{VaR}_p(X)$, Burr distribution with $\beta = -0.5$ for the left panel and $\beta = -2$ for the right panel. The simulated values are the averages over 100 repetitions with a sample size of 10^6 each.

Table 3.5: Simulated values $\text{SICE}_p(S_2)_{MC}$ versus the first-order asymptotic values $\text{SICE}_p(S_2)_{1st}$ and the second-order asymptotic values $\text{SICE}_p(S_2)_{2nd}$ with $\alpha = 2, a_{12} = 0.5$. The simulated values are the averages over 100 repetitions with a sample size of 10^6 each.

p	$\text{SICE}_{p,1}(S_2)_{MC}$	$\text{SICE}_{p,1}(S_2)_{1st}$	$\text{SICE}_{p,1}(S_2)_{2nd}$	$\frac{\text{SICE}_{p,1}(S_2)_{1st}}{\text{SICE}_{p,1}(S_2)_{MC}}$	$\frac{\text{SICE}_{p,1}(S_2)_{2nd}}{\text{SICE}_{p,1}(S_2)_{MC}}$
0.9500	4.4381	4.0390	4.6044	0.9101	1.0375
0.9600	4.9085	4.5284	5.0565	0.9226	1.0301
0.9700	5.6135	5.2424	5.7276	0.9339	1.0203
0.9800	6.7875	6.4482	6.8810	0.9500	1.0138
0.9900	9.4881	9.1552	9.5187	0.9649	1.0032
0.9990	29.4512	29.1007	29.4465	0.9881	0.9998
0.9999	93.8148	93.5710	93.7786	0.9974	0.9996

- Figures 3.4 and 3.6 represent that the second-order asymptotics of CTE, CE, SES and SICE with $\beta = -2$ are better than those with $\beta = -0.5$.

3.6 Diversification Benefits

Introduced by Bürgi et al. (2008), the concept of diversification benefit represents the retained capital gained by collectively managing all risks within a portfolio, in contrast to addressing each risk individually. For a fixed threshold of $0 < p < 1$, the diversification benefit is defined by

$$D_p^\rho(S_n) = 1 - \frac{\rho_p(S_n) - \mathbb{E}[S_n]}{\sum_{i=1}^n (\rho_p(X_i) - \mathbb{E}[X_i])}, \quad (6.1)$$

where ρ_p represents a risk measure at a specific confidence level p (e.g., VaR_p , ex_p , CTE_p , etc).

Constructed as such, for a fixed systemic risk measure ρ , $D_p^\rho > 0$ indicates that diversification is advantageous, potentially reducing an insurer's catastrophic risk by market engagement. Conversely, $D_p^\rho \leq 0$ suggests diversification is not advantageous for a single insurer. We are particularly interested in cases of heavy-tailed risks whether diversification

may be beneficial. Further, this aspect of whether $D_p^\rho > 0$ is technically linked to the risk measure ρ 's subadditivity. It further relates to the so-called coherence; e.g., see [Artzner et al. \(1999\)](#).

Using $D_p^\rho(S_n)$ aids in the portfolio selection for a company. By maximizing diversification benefits, the company can reduce the risk and boost the performance of a portfolio. It is worthwhile to mention that the usage of $D_p^\rho(S_n)$ is not always applicable; its value depends on the number of risks involved and the specific risk measures employed. Recent findings from [Dacorogna et al. \(2018\)](#) highlighted that diversification benefits vary notably based on the type of dependence and the risk measures.

Experts emphasize caution against careless diversification practices, especially when confronted with risks with heavy tails. By adopting the above results of risk measures and deriving formulas for diversification benefits, we can evaluate the performance of a portfolio S_n in contrast to individual risks operating independently. In the following, we first derive the second-order asymptotic of $D_p^\rho(S_n)$ with ρ based on VaR, expectile, CTE and CE. Additionally, we give an example with the Weiss distribution.

Theorem 3.6.1 *Under the conditions of Proposition 3.3.1 with $\alpha > 1$, we have, as $p \uparrow 1$,*

$$D_p^{\text{VaR}}(S_n) = 1 - n^{1/\alpha-1} \left(1 + \left(\frac{\mu_n^*(F^{\leftarrow}(p))}{n^{1/\alpha} F^{\leftarrow}(p)} + \frac{n^{\beta/\alpha} - 1}{\alpha\beta} A(F^{\leftarrow}(p)) \right) (1 + o(1)) \right) \\ + \frac{(1 - n^{1/\alpha-1}) \mu}{F^{\leftarrow}(p)} (1 + o(1)) + o \left(\sum_{i=1}^n (\phi_i(t) - d_i) \right),$$

$$D_p^{\text{ex}}(S_n) = 1 - n^{1/\alpha-1} \left(1 + \frac{(n^{\beta/\alpha} - 1)(\alpha - 1)^{1-\beta/\alpha}}{\alpha\beta(\alpha - \beta - 1)} A(F^{\leftarrow}(p)) (1 + o(1)) \right) \\ - \frac{(\alpha - 1)^{1/\alpha+1} \mu_n^*(F^{\leftarrow}(p))}{\alpha n F^{\leftarrow}(p)} (1 + o(1)) + o \left(\sum_{i=1}^n (\phi_i(t) - d_i) \right),$$

$$D_p^{\text{CTE}}(S_n) = 1 - n^{1/\alpha-1} \left(1 + \frac{1}{\alpha\beta} \left(\frac{n^{\beta/\alpha}(\alpha-1) - \beta}{\alpha - \beta - 1} - 1 \right) A(F^{\leftarrow}(p)) (1 + o(1)) \right) \\ - \frac{(\alpha-1)(\mu_n^*(F^{\leftarrow}(p)) - (n - n^{1/\alpha})\mu)}{n\alpha F^{\leftarrow}(p)} (1 + o(1)) + o\left(\sum_{i=1}^n (\phi_i(t) - d_i)\right),$$

and

$$D_p^{\text{CE}}(S_n) = 1 - n^{1/\alpha-1} \left(1 + \frac{(n^{\beta/\alpha} - 1)(\alpha-1)^{-\beta/\alpha}(\alpha + \beta - 1)}{\alpha\beta(\alpha - \beta - 1)} A(F^{\leftarrow}(p)) (1 + o(1)) \right) \\ - \frac{(\alpha-1)^{1/\alpha}(\alpha-2)\mu_n^*(F^{\leftarrow}(p))}{\alpha n F^{\leftarrow}(p)} (1 + o(1)) + o\left(\sum_{i=1}^n (\phi_i(t) - d_i)\right).$$

Obviously, the first-order asymptotics of $D_p^{\text{VaR}}(S_n)$, $D_p^{\text{ex}}(S_n)$, $D_p^{\text{CTE}}(S_n)$ and $D_p^{\text{CE}}(S_n)$ are $1 - n^{1/\alpha-1}$.

Proof. See Appendix A. ■

Example 3.6.1 (Weiss distribution) A Weiss distribution function F satisfies that

$$F(x) = 1 - x^\alpha (1 + x^{-\beta}), \quad x, \alpha > 1, \beta < 0.$$

It is easy to check that $\overline{F}(\cdot) \in 2\text{RV}_{-\alpha, \beta}$ with auxiliary function $A(t) = \beta t^\beta$. Let X_1 and X_2 have an identical Weiss distribution F . Suppose that the random vector (X_1, X_2) follows a Sarmanov distribution in (2.1) with $\phi_i(\cdot) = 1 - 2F(\cdot)$. Clearly, $d_i = -1, i = 1, 2$.

In the context of Example 3.6.1, we aim to present the asymptotic performance for diversification benefits D_p^ρ across four risk measures (i.e. VaR, expectile, CTE, and CE) based on the results obtained in Theorem 3.6.1. Here, we denote by $\hat{D}_p^\rho$ the second-order asymptotic and employ the ratio $\hat{D}_p^\rho/D_p^\rho$ to assess the asymptotic performance of diversification benefits. A value of $\hat{D}_p^\rho/D_p^\rho$ closer to 1 indicates a more accurate asymptotic result, while deviations from 1 imply poorer outcomes. Besides, according to Bürgi et al. (2008), $\hat{D}_p^\rho/D_p^\rho > 1$ signifies the overestimation of the diversification benefit, while $\hat{D}_p^\rho/D_p^\rho < 1$ corresponds to the underestimation.

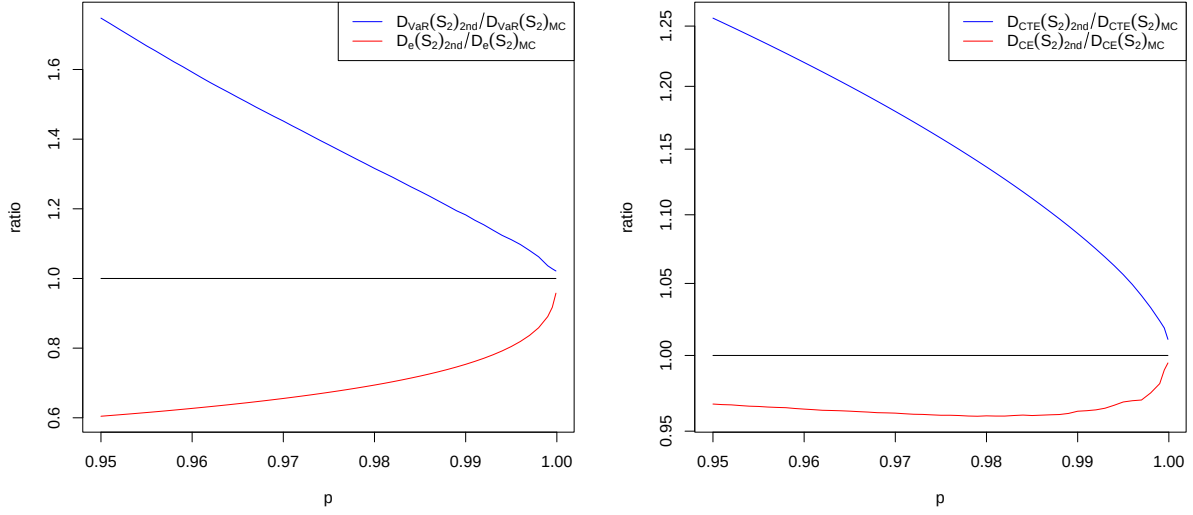


Figure 3.7: Weiss distribution with $\alpha = 2, k = 1$ and $a_{12} = 0.3$

The numerical experiment is shown in Figure 3.7. For the comparing purpose, we use blue lines to represent the outcomes derived from VaR-based systemic risk measures (i.e. VaR and CTE), while red lines represent the outcomes obtained from expectile-based systemic risk measures (i.e. expectile and CE).

We can observe that, in both plots, the $\hat{D}_p^\rho/D_p^\rho$ obtained from alternative expectile-based systemic risk measures (i.e. expectile and CE) generally exhibit higher accuracy compared to the $\hat{D}_p^\rho/D_p^\rho$ obtained from VaR-based systemic risk measures (i.e. VaR and CTE). Particularly noteworthy is the performance of CE, the ratio of $\hat{D}_p^\rho/D_p^\rho$ is consistently greater than 0.95 when $p \in [0.95, 1)$, indicating an error range of less than 5%. Moreover, it is worth noting that expectile inherently incorporates more information than VaR when estimated from the empirical dataset. Another interesting finding is that, when ρ is the selected expectile-based systemic risk measures, $\hat{D}_p^\rho/D_p^\rho < 1$ consistently with $p \in [0.95, 1)$, implying an underestimation of the diversification benefit and suggesting a conservative approximation. Conversely, when ρ is the selected VaR-based systemic risk measures, $\hat{D}_p^\rho/D_p^\rho > 1$ consistently with $p \in [0.95, 1)$, which overestimate the diversification benefit and reflecting an optimistic view of diversification. Such characteristics highlight

the potential of expectile-based and VaR-based risk measures as alternatives, each serving distinct purposes.

There are more new ways to quantify diversification. E.g., the diversification quotient, was proposed in [Han et al. \(2022, 2023\)](#). Our asymptotic treatment provides a framework to investigate these new quotients, which will be studied in future.

3.7 Conclusion

In this chapter, we study systemic risk measures with a multivariate Sarmanov distribution. We first classify them into two families of VaR- and expectile-based systemic risk measures. We have the second-order asymptotics of VaR, CTE, MES and SES in the first family. Furthermore, we obtain the second-order asymptotics of expectile, CE, ICE, and SICE in the second family. In addition, we give concrete analytical and numerical examples to illustrate the main results. We emphasize that the second-order asymptotics can provide a more accurate approximation as $p \uparrow 1$ than the first-order asymptotics. Moreover, we provide a comprehensive comparison among VaR- and expectile-based systemic risk measures. We find that expectile-based measures deduce a larger risk evaluation than VaR-based measures, suggesting a lower confidence level when the expectile is adopted. Finally, we apply the asymptotic treatment to financial diversification and provide instructive insights for risk management. We believe that our results consolidate future research in risk management, and our findings have implications for financial practitioners and regulators striving to better understand and mitigate systemic risks in complex financial systems.

Chapter 4

Catastrophe Risk Pooling

4.1 Introduction

Changing climate patterns and economic development continue to drive rising natural catastrophe losses worldwide. Global economic losses totaled \$368 billion USD in 2024, of which 60% was uninsured, known as the global ‘protection gap’ ([Aon, 2025](#)). To address this ongoing protection gap and improve economic resilience to physical climate risk, catastrophe risk pools have recently emerged as a promising addition to existing catastrophe risk financing mechanisms ([Ciullo et al., 2023](#)). Pooling catastrophe risks across different geographical regions and perils such that the risks are relatively independent in nature can reduce the risk faced by each participant through diversification. However, the well-known traditional diversification benefits of pooling may not be effectively realized for catastrophe risks due to unique characteristics like extremely heavy tails, creating challenges for the long-term viability and sustainability of catastrophe risk pools.

In this chapter, we study mechanisms for catastrophe risk pools to effectively provide diversification benefits for all participants. Prominent examples of catastrophe risk pools include the Caribbean Catastrophe Risk Insurance Facility (CCRIF), the first multi-country risk pool launched in 2007, which offers earthquake, tropical cyclone, and excess rainfall protection for 20 member countries, and the Pacific Catastrophe Risk Assessment and Financing Initiative Facility (PCRAFI Facility) established in 2016 to provide emer-

agency response liquidity for earthquakes and tropical cyclones in 15 countries; see [Bollmann and Wang \(2019\)](#) for more details and examples. Literature assessing diversification in catastrophe risk pools is sparse; empirical studies note the geographical diversification benefits across a wide array of countries ([Ciullo et al., 2023](#)) and states ([Dumm et al., 2015](#)) for existing pools, in part due to the low correlation between tail losses from different regions and perils ([Ciullo et al., 2023](#); [Bollmann and Wang, 2019](#)). In contrast, we adopt an extreme value analysis perspective and focus on the underlying pooling of natural catastrophe risks to provide theoretical insight into pool structures that can achieve more effective diversification.

According to [Ibragimov et al. \(2009\)](#), economic losses from natural disasters can be justified as having a power-law decay in the tail when the natural disaster exhibits the power-law decay property; examples include earthquakes ([Sornette et al., 1996](#)), hurricanes ([Hsieh, 1999](#); [Hogg and Klugman, 1983](#)), wildfires ([Malamud et al., 2005](#)), and river floods ([Woo, 2011](#)). Intuitively, a power-law decay in the tail of a risk X with distribution function F is

$$\Pr(X > x) \sim cx^{-\alpha}, \quad x > 0,$$

for some tail index $\alpha > 0$. Further empirical studies have shown that catastrophic risks are extremely heavy-tailed with a tail index $\alpha < 1$, meaning even their first moments are not finite (for example, [Woo, 2011](#); [Hofert and Wüthrich, 2012](#); [Brazauskas and Kleefeld, 2016](#)). Simply pooling such heavy-tailed risks together does not yield the traditional diversification effect, a phenomenon that has been recognized in the literature ([Ibragimov and Walden, 2007](#); [Ibragimov, 2009](#); [Ibragimov and Walden, 2011](#); [Cui et al., 2021](#)). Departing from existing studies, this chapter examines the diversification benefits of pools with extremely heavy-tailed, heterogeneous catastrophe risks to propose effective pool structures from the perspective of limited liability schemes and risk-sharing rules.

Central to our study is the measurement of the diversification benefit for a participant in the catastrophe risk pool. For n catastrophe risks $X_1, \dots, X_n$, the diversification benefit for a participant is measured by the diversification ratio (DR) based on the Value-at-Risk (VaR) measure,

$$\text{DR}_q(\mathbf{X}) = \frac{\text{VaR}_q(H_1(X_1, \dots, X_n))}{\text{VaR}_q(X_1)},$$

where $\text{VaR}_q(X) = \inf\{x : F(x) \geq q\}$ is the VaR of a risk X with distribution function F at confidence level $0 < q < 1$, and $H_1(X_1, \dots, X_n)$ is the risk shared by participant 1 through joining the pool (without loss of generality). Hence, DR is the ratio of the risk faced by a participant after joining the pool to the risk faced by the participant without joining the pool. In particular, $1 - \text{DR}$ can be used to measure the diversification benefit; see [Embrechts et al. \(2009\)](#), [Degen et al. \(2010\)](#), [Mao and Yang \(2015\)](#), [Cui et al. \(2021\)](#) and [Cui et al. \(2022\)](#). If the value of DR is less than 1, the participant's risk is reduced by joining the pool. We use DR as a criterion for determining a participant's willingness to join the pool, and develop conditions for which a catastrophe risk pool can successfully form by requiring all participants to have a DR value less than 1. Because the analytical expression for DR is generally unavailable and the risks considered in the pool are catastrophic in nature, we investigate the conditions under which participants are willing to join the pool when the confidence level of VaR approaches 1. That is, the derived conditions hold for all sufficiently high confidence levels.

One important feature for the pool to function is to set a maximum payout (i.e., limited liability) for each catastrophe risk since the pool would fail to diversify the risk if all of the participants bring in their unlimited, extremely heavy-tailed catastrophe risks. This is related to the subadditivity of VaR, discussed in greater detail in [Cui et al. \(2021\)](#). We further allow the liability level to vary with the level of tail risk as measured by VaR. That is, when the aggregated risk of the pool or the original individual risk is evaluated at a higher confidence level of VaR, the limited liability is allowed to increase accordingly. Limited liability for catastrophe risks has been justified by insurance literature such as [Cummins et al. \(2002\)](#) and [Cummins and Mahul \(2003\)](#), and features in existing pools like the CCRIF and PCRAFI. In practice, losses beyond the limit may be covered by government assistance programs such as the Federal Emergency Management Agency, international donors, reinsurance markets, or through securitization ([Bollmann and Wang, 2019](#)).

Since the pooled catastrophe risks may not be identically distributed, we consider different limited liability schemes imposed on the risks contributing to the pool, as well as different risk sharing rules for distributing the aggregated risk to the participants. The two liability schemes considered are uniform liability, which assumes the same limited liability level for all risks, and proportional liability, which allows different liability levels for risks

based on their scale. Various risk sharing rules have been discussed in the literature; see, for example, [Denuit et al. \(2022\)](#). The most intuitive one is the uniform risk sharing rule, which distributes the aggregate risk equally back to the participants. We begin our investigation with uniform risk sharing, and ultimately propose a proportional risk sharing rule to ensure fair diversification within the pool, where all participants achieve the same diversification benefit.

To preview our findings, we introduce and evaluate three different catastrophe risk pool structures by establishing conditions for the pools to work using asymptotic analysis. That is, at confidence levels of VaR close to 1, we derive the conditions for all participants to have DR less than 1. The first pool involves any heterogeneous catastrophe risks that are independent, e.g., the risks may exhibit varying degrees of heavy-tailedness. Under the uniform limited liability scheme and uniform risk sharing rule, we find that it is only possible for all participants to be willing to join the pool when the limited liability is set at a low level. This is intuitive, as participants may contribute substantially different tail risks to the pool. Only when the tail risks are controlled through a low level of limited liability is it possible for participants with lighter-tailed risks to avoid being distributed relatively more risk compared to their risk prior to joining the pool.

In the second pool structure, to form the pool at a higher liability level, we propose to pool risks that have the same heavy-tailedness. The risks may still be heterogeneous with different scales and therefore do not need to be identically distributed, a considerably less restrictive assumption for practical applications. Again under the uniform liability scheme and uniform risk sharing rule, we find that the pool can work for a much higher level of limited liability and thus a much larger aggregate risk can be diversified through the pool compared to the first structure.

In the third structure, we construct the pool to make the risk contribution and sharing fair between all of the participants. First, the proportional limited liability scheme based on the scales of the individual risks allows each participant to contribute (asymptotically) the same tail risk to the pool. Under the uniform risk sharing rule, we find that the pool can work for an even greater aggregated risk compared to the second structure. However, participants with risks of smaller scales receive less diversification benefit than those with risks of larger scales. We then propose the proportional risk sharing rule, which distributes

the aggregate risk to each participant based on the scale of their individual risk. The combination of the proportional liability scheme and proportional risk sharing rule allows all of the participants to achieve equal diversification benefit by joining the pool. This structure can be regarded as a fair and practically applicable risk pooling mechanism.

We emphasize that each structure has its own merits and drawbacks. The first pool structure applies to general catastrophe risks with varying degrees of heavy-tailedness. However, it can only form when the uniform limited liability level is relatively low. The second pool structure is suitable for catastrophe risks with the same heavy-tailedness but which differ in scale. This structure allows the pool to form under a more relaxed condition and at a much higher liability level. However, the diversification benefit is unevenly distributed among participants. Like the second structure, the third pool structure applies to risks with the same heavy-tailedness but different scales. Although the conditions for forming the pool are more stringent than those of the second structure, it is possible to diversify greater aggregated risk and have each participant obtain the same diversification benefit by joining the pool.

We conduct numerical experiments at each stage of the investigation to illustrate the theoretical results. In addition, we demonstrate the practical applicability of the proposed approach for a multi-peril, multi-state risk pool using US catastrophe loss data. Notably, the real data analysis indicates that the independence assumption in the theoretical results can be relaxed, since data limitations preclude formal statistical tests of independence, yet the pools are still shown to work for risks that satisfy all of the other conditions in the theorems.

The remainder of the chapter is organized as follows: Section 4.2 investigates the pool with general catastrophe risks under the uniform risk sharing rule and uniform liability scheme, Section 4.3 studies the pool with tail equivalent risks, Section 4.4 explores the pool structures with the proportional liability scheme and proportional risk sharing rule, Section 4.5 presents the real data analysis, and Section 4.6 concludes. All of the proofs are deferred to Appendix B.

4.2 Uniform Risk Sharing with General Heavy-tailed Risks

We begin with general heterogeneous, heavy-tailed catastrophe risks, and investigate the conditions under which a participant would obtain a diversification benefit and prefer to join the catastrophe risk pool. We further illustrate the theoretical results using simulation experiments.

4.2.1 Theory

For $n \geq 2$, let $\mathbf{X} = (X_1, \dots, X_n)$ be independent catastrophe risks with distribution functions $F_1, F_2, \dots, F_n$. These risks are assumed to be heavy-tailed in terms of regular variation (RV). By definition, a risk X with distribution function $F = 1 - \bar{F}$ is said to have a regularly varying tail with index $\alpha > 0$, denoted by $X \in \text{RV}_{-\alpha}$ or $\bar{F} \in \text{RV}_{-\alpha}$, if

$$\lim_{t \rightarrow \infty} \frac{\bar{F}(tx)}{\bar{F}(t)} = x^{-\alpha}, \quad x > 0.$$

The tail index α represents the heaviness of the tail, where the smaller the value of α , the heavier the tail. As mentioned in the Section 4.1, we assume that $\bar{F}_i \in \text{RV}_{-\alpha_i}$ with $0 < \alpha_i \leq 1$, which means that these X_i 's are extremely heavy-tailed without finite first moments. See, for example, [Haan and Ferreira \(2006\)](#) and [Resnick \(2008\)](#) for more details on RV functions.

We consider a uniform limited liability for all of the risks in this section, which is defined as

$$X_i(k) = \begin{cases} k, & X_i > k, \\ X_i, & X_i \leq k. \end{cases} \quad (2.1)$$

In this case, we denote $\mathbf{X}(k) = (X_1(k), X_2(k), \dots, X_n(k))$.

The uniform risk sharing rule is defined as follows. If participants with risks X_i join

the pool, then each of them shares the risk uniformly as

$$S_n = \frac{1}{n} (X_1 + X_2 + \cdots + X_n).$$

That is, before joining the pool, participant i is liable for the risk X_i , and after joining the pool, she and the other participants would each bear S_n . When the limited liability is imposed for each risk, we denote the uniformly shared risk by

$$S_n(k) = \frac{1}{n} (X_1(k) + X_2(k) + \cdots + X_n(k)). \quad (2.2)$$

To analyze the change of risk for a participant before and after joining the pool, the DR in this case is given by

$$\text{DR}_q(\mathbf{X}(k)) = \frac{\text{VaR}_q(S_n(k))}{\text{VaR}_q(X_1(k))}. \quad (2.3)$$

Note that without loss of generality, in the above definition we compare the risk for participant 1. The DR in (2.3) is understood as the ratio of the risk for participant 1 after joining the pool and the risk for participant 1 without joining the pool. Thus, if $\text{DR}_q(\mathbf{X}(k)) \leq 1$, it is preferred for participant 1 to join the pool; otherwise, it is not preferred.

Since our focus is how the pool can be formed for catastrophe risks, to measure the tail risk, the confidence level of the VaR should be set close to 1. As mentioned in the Section 4.1, the limited liability is allowed to increase with the confidence level, i.e., we allow $k = k(q)$.

In the following lemma, we explore the conditions such that it is preferred for a participant to join the catastrophe risk pool. Since we begin with general catastrophe risks where the risks can have different heavy-tailedness, intuitively, the preferred condition for an participant to join the pool would depend on the heavy-tailedness. Hence, without loss of a generality, we assume that all of the risks have different heavy-tailedness and rank the tail indices $\alpha_1, \dots, \alpha_n$ as $\alpha_{(1)} < \alpha_{(2)} < \cdots < \alpha_{(n)}$. Accordingly, for $i = 1, 2, \dots, n$, we denote $X_{(i)}$ to have tail index $\alpha_{(i)}$ and distribution function $\bar{F}_{(i)}$.

Lemma 4.2.1 *For $n \geq 2$, let $X_1, \dots, X_n$ be nonnegative and independent random variables with continuous distribution functions $F_1, \dots, F_n$ such that $\bar{F}_i \in \text{RV}_{-\alpha_i}$ with $0 < \alpha_i \leq 1$.*

Suppose each catastrophe risk X_i has the limited liability defined in (2.1) and follows the uniform risk sharing rule as in (2.2). Assume that $\lim_{q \rightarrow 1} \frac{k(q)}{\text{VaR}_q(X_1)} = \xi \geq 0$.

- (i) For $0 \leq \xi < \frac{n}{n-1}$, we have $\lim_{q \rightarrow 1} \text{DR}_q(\mathbf{X}(k)) \leq 1$.
- (ii) For $\frac{n}{n-i+1} < \xi < \frac{n}{n-i}$ with $i = 2, \dots, n-1$, if $\alpha_1 < \alpha_{(n-i+2)}$, then $\lim_{q \rightarrow 1} \text{DR}_q(\mathbf{X}(k)) < 1$.
- (iii) For $\xi > n$, if $\alpha_1 < \alpha_{(2)}$, then $\lim_{q \rightarrow 1} \text{DR}_q(\mathbf{X}(k)) < 1$.

In Lemma 4.2.1, the condition $\lim_{q \rightarrow 1} k(q)/\text{VaR}_q(X_1) = \xi$ means that the liability k for participant 1 is assumed to be ξ times their benchmark risk at high confidence levels, where the benchmark for participant 1 is measured by $\text{VaR}_q(X_1)$. That is, $k(q)$ is linearly proportional to the benchmark risk. Lemma 4.2.1 shows that when $0 \leq \xi < \frac{n}{n-1}$, or equivalently, when the liability level k is low relative to the benchmark $\text{VaR}_q(X_1)$ as in $0 < k < \frac{n}{n-1} \text{VaR}_q(X_1)$, then there are no other restrictions on the tail indices of the other risks in the pool for participant 1 to prefer to join. If the liability level k is increased to $\frac{n}{n-1} \text{VaR}_q(X_1) < k < \frac{n}{n-2} \text{VaR}_q(X_1)$, then participant 1 would prefer to join the pool, provided at least one other participant has a lighter-tailed risk than she. Intuitively, the presence of a lighter-tailed risk in the pool is necessary to help diversify the additional risk associated with the higher liability level. If the liability level k further increases to $\frac{n}{n-2} \text{VaR}_q(X_1) < k < \frac{n}{n-3} \text{VaR}_q(X_1)$, then at least two lighter-tailed risks are required in the pool to maintain a diversification benefit for participant 1. This pattern continues—each time k increases to a higher level, an additional lighter-tailed risk is needed in the pool. Eventually, when $k > n \text{VaR}_q(X_1)$, participant 1 must have the heaviest tail in the pool for joining to still reduce her individual risk.

Thus, when the uniform liability level k is set to be low relative to the lightest-tailed risk in the pool, the pool of n participants can be formed regardless of the extent of heavy-tailedness of the catastrophe risks. The pooling reduces the risk for each participant, making it beneficial for them to join and share the aggregated risk. We formalize this result in the next theorem. The following notation is adopted to denote the diversification benefit for the i th participant in the pool, $i = 1, \dots, n$,

$$\text{DR}_q^{(i)}(\mathbf{X}(k)) = \frac{\text{VaR}_q(S_n(k))}{\text{VaR}_q(X_i(k))}.$$

Thus, $\text{DR}_q^{(1)}(\mathbf{X}(k)) = \text{DR}_q(\mathbf{X}(k))$.

Theorem 4.2.1 *Under the conditions of Lemma 4.2.1, if $\lim_{q \rightarrow 1} \frac{k(q)}{\text{VaR}_q(X_{(n)})} < \frac{n}{n-1}$, then $\lim_{q \rightarrow 1} \text{DR}_q^{(i)}(\mathbf{X}(k)) \leq 1$ for any $i = 1, \dots, n$.*

There is room for improvement in the condition derived in Theorem 4.2.1. In fact, from the proof of Lemma 4.2.1, we observe that in most cases, $\text{DR}_q(\mathbf{X}(k)) \rightarrow 0$ as $q \rightarrow 1$, indicating that $\text{DR}_q(\mathbf{X}(k))$ is significantly below 1. This is further illustrated in the simulation study in the following subsection. However, for simplicity of presentation, we retain the current conditions. In the next two sections, we explore conditions under which all participants prefer to join the pool for relatively higher levels of limited liability k .

4.2.2 Simulation Studies

In this subsection, we illustrate the two theoretical results from Section 4.2.1 through simulation experiments. First, we illustrate Lemma 4.2.1 in Figure 4.1. Consider 10 independent catastrophe risks X_i , $i = 1, \dots, 10$. Each follows a Pareto distribution function as

$$F_i(x) = 1 - (1 + x)^{-\alpha_i},$$

with $0 < \alpha_i \leq 1$. We vary α_1 from 0.01 to 1 at a step size of 0.01, and set $\alpha_2 = 0.5$, $\alpha_3 = 0.55, \dots, \alpha_{10} = 0.9$. In addition, we set the liability level k to be $k = \xi \text{VaR}_q(X_1)$, where $q = 0.975$ and ξ takes values in the intervals $(0, \frac{n}{n-1}), (\frac{n}{n-1}, \frac{n}{n-2}), \dots, (n, \infty)$ based on Lemma 4.2.1. Specifically, ξ takes values 1.1, 1.2, 1.3, 1.6, 1.8, 2.2, 3, 4, 9, and 10.1. Then we explore how $\text{DR}_q(\mathbf{X}(k))$ changes by varying the tail index α_1 under various limited liability levels determined by ξ . We generate a sample of size 1,000 for each risk X_i and compute $\text{DR}_q(\mathbf{X}(k))$ using the empirical estimator

$$\widehat{\text{DR}}_q(\mathbf{X}(\hat{k})) = \frac{\widehat{\text{VaR}}_q(S_{10}(\hat{k}))}{\widehat{\text{VaR}}_q(X_1(\hat{k}))}, = \frac{S_{10}^{(\lfloor 1000q \rfloor)}(\hat{k})}{X_1^{(\lfloor 1000q \rfloor)}(\hat{k})},$$

where $X^{(1)} \leq \dots \leq X^{(n)}$ are the order statistics of X and limited liability $\hat{k} = \xi X_1^{(\lfloor 1000q \rfloor)}$ is empirically estimated. Repeating the experiment 100 times, we report the average $\widehat{\text{DR}}_q(\mathbf{X}(\hat{k}))$ in Figure 4.1.

Lemma 4.2.1 indicates that for different values of ξ , there is a constraint or an upper bound on α_1 in order for $\text{DR}_q(\mathbf{X}(k))$ to be less than 1. The upper bound is marked by a star on each curve in Figure 4.1. Clearly, we can observe that when α_1 takes values below the upper bound, $\text{DR}_q(\mathbf{X}(k))$ is strictly less than 1. As the value of ξ becomes larger, the upper bound becomes smaller, which means the condition for participant 1 to be willing to join the pool is stricter. In all of the cases, even when α_1 is larger than the upper bound, the value of $\text{DR}_q(\mathbf{X}(k))$ is not immediately greater than 1, which shows that there is still room to improve the upper bound.

Next, we illustrate Theorem 4.2.1. We still consider 10 risks, each following a Pareto distribution with fixed tail indices $(\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_{10}) = (0.999, 0.998, 0.997, \dots, 0.99)$. Note that X_1 has the lightest tailed in the pool and hence we set the liability $k = \xi \text{VaR}_q(X_1)$ with $\xi = 1.1$. The reason all the tail indices are very close to each other and close to 1 is to avoid the trivial case in which $\text{DR}_q^{(i)}(\mathbf{X}(k))$ overlaps at the low limited liability level. By varying the confidence level q from 0.95 to 0.999 with a step size of 0.001, we explore the change of $\text{DR}_q^{(i)}(\mathbf{X}(k))$ for each participant in Figure 4.2. As expected, all participants have a DR less than 1, meaning that they are all willing to join the pool.

4.3 Uniform Risk Sharing with Equivalent-tailed Risks under Uniform Liability

In Section 4.2, we showed that for catastrophe risks with different tails, if the uniform liability is higher than some threshold, then it is not optimal for the participants with lighter-tailed risks to join the pool. Theorem 4.2.1 suggests that one way for all participants to be willing to join the pool is to set a sufficiently low liability; however, this may not meet the needs for catastrophe risks in practice. Alternatively, we can pool the risks with the same heavy-tailedness, so that the risks have equivalent tails but are not necessarily identically distributed.

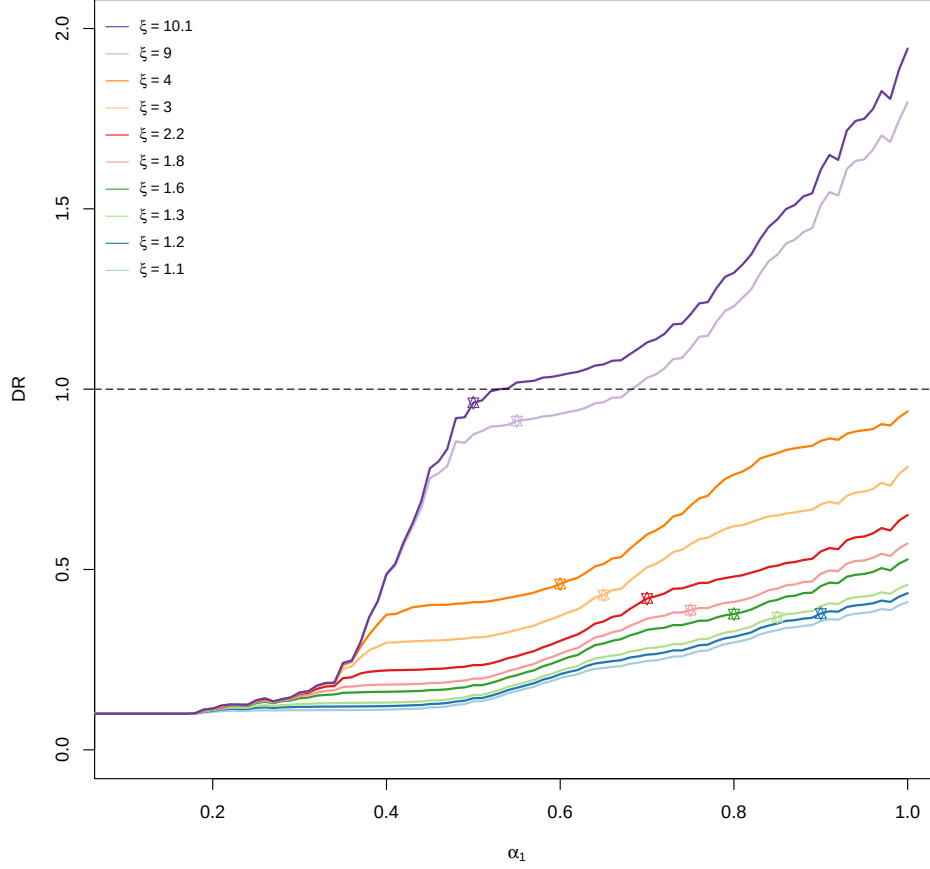


Figure 4.1: The value of $\text{DR}_{0.975}(\mathbf{X}(k))$ for 10 catastrophe risks following Pareto distributions. The tail index of X_1 is varying and all 9 other risks have fixed tail indices. The limited liability k is set as $\xi \text{VaR}_{0.975}(X_1)$.

4.3.1 Theory

Formally, X_1 and X_i have equivalent tails, or the same tail index, if there exists some $0 < \theta_i < \infty$ such that

$$\lim_{x \rightarrow \infty} \frac{\overline{F}_i(x)}{\overline{F}_1(x)} = \theta_i. \quad (3.4)$$

We emphasize that X_1 and X_i having equivalent tails does not mean that they have

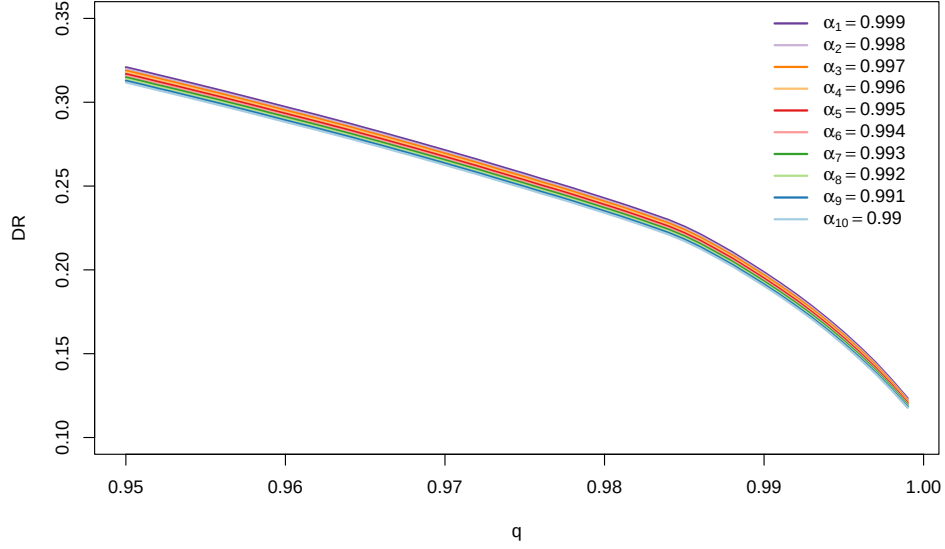


Figure 4.2: The values of $DR_q^{(i)}(\mathbf{X}(k))$ for 10 catastrophe risks following Pareto distributions with fixed tail indices. The limited liability k is set as $1.1\text{VaR}_q(X_1)$, where X_1 has the lightest tail in the pool.

identical distributions. The difference in their distributions is reflected by a constant θ_i , which we call the scale coefficient compared to X_1 . That is, X_1 and X_i can have the same heavy-tailedness, but different scales. When $\theta_i > 1$ (or < 1), it means that $\bar{F}_i(x)$ is θ_i times larger (or smaller) than $\bar{F}_1(x)$. Even when $\theta_i = 1$, X_1 and X_i may not necessarily be identically distributed. Thus, the assumption of tail equivalence is much easier to achieve in practice than requiring all risks to have identical distributions.

In the following lemma, we derive the conditions for a participant to join the pool when all of the participants bring tail equivalent risks to the pool. We assume uniform liability for the catastrophe risks and apply (2.1) for all X_i .

Lemma 4.3.1 *For $n \geq 2$, let $X_1, \dots, X_n$ be non-negative and independent random variables with continuous distribution functions $F_1, \dots, F_n$ such that $\bar{F}_i \in \text{RV}_{-\alpha}$ with $0 < \alpha \leq 1$. Assume that $\lim_{t \rightarrow \infty} \frac{\bar{F}_i(t)}{\bar{F}_1(t)} = \theta_i \in (0, \infty)$, $i = 2, 3, \dots, n$. Each X_i has the same limited*

liability as in (2.1) and follows the uniform risk sharing rule as in (2.2). Further assume that $\lim_{q \rightarrow 1} \frac{k(q)}{\text{VaR}_q(X_1)} = \xi > 0$.

