Risk Sharing with Distortion Risk Measures Beyond Risk Aversion

by

Qinghua Ren

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Examining Committee Membership

The following served on the Examining Committee for this thesis. The decision of the Examining Committee is by majority vote.

External Examiner:	Erick Delage Professor, Department of Decision Sciences HEC Montréal
Supervisor(s):	Mario Ghossoub Associate Professor, Department of Statistics and Actuarial Science University of Waterloo
	Ruodu Wang Professor, Department of Statistics and Actuarial Science University of Waterloo
Internal Member:	Jun Cai Professor, Department of Statistics and Actuarial Science University of Waterloo
Internal Member:	Fan Yang Associate Professor, Department of Statistics and Actuarial Science University of Waterloo
Internal-External Member:	Anqi Li Assistant Professor, Department of Economics University of Waterloo

Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

This thesis studies optimal risk sharing among multiple agents whose preferences are represented by distortion risk measures, equivalently Yaari dual utilities. The central question is how to characterize Pareto-optimal risk allocations when agents may have heterogeneous risk preferences, including risk-averse, risk-seeking, and behavioral attitudes toward risk. Particular attention is paid to the dependence structure of optimal allocations and to the geometry of the Pareto frontier.

The first part of the thesis develops counter-monotonic risk sharing as a counterpart to the classical comonotonic theory. Comonotonicity represents positive dependence and is fundamental in risk sharing among risk-averse agents, while counter-monotonicity represents an extreme form of negative dependence and arises naturally for risk-seeking agents. Chapters 2 and 3 analyze this counter-monotonic structure for distortion risk measures, moving from a homogeneous setting with a common distortion function to a heterogeneous setting with different distortion functions. Inf-convolution is an important tool for studying Pareto optimality. Using this tool, Chapters 2 and 3 compare the usual formulation with variants that restrict allocations to be comonotonic or counter-monotonic, derive explicit formulas for risk-seeking agents, and illustrate the formulas through a portfolio manager's problem.

Chapter 4 studies markets with mixed risk attitudes, where risk-averse and risk-seeking agents coexist. This setting is more challenging because neither the usual comonotonic arguments for risk-averse agents nor the counter-monotonic arguments for risk-seeking agents apply directly to the whole market. The chapter establishes a reduction theorem showing that the general multi-agent problem can be reduced to a two-agent problem between representative risk-averse and risk-seeking agents. Based on this reduction, the chapter further studies the existence of optimal allocations, identifies cases in which the inf-convolution is unbounded, and derives explicit solutions for piecewise linear distortion functions and Bernoulli-type aggregate risks.

Chapter 5 studies Pareto optimality beyond universal risk aversion, with emphasis on

constrained allocation problems. Feasibility constraints, such as nonnegative allocations, are natural in insurance and reinsurance, but they change the geometry of the risk-sharing problem and may limit the applicability of weighted-sum methods. The chapter reduces the two-agent Pareto problem to a one-parameter family of constrained optimization problems. For Bernoulli aggregate risks, the Pareto frontier admits a convex-envelope characterization and can be attained by three-atom allocations. For two-point aggregate risks, finite-atom structural results are developed, showing that efficient allocations can be represented by a bounded number of payment levels. These results provide both theoretical insight into non-risk-averse risk sharing and a tractable framework for numerical computation.

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Dedication

This thesis is dedicated to my family.

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Chapter 1

Introduction and preliminary

1.1 Introduction

A central problem in actuarial science, mathematical finance and economic decision theory is how a given pool of risk should be divided among a group of agents. This problem arises in many settings. An insurer transfers part of its underwriting risk to reinsurers, investors pool exposures through structured financial products, and participants in peer-to-peer insurance arrangements share aggregate losses among themselves. In each case, there is an aggregate risk faced by the group and a set of agents who must agree on how this risk is redistributed. Mathematically, a risk allocation is a vector of random variables whose sum equals the aggregate risk, with each component representing the share assigned to one agent. Classical references on risk sharing and insurance include Aase (2002), Denuit et al. (2006) and Albrecher et al. (2017). Recent developments in peer-to-peer insurance and decentralized risk-sharing markets have renewed interest in this problem; see, for instance, Denuit (2020), Abdikerimova and Feng (2022) and Feng et al. (2023).

The natural question is what it means for an allocation to be optimal. Among several possible criteria, Pareto optimality is a fundamental notion of efficiency. An allocation is Pareto optimal if no other feasible allocation can make one or more agents better off

without making any agent worse off. Characterizing Pareto-optimal allocations is thus a natural starting point for the analysis of risk sharing and is the central object of this thesis.

The classical theory of risk sharing assumes that agents evaluate random outcomes through expected utility. Under expected-utility preferences and risk aversion, the seminal works of Borch (1962) and Wilson (1968) showed that Pareto-optimal allocations admit a simple structure. Each agent's share is an increasing function of the aggregate risk, so that all agents' positions move in the same direction. Such allocations are called comonotonic. The associated first-order conditions are often referred to as the Borch rule, which equates marginal rates of substitution across agents and states. Related results in optimal insurance design, beginning with Arrow (1963), established the optimality of deductible-type contracts in classical insurance settings. These foundational results have shaped much of the subsequent theory of insurance, reinsurance and risk pooling.

The structural role of comonotonicity extends well beyond the expected-utility framework. In decision-making under objective risk, risk aversion is commonly understood as consistency with second-order stochastic dominance, or equivalently, in many settings, as Schur concavity of the preference functional. The comonotonic improvement theorem of Landsberger and Meilijson (1994) shows that, for risk-averse agents, every feasible allocation admits a comonotonic improvement in the convex order. Hence, efficient risk sharing can often be studied within the comonotonic subclass. The original result was developed for two agents in a discrete setting, and was later extended to atomless probability spaces by Dana (2004), to multi-agent and unbounded settings by Ludkovski and Rüschendorf (2008), and to vector-valued risks by Carlier et al. (2012). A recent constructive treatment of the general case is given by Denuit et al. (2025). Thus, in many risk-averse models, comonotonicity is not merely a tractable restriction but a structural feature of Pareto optimality. This perspective has also motivated the study of comonotone markets, in which only comonotonic allocations are admissible. Comonotone risk-sharing problems with heterogeneous beliefs are studied by Liu (2020), while competitive equilibria in comonotone markets are analyzed by Boonen et al. (2021). Recent work by Ghossoub and Zhu (2026) further shows that, for broad classes of risk-averse monetary utilities, comonotone Pareto

optima can be explicitly characterized.

This observation has motivated a broad literature on Pareto-optimal allocations under risk-averse preferences beyond expected utility. For ambiguity-sensitive preferences, optimal allocations have been studied under Choquet expected utility, maxmin expected utility, and variational preference models; see, for instance, Chateauneuf et al. (2000), Dana (2004), De Castro and Chateauneuf (2011), Maccheroni et al. (2006), Ravanelli and Svindland (2014), and Beißner and Werner (2023). For monetary utilities and convex risk measures, Pareto optimality is commonly formulated through inf-convolution, which computes the minimal aggregate risk value attainable over all feasible allocations. This approach provides a convenient link between Pareto optimality and optimization, and was developed in the risk-sharing literature by Barrieu and El Karoui (2005), Jouini et al. (2008), and Filipović and Svindland (2008). Further developments include non-monotone preferences (Acciaio, 2007), quasi-concave utilities (Mastrogiacomo and Rosazza Gianin, 2015), and SSD-preserving or law-invariant utility functionals (Ravanelli and Svindland, 2014; Mao and Wang, 2020; Liebrich, 2024; Ghossoub and Zhu, 2026). Although these models go beyond expected utility in different ways, they largely preserve the risk-averse structure under which comonotonicity remains central to Pareto optimality.

A particularly important class of non-expected-utility models is built on probability weighting. Models such as rank-dependent utility, cumulative prospect theory, and Yaari's dual theory of choice allow agents to distort probabilities rather than evaluate risks only through expected values; see Quiggin (1982), Tversky and Kahneman (1992), and Yaari (1987). In these models, probability weighting captures features such as the overweighting of small probabilities and the different treatment of tail events, which are important in insurance and behavioral decision-making. Equivalently, distortion risk measures correspond to Yaari dual utilities, up to a sign convention between losses and gains. They include widely-used risk measures, such as, Value-at-Risk, Expected Shortfall, and spectral risk measures; see Artzner et al. (1999), Acerbi (2002), and McNeil et al. (2015). Thus, they provide a natural and tractable framework for studying risk sharing under non-expected-utility preferences.

A key advantage of distortion risk measures is that the distortion function itself captures different attitudes toward risk. Concave distortions correspond to risk aversion and yield convex risk measures, convex distortions correspond to risk seeking and yield concave risk functionals, and inverse S-shaped distortions combine these two features by allowing risk attitudes to vary across probability levels. Thus, distortion risk measures provide a unified framework for risk-averse, risk-seeking, and mixed-attitude agents. Risk sharing under distortion risk measures has been studied in optimal insurance and reinsurance design (Cui et al., 2013; Chi and Tan, 2013; Assa, 2015; Zhuang et al., 2016), quantile-based and tail-risk sharing (Embrechts et al., 2018, 2020; Liu et al., 2022; Weber, 2018), and comonotonic risk sharing under heterogeneous beliefs (Liu, 2020). More recent work gives explicit characterizations for law-invariant coherent risk measures and robust distortion risk measures (Ghossoub and Zhu, 2026; Ghossoub et al., 2025). Across these studies, comonotonicity remains the dominant structure under risk aversion, convexity or second-order stochastic dominance consistency.

The theory is less developed when the market includes agents who are not uniformly risk averse. Risk-seeking behavior arises naturally in gambling, speculative investment and behavioral decision-making, while inverse S-shaped distortion functions allow risk attitudes to vary across probability levels. In such settings, classical convexity and comonotonicity arguments are no longer sufficient. Recent work by Lauzier et al. (2023) and Lauzier et al. (2026a) identifies counter-monotonicity as the natural counterpart of comonotonicity for risk-seeking agents. The resulting counter-monotonic improvement theorem connects risk seeking with an extreme form of negative dependence, in which agents' positions move in opposite directions. Counter-monotonic structures also arise in quantile-based risk sharing (Embrechts et al., 2018, 2020; Weber, 2018). These developments motivate a systematic study of counter-monotonic and non-comonotonic risk sharing.

This thesis develops a systematic study of optimal risk sharing with distortion risk measures beyond the classical risk-averse setting. In particular, the study shows how heterogeneity of risk preferences, risk-seeking behavior, mixed attitudes, and allocation constraints affect both the dependence structure and the shape of Pareto-optimal risk allo-

cations. The analysis focuses on four related problems: characterizing counter-monotonic allocations and their inf-convolutions for risk-seeking agents, understanding how heterogeneous distortion functions determine Pareto-optimal allocations, and describing the Pareto frontier when standard convexity arguments fail.

Chapters 2 and 3 study counter-monotonic risk sharing for agents with distortion risk measures, where allocations are restricted to be counter-monotonic. Chapter 2 begins with the homogeneous setting, in which all agents share a common distortion function, and Chapter 3 then introduces heterogeneity across agents' distortion functions. Together, these two chapters compare unconstrained, comonotonic, and counter-monotonic inf-convolutions and derive explicit formulas for risk-seeking agents. The results show how the agents' risk attitudes and the degree of preference heterogeneity shape the dependence structure of optimal allocations. The analysis also demonstrates that the effective risk attitude of the group need not vary monotonically with the individual agents' risk attitudes, and this phenomenon is illustrated through portfolio manager's problems in both the homogeneous and heterogeneous settings.

Chapter 4 studies markets with mixed risk attitudes, where risk-averse and risk-seeking agents coexist. This setting lies between the purely comonotonic structure of risk-averse sharing and the counter-monotonic structure of risk-seeking sharing, so neither classical argument applies directly to the whole market. The chapter shows that the multi-agent problem can be reduced to a two-agent problem between a representative risk-averse agent and a representative risk-seeking agent. This reduction provides a tractable way to analyze optimal allocations in mixed-attitude markets. Based on this reduction, the chapter studies the existence of optimal allocations, identifies cases in which the inf-convolution is unbounded, and derives explicit solutions for important cases, including piecewise-linear distortion functions and Bernoulli-type aggregate risks.

Chapter 5 studies constrained Pareto optimality beyond universal risk aversion. As shown in Chapter 4, the unconstrained inf-convolution may be unbounded for general distortion functions even when the aggregate risk is bounded, so Pareto-optimal allocations

need not exist. This motivates imposing natural feasibility constraints, such as nonnegative allocations. Under such constraints, the relevant object becomes the risk possibility set, whose boundary describes the constrained Pareto frontier. The chapter investigates whether the resulting risk possibility set always has convexity structure for weighted-sum methods to recover the full frontier. Although the general question remains open, positive results are obtained in important special cases. For Bernoulli aggregate risks, the frontier admits a convex-envelope characterization and can be attained by three-atom allocations. Chapter 6 concludes the thesis with a summary of contributions and a detailed discussion of potential directions for future research.

1.2 Preliminaries

Throughout the thesis, let $(\Omega, \mathcal{F}, \mathbb{P})$ be an atomless probability space. For $p \in [1, \infty)$, L^p is the space of random variables with finite p-th moment; L^∞ is the set of essentially bounded random variables; L^+ is the set of nonnegative bounded random variables. Write $X \overset{d}{=} Y$ if X and Y have the same distribution under $\mathbb{P}$. We treat almost surely equal random variables as identical. A random variable X is said to be smaller than a random variable Y in the convex order, denoted by $X \leqslant_{\mathrm{cx}} Y$, if $\mathbb{E}[f(X)] \leqslant \mathbb{E}[f(Y)]$ for every convex function $f : \mathbb{R} \mapsto \mathbb{R}$ provided that both expectations are well-defined (see Rüschendorf (2013)). Similarly, a random variable X is said to be smaller than a random variable Y in the concave order, denoted by $X \leqslant_{\mathrm{cv}} Y$, if $\mathbb{E}[f(X)] \leqslant \mathbb{E}[f(Y)]$ for every concave function $f : \mathbb{R} \mapsto \mathbb{R}$ provided that both expectations are well-defined. For a distribution F and a random variable X, $X \sim F$ means that X has distribution F. We denote by $\mathbb{1}_A$ the indicator function for an event $A \in \mathcal{F}$.

A risk measure is a mapping $\rho : \mathcal{X} \to [-\infty, \infty]$, where $\mathcal{X}$ is a set of random variables and will be specified in different contexts. Below we list some standard properties for a risk measure ρ. For any $X, Y \in \mathcal{X}$,

(a) Monotonicity: $\rho(X) \leqslant \rho(Y)$ if $X \leqslant Y$;

(b) Law-invariance: $\rho(X) = \rho(Y)$ if $X \overset{\mathrm{d}}{=} Y$;

(c) Positive homogeneity: $\rho(\lambda X) = \lambda\rho(X)$ for any $\lambda > 0$;

(d) Translation invariance: $\rho(X + c) = \rho(X) + c$ for $c \in \mathbb{R}$, and $X + c \in \mathcal{X}$;

(e) Uniform continuity:[1] for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $X, Y \in \mathcal{X}$, $\|X - Y\| \leqslant \delta$ implies $|\rho(X) - \rho(Y)| \leqslant \varepsilon$;

(f) Subadditivity: $\rho(X + Y) \leqslant \rho(X) + \rho(Y)$;

(g) Comonotonic additivity: $\rho(X + Y) = \rho(X) + \rho(Y)$ if X and Y are comonotonic;

(h) Convex order consistency: $\rho(X) \leqslant \rho(Y)$ if $X \leqslant_{\mathrm{cx}} Y$.

(i) Convexity: $\rho(\lambda X + (1 - \lambda)Y) \leqslant \lambda\rho(X) + (1 - \lambda)\rho(Y)$ for all $\lambda \in [0, 1]$.

In the framework of Artzner et al. (1999), a monetary risk measure is a functional that satisfies (a) and (d). A coherent risk measure is one that satisfies (a), (c), (d) and (f). In (f) and (g), we require that ρ do not take both values $-\infty$ and ∞. We allow for both ∞ and $-\infty$ in the range of risk measures because we would like to treat inf-convolutions as risk measures, and in some cases they can take infinite values. The definition of inf-convolution will be provided in Chapter 2 and Chapter 3.