(i) For $0 < \xi < n$, we have $\lim_{q \rightarrow 1} \text{DR}_q(\mathbf{X}(k)) \leq 1$.

(ii) For $\xi > n$, if $\sum_{i=1}^n \theta_i < n^\alpha$, then $\lim_{q \rightarrow 1} \text{DR}_q(\mathbf{X}(k)) < 1$.

Lemma 4.3.1 states that if the liability level k is lower than n times the benchmark so that $k < n\text{VaR}_q(X_1)$, then there are no additional restrictions on the risks and all the participants are willing to join the pool. This result is consistent with Theorem 3.1(i) in Cui et al. (2021). If the liability level k is higher than n times the benchmark so that $k > n\text{VaR}_q(X_1)$, then there is an additional condition for participant 1 to join the pool: $\sum_{i=1}^n \theta_i < n^\alpha$. This condition requires that the sum of the scale coefficients compared to X_1 not be too large. Since $0 < \alpha < 1$, we have $\sum_{i=1}^n \theta_i < n^\alpha < n$, which indicates that there is at least one risk in the pool with $\theta_i < 1$, a smaller scale coefficient than that of X_1 . Such a requirement prevents participants with smaller scale risks from joining the pool even when the risks have the same heavy-tailedness. Thus, when the limited liability is not too high, it is possible to form the pool. This result is presented in the following theorem.

Theorem 4.3.1 Under the conditions of Lemma 4.3.1, if $\theta_{\min}^{-\frac{1}{\alpha}} \xi < n$, where $\theta_{\min} = \min\{1, \theta_2, \dots, \theta_n\}$, then $\lim_{q \rightarrow 1} \text{DR}_q^{(i)}(\mathbf{X}(k)) < 1$ for any $i = 1, \dots, n$.

4.3.2 Simulation Studies

In this subsection, we illustrate the theoretical results from Section 4.3.1 through simulations. We first illustrate Lemma 4.3.1 in Figure 4.3. Again, we consider 10 independent catastrophe risks X_i , $i = 1, \dots, 10$, each following a Type II Pareto distribution with common tail index 0.9, but different scale coefficients. Specifically, for risk X_i , the distribution is

$$F_i(x) = 1 - \theta_i \left(x + \theta_i^{1/0.9} \right)^{-0.9}, \quad x, \theta_i > 0.$$

We set $\theta_1 = 1$ and consider the following three combinations of scale coefficients $(\theta_1, \dots, \theta_{10})$: $\Theta_1 := (1, 1.1, 1.3, 1.5, 1.7, 1.9, 2.1, 2.3, 2.5, 5)$, $\Theta_2 := (1.0, 0.1, 0.2, 0.3, 0.4, 0.5, 1.2, 1.3, 1.4, 1.5)$,

and $\Theta_3 := (1, 0.35, 0.45, 0.55, 0.65, 0.75, 0.85, 0.95, 1.05, 1.15)$. These combinations are chosen so that Θ_2 and Θ_3 satisfy the condition in Lemma 4.3.1 (ii) where $\sum_{i=1}^{10} \theta_i < 10^{0.9}$, while Θ_1 does not satisfy the condition. It is apparent that the Θ_1 combination contains many risks with large scales in the pool and both Θ_2 and Θ_3 include risks that have a smaller scale than X_1 . The limited liability is set as $k = \xi \text{VaR}_q(X_1)$, with ξ taking values 2 and 12, which corresponds to the two cases $\xi < n$ and $\xi > n$ in Lemma 4.3.1. Again, we simulate a sample of 1,000 observations for each distribution X_i and apply the empirical estimator of $\text{DR}_q(\mathbf{X}(k))$ introduced in Section 4.2.2. Repeating the experiment 100 times, we plot the average $\overline{\text{DR}}_q(\mathbf{X}(\hat{k}))$ in Figure 4.3 for different combinations of Θ and ξ as q increases from 0.95 to 0.999 with a step size of 0.001. We can see that when the limited liability is relatively low ($\xi = 2$), the diversification benefit for participant 1 is achieved starting from a lower confidence level, even for the pool with large scale risks (Θ_1). When the limited liability is higher ($\xi = 12$), the diversification benefit for participant 1 can still be achieved, but at a higher confidence level and generally only works for the pool with smaller scale risks (Θ_2 and Θ_3).

Next, we illustrate Theorem 4.3.1 in Figure 4.4 using the same 10 catastrophe risks, which follow Pareto distributions with common tail index 0.9 and different scale parameters Θ_1 . By varying the confidence level q from 0.95 to 0.999 with a step size of 0.001, we explore the change in $\text{DR}_q^{(i)}(\mathbf{X}(k))$ for each participant. The limited liability k is set at different values in the four subfigures. In Figure 4.4(a), we set $k = 2\text{VaR}_q(X_1)$ and it can be verified that Θ_1 satisfies the $\theta_{\min}^{-\frac{1}{\alpha}}\xi < n$ condition from Theorem 4.3.1. As expected, all participants have a DR less than 1, which means that they are all willing to join the pool. In Figure 4.4(b), we set $k = 2\text{VaR}_q(X_9)$, which is based on X_9 with the second largest scale in the pool. At this liability level, Θ_1 still satisfies the condition in Theorem 4.3.1. Although all participants have a DR less than 1 in Figure 4.4(b), the DR values are larger than those in Figure 4.4(a) due to the higher liability level for each participant and greater risk brought to the pool, indicating a lower diversification benefit for each participant. In Figure 4.4(c), we have $k = 1.5\text{VaR}_q(X_{10})$, which is based on X_{10} with the largest scale in the pool. This time the coefficient is reduced to 1.5 to make Θ_1 satisfy the condition in Theorem 4.3.1. Moreover, we observe that the two risks with the smallest scales, X_1 and X_2 , only prefer to join the pool at higher confidence levels. Lastly, in Figure 4.4(d) we

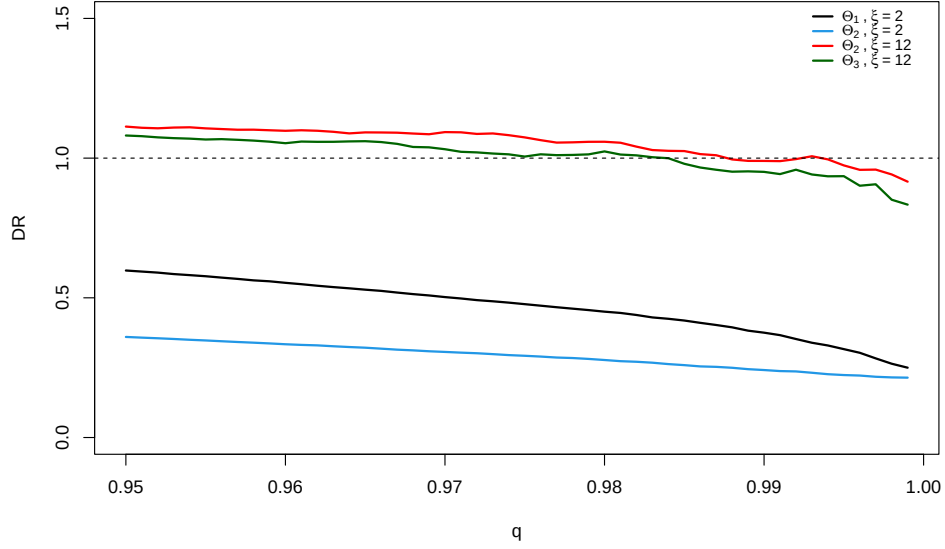


Figure 4.3: The value of $DR_q(\mathbf{X}(k))$ for 10 catastrophe risks following Pareto distributions with the tail index 0.9 and different scale parameters Θ_1 , Θ_2 and Θ_3 . The limited liability k is set as $2\text{VaR}_q(X_1)$ and $12\text{VaR}_q(X_1)$ with $\xi = 2$ and 12.

set $k = 2\text{VaR}_q(X_{10})$, which makes Θ_1 fail to satisfy the condition in Theorem 4.3.1. As a result, X_1 and X_2 no longer prefer to join the pool. These experiments demonstrate that under uniform limited liability levels, the choice of benchmark risk affects the preference of each participant, which is caused by the imbalanced risks contributed and shared among the participants. In the next section, we aim to address this issue.

4.4 Risk Sharing with Equivalent-tailed Risks under Proportional Liabilities

In Sections 4.2 and 4.3, we explored the conditions for participants with catastrophe risks to join the pool under the uniform limited liability scheme (2.1). Combined with the uniform risk sharing rule, the uniform liability scheme can create some inequality among

the participants. To see this, for equivalent tails, when the same liability k is applied to different X_i , we have

$$\overline{F}_i(k) \sim \theta_i \overline{F}_1(k), \text{ or } \text{VaR}_q(X_i) \sim \theta_i^{1/\alpha} \text{VaR}_q(X_1).$$

That is, the risk brought into the pool by participant i is $\theta_i^{1/\alpha}$ times the risk that participant 1 contributed. When the pooled risk is distributed back to each participant, participant i may be at a disadvantage if the scale of its risk, θ_i , is smaller than 1, that of participant 1. To address the inequality, in this section we consider setting proportional limited liability levels.

4.4.1 Theory

We consider proportional liabilities as follows. For $i = 2, \dots, n$,

$$X_i(k_i) = \begin{cases} k_i, & X_i > k_i, \\ X_i, & X_i \leq k_i, \end{cases} \quad (4.5)$$

where $k_i = \theta_i^{1/\alpha} k$ and k is the limited liability chosen for X_1 . By allowing different liabilities for different risks, each participant contributes equivalent risks to the pool. To see this, note that

$$\lim_{k \rightarrow \infty} \frac{\overline{F}_i(k_i)}{\overline{F}_1(k)} = \lim_{k \rightarrow \infty} \frac{\overline{F}_i(\theta_i^{1/\alpha} k)}{\overline{F}_1(\theta_i^{1/\alpha} k)} \frac{\overline{F}_1(\theta_i^{1/\alpha} k)}{\overline{F}_1(k)} = \theta_i (\theta_i^{1/\alpha})^{-\alpha} = 1.$$

That is, the tail probabilities for X_1 and X_i at their corresponding liability limits are equivalent:

$$\Pr(X_i > k_i) \sim \Pr(X_1 > k).$$

Such proportional liabilities are more appropriate with the uniform risk sharing rule when the risks are not identically distributed. Note that the proportional liability (4.5) does not apply to the general risks examined in Section 4.2. For example, if X_1 has a lighter tail than X_i so that $\alpha_1 > \alpha_i$, then $\theta_i = \infty$ and it is not possible to define k_i as $\theta_i^{1/\alpha} k$. In other words, it is not possible to find a k_i such that $\Pr(X_i > k_i) \sim \Pr(X_1 > k)$.

Let $\mathbf{k} = (k_1, k_2, \dots, k_n) = (k, \theta_2^{1/\alpha} k, \dots, \theta_n^{1/\alpha} k)$. We denote $\mathbf{X}(\mathbf{k}) = (X_1(k_1), X_2(k_2), \dots, X_n(k_n)) = (X_1(k), X_2(\theta_2^{1/\alpha} k), \dots, X_n(\theta_n^{1/\alpha} k))$ and

$$\begin{aligned} S_n(\mathbf{k}) &= \frac{1}{n} (X_1(k_1) + X_2(k_2) + \dots + X_n(k_n)) \\ &= \frac{1}{n} (X_1(k) + X_2(\theta_2^{1/\alpha} k) + \dots + X_n(\theta_n^{1/\alpha} k)). \end{aligned} \quad (4.6)$$

The following lemma explores the conditions for participant 1 to join the pool.

Lemma 4.4.1 *For $n \geq 2$, let $X_1, \dots, X_n$ be non-negative and independent random variables with continuous distribution functions $F_1, \dots, F_n$ such that $\bar{F}_i \in \text{RV}_{-\alpha}$ with $0 < \alpha \leq 1$ for $i = 1, \dots, n$. Assume that $\lim_{t \rightarrow \infty} \frac{\bar{F}_i(t)}{\bar{F}_1(t)} = \theta_i \in (0, \infty)$, $i = 2, \dots, n$. Each catastrophe risk X_i has a proportional limited liability as in (4.5) and follows the uniform risk sharing rule as in (4.6). Further assume that $\lim_{q \rightarrow 1} \frac{k(q)}{\text{VaR}_q(X_1)} = \xi > 0$.*

- (i) *For $0 < \xi < 1$, if $\sum_{i=1}^n \theta_i^{1/\alpha} \leq n$, then $\lim_{q \rightarrow 1} \text{DR}_q(\mathbf{X}(\mathbf{k})) \leq 1$.*
- (ii) *For $\xi > 1$, if $\sum_{j \in Z} \theta_j < n^\alpha$ where $Z = \{j = 2, \dots, n : \theta_j^{1/\alpha} \geq n/\xi\}$, then $\lim_{q \rightarrow 1} \text{DR}_q(\mathbf{X}(\mathbf{k})) < 1$.*

The set Z in Lemma 4.4.1 is the set of risks with relatively larger θ_j 's, where $\theta_j^{1/\alpha} \geq n/\xi$, and the condition for participant 1 to be willing to join the pool is that the sum of these θ_j 's is not too large.

Comparing Lemmas 4.3.1 and 4.4.1 regarding participant 1's willingness to join the pool with uniform liability and proportional liability respectively, Lemma 4.4.1 places additional restrictions on the scale coefficients θ_j in the $\xi < n$ case. This is because under the proportional liability scheme, risks with smaller scales can have a lower liability level, while risks with larger scales have a higher liability level. When comparing participant 1's share of the pooled risk to the original risk using $\text{DR}_q(\mathbf{X}(\mathbf{k}))$, even though the limited liability level is not very high ($\xi < n$), it is still necessary to control those high liability levels so that participant 1 is not distributed too much risk compared to its original risk.

In the following theorem, we derive the conditions such that it is preferred for all participants to join the pool with proportional limited liability levels under uniform risk sharing.

Theorem 4.4.1 *Under the conditions of Lemma 4.4.1, we have*

- (i) *For $0 < \xi < 1$, if $\theta_i = 1$ for all $i = 1, \dots, n$, then $\lim_{q \rightarrow 1} \text{DR}_q^{(i)}(\mathbf{X}(\mathbf{k})) \leq 1$ for any $i = 1, \dots, n$.*
- (ii) *For $1 < \xi < n$, if $\sum_{j \in Z_{\min}} \theta_j < n^\alpha \theta_{\min}$, where $\theta_{\min} = \min\{1, \theta_2, \dots, \theta_n\}$ and $Z_{\min} = \{j = 1, \dots, n : \theta_j^{1/\alpha} \geq \theta_{\min}^{1/\alpha} n / \xi\}$, then $\lim_{q \rightarrow 1} \text{DR}_q^{(i)}(\mathbf{X}(\mathbf{k})) < 1$ for any $i = 1, \dots, n$.*

Under uniform risk sharing, it is possible to pool larger risks with the proportional liability scheme than with the uniform liability scheme. However, a drawback of proportional liability under uniform risk sharing is that, despite all being willing to join the pool, the participants with smaller scales may not obtain as much diversification benefit as the participants with larger scales. One way to improve a fairer distribution of the diversification benefit is to modify the uniform risk sharing rule to achieve an equal diversification benefit for all participants. More specifically, for tail equivalent risks with proportional liability, after pooling $X_1(k_1), X_2(k_2), \dots$, and $X_n(k_n)$, the amount of risk distributed to participant i is

$$S_n^{(i)}(\mathbf{k}) = w_i \sum_{j=1}^n X_j(k_j), \quad \text{with } w_i = \frac{\theta_i^{1/\alpha}}{\sum_{j=1}^n \theta_j^{1/\alpha}}, \quad (4.7)$$

which we introduce as the proportional risk sharing rule. Then each participant i has the same diversification ratio as

$$\lim_{q \rightarrow 1} \text{DR}_q^*(\mathbf{X}(\mathbf{k})) = \lim_{q \rightarrow 1} \frac{\text{VaR}_q \left(w_i \sum_{j=1}^n X_j(k_j) \right)}{\text{VaR}_q(X_i(k_i))} = \lim_{q \rightarrow 1} \frac{\text{VaR}_q \left(\sum_{j=1}^n X_j(k_j) \right)}{\sum_{j=1}^n \text{VaR}_q(X_j(k_j))}.$$

To see this, first note that

$$\frac{\text{VaR}_q \left(w_i \sum_{j=1}^n X_j(k_j) \right)}{\text{VaR}_q(X_i(k_i))} = w_i \frac{\text{VaR}_q \left(\sum_{j=1}^n X_j(k_j) \right)}{\sum_{j=1}^n \text{VaR}_q(X_j(k_j))} \frac{\sum_{j=1}^n \text{VaR}_q(X_j(k_j))}{\text{VaR}_q(X_i(k_i))}.$$

With the proportional liability scheme, for any $i = 1, \dots, n$, we have

$$\lim_{q \rightarrow 1} \frac{k_i(q)}{\text{VaR}_q(X_i)} = \xi.$$

This implies that for $0 < \xi < 1$ or $\xi > 1$, we have

$$\lim_{q \rightarrow 1} \frac{\text{VaR}_q(X_j(k_j))}{\text{VaR}_q(X_i(k_i))} = \frac{\theta_j^{1/\alpha}}{\theta_i^{1/\alpha}}.$$

Thus, under proportional risk sharing, each participant can obtain the same diversification benefit by joining the pool. In the following theorem, we derive the conditions for every participant to be willing to join the pool with proportional liability under the proportional risk sharing rule.

Theorem 4.4.2 *For $n \geq 2$, let $X_1, \dots, X_n$ be non-negative and independent random variables with continuous distribution functions $F_1, \dots, F_n$ such that $\bar{F}_i \in \text{RV}_{-\alpha}$ with $0 < \alpha \leq 1$ for $i = 1, \dots, n$. Assume that $\lim_{t \rightarrow \infty} \frac{\bar{F}_i(t)}{\bar{F}_1(t)} = \theta_i \in (0, \infty)$, $i = 2, \dots, n$. Each catastrophe risk X_i has a proportional limited liability as in (4.5) and follows the proportional risk sharing rule as in (4.7). Further assume that $\lim_{q \rightarrow 1} \frac{k(q)}{\text{VaR}_q(X_1)} = \xi > 0$.*

- (i) *For $0 < \xi \leq 1$, $\lim_{q \rightarrow 1} \text{DR}_q^*(\mathbf{X}(\mathbf{k})) \leq 1$.*
- (ii) *For $\xi > 1$, if $\sum_{j \in W} \theta_j < \left(\sum_{i=1}^n \theta_i^{1/\alpha} \right)^\alpha$ where $W = \{j = 1, \dots, n : \theta_j^{1/\alpha} \xi \geq \sum_{i=1}^n \theta_i^{1/\alpha}\}$, then $\lim_{q \rightarrow 1} \text{DR}_q^*(\mathbf{X}(\mathbf{k})) < 1$.*

As the simulations in Section 4.4.2 will illustrate, with the proportional liability scheme where participants contribute equivalent risks to the pool, the uniform risk sharing rule allows participants with larger scale risks to obtain greater diversification benefit. On the other hand, participants with smaller scale risks are willing to join the pool only at higher confidence levels and obtain less diversification benefit. Combining the proportional liability scheme and the proportional risk sharing rule would allow all of the participants to obtain the same diversification benefit. This results from a redistribution of diversification benefit among participants so that larger scale risks have a reduced diversification benefit.

4.4.2 Simulation Studies

In this subsection we illustrate the theoretical results from Section 4.4.1 involving tail equivalent risks with proportional liability, under both uniform and proportional risk sharing.

We illustrate Lemma 4.4.1 in Figure 4.5 using a similar setup to that of Figure 4.3 for Lemma 4.3.1. We consider 10 independent catastrophe risks X_i , $i = 1, \dots, 10$, each following a Pareto distribution with equivalent tail index 0.9 and different scale coefficient combinations of $(\theta_1, \dots, \theta_{10})$ given by Θ_1 and Θ_2 defined in Section 4.3.2, where participant 1 is the smallest scale risk in Θ_1 , while there are smaller and larger scale risks in Θ_2 . We also introduce another combination $\Theta_4 := (1.0, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0, 1.1, 1.2, 1.3)$. To incorporate the proportional liability, we set the limited liability for X_1 as $k = \xi \text{VaR}_q(X_1)$, with ξ taking values 0.8 and 2, which corresponds to the two cases $0 < \xi < 1$ and $\xi > 1$ in Lemma 4.4.1. Then the limited liability for X_i is set as $k_i = \theta_i^{1/\alpha} k$ for $i = 2, \dots, 10$. These parameters are chosen so that Θ_2 and Θ_4 together with $\xi = 0.8$ satisfy the condition $\sum_{i=1}^{10} \theta_i^{1/0.9} \leq 10$ in Lemma 4.4.1 (i), and Θ_1 and Θ_2 together with $\xi = 2$ satisfy the condition $\sum_{i \in Z} \theta_i < 10^{0.9}$ in Lemma 4.4.1 (ii), where $Z = \{i = 2, \dots, 10 : \theta_i^{1/0.9} \geq 10/\xi\}$. We again simulate a sample of 1000 observations for each X_i and apply the empirical estimator of $\text{DR}_q(\mathbf{X}(k))$ introduced in Section 4.2.2. Repeating the experiment 100 times, Figure 4.5 plots the average $\widehat{\text{DR}}_q(\mathbf{X}(\hat{k}))$ for various combinations of Θ and ξ as q increases from 0.95 to 0.999 with a step size of 0.001.

We can similarly observe that the lower liability level with $\xi = 0.8$ results in a larger diversification benefit (lower DR) for participant 1 compared to the higher liability with $\xi = 2$ in Figure 4.5. However, at $\xi = 2$ with scale combination Θ_1 under uniform risk sharing (solid lines), the DR for participant 1 is higher in Figure 4.5 with proportional liability than in Figure 4.3 with uniform liability. Under the proportional liability scheme, larger scale risks have higher liability limits, so the pool in Figure 4.5 contains much larger aggregated risk than the pool in Figure 4.3 under uniform limited liability. Specifically, for Θ_1 , all of the risks in Figure 4.3 have the same limited liability $k = 2\text{VaR}_q(X_1)$, whereas in Figure 4.5, X_1 has the smallest scale coefficient and thus the lowest liability k , and the remaining risks have liabilities $\theta_i^{1/\alpha} k$, which are larger than k . For example, the liability level for X_{10} in Θ_1 of Figure 4.5 is about 6 times its level in Figure 4.3. Hence, to still

form the pool with the much larger aggregated risk in Figure 4.5, participants with smaller scale risks receive less diversification benefit.

Moreover, in Figure 4.5 we plot the empirical estimator of $\text{DR}_q^*(\mathbf{X}(\mathbf{k}))$ when the pool is constructed under the proportional risk sharing rule. According to Theorem 4.4.2, all participants are able to achieve the same diversification benefit by having the same $\text{DR}_q^*(\mathbf{X}(\mathbf{k}))$, which is shown by the dashed lines for Θ_1 , Θ_2 and Θ_4 , respectively. We observe that participant 1, the smallest scale in the pool with Θ_1 , obtains a greater diversification benefit (lower DR) under proportional risk sharing compared to uniform risk sharing. In contrast, Θ_2 and Θ_4 both include smaller and larger scale risks than that of participant 1, and the diversification benefit for participant 1 is reduced under proportional risk sharing (higher DR), indicating a redistribution of the diversification benefit so that all participants ultimately obtain the same level of diversification benefit.

Next, we illustrate Theorem 4.4.1 using a similar setup of 10 catastrophe risks following Pareto distributions with common tail index 0.9 and scale coefficient combination Θ_1 in Figure 4.6, which are plotted in solid lines for uniform risk sharing. The limited liability is set to be $k = 2\text{VaR}_q(X_1)$, which allows Θ_1 to satisfy the condition $\sum_{j \in Z_{\min}} \theta_j < n^\alpha \theta_{\min}$ in Theorem 4.4.1 and the condition in Theorem 4.4.2. Note that under the proportional liability scheme, setting $k = 2\text{VaR}_q(X_i)$ for any $i = 1, \dots, n$ means that $k_j = 2\text{VaR}_q(X_j)$ for any $j = 1, \dots, n$. That is, all participants have the same $\xi = k_j/\text{VaR}_q(X_j) = 2$ so that the liability level for each participant j is always 2 times its benchmark $\text{VaR}_q(X_j)$. In Figure 4.6, by varying the confidence level q from 0.95 to 0.999 with a step size of 0.001, we explore the change in $\text{DR}_q^{(i)}(\mathbf{X}(k))$ as a solid line for each participant. All of the participants are eventually willing to join the pool. To consider the difference in the diversification benefit for participants under the uniform versus proportional liability schemes, we compare Figure 4.4 and Figure 4.6. First, in Figure 4.4(a), all participants have a uniform liability level of $k = 2\text{VaR}_q(X_1)$, whereas in Figure 4.6, participant 1 has a liability of k and all other participants have liabilities of $\theta_i^{1/\alpha} k$, which are greater than k . Thus, larger risks are pooled in Figure 4.6, and participants with smaller scales (e.g., X_1 and X_2) forgo part of their diversification benefit to enable the formation of the pool. When the uniform liability for all participants is increased to $k = 2\text{VaR}_q(X_{10})$ in Figure 4.4(d), the pool fails to form because the aggregated risk is too large to be effectively diversified among the participants.

Under proportional liability in Figure 4.6, although the aggregated risk is not as large as in Figure 4.4(d), participant 10 (with the largest scale coefficient) is still able to contribute a liability of $2\text{VaR}_q(X_{10})$ to a successfully formed pool.

Furthermore, for the same group of risks Θ_1 and proportional limited liability scheme, we consider $\text{DR}_q^*(\mathbf{X}(\mathbf{k}))$ when the pool is constructed under the proportional risk sharing rule. According to Theorem 4.4.2, all participants have the same DR value $\text{DR}_q^*(\mathbf{X}(\mathbf{k}))$, which is shown as the multi-coloured dashed line in Figure 4.6. We can see that compared to the uniform risk sharing rule, the proportional risk sharing rule redistributes the diversification benefit such that participants with larger scale risks such as X_9 and X_{10} now have a reduced diversification benefit (higher DR), and participants with smaller scale risks such as X_1 and X_2 have an increased diversification benefit (lower DR). This is more fair for all of the participants from the diversification benefit perspective.

4.5 Real Data Analysis

In this section we use US natural catastrophe loss data to demonstrate the application of the theoretical insights and proposed mechanisms to the formation of risk pools for heavy-tailed catastrophe risks.

The dataset comes from the Storm Events Database maintained by the National Oceanic and Atmospheric Administration (NOAA, 2023), which documents storms and other severe weather phenomena such as tornados, floods, thunderstorm winds, and hail, that occurred in the US over the past 70 years. While the earliest records begin in 1950, our sample comprises severe weather events from 1996 (when reporting procedures were standardized across all peril types) through to 2023. Records are at the severe weather event level and include information on the date(s) and state of occurrence, type of peril, and economic losses associated with property damage. Thus, we can consider both peril-based and geographical diversification as advocated by existing literature on catastrophe risk pools. The data facilitate an analysis of pooling across different US states and perils, which we refer to as ‘state-perils,’ although the methodology can be applied to similar contexts for diversifying heavy-tailed losses, such as the formation of sovereign risk pools.

We consider risk pools for the annual economic property losses of various state-perils. Economic property losses include damage inflicted to private property and public infrastructure and facilities and are reported in US dollars, which we inflation adjust to 2023 dollars. Since only 28 years of historical annual losses are observed for each state-peril, we instead use the full dataset of event level records to construct the distribution of annual losses for a state-peril hierarchically. Using the empirical distribution of the event level losses, we obtain the distribution of annual losses by scaling the event level loss distribution up by the average annual event frequency. Specifically, we define a weather event based on the ‘episode’ identified by NOAA, where a single episode corresponds to an entire meteorological storm system that can potentially consist of multiple individual peril events (e.g., thunderstorm wind, hail, tornado).

The risk pool in Section 4.2 involving general heavy-tailed risks with uniform risk sharing only forms at a relatively low uniform liability level, so we focus our analyses on the pool structures in Sections 4.3 and 4.4. That is, we study independent and tail equivalent catastrophe risks. Due to the different numbers of observations of different state-perils, we cannot formally test the independence of state-perils; however, we can select risks from different perils and geographic regions. It is reasonable to believe that annual losses from different perils and in different, non-adjacent states may be relatively independent, a presumption that is supported by some empirical evidence of low correlation between adjacent country-perils for regional pools, e.g., in the Caribbean or Sub-Saharan Africa (Ciullo et al., 2023; Bollmann and Wang, 2019). To identify sets of tail equivalent risks with tail index less than 1, we first apply the classical Hill estimator (Hill, 1975) to estimate the tail index of each risk. Next, we apply the test of equality of tail indices proposed by Daouia et al. (2024) to examine the tail equivalence of groups of heavy-tailed risks. In the remaining analyses, we present results for two example pools identified using this procedure, which are summarized in Table 4.1. Pool 1 is constructed with risks from different perils and from different locations, and Pool 2 is constructed with risks from the same peril but are geographically separated from each other.

Figure 4.7 displays the p -values for the test of equality of tail indices for the two pools as the sample fraction increases from 0.5% to 20% with a step size of 0.5%. We can conclude that the risks in each pool have equal tail indices at the 0.05 significance level, aside from a

	State	Peril	Number of observations	$\widehat{\text{VaR}}_{0.95}(X)$ (USD, millions)
Pool 1				
	MN-TW	Minnesota Thunderstorm wind	632	42.7
	AL-TN	Alabama Tornado	539	72.3
	NE-H	Nebraska Hail	907	63.8
Pool 2				
	VA-TW	Virginia Thunderstorm wind	2,401	15.8
	IN-TW	Indiana Thunderstorm wind	1,606	21.2
	AL-TW	Alabama Thunderstorm wind	3,519	19.4

Table 4.1: Two candidate pools of heavy-tailed, tail equivalent catastrophe risks.

rejection point at the 1% sample fraction for Pool 1. In addition, the common tail index α is calculated based on the test statistic of equality of tail indices from [Daouia et al. \(2024\)](#), and is shown for varying fractions of the sample from 0.5% to 20% in Figure 4.8. Similar to the standard practice in extreme value theory for obtaining the Hill estimate of the tail index from the Hill plot, we visually infer the common tail index for each risk pool based on the first stable region of the plot. In Figure 4.8, the first stable region for Pool 1 is identified as beginning at 5.5% of the sample, for a common tail index estimate of 0.701. Similarly, the common tail index for Pool 2 is estimated as 0.974 at a 5% sample fraction.

Next, we determine the scale coefficients θ_i for the risks in each pool. Under the setup in Section 4.3, X_1 has $\theta_1 = 1$, and the scale coefficients for the remaining risks are calculated relative to X_1 . We propose the following empirical estimator of θ_i for any $i = 2, \dots, n$.

$$\hat{\theta}_i = \frac{\widehat{F}_i(X_1^{(h:n_1)})}{\widehat{F}_1(X_1^{(h:n_1)})} := \frac{n_1}{n_i h} \sum_{j=1}^{n_i} 1_{\{X_i^{(j)} > X_1^{(h:n_1)}\}}, \quad (5.8)$$

where n_i is X_i 's sample size, $X_i^{(j)}$ is j -th data point in the sample of X_i , and $X_1^{(1:n_1)} \leq X_1^{(2:n_1)} \leq \dots \leq X_1^{(n_1:n_1)}$ are order statistics of X_1 , so $X_1^{(h:n_1)}$ represents the h -th order statistic of X_1 with sample size n_1 . We estimate θ_i at sample fractions from 0.5% to 20% with a step size of 0.5%. Since each sample fraction corresponds to an order statistic indexed by an integer, we let the order index h vary from $(\lfloor 0.005 \times n_1 \rfloor)$ to $(\lfloor 0.2 \times n_1 \rfloor)$,

with increments corresponding to 0.5% increases in the sample fraction. Figure 4.9 plots $\hat{\theta}_i$ for each risk in Pool 1 and Pool 2 at increasing sample fractions. Without loss of generality, MN-TW and VA-TW are assigned the first risk in Pools 1 and 2 respectively so that their scale coefficients are both set as $\theta_1 = 1$. Similar to the estimation of the tail indices, we follow standard practice in extreme value theory by choosing the value of $\hat{\theta}_i$ from the first stable region in the plots. Then in Pool 1, the estimated scale coefficients for AL-TN and NE-H are 1.577 and 1.533 respectively, indicating that both AL-TN and NE-H are larger scale risks than MN-TW. In Pool 2, the estimated scale coefficients for IN-TW and AL-TW are 1.301 and 1.271 respectively, indicating that both IN-TW and AL-TW are larger scale risks than VA-TW.

Having estimated the common tail index and scale coefficients of the risks in Pool 1 and Pool 2, we now explore the performance of risk pools with various liability schemes and risk sharing rules. First, for the risks in Pool 1, Figure 4.10 displays the DR under three pool structures: uniform liability with uniform risk sharing (Theorem 4.3.1); proportional liability with uniform risk sharing (Theorem 4.4.1); and proportional liability with proportional risk sharing (Theorem 4.4.2), in rows 1, 2, and 3 respectively. In addition, we consider different levels of $\xi = 1.2, 2.2$, and 2.9 in columns 1, 2, and 3 respectively. Note that due to the rejection point observed at the 1% sample fraction in Figure 4.7 for Pool 1, we adopt a conservative approach by restricting the pooling to the range where tail equivalence holds at the 0.05 significance level, specifically for confidence levels q from 0.8 to 0.985 with a step size of 0.005.

The first row of Figure 4.10 depicts the uniform liability scheme and uniform risk sharing rule. All three values of $\xi = 1.2, 2.2$, and 2.9 satisfy the conditions of Theorem 4.3.1. We can correspondingly see that all of the risks obtain a DR less than 1 eventually, and would be willing to join the pool, although the smallest scale risk MN-TW obtains the lowest diversification benefit. In addition, as the uniform liability level increases with larger ξ , the smallest scale risk MN-TW only prefers to join the pool at a higher confidence level q , and their diversification benefit is even further reduced compared to the other risks.

The three graphs in the second row of Figure 4.10 are plotted under the proportional liability scheme and uniform risk sharing rule as in Theorem 4.4.1 for $\xi = 1.2, 2.2$ and 2.9 . In this case, only $\xi = 1.2$ and 2.2 satisfy the conditions of Theorem 4.4.1. Thus when

$\xi = 2.9$, MN-TW has a DR larger than 1 for all the plotted confidence levels. Under the proportional liability we see a redistribution of diversification benefits among the three risks. The diversification benefits for risks with larger scales are reduced.

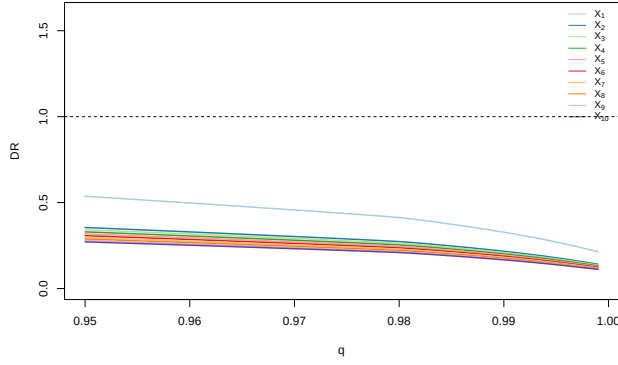
The three graphs in the third row of Figure 4.10 are plotted under the proportional liability scheme and proportional risk sharing rule as in Theorem 4.4.2 for $\xi = 1.2, 2.2$ and 2.9 . In this case, only $\xi = 1.2$ and 2.2 satisfy the conditions of Theorem 4.4.2. In this setting the diversification benefit in theory is the same for all three risks and it is verified with the real data as well. The risks with larger scales have reduced diversification benefit compared to the three graphs in the first row but participants may find it is more fair in this setting.

The similar plots are done for Pool 2 in Figure 4.11, where the confidence level q is ranging from 0.8 to 0.995 with a step size of 0.005. For Pool 2, all three values of $\xi = 1.2, 2.2$ and 2.9 satisfy the conditions in Theorems 4.3.1, 4.4.1 and 4.4.2. Similar observations can be found in Figure 4.11. Better performance is seen in Pool 2 than that in Pool 1.

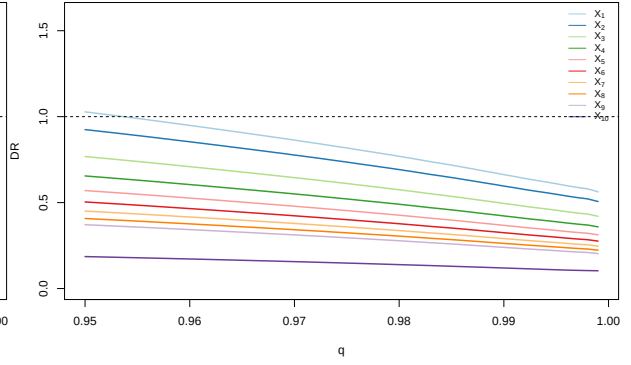
4.6 Conclusion

A catastrophe risk pool serves as an effective mechanism to enhance the financial resilience of firms or countries in response to the increasing incidence of weather-related natural disasters driven by climate change. For a risk pool to diversify participants' risks, it pools catastrophic risks from various geographies or perils such that the risks are (almost) independent. Due to the extremely heavy-tailed nature of catastrophe risks, the performance of the risk pool may deviate from expectations. In this chapter, we explore different structures for a catastrophe risk pool to work for all participants. DR is used as a criterion to evaluate the willingness of a participant to join the pool. All the catastrophe risks are independent and extremely heavy tailed with limited liabilities. We explore the conditions for the pool to form under various scenarios. When the risks have different heavy tailedness, under the uniform liability scheme and uniform risk sharing, the pool can form only when the liability level is very low. When the risks are tail equivalent but with different scales, the pool can form for a higher liability level under the uniform liability scheme and uniform risk sharing rule. By combining the proportional liability scheme and

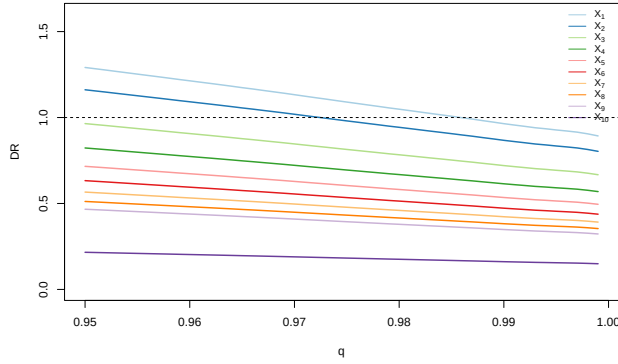
proportional risk sharing rule, the pool can form at a high level liability limit, and all the participants can achieve (asymptotically) the same amount of diversification benefit, which can be considered to be a fair pool. Simulations are conducted to illustrate the theoretical results, and a real-data analysis is presented to demonstrate the practical implementation of the proposed approach.



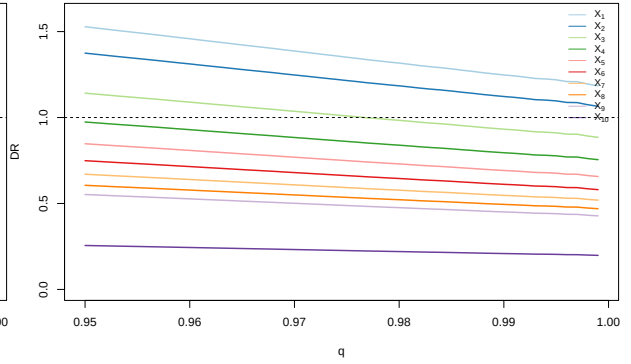
((a)) $k = 2\text{VaR}_q(X_1)$



((b)) $k = 2\text{VaR}_q(X_9)$



((c)) $k = 1.5\text{VaR}_q(X_{10})$



((d)) $k = 2\text{VaR}_q(X_{10})$

Figure 4.4: The values of $\widehat{\text{DR}}_q^{(i)}(\mathbf{X}(\hat{k}))$ for 10 catastrophe risks following Pareto distributions with tail index 0.9 and scale parameters Θ_1 . The limited liability k is set at different levels.

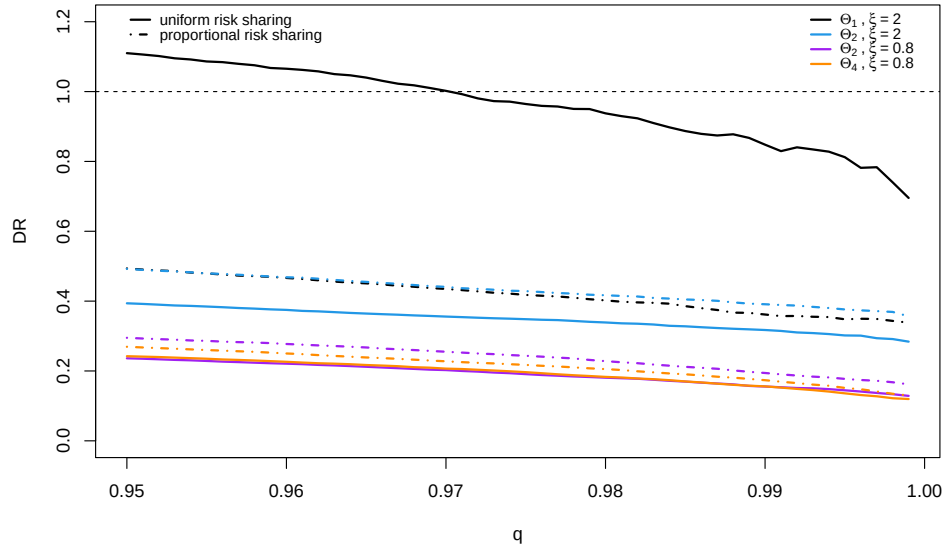


Figure 4.5: The value of $DR_q(\mathbf{X}(k))$ for 10 catastrophe risks following Pareto distributions with the tail index 0.9 and different scale parameters Θ_1 , Θ_2 and Θ_4 . The limited liability k is set as $0.8\text{VaR}_q(X_1)$ and $2\text{VaR}_q(X_1)$.

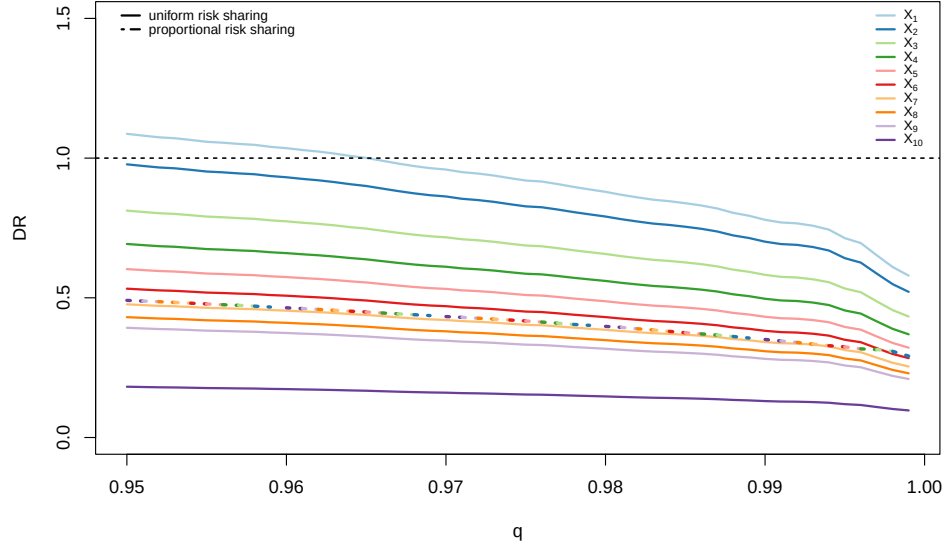


Figure 4.6: The values of $DR_q^{(i)}(\mathbf{X}(\mathbf{k}))$ for 10 catastrophe risks following Pareto distributions with the tail index 0.9 and scale parameters Θ_1 . The limited liability k is set as $2\text{VaR}_q(X_1)$. The dashed line represents the common $DR_q^*(\mathbf{X}(\mathbf{k}))$ for all risks under proportional risk sharing.

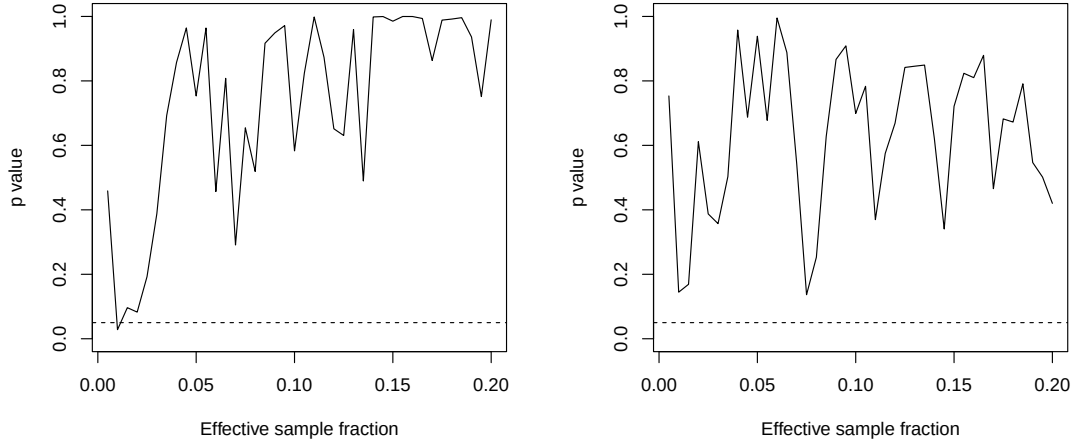


Figure 4.7: p -values from the test of equality of tail indices for Pool 1 (left) and Pool 2 (right), with the 0.05 significance level indicated by the dashed line.

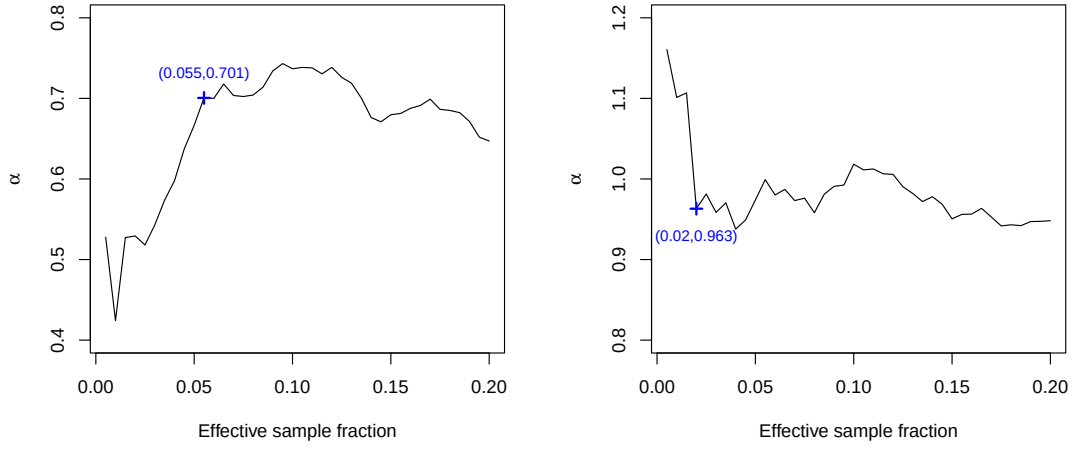


Figure 4.8: Estimation of the common tail index α for Pool 1 (left) and Pool 2 (right), with the chosen estimates indicated by ‘+’ markers.

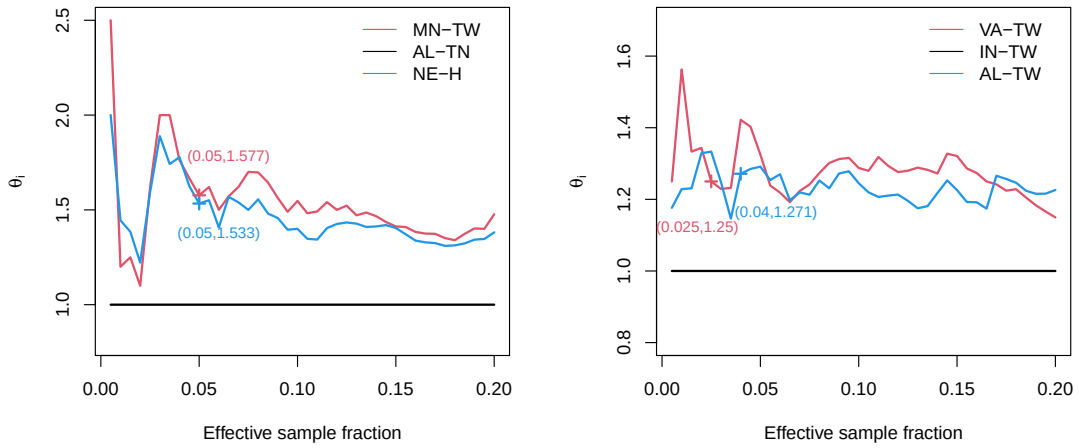


Figure 4.9: Estimation of the scale coefficients θ_i for each risk in Pool 1 (left) and Pool 2 (right), with the chosen estimates indicated by ‘+’ markers.

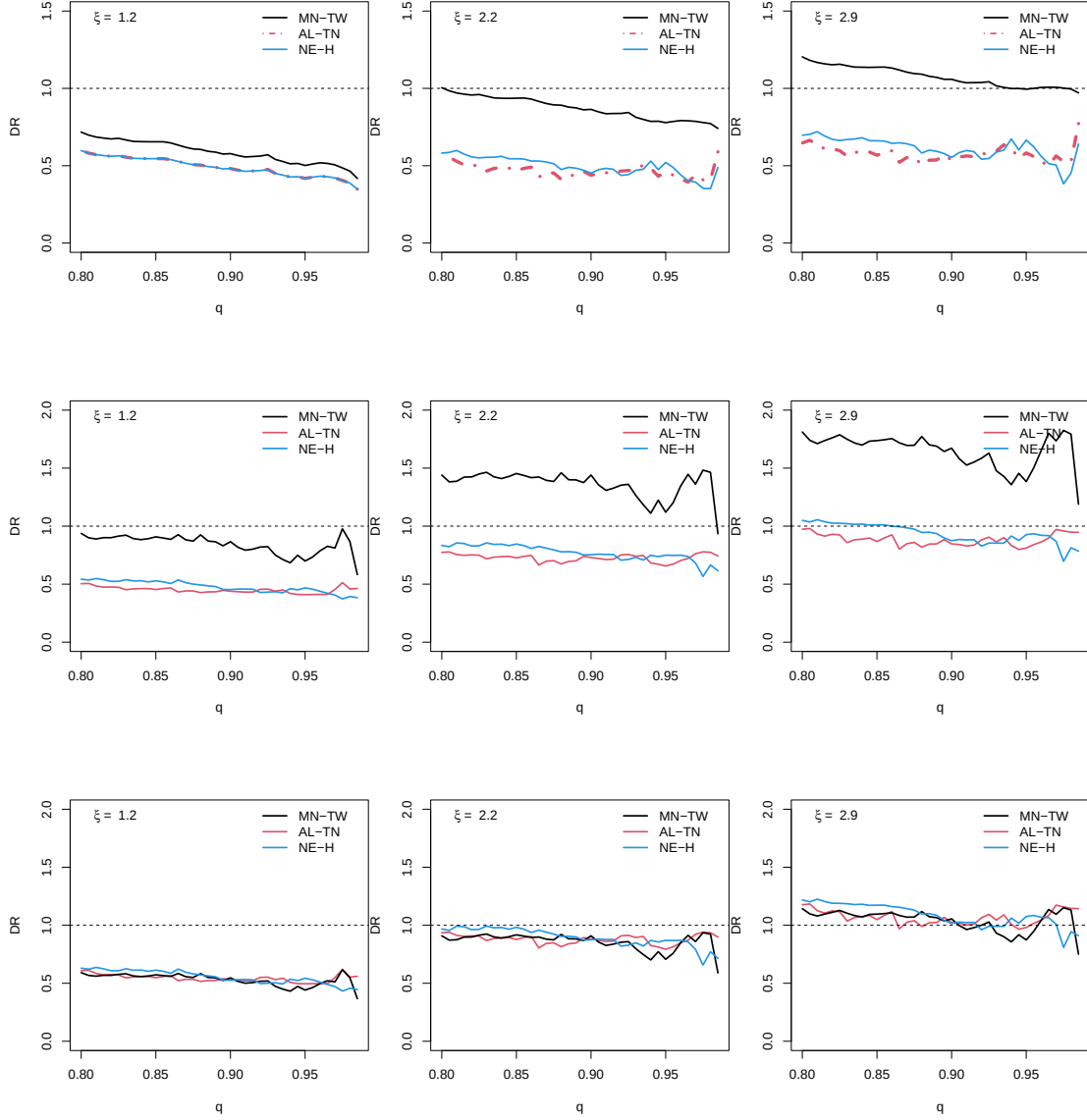


Figure 4.10: DRs for each risk in Pool 1. Row 1 is under uniform liability and uniform risk sharing as in Theorem 4.3.1. Row 2 is under the proportional liability scheme and uniform risk sharing rule as in Theorem 4.4.1. Row 3 is under the proportional liability scheme and proportional risk sharing rule as in Theorem 4.4.2.

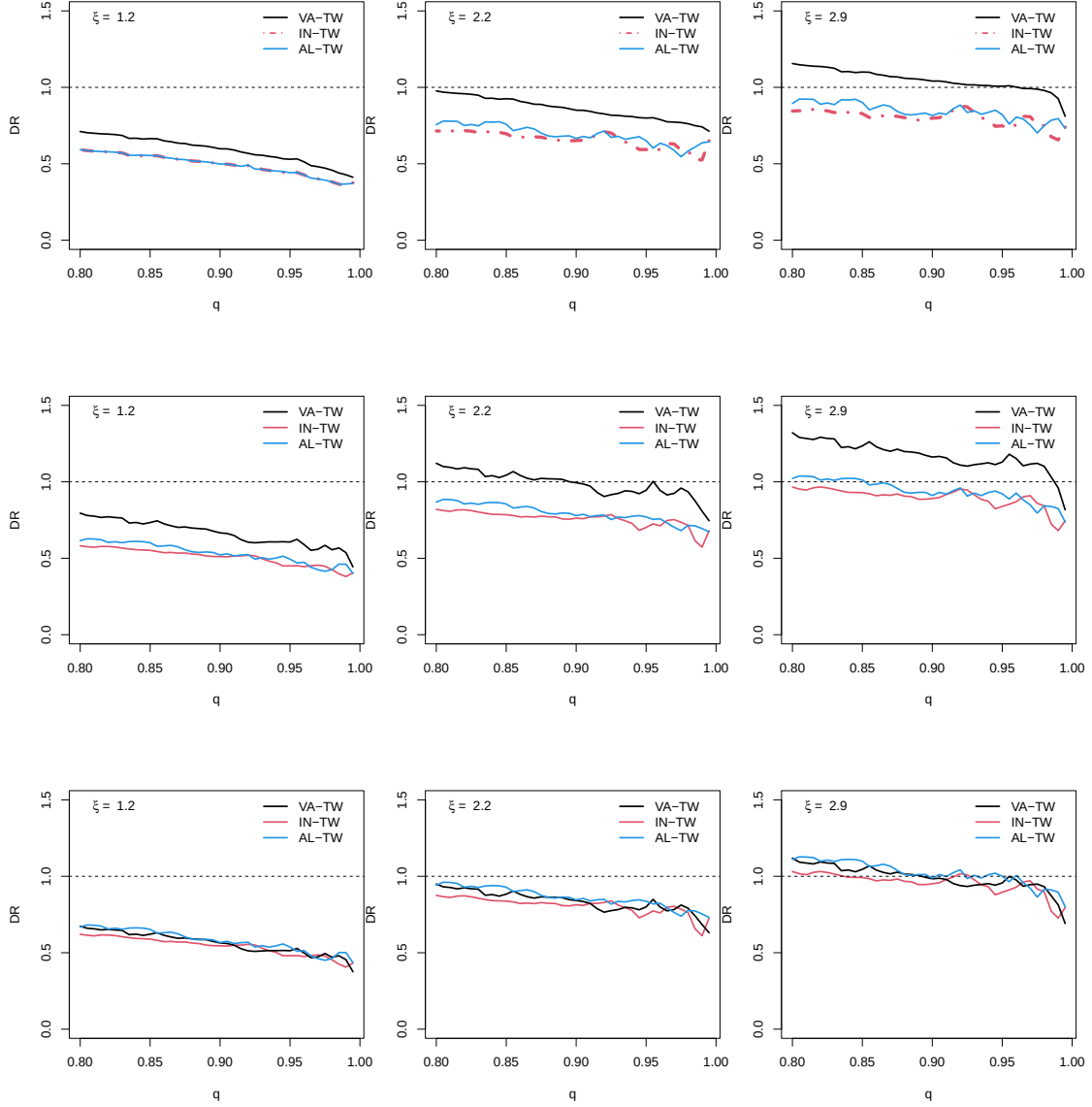


Figure 4.11: DRs for each risk in Pool 2. Row 1 is under uniform liability and uniform risk sharing as in Theorem 4.3.1. Row 2 is under the proportional liability scheme and uniform risk sharing rule as in Theorem 4.4.1. Row 3 is under the proportional liability scheme and proportional risk sharing rule as in Theorem 4.4.2.

Chapter 5

Standard and Comparative E-backtests based on Identifiability and Elicitability¹

5.1 Introduction

The evaluation of risk forecasts plays a central role in modern financial risk management and regulatory supervision. Financial institutions are required to report forecasts for various risk measures, such as Value-at-Risk (VaR) and Expected Shortfall (ES), to ensure adequate capital reserves and effective risk control. Consequently, rigorous backtesting procedures are essential to assess the accuracy and adequacy of these forecasts, forming a critical component of regulatory frameworks such as Basel III.

Standard backtests primarily aim to assess whether the realized losses are consistent with the risk levels forecasted by a single internal model. In practice, these tests often focus on identifying potential underestimation of risk, as this poses significant regulatory and financial concerns. Classical approaches, such as those proposed by [McNeil and Frey](#)

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Dr. Qiuqi Wang, Maurice R. Greenberg School of Risk Science, Georgia State University.

(2000) and [Acerbi and Szekely \(2014\)](#), provide formal statistical tools to evaluate the alignment between observed losses and predicted risk measures. To complement these methods, comparative backtesting has been introduced to facilitate direct performance comparison between competing risk forecasting models ([Nolde and Ziegel \(2017\)](#)). Rather than evaluating forecasts relative to an unknown “true” model, comparative backtests assess whether one forecasting method statistically outperforms another based on consistent scoring functions. Both standard and comparative backtests rely fundamentally on the concepts of elicibility and identifiability, which characterize whether a risk measure permits meaningful forecast evaluation through observable data and properly designed loss functions ([Gneiting, 2011](#); [Fissler and Ziegel, 2016](#)). While these backtesting procedures have been widely implemented in practice, they often depend on asymptotic arguments, restrictive parametric assumptions, and structural assumptions on forecast models. These limitations can compromise the robustness and reliability of the tests, particularly in complex, dynamic, or misspecified environments. Table 5.1 summarizes representative backtesting methods in the literature, highlighting key differences in testing scope and underlying assumptions.

Literature	Standard backtest/ comparative backtest/ static test	Risk measures tested	Parametric or dependence assumptions	Forecast structural assumptions
NZ17	standard, comparative	VaR, ES, expectiles	yes	yes
MF00, AS14, DE17, BD22; HD23	standard	VaR, ES	yes	yes
WWZ25	standard	VaR, ES	no	no
FJW23	static	$\mathbb{E}$, Var	no	yes
CLZ22	static	elicitable risk measures	no	yes
This chapter	standard, comparative	elicitable risk measures	no	no

Table 5.1: Comparison of existing backtesting methods for risk measures.

Notes: We use shortcuts NZ17 for [Nolde and Ziegel \(2017\)](#), MF00 for [McNeil and Frey \(2000\)](#), AS14 for [Acerbi and Szekely \(2014\)](#), DE17 for [Du and Escanciano \(2017\)](#), BD22 for [Bayer and Dimitriadis \(2022\)](#), and HD22 for [Hoga and Demetrescu \(2023\)](#), WWZ25 for [Wang et al. \(2025\)](#), FJW23 for [Fan et al. \(2023\)](#) and CLZ22 for [Casgrain et al. \(2022\)](#). Parametric or dependence assumptions refer to assumptions on loss distributions, time series models, stationarity, or strong mixing. Forecast structural assumptions refer to requirements on the form of risk forecasts.

To overcome the limitations of classical approaches, recent developments in model-free backtesting methodologies have introduced the concepts of e-values and e-processes, offering a flexible and robust alternative to p-value based hypothesis testing. The e-process framework, originally proposed by [Vovk and Wang \(2021\)](#), constructs sequential supermartingales that enable valid, non-asymptotic Type-I error control without requiring specific distributional assumptions. Building on this foundation, [Wang et al. \(2025\)](#) developed e-value based standard backtesting procedures for evaluating VaR and ES forecasts, demonstrating that e-processes provide a model-free, theoretically sound approach that addresses key limitations of traditional backtests, including reliance on asymptotic approximations, parametric assumptions, and rigid forecast structures.

Building on existing model-free testing approaches using e-values and e-processes, this chapter extends these ideas to develop a backtesting framework for identifiable and elicitable risk measures. The main contributions of this chapter are as follows. First, we develop model-free standard e-backtests for identifiable risk measures, as detailed in [Section 5.3](#). The proposed framework accommodates a broad class of widely used risk measures, including the mean, variance, VaR, ES, and expectiles, covering both one-dimensional and multi-dimensional settings. Second, we introduce a new class of comparative e-backtests for elicitable risk measures in [Section 5.4](#), enabling direct model comparison without imposing restrictive parametric or dependence assumptions. By leveraging e-processes, the proposed comparative backtests provide valid sequential inference and naturally extend to a three-zone interpretation, similar to the framework of [Fissler and Ziegel \(2016\)](#) and [Nolde and Ziegel \(2017\)](#), enhancing interpretability and practical relevance in regulatory applications. Third, we establish theoretical guarantees for Type-I error control and false discovery rate control of the proposed e-backtests in [Section 5.5](#), ensuring their statistical validity under realistic conditions. Finally, [Sections 5.6](#) and [5.7](#) demonstrate the practical performance of our methods through extensive simulation studies and real-world financial data analysis. The results illustrate the flexibility, robustness, and effectiveness of e-value based backtests, providing practical tools for reliable risk forecast evaluation in modern, data-driven financial environments.

5.2 Preliminaries

5.2.1 Elicitability, Identifiability, and Bayes pairs

We consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{X}$ and $\mathcal{M}$ be some convex sets of random variables and distributions, respectively. Let $\mathcal{X}_+$ be the set of all non-negative random variables. For $n > 0$, denote by $\mathcal{L}_n$ (resp. $\mathcal{M}_n$) the set of random variables (resp. distributions) with finite n -th moment. Let $\mathcal{L}_0$ and $\mathcal{L}_\infty$ (resp. $\mathcal{M}_0$ and $\mathcal{M}_\infty$) be the sets of all random variables (resp. distributions) and the sets of random variables (resp. distributions) with bounded supports. For a functional $\psi : \mathcal{M} \rightarrow \mathbb{R}$, write $\psi(X) = \psi(F)$ for a random variable $X \sim F$. For $x, y \in \mathbb{R}$, we define $x \wedge y = \min\{x, y\}$, $x \vee y = \max\{x, y\}$, $x_+ = x \vee 0$ and $x_- = -(-x)_+$.

Elicitability and identifiability are both important concepts for statistical inference. Following the definitions in [Fissler and Ziegel \(2016\)](#), for $d \in \mathbb{N}$, a set-valued functional $\psi : \mathcal{M} \rightarrow 2^{\mathbb{R}^d}$ is called $\mathcal{M}$ -*elicitable* (or *elicitable* for simplicity) if there exists a function $S : \mathbb{R}^{d+1} \rightarrow \mathbb{R}$ such that for all $F \in \mathcal{M}$,

$$\psi(F) = \arg \min_{a \in \mathbb{R}^d} \int_{\mathbb{R}} S(x, a) dF(x),$$

where S is called a *strictly consistent scoring function* for ψ . Here, “strict” means that the minimum of the expected score attains its minimum *uniquely* at the correct value $\psi(F)$. An integrable function $I : \mathbb{R}^{d+1} \rightarrow \mathbb{R}^d$ is said to be an $\mathcal{M}$ -*identification function* (or *identification function* for simplicity) for ψ if $\int_{\mathbb{R}} I(x, a) dF(x) = \mathbf{0}$ for all $F \in \mathcal{M}$ and $a \in \psi(F)$. Further, the functional ψ is called $\mathcal{M}$ -*identifiable* (or *identifiable* for simplicity) if there exists a *strict* identification function $I : \mathbb{R}^{d+1} \rightarrow \mathbb{R}^d$ such that for all $F \in \mathcal{M}$ and $y \in \mathbb{R}^d$,

$$a \in \psi(F) \iff \int_{\mathbb{R}} I(x, a) dF(x) = \mathbf{0}.$$

Here, “strict” means that the zero of the expected identification function *uniquely* determines $\psi(F)$; no other point in $\mathbb{R}^d$ yields zero expectation. Following the insights of elicibility and Bayes risk, in the single-dimensional case, [Embrechts et al. \(2021\)](#) introduced Bayes pairs defined as follows. Let $I(\mathbb{R})$ be the set of closed real intervals. A pair

of risk measures $(\rho, \phi) : \mathcal{M} \rightarrow \mathbb{R} \times I(\mathbb{R})$ is a *Bayes pair* if there exists a *scoring* (or *loss*) function $S : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that for all $F \in \mathcal{M}$,

$$\phi(F) = \arg \min_{a \in \mathbb{R}} \int_{\mathbb{R}} S(x, a) dF(x) \quad \text{and} \quad \rho(F) = \min_{a \in \mathbb{R}} \int_{\mathbb{R}} S(x, a) dF(x).$$

We first have the following lemma showing the link between Bayes pairs and identifiability, which will be useful for constructing our backtesting method.

Lemma 5.2.1 *Let $(\rho, \phi) : \mathcal{M} \rightarrow \mathbb{R} \times I(\mathbb{R})$ be a Bayes pair with a loss function $S : \mathbb{R}^2 \rightarrow \mathbb{R}$. If ϕ has (strict) identification function $v : \mathbb{R}^2 \rightarrow \mathbb{R}$, then (ρ, ϕ) has (strict) identification function $(x, r, z) \mapsto (v(x, z), h(r, z)(S(x, z) - r))^\top$ for every function $h : \mathbb{R}^2 \rightarrow \mathbb{R}$ which is not constantly 0.*

Proof. For each $F \in \mathcal{M}$, suppose that $r = \rho(F)$ and $z \in \phi(F)$. We have

$$\int_{\mathbb{R}} v(x, z) dF(x) = 0 \quad \text{and} \quad \int_{\mathbb{R}} h(r, z)(S(x, z) - r) dF(x) = h(r, z) \left(\int_{\mathbb{R}} S(x, z) dF(x) - r \right) = 0$$

by the definition of Bayes pairs. On the other hand, if the identification function v is strict, suppose that

$$\int_{\mathbb{R}} v(x, z) dF(x) = 0 \quad \text{and} \quad \int_{\mathbb{R}} h(r, z)(S(x, z) - r) dF(x) = 0.$$

We have $z \in \phi(F)$ as ϕ is identifiable. Next, we have $\int_{\mathbb{R}} (S(x, z) - r) dF(x) = 0$, which yields that $r = \int_{\mathbb{R}} S(x, z) dF(x) = \rho(F)$. Therefore, (ρ, ϕ) is identifiable with (strict) identification function $(x, r, z) \mapsto (v(x, z), h(r, z)(S(x, z) - r))^\top$. ■

5.2.2 E-values and E-processes for Model-free Tests

We construct our model-free testing approach using the concept of e-values and e-processes. Let H be a set of probability measures on $(\Omega, \mathcal{F})$, representing the (composite) hypothesis we are testing. The hypothesis H is simple if it is a singleton. Following the terminology of [Vovk and Wang \(2021\)](#), we say a non-negative random variable E is an

e-variable for H if $\mathbb{E}^Q[E] \leq 1$ for all $Q \in H$. A realized *e*-variable is called an *e*-value. In a dynamic setup, we say a non-negative stochastic process $(E_t)_{t \in K}$, $K \subseteq \mathbb{N}$, adapted to a filtration $\{\mathcal{F}_t\}_{t \in K}$, is an *e*-process for H if $\mathbb{E}^Q[E_\tau] \leq 1$ for any stopping time τ taking values in K and all $P \in H$. Following the definition in [Fan et al. \(2023\)](#), we say an *e*-variable E for H is *precise* if $\sup_{Q \in H} \mathbb{E}^Q[E] = 1$. Intuitively, a precise *e*-variable is not conservative in its realized values when the null hypothesis holds true.

To illustrate our idea of model-free tests using *e*-values and *e*-processes, we demonstrate two simple examples testing the mean $\mathbb{E}$.

Example 5.2.1 (Standard test) For $n \in \mathbb{N}$, suppose that we observe i.i.d. data $\{X_t\}_{t \in [n]}$ of a random variable $X \in \mathcal{X}_+$. The data arrive sequentially. We aim to test the following null hypothesis for $\mu \in \mathbb{R}$:

$$H_0 : \mu \geq \mathbb{E}[X].$$

We go through the following procedures to achieve the goal.

1. We know that $\mathbb{E}$ is identifiable. Choose a strict identification function $(x, a) \mapsto I(x, a)$ for $\mathbb{E}$ that is decreasing in a . For example, we can choose $I(x, a) = x/a - 1$, $x, a \geq 0$. This choice is apparently not unique.
2. We verify that $\mathbb{E}[1 + I(X, \mu)] \leq 1$ for all $X \in \mathcal{X}_+$ under H_0 . Thus $1 + I(X, \mu)$ is an *e*-variable for H_0 for any $X \in \mathcal{X}_+$. Moreover, for all $X \in \mathcal{X}_+$ and $\mu > \mathbb{E}[X]$, we have $\mathbb{E}[1 + I(X, \mu)] > 1$.
3. At time $t \in [n]$, we choose a value $\lambda_t \in [0, 1]$. This value can depend on past data $X_1, \dots, X_{t-1}$.
4. Construct the following stochastic process $(M_t)_{t \in [n]}$ with $M_0 = 1$:

$$M_t = (1 - \lambda_t + \lambda_t(1 + I(X_t, \mu)))M_{t-1} = \prod_{s=1}^t (1 + \lambda_s I(X_s, \mu)), \quad t \in [n].$$

By definition, the process (M_t) is a super-martingale and is an *e*-process under H_0 .

5. For a pre-specified significance level $\alpha \in (0, 1)$, we reject H_0 if the e-process (M_t) exceeds the threshold $1/\alpha$.

Example 5.2.2 (Comparative test) Suppose that we observe i.i.d. data $\{X_t\}_{t \in [n]}$ of a random variable $X \in \mathcal{L}_\infty$ which takes values within the interval $[-M, M]$, $M > 0$. Now we have two estimates of the mean $\mu_1, \mu_2 \in \mathbb{R}$ obtained by different methods. Our aim is now to compare which estimate is better, or is closer to the true value. It is clear that $\mathbb{E}$ is elicitable. We choose a strictly consistent scoring function $S : \mathbb{R}^2 \rightarrow \mathbb{R}$ for $\mathbb{E}$. For example, we can choose $S(x, a) = (x - a)^2$. To compare between the two estimates, we test the following hypothesis based on the nature of the elicibility of the mean:

$$H_0 : \mathbb{E}[S(X, \mu_1)] \leq \mathbb{E}[S(X, \mu_2)].$$

We then go through the following steps:

1. At time $t \in [n]$, we choose a value $\lambda_t \geq 0$ such that $\inf_{x \in [-M, M]} \{\lambda_t(S(x, \mu_1) - S(x, \mu_2))\} \geq -1$. This value can depend on past data $X_1, \dots, X_{t-1}$.
2. For all $t \in [n]$, we verify that $\mathbb{E}[1 + \lambda_t(S(X, \mu_1) - S(X, \mu_2))] \leq 1$ for all $-M \leq X \leq M$ under H_0 . Thus $1 + \lambda_t(S(X, \mu_1) - S(X, \mu_2))$ is an e-variable for H_0 for any $-M \leq X \leq M$. Moreover, for all $-M \leq X \leq M$ and $\mu_1, \mu_2 \in \mathbb{R}$ such that $\mathbb{E}[S(X, \mu_1)] > \mathbb{E}[S(X, \mu_2)]$, we have $\mathbb{E}[1 + \lambda_t(S(X, \mu_1) - S(X, \mu_2))] > 1$.
3. Construct the following stochastic process $(M_t)_{t \in [n]}$ with $M_0 = 1$:

$$M_t = (1 + \lambda_t(S(X_t, \mu_1) - S(X_t, \mu_2)))M_{t-1} = \prod_{s=1}^t (1 + \lambda_s(S(X_s, \mu_1) - S(X_s, \mu_2))), \quad t \in [n].$$

By definition, the process (M_t) is a super-martingale and is an e-process under H_0 .

4. For a pre-specified significance level $\alpha \in (0, 1)$, we reject H_0 if the e-process (M_t) exceeds the threshold $1/\alpha$.

Examples 5.2.1 and 5.2.2 are in a similar sense to Section 2 of Wang et al. (2025). However, they illustrate a more general framework of model-free tests using e-values and e-processes. Built on the discussion of Wang et al. (2025), we make the following remarks.

1. We construct e-values for all random variables X in a certain set. It means that the validity and consistency of our e-tests do not depend on the specific distribution of X . In other words, we achieve non-parametric or model-free tests based on e-values and e-processes.
2. We construct e-values based on the identifiability or elicibility of the mean. Naturally, we are able to construct e-values for our model-free tests for a general identifiable or elicitable risk measure. Clearly, the risk measure we are testing on can also be multi-dimensional.
3. The examples deal with i.i.d. observations. In a general setup, our tests are feasible for arbitrary dependence structures of the random losses.
4. We are able to do both standard and comparative tests depending on specific purposes. Our tests can also be extended to a dynamic setup where the estimates of the risk measures are predictable processes adapted to some filtration. In this case, we conduct standard e-backtests (see Section 5.3) and comparative e-backtests (see Section 5.4), both of which are of independent interests in financial regulation.

5.3 Standard E-backtests

5.3.1 Null Hypotheses

For backtesting purposes, we consider two risk measures to be tested denoted by $\rho : \mathcal{M} \rightarrow \mathbb{R}$ and $\phi : \mathcal{M} \rightarrow 2^{\mathbb{R}^{d-1}}$ for $d \in \mathbb{N}$. Write $\psi = (\rho, \phi)$. Here we call the single-dimensional risk measure ρ a *regulatory risk measure*, which usually represents the risk measure the financial institution uses to calculate financial reserves. We call the multi-dimensional risk measure ϕ a *statistic*, containing distributional information of the underlying random losses. Let $T \in \mathbb{N}$ be a finite time horizon. Let $\{L_t\}_{t \in [T]}$ be a sequence of random losses adapted to the filtration $\{\mathcal{F}_t\}_{t \in [T]}$, taking values in a measurable space $\mathcal{K} \subseteq \mathbb{R}$. The *conditional distribution* of L_t given $\mathcal{F}_{t-1}$, denoted by $L_t \mid \mathcal{F}_{t-1}$, is the family of random variables

$$F_{L_t \mid \mathcal{F}_{t-1}}(x) := \mathbb{P}(L_t \leq x \mid \mathcal{F}_{t-1}), \quad x \in \mathbb{R},$$

where $\mathbb{P}(\cdot \mid \mathcal{F}_{t-1})$ is a *regular conditional probability* on $(\Omega, \mathcal{F}_{t-1})$. The risk measures $\rho(L_t \mid \mathcal{F}_{t-1})$ and $\phi(L_t \mid \mathcal{F}_{t-1})$ are predicted by $R_t : \Omega \rightarrow \mathbb{R}$ and $Z_t : \Omega \rightarrow \mathbb{R}^{d-1}$, respectively, which are $\mathcal{F}_{t-1}$ -measurable for $t \in [T]$. That is, the processes $\{R_t\}_{t \in [T]}$ and $\{Z_t\}_{t \in [T]}$ are *predictable*. Depending on different practical situations, the financial institution and the regulator may care about either the regulatory risk measure ρ or the statistic ϕ . For standard backtests, we are first interested in the one-sided test for the following null hypothesis:

$$H(\psi) : R_t \geq \rho(L_t \mid \mathcal{F}_{t-1}) \text{ and } Z_t \in \phi(L_t \mid \mathcal{F}_{t-1}), \text{ for all } t \in [T]. \quad (3.1)$$

A test based on (3.1) focuses on whether the regulatory risk measure ρ is underestimated and ϕ provides auxiliary information. Here we are concerned of underestimation for ρ because underestimation of a regulatory risk measure usually leads to less capital reserves for financial risk management, which may result in severe problems such as bankruptcy. On the other hand, overestimation means that more conservative risk management strategies are taken, which is acceptable in the current setup. However, overestimation sometimes also causes practical problems due to model misspecification. Considering such a possibility, we may also conduct the two-sided test for the following null hypothesis:

$$\tilde{H}(\psi) : R_t = \rho(L_t \mid \mathcal{F}_{t-1}) \text{ and } Z_t \in \phi(L_t \mid \mathcal{F}_{t-1}), \text{ for all } t \in [T]. \quad (3.2)$$

The hypothesis above tests whether the financial institution provides exact forecasts of the statistic ϕ given that the forecasts for the regulatory risk measure ρ are accurate or close to their true values. In such a setup above, we are interested in whether a correct model is chosen, where we focus more on the statistic ϕ and the regulatory risk measure ρ serves as an auxiliary input.

5.3.2 E-variables and E-processes via Identification Functions

In a static setup, for some unknown distribution $F \in \mathcal{M}$ and real-valued $r \in \rho(\mathcal{M})$ and $z \in \phi(\mathcal{M})$, we first show the e-variables for the following one- and two-sided hypotheses:

$$H(\psi) : r \geq \rho(F) \text{ and } z \in \phi(F), \quad \tilde{H}(\psi) : r = \rho(F) \text{ and } z \in \phi(F). \quad (3.3)$$

The proof for the following lemma is straightforward by the definitions of identification functions and e-variables.

Lemma 5.3.1 *Suppose that $\phi : \mathcal{M} \rightarrow 2^{\mathbb{R}^{d-1}}$ has an identification function $v : \mathbb{R}^d \rightarrow \mathbb{R}^{d-1}$, and $\psi = (\rho, \phi) : \mathcal{M} \rightarrow \mathbb{R} \times \mathbb{R}^{d-1}$ has an identification function $(x, a, b) \mapsto (v(x, b), V(x, a, b))^\top$. If $\inf_{x \in \mathcal{K}, (a, b) \in \psi(\mathcal{M})} V(x, a, b) \geq -1$, then for all $F_L \in \mathcal{M}$ and $(r, z) \in \psi(\mathcal{M})$, $E = 1 + V(L, r, z)$ is a precise e-variable for $\tilde{H}(\psi)$ in (3.3). Moreover, if $V(x, a, b)$ is further decreasing in a , then for all $F_L \in \mathcal{M}$ and $(r, z) \in \psi(\mathcal{M})$, E is a precise e-variable for $H(\psi)$ in (3.3).*

Lemma 5.3.1 still holds when the risk measure ψ is single dimensional (i.e. $\psi = \rho$) without a statistic ϕ . However, for the single-dimensional case, we can achieve a stronger result by characterizing all possible forms of e-variables for the tests given the consistency of the one-sided test. Precisely, we have the following result under some mild assumptions from Dimitriadis et al. (2023) (stated in Appendix C).

Theorem 5.3.1 (Single dimension) *Let $\psi = [\psi^-, \psi^+] : \mathcal{M} \rightarrow I(\mathbb{R})$ be identifiable with a strict identifiable function $(x, a) \mapsto V(x, a)$ on $\mathbb{R}^2$ decreasing in a and $\inf_{x \in \mathcal{K}, a \in \psi(\mathcal{M})} V(x, a) \geq -1$. Suppose that $\mathcal{M}$ and V satisfies Assumptions C.0.2-C.0.3 in Appendix C. For a function $e : \mathbb{R}^2 \rightarrow [0, \infty)$ continuous almost everywhere, $E = e(L, r)$ is a precise e-variable for $H(\psi) : r \geq \psi^-(L)$ and $\tilde{H}(\psi) : r \in \psi(L)$ for all $F_L \in \mathcal{M}$ and $r \in \psi(\mathcal{M})$ if and only if*

$$e(x, a) = 1 + h(a)V(x, a) \tag{3.4}$$

for a non-negative function h on $\mathbb{R}$ with $0 < h \leq -1/\inf_{x \in \mathcal{K}, a \in \psi(\mathcal{M})} V(x, a)$. In this case, $\mathbb{E}[e(L, r)] > 1$ for all $F_L \in \mathcal{M}$ and $r < \psi^-(L)$.