[1] Continuity of ρ is defined with respect to sup-norm, i.e. $\|X\| = \text{ess-sup}(|X|)$ for $X \in \mathcal{X}$.

Chapter 2

Counter-Monotonic Risk Allocations and Distortion Risk Measures

2.1 Introduction

The literature on efficient allocations in pure exchange economies, or optimal risk sharing in risk-sharing markets, has hitherto mostly been interested in Pareto optimality (also called Pareto efficiency) for risk-averse agents, both within the classical expected-utility theory (EUT) framework and beyond. For decision-making under objective risk, risk aversion is commonly defined as consistency with respect to second-order stochastic dominance. The seminal work of Borch (1962) and Wilson (1968) on risk sharing for risk-averse agents within EUT provided an explicit characterization of Pareto-optimal allocations, and showed in particular that optimal allocations are *comonotonic*. This property of optima was extended beyond the EUT framework, and a cornerstone result in this literature is the so-called *comonotonic improvement theorem* (e.g., Landsberger and Meilijson (1994), Carlier and Dana (2003), Rüschendorf (2013), or Denuit et al. (2025)). Existence and comonotonicity of Pareto optima beyond EUT has been established widely. See, for instance, Chateauneuf et al. (2000), Dana (2004), Tsanakas and Christofides (2006), De Cas-

tro and Chateauneuf (2011), or Beißner and Werner (2023), for risk-sharing markets with ambiguity-sensitive agents; or Barrieu and El Karoui (2005), Jouini et al. (2008), and Filipović and Svindland (2008) for the popular class of law-invariant monetary utilities, or law-invariant convex risk measures. Ravanelli and Svindland (2014) showed the existence and comonotonicity of Pareto optima for a class of law-invariant variational preferences that are SSD-consistent. Recently, Ghossoub and Zhu (2026) provided an explicit characterization of Pareto optima and showed their comonotonicity, for a wide class of robust SSD-consistent concave utility functionals.

Pareto optimality in risk-sharing markets that include risk-seeking agents is less well-studied. A first step in this direction was taken by Araujo et al. (2018, 2022) and Herings and Yang (2022) in an equilibrium context. Beißner and Werner (2023) allowed for the existence of risk-loving and ambiguity-loving agents in exchange economies. Recently, Lauzier et al. (2023) provided a stochastic representation of counter-monotonicity. Using this result, Lauzier et al. (2026a) further derived the so-called *counter-monotonic improvement theorem*, a counterpart to the comonotonic improvement theorem. Just as risk aversion is linked to comonotonicity, risk seeking is linked to counter-monotonicity. The counter-monotonic improvement theorem states that for any random vector bounded from below or above, there exists a counter-monotonic random vector whose components are riskier than those of the given random vector. An important implication of this theorem is that counter-monotonic allocations will always be preferred by risk-seeking agents. In addition to risk-seeking agents, counter-monotonic allocations can also be optimal for quantile agents, who are neither risk-seeking nor risk-averse, as shown by Embrechts et al. (2018, 2020) and generalized by Weber (2018). These observations suggest that a systematic study of risk sharing problems when constrained to counter-monotonic allocations is useful, and this is central to this chapter.

Specifically, in this chapter, we study optimal risk sharing and inf-convolution of distortion risk measures. We consider not only risk-averse agents, but also risk-seeking agents and behavioral agents. Our study can be further motivated by a simple portfolio optimization problem. A fund collects an initial constant endowment W from a group of clients,

assumed to have the same risk preferences. These clients can either be risk-seeking or risk-averse, and their risk attitudes are modeled using distortion risk measures $\rho_1, \dots, \rho_n$. The fund is managed by a professional manager whose task is to invest the endowment in a manner that aligns with the clients' risk attitudes and minimizes the total risk. The manager intends to allocate a proportion λ of the fund to a risky asset X in a domain $\mathcal{X}$. In this context, Pareto optimality of an allocation $(X_1, \dots, X_n)$ is equivalent to optimality with respect to the sum of the risk measures; see Embrechts et al. (2018, Proposition 1). Therefore, the objective function of the fund manager is given by

$$\begin{aligned} &\text{minimize} \quad \sum_{i=1}^{n} \rho_i(-X_i) \\ &\text{subject to } X_1 + \cdots + X_n = W + \lambda X - c(\lambda) \text{ and } c(\lambda) \leqslant W, \end{aligned} \tag{2.1}$$

where c represents the corresponding cost function of investing λ in the fund, and hence $W+\lambda X-c(\lambda)$ represents the terminal wealth of the fund, to be allocated to the participants. Note that there are two layers of optimization involved in (2.1), that is, deciding the investment strategy λ and the allocation $(X_1, \dots, X_n)$. Investors interested in risky assets are typically not necessarily risk-averse, and hence (2.1) calls for a study of inf-convolution for non-risk-averse agents. As we will see in Section 2.6, our results on risk sharing for distortion risk measures will help to solve (2.1). A natural intuition is that less will be invested in the risky asset when risk-averse agents are involved, whereas more will be invested in the risky asset when risk-seeking agents are involved. This intuition checks out as justified by our results.

As mentioned above, we will focus on distortion risk measures, a popular class of risk measures widely used in finance and insurance; see McNeil et al. (2015) for distortion risk measures in risk management. To make our notion more specialized, now we use $\rho_{h_1}, \dots, \rho_{h_n}$ for distortion risk measures and $h_1, \dots, h_n$ for their distortion functions. We denote by $\square_{i=1}^{n} \rho_{h_i}(X)$, $\boxplus_{i=1}^{n} \rho_{h_i}(X)$ and $\boxminus_{i=1}^{n} \rho_{h_i}(X)$ the smallest value of $\rho_{h_1}(X_1) + \cdots + \rho_{h_n}(X_n)$ over allocations of X that are general, comonotonic, and counter-

monotonic, respectively; see Sections 2.2 and 2.3 for precise definitions.

The existing literature primarily focuses on the unconstrained inf-convolution $\square_{i=1}^{n} \rho_{h_i}$ and comontonic inf-convolution $\boxplus_{i=1}^{n} \rho_{h_i}$. The problem setting of counter-monotonic inf-convolution $\boxminus_{i=1}^{n} \rho_{h_i}$ is novel. For most results in this chapter, we assume $h_1 = \cdots = h_n$, that is, the homogeneous setting, for tractability. We aim to answer the following questions:

(a) It is known that the comonotonic inf-convolution has an explicit form, i.e., $\boxplus_{i=1}^{n} \rho_h = \rho_h$, which holds true regardless of whether h is concave or convex; see e.g., Embrechts et al. (2018, Proposition 5). This naturally leads to the question of whether the counter-monotonic inf-convolution exhibits similar properties. Specifically, does $\boxminus_{i=1}^{n} \rho_h$ share the same formula for both concave and convex h? Under which conditions does $\boxminus_{i=1}^{n} \rho_h = \rho_h$ hold true?

(b) We study three types of problems, i.e., unconstrained, comonotonic, and counter-monotonic risk sharing. Each type provides insights into the behavior of risk measures under different allocation constraints. What is the relationship among three variations of inf-convolution?

(c) Question (a) concerns the counter-monotonic inf-convolution of concave or convex distortion risk measures. Is there an explicit formula for the counter-monotonic inf-convolution if the underlying risk measures are neither concave nor convex?

(d) In the application of portfolio optimization problems, optimal asset allocations often have different structures for agents with varying risk attitudes. How will the optimal strategies change as agents' risk preferences vary, for example, becoming more risk-seeking? Additionally, how does the number of agents involved in the pool affect the optimal allocations?

The chapter is dedicated to answering the above four questions by offering general results on the counter-monotonic inf-convolution. The rest of the chapter is organized as follows. Sections 2.2 and 2.3 contain preliminaries on risk measures and on risk sharing

problems, respectively. In particular, Section 2.3 introduces the new concept of counter-monotonic inf-convolution, along with some related discussions on negative dependence. In Sections 2.4 and 2.5, we analyze the counter-monotonic risk sharing problem for risk-averse agents and risk-seeking agents, respectively. For risk-averse agents, all three forms of inf-convolution lead to the same optimal value (Theorem 2.1). We also characterize conditions for the distortion function that yields equality between the value of the inf-convolution and that of the risk measure (Theorems 2.2 and 2.3). Based on the counter-monotonic improvement theorem in the literature (reported as Theorem 2.4), explicit formulas for counter-monotonic inf-convolutions and optimal allocations are obtained for risk-seeking agents (Theorem 2.5). Applying these results, we solve the portfolio optimization problem, demonstrating how decision-making varies with agents' risk attitudes, as detailed in Section 2.6. In Section 2.7, we consider agents with inverse S-shaped distortion functions, who are neither risk-averse nor risk-seeking, and obtain an explicit formula for the corresponding counter-monotonic inf-convolution (Theorem 2.6) under mild conditions. Section 2.8 concludes the chapter.

2.2 Setting

2.2.1 Risk measures, comonotonicity, and counter-monotonicity

Fix an atomless probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{X}$ be a convex cone of random variables on $(\Omega, \mathcal{F}, \mathbb{P})$, which will be specified in subsequent sections to several different choices. For example, $\mathcal{X}$ may be the set L^1 of integrable random variables, the set L^∞ of bounded random variables, or the set L^+ of nonnegative bounded random variables. Almost surely equal random variables are treated as identical. We denote by $\mathbb{1}_A$ the indicator function for an event $A \in \mathcal{F}$. A risk measure is a mapping $\rho : \mathcal{X} \to [-\infty, \infty]$. We allow for both ∞ and $-\infty$ in the range of risk measures because we would like to treat inf-convolutions defined below as risk measures, and in some cases they can take infinite values.

We consider n agents, where n is a positive integer, and denote by $[n] = \{1, \dots, n\}$. A random vector (X, Y) is said to be comonotonic if $(X(\omega) - X(\omega'))(Y(\omega) - Y(\omega')) \geqslant 0$ for all $\omega, \omega' \in \Omega$. We refer to Dhaene et al. (2002) for an overview on comonotonicity. A pair (X, Y) of random variables is said to be counter-monotonic if $(X, -Y)$ is comonotonic, i.e., $(X(\omega) - X(\omega'))(Y(\omega) - Y(\omega')) \leqslant 0$ for all $\omega, \omega' \in \Omega$. A random vector $(X_1, \dots, X_n)$ is comonotonic (resp. counter-monotonic) if each pair of its components is comonotonic (resp. counter-monotonic). Comonotonicity of $(X_1, \dots, X_n)$ is equivalent to the existence of increasing functions $f_i : \mathbb{R} \mapsto \mathbb{R}$, $i \in [n]$, such that $X_i = f_i(\sum_{i=1}^n X_i)$ for $i \in [n]$. Terms like "increasing" or "decreasing" are in the non-strict sense. Although comonotonicity for $n \geqslant 3$ is a straightforward extension of the case $n = 2$ and well understood, counter-monotonicity for $n \geqslant 3$ is highly restrictive on the marginal distributions, and quite different from the case $n = 2$. A stochastic representation of counter-monotonicity for $n \geqslant 3$ is provided in Lauzier et al. (2023), which we present in Proposition 2.1 below. To emphasize this difference, we sometimes mention "pairwise counter-monotonicity" for counter-monotonicity in dimension 3 or higher.

We will model agents' preferences by the class of distortion risk measures. Equivalently, these agents are modelled by the dual utility of Yaari (1987), with minimization problems switched to maximization problems. To analyze the risk sharing problems, it will be convenient to work with a more general class than distortion risk measures, called *distortion riskmetrics* by Wang et al. (2020a), which we explain below. Let

$$\mathcal{H}^{\mathrm{BV}} = \{h : [0, 1] \to \mathbb{R} \mid h \text{ is of bounded variation and } h(0) = 0\}.$$

For $h \in \mathcal{H}^{\mathrm{BV}}$, we first define the Choquet integral for a random variable X by

$$\int X \mathrm{d}\,(h \circ \mathbb{P}) = \int_0^\infty h(\mathbb{P}(X > x))\mathrm{d}x + \int_{-\infty}^0 (h(\mathbb{P}(X > x)) - h(1))\mathrm{d}x,$$

provided that the above is well-defined. A distortion riskmetric $\rho_h : \mathcal{X} \to \mathbb{R}$ is $\rho_h(X) = \int X \mathrm{d}\,(h \circ \mathbb{P})$, where $\mathcal{X}$ is such that the Choquet integral is well-defined. Elements of $\mathcal{H}^{\mathrm{BV}}$

are called distortion functions, and they are not necessarily monotone. Denote by

$$\mathcal{H} = \{h : [0,1] \to [0,1] \mid h \text{ is increasing and } h(0) = 1 - h(1) = 0\},$$

which is a subset of $\mathcal{H}^{\mathrm{BV}}$. In case of $h \in \mathcal{H}$, ρ_h is a *distortion risk measure*. Distortion riskmetrics satisfy many useful properties. For any $h \in \mathcal{H}^{\mathrm{BV}}$, the following properties are satisfied by $\rho = \rho_h$: For all $X, Y \in \mathcal{X}$,

(a) Law-invariance: $\rho(X) = \rho(Y)$ if X and Y have the same distribution, i.e., $X \overset{\mathrm{d}}{=} Y$;

(b) Positive homogeneity: $\rho(\lambda X) = \lambda \rho(X)$ for all $\lambda > 0$;

(c) Comonotonic additivity: $\rho(X + Y) = \rho(X) + \rho(Y)$ if X and Y are comonotonic.

If $h \in \mathcal{H}$, then the following two additional properties hold:

(d) Monotonicity: $\rho(X) \leqslant \rho(Y)$ if $X \leqslant Y$;

(e) Translation invariance: $\rho(X + c) = \rho(X) + c$ for $c \in \mathbb{R}$, and $X + c \in \mathcal{X}$.

If $h \in \mathcal{H}^{\mathrm{BV}}$ is concave, $\rho = \rho_h$ further satisfies

(f) Subadditivity: $\rho(X + Y) \leqslant \rho(X) + \rho(Y)$;

(g) Convex order consistency: $\rho(X) \leqslant \rho(Y)$ if $X \leqslant_{\mathrm{cx}} Y$, where the inequality is the convex order, meaning $\mathbb{E}[u(X)] \leqslant \mathbb{E}[u(Y)]$ for all convex functions u such that the two expectations are well-defined.

A coherent risk measure (Artzner et al. (1999)) is one that satisfies (b), (d), (e) and (f). In (c) and (f), we require that ρ do not take both values $-\infty$ and ∞. The following are equivalent for ρ_h (see Wang et al. (2020b, Theorem 3)): (i) h is concave; (ii) ρ_h is subadditive; (iii) ρ_h is convex; (iv) ρ_h is convex order consistent. Moreover, comonotonic additivity and law-invariance plus some continuity characterize the class of ρ_h for $h \in \mathcal{H}^{\mathrm{BV}}$.

2.2.2 Risk sharing and inf-convolution

In risk management and game theory, the concept of risk sharing, often referred to as risk allocation, involves distributing the aggregate risk or wealth among multiple agents. A common approach to optimally distribute the risk is by minimizing the overall value of the aggregate risk. We assume there are n agents sharing a total loss $X \in \mathcal{X}$ in the market. Suppose that agent $i \in [n]$ has a risk preference modelled by a risk measure ρ_i. Given $X \in \mathcal{X}$, we define the set of allocations of X as

$$\mathbb{A}_n(X) = \left\{ (X_1, \dots, X_n) \in \mathcal{X}^n : \sum_{i=1}^n X_i = X \right\}. \tag{2.2}$$

The allocation set consists of all potential ways to distribute the total risk X among n agents and the associated aggregate risk value is $\sum_{i=1}^n \rho_i(X_i)$. Note that the definition of allocations crucially depends on the specification of $\mathcal{X}$, which will vary across different applications in the later sections.

Using (2.2), the inf-convolution $\square_{i=1}^n \rho_i$ of n risk measures $\rho_1, \dots, \rho_n$ is defined as

$$\square_{i=1}^n \rho_i(X) := \inf \left\{ \sum_{i=1}^n \rho_i(X_i) : (X_1, \dots, X_n) \in \mathbb{A}_n(X) \right\}, \quad X \in \mathcal{X}. \tag{2.3}$$

That is, the inf-convolution of n risk measures is the infimum over aggregate risk values for all possible allocations. Like $\mathbb{A}_n(X)$, we make the reliance of $\square_{i=1}^n \rho_i$ on $\mathcal{X}$ implicit, but it is useful to keep in mind that $\mathcal{X}$ matters in this definition; that is, for $\mathcal{X} = L^\infty$ and $\mathcal{X} = L^+$, the inf-convolution may differ for the same set of risk measures well-defined on L^∞.

An allocation $(X_1, \dots, X_n)$ is sum-optimal in $\mathbb{A}_n(X)$ if $\square_{i=1}^n \rho_i(X) = \sum_{i=1}^n \rho_i(X_i)$, i.e., it attains the best total risk value. An allocation $(X_1, \dots, X_n) \in \mathbb{A}_n(X)$ is Pareto optimal in $\mathbb{A}_n(X)$ if for any $(Y_1, \dots, Y_n) \in \mathbb{A}_n(X)$ satisfying $\rho_i(Y_i) \leqslant \rho_i(X_i)$ for all $i \in [n]$, we have $\rho_i(Y_i) = \rho_i(X_i)$ for all $i \in [n]$. Pareto optimality means that the allocation cannot be

improved for all agents with one agent being strictly improved.

It is well-known that for finite monetary risk measures that satisfies monotonicity and translation-invariance, Pareto optimality is equivalent to sum-optimality; see Embrechts et al. (2018, Proposition 1). Since we focus on monetary risk measures (particularly, distortion risk measures) in this chapter, we will simply say that an allocation is optimal if sum-optimality holds.

2.3 Comonotonic and counter-monotonic risk sharing

As in other multivariate models in risk management, the dependence structure among the components of the risk allocation is important for interpreting the economic incentives created by the allocation. For instance, if the allocation is comonotonic, then all agents receive gains and losses together; if the allocation is counter-monotonic, then all agents are essentially gambling against each other. In some situations, e.g., in an insurance setting, it may be preferred or mandatory to allocate the aggregate risk in a comonotonic way among agents, who are insureds and insurers in this context. On the other hand, in a lottery setting, the allocation of payoffs may be counter-monotonic.

We will consider risk sharing problems constrained to comonotonic or counter-monotonic allocations. The comonotonic risk sharing problem is well studied in the risk management and insurance literature; see e.g., Jouini et al. (2008), Cui et al. (2013), and Boonen et al. (2021). The counter-monotonic problem would be the focus of this chapter, though we will compare these three types of risk sharing problems throughout. The set of comonotonic allocations is defined as

$$\mathbb{A}_n^+(X) = \{(X_1, \dots, X_n) \in \mathbb{A}_n(X) : X_1, \dots, X_n \text{ are comonotonic}\}.$$

The corresponding comonotonic inf-convolution $\boxplus_{i=1}^n \rho_i$ of risk measures $\rho_1, \dots, \rho_n$ is defined

as

$$\boxplus_{i=1}^{n} \rho_i(X) := \inf\left\{\sum_{i=1}^{n} \rho_i\left(X_i\right) : (X_1, \dots, X_n) \in \mathbb{A}_n^+(X)\right\}.$$

An allocation $(X_1, \dots, X_n) \in \mathbb{A}_n^+(X)$ is called an optimal allocation of X for $(\rho_1, \dots, \rho_n)$ within $\mathbb{A}_n^+$ if $\sum_{i=1}^n \rho_i\left(X_i\right) = \boxplus_{i=1}^n \rho_i(X)$. By definition, it is clear that $\square_{i=1}^n \rho_i(X) \leqslant \boxplus_{i=1}^n \rho_i(X)$. Hence, if an optimal allocation of X is comonotonic, then it is also an optimal allocation within $\mathbb{A}_n^+$, and further we have $\square_{i=1}^n \rho_i(X) = \boxplus_{i=1}^n \rho_i(X)$. For law-invariant convex risk measures on L^∞, optimal constrained allocations are also optimal allocations; see Jouini et al. (2008). This statement remains true if the underlying risk measures preserve convex order, and this is based on the comonotone improvement of Landsberger and Meilijson (1994).

Our new invention is the risk sharing framework where allocations are restricted in the set of counter-monotonic allocations, which can be rigorously formulated below, by

$$\mathbb{A}_n^-(X) = \left\{(X_1, \dots, X_n) \in \mathbb{A}_n(X) : X_1, \dots, X_n \text{ are counter-monotonic}\right\}.$$

The corresponding counter-monotonic inf-convolution $\boxminus_{i=1}^n \rho_i$ is thus defined as

$$\boxminus_{i=1}^{n} \rho_i(X) = \inf\left\{\sum_{i=1}^{n} \rho_i(X_i) : (X_1, \dots, X_n) \in \mathbb{A}_n^-(X)\right\}.$$

Similarly, an allocation $(X_1, \dots, X_n) \in \mathbb{A}_n^-(X)$ is called an optimal allocation of X within $\mathbb{A}_n^-$ if $\sum_{i=1}^n \rho_i\left(X_i\right) = \boxminus_{i=1}^n \rho_i(X)$. It is clear that $\square_{i=1}^n \rho_i(X) \leqslant \boxminus_{i=1}^n \rho_i(X)$. In contrast to the rich literature on comonotonic risk sharing, research on counter-monotonic risk sharing problem is quite limited. Counter-monotonic allocations have been recently explored by Lauzier et al. (2026a), but with a different approach than ours. More precisely, instead of working with $\mathbb{A}_n^-(X)$ directly, Lauzier et al. (2026a) considered conditions under which the original problem within $\mathbb{A}_n$ has an optimal solution within $\mathbb{A}_n^-$.

Before we analyze the counter-monotonic risk sharing problem, we first recall the stochastic representation of counter-monotonicity given by Lauzier et al. (2026a), which

will be useful throughout our analysis. Let Π_n be the set of all n-compositions (ordered partitions) of Ω, that is,

$$\Pi_n = \left\{ (A_1, \dots, A_n) \in \mathcal{F}^n : \bigcup_{i \in [n]} A_i = \Omega \text{ and } A_1, \dots, A_n \text{ are disjoint} \right\}.$$

Proposition 2.1 (Lauzier et al. (2026a)). *For $X \in \mathcal{X}$ and $n \geqslant 3$, suppose that at least three of $(X_1, \dots, X_n) \in \mathbb{A}_n(X)$ are non-degenerate. Then $(X_1, \dots, X_n)$ is counter-monotonic if and only if there exist constants $m_1, \dots, m_n$ and $(A_1, \dots, A_n) \in \Pi_n$ such that*

$$X_i = (X - m)\mathbb{1}_{A_i} + m_i \quad \text{for all } i \in [n] \text{ with } m = \sum_{i=1}^{n} m_i \leqslant \operatorname{ess\text{-}inf} X \tag{2.4}$$

or

$$X_i = (X - m)\mathbb{1}_{A_i} + m_i \quad \text{for all } i \in [n] \text{ with } m = \sum_{i=1}^{n} m_i \geqslant \operatorname{ess\text{-}sup} X. \tag{2.5}$$

By taking $m_1 = \ldots = m_n = 0$, i.e., $X \geqslant 0$ or $X \leqslant 0$, a simple counter-monotonic allocation in the form of (2.4) and (2.5) is given by

$$X_i = X\mathbb{1}_{A_i} \text{ for each } i \in [n], \text{ where } (A_1, \dots, A_n) \in \Pi_n.$$

Specifically, such an allocation is called a *jackpot allocation* if $X \geqslant 0$ and a *scapegoat allocation* if $X \leqslant 0$ by Lauzier et al. (2026a). It is clear to see that there is a "winner-takes-all" structure in a jackpot allocation and a "loser-loses-all" structure in a scapegoat allocation. In real life, such $(X_1, \dots, X_n)$ may represent the outcome of n lottery tickets, exactly one of which wins a random reward of X. "Roulette wheel decisions"[1] is an example for scapegoat allocations.

[1]For tasks no one wants to do, a roulette wheel with everyone's names on it can be spun to determine who has to undertake the task.

A special form of jackpot and scapegoat allocations will be useful in our analysis. The allocation $(X_1, \dots, X_n)$ of X with either $X \geqslant 0$ or $X \leqslant 0$ is called a *uniform counter-monotonic allocation* if $X_i = X\mathbb{1}_{A_i}$ with $\mathbb{P}(A_i) = 1/n$ for each $i \in [n]$, where $(A_1, \dots, A_n) \in \Pi_n$ is independent of X. For any $X \geqslant 0$ or $X \leqslant 0$, a uniform counter-monotonic allocation exists as soon as there exists a (nondegenerate) uniform random variable independent of X.

Both the set of comonotonic allocations $\mathbb{A}_n^+(X)$ and the set of counter-monotonic allocations $\mathbb{A}_n^-(X)$ are strict subsets of the set of all potential allocations $\mathbb{A}_n(X)$. Hence, the sequel refers to the problem of sharing risk in $\mathbb{A}_n(X)$, $\mathbb{A}_n^+(X)$ and $\mathbb{A}_n^-(X)$ as unconstrained, comonotonic and counter-monotonic risk sharing, respectively. A first and natural question is whether $\boxplus_{i=1}^n \rho_i(X)$ and $\boxminus_{i=1}^n \rho_i(X)$ admit a clear relation in terms of their values. In the next few sections, we will answer this question and several others mentioned in the Introduction.

Although we have introduced the framework of counter-monotonic risk sharing with possibly different risk measures for each agent, in the rest of the chapter, we will always consider the case of homogeneous agents; that is, all agents have the same risk measure, and it is a distortion risk measure. This homogeneous setting, although being restrictive, already gives rise to interesting mathematical results. We leave the heterogeneous case, as well as other decision models, for future study.

2.4 Risk-averse agents and pseudo-concave capacities

In this section, we take $\mathcal{X}$ as the set L^∞ of bounded random variables. Assume that all agents use the same distortion risk measure ρ_h. We first present a general result that holds for all types of distortion functions h, in which a clear ordering among the unconstrained, comonotonic and counter-monotonic inf-convolutions is provided, and they become equivalent when h is concave.

Theorem 2.1. *For $h \in \mathcal{H}$, the following always holds:*

$$\boxdot_{i=1}^{n} \rho_h \leqslant \boxminus_{i=1}^{n} \rho_h \leqslant \boxplus_{i=1}^{n} \rho_h = \rho_h. \tag{2.6}$$

Moreover, if h is concave, then

$$\boxdot_{i=1}^{n} \rho_h = \boxminus_{i=1}^{n} \rho_h = \boxplus_{i=1}^{n} \rho_h = \rho_h. \tag{2.7}$$

Proof. For the first inequality in (2.6), we have seen above that the unconstrained inf-convolution is the smallest since there are no constraints on the allocation set. This relation also holds for agents with other risk preferences than distortion risk measures. The second inequality in (2.6) holds by taking allocation $(X_1, \dots, X_n) = (X, 0, \dots, 0)$, that is both counter-monotonic and comonotonic, thus leading to $\boxminus_{i=1}^{n} \rho_h \leqslant \rho_h$ and $\boxplus_{i=1}^{n} \rho_h \leqslant \rho_h$. The last equality in (2.6) follows directly from comonotonic additivity of ρ_h. To show (2.7), the inequality $\boxdot_{i=1}^{n} \rho_h \geqslant \rho_h$ follows from subadditivity of ρ_h. □

The general chains of relations in (2.6) and (2.7) do not necessarily generalize to risk measures that are not comonotonic additive, or to heterogeneous risk preferences among agents.

In the following examples, we compare the three inf-convolutions in (2.6) for the Value-at-Risk (VaR) and the Expected Shortfall (ES, also known as CVaR and TVaR), defined below. For a random variable X, VaR at level $\alpha \in \mathbb{R}^+ := [0, \infty)$ is defined as

$$\mathrm{VaR}_\alpha(X) = \inf\{x \in [-\infty, \infty] : \mathbb{P}(X \leqslant x) \geqslant 1 - \alpha\}, \tag{2.8}$$

and ES at level $\beta \in \mathbb{R}^+ := [0, 1)$ is defined as

$$\mathrm{ES}_\beta(X) = \frac{1}{\beta} \int_0^\beta \mathrm{VaR}_\gamma(X) \mathrm{d}\gamma,$$

where VaR_γ is defined in (2.8). Here we allow $\alpha > 1$ following the convention in Embrechts

et al. (2018). If $\alpha \in [0,1)$, VaR_α and ES_α are distortion risk measures. They are well-defined on the set of all random variables, and they are associated with the distortion functions $h(t) = \mathbb{1}_{\{t>\alpha\}}$ and $h(t) = \min\{t/\alpha, 1\}$, respectively.

Example 2.1 (VaR)**.** We now analyze the relationship among counter-monotonic, comonotonic and unconstrained optimal allocations in the risk sharing problem for VaR agents with levels $\alpha_1, \dots, \alpha_n, \alpha \in (0,1)$. Here, we assume $\alpha_i = \alpha$ for each $i \in [n]$. In particular, the values of $\square_{i=1}^n \mathrm{VaR}_\alpha(X)$ and $\boxplus_{i=1}^n \mathrm{VaR}_\alpha(X)$ are given by Theorem 2 and Proposition 5 of Embrechts et al. (2018). They yield

$$\operatorname*{\square}_{i=1}^{n} \mathrm{VaR}_\alpha(X) = \mathrm{VaR}_{n\alpha}(X) \leqslant \operatorname*{\boxplus}_{i=1}^{n} \mathrm{VaR}_\alpha(X) = \mathrm{VaR}_\alpha(X). \tag{2.9}$$

Moreover, if $\alpha < 1/n$, by Embrechts et al. (2018, Theorem 2), there exists a Pareto-optimal allocation $(X_1, \dots, X_n)$ of X with the form of

$$X_i = (X - m)\mathbb{1}_{A_i}, i \in [n-1] \quad \text{and} \quad X_n = (X - m)\mathbb{1}_{A_n} + m$$

for some $(A_1, \dots, A_n) \in \Pi_n$, where $m \in (-\infty, \mathrm{VaR}_{n\alpha}(X)]$ is a constant. By taking $m = \text{ess-inf}X$, $(X_1, \dots, X_n)$ is counter-monotonic, as noticed by Lauzier et al. (2023, Theorem 1). Hence, $\boxminus_{i=1}^n \mathrm{VaR}_\alpha(X) = \square_{i=1}^n \mathrm{VaR}_\alpha(X)$, and thus the inf-convolution of several VaRs is equivalent to the counter-monotonic one. Therefore, for any $X \in \mathcal{X}$, the following always holds true

$$\operatorname*{\boxminus}_{i=1}^{n} \mathrm{VaR}_\alpha(X) = \operatorname*{\square}_{i=1}^{n} \mathrm{VaR}_\alpha(X) \leqslant \operatorname*{\boxplus}_{i=1}^{n} \mathrm{VaR}_\alpha(X) \tag{2.10}$$

and generally the inequality in (2.10) is not an equality. Furthermore, we can analogously show that the result (2.10) also holds true for RVaR agents. It is notable that the distortion risk function corresponding to VaR is neither concave nor convex, which implies that concavity or convexity of h is not necessary for $\boxminus_{i=1}^n \rho_h$ being equivalent to $\square_{i=1}^n \rho_h$, although concavity is sufficient as we see in (2.7).

Example 2.2 (ES)**.** It is well known that ES has a concave distortion function, and hence,

by Theorem 2.1, $\square_{i=1}^{n} \mathrm{ES}_\beta = \boxminus_{i=1}^{n} \mathrm{ES}_\beta = \boxplus_{i=1}^{n} \mathrm{ES}_\beta = \mathrm{ES}_\beta$.

Without assuming that h is concave, solving the counter-monotonic inf-convolution poses a challenge, as is the case of the unconstrained problem. Following Theorem 1 of Aouani et al. (2021), which shows that counter-monotonic superadditive Choquet functional can be characterized by pseudo-convex capacity, we provide some necessary and sufficient conditions for the equality $\boxminus_{i=1}^{n} \rho_h = \rho_h$. We first review some useful concepts in the theory of capacity.