Proof. “ $\Leftarrow$ ”: Suppose that $e(x, a) = 1 + h(a)V(x, a)$ for $0 < h \leq -1/\inf_{x \in \mathcal{K}, a \in \psi(\mathcal{M})} V(x, a)$. As V is a strict identification function for ψ , we have

$$\mathbb{E}[V(L, r)] = 0 \iff r \in \psi(L) \tag{3.5}$$

for all $F_L \in \mathcal{M}$ and $r \in \psi(\mathcal{M})$. Because $a \mapsto V(x, a)$ is decreasing in a , $\mathbb{E}[e(L, r)] = 1 + h(r)\mathbb{E}[V(L, r)] \leq 1$ for all $r \geq \psi^-(L)$ and $F_L \in \mathcal{M}$, and the equality holds when

$r \in \psi(L)$. Hence, $e(L, r)$ is a precise e-variable for $H(\psi) : r \geq \psi^-(L)$ and $\tilde{H}(\psi) : r \in \psi(L)$ for all $F_L \in \mathcal{M}$. Moreover, we have $\mathbb{E}[V(L, r)] > 0$ for all $r < \psi^-(L)$ and $F_L \in \mathcal{M}$ by (3.5) and monotonicity of V . Thus, $\mathbb{E}[e(L, r)] > 1$ for all $F_L \in \mathcal{M}$ and $r < \psi^-(L)$.

“ $\Rightarrow$ ”: As $e(L, r)$ is a precise e-variable for $\tilde{H}_0 : r \in \psi(L)$, we have $\mathbb{E}[e(L, r) - 1] = 0$ for all $r \in \psi(L)$ and $F_L \in \mathcal{M}$. Thus $(x, a) \mapsto e(x, a) - 1$ is an identification function for ψ . As e is continuous almost everywhere on $\mathbb{R}^2$, it satisfies Assumption C.0.3 in Appendix C. Hence, we have by Theorem 4 of Dimitriadis et al. (2023) that

$$e(x, a) - 1 = h(a)V(x, a)$$

for h defined on $\mathbb{R}$. As $e(L, r)$ is a precise e-variable for $H(\psi) : r \geq \psi^-(L)$, we have

$$\mathbb{E}[e(L, r) - 1] = h(r)\mathbb{E}[V(L, r)] \leq 0 \quad \text{for all } F_L \in \mathcal{M} \text{ and } r \geq \psi^-(L).$$

Monotonicity of V yields that $h > 0$. The condition $h \leq -1/\inf_{x \in \mathcal{K}, a \in \psi(\mathcal{M})} V(x, a)$ arises from non-negativity of e . ■

Remark 5.3.1 *Under the assumptions in Theorem 5.3.1, if $(x, a) \mapsto V(x, a)$ is increasing in a instead, then $e(L, r)$, where e is in (3.4), is a precise e-variable for the hypothesis $H'(\psi) : r \leq \psi^+(L)$ for all $F_L \in \mathcal{M}$ and $r \in \psi(\mathcal{M})$. This test is usually not practically interesting since overestimation is not considered a severe problem in risk management. However, based on this result, we can construct an e-process for a two-sided test of a single-dimensional identifiable risk measure (see Theorem 5.3.2 below).*

When a statistic ϕ is involved (i.e., $d > 1$), Lemma 5.3.1 does not guarantee consistency of both tests based on H_0 or $\tilde{H}_0$ in (3.3) when the regulatory risk measure ρ is underestimated or the statistic ϕ is inaccurately estimated. The consistency of such tests can be achieved for a Bayes pair, as shown by the following result.

Proposition 5.3.1 *Let $\psi = (\rho, \phi) : \mathcal{M} \rightarrow \mathbb{R} \times I(\mathbb{R})$ be a Bayes pair with loss function $S : \mathbb{R}^2 \rightarrow \mathbb{R}$. Suppose that ϕ is identifiable with identification function $v : \mathbb{R}^2 \rightarrow \mathbb{R}$, and ψ is identifiable with identification function $(x, a, b) \mapsto (v(x, b), V(x, a, b))^\top$.*

- (i) If $\inf_{x \in \mathcal{K}, (a,b) \in \psi(\mathcal{M})} h(a,b)(S(x,b) - a) \geq -1$ for some function $h : \mathbb{R}^2 \rightarrow [0, \infty)$ that is not constantly 0, then for all $F_L \in \mathcal{M}$ and $(r, z) \in \psi(\mathcal{M})$, $E = 1 + h(r, z)(S(L, z) - r)$ is a precise e -variable for $\tilde{H}(\psi)$ in (3.3). Moreover, $\mathbb{E}[E] > 1$ for all $F_L \in \mathcal{M}$, $r = \rho(L)$ and $z \notin \phi(L)$.
- (ii) Under the assumptions in (i), if $h(a,b)(S(x,b) - a)$ is decreasing in a , then for all $F_L \in \mathcal{M}$ and $(r, z) \in \psi(\mathcal{M})$, E is a precise e -variable for $H(\psi)$ in (3.3). Moreover, $\mathbb{E}[E] > 1$ for all $F_L \in \mathcal{M}$, $r < \rho(L)$ and $z \in \phi(\mathcal{M})$.

Proof. The first statements of (i) and (ii) follow directly from Lemmas 5.2.1 and 5.3.1. We prove the second statements of (i) and (ii).

(i) For all $F_L \in \mathcal{M}$, $r = \rho(L)$, and $z \notin \phi(L)$, because

$$\phi(L) = \arg \min_{a \in \mathbb{R}} \int_{\mathbb{R}} S(x, a) dF_L(x) \quad \text{and} \quad r = \min_{a \in \mathbb{R}} \int_{\mathbb{R}} S(x, a) dF_L(x),$$

we have $r < \int_{\mathbb{R}} S(x, z) dF_L(x)$. As $h(r, z) \geq 0$, we have

$$\mathbb{E}[E] = 1 + h(r, z) \left(\int_{\mathbb{R}} S(x, z) dF_L(x) - r \right) > 1.$$

(ii) For all $F_L \in \mathcal{M}$, $r < \rho(L)$, and $z \in \phi(\mathcal{M})$, because

$$\phi(L) = \arg \min_{a \in \mathbb{R}} \int_{\mathbb{R}} S(x, a) dF_L(x) \quad \text{and} \quad \rho(L) = \min_{a \in \mathbb{R}} \int_{\mathbb{R}} S(x, a) dF_L(x),$$

we have $r < \min_{a \in \mathbb{R}} \int_{\mathbb{R}} S(x, a) dF_L(x) \leq \int_{\mathbb{R}} S(x, z) dF_L(x)$, and thus $\mathbb{E}[E] > 1$ similarly to (i). ■

Remark 5.3.2 The e -variable $E = 1 + h(r, z)(S(L, z) - r)$ we construct in Proposition 5.3.1 is a real function of L , r , and z . Wang et al. (2025) called such a function in Proposition 5.3.1 (ii) a model-free e -statistic for $\psi = (\rho, \phi)$ strictly testing ρ , where validity and consistency of the test are both guaranteed. We show by Proposition 5.3.1 that we can naturally construct a model-free e -statistic with a Bayes pair.

In light of Proposition 5.3.1, when (ρ, ϕ) is a Bayes pair and we choose an identification function $h(a, b)(S(x, b) - a)$ decreasing in a , suppose that we detect a rejection through a realized e-value exceeding some pre-specified threshold. Then the following conclusions can be made: (i) It is possible that the regulatory risk measure ρ is underestimated no matter what the forecasts for the statistic ϕ are reported. (ii) If the regulatory risk measure ρ is accurately estimated (or not underestimated), then the forecast for the statistic ϕ is misspecified possibly due to a wrong model chosen by the financial institution.

Moving to the dynamic framework, we construct e-processes for our standard backtesting hypotheses (3.1) and (3.2).

Theorem 5.3.2 *Let $\psi = (\rho, \phi) : \mathcal{M} \rightarrow \mathbb{R} \times \mathbb{R}^{d-1}$ and $\lambda = \{\lambda_t\}_{t \in [T]}$ be a predictable process taking values in $[0, 1]$. Suppose that ϕ is identifiable with identification function $v : \mathbb{R}^d \rightarrow \mathbb{R}^{d-1}$, and ψ is identifiable with identification functions $(x, a, b) \mapsto (v(x, b), V(x, a, b))^\top$ and $(x, a, b) \mapsto (v(x, b), V'(x, a, b))^\top$. The following statements hold.*

(i) *If $\inf_{x \in \mathcal{K}, (a, b) \in \psi(\mathcal{M})} V(x, a, b) \geq -1$, then the process $\{M_t\}_{t \in \{0\} \cup [T]}$ with $M_0 = 1$ and*

$$M_t(\lambda) = (1 + \lambda_t V(L_t, R_t, Z_t)) M_{t-1}(\lambda) = \prod_{s=1}^t (1 + \lambda_s V(L_s, R_s, Z_s)) \quad (3.6)$$

is a non-negative martingale under $\tilde{H}(\psi)$ in (3.2), and is thus an e-process. Moreover, if $V(x, a, b)$ is further decreasing in a , then $\{M_t\}_{t \in \{0\} \cup [T]}$ is a non-negative supermartingale under $H(\psi)$ in (3.1).

(ii) *(Single dimension) When $d = 1$, if $V(x, a)$ and $V'(x, a)$ are decreasing in a for all $x \in \mathcal{K}$, $\inf_{x \in \mathcal{K}, a \in \psi(\mathcal{M})} V(x, a) \geq -1$ and $\sup_{x \in \mathcal{K}, a \in \psi(\mathcal{M})} V'(x, a) \leq 1$, then the process $\{M_t\}_{t \in \{0\} \cup [T]}$ with $M_0 = 1$ and*

$$M_t(\lambda) = \frac{1}{2} \prod_{s=1}^t (1 + \lambda_s V(L_s, R_s)) + \frac{1}{2} \prod_{s=1}^t (1 - \lambda_s V'(L_s, R_s)) \quad (3.7)$$

is a non-negative martingale under

$$\tilde{H}(\psi) : R_t = \psi(L_t | \mathcal{F}_{t-1}) \text{ for all } t \in [T],$$

and is thus an e -process.

Proof. (i) For all $s \in [t]$, as $\inf_{x \in \mathcal{K}, (a,b) \in \psi(\mathcal{M})} V(x, a, b) \geq -1$ and $0 \leq \lambda_s \leq 1$, we have $1 + \lambda_s V(L_s, R_s, Z_s) \geq 0$. Therefore, $M_t(\boldsymbol{\lambda}) \geq 0$ for all $t \in [T]$.

For $t \in [T]$, when $\tilde{H}(\psi)$ in (3.2) holds true, as R_t and Z_t are $\mathcal{F}_{t-1}$ -measurable, we have

$$\mathbb{E}[V(L_t, R_t, Z_t) | \mathcal{F}_{t-1}] = 0.$$

Because L_{t-1} and λ_t are $\mathcal{F}_{t-1}$ -measurable, it follows that

$$\mathbb{E}[M_t(\boldsymbol{\lambda}) | \mathcal{F}_{t-1}] = M_{t-1}(\boldsymbol{\lambda})(1 + \lambda_t \mathbb{E}[V(L_t, R_t, Z_t) | \mathcal{F}_{t-1}]) = M_{t-1}(\boldsymbol{\lambda}).$$

(ii) Suppose that $V(x, a, b)$ is decreasing in a . We have $\{M_t\}_{t \in \{0\} \cup [T]}$ being a non-negative process following the same argument as (i). For $t \in [T]$, when $H(\psi)$ in (3.1) holds true, we have

$$\mathbb{E}[V(L_t, R_t, Z_t) | \mathcal{F}_{t-1}] \leq 0$$

by monotonicity of V . Hence,

$$\mathbb{E}[M_t(\boldsymbol{\lambda}) | \mathcal{F}_{t-1}] = M_{t-1}(\boldsymbol{\lambda})(1 + \lambda_t \mathbb{E}[V(L_t, R_t, Z_t) | \mathcal{F}_{t-1}]) \leq M_{t-1}(\boldsymbol{\lambda}).$$

(iii) The assertion holds by noting that the mixture of two martingales is still a martingale. ■

As shown in Lemma 5.3.1 and Theorem 5.3.2, constructing e -variables and e -processes requires the uniform boundedness of the corresponding identification functions. This is in a similar sense to Lemma 3.2 of Casgrain et al. (2022). Differently from their result, the e -process $\{M_t\}_{t \in \{0\} \cup [T]}$ we construct is a function of the predictable process $\{\lambda_t\}_{t \in [T]}$. We call the process $\{\lambda_t\}_{t \in [T]}$ a *betting process*. Moreover, we construct the e -process using only the function $(x, a, b) \mapsto V(x, a, b)$ instead of the whole identification function $(x, a, b) \mapsto (v(x, b), V(x, a, b))^\top$ for ψ . This reduces computational complexity a lot.

In Theorem 5.3.2 (i) and (ii), when $\psi = (\rho, \phi)$ is a Bayes pair with loss function $S : \mathbb{R}^2 \rightarrow \mathbb{R}$ and $\inf_{x \in \mathcal{K}, (a,b) \in \psi(\mathcal{M})} h(a, b)(S(x, b) - a) \geq -1$ for some $h : \mathbb{R}^2 \rightarrow [0, \infty)$ that

is not constantly zero (and $h(a, b)(S(x, b) - a)$ is further decreasing in a for (ii)), we can take $V(x, a, b) = h(a, b)(S(x, b) - a)$.

Example 5.3.1 *We show examples of e-variables we get via the identification functions of some common risk measures in the financial industry. The corresponding e-processes can be constructed in a similar sense by taking mixtures with the form of (3.6).*

1. (The mean) For $L \in \mathcal{X}_+ \cap \mathcal{L}_1$, the mean $\mathbb{E}[L]$ is identifiable with identification function $V(x, a) = x/a - 1$, $x, a \geq 0$, which is decreasing in a . By Theorem 5.3.1, L/r is a precise e-variable for the null hypothesis $H(\mathbb{E}) : r \geq \mathbb{E}[L]$ for all $L \in \mathcal{X}_+$ and $r \geq 0$.

Now consider bounded losses $L \in \mathcal{L}_\infty$ so that $L : \Omega \rightarrow [-M, M]$ for $M > 0$. Another identification function for the mean $\mathbb{E}[L]$ is given by $V'(x, a) = (x - a)/(2M)$, $x, a \in [-M, M]$, which is decreasing in a . It is clear that $\inf_{x, a \in [0, 1]} V'(x, a) \geq -1$ and $\sup_{x, a \in [0, 1]} V'(x, a) \leq 1$, thus we have by Theorem 5.3.1 that $1 + (L - r)/(2M)$ and $1 - (L - r)/(2M)$ are precise e-variables for the null hypotheses $H(\mathbb{E}) : r \geq \mathbb{E}[L]$ and $H'(\mathbb{E}) : r \leq \mathbb{E}[L]$, respectively, for all $L : \Omega \rightarrow [-M, M]$ and $r \in [0, 1]$.²

2. (The mean and the variance) For $L \in \mathcal{L}_2$, the variance $\text{Var}(L)$ is jointly identifiable with the mean $\mathbb{E}[L]$. The risk measure $(\text{Var}, \mathbb{E})$ is a Bayes pair. The identification function for $(\text{Var}, \mathbb{E})$ is

$$(v(x, b), V(x, a, b)) = \left(\frac{x}{b} - 1, \frac{(x - b)^2}{a} - 1 \right), \quad x, b \in \mathbb{R}, \quad a \geq 0.$$

The function $V(x, a, b) = (x - b)^2/a$ is decreasing in a . By Lemma 5.3.1, $(L - z)^2/r$ is a precise e-variable for the following null hypotheses $H(\mathbb{E}, \text{Var})$ and $\tilde{H}(\mathbb{E}, \text{Var})$ for all $L \in \mathcal{X}_2$, $z \in \mathbb{R}$, and $r \geq 0$:

$$H(\text{Var}, \mathbb{E}) : r \geq \text{Var}(L) \text{ and } z = \mathbb{E}[L], \quad \tilde{H}(\text{Var}, \mathbb{E}) : r = \text{Var}(L) \text{ and } z = \mathbb{E}[L].$$

Here, the mean serves as the statistic and the variance is the regulatory risk measure.

²When constructing the corresponding e-process, dividing the identification functions by $2M$ does not affect the calculation of betting processes because betting processes are usually very small in practical standard backtests and thus the condition $\lambda \leq (2M)^{-1}$ is easy to satisfy.

We refer to the work of [Fan et al. \(2023\)](#) for details of testing the mean and the variance by e -values.

3. (The Value-at-Risk) For $L \in \mathcal{L}_0$ and $p \in (0, 1)$, we define the lower (VaR^-) and upper Value-at-Risk (VaR^+) as the generalized left- and right-quantiles, respectively, given by

$$\text{VaR}_p^-(L) = \inf \{x \in \mathbb{R} : F_L(x) \geq p\} \quad \text{and} \quad \text{VaR}_p^+(L) = \inf \{x \in \mathbb{R} : F_L(x) > p\}$$

with the convention $\inf(\emptyset) = \infty$. We denote the interval $\text{VaR}_p = [\text{VaR}_p^-, \text{VaR}_p^+]$. It is well-known that VaR_p is identifiable with the following identification functions decreasing in a :

$$V(x, a) = \frac{\mathbb{1}_{\{x > a\}}}{1 - p} - 1, \quad \text{and} \quad V'(x, a) = 1 - \frac{\mathbb{1}_{\{x \leq a\}}}{p}, \quad x, a \in \mathbb{R}.$$

By Theorem [5.3.1](#), $\mathbb{1}_{\{L > r\}}/(1 - p)$ and $\mathbb{1}_{\{L \leq r\}}/p$ are precise e -variables for the following null hypotheses $H(\text{VaR}_p)$ and $H'(\text{VaR}_p)$, respectively, for all $L \in \mathcal{L}_0$ and $r \in \mathbb{R}$:

$$H(\text{VaR}_p) : r \geq \text{VaR}_p^-(L) \quad \text{and} \quad H'(\text{VaR}_p) : r \leq \text{VaR}_p^+(L).$$

4. (The Value-at-Risk and the Expected Shortfall) For $L \in \mathcal{L}_1$ and $p \in (0, 1)$, the Expected Shortfall (ES) is defined by

$$\text{ES}_p = \frac{1}{1 - p} \int_p^1 \text{VaR}_t^-(L) dt.$$

ES itself is neither elicitable nor identifiable (see e.g., [Gneiting, 2011](#)), but is jointly identifiable with VaR. The risk measure $(\text{ES}_p, \text{VaR}_p)$ is a Bayes pair. The identification function for $(\text{ES}_p, \text{VaR}_p)$ is

$$(v(x, b), V(x, a, b)) = \left(\frac{\mathbb{1}_{\{x > b\}}}{1 - p} - 1, \frac{(x - b)_+}{(1 - p)(a - b)} - 1 \right), \quad x \in \mathbb{R}, \quad a \geq b.$$

As V is uniformly bounded below by -1 and $V(x, a, b)$ is decreasing in a , we have by

Lemma 5.3.1 that $(L - z)_+ / ((1 - p)(r - z))$ is a precise e -variable for the following null hypotheses H_0 and $\tilde{H}_0$ for all $L \in \mathcal{L}_1$, $x \in \mathbb{R}$, and $r \geq z$:

$$H(\text{ES}_p, \text{VaR}_p) : r \geq \text{ES}_p(L) \text{ and } z \in \text{VaR}_p(L), \quad \tilde{H}(\text{ES}_p, \text{VaR}_p) : r = \text{ES}_p(L) \text{ and } z \in \text{VaR}_p(L).$$

We regard ES as the regulatory risk measure and VaR as the statistic for their joint tests. Wang et al. (2025) discussed the detailed model-free procedures backtesting VaR and ES using e -values and e -processes.

5. (The expectile) For $L \in \mathcal{L}_1$ and $p \in (0, 1)$, the expectile (ex_p , Newey and Powell, 1987) is defined as

$$\text{ex}_p(L) = \arg \min_{y \in \mathbb{R}} \mathbb{E} [p(L - y)_+^2 + (1 - p)(L - y)_-^2].$$

The expectile ex_p (with $p \geq 1/2$) is the only monetary risk measure that is elicitable and coherent according to the results of Weber (2006) and Bellini and Big-nozzi (2015). Focusing on the space of non-negative random variables, the expectile $\text{ex}_p : \mathcal{X}_+ \cap \mathcal{L}_1 \rightarrow \mathbb{R}$ is identifiable with identification function

$$V(x, a) = |1 - p - \mathbb{1}_{\{x > a\}}| \left(\frac{x}{a} - 1 \right), \quad x, a \geq 0.$$

We can check that $\inf_{x, a \geq 0} V(x, a) \geq -1$ and $V(x, a)$ is decreasing in r . By Theorem 5.3.1,

$$|1 - p - \mathbb{1}_{\{L > r\}}| \left(\frac{L}{r} - 1 \right) + 1$$

is a precise e -variable for the null hypotheses $H(\text{ex}_p) : r \geq \text{ex}_p(L)$ for all $L \in \mathcal{X}_+$ and $r \geq 0$.

Similarly to point 1, if we consider bounded losses $L : \Omega \rightarrow [-M, M]$, then the identification function

$$V'(x, a) = |1 - p - \mathbb{1}_{\{x > a\}}| (x - a)$$

for ex_p is decreasing in a and uniformly bounded by $-2M$ and $2M$. Thus we have by

Theorem 5.3.1 that

$$1 + \frac{1}{2M}|1 - p - \mathbb{1}_{\{L > r\}}|(L - r) \quad \text{and} \quad 1 - \frac{1}{2M}|1 - p - \mathbb{1}_{\{L > r\}}|(L - r)$$

are precise e -variables for the null hypotheses $H(\text{ex}_p) : r \geq \text{ex}_p(L)$ and $H'(\text{ex}_p) : r \leq \text{ex}_p(L)$, respectively, for all $L : \Omega \rightarrow [-M, M]$ and $r \in [-M, M]$.

A model-free procedure backtesting expectiles using e -values and e -processes remains unsolved in the literature.

6. (The expectile and the variantile) For $L \in \mathcal{L}_1$ and $p \in (0, 1)$, the variantile (var_p , Frongillo and Kash, 2020) is defined as

$$\text{var}_p(L) = \min_{y \in \mathbb{R}} \mathbb{E} [p(L - y)_+^2 + (1 - p)(L - y)_-^2].$$

The risk measure $(\text{var}_p, \text{ex}_p)$ is a Bayes pair. It is identifiable with identification function

$$(v(x, b), V(x, a, b)) = \left(|1 - p - \mathbb{1}_{\{x > a\}}| \left(\frac{x}{a} - 1 \right), \frac{p(x - b)_+^2 + (1 - p)(x - b)_-^2}{a} - 1 \right)$$

for $x, b \in \mathbb{R}$ and $a \geq 0$. By uniform boundedness and monotonicity of V similar to the examples above, we get by Lemma 5.3.1 that

$$\frac{p(L - z)_+^2 + (1 - p)(L - z)_-^2}{r}$$

is a precise e -variable for the following null hypotheses $H(\text{var}_p, \text{ex}_p)$ and $\tilde{H}(\text{var}_p, \text{ex}_p)$ for all $L \in \mathcal{X}$, $z \in \mathbb{R}$ and $r \geq 0$:

$$H(\text{var}_p, \text{ex}_p) : r \geq \text{var}_p(L) \text{ and } z = \text{ex}_p(L), \quad \tilde{H}(\text{var}_p, \text{ex}_p) : r = \text{var}_p(L) \text{ and } z = \text{ex}_p(L).$$

In this case, the variantile is the regulatory risk measure and the expectile is the statistic.

5.4 Comparative E-backtests

5.4.1 Null Hypotheses

Standard backtests can detect underestimation of risk and model misspecification for financial regulation. In other words, they test whether the predictions reported by the financial institution deviate too much from a correct or reference model that is usually unknown in practice. However, if the financial institution provides multiple predictions based on different forecasting methods, standard backtests cannot help regulators or risk managers choose the best method among all the alternatives. To compare different alternative models, we resort to comparative backtests. As introduced by [Nolde and Ziegel \(2017\)](#), the evaluation for risk forecasts can be formulated by the so-called S -dominance defined below. The risk measure $\psi : \mathcal{M} \rightarrow 2^{\mathbb{R}^d}$ is forecasted by the internal process $\{R_t\}_{t \in [T]}$ and the alternative process $\{R_t^*\}_{t \in [T]}$ that are both predictable, where $R_t, R_t^* : \Omega \rightarrow \mathbb{R}^d$ for all $t \in [T]$. Instead of an unknown correct process, we regard the alternative process $\{R_t^*\}_{t \in [T]}$ as the standard or reference process. A comparative backtest should give a conclusion whether the internal process $\{R_t\}_{t \in [T]}$ passes or not compared with the reference $\{R_t^*\}_{t \in [T]}$.

Definition 5.4.1 *Let $\psi : \mathcal{M} \rightarrow \mathbb{R}^d$ be elicitable with scoring function $S : \mathbb{R}^{d+1} \rightarrow \mathbb{R}$. We say $\{R_t\}_{t \in \mathbb{N}}$ S -dominates $\{R_t^*\}_{t \in \mathbb{N}}$ if $\mathbb{E}[S(L_t, R_t) - S(L_t, R_t^*)] \leq 0$ for all $t \in [T]$. We say $\{R_t\}_{t \in \mathbb{N}}$ conditionally S -dominates $\{R_t^*\}_{t \in \mathbb{N}}$ if $\mathbb{E}[S(L_t, R_t) - S(L_t, R_t^*) | \mathcal{F}_{t-1}] \leq 0$ for all $t \in [T]$.*

In Definition 5.4.1, S -dominance is weaker than conditional S -dominance which is more suitable for sequential tests under a dynamic setup. Here, we assume predictions $\{R_t\}_{t \in [T]}$ and $\{R_t^*\}_{t \in [T]}$ are based on the same information set $\{\mathcal{F}_{t-1}\}_{t \in [T]}$. That is, the competing models are produced by the same financial institution without information gaps. We made this assumption because we aim to find the best forecasting method itself whose dominance over the other does not result from a better information set.

We consider the following null hypotheses for our comparative backtests based on con-

ditional S -dominance.

$$H^-(\psi) : \{R_t\}_{t \in [T]} \text{ conditionally } S\text{-dominates } \{R_t^*\}_{t \in [T]}. \quad (4.8)$$

$$H^+(\psi) : \{R_t^*\}_{t \in [T]} \text{ conditionally } S\text{-dominates } \{R_t\}_{t \in [T]}. \quad (4.9)$$

In practice, merely considering one of the two tests above does not always give a safe conclusion. For instance, if hypothesis $H^-(\psi)$ is not rejected, we are not able to say predictions $\{R_t\}_{t \in [T]}$ pass the comparative backtest without further justification. Instead, another test should be conducted for hypothesis $H^+(\psi)$. However, the null hypotheses we considered are testing conditional S -dominance. This is much stricter than those based on S -dominance (on average) in [Nolde and Ziegel \(2017\)](#) because conditional S -dominance is not usually satisfied for two competing stochastic processes in the real world. In this sense, it is possible that both null hypotheses (4.8) and (4.9) are rejected simultaneously. However, by the nature of e-processes as supermartingales, we can still tell valuable information by comparing the pair of e-processes testing (4.8) and (4.9) even though both of them might be rejected at the end. As a result, we can obtain a three-zone approach for comparative backtests similar to that proposed by [Fissler et al. \(2016\)](#) and [Nolde and Ziegel \(2017\)](#): Suppose that the e-processes we obtained for $H^-(\psi)$ and $H^+(\psi)$ are $\{M_t^-\}_{t \in [T]}$ and $\{M_t^+\}_{t \in [T]}$, respectively, and the rejection threshold is $1/\alpha$ for $\alpha \in (0, 1)$.

- (i) When $H^-(\psi)$ is rejected but $H^+(\psi)$ is not, the internal process is in the red region and does not pass the backtest.
- (ii) When $H^+(\psi)$ is rejected but $H^-(\psi)$ is not, the internal process is in the green region and passes the backtest.
- (iii) When both $H^-(\psi)$ and $H^+(\psi)$ are rejected, the internal process is in the yellow region. In this case, we further compare the e-processes $\{M_t^-\}_{t \in [T]}$ and $\{M_t^+\}_{t \in [T]}$. If $M_T^- < M_T^+$, then we say $\{R_t\}_{t \in [T]}$ *weakly dominates* $\{R_t^*\}_{t \in [T]}$ *by magnitude* (the converse holds true by changing “<” to “>”). Here, an e-value serves as an average overall “score” for our comparative test. On the other hand, if $\tau^- = \inf\{t \in [T] : M_t^- \geq 1/\alpha\} > \inf\{t \in [T] : M_t^+ \geq 1/\alpha\} = \tau^+$, then we say $\{R_t\}_{t \in [T]}$ *weakly dominates* $\{R_t^*\}_{t \in [T]}$ *by speed* (the converse holds true by changing “>” to “<”). Here,

we observe which null hypothesis is rejected faster than the other by comparing the stopping times when the two e-processes exceed a prespecified threshold respectively. Of course, the larger the differences $|M_T^- - M_T^+|$ and $|\tau^- - \tau^+|$ we observe, the stronger the conclusions we get.

- (iv) When neither $H^-(\psi)$ nor $H^+(\psi)$ is rejected, no clear conclusion is obtained and further investigation is required. However, Theorem 5.5.3 below shows that under an iid setup, if neither $H^-(\psi)$ nor $H^+(\psi)$ is true, both $H^-(\psi)$ and $H^+(\psi)$ will be rejected with asymptotic power 1 under properly chosen betting processes. Moreover, we seldom observe case (iv) in practice. Therefore, we do not consider this case specifically.

5.4.2 E-variables and E-processes via Scoring Functions

In this subsection, we suppose that $\psi : \mathcal{M} \rightarrow 2^{\mathbb{R}^d}$ is elicitable with scoring function $S : \mathbb{R}^{d+1} \rightarrow \mathbb{R}$. As a building block for the e-process we construct for the comparative backtest, we first consider the following static testing hypotheses for some unknown distribution $F \in \mathcal{M}$ and real-valued $r, r^* \in \psi(\mathcal{M}) \times \psi(\mathcal{M})$:

$$H^-(\psi) : \int_{\mathbb{R}} S(x, r) - S(x, r^*) dF(x) \leq 0, \quad H^+(\psi) : \int_{\mathbb{R}} S(x, r) - S(x, r^*) dF(x) \geq 0. \quad (4.10)$$

Similarly to the construction of e-variables for standard backtests, we require boundedness conditions of the scoring functions to construct e-variables for comparative backtests. However, unlike identification functions, the uniform boundedness property for the difference between two scoring functions is usually hard to satisfy in practice (see Example 5.4.1 for usual choices of scoring functions of common risk measures). As shown in the following result, we can achieve a weaker uniform boundedness condition by the following: (i) We restrict the domain $\mathcal{K}$ to a compact space (usually we can take $\mathcal{K} = [-M, M]$ for $M > 0$). (ii) We find a non-negative function h of given risk forecasts such that the boundedness condition is satisfied for all losses. The proof of the following result is straightforward.

Lemma 5.4.1 For $r, r^* \in \psi(\mathcal{M}) \times \psi(\mathcal{M})$, suppose that $\inf_{x \in \mathcal{K}} \{h(r, r^*)(S(x, r) - S(x, r^*))\} \geq -1$ for some function $h : \mathbb{R}^d \times \mathbb{R}^d \rightarrow [0, 1]$. We have $E = 1 + h(r, r^*)(S(L, r) - S(L, r^*))$ is a precise e-variable for $H^-(\psi)$ in (4.10) for all $F_L \in \mathcal{M}$. Moreover, $\mathbb{E}[E] > 1$ for all $F_L \in \mathcal{M}$ and $r, r^* \in \psi(\mathcal{M}) \times \psi(\mathcal{M})$ such that $\int_{\mathbb{R}} S(x, r) - S(x, r^*) dF_L(x) > 0$.

If we switch the positions of r and r^* in Lemma 5.4.1 (i), we can also get an e-variable $1 - (S(L, r) - S(L, r^*))$ for $H^+(\psi)$ in (4.10) under a proper uniform boundedness assumption. Based on the rationale of Lemma 5.4.1, we construct the e-process for comparative backtests as shown by the following result.

Theorem 5.4.1 Let $\{R_t\}_{t \in [T]}$ and $\{R_t^*\}_{t \in [T]}$ be predictable processes. Let $\{\lambda_t\}_{t \in [T]}$ be a non-negative betting process such that $\inf_{x \in \mathcal{K}} \{\lambda_t(S(x, R_t) - S(x, R_t^*))\} \geq -1$ for all $t \in [T]$. The process $\{M_t^-\}_{t \in \{0\} \cup [T]}$ with $M_0^- = 1$ and

$$M_t^-(\boldsymbol{\lambda}) = (1 + \lambda_t(S(L_t, R_t) - S(L_t, R_t^*)))M_{t-1}^-(\boldsymbol{\lambda}) = \prod_{s=1}^t (1 + \lambda_s(S(L_s, R_s) - S(L_s, R_s^*))) \quad (4.11)$$

is a non-negative supermartingale under $H^-(\psi)$ in (4.8), and is thus an e-process.

Proof. For all $s \in [t]$, as $\lambda_s(S(L_s, R_s) - S(L_s, R_s^*)) \geq -1$, we have M_t^- is non-negative.

For $t \in [T]$, under $H^-(\psi)$ in (4.8), we have

$$\mathbb{E}[S(L_t, R_t) - S(L_t, R_t^*) | \mathcal{F}_{t-1}] \leq 0.$$

As λ_t, L_{t-1}, R_t and R_t^* are $\mathcal{F}_{t-1}$ -measurable, we have

$$\mathbb{E}[M_t^-(\boldsymbol{\lambda}) | \mathcal{F}_{t-1}] = M_{t-1}^-(\boldsymbol{\lambda})(1 + \lambda_t \mathbb{E}[S(L_t, R_t) - S(L_t, R_t^*) | \mathcal{F}_{t-1}]) \leq M_{t-1}^-(\boldsymbol{\lambda}).$$

■

Remark 5.4.1 The corresponding result for $H^+(\psi)$ in (4.9) holds similarly by switching the positions of $\{R_t\}_{t \in [T]}$ and $\{R_t^*\}_{t \in [T]}$. That is, if $\inf_{x \in \mathcal{K}} \{\lambda_t(S(x, R_t^*) - S(x, R_t))\} \geq -1$

for all $t \in [T]$, then we define the process $\{M_t^+\}_{t \in \{0\} \cup [T]}$ with $M_0^+ = 1$ and

$$M_t^+(\boldsymbol{\lambda}) = (1 + \lambda_t(S(L_t, R_t^*) - S(L_t, R_t)))M_{t-1}^+(\boldsymbol{\lambda}) = \prod_{s=1}^t (1 + \lambda_s(S(L_s, R_s^*) - S(L_s, R_s))). \quad (4.12)$$

Similarly to Theorem 5.4.1, $\{M_t^+\}_{t \in \{0\} \cup [T]}$ is a non-negative supermartingale under $H^+(\psi)$ in (4.9).

Theorem 5.4.1 constructs a valid e-process under the hypothesis $H^-(\psi)$ in (4.11) for comparative backtests. It allows inputs of forecasts $\{R_t\}_{t \in [T]}$ and $\{R_t^*\}_{t \in [T]}$ that are processes instead of prespecified fixed parameters. Moreover, the growth rate of the e-process in Theorem 5.4.1 is controlled by the betting process $\boldsymbol{\lambda}$, offering flexibility to guarantee the performance of the tests.

Next, we show examples of e-variables constructed through the scoring functions for comparative backtests with the risk measures discussed in Example 5.3.1. We calculate the bounds for the h functions in Lemma 5.4.1 so that the weak uniform boundedness condition is satisfied. Direct observation tells that the bounds for h rely on the choice of $M > 0$ and the differences between the comparing risk forecasts. The upper bounds of h boil down to 1 if the comparing risk forecasts do not differ too much from each other. We can also construct corresponding e-processes in a similar sense, where we should choose the betting processes $\{\lambda_t\}_{t \in [T]}$ with $\lambda_t = h(R_t, R_t^*)$ for $t \in [T]$.

Example 5.4.1 *Here, we only show the e-variables for $H^-(\psi)$ and the corresponding h functions for all risk measures below. Those for $H^+(\psi)$ are similar.*

1. (The mean) The mean $\mathbb{E}$ is elicitable with strictly consistent scoring function $S_{\mathbb{E}}(x, a) = (x - a)^2$, $x, a \in [-M, M]$. For $r, r^* \in [-M, M]$, we have

$$\inf_{x \in [-M, M]} ((x - r)^2 - (x - r^*)^2) = -2M|r - r^*| + r^2 - r^{*2} \leq 0.$$

By Lemma 5.4.1, $1 + h(r, r^*)((L - r)^2 - (L - r^*)^2)$ is a precise e-variable for the following null hypothesis $H^-(\mathbb{E})$ for all $L : \Omega \rightarrow [-M, M]$:

$$H^-(\mathbb{E}) : \mathbb{E}[(L - r)^2 - (L - r^*)^2] \leq 0,$$

where

$$0 \leq h(r, r^*) \leq \frac{1}{2M|r - r^*| - r^2 + r^{*2}}.$$

2. (The mean and the variance) The mean and the variance $(\text{Var}, \mathbb{E})$ is jointly elicitable with a strictly consistent scoring function

$$S_{\text{Var}, \mathbb{E}}(x, a, b) = b(b - 2x) + a(a - 2x^2), \quad x, b \in [-M, M], \quad a \in [0, M^2].$$

For $r, r^* \in [0, M^2]$ and $z, z^* \in [-M, M]$, we have

$$\begin{aligned} \gamma_1 &:= \inf_{x \in [-M, M]} (S_{\text{Var}, \mathbb{E}}(x, r, z) - S_{\text{Var}, \mathbb{E}}(x, r^*, z^*)) \\ &= \frac{(z - z^*)^2}{2(r - r^*)} \mathbf{1}_{\left\{-\frac{z^* - z}{2(r^* - r)} \in [-M, M] \text{ and } r^* > r\right\}} \\ &\quad - 2M(M(r - r^*) + |z - z^*|) \mathbf{1}_{\left\{-\frac{z^* - z}{2(r^* - r)} \notin [-M, M] \text{ or } r^* \leq r\right\}} + r^2 - r^{*2} + z^2 - z^{*2}. \end{aligned}$$

By Lemma 5.4.1,

$$1 + h(r, r^*, z, z^*) (z(z - 2L) + r(r - 2L^2)) - h(r, r^*, z, z^*) (z^*(z^* - 2L) + r^*(r^* - 2L^2))$$

is a precise e -variable for the following null hypothesis $H^-(\text{Var}, \mathbb{E})$ for all $L : \Omega \rightarrow [-M, M]$, $r, r^* \in [0, M^2]$ and $z, z^* \in [-M, M]$:

$$H^-(\text{Var}, \mathbb{E}) : \mathbb{E}[S_{\text{Var}, \mathbb{E}}(L, r, z) - S_{\text{Var}, \mathbb{E}}(L, r^*, z^*)] \leq 0,$$

where

$$0 \leq h(r, r^*, z, z^*) \leq \begin{cases} \infty, & \gamma_1 \geq 0, \\ -1/\gamma_1, & \gamma_1 < 0. \end{cases}$$

3. (The Value-at-Risk) The Value-at-Risk $\text{VaR}_p = [\text{VaR}_p^-, \text{VaR}_p^+]$ is elicitable with a strictly consistent scoring function

$$S_{\text{VaR}}(x, a) = (1 - p)a + \mathbf{1}_{\{x > a\}}(x - a), \quad x, a \in \mathbb{R}.$$

For $r, r^* \in \mathbb{R}$, we have

$$\inf_{x \in \mathbb{R}} (S_{\text{VaR}}(x, r) - S_{\text{VaR}}(x, r^*)) = (1 - p)(r - r^*)\mathbb{1}_{\{r \leq r^*\}} - p(r - r^*)\mathbb{1}_{\{r > r^*\}} \leq 0.$$

By Lemma 5.4.1,

$$1 + h(r, r^*)((1 - p)r + \mathbb{1}_{\{L > r\}}(L - r)) - h(r, r^*)((1 - p)r^* + \mathbb{1}_{\{L > r^*\}}(L - r^*))$$

is a precise e -variable for the following null hypothesis $H^-(\text{VaR}_p)$ for all $L : \Omega \rightarrow \mathbb{R}$ and $r, r^* \in \mathbb{R}$:

$$H^-(\text{VaR}_p) : \mathbb{E}[S_{\text{VaR}}(L, r) - S_{\text{VaR}}(L, r^*)] \leq 0,$$

where

$$0 \leq h(r, r^*) \leq \frac{1}{p(r - r^*)\mathbb{1}_{\{r > r^*\}} - (1 - p)(r - r^*)\mathbb{1}_{\{r \leq r^*\}}}.$$

4. (The Value-at-Risk and the Expected Shortfall) The Value-at-Risk and the Expected Shortfall $(\text{VaR}_p, \text{ES}_p)$ is jointly elicitable with a strict consistent scoring function

$$S_{\text{ES}, \text{VaR}}(x, a, b) = \mathbb{1}_{\{x > b\}} \frac{x - b}{2\sqrt{a}} + (1 - p) \frac{a + b}{2\sqrt{a}}, \quad x \in [-M, M], \quad 0 < b \leq a \leq M.$$

Here, we assume that the risk forecasts take positive values since we focus on losses. For $0 < z \leq r \leq M$ and $0 < z^* \leq r^* \leq M$, we have

$$\begin{aligned} \gamma_2 &:= \inf_{x \in [-M, M]} (S_{\text{ES}, \text{VaR}}(x, r, z) - S_{\text{ES}, \text{VaR}}(x, r^*, z^*)) \\ &= \left\{ \mathbb{1}_{\{r > r^*\}} \left(\frac{M - z}{2\sqrt{r}} - \frac{M - z^*}{2\sqrt{r^*}} \right) - \mathbb{1}_{\{r \leq r^*\}} (\mathbb{1}_{\{z > z^*\}} \frac{z - z^*}{2\sqrt{r^*}}) \right\} \wedge 0 \\ &\quad + (1 - p) \left(\frac{r + z}{2\sqrt{r}} - \frac{r^* + z^*}{2\sqrt{r^*}} \right). \end{aligned}$$

Thus we have by Lemma 5.4.1 that

$$1 + h(r, r^*, z, z^*) \left(\mathbb{1}_{\{L > z\}} \frac{L - z}{2\sqrt{r}} + (1 - p) \frac{r + z}{2\sqrt{r}} \right) \\ - h(r, r^*, z, z^*) \left(\mathbb{1}_{\{L > z^*\}} \frac{L - z^*}{2\sqrt{r^*}} + (1 - p) \frac{r^* + z^*}{2\sqrt{r^*}} \right)$$

is a precise e -variable for the following null hypothesis $H^-(\text{ES}_p, \text{VaR}_p)$ for all $L : \Omega \rightarrow [-1, 1]$:

$$H^-(\text{ES}_p, \text{VaR}_p) : \mathbb{E}[S_{\text{ES}, \text{VaR}}(L, r, z) - S_{\text{ES}, \text{VaR}}(L, r^*, z^*)] \leq 0,$$

where

$$0 \leq h(r, r^*, z, z^*) \leq \begin{cases} \infty, & \gamma_2 \geq 0, \\ -1/\gamma_2, & \gamma_2 < 0. \end{cases}.$$

5. (The expectile) The expectile ex_p is elicitable with a strict consistent scoring function

$$S_{\text{ex}}(x, a) = -\mathbb{1}_{\{x > a\}}(1 - 2p)(x - a)^2 + (1 - p)a(a - 2x), \quad x, a \in [-M, M].$$

For all $r, r^* \in [-M, M]$, we have

$$\inf_{x \in [-M, M]} (S_{\text{ex}}(x, r) - S_{\text{ex}}(x, r^*)) = \left\{ (p\mathbb{1}_{\{r > r^*\}} + (1 - p)\mathbb{1}_{\{r \leq r^*\}})(-2|r - r^*|M + r^2 - r^{*2}) \right\} \\ \wedge \left\{ (-\mathbb{1}_{\{r > r^*\}} + \mathbb{1}_{\{r \leq r^*\}}) \frac{p(1 - p)}{1 - 2p} (r - r^*)^2 \right\} \leq 0. \quad (4.13)$$

By Lemma 5.4.1,

$$1 + h(r, r^*)(-\mathbb{1}_{\{L > r\}}(1 - 2p)(L - r)^2 + (1 - p)r(r - 2L)) \\ - h(r, r^*)(-\mathbb{1}_{\{L > r^*\}}(1 - 2p)(L - r^*)^2 + (1 - p)r^*(r^* - 2L))$$

is a precise e -variable for the following null hypothesis $H^-(\text{ex}_p)$ for all $L : \Omega \rightarrow [-M, M]$ and $r, r^* \in [-M, M]$:

$$H^-(\text{ex}_p) : \mathbb{E}[S_{\text{ex}}(L, r) - S_{\text{ex}}(L, r^*)] \leq 0,$$

where

$$0 \leq h(r, r^*) \leq \frac{1}{-\inf_{x \in [-M, M]} (S_{\text{ex}}(x, r) - S_{\text{ex}}(x, r^*))},$$

and $\inf_{x \in [-M, M]} (S_{\text{ex}}(x, r) - S_{\text{ex}}(x, r^*))$ is given by (C.1).