A set function $\nu : \mathcal{F} \mapsto \mathbb{R}$ is called a *capacity* if $\nu(\emptyset) = 0$ and ν is *monotone*, i.e., $A \subseteq B \Rightarrow v(A) \leqslant v(B)$. Moreover, ν is *normalized* if $\nu(\Omega) = 1$. In particular, a capacity ν on $\mathcal{F}$ is called

(i) *concave* if

$$\nu(A \cup B) + \nu(A \cap B) \leqslant \nu(A) + \nu(B) \tag{2.11}$$

holds for all $A, B \in \mathcal{F}$;

(ii) *subadditive* if (2.11) holds for all disjoint $A, B \in \mathcal{F}$;

(iii) *concave at the sure event* if (2.11) holds for all $A, B \in \mathcal{F}$ with $A \cup B = \Omega$;

(iv) *pseudo-concave* if it is subadditive and concave at the sure event.

In the literature, concavity (convexity) of ν is also called submodularity (supremodularity). Clearly, concavity is stronger than the other three properties.

Let ν be a capacity. The Choquet integral of a bounded random variable X with respect to ν is defined as

$$\int X \, \mathrm{d}\nu = \int_0^\infty \nu(X > x)\mathrm{d}x + \int_{-\infty}^0 (\nu(X > x) - \nu(\Omega))\mathrm{d}x$$

The distortion risk measure ρ_h is a Choquet integral with $\nu = h \circ \mathbb{P}$ where ν is normalized. The properties of ν are closely related to those of h. For instance, concavity of ν is

equivalent to concavity of h; see Föllmer and Schied (2016) or Marinacci and Montrucchio (2004). Next, we introduce a property for $h \in \mathcal{H}$, which we refer to as dual subadditivity, that turns out to correspond to pseudo-concavity of ν.

Definition 2.1. Let $h \in \mathcal{H}$.

(i) The *dual function* of h is defined by $\tilde{h}(t) = 1 - h(1-t)$ for $t \in [0,1]$. It is clear that $\tilde{h} \in \mathcal{H}$.

(ii) The function h is *dually subadditive* if h satisfies subadditivity (i.e., $h(x+y) \leqslant h(x)+h(y)$ for $x, y \in [0,1]$ with $x+y \leqslant 1$) and $\tilde{h}(x)$ is superadditive (i.e., $\tilde{h}(x+y) \geqslant \tilde{h}(x)+\tilde{h}(y)$ for $x, y \in [0,1]$ with $x+y \leqslant 1$).

Note that h is concave if and only $\tilde{h}$ is convex. It is straightforward to check that any concave function $h \in \mathcal{H}$ is dually subadditive. The following theorem gives a necessary and sufficient condition for $\rho_h \boxminus \rho_h = \rho_h$.

Theorem 2.2. *Suppose $h \in \mathcal{H}$ and denote by $\nu = h \circ \mathbb{P}$. The following are equivalent.*

(i) $\rho_h \boxminus \rho_h(X) = \rho_h(X)$ *holds for all* $X \in \mathcal{X}$.

(ii) h *satisfies dual subadditivity.*

(iii) ν *is pseudo-concave.*

Proof. We first show the implication (i) $\Rightarrow$ (ii). Clearly, statement (i) implies $\rho_h(X) \leqslant \rho_h(X_1) + \rho_h(X_2)$ for $(X_1, X_2) \in \mathbb{A}_2^-(X)$. Let $A, B \in \mathcal{F}$ be such that $A \cap B = \emptyset$. It is straightforward to show h is subadditive by taking $X_1 = \mathbb{1}_A$ and $X_2 = \mathbb{1}_B$. Moreover, for such A and B, it can be verified that $X_1 = \mathbb{1}_{A \cup B}$ and $X_2 = \mathbb{1}_{A^{\mathrm{c}}}$ are counter-monotonic (see Table 2.1). It follows that $h(x+y-1)+1 \leqslant h(x)+h(y)$ for $x, y \in [0,1]$ with $x+y \geqslant 1$, which is equivalent to superadditivity of $\tilde{h}$ by some simple manipulations.

	A	B	$A^{\mathsf{c}} \cap B^{\mathsf{c}}$
$\mathbb{1}_{A\cup B}$	1	1	0
$\mathbb{1}_{A^{\mathsf{c}}}$	0	1	1
$\mathbb{1}_{A\cup B} + \mathbb{1}_{A^{\mathsf{c}}}$	1	2	1

Table 2.1: A counter-monotonic allocation

Next, we show (ii) $\Rightarrow$ (iii). Statement (ii) can be rewritten as h being subadditive and satisfying $h(x+y-1)+1 \leqslant h(x)+h(y)$ for $x, y \in [0,1]$ with $x+y \geqslant 1$, which is equivalent to pseudo-concavity of ν; see Principi et al. (2025, Example 14).

For a proof of (iii) $\Rightarrow$ (i), first note that $\rho_h \boxminus \rho_h(X) \leqslant \rho_h(X)$ by Theorem 2.1. Thus it suffices to show ρ_h is counter-monotonic subadditive, which directly follows Theorem 1 of Aouani et al. (2021). □

As we know, pairwise counter-monotonicity is the generalization of counter-monotonicity for the case of $n \geqslant 3$. The next result shows that Theorem 2.2 can be generalized to the case of n agents with identical distortion risk functions.

Theorem 2.3. *Let $h \in \mathcal{H}$ and $n \geqslant 3$.*

(i) $\boxminus_{i=1}^{n} \rho_h = \rho_h$ holds true if and only if h is dually subadditive.

(ii) $\square_{i=1}^{n} \rho_h = \rho_h$ holds true if and only if h is concave.

(iii) $\boxplus_{i=1}^{n} \rho_h = \rho_h$ holds true for all h.

Proof. (i) It is straightforward to show the "only if" part by Theorem 2.2. Next, we demonstrate the "if" statement. To prove whether the theorem can be generalized to n agents for $n \geqslant 3$, it suffices to show whether the inf-convolution of n risk measures can be understood as applying the inf-convolutions of two risk measures repeatedly. Specifically, we aim to show $\boxminus_{i=1}^{n} \rho_h = \rho_h \boxminus \ldots \boxminus \rho_h$, where the right-hand side has n terms of ρ_h. For simplicity, we denote the repeated inf-convolution of n risk measures from left to right as $\rho_h \boxminus \ldots \boxminus \rho_h$.

For a given $X \in \mathcal{X}$, we define a new set of allocations of X, that is,

$$\mathbb{A}_n^*(X) = \left\{ (X_1, \dots, X_n) \in \mathbb{A}_n(X) : \sum_{i=1}^{j-1} X_i \text{ and } X_j \text{ are counter-monotonic for } j \in [n] \right\}.$$

Then $\rho_h \boxminus \dots \boxminus \rho_h$ can be expressed as:

$$\rho_h \boxminus \dots \boxminus \rho_h(X) = \inf \left\{ \sum_{i=1}^{n} \rho_h(X_i) : (X_1, \dots, X_n) \in \mathbb{A}_n^*(X) \right\}. \tag{2.12}$$

We first show that $\rho_h \boxminus \dots \boxminus \rho_h \leqslant \boxminus_{i=1}^n \rho_h$. In fact, the inequality also holds true when applying multiple ρ_i for $i \in [n]$. The result directly follows part (ii) of Theorem 2 in Lauzier et al. (2023), implying $\mathbb{A}_n^-(X)$ is a subset of $\mathbb{A}_n^*(X)$.

Next, we show the converse direction. For any $(X_1, \dots, X_n) \in \mathbb{A}_n^*(X)$, by repeatedly using dually subadditivity of h and Theorem 2.2, we have

$$\begin{aligned}
\sum_{i=1}^{n} \rho_h(X_i) &= \rho_h(X_1) + \rho_h(X_2) + \sum_{i=3}^{n} \rho_h(X_i) \\
&\geqslant \rho_h(X_1 + X_2) + \sum_{i=3}^{n} \rho_h(X_i) \\
&\geqslant \rho_h(X_1 + \dots + X_{n-1}) + \rho_h(X_n) \\
&\geqslant \rho_h(X) \geqslant \boxminus_{i=1}^{n} \rho_h(X),
\end{aligned}$$

which implies that $\rho_h \boxminus \dots \boxminus \rho_h \geqslant \boxminus_{i=1}^n \rho_h$. Combining the above results, we can conclude that $\boxminus_{i=1}^n \rho_h = \rho_h \boxminus \dots \boxminus \rho_h = \rho_h$.

(ii) The "if" part directly follows Theorem 2.1. It is also straightforward to prove the "only if" part since concavity of h is equivalent to subadditivity of ρ_h.

(iii) The result holds trivially from comonotonic additivity of ρ_h. □

Remark 2.1. The inf-convolution of n risk measures $\square_{i=1}^n \rho_i$ is equal to the repeated appli-

cation of inf-convolutions of two risk measures; see Liu et al. (2020, Lemma 2). That is, $\square_{i=1}^{n} \rho_i = \rho_1 \square \ldots \square \rho_n$. Similarly, it also holds that $\boxplus_{i=1}^{n} \rho_i = \rho_1 \boxplus \ldots \boxplus \rho_n$. The relation does not hold true for the counter-monotonic inf-convolution. A counter-example is given by Example 2.6 in Section 2.10, showing

$$\boxminus_{i=1}^{n} \rho_i(X) > \rho_1 \boxminus \cdots \boxminus \rho_n(X) \quad \text{for some } X \in \mathcal{X}.$$

This is due to the fact that counter-monotonicity behaves quite differently in the cases $n = 2$ and $n \geqslant 3$. Nevertheless, $\boxminus_{i=1}^{n} \rho_i = \rho_1 \boxminus \cdots \boxminus \rho_n$ can be expected in many relevant cases. For instance, it holds true for agents with dually subadditive distortion functions, as shown in Theorem 2.3. Additionally, it also holds when risks are measured with VaRs because the unconstrained inf-convolution and counter-monotonic inf-convolution coincide, as shown in Example 2.1.

We can immediately obtain the following corollary of Theorem 2.3.

Corollary 2.1. *Suppose $\mathcal{X} = L^\infty$. If $h \in \mathcal{H}$ is convex and not the identity, then a comonotonic allocation of X is never Pareto optimal.*

Proof. Let $\mathcal{X} = L^\infty$. We first assume some comonotonic allocations of X are Pareto optimal. Thus, there exists an allocation $(X_1, \ldots, X_n) \in \mathbb{A}_n^+$ such that

$$\rho_h(X) = \sum_{i=1}^{n} \rho_h(X_i) = \boxplus_{i=1}^{n} \rho_h(X).$$

The first equality is from the comonotonic additivity of ρ_h and the second is from the Pareto optimality of $(X_1, \ldots, X_n)$. By Theorem 2.3, the second equality implies that h is concave, a contradicting h being convex and not the identity. Therefore, a comonotonic allocation $(X_1, \ldots, X_n) \in \mathbb{A}_n^+$ cannot be Pareto optimal. □

So far, we have seen that for concave distortion functions, the counter-monotonic inf-convolution gives equivalent results to the comonotonic and unconstrained ones. The situ-

ation is drastically different when risk-seeking agents are involved. This will be discussed in the next section.

2.5 Risk-seeking agents

In this section, our aim is to investigate the risk sharing problem for risk-seeking agents and further determine the explicit form of the corresponding counter-monotonic inf-convolutions. The set of random variables $\mathcal{X}$ in this section may be the set L^∞ of bounded random variables, the set L^+ of nonnegative bounded random variables, or the set L^- of nonpositive bounded random variables. We will specify which set is used in each case. As we know, it always holds that $\boxminus_{i=1}^n \rho_h \leqslant \rho_h$. Counter-monotonic risk sharing among risk-averse agents achieves the upper bound ρ_h, as shown in Theorem 2.3.

We now take a small detour to consider sup-convolution instead of inf-convolution. The following proposition shows ρ_h is indeed the supremum of aggregate risks for risk-seeking agents over all possible allocations, whether in the general sense or in the counter-monotonic sense.

Proposition 2.2. *Suppose $\mathcal{X} = L^\infty$ and $X \in \mathcal{X}$. If $h \in \mathcal{H}$ is convex, then*

$$\begin{aligned}\rho_h(X) &= \sup\left\{\sum_{i=1}^n \rho_h(X_i) : (X_1,\dots,X_n) \in \mathbb{A}_n(X)\right\}\\ &= \sup\left\{\sum_{i=1}^n \rho_h(X_i) : (X_1,\dots,X_n) \in \mathbb{A}_n^-(X)\right\}.\end{aligned}$$

Proof. If h is convex, then ρ_h is superadditive (Wang et al. (2020b, Theorem 3)), implying

$$\begin{aligned}\rho_h(X) &\geqslant \sup\left\{\sum_{i=1}^n \rho_h(X_i) : (X_1,\dots,X_n) \in \mathbb{A}_n(X)\right\}\\ &\geqslant \sup\left\{\sum_{i=1}^n \rho_h(X_i) : (X_1,\dots,X_n) \in \mathbb{A}_n^-(X)\right\} \geqslant \rho_h(X),\end{aligned}$$

where the last inequality is due to $(X, 0, \dots, 0) \in \mathbb{A}_n^-(X)$. This gives the desired equalities. □

Proposition 2.2 shows that ρ_h is indeed the sup-convolution of several concave distortion risk measures. Unlike the comonotonic risk sharing problem, where we always have $\boxplus_{i=1}^n \rho_h = \rho_h$ for any h, regardless of h is concave or convex, the counter-monotonic risk sharing is more complicated. In particular, for risk-seeking agents, the equality in $\boxminus_{i=1}^n \rho_h \leqslant \rho_h$ generally does not hold true, which can also be verified by some numerical results in Table 2.1.

The technique of counter-monotonic improvement theorem, a converse to the comonotonic improvements, as introduced in Lauzier et al. (2026a), will be helpful for our next results. In what follows, a uniform random variable is nondegenerate; that is, we do not consider a constant random variable as uniform.

Theorem 2.4 (Lauzier et al. (2026a))**.** *Let $X_1, \dots, X_n \in L^1$ be nonnegative and $X = \sum_{i=1}^n X_i$. Assume that there exists a uniform random variable U independent of X. Then, there exists $(Y_1, \dots, Y_n) \in \mathbb{A}_n(X)$ such that (i) $(Y_1, \dots, Y_n)$ is counter-monotonic; (ii) $Y_i \geqslant_{\mathrm{cx}} X_i$ for $i \in [n]$; (iii) $Y_1, \dots, Y_n$ are nonnegative. Moreover, $(Y_1, \dots, Y_n)$ can be chosen as a jackpot allocation.*

The counter-monotonic improvement theorem indicates the jackpot allocation is always be preferred by risk-seeking agents. To apply the counter-monotonic improvement theorem, it is important to emphasize the technical assumption that there exists a uniform random variable U independent of X. To formalize this, we introduce the following set:

$$\mathcal{X}^{\perp} = \{X \in \mathcal{X} : \text{there exists a uniform random variable } U \text{ independent of } X\}.$$

Note that in any atomless probability space, $\mathcal{X}^{\perp}$ is non-empty. Indeed, we can find independent uniform random variables U and V on an atomless probability space, and $f(U) \in \mathcal{X}^{\perp}$ for all measurable functions f, and hence any distribution is allowed for elements of $\mathcal{X}^{\perp}$.

The next lemma provides the optimal value for the inf-convolution when the allocation is constrained to be jackpot allocations, given that the total risk X is nonpositive or nonnegative. The lemma would be helpful to establish our main result (Theorem 2.5).

Lemma 2.1. *Suppose $X \in \mathcal{X}^{\perp}$ and $h \in \mathcal{H}$ is convex. Denote by*

$$I(X) = \inf\left\{\sum_{i=1}^{n} \rho_h(X\mathbb{1}_{A_i}) : (A_1, \dots, A_n) \in \Pi_n\right\}.$$

The following statements hold.

(i) If $\mathcal{X} = L^+$, then $I = \rho_g$, where $g(t) = nh(t/n)$.

(ii) If $\mathcal{X} = L^-$, then $I = \rho_g$, where $g(t) = nh(1-(1-t)/n) - nh(1-1/n)$.

Proof. (i) Let $\mathcal{X} = L^+$ and $X \in \mathcal{X}^{\perp}$. It follows that

$$\begin{aligned} I(X) &= \inf\left\{\sum_{i=1}^{n} \int_0^{\infty} h(\mathbb{P}(X\mathbb{1}_{A_i} > x))\mathrm{d}x : (A_1, \dots, A_n) \in \Pi_n\right\} \\ &\geqslant \int_0^{\infty} \inf\left\{\sum_{i=1}^{n} h(x_i) : \sum_{i=1}^{n} x_i = \mathbb{P}(X > x)\right\} \mathrm{d}x \\ &= \int_0^{\infty} nh\left(\frac{1}{n}\mathbb{P}(X > x)\right) \mathrm{d}x. \end{aligned}$$

The last equality holds because of the convexity of h.

Furthermore, since we assume that there exists a uniform U independent of X, we can always find a composition $(A_1, \dots, A_n) \in \Pi_n$ independent of X such that $\mathbb{P}(A_i) = 1/n$ for all $i \in [n]$. Then we can obtain

$$\sum_{i=1}^{n} \rho_h(X\mathbb{1}_{A_i}) = \int_0^{\infty} nh\left(\frac{1}{n}\mathbb{P}(X > x)\right) \mathrm{d}x.$$

This implies that $I(X) \leqslant \rho_g(X)$, where $g(t) = nh(t/n)$, $t \in [0,1]$. Combining the above inequalities, the desired result is obtained.

(ii) Let $\mathcal{X} = L^-$ and $X \in \mathcal{X}^\perp$. It follows that

$$\begin{aligned} I(X) &= \inf\left\{\sum_{i=1}^n \int_{-\infty}^0 (h(\mathbb{P}(X\mathbb{1}_{A_i} > x)) - 1)\mathrm{d}x : (A_1,\dots,A_n) \in \Pi_n\right\} \\ &= \inf\left\{-\sum_{i=1}^n \int_{-\infty}^0 \tilde{h}(\mathbb{P}(X\mathbb{1}_{A_i} \leqslant x))\mathrm{d}x : (A_1,\dots,A_n) \in \Pi_n\right\} \\ &\geqslant -\int_{-\infty}^0 \sup\left\{\sum_{i=1}^n \tilde{h}(\mathbb{P}(X\mathbb{1}_{A_i} \leqslant x)) : (A_1,\dots,A_n) \in \Pi_n\right\}\mathrm{d}x \\ &= -n\int_{-\infty}^0 \tilde{h}\left(\frac{1}{n}\mathbb{P}(X \leqslant x)\right)\mathrm{d}x, \end{aligned}$$

where $\tilde{h}(t) = 1 - h(1-t)$. The last equality can be obtained by convexity of h, implying $\tilde{h}$ is concave. Furthermore, the above inequality can be rewritten as

$$\begin{aligned} I(X) &\geqslant -n\int_{-\infty}^0 \tilde{h}\left(\frac{1}{n}\mathbb{P}(X \leqslant x)\right)\mathrm{d}x \\ &= -n\int_0^\infty \tilde{h}\left(\frac{1}{n}\mathbb{P}(-X \geqslant x)\right)\mathrm{d}x = -\rho_{\tilde{h}^*}(-X) = \rho_g(X), \end{aligned}$$

where $\tilde{h}^*(t) = n\tilde{h}(t/n)$. The last equation directly follows from Wang et al. (2020b, Lemma 2), and $g(x)$ is given by

$$g(t) = \tilde{h}^*(1) - \tilde{h}^*(1-t) = n\tilde{h}\left(\frac{1}{n}\right) - n\tilde{h}\left(\frac{1-t}{n}\right) = nh\left(1 - \frac{1-t}{n}\right) - nh\left(1 - \frac{1}{n}\right).$$

Similar as (i), we can always find a composition $(A_1,\dots,A_n) \in \Pi_n$ independent of X such that $\mathbb{P}(A_i) = 1/n$ for all $i \in [n]$. Then we can obtain that

$$\sum_{i=1}^n \rho_h(X\mathbb{1}_{A_i}) = -\int_{-\infty}^0 n\tilde{h}\left(\frac{1}{n}\mathbb{P}(X \leqslant x)\right)\mathrm{d}x = \rho_g(X),$$

which implies that $I(X) \leqslant \rho_g(X)$. Therefore, the desired results are obtained. $\square$

The next result derives explicit formulas of the unconstrained and counter-monotonic inf-convolution of n concave distortion risk measures, as well as the optimal allocation. These formulas vary depending on the set of bounded random variables considered. In particular, when total risk X are nonnegative or nonpositive and we restrict the set of allocations to allocations satisfying $X_i \geqslant 0$ or $X_i \leqslant 0$, both the constrained inf-convolution and the counter-monotonic inf-convolution are identified as distortion riskmetrics. Furthermore, when both X and X_i are general bounded random variables, the counter-monotonic inf-convolution is determined to be negative infinity. The above results are mainly proved by using the technique of counter-monotonic theorem and Lemma 2.1.

Theorem 2.5. *Let $h \in \mathcal{H}$ be convex. The following hold.*

(i) If $\mathcal{X} = L^+$ and $X \in \mathcal{X}^\perp$, then

$$\square_{i=1}^{n} \rho_h(X) = \boxminus_{i=1}^{n} \rho_h(X) = \rho_g(X),$$

where $g(t) = nh(t/n)$ for $t \in [0,1]$.

(ii) If $\mathcal{X} = L^-$ and $X \in \mathcal{X}^\perp$, then

$$\square_{i=1}^{n} \rho_h(X) = \boxminus_{i=1}^{n} \rho_h(X) = \rho_g(X),$$

where $g(t) = nh(1-(1-t)/n) - nh(1-1/n)$ for $t \in [0,1]$.

(iii) If $\mathcal{X} = L^\infty$, $X \in \mathcal{X}^\perp$ and h is not the identity, then

$$\square_{i=1}^{n} \rho_h(X) = \boxminus_{i=1}^{n} \rho_h(X) = -\infty.$$

Moreover, in (i) and (ii), any uniform counter-monotonic allocation is Pareto-optimal.

Proof. (i) Suppose $\mathcal{X} = L^+$ and $X \in \mathcal{X}^\perp$. Let $(X_1, \dots, X_n) \in \mathbb{A}_n(X)$. By counter-monotonic improvement theorem, there always exists a jackpot allocation $(Y_1, \dots, Y_n) \in \mathbb{A}_n(X)$ such that $Y_i = X\mathbb{1}_{A_i}$, $Y_i \leqslant_{\mathrm{cv}} X_i$ and $Y_i \geqslant 0$ for each $i \in [n]$ and $(A_1, \dots, A_n) \in \Pi_n$. If $h \in \mathcal{H}$ is convex, then it holds $\rho_h(X\mathbb{1}_{A_i}) \leqslant \rho_h(X_i)$ for all $i \in [n]$. Denote by $I(X) = \inf\{\sum_{i=1}^n \rho_h(X\mathbb{1}_{A_i}), (A_1, \dots, A_n) \in \Pi_n\}$. It follows that

$$\boxminus_{i=1}^n \rho_h(X) \leqslant I(X) \leqslant \boxplus_{i=1}^n \rho_h(X). \tag{2.13}$$

Also, it is straightforward to verify $\boxplus_{i=1}^n \rho_h(X) \leqslant \boxminus_{i=1}^n \rho_h(X)$. From Lemma 2.1, $I(X)$ is determined by $\rho_{h^*}(X)$, where $h^*(t) = nh(t/n)$, $t \in [0, 1]$.

(ii) Next we assume $\mathcal{X} = L^-$ and $X \in \mathcal{X}^\perp$. By counter-monotonic improvement theorem, for any $(X_1, \dots, X_n) \in \mathbb{A}_n(X)$, there exists a jackpot allocation $(Y_1, \dots, Y_n) \in \mathbb{A}_n(X)$ such that $Y_i \leqslant_{\mathrm{cv}} X_i$ and $Y_i \leqslant 0$ for each $i \in [n]$. Similarly with the analysis in (i), the inequality (2.13) also holds true in the case of $\mathcal{X} = L^-$ but with different $I(X)$. From (ii) of Lemma 2.1, we can obtain

$$\boxminus_{i=1}^n \rho_h(X) = \boxplus_{i=1}^n \rho_h(X) = \rho_g(X). \tag{2.14}$$

where $g(t) = nh(1 - (1-t)/n) - nh(1 - 1/n)$.

(iii) In this case, we consider $\mathcal{X} = L^\infty$ and $X \in \mathcal{X}^\perp$. Take $m \in \mathbb{R}_+$ large enough such that $X + m \geqslant 0$. By translation invariance of ρ_h, it holds that

$$\begin{aligned}
\boxminus_{i=1}^n \rho_h(X) &= \inf\left\{\sum_{i=1}^n \rho_h\left(X_i + \frac{m}{n}\right) : (X_1, \dots, X_n) \in \mathbb{A}_n^-(X)\right\} - m \\
&= \inf\left\{\sum_{i=1}^n \rho_h(Y_i) : (Y_1, \dots, Y_n) \in \mathbb{A}_n^-(X+m)\right\} - m \\
&\leqslant \inf\left\{\sum_{i=1}^n \rho_h(Y_i) : (Y_1, \dots, Y_n) \in \mathbb{A}_n^-(X+m),\ Y_1, \dots, Y_n \geqslant 0\right\} - m \\
&= \rho_g(X+m) - m,
\end{aligned}$$

where g is given by $g(t) = nh(t/n)$ for $t \in [0,1]$, and the last equality is due to part (i). Since h is convex and not the identity, we have $g(1) < 1$, and hence $\rho_g(X+m) - m = \rho_g(X) + mg(1) - m \to -\infty$ as $m \to \infty$. Therefore, $\boxminus_{i=1}^n \rho_h(X) = -\infty$. □

Remark 2.2. The uniform counter-monotonic allocation identified in Theorem 2.5 also admits a natural fairness interpretation. Since $(A_1, \dots, A_n) \in \Pi_n$ is independent of X and $\mathbb{P}(A_i) = 1/n$, the random variables $X_i = X\mathbb{1}_{A_i}$ are identically distributed. Hence, by law-invariance of ρ_h, we have $\rho_h(X_1) = \cdots = \rho_h(X_n) = 1/n\rho_g(X)$. Thus, under the conditions of Theorem 2.5, the aggregate risk admits an allocation that is both sum-optimal and equal-risk. This observation connects the risk sharing problem with the equal-risk pricing and hedging approach of Marzban et al. (2022), where prices and hedging strategies are chosen so that the two sides of the contract have equal residual risk, with risks evaluated by convex risk measures. A general characterization of when Pareto optimality and the equal-risk property can hold simultaneously, beyond this symmetric homogeneous construction, remains an interesting direction for future work.

When the total risk X is either nonnegative or nonpositive, we observe that ρ_g in both (i) and (ii) of Theorem 2.5 is no longer a distortion risk measure as $g(1) \neq 1$, but it generally belongs to the class of distortion riskmetrics. Given a convex h, it directly follows from Theorem 2.5 that $\rho_g(X) \leqslant \rho_h(X)$ since $\boxplus_{i=1}^n \rho_h(X) = \boxminus_{i=1}^n \rho_h(X) \leqslant \rho_h(X)$. Notably, in the case of X being nonpositive, $\rho_g(X) \leqslant \rho_h(X)$ holds even though $g \geqslant h$; recall that if $f, h \in \mathcal{H}$, then Lemma 1 of Wang et al. (2020b) implies that $\rho_f \leqslant \rho_h$ if and only if $f \leqslant h$, and this emphasizes $g \notin \mathcal{H}$. We provide a numerical example to show the relation of $\boxplus_{i=1}^n \rho_h \leqslant \rho_h$ with h being convex. In this example, we consider a scenario with two agents in the pool. Take $h(t) = \Phi(\Phi^{-1}(t) + \lambda)$ and $\lambda = -0.6$. Some numerical results are presented in Table 2.2.

Example 2.3. We provide an example to illustrate how inf-convolutions vary with different risk preferences by specifying the distortion function as $h(t) = 1 - (1-t)^\alpha, \alpha \in \mathbb{R}_+$. It is clear that agents exhibit risk-seeking (RS) behavior with $\alpha < 1$ due to the convexity of h, whereas they are risk-averse (RA) with $\alpha > 1$ and risk-neutral (RN) when $\alpha = 1$, indicating

	X	$\mathcal{X}$	$\rho_h(X) = \boxplus_{i=1}^2 \rho_h(X)$	$\boxminus_{i=1}^2 \rho_h(X) = \square_{i=1}^2 \rho_h(X)$
$Y \sim \text{Uniform}(0,1)$	Y	L^+	0.3317	0.1903
	$-Y$	L^-	-0.6609	-0.8776
$Y \sim \text{Pareto}(3,2)$	Y	L^+	2.4743	1.4406
	$-Y$	L^-	-3.6044	-4.9292
$Y \sim \text{logN}(0,1)$	Y	L^+	0.92704	0.6408
	$-Y$	L^-	-3.0062	-3.5515

Table 2.2: Comparison of the three inf-convolutions.

a linear h. It is well-known that $\boxplus_{i=1}^n \rho_h$ is always equal to ρ_h, regardless of whether h is concave or convex, as shown in the blue line in Figure 2.1. From Theorem 2.2, these three inf-convolutions are equal when h is concave, i.e., $\alpha > 1$ in this case. Consequently, they share the same blue line in scenarios where agents are risk-averse. Furthermore, Theorem 2.5 shows that ρ_h is consistently greater than both the counter-monotonic inf-convolution and the unconstrained one. This leads to the depicted deviation, represented by the red line in Figure 2.1, for risk-seeking agents.

It is natural to wonder about the connections among comonotonicity, counter-monotonicity and Pareto optimality. Next, we characterize Pareto-optimal allocations of the risk sharing problem with agents exhibiting risk-averse or risk-seeking behaviors. The following proposition shows that, within the framework of distortion risk measures, every comonotonic allocation is Pareto optimal for all risk-averse agents, and the converse holds true for strictly concave distortion functions. In contrast, for risk-seeking agents, Pareto-optimal allocations must be jackpot (scapegoat) allocations if the total risk X is nonnegative (nonpositive), complementing the result in Theorem 2.5.

Proposition 2.3. *Assume $h \in \mathcal{H}$. The following statements hold.*

(i) Suppose $\mathcal{X} = L^\infty$ and $X \in \mathcal{X}$. If h is concave, then all comonotonic allocations of

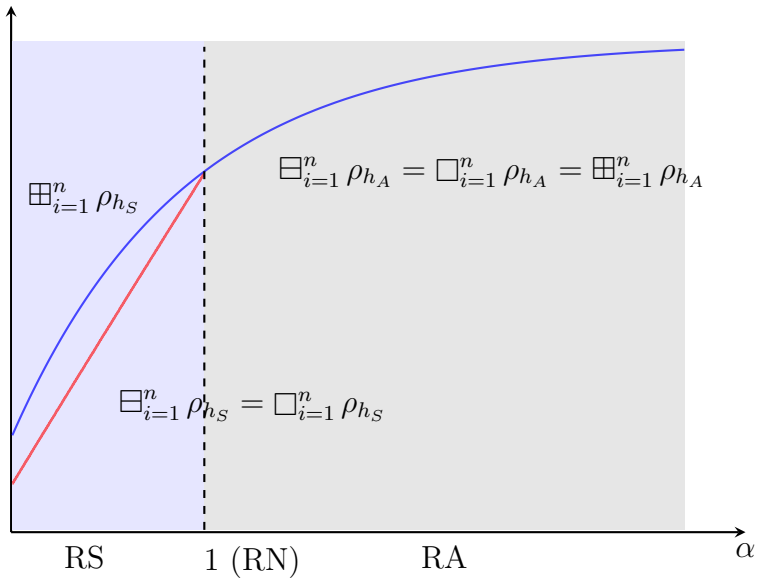


Figure 2.1: Comparison of inf-convolutions evaluated at $X \sim \text{Uniform}[0,1]$ for agents with different risk attitudes.

X are Pareto optimal. If h is strictly concave, then all Pareto-optimal allocations of X are comonotonic.

(ii) Suppose $\mathcal{X} = L^+$ (resp. $\mathcal{X} = L^-$) and $X \in \mathcal{X}^\perp$. If h is convex, then all Pareto-optimal allocations are jackpot (resp. scapegoat) allocations.