5.5 Type-I Error, FDR, Consistency, and Choosing Betting Processes

This section demonstrates necessary technical details for our backtesting method including the Type-I error control, consistency, and an established method for choosing the betting processes.

5.5.1 Type-I Error Control

By constructing the e-processes as (super)martingales, we control the Type-I error of our sequential tests at all stopping times by the following well-known Ville's inequality.

Theorem 5.5.1 (Ville's inequality) *For each $\alpha \in (0, 1)$, we have the e-processes $\{M_t\}_{t \in \{0\} \cup [T]}$ in (3.6), $\{M_t^-\}_{t \in \{0\} \cup [T]}$ in (4.11), and $\{M_t^+\}_{t \in \{0\} \cup [T]}$ in (4.12), under their corresponding null hypotheses, satisfy*

$$\mathbb{P} \left(\sup_{t \in \{0\} \cup [T]} \widetilde{M}_t \geq \frac{1}{\alpha} \right) \leq \alpha, \quad \widetilde{M}_t = M_t, M_t^-, \text{ or } M_t^+.$$

Theorem 5.5.1 guarantees that the e-processes we define for our standard and comparative backtests control the Type-I error at α if the rejection threshold we choose is $1/\alpha$. The supremum in the inequality makes our test valid for all stopping times and thus no asymptotic models are required.

5.5.2 False Discovery Rate (FDR) Control for Multiple Rejections

For our sequential multiple testing problem, we are particularly interested in the case where we find multiple rejections. The control of the false discovery rate (FDR) of our e-test has become an essential issue.

FDR control for deterministic number of runs

Suppose that we deliberately divide our time frame into $n \in [T]$ parts in advance. That is, we choose a prespecified partition $(t_0, t_1, \dots, t_{n-1}, t_n)$ for $t_i \in [T]$, $t_{i-1} < t_i$, $i \in [n] \cup \{0\}$, where $t_0 = 0$ and $t_n = T$, and restart the e-process $(\widetilde{M}_t)$ from the initial value 1 at all points t_i , $i \in [n-1]$. In real life, the points $\{t_i\}_{i=1}^{n-1}$ can represent well-known structural changes such as pandemic periods and financial crises. They can also be even time windows set by the regulators to observe seasonal patterns of the backtesting results.

Following the above-mentioned testing approach, we can easily check that the false discovery rate of our e-test is controlled under the level of $\alpha \in (0, 1)$ if we reject our null hypothesis at the threshold $1/\alpha$:

Theorem 5.5.2 (FDR control) *Let $(\widetilde{M}_t)$ be the e-process for hypothesis H . For all $Q \in H$ and $\alpha \in (0, 1)$, we have*

$$\mathbb{E}^Q \left[\frac{1}{n} \sum_{i=1}^n \mathbb{1}_{\left\{ \sup_{t_{i-1} < t \leq t_i} \widetilde{M}_t \geq \frac{1}{\alpha} \right\}} \right] \leq \alpha.$$

Proof. The result holds by noting that

$$\mathbb{E}^Q \left[\frac{1}{n} \sum_{i=1}^n \mathbb{1}_{\left\{ \sup_{t_{i-1} < t \leq t_i} \widetilde{M}_t \geq \frac{1}{\alpha} \right\}} \right] = \frac{1}{n} \sum_{i=1}^n Q \left(\sup_{t_{i-1} < t \leq t_i} \widetilde{M}_t \geq \frac{1}{\alpha} \right) \leq \alpha,$$

where the last inequality holds by Theorem 5.5.1. ■

Using Theorem 5.5.2, we can guarantee the false discovery rate of our e-test to stay

under a certain level if the number of runs is deterministic. See Section 5.6.2 for a simulation study of multiple-run comparative e-tests with structural changes on the dataset.

FDR control for random number of runs

Suppose that we restart our e-process $(\widetilde{M}_t)$ from the initial value 1 whenever it exceeds the prespecified threshold $1/\alpha$ for $\alpha \in (0, 1)$. Let the random variable N , taking values between 0 and $T \in \mathbb{N}$, be the N -th rejection we get. For $i = 0, \dots, N$, define stopping times

$$\tau_i = \begin{cases} 0, & i = 0, \\ \inf \left\{ \tau_{i-1} < t \leq N : \widetilde{M}_t \geq \frac{1}{\alpha} \right\}, & 0 < i \leq N. \end{cases}$$

In such a case with random restarts, the general FDR control result may not hold.³ However, in the following result, we show that following the above-mentioned testing procedure, the average number of false rejections for our e-test is controlled at the level of $\alpha \in (0, 1)$ multiplied by the expected number of total rejections after choosing the rejection threshold $1/\alpha$.

Proposition 5.5.1 *Let $(\widetilde{M}_t)$ be the e-process for hypothesis H . For any $\alpha \in (0, 1)$, we have*

$$\mathbb{E}^Q \left[\sum_{i=1}^N \mathbb{1}_{\left\{ \sup_{\tau_{i-1} < t \leq \tau_i} \widetilde{M}_t \geq \frac{1}{\alpha} \right\}} \right] \leq \alpha \mathbb{E}^Q[N], \quad \text{for all } Q \in H.$$

Proof. Define

$$A_i = \left\{ \sup_{\tau_{i-1} < t \leq \tau_i} \widetilde{M}_t \geq \frac{1}{\alpha} \right\}, \quad i = 1, \dots, N.$$

Define the σ -algebras

$$\Sigma_0 = \{\emptyset, \Omega\}, \quad \Sigma_i = \sigma(A_1, \dots, A_i), \quad i = 1, \dots, N.$$

³By general FDR control result, we mean that

$$\mathbb{E}^Q \left[\frac{1}{N} \sum_{i=1}^N \mathbb{1}_{\left\{ \sup_{\tau_{i-1} < t \leq \tau_i} \widetilde{M}_t \geq \frac{1}{\alpha} \right\}} \right] \leq \alpha, \quad \text{for all } Q \in H, \alpha \in (0, 1).$$

We leave it as an open question.

For all $i = 1, \dots, N$, $Q \in H$, and $\omega \in \Omega$, we have

$$\mathbb{E}^Q[\mathbb{1}_{A_i}|\Sigma_{i-1}] = Q(A_i|\Sigma_{i-1})(\omega) = \begin{cases} Q(A_i|A_{i-1}), & \mathbb{1}_{A_{i-1}}(\omega) = 1, \\ 0, & \mathbb{1}_{A_{i-1}}(\omega) = 0 \end{cases} \leq \alpha, \quad (5.14)$$

Q -almost surely. Indeed, to obtain the last inequality in (5.14), we use the fact that $(\widetilde{M}_t)$ is a supermartingale. Hence by Theorem 5.5.1, $Q(A_i|A_{i-1}) \leq \alpha$ and thus $\mathbb{E}^Q[\mathbb{1}_{A_i}|\Sigma_{i-1}] \leq \alpha$ holds Q -almost surely. Define the following process with $Z_0 = 0$:

$$Z_n = \frac{1}{n} \sum_{i=1}^n (\mathbb{1}_{A_i} - \alpha), \quad n \in [T].$$

For all $n \in [T] \setminus \{1\}$, we have

$$\mathbb{E}^Q[nZ_n|\Sigma_{n-1}] = \left(\sum_{i=1}^{n-1} (\mathbb{1}_{A_i} - \alpha) + \mathbb{E}^Q[\mathbb{1}_{A_n}|\Sigma_{n-1}] - \alpha \right) \leq \left(\sum_{i=1}^{n-1} (\mathbb{1}_{A_i} - \alpha) \right) = (n-1)Z_{n-1},$$

Q -almost surely. Moreover, it is straightforward that $\mathbb{E}^Q[Z_1] = Q(A_1) - \alpha \leq 0$ by (5.14). Therefore, $(nZ_n)_n^T$ is a supermartingale adapted to the filtration $(\Sigma_n)_{n=0}^T$. As $\{N = n\} \in \Sigma_n$ for all $n \in [T]$, N is a stopping time. By Doob's optional stopping theorem, we have

$$\mathbb{E}^Q \left[\sum_{i=1}^N (\mathbb{1}_{A_i} - \alpha) \right] = \mathbb{E}^Q[NZ_N] \leq \mathbb{E}^Q[0 \cdot Z_0] = 0.$$

Therefore, we have

$$\mathbb{E}^Q \left[\sum_{i=1}^N \mathbb{1}_{\left\{ \sup_{\tau_{i-1} < t \leq \tau_i} \widetilde{M}_t \geq \frac{1}{\alpha} \right\}} \right] = \mathbb{E}^Q \left[\sum_{i=1}^N \mathbb{1}_{A_i} \right] \leq \alpha \mathbb{E}^Q[N].$$

The proof is complete. ■

Proposition 5.5.1 provides a theoretical foundation for our e-backtest if we restart the test as soon as the e-process hits a prespecified threshold, and thus multiple rejections can be found sequentially over time with the average number of rejections being controlled under a certain level when the expected number of rejections is finite. We refer to Section

5.7 for the real data analysis of such a backtesting method.

5.5.3 Test Power

Next, we formally present the consistency result for our testing method under an iid setting. This is in a similar sense to but more general than Proposition 1 of [Fan et al. \(2023\)](#).

Theorem 5.5.3 (Consistency) *Let $E_1, E_2, \dots$ be iid non-negative random variables generated from an alternative probability measure Q . Let*

$$\lambda^* = \arg \max_{\lambda \in [0,1]} \mathbb{E}^Q[\log(1 - \lambda + \lambda E_1)] \quad (5.15)$$

and $\boldsymbol{\lambda} = \{\lambda_t\}_{t \in \mathbb{N}}$ be a betting process such that

$$\frac{1}{T}(\log M_T(\boldsymbol{\lambda}) - \log M_T(\lambda^*)) \xrightarrow{L^1} 0 \quad \text{as } T \rightarrow \infty, \quad (5.16)$$

where $M_t(\mathbf{a}) = \prod_{s=1}^t (1 - a_s + a_s E_s)$ for $\mathbf{a} = \{a_s\}_{s \in [t]}$. We have $Q(\sup_{t \in [T]} M_t(\boldsymbol{\lambda}) \geq 1/\alpha) \rightarrow 1$ as $T \rightarrow \infty$ for all $\alpha \in (0, 1)$ if and only if $\mathbb{E}^Q[E_1] > 1$. In other words, the e-test with betting process $\boldsymbol{\lambda}$ has asymptotic power 1 as $T \rightarrow \infty$.

Proof. “ $\Rightarrow$ ”: Suppose that $\mathbb{E}^Q[E_1] \leq 1$. It follows that

$$\mathbb{E}^Q[M_t(\boldsymbol{\lambda}) | \mathcal{F}_{t-1}] = M_{t-1}(\boldsymbol{\lambda}) \mathbb{E}^Q[(1 - \lambda_t + \lambda_t E_1) | \mathcal{F}_{t-1}] \leq M_{t-1}(\boldsymbol{\lambda}), \quad t \in \mathbb{N}$$

and thus $\{M_t(\boldsymbol{\lambda})\}_{t \in \mathbb{N}}$ is a supermartingale. Hence, we have by Ville’s inequality that

$$Q\left(\sup_{t \in [T]} M_t(\boldsymbol{\lambda}) \geq 1/\alpha\right) \leq \alpha, \quad \text{for all } \alpha \in (0, 1).$$

This implies that if $Q(\sup_{t \in [T]} M_t(\boldsymbol{\lambda}) \geq 1/\alpha) \rightarrow 1 > \alpha$ as $T \rightarrow \infty$ for all $\alpha \in (0, 1)$, then $\mathbb{E}^Q[E_1] > 1$.

“ $\Leftarrow$ ”: Suppose that $\mathbb{E}^Q[E_1] > 1$. Write $f(\lambda) = \mathbb{E}^Q[\log(1 - \lambda + \lambda E_1)]$, $\lambda \in [0, 1]$. We have

$$f'(0) = \mathbb{E}^Q \left[\frac{E_1 - 1}{1 - \lambda + \lambda E_1} \right] \Big|_{\lambda=0} = \mathbb{E}^Q[E_1 - 1] > 0.$$

Thus we have $\max_{\lambda \in [0,1]} f(\lambda) > f(0) = 0$. By the strong law of large numbers,

$$\frac{1}{T} \log M_T(\lambda^*) = \frac{1}{T} \sum_{t=1}^T \log(1 - \lambda^* + \lambda^* E_t) \rightarrow \max_{\lambda \in [0,1]} \mathbb{E}^Q[\log(1 - \lambda + \lambda E_1)] > 0, \quad Q\text{-almost surely.}$$

Hence, we have $Q(\sup_{t \in [T]} M_t(\lambda^*) \geq 1/\alpha) \rightarrow 1$ as $T \rightarrow \infty$ for all $\alpha \in (0, 1)$. Further because

$$\frac{1}{T}(\log M_T(\boldsymbol{\lambda}) - \log M_T(\lambda^*)) \xrightarrow{L^1} 0 \quad \text{as } T \rightarrow \infty,$$

we have $Q(\sup_{t \in [T]} M_t(\boldsymbol{\lambda}) \geq 1/\alpha) \rightarrow 1$ as $T \rightarrow \infty$ for all $\alpha \in (0, 1)$. ■

In Theorem 5.5.3, the optimal λ^* obtained in (5.15) maximizes the log-capital growth rate of the e-process, and thus can be considered as a benchmark betting process calculated using the true distribution Q . Using Theorem 5.5.3, we can achieve consistency with the e-process we constructed given that (i) a betting process $\boldsymbol{\lambda}$ is properly chosen in the sense of (5.16), where the log-capital growth rate for $\boldsymbol{\lambda}$ is asymptotically the same as that for λ^* ; and (ii) the consistency of a single e-variable is satisfied (see Theorem 5.3.1, Proposition 5.3.1, and Lemma 5.4.1 above).

5.5.4 Choosing Betting Process

Sections 5.5.1 and 5.5.2 show that our e-test is always valid no matter how the betting process $\boldsymbol{\lambda}$ is chosen as long as it stays in a certain range such that the e-process is non-negative. However, as shown by Section 5.5.3, the power of the e-test is guaranteed only if the betting process $\boldsymbol{\lambda}$ is “properly” chosen. Therefore, one of the important missions for calculating the e-processes is to find good choices of betting process $\boldsymbol{\lambda}$. In this chapter, we use the *GREL* (*growth-rate for empirical losses*) method introduced by Wang et al. (2025): Let L follow the empirical distribution of the sample losses $L_1, L_2, \dots, L_T$. Let $f(X, r)$ be

some function of the losses $X \in \mathcal{X}$ and forecasts $r \in \mathbb{R}^d$. We have

$$\lambda_t = \lambda_t(r) = \arg \max_{\lambda \in [0, \gamma]} \mathbb{E}[\log(1 - \lambda + \lambda f(L, r))] = \arg \max_{\lambda \in [0, \gamma]} \frac{1}{t-1} \sum_{s=1}^{t-1} \log(1 - \lambda + \lambda f(L_s, r)), \quad t \in [T], \quad (5.17)$$

where $\gamma > 0$ is some bound of the betting process to guarantee the e-process to stay above 0. Specifically, it is defined as

$$\gamma = \gamma(r) = \begin{cases} \infty, & \inf_{x \in \mathcal{K}} f(x, r) \geq 1, \\ -1/(\inf_{x \in \mathcal{K}} f(x, r) - 1), & \inf_{x \in \mathcal{K}} f(x, r) < 1. \end{cases}$$

To facilitate the calculation of the optimization problem stated above, we approximate these expressions by using the Taylor expansion $\log(1+x) \approx x - x^2/2$ at $x = 0$, and apply the first-order condition. Then, (5.17) can be approximated by

$$\lambda_t \approx 0 \vee \frac{\sum_{s=1}^{t-1} (f(L_s, r) - 1)}{\sum_{s=1}^{t-1} (f(L_s, r) - 1)^2} \wedge c\gamma, \quad (5.18)$$

where we set a truncation level $0 < c \leq 1$ to avoid extreme ill-behaved scenarios. We say it is a “proper” choice of a betting process because the GREL method is asymptotically optimal when $L_1, L_2, \dots, L_T$ are iid as shown by Theorem 3 of Wang et al. (2025). We only adopt the GREL method to determine our betting processes because the focus of this chapter is not exploring betting processes with the best performance. One can use alternative methods for choosing the betting processes (see e.g., Grünwald et al., 2020; Fan et al., 2023; Waudby-Smith and Ramdas, 2024; Wang et al., 2025).

5.6 Simulation Studies

In this section, we present several simulation studies to compare e-backtesting of different risk measures, including Value-at-Risk, Expected Shortfall, and expectile. We show the backtesting results and evaluate different forecasting methods based on under-reporting, over-reporting, and exact reporting by the risk forecaster. Our main proposal is to use e-processes for comparative e-backtesting among different forecasting methods. Recall that

for each comparison between two forecasting methods, we need to do two sides of testing, referred to as (4.8) and (4.9), whose corresponding e-processes are constructed by (4.11) and (4.12) respectively.

For comparative backtests, we use the Taylor approximated betting process via the GREL method in (5.18). Specifically, we get

$$\lambda_t^- \approx 0 \vee \frac{\sum_{s=1}^{t-1} (S(L_s, R_t) - S(L_s, R_t^*))}{\sum_{s=1}^{t-1} (S(L_s, R_t) - S(L_s, R_t^*))^2} \wedge c\gamma, 0 < c \leq 1, \quad (6.19)$$

for $H^-(\psi)$ in (4.8), where

$$\gamma = \begin{cases} \infty, & \inf_{x \in \mathcal{K}} \{S(x, R_t) - S(x, R_t^*)\} \geq 0, \\ -1/(\inf_{x \in \mathcal{K}} \{S(x, R_t) - S(x, R_t^*)\}), & \inf_{x \in \mathcal{K}} \{S(x, R_t) - S(x, R_t^*)\} < 0. \end{cases}$$

Switching the positions of $S(L_s, R_t)$ and $S(L_s, R_t^*)$ in (6.19) yields the approximation of λ_t^+ for $H^-(\psi)$ in (4.9). We replace $S(L_s, R_t)$ and $S(L_s, R_t^*)$ by $S(L_s, R_t, Z_t)$ and $S(L_s, R_t^*, Z_t^*)$ for the case of two-dimensional risk measures; e.g., (ES, VaR).

5.6.1 Comparative Backtests via Time Series Data

We apply our e-backtesting procedure to evaluate the relative performance of different forecasting methods on time series data. The data are simulated from an AR(1)-GARCH(1,1) process:

$$L_t = \mu_t + \sigma_t Z_t, \quad \mu_t = -0.05 + 0.3L_{t-1}, \quad \sigma_t^2 = 0.01 + 0.1\sigma_{t-1}^2 Z_{t-1}^2 + 0.85\sigma_{t-1}^2 \quad t \in \mathbb{N} \quad (6.20)$$

where $\{Z_t\}_{t \in \mathbb{N}}$ is a sequence of iid innovations following a skewed-t distribution with shape parameter $v = 5$ and skewness parameter $\gamma = 1.5$. We adopt the same setup as in [Nolde and Ziegel \(2017\)](#) to enable a direct comparison, and the same applies to the forecast models. Following [Nolde and Ziegel \(2017\)](#), the estimation of ρ is carried out using a two-stage procedure. We assume that the innovations follow either a normal, t, or skewed-t distribution. For each distribution, we apply three estimation methods: Fully Parametric (FP), Filtered Historical Simulation (FHS), and Extreme Value Theory-based (EVT) estimation.

This yields a total of nine estimation approaches, denoted as n-FP, n-FHS, n-EVT, t-FP, t-FHS, t-EVT, st-FP, st-FHS, and st-EVT. In addition, we supplement the results with optimal forecasts that utilize knowledge of the data-generating process. The estimation is performed using a moving window of size 500, and the forecasts are evaluated over an out-of-sample period comprising 5,000 verification observations.

We conduct comparative backtests for each of the ten models, reporting the e-values and e-processes for each case. We implement the Taylor-approximated betting process via the GREL method as described in (6.19), with a truncation level $c = 0.1$ to prevent ill-behaved e-processes. Evidence against the null is detected when e-values exceed thresholds 2, 5, and 10. In particular, e-values exceeding 5 or 10 provide substantial evidence to reject the null hypothesis.⁴ While exceeding threshold of 2 may not be substantial enough to reject the null hypothesis, it can still serve as an early warning, suggesting that one set of risk predictions may not be more accurate than another.

Additionally, we present heat maps for these comparative backtests, similar to the traffic light matrices in [Nolde and Ziegel \(2017\)](#). Green cells indicate that the comparative backtest is passed, while yellow cells indicate inconclusive evidence, occurring either when neither $H^-(\psi)$ nor $H^+(\psi)$ is rejected, or when both are rejected (further details are provided in Section 5.4.1). In our comparative backtest heat maps, red cells indicate a failure in the backtest. Since the e-value in this chapter also serve as an average overall “score” for the comparative test, we use three shades of red to distinguish different levels of failure based on the e-values: light red indicates an e-value greater than 2 but less than 5 (early warning); medium red indicates an e-value greater than 5 but less than 10 (substantial evidence); and dark red denotes an e-value greater than 10 (strong evidence).

In the following, we present in Figure 5.1 a set of representative results corresponding to the cases discussed in Section 5.4.1. The figure displays six comparative backtests for VaR_α , ex_τ , and $(\text{ES}_\nu, \text{VaR}_\nu)$ at typical internal risk management levels ($\alpha = 0.9$, $\tau = 0.96567$, $\nu = 0.754$) and the Basel standard levels ($\alpha = 0.99$, $\tau = 0.99855$, $\nu = 0.975$). Each plot shows the e-processes for both test directions (M_t^+ and M_t^-), along with their difference ($M_t^+ - M_t^-$).

⁴In accordance with Jeffrey’s rule of thumb about e-values (see [Jeffreys \(1961\)](#)), the evidence against the null hypothesis is considered *substantial* if the e-value falls within the interval of $(10^{1/2}, 10)$; The evidence against the null hypothesis is regarded as *strong*, if the e-value falls within the interval of $(10, 10^{3/2})$.

For VaR_α , we present the result for t-FHS (internal model) against t-EVT (standard model). At $\alpha = 0.9$, both M_t^+ and M_t^- exceed the early warning threshold of 2. This indicates rejections on both sides of the test, which is not a case under traditional p-value based comparative backtesting. As discussed in Section 5.4.1, such both sides rejections can be interpreted in terms of “magnitude” (via the final e-values) and “speed” (via the growth rate of M_t^+ and M_t^-). A faster rise in either M_t^+ or M_t^- indicates stronger evidence against $H^+(\psi)$ or $H^-(\psi)$, thus favoring the internal or standard model. Additionally, the trend of $M_t^+ - M_t^-$ can also offers insight into relative model performance: an increasing (decreasing) trend suggests outperformance of the internal (standard) model. We will provide more detailed examples based on real data analysis in section 5.7. At $\alpha = 0.99$, we observe a more standard rejection pattern, where $H^-(\psi)$ is rejected with M_t^- between 2 and 5, while $H^+(\psi)$ is not rejected. This suggests that st-EVT outperforms st-FHS under this scenario, indicating the superior performance of EVT-based methods over FHS-based methods at higher confidence levels.

For expectile, we compare the st-EVT method with the optimal (opt) method. The case at the lower level $\tau = 0.96561$ serves as a clear example of a rejection with substantial evidence. Specifically, $H^-(\psi)$ is rejected with M_t^- stopping in the range of 5 to 10, while $H^+(\psi)$ is not rejected. At a more extreme level, $\tau = 0.99855$, we obtain early warning evidence that the optimal method outperforms st-EVT, as $H^-(\psi)$ is rejected in the range of 2 to 5 and $H^+(\psi)$ remains unrejected. Both results suggest that the optimal method outperforms st-EVT, which is consistent with our expectation. For (ES, VaR), we compare the n-FP method with the n-EVT method. Both cases serve as examples of rejection with strong evidence. Specifically, M_t^- exhibits a faster rate of increase and exceeds 10, while M_t^+ remains within the range of 0 to 1. These results indicate that n-EVT outperforms n-FP in both scenarios. Moreover, at the higher confidence level $\nu = 0.975$, the even steeper growth and larger final value of M_t^- provide additional evidence in terms of both speed and magnitude, supporting the conclusion that n-EVT tends to outperform n-FP at more extreme levels.

The final comparative backtesting results for VaR, (ES, VaR), and expectile are displayed as heat maps in Figures 5.2, 5.3 and 5.4. The results are generally consistent with [Nolde and Ziegel \(2017\)](#) and provide additional statistical insight. Across all three mea-

sures, the n-FP and t-FP methods consistently fail the comparative backtests in most cases when used as internal models, and are therefore regarded as the two worst forecasters. The optimal method passes the comparative backtest against all other methods, with the exception of st-FP and st-EVT in the VaR case, which is again consistent with expectations. For VaR and (ES, VaR), skewed t-based methods generally outperform t-based methods, which in turn outperform n-based methods. Under certain distributional assumptions on the innovations, the performance of the three estimation methods (i.e. FP, FHS and EVT) varies depending on the specific case. For expectiles, aside from the best method (i.e., opt) and the worst methods (i.e., n-FP and t-FP), only limited clear evidence is observed, indicating that the remaining methods are all suboptimal with indistinguishable performance. Visualizing the e-processes of the comparative backtests provide further insight into the strength and timing of the test outcomes.

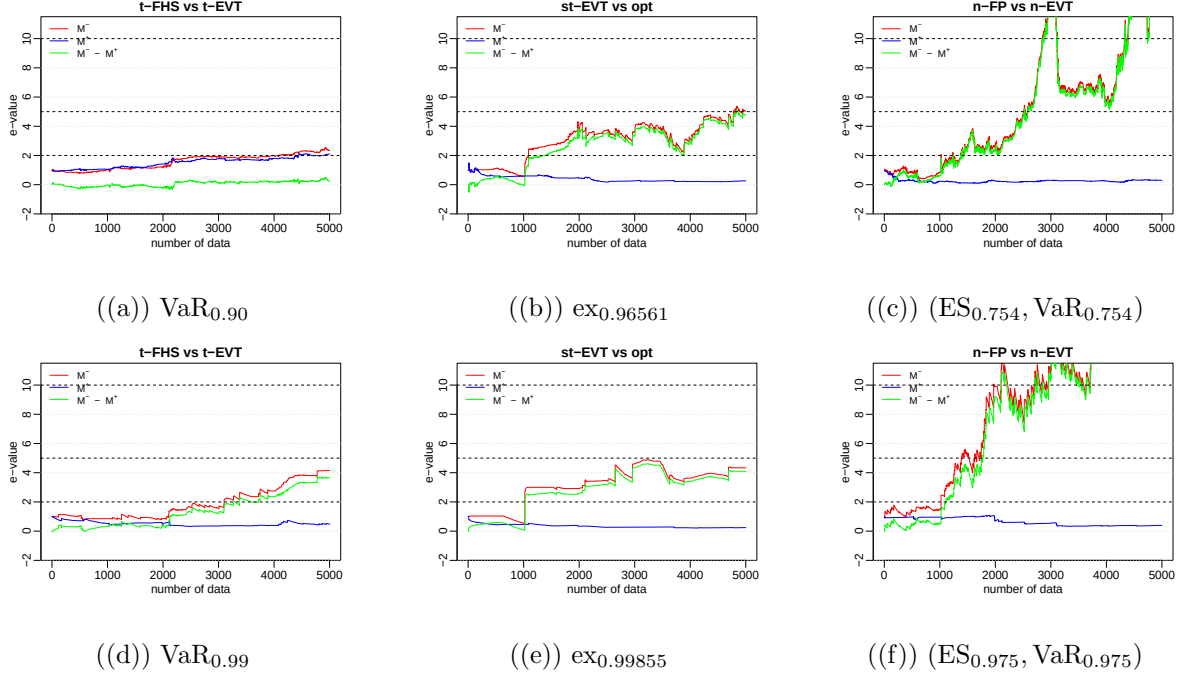


Figure 5.1: E-process of comparative backtests for VaR, ex and (ES, VaR) forecasts at various confidence levels; see Section 5.6.1 for details. The tests employ the GREL betting process with $c = 0.1$, based on the scoring functions in Example 5.4.1. The title of each plot indicates the comparison: “internal model” vs. “standard model”.

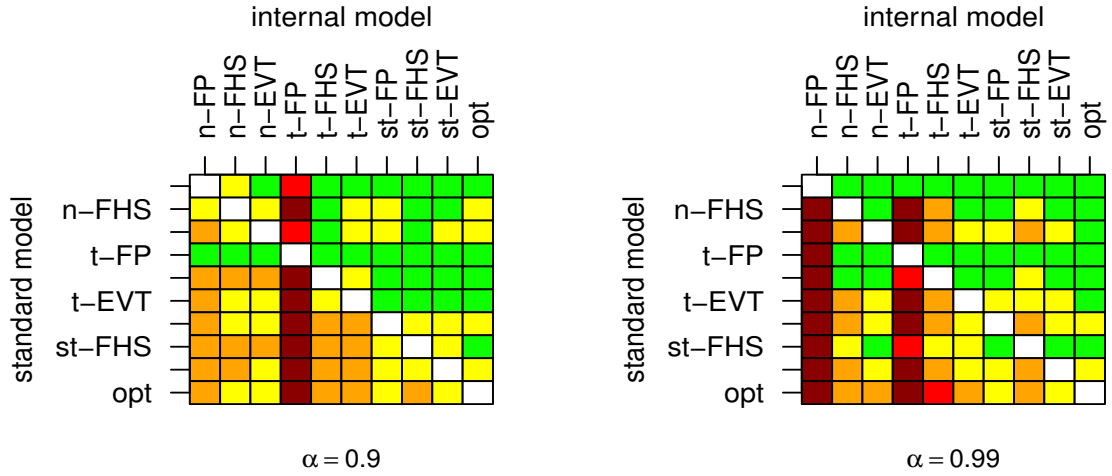


Figure 5.2: Heat map matrices for VaR_{α} forecasts. The test employs the GREL betting process with $c = 0.1$, based on the score function in Example 5.4.1. This figure is comparable to Figure 1 in [Nolde and Ziegel \(2017\)](#)

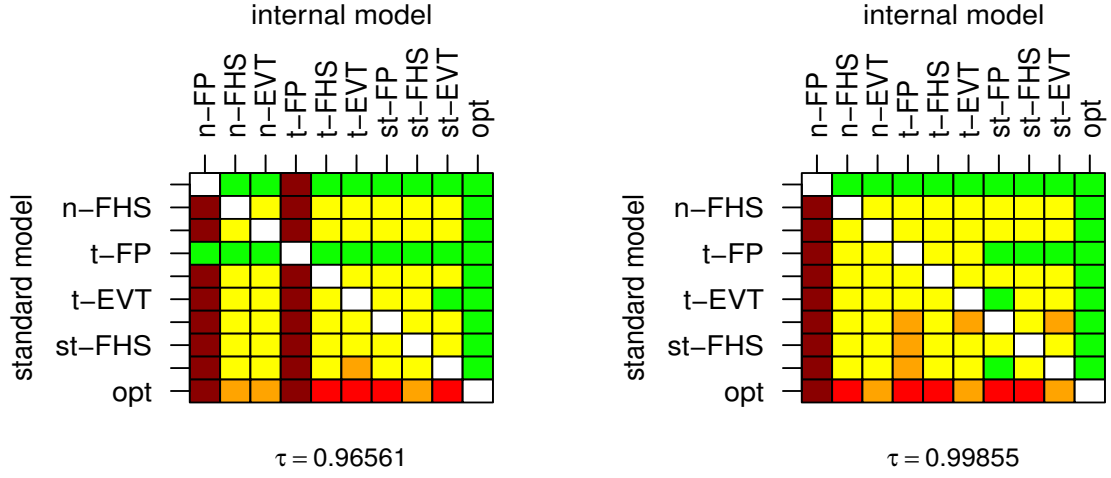


Figure 5.3: Heat map matrices for ex_τ forecasts. The test employs the GREL betting process with $c = 0.1$, based on the score function in Example 5.4.1. This figure is comparable to Figure 2 in [Nolde and Ziegel \(2017\)](#)

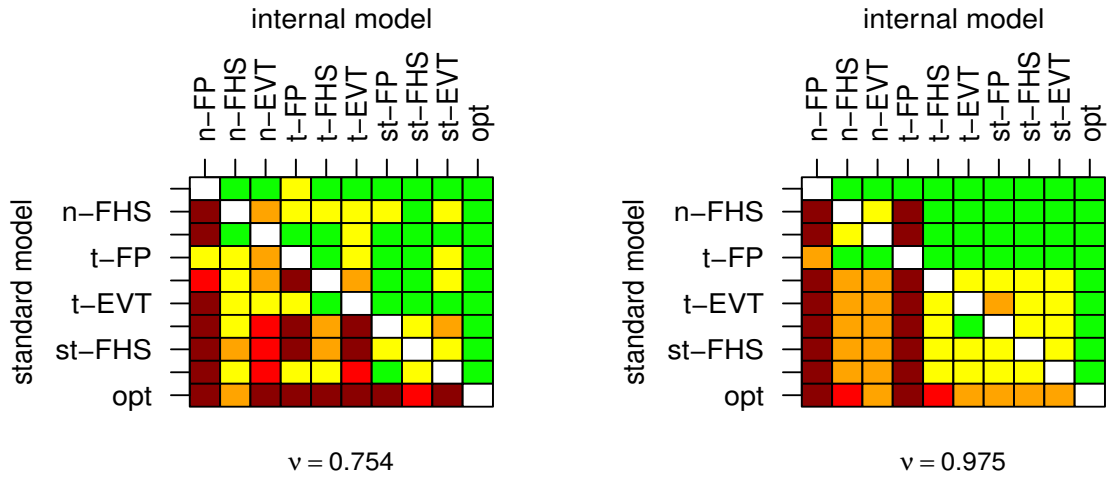


Figure 5.4: Heat map matrices for (ES_ν, VaR_ν) forecasts. The test employs the GREL betting process with $c = 0.1$, based on the score function in Example 5.4.1. This figure is comparable to Figure 3 in [Nolde and Ziegel \(2017\)](#)

5.6.2 Comparative Backtests via Structural Change of Time Series

We examine the power of comparative backtests for risk forecasts under structural changes in simulated time series, focusing on a deterministic number of runs mentioned in section 5.5.2. We simulate the loss $\{L_t\}_{t \in \mathbb{N}}$ following the AR(1)-GARCH(1,1) process:

$$L_t = \mu_t + \sigma_t Z_t, \quad \mu_t = -0.05 + 0.3L_{t-1}, \quad \sigma_t^2 = 0.3 + 0.05\sigma_t^2 Z_t^2 + \beta_t \sigma_{t-1}^2 \quad t \in \mathbb{N} \quad (6.21)$$

where $\{Z_t\}_{t \in \mathbb{N}}$ is a sequence of iid innovations following a skewed-t distribution with shape parameter $v = 5$ and skewness parameter $\gamma = 1.5$. The volatility coefficient shifts at the structural change point $b^* = 1500$, with $\beta_t = 0.4 + 0.5\mathbb{1}_{\{t > b^*\}}$. We simulate 500 presampled data for forecasting risk measures and another 2000 data for backtesting. The change point is placed at the midpoint of the backtesting period, creating a contrast between a normal-volatility regime (first half) and a high-volatility regime (second half).

We again consider a two-stage estimation procedure for a risk measure ρ . We assume the innovations follow either a normal or skewed-t distribution and apply three estimation methods for each distribution: FP, FHS, and EVT. This results in a total of six estimation approaches: n-FP, n-FHS, n-EVT and st-FP, st-FHS, and st-EVT. Then, we conduct comparative backtests between these six estimation approaches. In comparative backtests, we restart the e-process at the structural change point to account for structural shifts, reflecting realistic scenarios where regime changes are known. We implement the Taylor-approximated betting process via the GREL method as described in (6.19), with a truncation level of $c = 0.1$ to prevent ill-behaved e-processes. Both risk forecasts and the betting procedure use a rolling window of size $n = 500$. We detect evidence against the null hypothesis when the e-processes exceed threshold 2.

Figures C.1, C.2, and C.3 present the comparative backtesting results for six estimation methods of VaR, expectiles, and (ES, VaR) at their respective Basel levels, with the e-process restarted at the structural change point.

For $\text{VaR}_{0.99}$, n-FP and n-FHS emerge as the weakest methods, as evidenced by M^- exceeding 2 when treated as internal models and compared against others, while M^+ remains stable and well below the rejection threshold of 2. Among the remaining methods,

we observe that n-EVT performs worse than the skew-t based estimation methods (st-FP, st-FHS, and st-EVT) after the structural change point. Before the change, however, n-EVT demonstrates comparable performance to the skew-t methods and even trending better than st-FHS. This relationship reverses following the structural change. We conclude that st-FP and st-EVT both outperform st-FHS over the entire period. Moreover, st-EVT consistently shows a stronger performance than st-FP, although both M^- and M^+ exceed 2 after the structural change, indicating a rejection that warrants further investigation.

For $\text{ex}_{0.99855}$, aside from the strong evidence identifying n-FP as the weakest method throughout the period, as indicated by a sharp increase in M^- and a consistently stable M^+ , there is no clear evidence of superiority among the remaining methods in either period, suggesting comparable accuracy.

The results for $(\text{ES}_{0.975}, \text{VaR}_{0.975})$ once again highlight the substantially poor performance of n-FP in forecasting. Although no rejection evidence is found in M^- , a clear underperformance trend is observed. With a more aggressive truncation level (i.e., higher c), evidence of rejection becomes apparent. Most of the remaining methods do not show clear differences in performance, except for two notable comparisons. n-EVT performs comparably to st-FHS before the structural change and outperforms it afterward. Meanwhile, st-FP initially shows better performance than st-EVT, but this relationship reverses after the structural change, with a clear rejection observed.

5.7 Real Data Analysis

We fit the negated percentage log-returns of the NASDAQ Composite Index using an AR(1)-GARCH(1,1) model over the period Jan 3, 2003, to May 30, 2025. To account for recent market and economic conditions, we use a rolling estimation window of 500 data points, resulting in an out-of-sample size of $n = 5138$ for backtesting. The same forecasting methods described in Section 5.6.2 are employed, consistent with methods in real data analysis of [Nolde and Ziegel \(2017\)](#). We conduct comparative backtests between methods using the e-processes defined in (4.11). The betting process λ_t is selected based on GREL. While GREE, as introduced in [Wang et al. \(2025\)](#) is also applicable in comparative backtests, its range for λ is constrained by both a predefined upper bound and the minimum

value of $h(R_t, R_t^*)$ before time t . This constraint leads to a shrinking bound, causing inconsistent bound ranges over time. In contrast, this issue does not arise in GREL, which supports its use as our recommended approach. Additionally, as previously mentioned, the boundedness condition is required to construct e-variables for comparative backtests. Different choices of truncation level c reflect regulators' preferences, with a larger value of c can lead to more aggressive test outcomes. Through our extensive numerical experiments, we find that the choice of c is highly flexible in practical applications. To illustrate this flexibility and to avoid the illness behavior caused by highly uncertainty real-world data, we set a conservative truncation level of $c = 0.01$ for all experiments without restart.

Figures C.4, C.5, and C.6 present the comparative backtesting results of the six forecasting methods for VaR, expectile, and the (VaR, ES) pair at their respective Basel levels. The experiments are conducted based on the two-sided conditional S-dominance test proposed in (4.10). Each plot displays the comparative backtesting results for two models along with their corresponding e-processes for M_t^- , M_t^+ and the difference $(M_t^+ - M_t^-)$. We detect evidence against the null when e-values exceed thresholds 2. To determine the superior method in a comparative backtest, we assess through the e-process of M_t^- and M_t^+ and the trend of $(M_t^+ - M_t^-)$. Specifically, if M_t^- exceeds and is detected before M_t^+ over the testing period, the internal model outperforms the alternative model, and vice versa. In some cases, no clear detection occurs, but valuable insights can still be obtained by analyzing the trend of $(M_t^- - M_t^+)$. An upward trend in $(M_t^- - M_t^+)$ suggests the internal model outperforms the alternative, while a downward trend indicates the opposite. More complex cases are addressed later in the discussion.

For $\text{VaR}_{0.99}$, strong evidence suggests that n-FP is the least accurate method, and n-EVT and n-FHS perform worse than t-FHS and st-FHS, respectively. In the remaining comparative backtests, both sides of the two-sided test are rejected; however, meaningful insights can still be drawn in some cases by examining the trend of the $M^- - M^+$ process and comparing the behaviors of M^- and M^+ . For instance, in the comparison between st-FP and st-EVT, the M^- process shows stronger evidence than M^+ in terms of both speed and magnitude, suggesting that st-EVT outperforms st-FP. Another noteworthy example is the comparison between n-EVT and st-FHS, where n-EVT outperforms st-FHS prior to 2016, as indicated by a one-sided rejection in $H^-(\psi)$. However, the $M^- - M^+$

trend reverses and decreases thereafter. Although both $H^-(\psi)$ and $H^+(\psi)$ eventually are rejected, M^- shows stronger evidence in both magnitude and speed, indicating a weak dominance of n-EVT on st-FHS. Similar patterns can be observed in the comparisons of n-FHS vs. st-FP, n-EVT vs. st-EVT, and st-FP vs. st-FHS. Additionally, there are more complex scenarios in which both sides of the two-sided test are rejected and the $M^- - M^+$ process appears ambiguous in both direction and magnitude, making it difficult to draw firm conclusions. Examples include n-FHS vs st-EVT, n-EVT vs. st-FP and n-EVT vs. st-FHS, highlighting the need for further investigation.

To address the above-mentioned issue and facilitate timely detection, we restart the e-processes upon each detection, enabling more dynamic monitoring of the relative performance of competing models over time. To further enhance sensitivity, we adopt a more aggressive betting strategy by increasing the truncation level to $c = 0.1$. Figure C.7 shows the comparative backtests for $\text{VaR}_{0.99}$ with e-processes restarted at 2 across the considered methods. In the comparison between n-EVT and st-FP, M^+ exceeds 2 more frequently than M^- , indicating overall superior performance of n-EVT. Furthermore, since 2018, n-EVT continues to outperform st-FP, as shown by the persistent rejection of $H^+(\psi)$, suggesting its potential for superior future performance. The comparisons n-FHS vs. st-EVT and n-EVT vs. st-FP show considerable temporal variation. Notably, both st-EVT and n-EVT outperform n-FHS and st-FP during the 2008 financial crisis and the COVID-19 pandemic period between 2020 and 2024, highlighting the advantage of EVT-based methods under extreme scenarios. In contrast, n-FHS tends to outperform st-EVT during more stable periods. These findings demonstrate that the relative performance of the methods considered varies between different time intervals.

For $\text{ex}_{0.99855}$, we can first establish a statistically supported ranking of the methods based on Figure C.5:

$$\text{n-FP} < \text{n-EVT} < \text{st-EVT} < \text{n-FHS} < \text{st-FHS} < \text{st-FP}$$

This ranking clearly reflects the overall relative performance of the methods over the entire period. Moreover, the results from the restarted comparative backtest are presented in Figure C.8. With a more aggressive betting strategy, the test becomes more sensitive to changes, allowing us to capture additional information. For instance, we observe a reversal

in the relative performance between the comparison of skewed- t based methods after 2016, providing further insight for evaluating model behavior under the current scenario.

For $(ES_{0.975}, VaR_{0.975})$, Figure C.6 supports sufficient statistical evidence to obtain the following ranks :

$$n-FP < st-FP < st-FHS < n-EVT$$

The ranks of n-FHS and st-EVT remain to be determined. There is clear evidence that n-EVT, n-FHS, and st-EVT outperform n-FP, st-EVT, and st-FHS, respectively. Besides, although both sides of the test are rejected in n-FHS vs. st-FP, M^+ weakly dominates M^- in both magnitude and speed, suggesting that n-FHS outperforms st-FP. To complete the ranking, we further consider the results of n-FHS vs. st-FHS, n-EVT vs. st-EVT, and st-FHS vs. st-EVT. In both n-FHS vs. st-FHS and n-EVT vs. st-EVT, we observe a reversal between M^- and M^+ around 2013, indicating improved performance of normal distribution-based methods in the subsequent period. Furthermore, combining the restarted backtests in Figure C.9, we find that n-based methods (n-FHS and n-EVT) consistently outperform their skewed- t counterparts after 2006, except at the beginning of the pandemic. This suggests n-FHS and n-EVT are likely to maintain their superior performance in the near future. Finally, for st-FHS vs. st-EVT, the M^- and M^+ processes show comparable trends after 2009 (as seen in Figures C.6 and C.9), with no rejections in either direction, indicating similar performance. Thus, for forecasting $(ES_{0.975}, VaR_{0.975})$, the above findings support the following ranking of methods:

$$n-FP < st-FP < st-FHS \approx st-EVT < n-FHS < n-EVT$$

5.8 Conclusion

This chapter develops a model-free framework for standard and comparative backtests of identifiable and elicitable risk measures, building on the recently introduced concepts of e-values and e-processes. By leveraging these tools, we construct valid e-tests applicable to a broad class of commonly used risk measures, including the mean, variance, Value-

at-Risk, Expected Shortfall, and expectile. Our approach accommodates both standard backtesting, which detects underestimation and model misspecification, and comparative backtesting, which facilitates rigorous performance comparison between competing forecasting models. The proposed methodology provides valid Type-I error and false discovery rate control in sequential testing, even under structural change and without restrictive assumptions on the underlying data-generating process. The framework enhances existing backtesting techniques by incorporating the fundamental properties of identifiability and elicibility, offering a systematic and flexible approach to evaluating risk forecasts. Through comprehensive simulation studies and empirical analysis with real financial data, we demonstrate the practical effectiveness of our methods. These findings provide valuable insights for improving risk model evaluation in both financial risk management and regulatory applications.

Chapter 6

Future work

In this chapter, we conclude with some potential extensions to consider in the future.

- In chapter 3, we improve the approximation accuracy for two families of systemic risk measures by developing second-order asymptotic results. The next step is to construct unified estimation methods for systemic risk measures based on conditional distortion risk measures, encompassing a wider range of classic risk metrics. We will develop first- and second-order asymptotics and corresponding estimators for ease and accurate use. Given the strong dependence among firm losses during systemic events, our goal is to create estimators that can capture tail dependence within the data range and further extrapolate these estimations to extreme levels. Meanwhile, the asymptotic normality of the proposed estimators will be established. Our findings expect to enhance understanding of systemic risk management in finance and actuarial science. Preliminary results have already been obtained.
- In chapter 4, we establish a systematic diversification framework for managing extremely heavy-tailed catastrophe risks. Building on these findings, future research offers promising directions for further enhancing insurers' diversification efficiency within the constraints of insurability (e.g. business types & location) and capital capacity. Currently, we have utilized the factor copula to model multidimensional general dependence and tail dependence while adapting mixture models to

capture the heavy-tailed nature of catastrophic losses. The optimization criterion will be studied to assist insurers in optimizing risk selection and evaluating new policy acceptances. Next, we will derive sharp asymptotics for diversification ratios, focusing on risks with varying heavy-tailedness and incorporating tail dependence to establish refined criteria for enhancing catastrophe risk management. Considering varying heavy-tailedness improves the pooling efficiency by diversifying across a broader range, including those beyond natural disasters. Adapting Archimedean copulas with Williamson transform-based generators will enable us to analyze how tail dependence, particularly negative dependence, affects pooling outcomes. Furthermore, various risk-sharing rules can enhance diversification from different perspectives: conditional-mean and mean-proportional rules improve actuarial fairness, quantile-based rules improve Pareto efficiency, and order-statistics and multi-layer rules help in loss allocation. Exploring diversification strategies within various risk-sharing rules provides a robust foundation for advancing risk-sharing practices and offers promising directions for future research.

- In chapter 5, the e-value based backtesting framework tests the null hypothesis through betting, where the strategy directly impacts the power of the e-backtest. Exploring optimal betting strategies is a promising research direction to enhance the power of the e-test and help market practitioners improve timely risk awareness. Beyond the established connection between elicitable risk measures and backtestability, distortion risk measures offer another promising class for backtesting, with significant practical benefits for evaluating a variety of risk measures. An empirical project could apply our developed e-backtesting model to financial and insurance data, identifying the most accurate methods in extensive literature across various datasets, refining evaluations, adjusting mitigation strategies and further examining the strategies' effectiveness. For extreme risks, determining the maximum domain of attraction (MDA) is crucial for statistical inference. While many risks often exhibit heavy tails, heteroscedasticity can vary the tail index over time. Leveraging e-values for single and multiple hypotheses offers a practical approach to monitoring and assessing the MDA and tail index homogeneity across single and multiple risks. This enables the timely detection of shifts and misconceptions in extreme risk assess-

ment. The above findings enhance understanding, guide advancements, and provide iterative insights for the first two research paths.

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Appendix A

Value-at-Risk- and Expectile-based Systemic Risk Measures and Second-order Asymptotics: With Applications to Diversification

Lemma A.1 *Assume that $\bar{F}(\cdot) \in 2\text{RV}_{-\alpha, \beta}$ with $\alpha > 0, \beta \leq 0$ and auxiliary function $A(\cdot)$. Define $\tilde{F}(t) = \int_0^t \left(1 - \frac{\phi(x)}{c}\right) dF(x)$, where $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is a function with $\mathbb{E}[\phi(X)] = 0$, $|\phi(x)| \leq c - 1$ for all $x \geq 0$, $\lim_{t \rightarrow \infty} \phi(t) = b$ and $\phi(\cdot) - b \in \text{RV}_\rho$ with $b \in \mathbb{R}$ and $\rho \leq 0$. We have $\tilde{F}(\cdot) \in 2\text{RV}_{-\alpha, \gamma}$ with $\gamma = \max\{\beta, \rho\}$ and $\tilde{A}(\cdot) = A(\cdot) - \frac{\rho\alpha}{(c-b)(\alpha-\rho)} (\phi(\cdot) - b)$. In addition, as $t \rightarrow \infty$, we have*

$$\frac{\tilde{F}(t)}{\bar{F}(t)} = \left(1 - \frac{b}{c}\right) \left(1 - \frac{\alpha}{(c-b)(\alpha-\rho)} (\phi(t) - b) (1 + o(1))\right). \quad (\text{A.1})$$

Proof. Firstly, we need to prove that $\tilde{F}(\cdot) \in \text{RV}_{-\alpha}$. According to $\lim_{t \rightarrow \infty} \phi(t) = b$, for any $\epsilon > 0$, there exists $t_0 > 0$ such that

$$b - \epsilon \leq \phi(t) \leq b + \epsilon, \quad \forall t > t_0.$$

Fix any $x > 0$. we have

$$\lim_{t \rightarrow \infty} \frac{\widetilde{\overline{F}}(tx)}{\widetilde{\overline{F}}(t)} = \lim_{t \rightarrow \infty} \frac{\int_{tx}^{\infty} \left(\frac{\phi(y)-b}{c} \right) dF(y) + \left(1 - \frac{b}{c}\right) \overline{F}(tx)}{\int_t^{\infty} \left(\frac{\phi(y)-b}{c} \right) dF(y) + \left(1 - \frac{b}{c}\right) \overline{F}(t)} = x^{-\alpha}.$$

Secondly, according to $\overline{F} \in \text{RV}_{-\alpha}$, $\phi(\cdot) - b \in \text{RV}_{\rho}$ and Potter's inequality (Proposition B.1.9 (5) of [De Haan and Ferreira \(2006\)](#)), for any $\delta > 0$, there exists $t_1 > t_0$ such that for $t, ty > t_1$,

$$\left| \frac{\overline{F}(ty)}{\overline{F}(t)} - y^{-\alpha} \right| \leq \epsilon \max\{y^{-\alpha+\delta}, y^{-\alpha-\delta}\},$$

and

$$\left| \frac{\phi(ty) - b}{\phi(t) - b} - x^{\rho} \right| \leq \epsilon \max\{y^{\rho+\delta}, y^{\rho-\delta}\}.$$

For any $t > t_1$, by the dominated convergence theorem, we have

$$\begin{aligned} \frac{\widetilde{\overline{F}}(t)}{\widetilde{\overline{F}}(t)} &= \left(1 - \frac{b}{c}\right) - \int_t^{\infty} \frac{\phi(x) - b}{c\overline{F}(t)} dF(x) \\ &= \left(1 - \frac{b}{c}\right) - \left(1 + \int_1^{\infty} x^{-\alpha} dx^{\rho}\right) \frac{\phi(t) - b}{c} (1 + o(1)) \\ &= \left(1 - \frac{b}{c}\right) \left(1 - \frac{\alpha}{(c-b)(\alpha-\rho)} (\phi(t) - b) (1 + o(1))\right) \end{aligned}$$

By $\overline{F}(\cdot) \in 2\text{RV}_{-\alpha, \beta}$ and Drees-type inequality in [Mao \(2013\)](#), there exists $t_2 > t_1$ such that for all $y > 0$ and $t > \max\{t_2, \frac{t_2}{y}\}$,

$$\left| \frac{1}{A(t)} \left(\frac{\overline{F}(ty)}{\overline{F}(t)} - y^{-\alpha} \right) - H_{-\alpha, \beta}(y) \right| \leq \epsilon y^{-\alpha+\rho} \max\{y^{\delta}, y^{-\delta}\}.$$

For any $t > \max\{t_2, \frac{t_2}{x}\}$, according to the dominated convergence theorem, for all $x > 0$,

it follows that

$$\begin{aligned}
\frac{\widetilde{F}(tx)}{\widetilde{F}(t)} - x^{-\alpha} &= \frac{\left(1 - \frac{b}{c}\right) \overline{F}(tx) \left(1 - \frac{\alpha}{(c-b)(\alpha-\rho)} (\phi(tx) - b) (1 + o(1))\right)}{\left(1 - \frac{b}{c}\right) \overline{F}(t) \left(1 - \frac{\alpha}{(c-b)(\alpha-\rho)} (\phi(t) - b) (1 + o(1))\right)} - x^{-\alpha} \\
&= \left(x^{-\alpha} + x^{-\alpha} \frac{x^\beta - 1}{\beta} A(t) (1 + o(1))\right) \left(1 - \frac{\alpha(x^\rho - 1)}{(c-b)(\alpha-\rho)} (\phi(t) - b) (1 + o(1))\right) - x^{-\alpha} \\
&= x^{-\alpha} \frac{x^\beta - 1}{\beta} A(t) (1 + o(1)) - x^{-\alpha} \frac{x^\rho - 1}{\rho} \frac{\rho\alpha}{(c-b)(\alpha-\rho)} (\phi(t) - b) (1 + o(1)).
\end{aligned}$$

Thus, this ends the proof of Lemma A.1. ■

Proof of Proposition 3.4.1. Because of equation (2.1.1), for large enough $p \uparrow 1$ satisfying $\text{ex}_p(X) > 0$, we have

$$1 - \frac{\mathbb{E}(X)}{\text{ex}_p(X)} = \frac{2p-1}{1-p} \mathbb{E} \left(\left[\frac{X}{\text{ex}_p(X)} - 1 \right] \mathbf{1}_{\{X/\text{ex}_p(X) \geq 1\}} \right).$$

Applying integration by parts, we have

$$\begin{aligned}
\mathbb{E} \left(\left[\frac{X}{\text{ex}_p(X)} - 1 \right] \mathbf{1}_{\{X/\text{ex}_p(X) \geq 1\}} \right) &= \overline{F}(\text{ex}_p(X)) \left(\int_1^\infty x^{-\alpha} dx + \int_1^\infty \frac{\overline{F}(x \text{ex}_p(X))}{\overline{F}(\text{ex}_p(X))} - x^{-\alpha} dx \right) \\
&= \overline{F}(\text{ex}_p(X)) \left(\frac{1}{\alpha-1} + \int_1^\infty H_{-\alpha,\beta}(x) A(\text{ex}_p(X)) (1 + o(1)) dx \right) \\
&= \frac{\overline{F}(\text{ex}_p(X))}{\alpha-1} \left(1 + \frac{1}{\alpha-\beta-1} A(\text{ex}_p(X)) (1 + o(1)) \right),
\end{aligned}$$

where the third step is due to the dominated convergence theorem ensured by Theorem 2.3.9 of De Haan and Ferreira (2006). In particular, Bellini et al. (2014) shows that

$$\text{ex}_p(X) \sim (\alpha-1)^{-1/\alpha} F^{\leftarrow}(p), \quad p \uparrow 1.$$

Since $\text{ex}_p(X) \rightarrow \infty$, $1-p \downarrow 0$ and $A(\text{ex}_p(X)) \downarrow 0$ as $p \uparrow 1$, by the first-order Taylor

expansion, we have that

$$\begin{aligned}
\frac{1-p}{\overline{F}(\text{exp}_p(X))} &= \frac{1}{\alpha-1}(1-2(1-p)) \left(1 - \frac{\mu}{\text{exp}_p(X)}\right)^{-1} \left(1 + \frac{1}{\alpha-\beta-1}A(\text{exp}_p(X))(1+o(1))\right) \\
&= \frac{1}{\alpha-1}(1-2(1-p)) \left(1 + \frac{\mu}{\text{exp}_p(X)}(1+o(1))\right) \left(1 + \frac{1}{\alpha-\beta-1}A(\text{exp}_p(X))(1+o(1))\right) \\
&= \frac{1}{\alpha-1} \left(1 + \frac{(\alpha-1)^{-\beta/\alpha}}{\alpha-\beta-1}A(F^{\leftarrow}(p))(1+o(1)) + \frac{(\alpha-1)^{1/\alpha}\mu}{F^{\leftarrow}(p)}(1+o(1))\right),
\end{aligned}$$

where in the third step we use $1-p \sim \overline{F}(F^{\leftarrow}(p))$ as $p \uparrow 1$. Notably, in the second last step, we use $\lim_{p \uparrow 1} \overline{F}(F^{\leftarrow}(p))F^{\leftarrow}(p) = 0$, and thus $\overline{F}(F^{\leftarrow}(p)) = o(1/F^{\leftarrow}(p))$. In addition, due to the fact that $U_F(\cdot) \in \text{RV}_{1/\alpha, \beta/\alpha}$ with auxiliary function $\alpha^{-2}A \circ U_F(\cdot)$, it follows that

$$\begin{aligned}
\frac{\text{exp}_p(X)}{F^{\leftarrow}(p)} &= \left(\frac{1-p}{\overline{F}(\text{exp}_p(X))}\right)^{1/\alpha} \left(1 + \frac{\left(\frac{1-p}{\overline{F}(\text{exp}_p(X))}\right)^{\beta/\alpha} - 1}{\beta/\alpha} \alpha^{-2}A \circ U_F(1/(1-p))\right) \\
&= (\alpha-1)^{-1/\alpha} \left(1 + \frac{(\alpha-1)^{-\beta/\alpha}}{\alpha(\alpha-\beta-1)}A(F^{\leftarrow}(p))(1+o(1)) + \frac{(\alpha-1)^{1/\alpha}\mu}{\alpha F^{\leftarrow}(p)}(1+o(1))\right) \\
&\quad \times \left(1 + \frac{(\alpha-1)^{-\beta/\alpha} - 1}{\alpha\beta}A(F^{\leftarrow}(p))(1+o(1))\right) \\
&= (\alpha-1)^{-1/\alpha} \left(1 + \left(\frac{1}{\alpha\beta} \left(\frac{(\alpha-1)^{1-\beta/\alpha}}{\alpha-\beta-1} - 1\right)A(F^{\leftarrow}(p)) + \frac{(\alpha-1)^{1/\alpha}\mu}{\alpha F^{\leftarrow}(p)}\right)(1+o(1))\right).
\end{aligned}$$

Thus, we obtain the desired results.

Lemma A.2 *Let Y be the nonnegative rv with a distribution H satisfying that $\overline{H}(\cdot) \in 2\text{RV}_{-\alpha, \rho}$ with $\alpha > 1$, $\rho < 0$ and auxiliary function $A_H(\cdot)$. As $t \rightarrow \infty$, we have*

$$\mathbb{E}[Y|Y > t] = \frac{\alpha t}{\alpha-1} \left(1 + \frac{1}{\alpha(\alpha-\rho-1)}A_H(t)(1+o(1))\right).$$

Proof. By the dominated convergence theorem ensured by Theorem 2.3.9 of [De Haan and](#)

Ferreira (2006), it follows that, as $t \rightarrow \infty$,

$$\begin{aligned}
\mathbb{E}[Y|Y > t] &= \int_0^\infty \frac{\mathbb{P}(Y > z, Y > t)}{\mathbb{P}(Y > t)} dz \\
&= t \left(1 + \int_1^\infty z^{-\alpha} \left(1 + \frac{z^\rho - 1}{\rho} A_H(t)(1 + o(1)) \right) dz \right) \\
&= t \left(\frac{\alpha}{\alpha - 1} + \frac{1}{(\alpha - \rho - 1)(\alpha - 1)} A_H(t)(1 + o(1)) \right) \\
&= \frac{\alpha t}{\alpha - 1} \left(1 + \frac{1}{\alpha(\alpha - \rho - 1)} A_H(t)(1 + o(1)) \right).
\end{aligned}$$

Thus, we prove this lemma. ■

The next lemma extends Lemma 2.4 of Mao and Hu (2013).

Lemma A.3 *Let F be the distribution function of a nonnegative random variable satisfying $\bar{F}(\cdot) \in \text{RV}_{-\alpha}$ with $\alpha > 1$. For any fixed $z \in (0, 1)$ and $\beta > 0$, define*

$$V_\beta(zt) = \int_0^{zt} \left(\left(1 - \frac{y}{t} \right)^{-\beta} - 1 \right) dF(y), \quad t > 0.$$

Then, as $t \rightarrow \infty$, we have

$$V_\beta(zt) \sim \beta t^{-1} \mu(t).$$

Proof. Since $\alpha > 1$ and $\mu(t) \rightarrow \mu$ as $t \rightarrow \infty$, we have that $\mu(t) < \infty$ and $\frac{\mu(t)}{t} \in \text{RV}_{-1}$. We have

$$\mu(t) = \int_0^t x dF(x) = - \int_0^t x d\bar{F}(x) = -t\bar{F}(t) + \int_0^t \bar{F}(x) dx.$$

According to Karamata's theorem, it is easy to check that

$$\mu(t) \sim \int_0^t \bar{F}(x) dx \quad \text{as } t \rightarrow \infty, \quad \text{and} \quad \lim_{t \rightarrow \infty} \frac{t\bar{F}(t)}{\int_0^t \bar{F}(y) dy} = 0. \quad (\text{A.2})$$

By the integration by parts, it follows that

$$\begin{aligned} V_\beta(zt) &= - \int_0^{zt} \left(\left(1 - \frac{y}{t}\right)^{-\beta} - 1 \right) d\bar{F}(y) \\ &= -(1-z)^{-\beta} \bar{F}(zt) + \frac{\beta}{t} \int_0^{zt} \bar{F}(y) \left(1 - \frac{y}{t}\right)^{-\beta-1} dy. \end{aligned}$$

For any fixed $z \in (0, 1)$ and (A.2),

$$\lim_{t \rightarrow \infty} \frac{t\bar{F}(zt)}{\int_0^{zt} \bar{F}(y) \left(1 - \frac{y}{t}\right)^{-\beta-1} dy} \leq \lim_{t \rightarrow \infty} \frac{t\bar{F}(zt)}{\int_0^{zt} \bar{F}(y) dy} = 0.$$

Since (A.2) holds for all $\alpha > 1$ and $1 + (\beta + 1)x \leq (1 - x)^{-\beta-1} \leq 1 + (\beta + 1)(1 - z)^{-\beta-2}x$ for $x \in (0, z)$, we obtain

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{V_\beta(zt)}{\mu(zt)} &= \beta t^{-1} \lim_{t \rightarrow \infty} \frac{\int_0^{zt} \bar{F}(y) \left(1 - \frac{y}{t}\right)^{-\beta-1} dy}{\int_0^{zt} \bar{F}(y) dy} \\ &\geq \beta t^{-1} \lim_{t \rightarrow \infty} \frac{\int_0^{zt} \bar{F}(y) dy + (\beta + 1) \int_0^{zt} \bar{F}(y) y/t dy}{\int_0^{zt} \bar{F}(y) dy} \\ &= \beta t^{-1} \left(1 + (\beta + 1) \lim_{t \rightarrow \infty} \frac{zt\bar{F}(zt)}{\int_0^{zt} \bar{F}(y) dy + t\bar{F}(zt)} \right) \\ &= \beta t^{-1}, \end{aligned}$$

and

$$\lim_{t \rightarrow \infty} \frac{V_\alpha(zt)}{\mu(zt)} \leq \beta t^{-1} \left(1 + (\beta + 1)(1 - z)^{-\beta-2} \lim_{t \rightarrow \infty} \frac{\int_0^{zt} \bar{F}(y) y dy}{t \int_0^{zt} \bar{F}(y) dy} \right) = \beta t^{-1}.$$

By $\frac{\mu(t)}{t} \in \text{RV}_{-1}$, it follows that

$$V_\beta(zt) \sim \beta t^{-1} \mu(zt) \sim \beta t^{-1} \mu(t), \quad \text{as } t \rightarrow \infty.$$

This ends the proof. ■

Lemma A.4 *Under the conditions of Theorem 3.3.2, as $t \rightarrow \infty$, we have that*

$$\mathbb{E} [X_m | S_n > t] = \frac{\alpha t}{(\alpha - 1)n} \left(1 + \tilde{B}(t)(1 + o(1)) \right) + o \left(\sum_{i=1}^n |\phi_i(t) - d_i| \right),$$

where

$$\tilde{B}(t) = \frac{1}{\alpha(\alpha - \beta - 1)} A(t) - \frac{\mu_n^*(t)}{t}. \quad (\text{A.3})$$

Proof. It follows that, as $t \rightarrow \infty$,

$$\begin{aligned} \mathbb{E} [X_m | S_n > t] &= \int_0^\infty \frac{\mathbb{P}(S_n > t, X_m > z)}{\mathbb{P}(S_n > t)} dz \\ &= t \left(1 + \frac{\mathbb{P}(X > t)}{\mathbb{P}(S_n > t)} \left(\int_1^\infty \frac{\mathbb{P}(X_m > zt)}{\mathbb{P}(X > t)} dz - \int_0^1 \frac{\mathbb{P}(S_n > t, X_m \leq zt)}{\mathbb{P}(X > t)} dz \right) \right) \\ &:= t \left(1 + \frac{\mathbb{P}(X > t)}{\mathbb{P}(S_n > t)} (Q_1(t) - Q_2(t)) \right). \end{aligned}$$

For $Q_1(t)$, using the fact that $\bar{F}(\cdot) \in 2\text{RV}_{-\alpha, \beta}$ with $\alpha > 1$ $\beta < 0$ and the auxiliary function $A(t)$, as $t \rightarrow \infty$, we have

$$\begin{aligned} Q_1(t) &= \int_1^\infty z^{-\alpha} \left(1 + \frac{z^\beta - 1}{\beta} A(t)(1 + o(1)) \right) dz \\ &= \frac{1}{\alpha - 1} \left(1 + \frac{1}{\alpha - \beta - 1} A(t)(1 + o(1)) \right). \end{aligned}$$

For $t > 0$ and $z \in (0, 1)$, write $\Omega_{t,z} := \{(x_1, \dots, x_n) \in \mathbb{R}_+^n : \sum_{i=1}^n x_i > t, x_m \leq zt\}$. For $Q_2(t)$, the key idea is to connect $\mathbb{P}(S_n > t, X_m \leq zt)$. Similar to the proof of Proposition

3.3.1, we have that

$$\begin{aligned}
\mathbb{P}(S_n > t, X_m \leq zt) &= \int_{\Omega_{t,z}} \prod_{k=1}^n dF(x_k) + \sum_{1 \leq i < j \leq n} a_{ij} c_i c_j \left(\int_{\Omega_{t,z}} \prod_{k=1}^n dF(x_k) - \int_{\Omega_{t,z}} \prod_{k=1, k \neq i}^n dF(x_k) d\tilde{F}_i(x_i) \right. \\
&\quad \left. - \int_{\Omega_{t,z}} \prod_{k=1, k \neq j}^n dF(x_k) d\tilde{F}_j(x_j) + \int_{\Omega_{t,z}} \prod_{k=1, k \neq i, j}^n dF(x_k) d\tilde{F}_i(x_i) d\tilde{F}_j(x_j) \right) \\
&:= J(t, z) + \sum_{1 \leq i < j \leq n} a_{ij} c_i c_j (J(t, z) - K_i(t, z) - K_j(t, z) + K_{i,j}(t, z)).
\end{aligned} \tag{A.4}$$

For simplicity, denote $S_n^{(m)} = \sum_{i \neq m}^n X_i^*$ which has the distribution G_m . By Theorem 3.5 of [Mao and Hu \(2013\)](#) with $\alpha > 1$ and the induction assumption, it is easy to check that

$$\frac{\overline{G_m}(t)}{\overline{F}(t)} = (n-1) (1 + (n-2)\alpha t^{-1} \mu(t) (1 + o(1))) + o(A(t)).$$

Obviously, $\overline{G_m}(\cdot) \in 2\text{RV}_{-\alpha, \lambda}$ with auxiliary function $B(t)$, where $\lambda = -\min\{1, -\beta\}$ and $B(t)$ is given by

$$B(t) = A(t) - (n-2)\alpha t^{-1} \mu(t).$$

For $J(t, z)$, by the dominated convergence theorem, it follows that

$$\begin{aligned}
J(t, z) &= \int_0^{zt} \overline{G_m}(t-y) dF(y) \\
&= \overline{G_m}(t) \int_0^{zt} \left(1 - \frac{y}{t}\right)^{-\alpha} + H_{-\alpha, \lambda} \left(1 - \frac{y}{t}\right) B(t) (1 + o(1)) dF(y) \\
&:= \overline{G_m}(t) (J_1(t, z) + J_2(t, z)).
\end{aligned}$$

For $J_1(t, z)$, by Lemma A.3 and $\overline{F}(t) = o(\frac{\mu(t)}{t})$, we have that

$$\begin{aligned} J_1(t, z) &= \int_0^{zt} \left(\left(1 - \frac{y}{t}\right)^{-\alpha} - 1 \right) dF(y) + F(zt) \\ &= \alpha t^{-1} \mu(t) + 1 - \overline{F}(zt) \\ &= 1 + \alpha t^{-1} \mu(t) (1 + o(1)). \end{aligned}$$

For $J_2(t, z)$, since $H_{\alpha, \lambda} \left(1 - \frac{y}{t}\right) \leq \frac{(1-z)^{-\alpha}}{|\lambda|} \left(\left(1 - \frac{y}{t}\right)^\lambda - 1 \right)$ for any $y \in (0, zt)$ and $z \in (0, 1)$ and Lemma 5.6 in Barbe and McCormick (2005), we have

$$\int_0^{zt} H_{\alpha, \lambda} \left(1 - \frac{y}{t}\right) dF(y) \leq \int_0^{zt} \frac{(1-z)^{-\alpha}}{|\lambda|} \left(\left(1 - \frac{y}{t}\right)^\lambda - 1 \right) dF(y) = 0.$$

Then,

$$J_2(t, z) = o(B(t)).$$

Thus, according to $\overline{G}_m(t)$, $J_1(t, z)$ and $J_2(t, z)$, it follows that

$$\begin{aligned} \frac{J(t, z)}{\overline{F}(t)} &= \frac{\overline{G}_m(t)}{\overline{F}(t)} (1 + \alpha t^{-1} \mu(t) (1 + o(1)) + o(B(t))) \\ &= (n-1) (1 + (n-1) \alpha t^{-1} \mu(t) (1 + o(1)) + o(|A(t)|)). \end{aligned}$$

Similarly, it is easy to see that

$$\begin{aligned} \frac{K_i(t, z)}{\overline{F}(t)} &= \left(n - 1 - \frac{d_i}{c_i} \right) (1 + \alpha t^{-1} (n-1) \mu(t) (1 + o(1))) + o(|A(t)|) \\ &\quad - \left(\frac{\alpha(n-1) \mu_i(t)}{c_i t} + \frac{\alpha(\phi_i(t) - d_i)}{c_i(\alpha - \rho_i)} \right) (1 + o(1)), \end{aligned}$$

and

$$\begin{aligned} \frac{K_{i,m}(t, z)}{\bar{F}(t)} &= \left(n - 1 - \frac{d_i}{c_i} - \frac{d_m}{c_m} \right) (1 + \alpha t^{-1}(n-1)\mu(t)(1 + o(1))) + o(|A(t)|) \\ &\quad - \left(\frac{\alpha(n-1)\mu_i(t)}{c_i t} + \frac{\alpha(\phi_i(t) - d_i)}{c_i(\alpha - \rho_i)} + \frac{\alpha(n-1)\mu_m(t)}{c_m t} + \frac{\alpha(\phi_m(t) - d_m)}{c_m(\alpha - \rho_m)} \right) (1 + o(1)). \end{aligned}$$

Pulling $J(t, z)$, $K_i(t, z)$, $K_m(t, z)$ and $K_{i,m}(t, z)$ into (A.4) yields that

$$\frac{\mathbb{P}(S_n > t, X_m \leq zt)}{\bar{F}(t)} = (n-1)(1 + \alpha t^{-1}\mu_n^*(t)(1 + o(1))) + o\left(|A(t)| + \sum_{i=1}^n |\phi_i(t) - d_i|\right).$$

Thus, we have that

$$Q_2(t) = \int_0^1 \frac{\mathbb{P}(S_n > t, X_m \leq zt)}{\bar{F}(t)} dz = (n-1)(1 + \alpha t^{-1}\mu_n^*(t)) + o\left(|A(t)| + \sum_{i=1}^n |\phi_i(t) - d_i|\right).$$

According to the dominated convergence theorem, it follows that as $t \rightarrow \infty$,

$$\begin{aligned} \mathbb{E}[X_m | S_n > t] &= t \left(1 + \left(n \left(1 + \tilde{A}_n(t)(1 + o(1)) \right) \right)^{-1} (Q_1(t) - Q_2(t)) \right) \\ &= t \left(1 + \frac{1}{n} \left(1 - \alpha t^{-1}\mu_n^*(t)(1 + o(1)) + o\left(|A(t)| + \sum_{i=1}^n |\phi_i(t) - d_i|\right) \right) \right) \\ &\quad \cdot \left(\frac{1}{\alpha - 1} \left(1 + \frac{1}{\alpha - \beta - 1} A(t)(1 + o(1)) \right) - (n-1)(1 + \alpha t^{-1}\mu_n^*(t) + o(|A(t)|)) \right) \\ &= \frac{\alpha t}{(\alpha - 1)n} \left(1 + \left(\frac{1}{\alpha(\alpha - \beta - 1)} A(t) - \frac{\mu_n^*(t)}{t} \right) (1 + o(1)) \right) + o\left(\sum_{i=1}^n |\phi_i(t) - d_i|\right). \end{aligned}$$

This completes the proof of the lemma. ■

Lemma A.5 *Under the conditions of Theorem 3.3.2, there exist some large enough t_1 such that for any $t \geq t_1$,*

$$\mathbb{E}\left[(X_m - t_1)_+ | S_n > t\right] = \frac{\alpha t}{(\alpha - 1)n} \left(1 + \tilde{B}(t)(1 + o(1)) \right) - \frac{t_1}{n} + o\left(\sum_{i=1}^n |\phi_i(t) - d_i|\right),$$

where $\tilde{B}(t)$ is defined in (A.3).