Proof. (i) The first statement follows from Theorem 2.1 and comonotonic additivity of ρ_h. The second statement follows from Proposition 4 of Lauzier et al. (2026b). (ii) When $\mathcal{X} = L^+$, the result directly follows Theorem 2 of Lauzier et al. (2026a). The proof for nonpositive case is analogous and is thus omitted. □

2.6 Portfolio manager's problem

In this section, we analyze a portfolio optimization problem, showing how the risk preferences of agents influence their decision-making regarding risky investments. In the

market, there is a portfolio manager and a group of agents who collectively possess a constant initial endowment of W and homogeneous individual risk preferences, which can vary from highly risk-averse to potentially risk-seeking. The task of a portfolio manager is to manage the investments for a group of clients, aiming to optimize the collective portfolio in a way that aligns with the risk preference of the agents and to minimize the aggregate risks. The investment payoff from a risky asset, such as a stock or mutual fund, after the investment period at time 1 is modelled by a random variable $X \geqslant 0$. The first question for the manager is to construct an investment strategy λX, where λ represents the total proportion of the total investment to allocate to the risky asset. This investment has a cost $c(\lambda)$ incurred from investing in the risky asset and c is assumed to be increasing and convex, meaning that a larger leverage is marginally more costly (see examples in Föllmer and Schied (2002) and Castagnoli et al. (2022)). The second task is to allocate the total investment wealth $W + \lambda X - c(\lambda)$ at time 1 to the participants via an allocation $(X_1, \dots, X_n)$ of wealth.

To summarize, the goal of the manager, taking into account both the investment problem and the allocation problem, is to optimize the following objective function

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^{n} \rho_h(-X_i) \\ \text{subject to} \quad & \lambda \in [0,1], \quad c(\lambda) \leqslant W; \\ & X_1 + \cdots + X_n = W + \lambda X - c(\lambda); \quad X_1, \dots, X_n \geqslant 0. \end{aligned} \tag{2.15}$$

In this model, the constraint $X_1, \dots, X_n \geqslant 0$ means that the manager does not give additional loss to the participants at time 1. In other words, the participation fund represented by W has been collected in the beginning of the investment period. To avoid infeasibility, we assume $c(\lambda) \leqslant W$ to ensure the total wealth $W + \lambda X - c(\lambda)$ remains non-negative. We will determine the optimal proportion λ of risky investments in the portfolio, which is dependent on the risk attitudes of agents, as shown in the subsequent result. We aim to investigate how these risk preferences influence the allocation of risky investments.

Specifically, we consider scenarios where agents are either risk-averse or risk-seeking.

Proposition 2.4. *Suppose $\mathcal{X} = L^+$ and $h \in \mathcal{H}$. For $X \in \mathcal{X}^\perp$, the following hold.*

(i) If h is concave, the optimal value for problem (2.15) is

$$\lambda^* = \min\left\{c'^{-1}(\rho_{\tilde{h}}(X)), c^{-1}(W)\right\}, \quad \textit{where } \tilde{h}(t) = 1 - h(1-t).$$

(ii) If h is convex, the optimal value is

$$\lambda^* = \min\left\{c'^{-1}(\rho_g(X)), c^{-1}(W)\right\}, \quad \textit{where } g(t) = \frac{1 - h(1 - t/n)}{1 - h(1 - 1/n)}.$$

Proof. (i) Denote by $f(\lambda) = \rho_h(-W - \lambda X + c(\lambda))$. From concavity of h and Theorem 2.3, problem (2.15) becomes finding the minimum of $f(\lambda)$ over $\lambda \in [0, 1)$ with constraint of $c(\lambda) \leqslant W$. Furthermore, the function $f(\lambda)$ can be rewritten as

$$f(\lambda) = -\lambda\rho_{\tilde{h}}(X) + c(\lambda) - W.$$

Take derivative with respect to λ, we have $f'(\lambda) = -\rho_{\tilde{h}}(X) + c'(\lambda) = 0$. Consider the restriction of $c(\lambda) \leqslant W$, the optimal value λ^* is then determined by $\lambda^* = \min\{c'^{-1}(\rho_{\tilde{h}}(X)), c^{-1}(W)\}$, where $\tilde{h}(t) = 1 - h(1-t)$.

(ii) The result can be proved using Theorem 2.5 and convexity of h. Its proof is similar to (i) and thus omitted. □

An immediate implication of the above results is that for risk-averse agents, the optimal proportion of risky investment remains unaffected by the number of agents involved in the pool, whereas this is not the case for risk-seeking agents. In the following proposition, we present a necessary and sufficient condition for the optimal strategy to be independent with n for risk-seeking agents. The assumptions in Proposition 2.4 are maintained in the following results.

Proposition 2.5. *If h is convex, the optimal value λ^* is independent of n if and only if h is the dual-power transform, $h(t) = 1 - (1-t)^\alpha$ with $\alpha \in (0, 1]$.*

Proof. For the "if" part, it is trivial to show the optimal value λ^* is unrelated to n by (ii) of Proposition 2.4. Furthermore, the optimal value λ^* is determined as

$$\lambda^* = c'^{-1}\left(\rho_g(X)\right), \quad \text{where } g(t) = \tilde{h}(t) = x^\alpha.$$

Next, we show the "only if" part. From Proposition 2.4, independence between the optimal value λ^* and n for h being convex is equivalent to independence between $g(t) = (1 - h(1 - t/n))/(1 - h(1 - 1/n))$ and n. The function $g(x)$ can be rewritten as $g(t) = \tilde{h}(t/n)/\tilde{h}(1/n)$. It follows that the equality

$$\frac{\tilde{h}\left(\frac{t}{n_1}\right)}{\tilde{h}\left(\frac{1}{n_1}\right)} = \frac{\tilde{h}\left(\frac{t}{n_2}\right)}{\tilde{h}\left(\frac{1}{n_2}\right)}$$

holds for any $n_1 \neq n_2 \in \mathbb{Z}$. Take $n_1 = 1$ and $n_2 = n$. It follows that

$$\tilde{h}\left(\frac{t}{n}\right) = \tilde{h}\left(\frac{1}{n}\right)\tilde{h}(t). \tag{2.16}$$

Taking the derivative with respect to t on both sides, we get

$$\frac{1}{n}\tilde{h}'\left(\frac{t}{n}\right) = \tilde{h}\left(\frac{1}{n}\right)\tilde{h}'(t). \tag{2.17}$$

Dividing (2.16) by (2.17), we get

$$\ell(t) = n\ell\left(\frac{t}{n}\right), \quad \text{where } \ell(t) = \frac{\tilde{h}(t)}{\tilde{h}'(t)}.$$

Hence, the above property implies that ℓ has the form of $\ell(t) = \beta t$ for some $\beta \geqslant 0$ and all

$t \in (0, 1]$. As a consequence,

$$\frac{\tilde{h}'(t)}{\tilde{h}(t)} = \frac{\alpha}{t}, \quad \text{where } \alpha = \frac{1}{\beta}$$

and thus $\log \tilde{h}(t) = \alpha \log t$. Therefore, the distortion function $h(t)$ has the form of $h(t) = 1 - (1-t)^\alpha$ where $\alpha \in (0, 1]$. □

We give two simple examples to see how the optimal proportion of risky investments varies with the number of agents and the risk attitudes of agents. Here the cost function is specified as $c(\lambda) = \lambda^2/2$.

Example 2.4. Consider agents are associated with $h(t) = 1-(1-t)^\alpha$. It is clear that h is concave with $\alpha \in [1, \infty)$ and is convex with $\alpha \in (0, 1]$. The investment X here is assumed to follow the uniform distribution $\mathrm{U}(0, 1)$. We aim to explore the relationship between the optimal value λ^* and the risk preference characterized by α. Some numerical results are presented in Figure 2.2. Proposition 2.5 states the optimal value λ^* would not vary with

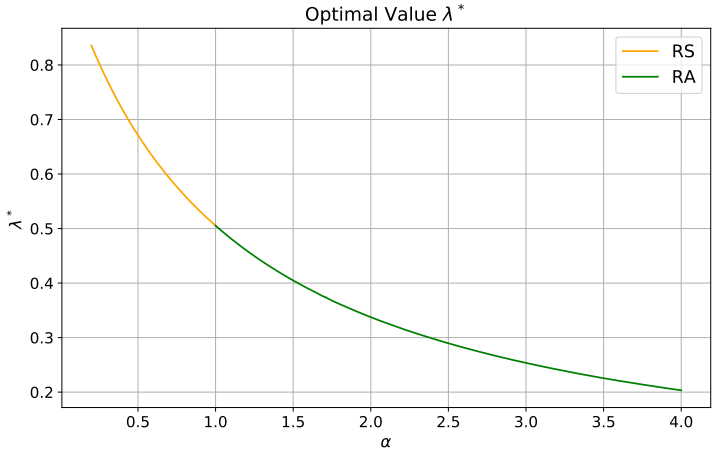


Figure 2.2: Optimal value of λ^* with $h(t) = 1 - (1-t)^\alpha$.

n, and that is the reason why we only have one single line for $\alpha \in (0, 1]$. From Figure 2.2, we observe that the optimal value λ^* decreases as α increases, implying that the manager

on behalf of risk-averse agents tends to invest less in risky assets compared to the case of risk-seeking agents.

Example 2.5. Consider a distortion function, $h(t) = \Phi(\Phi^{-1}(t) + \alpha)$, known as a transform of Wang (2000). Agents are risk-averse if $\alpha \in [0, \infty)$, while they are risk-seeking if $\alpha \in (-\infty, 0]$. Take a uniform random variable $X \sim \mathrm{U}(0, 1)$. The optimal value of λ is computed by varying the number of agents within the group and adjusting the risk preference parameter α from negative to positive values. Numerical results are presented in the Figure 2.3. Similar to Example 2.4, the more risk-seeking the agents are, the greater the

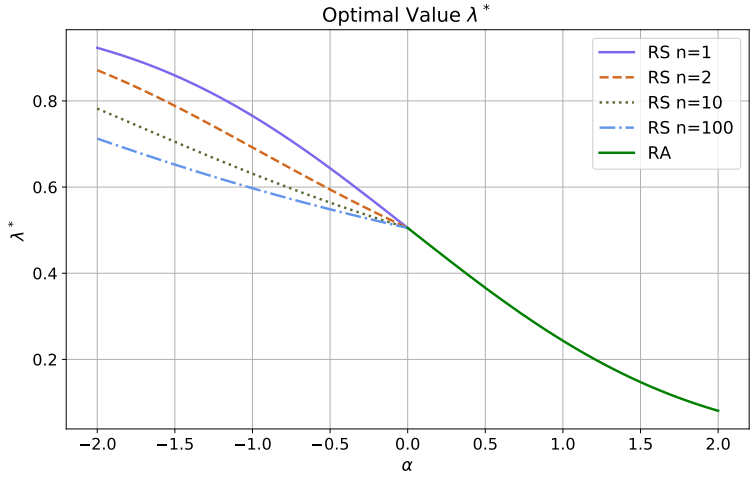


Figure 2.3: Optimal value of λ^* with $h(t) = \Phi(\Phi^{-1}(t) + \alpha)$.

allocation of risky investments. In this case, the optimal value λ^* varies with different sizes of participants and decreases as the number of agents increases. The economic intuition is that as more risk-seeking agents join the group, they tend to engage more in gambling with each other. Consequently, the randomness in outcomes arises from two sources: the gambling itself and the risky investment. With an increase in risk-seeking agents in the group, there is a tendency to focus on gambling among themselves rather than increasing investment in the financial market due to its increasing marginal cost; recall that gambling among participants themselves incur no additional cost for the group. It is natural to

wonder if this is always the case; that is, whether the optimal proportion of risky investments will always decrease with an increasing number of agents in the pool. The answer is negative. We provide a counter-example in Section 2.9 that shows the optimal proportion of risky investments can actually increase with more agents involved.

2.7 Inverse S-shaped distortion functions

We now consider another risk sharing problem among agents with a special distortion risk measure which is neither concave nor convex. Tversky and Kahneman (1992) introduced an inverse S-shaped distorted probability function expressed as

$$h_{\mathrm{TK}}(t) = \frac{t^{\gamma}}{(t^{\gamma} + (1-t)^{\gamma})^{1/\gamma}},$$

where $\gamma \in (0,1)$. This weighting function, also known as a TK distortion function, exhibits a concave-convex shape, indicating a mix of risk-seeking and risk-averse preferences.

Assumption 2.1. The distortion risk functional $h : [0,1] \mapsto [0,1]$ is concave-convex.

Denoting by $\underline{h} : [0,1] \to [0,1]$ the convex envelope of h, which is defined as the largest convex function such that $\underline{h}(t) \leqslant h(t)$ for all $t \in [0,1]$. Clearly, $\underline{h}$ is a distorted probability function being dominated pointwise by h. Thus, following from the Lebesgue–Stieltjes integral representation of $\rho_h(X)$ (see Dhaene et al. (2012)), for any $X \in \mathcal{X}$ we have

$$\rho_{\underline{h}}(X) = \int_0^1 F_X^{-1}(1-t)\mathrm{d}\underline{h}(t) \leqslant \int_0^1 F_X^{-1}(1-t)\mathrm{d}h(t) = \rho_h(X).$$

Next, we explore the risk sharing problem for agents whose risk preferences are modeled by concave-convex distortion functions. The subsequent theorem presents an explicit formula for the corresponding counter-monotonic inf-convolution, which depends on the convex envelope of the distortion functions. This result has a condition that holds if the

number of agents is involved in the risk sharing problem is larger than a constant $1/(1-t_0)$, discussed later.

Theorem 2.6. *Suppose $\mathcal{X} = L^-$ and $X \in \mathcal{X}^\perp$. Assume $h \in \mathcal{H}$ satisfies Assumption 2.1. If $n \geqslant 1/(1-t_0)$, where $t_0 = \sup\{t \in (0,1] : h'(t) < h(t)/t\}$, then it holds*

$$\boxminus_{i=1}^{n} \rho_h(X) = \boxminus_{i=1}^{n} \rho_{\underline{h}}(X) = \rho_g(X),$$

where $g(t) = n\underline{h}(1-(1-t)/n) - n\underline{h}(1-1/n)$ for $t \in [0,1]$.

Proof. Assume $h \in \mathcal{H}$ is concave-convex with $t_0 = \sup\{t \in (0,1] : h'(t) < h(t)/t\}$. One can check that h is convex on $[t_0, 1]$ and identical to $\underline{h}$; see e.g., Ghossoub (2019, Lemma A.8). Thus, the convex envelope $\underline{h}(t)$ is determined as

$$\underline{h}(t) = \begin{cases} \frac{h(t_0)}{t_0} t & \text{if } t \in [0, t_0] \\ h(t) & \text{if } t \in [t_0, 1] \end{cases}. \tag{2.18}$$

If $n \geqslant 1/(1-t_0)$, it is trivial to verify $h(t) = \underline{h}(t)$ for $t \in [1-1/n, 1]$.

Let $\mathcal{X} = L^-$. For $X \in \mathcal{X}^\perp$, it is trivial to verify $\boxminus_{i=1}^n \rho_{\underline{h}}(X) \leqslant \boxminus_{i=1}^n \rho_h(X)$, which directly follows from $\underline{h}$ being dominated by h pointwisely. Conversely, let $(X_1, \dots, X_n) \in \mathbb{A}_n^-(X)$ be a jackpot allocation, i.e., $X_i = X\mathbb{1}_{A_i}$ with $\mathbb{P}(A_i) = 1/n$ for all $i \in [n]$ and $(A_1, \dots, A_n) \in \Pi_n$ independent of X. For such $(A_1, \dots, A_n) \in \Pi_n$ and $X \in \mathcal{X}^\perp$, it follows that

$$\begin{aligned} \sum_{i=1}^{n} \rho_h(X\mathbb{1}_{A_i}) &= n \int_{-\infty}^{0} \left(h\left(\frac{1}{n}\mathbb{P}(X > x) + 1 - \frac{1}{n} \right) - 1 \right) \mathrm{d}x \\ &= n \int_{-\infty}^{0} \left(\underline{h}\left(1 - \frac{1}{n}\mathbb{P}(X \leqslant x) \right) - 1 \right) \mathrm{d}x \\ &= n \int_{0}^{\infty} \left(\underline{h}\left(1 - \frac{1}{n}\mathbb{P}(-X \geqslant x) \right) - 1 \right) \mathrm{d}x \\ &= -n \int_{0}^{\infty} \tilde{\underline{h}}\left(\frac{1}{n}\mathbb{P}(-X \geqslant x) \right) \mathrm{d}x \end{aligned}$$

$$= -\rho_{\tilde{\underline{h}}^*}(-X) = \rho_g(X),$$

where $\tilde{\underline{h}}(t) = 1 - \underline{h}(1-t)$, $\tilde{\underline{h}}^*(t) = n\tilde{\underline{h}}(t/n)$ and $g(t) = \tilde{\underline{h}}^*(1) - \tilde{\underline{h}}^*(1-t)$, which can be further simplified as $g(t) = n\underline{h}(1-(1-t)/n) - n\underline{h}(1-1/n)$. Furthermore, we can verify $\rho_g(X) = \boxminus_{i=1}^n \rho_{\underline{h}}(X)$ from convexity of $\underline{h}$ and (ii) of Theorem 2.5. Thus, $\boxminus_{i=1}^n \rho_h(X) \leqslant \boxminus_{i=1}^n \rho_{\underline{h}}(X)$ for $X \in \mathcal{X}^\perp$. Combining the above arguments, the desired result follows. □

The typically value t_0 in Theorem 2.6 is not close to 1 in behaviour economics, and hence $1/(1-t_0)$ is not very large. For instance, in the classic work of Wu and Gonzalez (1996), the TK distortion function has a best estimated parameter $\gamma = 0.71$, which corresponds to $t_0 \approx 0.768$; see Figure 6 of Wu and Gonzalez (1996). Therefore, it suffices for $n \geqslant 5$ to apply the result of Theorem 2.6 in that model.

An immediate consequence of Theorem 2.6 is that when a good number of agents are involved in the risk pool and their preferences are modeled by inverse S-shaped distortion risk measures, these agents will behave like risk-seeking agents discussed in Section 2.5; that is, they will achieve the same optimal value and share the same optimal allocations. The intuitive economic explanation is that when the number of agents in the pool is not too small, agents may prefer gambling against each other to improve the outcome of the risk minimization problem since those agents in Theorem 2.6 are non-risk-averse.

Theorem 2.6 implies that, with the space of risks being nonpositive random variables, scapegoat allocations are Pareto optimal for agents with inverse S-shaped distortion functions under some mild conditions. As we have seen from Proposition 2.3, this is the same situation for risk-seeking agents.

2.8 Conclusion

The comonotonic risk sharing problem has been well studied, and the comonotonic inf-convolution of n distortion risk measures can be determined explicitly, as studied extensive

in the literature. It is well-known that comonotonicity being optimal is a consequence of the concavity of distortion functions. This chapter addresses different situations from the classic literature, where the agents are possibly associated with convex distortion functions.

Our study mainly focuses on counter-monotonic risk sharing problems for agents with homogeneous risk distortion functions and provides a comparative analysis of three types of risk sharing problems: unconstrained, comonotonic, and counter-monotonic. This analysis is conducted for cases where the risk is pooled among risk-averse agents, risk-seeking agents, and agents with inverse S-shaped distortion functions. We provide explicit formulas of counter-monotonic inf-convolution for each of these three scenarios. In addition, our results provide insights into solving the unconstrained risk sharing problem for some non-concave distortion functions, i.e., convex or inverse S-shaped functions, typically leading to counter-monotonic Pareto-optimal allocations (under some mild conditions for the latter one).

By combining the results for risk-averse agents and risk-seeking agents, we are able to solve the portfolio optimization problem as described in Section 2.6, and the optimal strategies for risky assets can be determined explicitly. Some examples are presented in the Section 2.6 and Section 2.9.

The explicit results obtained in this chapter assumed that the preferences are the agents are homogeneous, that is, using the same risk measure. The case of heterogeneous risk measures imposes substantial technical challenges and will need separate analysis.

2.9 Another example of a portfolio manager's problem

As demonstrated in Example 2.5 from Section 2.6, agents' risk preferences are modeled using Wang transform. The model indicates that the optimal value of λ^* for a group with more agents tends to be smaller than that for a group with fewer agents. However, this is not always the case; the opposite outcome can also occur, depending on the specific

distortion function applied. In this section, we construct a new distortion function to model the agents' risk preferences, showing that, in contrast to the previous model, the optimal value of λ^* for a group with more agents tends to be larger than that for a group with fewer agents.

The dual distortion function $\tilde{h}$ is constructed as:

$$\tilde{h}(t) = \frac{k^\alpha}{1+(k-1)\alpha} t^\alpha \mathbb{1}_{\{t \leqslant \frac{1}{k}\}} + \left(\frac{k\alpha}{1+(k-1)\alpha} t + \frac{1-\alpha}{1+(k-1)\alpha} \right) \mathbb{1}_{\{t > \frac{1}{k}\}},$$

where $\alpha \in \mathbb{R}_+$ and k is a positive integer. The distortion function is thus defined as $h(t) = 1 - \tilde{h}(1-t)$. It is trivial to see h is convex with $\alpha \in (0, 1]$ and h is concave with $\alpha \in [1, \infty)$.

Here we choose $k = 10$. Numerical results are presented in Figure 2.4.

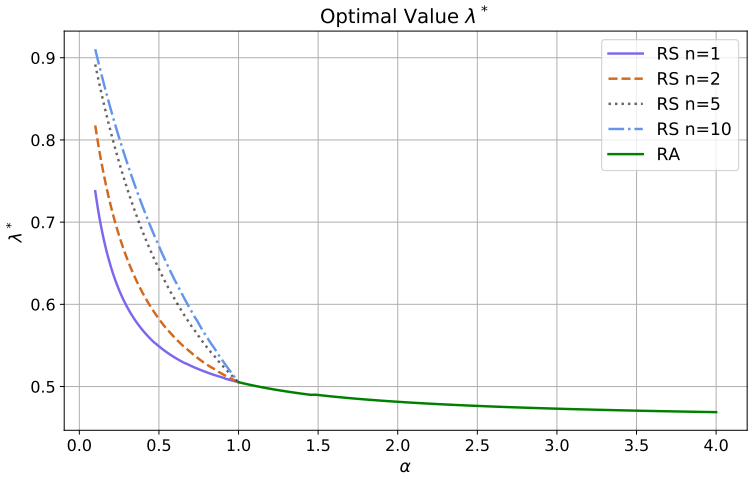


Figure 2.4: Optimal value of λ^* with $1 - \tilde{h}(1-t)$.

2.10 A counter-example

Below we present a counter-example showing that $\boxminus_{i=1}^3 \rho_i \neq \rho_1 \boxminus \rho_2 \boxminus \rho_3$.

Example 2.6. Define a probability space $(\Omega', \mathcal{F}', \mathbb{P}')$ where $\Omega' = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, $\mathcal{F}'$ is the power set of Ω, and $\mathbb{P}'$ is such that $\mathbb{P}'(\omega_i) = 1/4$ for $i = 1, \ldots, 4$. Define three distributions $F_1 = \text{Bernoulli}(1/2)$, and $F_2 = F_3 = 1/2 \times \text{Bernoulli}(1/4)$. Suppose that three agents have the risk measures given by

$$\rho_i(X) = 1 - \mathbb{1}_{\{X \sim F_i\}}, \ i = 1, 2, 3 \text{ and } X \in \mathcal{X}.$$

Then we define four random variables $X = \mathbb{1}_{\{\omega_1, \omega_2, \omega_3\}}$, $X_1 = \mathbb{1}_{\{\omega_1, \omega_2\}}$, and $X_2 = X_3 = 1/2 \times \mathbb{1}_{\{\omega_3\}}$. Clearly, (X_1, X_2, X_3) is an allocation of X. It is straightforward to show both (X_1, X_2) and $(X_1 + X_2, X_3)$ are counter-monotonic, whereas (X_2, X_3) is comonotonic. It follows that

$$\rho_1 \boxminus \rho_2 \boxminus \rho_3(X) \leqslant \rho_1(X_1) + \rho_2(X_2) + \rho_3(X_3) = 0.$$

Hence, we have $\rho_1 \boxminus \rho_2 \boxminus \rho_3(X) = 0$ since ρ_i are non-negative for all $i \in [3]$. Furthermore, for any $(Y_1, Y_2, Y_3) \in \mathbb{A}_3^-(X)$, at least one of $\rho_i(Y_i)$ is equal to 1 as there does not exist a counter-monotonic allocation such that $Y_i \sim F_i$, $i \in [3]$. Hence, $\boxminus_{i=1}^3 \rho_i(X) \neq 0 > \rho_1 \boxminus \rho_2 \boxminus \rho_3(X)$.

Chapter 3

Counter-Monotonic Risk Sharing with Heterogeneous Distortion Risk Measures

3.1 Introduction

Risk-exchange markets, such as insurance, reinsurance, or financial markets, are central to modern economics. The primary focus of studying such markets has traditionally been the determination of an optimal, or efficient redistribution of the aggregate market risk, through contracts or trading mechanisms, among market participants, henceforth referred to as agents. The seminal work of Borch (1962) and Wilson (1968) showed that within the framework of Expected-Utility Theory (EUT), Pareto-optimal allocations between risk-averse agents are comonotonic, and they can therefore be expressed as nondecreasing functions of the aggregate market risk. This is a cornerstone result in the theory of risk sharing, and it is often seen as a foundational justification for risk pooling, since each agent's risk allocation at an optimum depends only on the realization of the aggregate risk. Numerous extensions beyond EUT have been studied in the literature, with the perennial

assumption of (strong) risk aversion, that is, monotonicity with respect to the concave order. The literature is vast, and we refer for instance to the work of Chateauneuf et al. (2000), Dana (2002, 2004), Tsanakas and Christofides (2006), De Castro and Chateauneuf (2011), Beißner and Werner (2023), and Ravanelli and Svindland (2014) for several models of ambiguity-sensitive preferences.

A milestone result in this direction is the so-called *comonotonic improvement theorem* (e.g., Landsberger and Meilijson (1994), Dana (2004), Ludkovski and Rüschendorf (2008), Carlier et al. (2012), or Denuit et al. (2025)), an important implication of which is that risk-averse agents always prefer comonotone allocations, and that Pareto optima are comonotonic under strict risk aversion. This naturally led Boonen et al. (2021) to examine the so-called comonotone market, an incomplete market in which only comonotonic allocations are feasible. Pareto-optimal risk sharing in comonotone markets was also studied by Liu (2020) and Ghossoub and Zhu (2026).

In the risk measurement literature, Pareto-optimal risk sharing between agents with convex or coherent risk measures has been widely studied as well. We refer to Barrieu and El Karoui (2005), Acciaio (2007), Jouini et al. (2008), Filipović and Svindland (2008), Mastrogiacomo and Rosazza Gianin (2015), and the references therein, for instance. Additionally, Pareto-optimal risk sharing between agents with quantile-based risk measures that are not necessarily convex was examined by Embrechts et al. (2018, 2020), Liu (2020), Liebrich (2024) and Ghossoub et al. (2025), for instance.

The characterization of optimal allocations in risk sharing markets involving agents who are not risk-averse remains relatively underexplored. Recent studies on quantile-based risk sharing, including Embrechts et al. (2018, 2020) and Weber (2018), identified a pairwise counter-monotonic structure — the opposite of comonotonicity — in an optimal allocation. Furthermore, Lauzier et al. (2026b) provided explicit Pareto-optimal allocations among agents using the inter-quantile difference, demonstrating that an optimal allocation exhibits a mixture of pairwise counter-monotonic structures. As a dependence concept, pairwise counter-monotonicity has been studied by Dall'Aglio (1972), Dhaene and Denuit

(1999) and Cheung and Lo (2014). Parallel to the comonotone improvement theorem, Lauzier et al. (2026a) established the so-called *counter-monotonic improvement theorem*, leading to an implication that counter-monotonic allocations will always be preferred by risk-seeking agents. Based on the counter-monotonic improvement theorem, Ghossoub et al. (2026a) provided a systematic study of risk sharing in markets where only counter-monotonic allocations are allowed, and they gave an explicit characterization of optimal allocations when agents are risk-averse and risk-seeking. Their analysis assumes that the preferences of the agents are modelled by a common distortion risk measure.

This chapter extends the previous work by examining a market where agents may have heterogeneous risk preferences, and they are not necessarily risk averse. It is notable that although agents within a group may differ in their levels of risk aversion, we assume that all agents in the same group are either all risk-averse or all risk-seeking. We do not consider cases where both risk-averse and risk-seeking agents are combined in a single group. The key novel insights and extensions are summarized as follows:

(i) In the homogeneous case, Ghossoub et al. (2026a) established a universal ordering among the three versions of inf-convolution: unconstrained, counter-monotonic, and comonotonic, from the smallest to the largest. In contrast to the homogeneous case, the ordering between the comonotonic and counter-monotonic versions of the inf-convolution depends on the distortion functions.

(ii) When agents have identical concave distortion functions (the case of risk aversion), the three versions of inf-convolution have identical values, as shown in Theorem 3 of Ghossoub et al. (2026a). However, for a group of risk-averse agents with different levels of risk aversion, counter-monotonic allocations are generally not Pareto optimal, leading to a gap between the three versions of inf-convolution.

(iii) Under some mild conditions, we derive an explicit formula for the counter-monotonic inf-convolution in the case where agents are risk seeking, characterized by different convex distortion functions.

The rest of the chapter is organized as follows. Sections 3.2 and 3.3 contain preliminaries on risk measures and on risk sharing problems, respectively. In particular, Section 3.3 recalls the counter-monotonic improvement theorem (reported as Theorem 3.1) and some related discussions on counter-monotonicity. In Section 3.4, we analyze the counter-monotonic risk sharing problem and obtain general relations for different choices of the distortion functions (Theorem 3.2). We specialize to risk-seeking agents in Section 3.5. Based on the counter-monotonic improvement theorem, counter-monotonic inf-convolutions are determined explicitly for risk-seeking agents (Theorem 3.3). Section 3.6 concludes the chapter.

3.2 Preliminaries

3.2.1 Risk measures and basic terminology

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be an atomless probability space and $\mathcal{X}$ a convex cone of random variables on this space. Section 3.4 considers $\mathcal{X} = L^\infty$, the set of all essentially bounded random variables, and Section 3.5 considers $\mathcal{X} = L^+$ or $\mathcal{X} = L^-$, where L^+ (resp. L^-) represents the sets of nonnegative (resp. nonpositive) essentially bounded random variables. Almost surely equal random variables are treated as identical. Throughout, the random variable $X \in \mathcal{X}$ represents losses, and its negative values represent gains. We denote by $\mathbb{1}_A$ the indicator function for an event $A \in \mathcal{F}$. Let

$$\mathcal{H}^{\mathrm{BV}} = \{h : [0,1] \to \mathbb{R} \mid h \text{ is of bounded variation and } h(0) = 0\}.$$

Next, we present the definition of a distortion riskmetric.

Definition 3.1. A *distortion riskmetric* is a mapping $\rho_h : \mathcal{X} \to \mathbb{R}$ given by

$$\rho_h(X) = \int X \mathrm{d}\,(h \circ \mathbb{P}) = \int_0^\infty h(\mathbb{P}(X \geqslant x))\,\mathrm{d}x + \int_{-\infty}^0 (h(\mathbb{P}(X \geqslant x)) - h(1))\,\mathrm{d}x,$$

for some $h \in \mathcal{H}^{\mathrm{BV}}$.

We note that the elements of $\mathcal{H}^{\mathrm{BV}}$ are not necessarily monotone. If we constrain $h \in \mathcal{H}^{\mathrm{BV}}$ to be increasing and normalized, that is, $h \in \mathcal{H}$, where

$$\mathcal{H} = \{h : [0,1] \to [0,1] \mid h \text{ is increasing and } h(0) = 1 - h(1) = 0\},$$

then the distortion riskmetric ρ_h for $h \in \mathcal{H}$ is a distortion risk measure. Here and throughout, terms like "increasing" or "decreasing" are in the non-strict sense. In this chapter, the agents' risk preferences are modeled by the class of distortion risk measures. The more general class of distortion riskmetrics is introduced since it will be useful in our further analysis. In particular, we show in the setting of Section 3.5 that the comonotonic inf-convolution of distortion risk measures is a distortion risk measure, whereas their counter-monotonic inf-convolution is a distortion riskmetric.