Proof. This proof proceeds along similar lines as in the proof of Proposition 3.3.1. Applying the integration by part, we conclude that

$$\begin{aligned}\mathbb{E} \left[(X_m - t_1)_+ \middle| S_n > t \right] &= t \int_{\frac{t_1}{t}}^{\infty} \frac{\mathbb{P}(X_m > zt, S > t)}{\mathbb{P}(S > t)} dz \\ &= t \left(1 - \frac{t_1}{t} + \frac{\mathbb{P}(X > t)}{\mathbb{P}(S_n > t)} \left(\int_1^{\infty} \frac{\mathbb{P}(X_m > zt)}{\mathbb{P}(X > t)} dz - \int_{\frac{t_1}{t}}^1 \frac{\mathbb{P}(S_n > t, X_m \leq zt)}{\mathbb{P}(X > t)} dz \right) \right).\end{aligned}$$

Following a similar analysis of $Q_2(t)$ of Proposition 3.3.1, we have that

$$\int_{\frac{t_1}{t}}^1 \frac{\mathbb{P}(S_n > t, X_m \leq zt)}{\mathbb{P}(X > t)} dz = (1 - t_1/t) (n - 1) (1 + \alpha t^{-1} \mu_n^*(t)) + o \left(A(t) + \sum_{i=1}^n (\phi_i(t) - d_i) \right).$$

Thus, by the Taylor expansion, as $t \rightarrow \infty$, we have

$$\begin{aligned}&\mathbb{E} \left[(X_m - t_1)_+ \middle| S_n > t \right] \\ &= t \left(1 - \frac{t_1}{t} + \left(n \left(1 + \tilde{A}_n(t)(1 + o(1)) \right) \right)^{-1} \left(\frac{1}{\alpha - 1} \left(1 + \frac{1}{\alpha - \beta - 1} A(t) \right) \right. \right. \\ &\quad \left. \left. - (1 - t_1/t) (n - 1) \left(1 + \alpha t^{-1} \mu_n^*(t) + o \left(A(t) + \sum_{i=1}^n (\phi_i(t) - d_i) \right) \right) \right) \right) \\ &= \frac{\alpha t}{(\alpha - 1)n} \left(1 + \left(\frac{1}{\alpha(\alpha - \beta - 1)} A(t) - \frac{\mu_n^*(t)}{t} \right) (1 + o(1)) \right) - \frac{t_1}{n} + o \left(\sum_{i=1}^n (\phi_i(t) - d_i) \right).\end{aligned}$$

We complete this proof of Lemma. ■

Proof of Theorem 3.6.1. By Theorem 3.3.1 and (6.1), as $p \uparrow 1$, based on VaR, we obtain

$$\begin{aligned} D_p^{\text{VaR}}(S_n) &= 1 - \frac{\text{VaR}_p(S_n) - n\mu}{nF^{\leftarrow}(p)} \left(1 - \frac{\mu}{F^{\leftarrow}(p)}\right)^{-1} \\ &= 1 - n^{1/\alpha-1} \left(1 + \left(\frac{\mu_n^*(F^{\leftarrow}(p))}{n^{1/\alpha}F^{\leftarrow}(p)} + \frac{n^{\beta/\alpha} - 1}{\alpha\beta} A(F^{\leftarrow}(p))\right) (1 + o(1))\right) \\ &\quad + \frac{(1 - n^{1/\alpha-1})\mu}{F^{\leftarrow}(p)} (1 + o(1)) + o\left(\sum_{i=1}^n (\phi_i(t) - d_i)\right). \end{aligned}$$

According to Proposition 3.4.1 and Theorem 3.4.1, as $p \uparrow 1$, we have

$$\begin{aligned} D_p^{\text{ex}}(S_n) &= 1 - \frac{\text{exp}_p(S_n)/n - \mu}{\text{exp}_p(X) - \mu} \\ &= 1 - n^{1/\alpha-1} \left(1 + \frac{(n^{\beta/\alpha} - 1)(\alpha - 1)^{1-\beta/\alpha}}{\alpha\beta(\alpha - \beta - 1)} A(F^{\leftarrow}(p)) (1 + o(1))\right) \\ &\quad - \frac{(\alpha - 1)^{1/\alpha+1} \mu_n^*(F^{\leftarrow}(p))}{\alpha n F^{\leftarrow}(p)} (1 + o(1)) + o\left(\sum_{i=1}^n (\phi_i(t) - d_i)\right). \end{aligned}$$

Based on CTE and Lemma A.2, as $p \uparrow 1$, we have

$$\begin{aligned} D_p^{\text{CTE}}(S_n) &= 1 - \frac{\text{CTE}_p(S_n) - n\mu}{\frac{n\alpha F^{\leftarrow}(p)}{\alpha-1} \left(1 + \frac{1}{\alpha(\alpha-\beta-1)} A(F^{\leftarrow}(p)) - \frac{(\alpha-1)\mu}{\alpha F^{\leftarrow}(p)}\right)} \\ &= 1 - n^{1/\alpha-1} \left(1 + \left(\zeta_{\alpha,\beta}^n - \frac{1}{\alpha(\alpha-\beta-1)}\right) A(F^{\leftarrow}(p)) (1 + o(1))\right) + o\left(\sum_{i=1}^n (\phi_i(t) - d_i)\right) \\ &\quad - \left(\frac{(\alpha-1)\mu_n^*(F^{\leftarrow}(p)) - n\mu}{n\alpha F^{\leftarrow}(p)} + \frac{(\alpha-1)n^{1/\alpha-1}\mu}{\alpha F^{\leftarrow}(p)}\right) (1 + o(1)) \\ &= 1 - n^{1/\alpha-1} \left(1 + \frac{1}{\alpha\beta} \left(\frac{n^{\beta/\alpha}(\alpha-1) - \beta}{\alpha - \beta - 1} - 1\right) A(F^{\leftarrow}(p)) (1 + o(1))\right) \\ &\quad - \frac{(\alpha-1)(\mu_n^*(F^{\leftarrow}(p)) - (n - n^{1/\alpha})\mu)}{n\alpha F^{\leftarrow}(p)} (1 + o(1)) + o\left(\sum_{i=1}^n (\phi_i(t) - d_i)\right). \end{aligned}$$

In addition, applying Lemma A.2, it follows that

$$\begin{aligned}
D_p^{\text{CE}}(S_n) &= 1 - \frac{\text{CE}_p(S_n) - n\mu}{\frac{n\alpha \text{ex}_p(X)}{\alpha-1} \left(1 + \frac{1}{\alpha(\alpha-\beta-1)} A(\text{ex}_p(X))\right) - n\mu} \\
&= 1 - \frac{\text{CE}_p(S_n)/n - \mu}{\frac{\alpha \text{ex}_p(X)}{\alpha-1} \left(1 + \frac{(\alpha-1)^{-\beta/\alpha}}{\alpha(\alpha-\beta-1)} A(F^{\leftarrow}(p)) (1 + o(1))\right) - \mu} \\
&\quad - \frac{(\alpha-1)^{1/\alpha}(\alpha-2)\mu_n^*(F^{\leftarrow}(p))}{\alpha n F^{\leftarrow}(p)} (1 + o(1)) + o\left(\sum_{i=1}^n (\phi_i(t) - d_i)\right).
\end{aligned}$$

Thus, this completes the proof of Theorem 3.6.1.

Appendix B

Catastrophe Risk Pooling

Proof of Lemma 4.2.1 . By the similar argument in the proof of Theorem 3.1 of [Cui et al. \(2021\)](#), we can show that if $0 \leq \xi \leq 1$, then there exists $0 < q_0 < 1$ such that for all $q_0 < q < 1$, $\text{DR}_q(\mathbf{X}(k)) \leq 1$. It then follows that $\lim_{q \rightarrow 1} \text{DR}_q(\mathbf{X}(k)) \leq 1$. We are left to show the case of $\xi > 1$.

Let $t = \text{VaR}_q(X_1)$. To study $\text{DR}_q(\mathbf{X}(k)) \leq 1$, it is sufficient to analyze as $t \rightarrow \infty$,

$$\frac{\Pr(X_1(k) + \dots + X_n(k) > nt)}{\Pr(X_1(k) > t)} \leq 1 \quad (\text{B.1})$$

First we denote the index set $I = \{2, 3, \dots, n\}$. Let $A \subseteq I$ with the cardinality denoted by $|A|$. Let $x_I = \{x_1, \dots, x_{n-1}\}$ and $x_{I \setminus A} = \{x_2, \dots, x_n\} \setminus \{x_i, i \in A\}$. To simplify the notation, we denote

$$\Delta_{n-1} = \{x_i, i \in I : 0 \leq x_i \leq k\},$$

which is a unique set, and for any set A with $|A| = j$,

$$\Delta_{n-j} = \{x_i, i \in I \setminus A : 0 \leq x_i \leq k\},$$

which in fact represents C_{n-1}^{n-1-j} different sets. We rewrite

$$\begin{aligned}
& \Pr(X_1(k) + \dots + X_n(k) > nt) \\
&= \int_{\Delta_{n-1}} \Pr\left(X_1(k) > nt - \sum_{i=2}^n y_i\right) d^{n-1}F_i(y_i) \\
&+ \sum_{j_1=2}^n \int_{\Delta_{n-2}} \Pr\left(X_1(k) > nt - \sum_{i=2, i \neq j_1}^n y_i - k\right) \bar{F}_{j_1}(k) d_{\setminus j_1}^{n-2} F_i(y_i) \\
&+ \dots \\
&+ \sum_{j_1, \dots, j_l=2, j_1 \neq \dots \neq j_l}^n \int_{\Delta_{n-l-1}} \Pr\left(X_1(k) > nt - \sum_{i=2, i \neq j_1, \dots, j_l}^n y_i - lk\right) \bar{F}_{j_1}(k) \dots \bar{F}_{j_l}(k) d_{\setminus j_1, \dots, j_l}^{n-l-1} F_i(y_i) \\
&\dots + \Pr(X_1(k) > nt - (n-1)k) \bar{F}_2(k) \dots \bar{F}_n(k), \tag{B.2}
\end{aligned}$$

where $d^{n-1}F_i(y_i) = \prod_{i=2}^n dF_i(y_i)$, and $d_{\setminus j_1, \dots, j_l}^{n-l-1} F_i(y_i) = \prod_{i=2, i \neq j_1, \dots, j_l}^n dF_i(y_i)$ for $l = 1, 2, \dots, n-2$.

In total there are $C_{n-1}^0 + C_{n-1}^1 + \dots + C_{n-1}^{n-1} = 2^{n-1}$ terms on the right-hand side of (B.2). Denote the $(l+1)$ th term by I_l , $l = 0, 1, \dots, 2^{n-1} - 1$. Next we analyze these terms by different range of ξ .

(a) $1 < \xi < \frac{n}{n-1}$. That is there is $t_0 > 0$ such that for all $t > t_0$, $(n-1)k < nt < nk$. Note that

$$\frac{I_0}{\bar{F}_1(t)} = \int_{\{nt-k \leq \sum_{i=2}^n y_i\} \cap \Delta_{n-1}} \frac{\bar{F}_1(nt - \sum_{i=2}^n y_i)}{\bar{F}_1(t)} d^{n-1}F_i(y_i).$$

and $\sum_{i=2}^n y_i \leq (n-1)k$. We have

$$\begin{aligned}
\lim_{t \rightarrow \infty} \frac{I_0}{\bar{F}_1(t)} &\leq \lim_{t \rightarrow \infty} \frac{\bar{F}_1(nt - (n-1)k)}{\bar{F}_1(t)} \int_{\{nt-k \leq \sum_{i=2}^n y_i\} \cap \Delta_{n-1}} d^{n-1}F_i(y_i) \\
&= (n - (n-1)\xi)^{-\alpha} \lim_{t \rightarrow \infty} \int_{\{nt-k \leq \sum_{i=2}^n y_i\} \cap \Delta_{n-1}} d^{n-1}F_i(y_i) \\
&= 0, \tag{B.3}
\end{aligned}$$

where in the last step we used the fact that $nt - k \rightarrow \infty$ as $t \rightarrow \infty$ for $n \geq 2$. Similarly,

for $l = 1, \dots, 2^{n-1} - 2$,

$$\lim_{t \rightarrow \infty} \frac{I_l}{\overline{F}_1(t)} = 0.$$

Now for the last term, since

$$\lim_{t \rightarrow \infty} \frac{\Pr(X_1(k) > nt - (n-1)k)}{\overline{F}_1(t)} = (n - (n-1)\xi)^{-\alpha_1} < \infty,$$

we have $\lim_{t \rightarrow \infty} \frac{I_{n-1}}{\overline{F}_1(t)} = 0$ as well. Combining all the terms, we obtain $\lim_{q \rightarrow 1} \text{DR}_q(\mathbf{X}(k)) < 1$ with no further restriction on α_1 .

(b) $\frac{n}{n-1} < \xi < \frac{n}{n-2}$. That is there is $t_0 > 0$ such that for all $t > t_0$, $(n-2)k < nt < (n-1)k$. In this case, it requires α_1 to satisfy $\alpha_1 < \alpha_{(n)}$. For the first term,

$$\begin{aligned} I_0 &= \int_{\{nt-k \leq \sum_{i=2}^n y_i \leq nt\} \cap \Delta_{n-1}} \overline{F}_1 \left(nt - \sum_{i=2}^n y_i \right) d^{n-1} F_i(y_i) + \int_{\{nt \leq \sum_{i=2}^n y_i\} \cap \Delta_{n-1}} 1 d^{n-1} F_i(y_i) \\ &= \int_{\{nt-k \leq \sum_{i=2}^n y_i \leq (n-\varepsilon)t\} \cap \Delta_{n-1}} \overline{F}_1 \left(nt - \sum_{i=2}^n y_i \right) d^{n-1} F_i(y_i) \\ &\quad + \int_{\{(n-\varepsilon)t \leq \sum_{i=2}^n y_i \leq nt\} \cap \Delta_{n-1}} \overline{F}_1 \left(nt - \sum_{i=2}^n y_i \right) d^{n-1} F_i(y_i) \\ &\quad + \int_{\{nt \leq \sum_{i=2}^n y_i\} \cap \Delta_{n-1}} 1 d^{n-1} F_i(y_i) \\ &=: J_1^{(0)} + J_2^{(0)} + J_3^{(0)}. \end{aligned} \tag{B.4}$$

Let ε be chosen so that $0 < \varepsilon < n - (n-2)\xi$. Similarly as in (B.3),

$$\lim_{t \rightarrow \infty} \frac{J_1^{(0)}}{\overline{F}_1(t)} \leq \lim_{t \rightarrow \infty} \frac{\overline{F}_1(\varepsilon t)}{\overline{F}_1(t)} \int_{\{nt-k \leq \sum_{i=2}^n y_i \leq (n-\varepsilon)t\} \cap \Delta_{n-1}} d^{n-1} F_i(y_i) = 0.$$

For $J_2^{(0)}$, note that

$$\begin{aligned}
J_2^{(0)} &\leq \int_{\{(n-\varepsilon)t \leq \sum_{i=2}^n y_i \leq nt\}} d^{n-1} F_i(y_i) \\
&= \Pr \left((n-\varepsilon)t \leq \sum_{i=2}^n X_i \leq nt \right) \\
&\leq \bar{F}_{(n)}((n-\varepsilon)t - (n-2)k).
\end{aligned}$$

Since $0 < \varepsilon < n - (n-2)\xi$, we have $(n-\varepsilon)t - (n-2)k \rightarrow \infty$ as $t \rightarrow \infty$. Then

$$\begin{aligned}
\lim_{t \rightarrow \infty} \frac{J_2^{(0)}}{\bar{F}_1(t)} &= \lim_{t \rightarrow \infty} \frac{J_2^{(0)}}{\bar{F}_{(n)}(t)} \frac{\bar{F}_{(n)}(t)}{\bar{F}_1(t)} \\
&\leq \lim_{t \rightarrow \infty} \frac{\bar{F}_{(n)}((n-\varepsilon)t - (n-2)k)}{\bar{F}_{(n)}(t)} \frac{\bar{F}_{(n)}(t)}{\bar{F}_1(t)} \\
&= 0,
\end{aligned}$$

where the last step is due to that $\alpha_1 < \alpha_{(n)}$. Then, $J_3^{(0)}$ is $\Pr(X_2 + \dots + X_n > nt, X_2 \leq k, \dots, X_n \leq k)$, which can be analyzed similar to I_0 in (B.3) for case (a). That is for $(n-2)k < nt < (n-1)k$,

$$\lim_{t \rightarrow \infty} \frac{J_3^{(0)}}{\bar{F}_{(n)}(t)} = \lim_{t \rightarrow \infty} \int_{\{nt-k \leq \sum_{i=2, i \neq (n)}^n y_i\} \cap \Delta_{n-2}} \frac{\bar{F}_{(n)}(nt - \sum_{i=2, i \neq (n)}^n y_i)}{\bar{F}_{(n)}(t)} d_{\setminus(n)}^{n-2} F_i(y_i) = 0.$$

Thus, if $\alpha_1 < \alpha_{(n)}$, then $\lim_{t \rightarrow \infty} \frac{J_3^{(0)}}{\bar{F}_1(t)} = \lim_{t \rightarrow \infty} \frac{J_3^{(0)}}{\bar{F}_{(n)}(t)} \frac{\bar{F}_{(n)}(t)}{\bar{F}_1(t)} = 0$. Combining the terms $J_1^{(0)}$, $J_2^{(0)}$ and $J_3^{(0)}$, it leads to $\lim_{t \rightarrow \infty} \frac{I_0}{\bar{F}_1(t)} = 0$.

Next we turn to term I_l , $l = 1, 2, 3, \dots, 2^{n-1} - 2$,

I_l

$$\begin{aligned}
&= \sum_{\substack{j_1, \dots, j_l=2 \\ j_1 \neq \dots \neq j_l}}^n \int \left\{ \sum_{\substack{i=2 \\ i \neq j_1, \dots, j_l}}^n y_i \leq nt - lk \right\} \cap \Delta_{n-l-1} \Pr \left(X_1(k) > nt - \sum_{\substack{i=2 \\ i \neq j_1, \dots, j_l}}^n y_i - lk \right) \bar{F}_{j_1}(k) \cdots \bar{F}_{j_l}(k) d_{\setminus j_1, \dots, j_l}^{n-l-1} F_i(y_i) \\
&+ \sum_{\substack{j_1, \dots, j_l=2 \\ j_1 \neq \dots \neq j_l}}^n \int \left\{ \sum_{\substack{i=2 \\ i \neq j_1, \dots, j_l}}^n y_i > nt - lk \right\} \cap \Delta_{n-l-1} \bar{F}_{j_1}(k) \cdots \bar{F}_{j_l}(k) d_{\setminus j_1, \dots, j_l}^{n-l-1} F_i(y_i) \\
&:= J_1^{(l)} + J_2^{(l)}.
\end{aligned}$$

For $J_1^{(l)}$, if $\bar{F}_{(n)}(k)$ is one of $\bar{F}_{j_1}(k), \dots, \bar{F}_{j_l}(k)$, then noting that $\lim_{t \rightarrow \infty} \frac{\bar{F}_{(n)}(k)}{\bar{F}_1(t)} = 0$ and

$$\int \left\{ \sum_{\substack{i=2 \\ i \neq j_1, \dots, j_l}}^n y_i \leq nt - lk \right\} \cap \Delta_{n-l-1} \Pr \left(X_1(k) > nt - \sum_{\substack{i=2 \\ i \neq j_1, \dots, j_l}}^n y_i - lk \right) d_{\setminus j_1, \dots, j_l}^{n-l-1} F_i(y_i) < \infty,$$

we have $\lim_{t \rightarrow \infty} \frac{J_1^{(l)}}{\bar{F}_1(t)} = 0$. If $\bar{F}_{(n)}(k)$ is not one of $\bar{F}_{j_1}(k), \dots, \bar{F}_{j_l}(k)$, then the analysis is similar to that for $J_1^{(0)} + J_2^{(0)}$ in I_0 above. Thus, again we have $\lim_{t \rightarrow \infty} \frac{J_1^{(l)}}{\bar{F}_1(t)} = 0$. For $J_2^{(l)}$, it is essentially $\sum_{j_1, \dots, j_l=2, j_1 \neq \dots \neq j_l}^n \Pr \left(\sum_{i=2, i \neq j_1, \dots, j_l}^n X_i(k) > nt, X_{j_1} < k, \dots, X_{j_l} < k \right) \bar{F}_{j_1}(k) \cdots \bar{F}_{j_l}(k)$, which follows the analysis of $J_3^{(0)}$ in I_0 above. Thus, if $\alpha_1 < \alpha_{(n)}$, then $\frac{I_l}{\bar{F}_1(t)} \rightarrow 0$.

For the last term, as $t \rightarrow \infty$,

$$\frac{I_{n-1}}{\bar{F}_1(t)} = \Pr(X_1(k) > nt - (n-1)k) \frac{\prod_{j=2}^n \bar{F}_j(k)}{\bar{F}_{(n)}(t)} \frac{\bar{F}_{(n)}(t)}{\bar{F}_1(t)} \rightarrow 0.$$

Combining all the above, $\lim_{q \rightarrow 1} \text{DR}_q(\mathbf{X}(k)) < 1$.

(c) $\frac{n}{n-s+1} < \xi < \frac{n}{n-s}$ with $s = 3, \dots, n-1$. That is there is $t_0 > 0$ such that for all $t > t_0$, $(n-s)k < nt < (n-s+1)k$. In this case, we require that $\alpha_1 < \alpha_{(n-s+2)}$. The analysis is similar to that in (b) by repeatedly splitting the lightest risk out until the number of

random variables is reduced to $n - s + 2$. In this process, if $\alpha_1 < \alpha_{(n-s+2)}$, then each term can be shown to be $o(\bar{F}_1(t))$. Then it leads to $\lim_{q \rightarrow 1} \text{DR}_q(\mathbf{X}(k)) < 1$.

(d) Lastly, we turn to the case $\xi > n$. By the similar arguments of Theorem 3.1(ii) in Cui et al. (2021), we have that

$$\lim_{t \rightarrow \infty} \frac{\Pr(S_n(k) > t)}{\Pr(X_1(k) > t)} = \lim_{t \rightarrow \infty} \frac{\Pr(S_n > t)}{\Pr(X_1 > t)} = \lim_{t \rightarrow \infty} \frac{\sum_{i=1}^n \bar{F}_i(nt)}{\bar{F}_1(t)} = n^{-\alpha} < 1,$$

provided that $\alpha_1 < \alpha_{(2)}$. The second equality above is due to Feller's theorem; see page 278 of Feller (1991).

Combining the above analysis, the desired results follow. ■

Proof of Theorem 4.2.1 . Note that $X_{(n)}$ has the lightest tail in the pool and $\alpha_{(n)} = \max\{\alpha_1, \alpha_2, \dots, \alpha_n\}$. Then for any i with $\alpha_i < \alpha_{(n)}$, we have

$$\lim_{q \rightarrow 1} \frac{k(q)}{\text{VaR}_q(X_i)} = \lim_{q \rightarrow 1} \frac{k(q)}{\text{VaR}_q(X_{(n)})} \frac{\text{VaR}_q(X_{(n)})}{\text{VaR}_q(X_i)} = \xi \cdot 0 = 0.$$

By Lemma 4.2.1 (i), we have $\lim_{q \rightarrow 1} \text{DR}_q^{(i)}(\mathbf{X}(k)) \leq 1$. ■

Proof of Lemma 4.3.1. (i) If $0 < \xi < 1$, then there exists $0 < q_0 < 1$ such that for all $q_0 < q < 1$, $\text{VaR}_q(X_1(k)) = k$. Note that $\text{VaR}_q(S_n(k)) \leq k$ for any $0 < q < 1$. Thus

$$\lim_{q \uparrow 1} \text{DR}_q(\mathbf{X}(k)) \leq 1.$$

When $1 < \xi < n$, consider the decomposition

$$\begin{aligned} \Pr(S_n(k) > t) &= \Pr(S_n(k) > t, \text{none of } X_i\text{'s} > k) + \Pr(S_n(k) > t, \text{at least one of } X_i\text{'s} > k) \\ &= \Pr(S_n > t) - \Pr\left(S_n > t, \bigcup_{i=1}^n \{X_i > k\}\right) + \Pr\left(S_n(k) > t, \bigcup_{i=1}^n \{X_i > k\}\right) \\ &:= I_1 - I_2 + I_3. \end{aligned} \tag{B.5}$$

Next we analyze each term separately.

For I_1 , since X_i 's are independent and $\bar{F}_i \in \text{RV}_{-\alpha}$, by Feller's Theorem (see page 278 of [Feller, 1991](#)) we have

$$\lim_{t \rightarrow \infty} \frac{I_1}{\bar{F}_1(t)} = \lim_{t \rightarrow \infty} \frac{\Pr(\sum_{i=1}^n X_i > nt)}{\bar{F}_1(t)} = \lim_{t \rightarrow \infty} \frac{\sum_{i=1}^n \Pr(X_i > nt)}{\bar{F}_1(t)} = n^{-\alpha} \sum_{i=1}^n \theta_i. \quad (\text{B.6})$$

For I_2 , on one hand

$$I_2 \leq \sum_{i=1}^n \Pr(S_n > t, X_i > k).$$

On the other hand,

$$I_2 \geq \sum_{i=1}^n \Pr(S_n > t, X_i > k) - \sum_{i \neq j} \Pr(S_n > t, X_i > k, X_j > k).$$

Since X_i 's are independent and tail equivalent, we have

$$\lim_{t \rightarrow \infty} \frac{\sum_{i \neq j} \Pr(S_n > t, X_i > k, X_j > k)}{\bar{F}_1(t)} \leq \lim_{t \rightarrow \infty} \frac{\sum_{i \neq j} \Pr(X_i > k, X_j > k)}{\bar{F}_1(t)} = 0.$$

Thus,

$$\lim_{t \rightarrow \infty} \frac{I_2}{\bar{F}_1(t)} = \lim_{t \rightarrow \infty} \sum_{i=1}^n \frac{\Pr(S_n > t, X_i > k)}{\bar{F}_1(t)}. \quad (\text{B.7})$$

Note that for $\xi < n$, which is equivalent to that there is some $t_0 > 0$ such that for all $t > t_0$ we have $k < nt$,

$$\begin{aligned} \Pr(S_n > t, X_i > k) &= \int_k^\infty \Pr\left(\sum_{j \neq i} X_j > nt - y_i\right) dF_i(y_i) \\ &= \left(\int_k^{(n-\varepsilon)t} + \int_{(n-\varepsilon)t}^{nt}\right) \Pr\left(\sum_{j \neq i} X_j > nt - y_i\right) dF_i(y_i) + \int_{nt}^\infty dF_i(y_i) \\ &:= I_{21} + I_{22} + I_{23}. \end{aligned} \quad (\text{B.8})$$

First, we have

$$\lim_{t \rightarrow \infty} \frac{I_{21}}{\overline{F}_1(t)} \leq \lim_{t \rightarrow \infty} \Pr \left(\sum_{j \neq i} X_j > \varepsilon t \right) \frac{(\overline{F}_i(k) - \overline{F}_i((n - \varepsilon)t))}{\overline{F}_1(t)} = 0.$$

By the arbitrariness of ε ,

$$\lim_{t \rightarrow \infty} \frac{I_{22}}{\overline{F}_1(t)} \leq \lim_{t \rightarrow \infty} \frac{(\overline{F}_i((n - \varepsilon)t) - \overline{F}_i(nt))}{\overline{F}_1(t)} = ((n - \varepsilon)^{-\alpha} - n^{-\alpha}) \theta_i \rightarrow 0.$$

Lastly,

$$\lim_{t \rightarrow \infty} \frac{I_{23}}{\overline{F}_1(t)} = \lim_{t \rightarrow \infty} \frac{\overline{F}_i(nt)}{\overline{F}_1(t)} = n^{-\alpha} \theta_i.$$

Combining I_{21} , I_{22} and I_{23} leads to

$$\lim_{t \rightarrow \infty} \frac{I_2}{\overline{F}_1(t)} = n^{-\alpha} \sum_{i=1}^n \theta_i.$$

Now we turn to I_3 . Following similar arguments for I_2 , we have

$$\lim_{t \rightarrow \infty} \frac{I_3}{\overline{F}_1(t)} = \lim_{t \rightarrow \infty} \sum_{i=1}^n \frac{\Pr(S_n(k) > t, X_i > k)}{\overline{F}_1(t)}. \quad (\text{B.9})$$

Note that $nt - k = (n - \frac{k}{t})t \rightarrow \infty$ as $t \rightarrow \infty$ for $\xi < n$, we have $\Pr \left(\sum_{j \neq i} X_j(k_j) > nt - k \right) \rightarrow 0$ as $t \rightarrow \infty$. Then

$$\frac{\Pr(S_n(k) > t, X_i > k)}{\overline{F}_1(t)} = \Pr \left(\sum_{j \neq i} X_j(k) > nt - k \right) \frac{\overline{F}_i(k) \overline{F}_i(t)}{\overline{F}_i(t) \overline{F}_1(t)} \rightarrow 0.$$

Thus, $\lim_{t \rightarrow \infty} \frac{I_3}{\overline{F}_1(t)} = 0$.

Combining I_1 , I_2 and I_3 yields

$$\lim_{t \rightarrow \infty} \frac{\Pr(S_n(k) > t)}{\overline{F}_1(t)} = n^{-\alpha} \sum_{i=1}^n \theta_i - n^{-\alpha} \sum_{i \in Z^c} \theta_i + 0 = 0.$$

(ii) Now we consider the case $\xi > n$ by following the same decomposition in (B.5). The relations (B.6), (B.7) and (B.9) still hold. Note that $\xi > n$ is equivalent to that there is some $t_0 > 0$ such that for all $t > t_0$ we have $k > nt$. Then $\Pr(S_n(k) > t, X_i > k) = \Pr(S_n > t, X_i > k) = \Pr(X_i > k)$. It follows that

$$\lim_{t \rightarrow \infty} -\frac{I_2}{\overline{F}_1(t)} + \frac{I_3}{\overline{F}_1(t)} = 0.$$

Hence when $\xi > n$,

$$\lim_{t \rightarrow \infty} \frac{\Pr(S_n(k) > t)}{\overline{F}_1(t)} = \lim_{t \rightarrow \infty} \frac{I_1}{\overline{F}_1(t)} = n^{-\alpha} \sum_{i=1}^n \theta_i.$$

If $n^{-\alpha} \sum_{i=1}^n \theta_i < 1$, then $\lim_{q \rightarrow 1} \text{DR}_q(\mathbf{X}(\mathbf{k})) < 1$. ■

Proof of Theorem 4.3.1. Given $\lim_{q \rightarrow 1} \frac{k(q)}{\text{VaR}_q(X_1)} = \xi$ and $\lim_{q \rightarrow 1} \frac{\text{VaR}_q(X_k)}{\text{VaR}_q(X_1)} = \theta_i^{\frac{1}{\alpha}}$, we have

$$\lim_{q \rightarrow 1} \frac{k(q)}{\text{VaR}_q(X_i)} = \lim_{q \rightarrow 1} \frac{k(q)}{\text{VaR}_q(X_1)} \frac{\text{VaR}_q(X_1)}{\text{VaR}_q(X_i)} = \theta_i^{-\frac{1}{\alpha}} \xi.$$

Lemma 4.3.1 (i) implies, if $\theta_i^{-\frac{1}{\alpha}} \xi < n$, we have $\lim_{q \rightarrow 1} \text{DR}_q^{(i)}(\mathbf{X}(k)) < 1$. Then, if $\max(\xi, \theta_2^{-\frac{1}{\alpha}} \xi, \dots, \theta_n^{-\frac{1}{\alpha}} \xi) = \theta_{\min}^{-\frac{1}{\alpha}} \xi < n$, we have $\lim_{q \rightarrow 1} \text{DR}_q^{(i)}(\mathbf{X}(k)) < 1$ for any $i = 1, \dots, n$. ■

Proof of Lemma 4.4.1. (i) If $0 < \xi < 1$, then there exists $0 < q_0 < 1$ such that for all $q_0 < q < 1$, $\text{VaR}_q(X_1(k)) = k$. Note that $\text{VaR}_q(S_n(\mathbf{k})) \leq \frac{1}{n} \sum_{i=1}^n k_i$ for any $0 < q < 1$. If

$\sum_{i=2}^n \theta_i^{1/\alpha} < n - 1$, then

$$\lim_{q \uparrow 1} \text{DR}_q(\mathbf{X}(\mathbf{k})) \leq \frac{\frac{1}{n} \sum_{i=1}^n k_i}{k_1} = \frac{1}{n} \sum_{i=1}^n \theta_i^{1/\alpha} < 1.$$

(ii) When $\xi > 1$, following the similar decomposition (B.5) as in Theorem 4.3.1, we have

$$\begin{aligned} \Pr(S_n(\mathbf{k}) > t) &= \Pr(S_n(\mathbf{k}) > t, \text{none of } X_i\text{'s} > k_i) + \Pr(S_n(\mathbf{k}) > t, \text{at least one of } X_i\text{'s} > k_i) \\ &= \Pr(S_n > t) - \Pr\left(S_n > t, \bigcup_{i=1}^n \{X_i > k_i\}\right) + \Pr\left(S_n(\mathbf{k}) > t, \bigcup_{i=1}^n \{X_i > k_i\}\right) \\ &:= I_1 - I_2 + I_3. \end{aligned}$$

The analysis of I_1 as in (B.6) still holds. Similar to (B.7) and (B.9) we have

$$\lim_{t \rightarrow \infty} \frac{I_2}{\overline{F}_1(t)} = \lim_{t \rightarrow \infty} \sum_{i=1}^n \frac{\Pr(S_n > t, X_i > k_i)}{\overline{F}_1(t)},$$

and

$$\lim_{t \rightarrow \infty} \frac{I_3}{\overline{F}_1(t)} = \lim_{t \rightarrow \infty} \sum_{i=1}^n \frac{\Pr(S_n(\mathbf{k}) > t, X_i > k_i)}{\overline{F}_1(t)}.$$

Next we focus on I_2 and I_3 .

If for some i , $\theta_i^{1/\alpha} \xi > n$, which is equivalent to there is some $t_0 > 0$ such that $k_i > nt$ for all $t > t_0$, then $\Pr(S_n(\mathbf{k}) > t, X_i > k_i) = \Pr(S_n > t, X_i > k_i) = \Pr(X_i > k_i)$. We denote $Z = \{i = 2, \dots, n : \theta_i^{1/\alpha} \xi \geq n\}$. Then $Z^c = \{i = 2, \dots, n : \theta_i^{1/\alpha} \xi < n\}$. Now

$$\begin{aligned} \lim_{t \rightarrow \infty} -\frac{I_2}{\overline{F}_1(t)} + \frac{I_3}{\overline{F}_1(t)} &= \lim_{t \rightarrow \infty} \sum_{i=1}^n -\frac{\Pr(S_n > t, X_i > k_i)}{\overline{F}_1(t)} + \sum_{i=1}^n \frac{\Pr(S_n(\mathbf{k}) > t, X_i > k_i)}{\overline{F}_1(t)} \\ &= \lim_{t \rightarrow \infty} \sum_{i \in Z^c} -\frac{\Pr(S_n > t, X_i > k_i)}{\overline{F}_1(t)} + \frac{\Pr(S_n(\mathbf{k}) > t, X_i > k_i)}{\overline{F}_1(t)} \\ &:= \lim_{t \rightarrow \infty} -J_1 + J_2. \end{aligned}$$

For J_1 , note that for $\theta_i^{1/\alpha}\xi < n$, which is equivalent to that there is some $t_0 > 0$ such that $k_i < nt$ for all $t > t_0$, we have a similar decomposition as in (B.8) that

$$\begin{aligned}\Pr(S_n > t, X_i > k_i) &= \int_{k_i}^{\infty} \Pr\left(\sum_{j \neq i} X_j > nt - y_i\right) dF_i(y_i) \\ &= \left(\int_{k_i}^{(n-\varepsilon)t} + \int_{(n-\varepsilon)t}^{nt}\right) \Pr\left(\sum_{j \neq i} X_j > nt - y_i\right) dF_i(y_i) + \int_{nt}^{\infty} dF_i(y_i) \\ &:= J_{11} + J_{12} + J_{13}.\end{aligned}$$

Then following the similar line of proofs in Lemma 4.3.1, we can show that

$$\lim_{t \rightarrow \infty} \frac{J_{11}}{\bar{F}_1(t)} = \lim_{t \rightarrow \infty} \frac{J_{12}}{\bar{F}_1(t)} = 0,$$

and

$$\lim_{t \rightarrow \infty} \frac{J_{13}}{\bar{F}_1(t)} = \lim_{t \rightarrow \infty} \frac{\bar{F}_i(nt)}{\bar{F}_1(t)} = n^{-\alpha}\theta_i.$$

This leads to

$$\lim_{t \rightarrow \infty} J_1 = \sum_{i \in Z^c} \lim_{t \rightarrow \infty} (J_{11} + J_{12} + J_{13}) = \sum_{i \in Z^c} n^{-\alpha}\theta_i.$$

For J_2 , note that $nt - k_i = (n - \frac{k_i}{t})t \rightarrow \infty$ as $t \rightarrow \infty$ for $\theta_i^{1/\alpha}\xi < n$. Applying similar arguments as in Lemma 4.3.1 yields $\lim_{t \rightarrow \infty} J_2 = 0$.

Combining the above analysis,

$$\lim_{t \rightarrow \infty} \frac{\Pr(S_n(\mathbf{k}) > t)}{\bar{F}_1(t)} = n^{-\alpha} \sum_{i=1}^n \theta_i - n^{-\alpha} \sum_{i \in Z^c} \theta_i + 0 = n^{-\alpha} \sum_{i \in Z} \theta_i.$$

If $\sum_{i \in Z} \theta_i < n^\alpha$, then $\lim_{q \rightarrow 1} \text{DR}_q(\mathbf{X}(\mathbf{k})) < 1$. ■

Proof of Theorem 4.4.1. First note that for any $i, j = 1, \dots, n$,

$$\lim_{x \rightarrow \infty} \frac{\bar{F}_j(x)}{\bar{F}_i(x)} = \lim_{x \rightarrow \infty} \frac{\bar{F}_j(x)/\bar{F}_1(x)}{\bar{F}_i(x)/\bar{F}_1(x)} = \frac{\theta_j}{\theta_i}.$$

When $0 < \xi < 1$, Lemma 4.4.1 (i) implies for participant i , if $\sum_{j=1}^n \left(\frac{\theta_j}{\theta_i}\right)^{1/\alpha} \leq n$ or equivalently

$$\sum_{j=1}^n \theta_j^{1/\alpha} \leq n\theta_i^{1/\alpha},$$

then $\lim_{q \uparrow 1} \text{DR}_q^{(i)}(\mathbf{X}(\mathbf{k})) \leq 1$. Thus, for all participants willing to join the pool, it requires $\sum_{j=1}^n \theta_j^{1/\alpha} \leq n\theta_{\min}^{1/\alpha}$. Given $\theta_1 = 1$, all θ_j 's have to equal 1.

When $1 < \xi < n$, Lemma 4.4.1 (ii) implies for participant i , if $\sum_{j \in Z} \left(\frac{\theta_j}{\theta_i}\right) < n^\alpha$ or equivalently

$$\sum_{j \in Z_i} \theta_j < n^\alpha \theta_i,$$

where $Z_i = \{j = 1, \dots, n, j \neq i : \theta_j^{1/\alpha} \geq \theta_i^{1/\alpha} n / \xi\}$, we have $\lim_{q \uparrow 1} \text{DR}_q^{(i)}(\mathbf{X}(\mathbf{k})) < 1$. Thus, for all participants willing to join the pool, it requires $\sum_{j \in Z_{\min}} \theta_j < n^\alpha \theta_{\min}$.

When $\xi > n$, the set $Z_{\min}$ becomes $\{1, \dots, n\}$ and the condition reduces to $\sum_{j=1}^n \theta_j < n^\alpha \theta_{\min}$, which is impossible to satisfy. Thus, only when $1 < \xi < n$, it is possible for all participants to join the pool. ■

The proof of Theorem 4.4.2 follows from similar lines as in Lemma 4.4.1 and is omitted for simplicity.

Appendix C

Standard and Comparative E-backtests based on Identifiability and Elicitability

Assumptions

Assumption C.0.1 *For all $a \in \mathbb{R}^d$, there exist $F_1, \dots, F_{d+1} \in \mathcal{M}$, such that*

$$0 \in \text{int} \left(\left\{ \text{conv} \left(\left\{ \int_{\mathbb{R}} I(x, a) \, dF_1(x), \dots, \int_{\mathbb{R}} I(x, a) \, dF_{d+1}(x) \right\} \right) \right\} \right),$$

where int denotes interior and conv stands for convex hull.

Assumption C.0.2 *For all $a \in \mathbb{R}$, there exists a sequence $\{F_n\}_{n \in \mathbb{N}}$ such that $F_n \in \mathcal{M}$, the support of F_n is contained in a compact set K for all $n \in \mathbb{N}$, and $\{F_n\}_{n \in \mathbb{N}}$ converge weakly to the Dirac measure δ_a .*

Assumption C.0.3 *$a \mapsto I(x, a)$ is locally bounded for almost all $x \in \mathbb{R}^d$ in the Lebesgue sense. Moreover, the complement of the set $\{(x, a) \in \mathbb{R}^{d+1} : I(x, a) \text{ is continuous at } a\}$ has $(d + 1)$ -dimensional Lebesgue measure zero.*

E-variable and Bound under 0-Homogeneous Scoring Functions

Example C.0.1 *This example shows the e-variables constructed by 0-homogeneous scoring functions for some common risk measures.*

1. *The Value-at-Risk VaR_p under a strictly consistent scoring function*

$$S_{\text{VaR}}(x, a) = (1 - p) \log(a) + \mathbb{1}_{\{x > a\}} \log\left(\frac{x}{a}\right)$$

For $r, r^* \in \mathbb{R}$, we have

$$\inf_{x \in \mathbb{R}} (S_{\text{VaR}}(x, r) - S_{\text{VaR}}(x, r^*)) = (1 - p) \log\left(\frac{r}{r^*}\right) \mathbb{1}_{\{r \leq r^*\}} - p \log\left(\frac{r}{r^*}\right) \mathbb{1}_{\{r > r^*\}}.$$

By Lemma 5.4.1,

$$1 + h(r, r^*)((1 - p) \log(r) + \mathbb{1}_{\{L > r\}} \log\left(\frac{L}{r}\right)) - h(r, r^*)((1 - p) \log(r^*) + \mathbb{1}_{\{L > r^*\}} \log\left(\frac{L}{r^*}\right))$$

is a precise e-variable for the following null hypothesis $H^-(\text{VaR}_p)$ for all $L : \Omega \rightarrow \mathbb{R}$ and $r, r^* \in \mathbb{R}$:

$$H^-(\text{VaR}_p) : \mathbb{E}[S_{\text{VaR}}(L, r) - S_{\text{VaR}}(L, r^*)] \leq 0,$$

where

$$0 \leq h(r, r^*) \leq \frac{1}{p \log\left(\frac{r}{r^*}\right) \mathbb{1}_{\{r > r^*\}} - (1 - p) \log\left(\frac{r}{r^*}\right) \mathbb{1}_{\{r \leq r^*\}}}.$$

2. *(The Value-at-Risk and the Expected Shortfall) The Value-at-Risk and the Expected Shortfall $(\text{VaR}_p, \text{ES}_p)$ is jointly elicitable with a strict consistent scoring function*

$$S_{\text{ES}, \text{VaR}}(x, a, b) = \mathbb{1}_{\{x > b\}} \frac{x - b}{a} + (1 - p) \left(\frac{b}{a} - 1 + \log(a) \right), \quad x \in [-M, M], \quad 0 < b \leq a \leq M.$$

Here, we assume that the risk forecasts take positive values since we focus on losses.

For $0 < z \leq r \leq M$ and $0 < z^* \leq r^* \leq M$, we have

$$\begin{aligned}\gamma_2 &:= \inf_{x \in [-M, M]} (S_{\text{ES}, \text{VaR}}(x, r, z) - S_{\text{ES}, \text{VaR}}(x, r^*, z^*)) \\ &= \left\{ \mathbb{1}_{\{r > r^*\}} \left(\frac{M - z}{r} - \frac{M - z^*}{r^*} \right) \wedge \left(\frac{z^* - z}{r^*} \right) - \mathbb{1}_{\{r \leq r^*\}} \frac{z - z^*}{r^*} \wedge 0 \right\} \\ &\quad + (1 - p) \left(\frac{r}{z} - \frac{r^*}{z^*} + \frac{r}{r^*} \right).\end{aligned}$$

Thus we have by Lemma 5.4.1 that

$$\begin{aligned}1 + h(r, r^*, z, z^*) &\left(\mathbb{1}_{\{L > z\}} \frac{L - z}{r} + (1 - p) \left(\frac{z}{r} - 1 + \log(r) \right) \right) \\ &- h(r, r^*, z, z^*) \left(\mathbb{1}_{\{L > z^*\}} \frac{L - z^*}{r^*} + (1 - p) \left(\frac{z^*}{r^*} - 1 + \log(r^*) \right) \right)\end{aligned}$$

is a precise e -variable for the following null hypothesis $H^-(\text{ES}_p, \text{VaR}_p)$ for all $L : \Omega \rightarrow [-1, 1]$:

$$H^-(\text{ES}_p, \text{VaR}_p) : \mathbb{E}[S_{\text{ES}, \text{VaR}}(L, r, z) - S_{\text{ES}, \text{VaR}}(L, r^*, z^*)] \leq 0,$$

where

$$0 \leq h(r, r^*, z, z^*) \leq \mathbb{1}_{\{\gamma_2 \geq 0\}} - \frac{\mathbb{1}_{\{\gamma_2 < 0\}}}{\gamma_2} \wedge 1.$$

3. (The expectile) The expectile ex_p is elicitable with a strict consistent scoring function

$$S_{\text{ex}}(x, a) = \mathbb{1}_{\{x > a\}} (1 - 2p) \left(\log \frac{x}{a} + 1 - \frac{x}{a} \right) + (1 - p) \left(\log a - 1 + \frac{x}{a} \right), \quad x, a \in [-M, M].$$

For all $r, r^* \in [-M, M]$, we have

$$\inf_{x \in [-M, M]} (S_{\text{ex}}(x, r) - S_{\text{ex}}(x, r^*)) = (p \mathbb{1}_{\{r > r^*\}} + (1 - p) \mathbb{1}_{\{r \leq r^*\}}) \left(\log \frac{r}{r^*} - \left| \frac{1}{r} - \frac{1}{r^*} \right| M \right). \quad (\text{C.1})$$

By Lemma 5.4.1,

$$1 + h(r, r^*) S_{\text{ex}}(r, L) - h(r, r^*) S_{\text{ex}}(r^*, L)$$

is a precise e -variable for the following null hypothesis $H^-(\text{ex}_p)$ for all $L : \Omega \rightarrow [-M, M]$ and $r, r^* \in [-M, M]$:

$$H^-(\text{ex}_p) : \mathbb{E}[S_{\text{ex}}(L, r) - S_{\text{ex}}(L, r^*)] \leq 0,$$

where

$$0 \leq h(r, r^*) \leq \frac{1}{-\inf_{x \in [-M, M]}(S_{\text{ex}}(x, r) - S_{\text{ex}}(x, r^*))} \wedge 1,$$

and $\inf_{x \in [-M, M]}(S_{\text{ex}}(x, r) - S_{\text{ex}}(x, r^*))$ is given by (C.1).

Supplementary Results for Comparative Backtests via Structural Change of Time Series

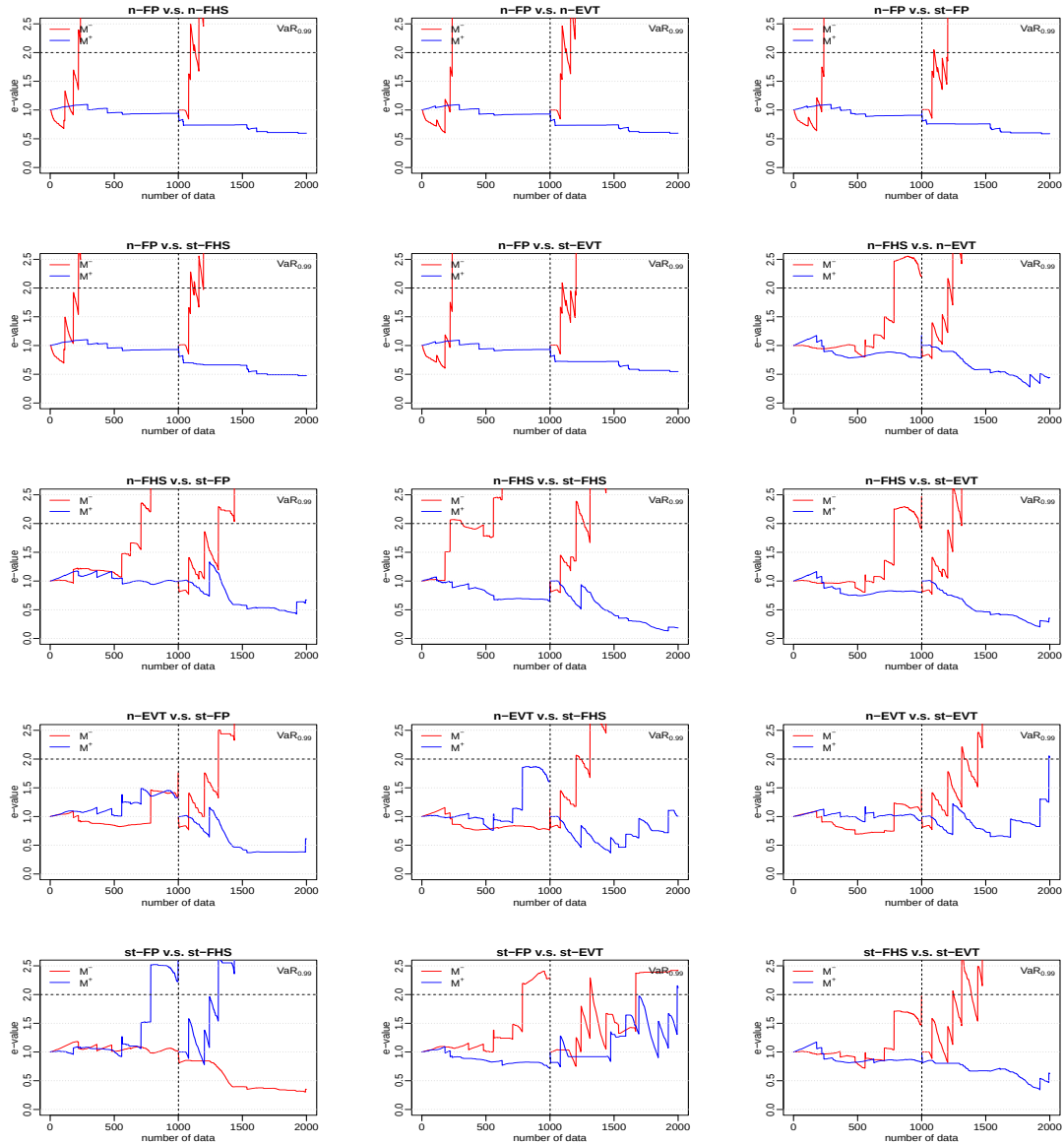


Figure C.1: E-processes of comparative backtests for $\text{VaR}_{0.99}$ rejecting at 2 and restarting at the structural change point, using the $h-1$ score function. Betting process is GREL with $c = 0.1$. The title of each plot represents "Internal model vs Alternative model".

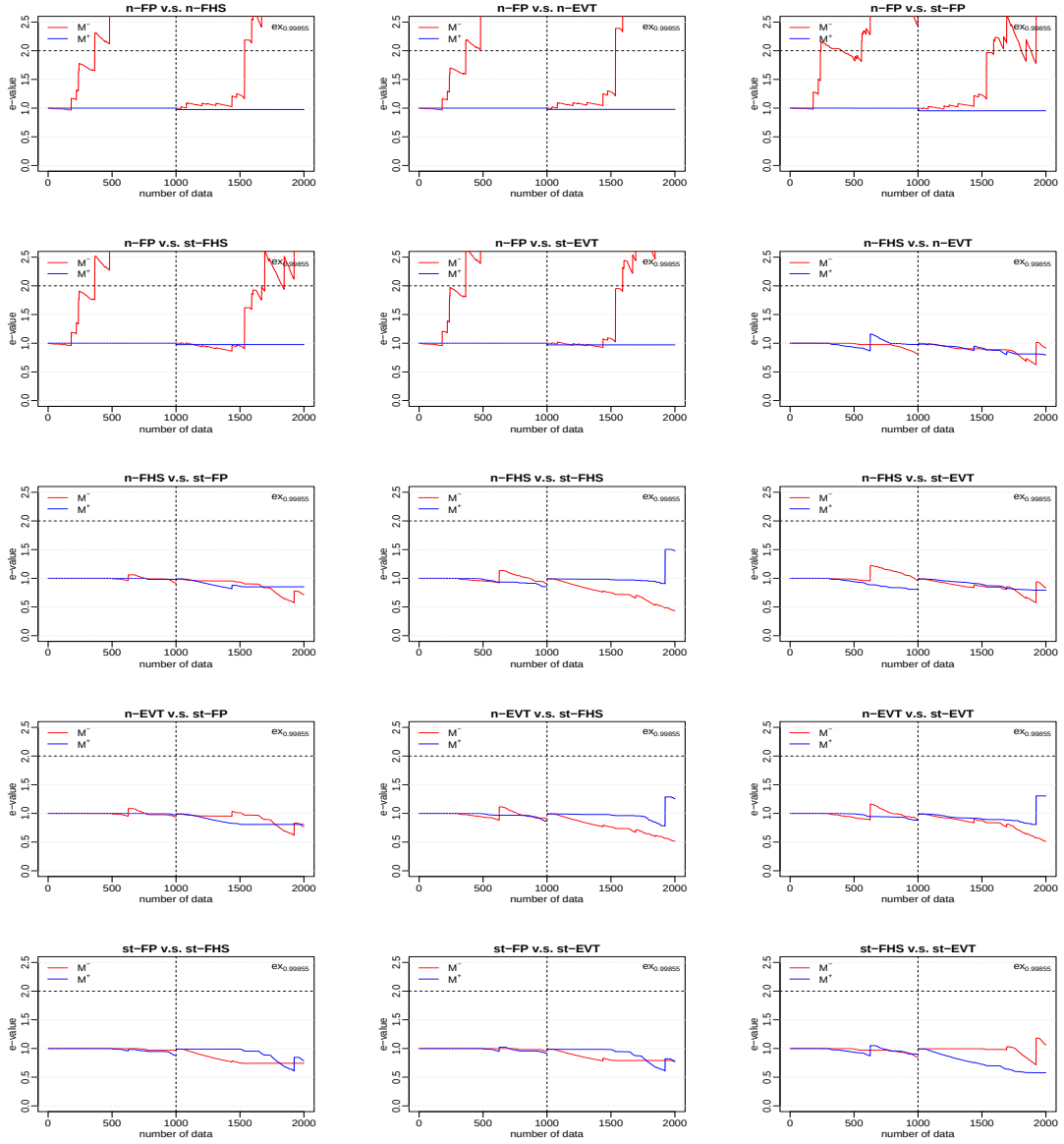


Figure C.2: E-processes of comparative backtests for $ex_{0.99855}$ rejecting at 2 and restarting at the structural change point, based on the score function in Example 5.4.1. Betting process is GREL with $c = 0.1$. The title of each plot represents "Internal model vs Alternative model".

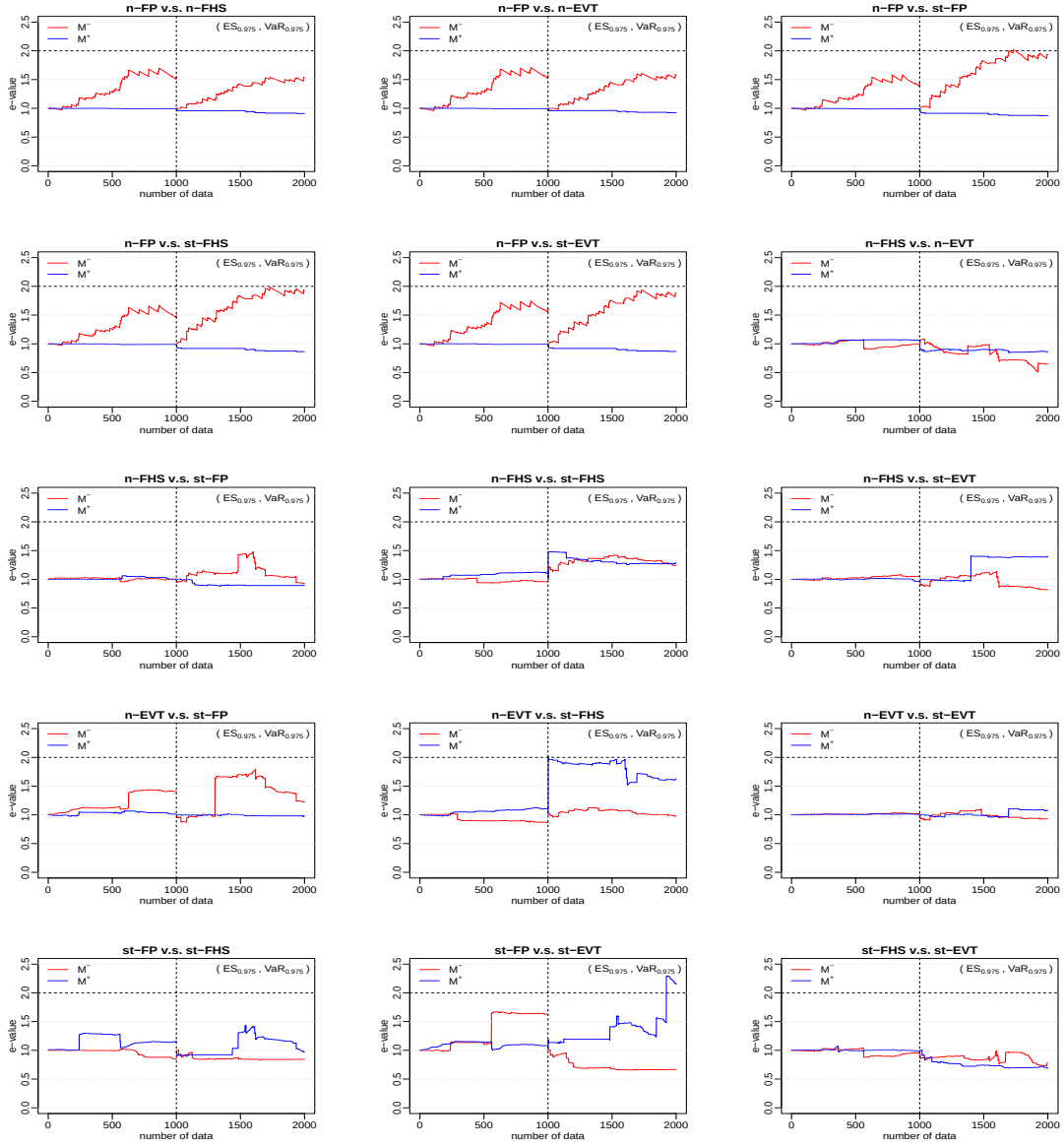


Figure C.3: E-processes of comparative backtests for $(ES_{0.975}, VaR_{0.975})$ rejecting at 2 and restarting at the structural change point, based on the score function in Example 5.4.1. Betting process is GREL with $c = 0.1$. The title of each plot represents "Internal model vs Alternative model".

Supplementary Results for Real Data Analysis

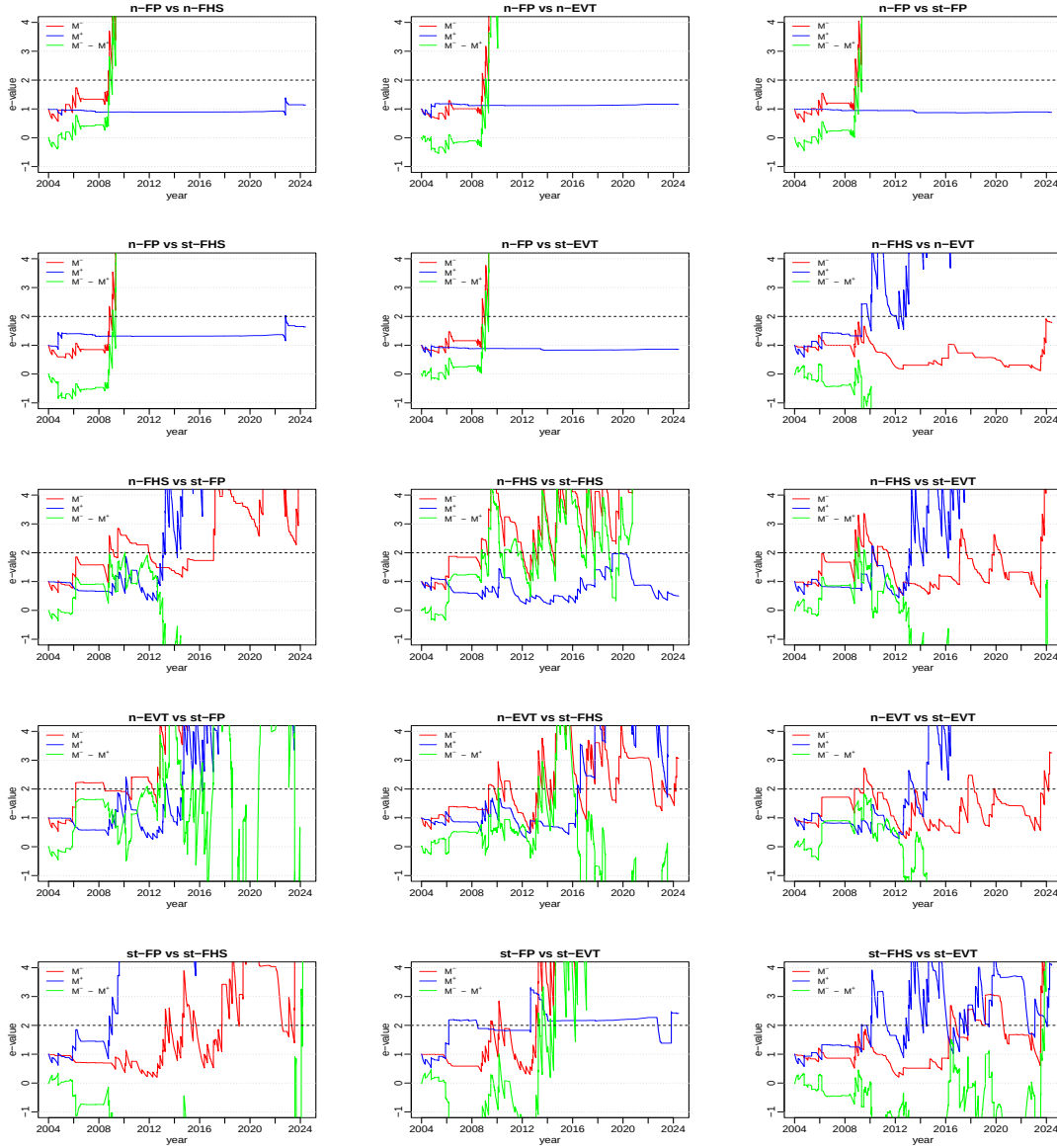


Figure C.4: E-processes for comparative backtests of $\text{VaR}_{0.99}$ on the NASDAQ index with respect to the number of days from January 3, 2003 to May 30, 2025. Adopting GREL betting process with $c = 0.01$ and the score function in Example 5.4.1. The title of each plot represents "Internal model vs Alternative model".

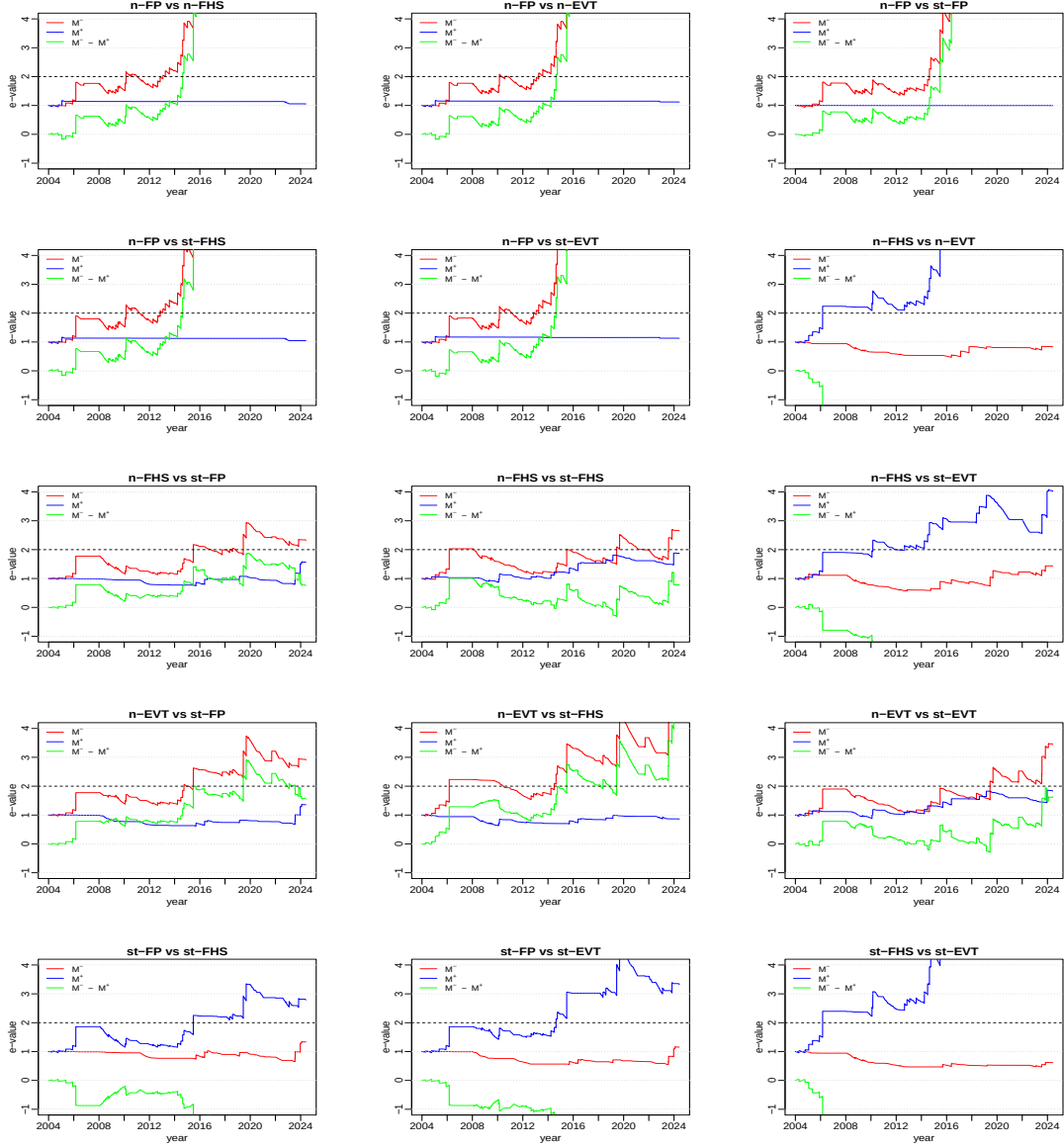


Figure C.5: E-processes of comparative backtests of $ex_{0.99855}$ on the NASDAQ index with respect to the number of days from January 3, 2003 to May 30, 2025. Adopting GREL betting process with $c = 0.01$ and the score function in Example 5.4.1. The title of each plot represents "Internal model vs Alternative model".

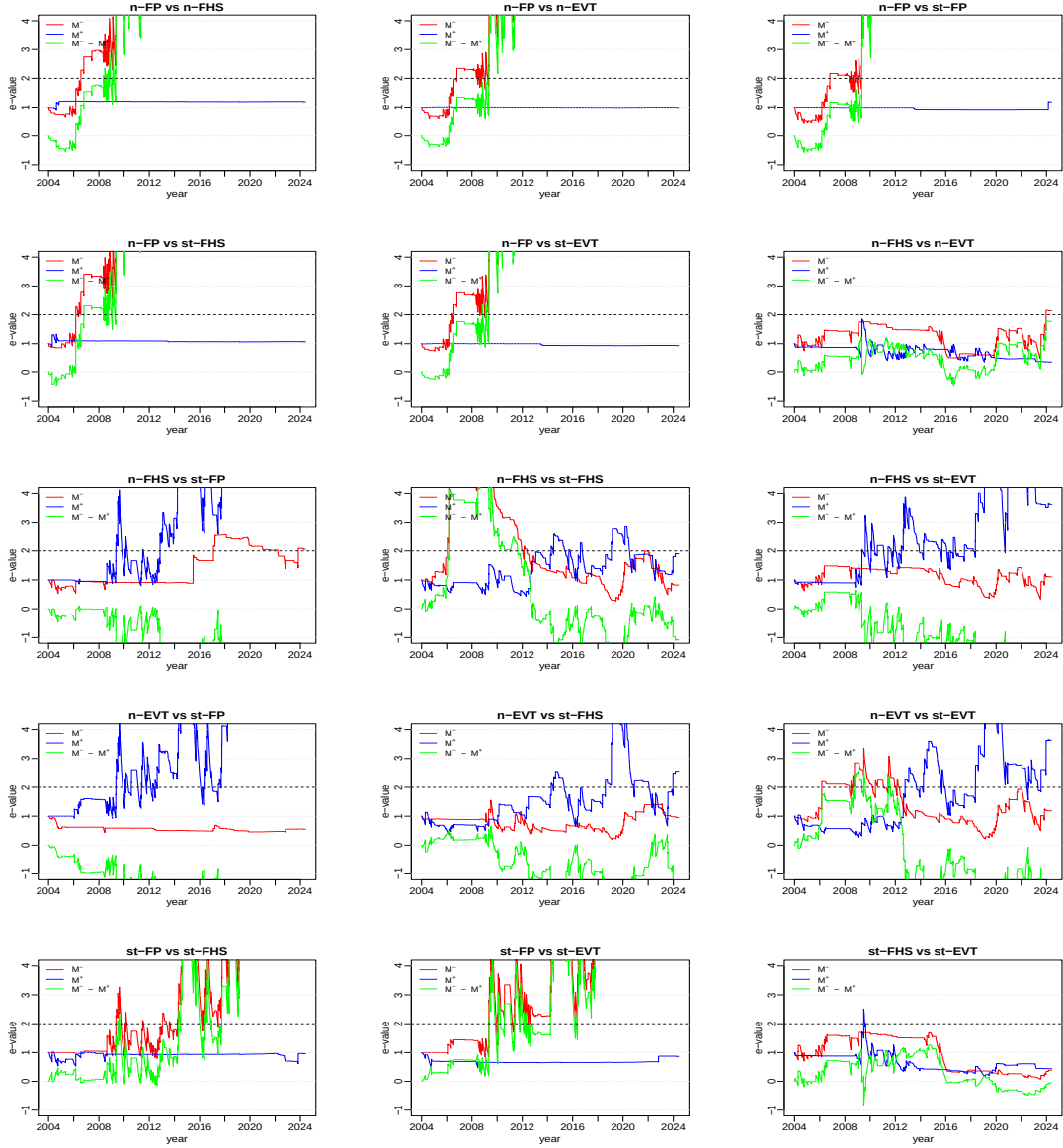


Figure C.6: E-processes for comparative backtests of $(ES_{0.975}, VaR_{0.975})$ on the NASDAQ index with respect to the number of days from January 3, 2003 to May 30, 2025. Adopting GREL betting process with $c = 0.01$ and the score function in Example 5.4.1. The title of each plot represents "Internal model vs Alternative model".

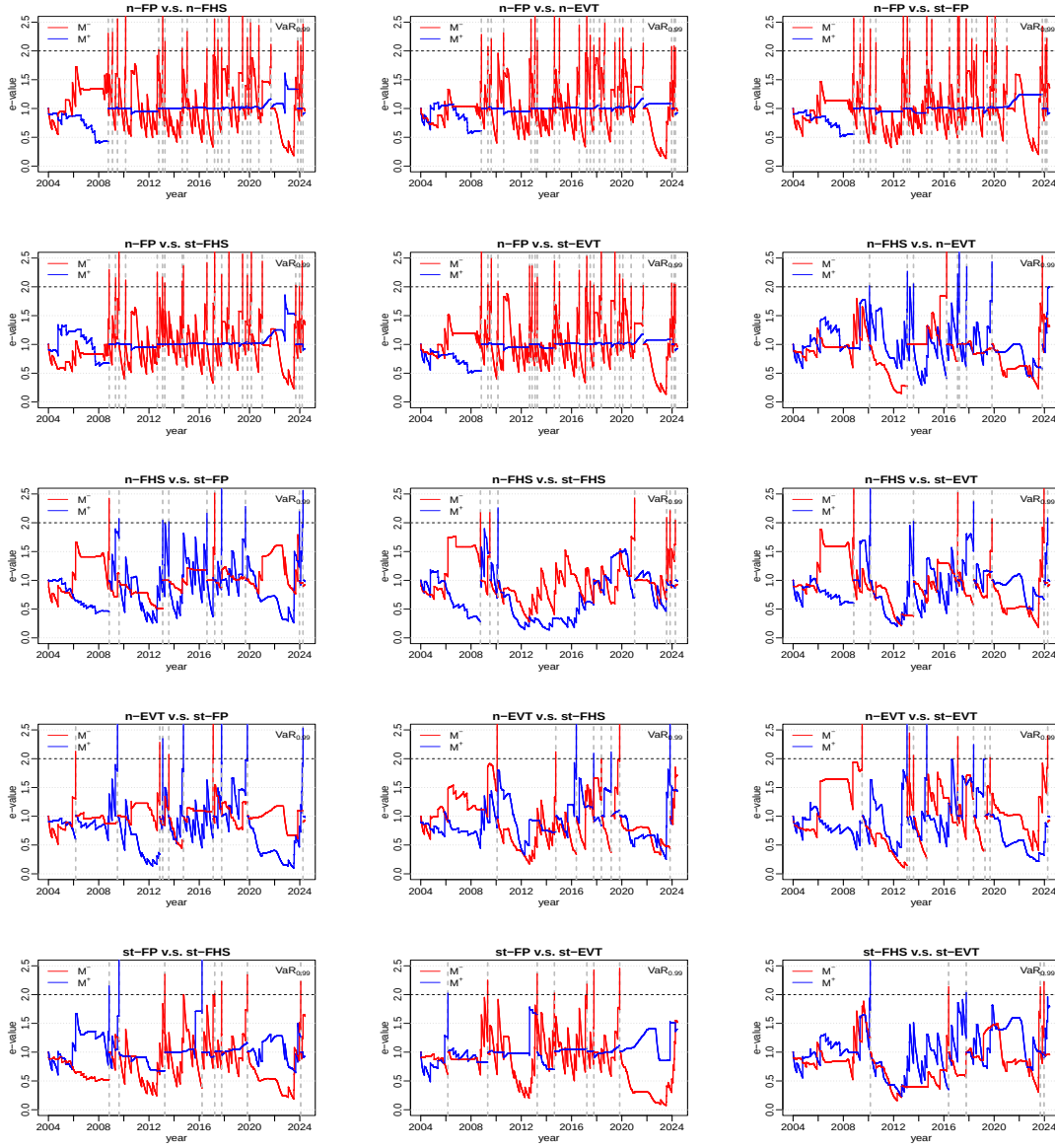


Figure C.7: E-processes for comparative backtests of $\text{VaR}_{0.975}$ on the NASDAQ index, rejecting at 2 and restarting thereafter. The x-axis represents the number of days from January 3, 2003, to May 30, 2025. The GREL betting process with $c = 0.1$ is adopted, using the score function specified in Example 5.4.1. The title of each plot indicates the comparison: “Internal model vs Alternative model”.

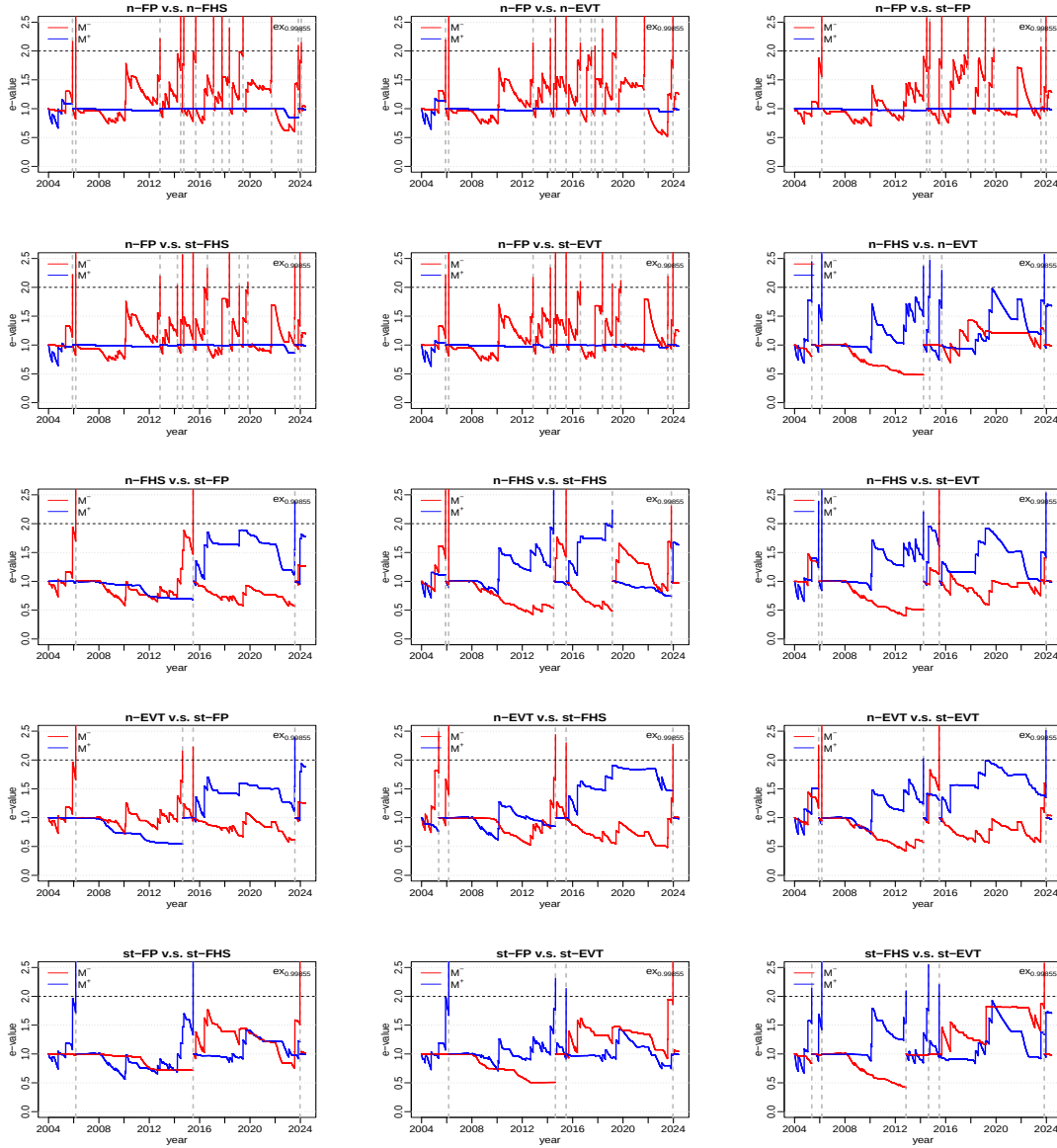


Figure C.8: E-processes for comparative backtests of $ex_{0.99855}$ on the NASDAQ index, rejecting at 2 and restarting thereafter. The x-axis represents the number of days from January 3, 2003, to May 30, 2025. The GREL betting process with $c = 0.1$ is adopted, using the score function specified in Example 5.4.1. The title of each plot indicates the comparison: “Internal model vs Alternative model”.

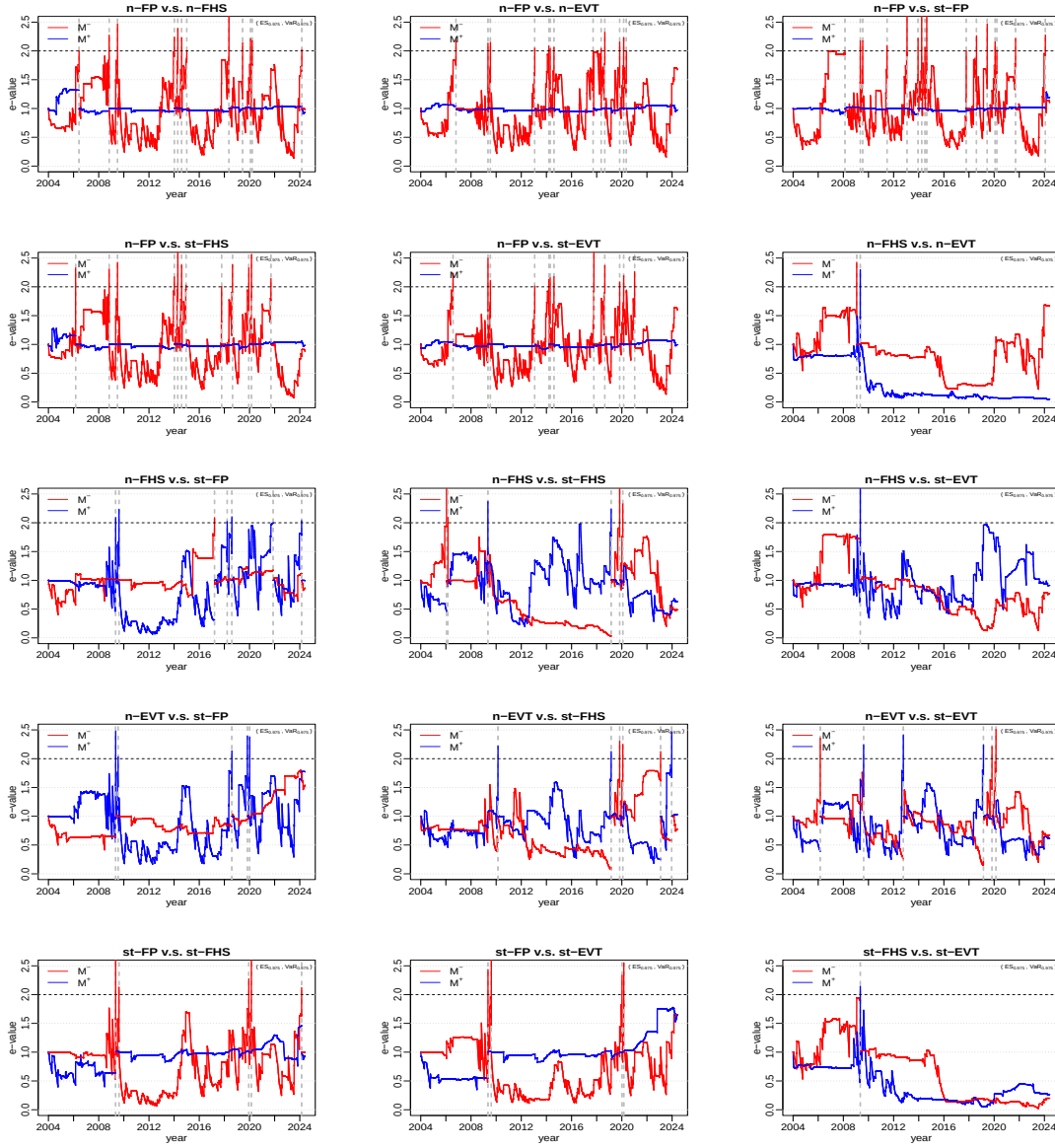


Figure C.9: E-processes for comparative backtests of $(ES_{0.975}, VaR_{0.975})$ on the NASDAQ index, rejecting at 2 and restarting thereafter. The x-axis represents the number of days from January 3, 2003, to May 30, 2025. The GREL betting process with $c = 0.1$ is adopted, using the score function specified in Example 5.4.1. The title of each plot indicates the comparison: “Internal model vs Alternative model”.