The *dual* $\tilde{h}$ of a given $h \in \mathcal{H}^{\mathrm{BV}}$, which will be useful in many of our results, is defined as

$$\tilde{h}(t) = h(1) - h(1-t), \qquad \text{for all } t \in [0,1],$$

and it is an element of $\mathcal{H}^{\mathrm{BV}}$. If h is in $\mathcal{H}$, then so is $\tilde{h}$. The dual of $\tilde{h}$ is equal to h, and the two corresponding distortion riskmetrics are connected via the equality

$$\rho_h(X) = -\rho_{\tilde{h}}(-X), \qquad \text{for all } X \in \mathcal{X}.$$

We now recall some properties of distortion riskmetrics that we use throughout. A distortion riskmetric ρ_h may have the following properties as a functional $\rho : \mathcal{X} \mapsto \mathbb{R}$.

(a) Law-invariance: $\rho(X) = \rho(Y)$ if X and Y have the same distribution, i.e., $X \overset{\mathrm{d}}{=} Y$;

(b) Positive homogeneity: $\rho(\lambda X) = \lambda \rho(X)$ for any $\lambda > 0$;

(c) Translation invariance: $\rho(X + c) = \rho(X) + c$ for $c \in \mathbb{R}$ and $X + c \in \mathcal{X}$;

(d) Comonotonic additivity: $\rho(X+Y)=\rho(X)+\rho(Y)$ if X and Y are comonotonic;

(e) Subadditivity: $\rho(X+Y)\leqslant\rho(X)+\rho(Y)$;

(f) Convex order consistency: $\rho(X)\leqslant\rho(Y)$ if $X\leqslant_{\mathrm{cx}}Y$, where the inequality is the convex order, meaning $\mathbb{E}[u(X)]\leqslant\mathbb{E}[u(Y)]$ for all convex functions u such that the two expectations are well-defined;

(g) Monotonicity: $\rho(X)\leqslant\rho(Y)$ if $X\leqslant Y$.

In fact, these properties do not always hold for a distortion riskmetric ρ_h. To be more specific, all distortion riskmetric ρ_h satisfy (a), (b), and (d). Property (c) holds true if $h(1)=1$. By Wang et al. (2020b, Theorem 3), conditions (e) and (f) are equivalent to the concavity of h. Condition (g) is equivalent to increasing monotonicity of h. The four properties (b), (c), (e) and (g) together define a coherent risk measure in the sense of Artzner et al. (1999), corresponding to an increasing and concave h with $h(1)=1$ for ρ_h. For various characterizations and properties of distortion riskmetrics, see Wang et al. (2020b) on L^∞ and Wang et al. (2020a) on more general spaces. Convex order in (f) and its related notions are popular for modeling risk aversion in decision theory (Rothschild and Stiglitz, 1970), and it is also widely studied in actuarial science and risk management (Rüschendorf, 2013; He et al., 2016).

Many popular risk measures belong to the family of distortion risk measures, including the regulatory risk measures used in the banking and insurance sectors, namely, the Value-at-Risk (VaR) and the Expected Shortfall (ES, also known as CVaR and TVaR), which are defined below. For a random variable X, the VaR at level $\alpha\in[0,1)$ is defined as

$$\mathrm{VaR}_\alpha(X)=\inf\{x\in[-\infty,\infty]:\mathbb{P}(X\leqslant x)\geqslant 1-\alpha\},\tag{3.1}$$

and the ES at level $\beta\in[0,1)$ is defined as

$$\mathrm{ES}_\beta(X)=\frac{1}{\beta}\int_0^\beta \mathrm{VaR}_\gamma(X)\mathrm{d}\gamma,$$

where VaR_γ is defined in (3.1). Here we use the convention of "small α" as in Embrechts et al. (2018). If $\alpha \in [0,1)$, VaR_α and ES_α are distortion risk measures, and they are associated with the distortion functions $h(t) = \mathbb{1}_{\{t>\alpha\}}$ and $h(t) = \min\{t/\alpha, 1\}$, respectively.

In this chapter, we write $X \sim F_X$ for $X \in \mathcal{X}$ having cumulative distribution F_X and survival distribution S_X. Since the space $(\Omega, \mathcal{F}, \mathbb{P})$ is atomless, for each $X \in \mathcal{X}$ there exists a random variable U_X with a uniform distribution on $[0,1]$ such that $F_X^{-1}(U_X) = X$, almost surely. The existence of such a U_X for any random variable X follows from Föllmer and Schied (2016, Lemma A.32). For $x, y \in \mathbb{R}$, write $x \vee y = \max\{x, y\}$ and $x \wedge y = \min\{x, y\}$.

3.2.2 Risk sharing and inf-convolution

Given $X \in \mathcal{X}$, the set of feasible allocations of X is defined as

$$\mathbb{A}_n(X) = \left\{ (X_1, \dots, X_n) \in \mathcal{X}^n : \sum_{i=1}^n X_i = X \right\}. \tag{3.2}$$

We consider a risk-sharing market, in which $n \in \mathbb{N}$ agents wish to share an aggregate risk $X \in \mathcal{X}$. All throughout, we let $[n] = \{1, \dots, n\}$, and we assume that agent $i \in [n]$ has a risk preference modeled by a risk measure ρ_i. The market redistributes the aggregate risk X into an allocation $(X_1, \dots, X_n) \in \mathbb{A}_n(X)$, and we refer to $\sum_{i=1}^n \rho_i(X_i)$ as the associated aggregate risk value. Note that the definition of allocations depends on the specification of $\mathcal{X}$, which will vary across different applications in the later sections.

Using (3.2), the inf-convolution $\square_{i=1}^n \rho_i$ of n risk measures $\rho_1, \dots, \rho_n$ is defined as

$$\square_{i=1}^n \rho_i(X) := \inf \left\{ \sum_{i=1}^n \rho_i(X_i) : (X_1, \dots, X_n) \in \mathbb{A}_n(X) \right\}, \quad X \in \mathcal{X}.$$

That is, the inf-convolution of n risk measures is the infimum over aggregate risk values for all possible allocations.

An allocation $(X_1, \dots, X_n)$ is sum-optimal in $\mathbb{A}_n(X)$ if $\square_{i=1}^n \rho_i(X) = \sum_{i=1}^n \rho_i(X_i)$, i.e., it attains the optimal total risk value. An allocation $(X_1, \dots, X_n) \in \mathbb{A}_n(X)$ is Pareto optimal in $\mathbb{A}_n(X)$ if for any $(Y_1, \dots, Y_n) \in \mathbb{A}_n(X)$ satisfying $\rho_i(Y_i) \leqslant \rho_i(X_i)$ for all $i \in [n]$, we have $\rho_i(Y_i) = \rho_i(X_i)$ for all $i \in [n]$. Pareto optimality means that the allocation cannot be improved upon for all agents while providing a strict improvement for at least one agent. For distortion risk measures, the equivalence between Pareto optimality and sum-optimality is guaranteed when $\mathcal{X} = L^\infty$, as obtained in Embrechts et al. (2018, Proposition 1). We will focus on sum-optimal allocations in this chapter. However, although sum-optimal allocations are always Pareto optimal, the converse may not hold in case $\mathcal{X} = L^+$ or $X = L^-$, which are examined in Sections 3.5.

3.3 Comonotonic and counter-monotonic risk sharing

The elements in the allocation set $\mathbb{A}_n(X)$ can exhibit different dependence structures, with comonotonicity and counter-monotonicity being the two extreme cases. We first define comonotonicity and counter-monotonicity for bivariate random variables.

Definition 3.2. Two random variables X and Y on $(\Omega, \mathcal{F}, \mathbb{P})$ are said to be comonotonic (resp. counter-monotonic) if

$$(X(\omega) - X(\omega'))(Y(\omega) - Y(\omega')) \geqslant 0 \text{ (resp. } \leqslant 0), \text{ for } (\mathbb{P} \times \mathbb{P})\text{-almost all } (\omega, \omega') \in \Omega^2.$$

Note that X and Y are counter-monotonic if and only if X and $-Y$ are comonotonic. Also, comonotonicity of (X, Y) is equivalent to the existence of increasing functions f and g and a random variable Z, such that $(X, Y) = (f(Z), g(Z))$ (almost surely), which follows from the stochastic representation of comonotonicity given in Denneberg (1994, Proposition 4.5). Since we treat almost surely identical random variables as equal, we will omit "almost surely" in statements like the one above. Comonotonicity is foundational to modern ambiguity models in decision theory (Schmeidler, 1989) and it is widely studied in

actuarial science and risk management (Dhaene et al., 2002, 2006). Counter-monotonicity also has special roles, quite different from comonotonicity, in decision theory (Principi et al., 2025) and actuarial science (Cheung et al., 2014; Chaoubi et al., 2020).

Next, we define these concepts in higher dimensions. A random vector $(X_1, \dots, X_n)$ is (pairwise) comonotonic (resp. counter-monotonic) if each pair of its components is comonotonic (resp. counter-monotonic). Pairwise counter-monotonicity is the generalization of counter-monotonicity to the case $n \geqslant 3$, and we will hereafter use the simpler term counter-monotonicity throughout. Although comonotonicity for $n \geqslant 3$ is a straightforward extension of the case $n = 2$, counter-monotonicity for $n \geqslant 3$ imposes strong constraints on the marginal distributions, and it is quite different from the case $n = 2$ (Dall'Aglio, 1972; Dhaene and Denuit, 1999; Cheung and Lo, 2014). Below we provide some facts about counter-monotonicity. Let Π_n be the set of all n-compositions of Ω, that is,

$$\Pi_n = \left\{ (A_1, \dots, A_n) \in \mathcal{F}^n : \bigcup_{i \in [n]} A_i = \Omega \text{ and } A_1, \dots, A_n \text{ are disjoint} \right\}.$$

Formally, compositions are partitions in which the order matters.

We quote below a stochastic representation of counter-monotonicity, which will be useful throughout our analysis. In what follows, ess-sup$X$ and ess-inf$X$ are the essential supremum and the essential infimum of X, respectively. Additionally, a random variable $X \in \mathcal{X}$ is said to be degenerate if there exists $c \in \mathbb{R}$ such that $X = c$ (almost surely).

Proposition 3.1 (Lauzier et al. (2023))**.** *For $X \in \mathcal{X}$ and $n \geqslant 3$, suppose that at least three components of $(X_1, \dots, X_n) \in \mathbb{A}_n(X)$ are non-degenerate. Then $(X_1, \dots, X_n)$ is counter-monotonic if and only if there exist constants $m_1, \dots, m_n$ and $(A_1, \dots, A_n) \in \Pi_n$ such that*

$$\textit{either} \quad X_i = (X - m)\mathbb{1}_{A_i} + m_i \textit{ for all } i \in [n], \textit{ with } m = \sum_{i=1}^{n} m_i \leqslant \text{ess-inf}X, \tag{3.3}$$

$$\text{or} \quad X_i = (X-m)\mathbb{1}_{A_i} + m_i \text{ for all } i \in [n], \text{ with } m = \sum_{i=1}^{n} m_i \geqslant \text{ess-sup}X. \tag{3.4}$$

By taking $m_1 = \ldots = m_n = 0$, which is possible when $X \geqslant 0$ or $X \leqslant 0$, a simple counter-monotonic allocation in the form of (3.3) and (3.4) is given by

$$X_i = X\mathbb{1}_{A_i}, \text{ for each } i \in [n], \text{ where } (A_1, \ldots, A_n) \in \Pi_n.$$

Specifically, such an allocation is called a *jackpot allocation* if $X \geqslant 0$ and a *scapegoat allocation* if $X \leqslant 0$ by Lauzier et al. (2026a). It is clear that there is a "winner-takes-all" structure in a jackpot allocation and a "loser-loses-all" structure in a scapegoat allocation.

For a given X, we denote by $\mathbb{A}_n^+(X)$ the set of comonotonic allocations and by $\mathbb{A}_n^-(X)$ the set of counter-monotonic allocations, introduced by Ghossoub et al. (2026a). These two sets of allocations impose restrictions on the dependence structures of allocations, and both are strict subsets of the set $\mathbb{A}_n(X)$ of all possible allocations.

It is well known that for any $(X_1, \ldots, X_n) \in \mathbb{A}_n(X)$, there exists $(Y_1, \ldots, Y_n) \in \mathbb{A}_n^+(X)$ such that $Y_i \leqslant_{\text{cx}} X_i$ for each $i \in [n]$. This is known as the comonotonic improvement theorem (e.g., Landsberger and Meilijson (1994), Rüschendorf (2013), or Denuit et al. (2025)). Recently, Lauzier et al. (2026a) provided a counter-monotonic improvement theorem, which states that under mild conditions, for any allocation $(X_1, \ldots, X_n) \in \mathbb{A}_n(X)$, there exists $(Y_1, \ldots, Y_n) \in \mathbb{A}_n^-(X)$ such that $Y_i \geqslant_{\text{cx}} X_i$, for each $i \in [n]$. The formal statement is summarized below, and it will be useful for the results of this chapter.

Theorem 3.1 (Lauzier et al. (2026a))**.** *Let $X_1, \ldots, X_n$ be nonnegative integrable random variables and $X = \sum_{i=1}^n X_i$. Assume that there exists a uniform random variable U independent of X. Then, there exists $(Y_1, \ldots, Y_n) \in \mathbb{A}_n(X)$ such that (i) $(Y_1, \ldots, Y_n)$ is counter-monotonic; (ii) $Y_i \geqslant_{\text{cx}} X_i$, for each $i \in [n]$; and (iii) $Y_1, \ldots, Y_n$ are nonnegative. Moreover, $(Y_1, \ldots, Y_n)$ can be chosen as a jackpot allocation.*

The counter-monotonic improvement theorem indicates that jackpot allocations are

always preferred by risk-seeking agents. To apply the counter-monotonic improvement theorem, the technical assumption that there exists a (nondegenerate) uniform random variable U independent of X is used to generate a random lottery, which has utility for risk-seeking agents. To formalize this, we introduce the following set:

$$\mathcal{X}^{\perp} = \{X \in \mathcal{X} : \text{there exists a uniform random variable } U \text{ independent of } X\}.$$

Due to the comonotonic improvement theorem and the counter-monotonic improvement theorem, Pareto-optimal risk allocations can be constrained in the sets of comonotonic or counter-monotonic allocations, for agents with suitable risk attitudes. Hereafter, we consider risk-sharing problems constrained to these specific allocation structures. The comonotonic inf-convolution $\boxplus_{i=1}^{n}\rho_i$ of risk measures $\rho_1, \ldots, \rho_n$ is defined as

$$\boxplus_{i=1}^{n} \rho_i(X) = \inf\left\{\sum_{i=1}^{n} \rho_i\left(X_i\right) : (X_1, \ldots, X_n) \in \mathbb{A}_n^{+}(X)\right\}.$$

Similarly, the counter-monotonic inf-convolution $\boxminus_{i=1}^{n} \rho_i$ is defined as

$$\boxminus_{i=1}^{n} \rho_i(X) = \inf\left\{\sum_{i=1}^{n} \rho_i(X_i) : (X_1, \ldots, X_n) \in \mathbb{A}_n^{-}(X)\right\}.$$

An allocation $(X_1, \ldots, X_n)$ in $\mathbb{A}_n^{+}(X)$ (resp. $\mathbb{A}_n^{-}(X)$) is called an optimal allocation of X for $(\rho_1, \ldots, \rho_n)$ within $\mathbb{A}_n^{+}$ (resp. $\mathbb{A}_n^{-}$) if

$$\sum_{i=1}^{n} \rho_i\left(X_i\right) = \boxplus_{i=1}^{n} \rho_i(X) \qquad \left(\text{resp. } \sum_{i=1}^{n} \rho_i\left(X_i\right) = \boxminus_{i=1}^{n} \rho_i(X)\right).$$

By definition, $\square_{i=1}^{n}\rho_i(X) \leqslant \boxplus_{i=1}^{n}\rho_i(X)$ and $\square_{i=1}^{n}\rho_i(X) \leqslant \boxminus_{i=1}^{n} \rho_i(X)$. Hence, if an optimal allocation of X is comonotonic, then it is also an optimal allocation within $\mathbb{A}_n^{+}$ and $\square_{i=1}^{n}\rho_i(X) = \boxplus_{i=1}^{n}\rho_i(X)$. A similar implication holds if an optimal allocation of X is counter-monotonic. Unconstrained, comonotonic, and counter-monotonic risk sharing

problems correspond to those problems over $\mathbb{A}_n(X)$, $\mathbb{A}_n^+(X)$, and $\mathbb{A}_n^-(X)$, respectively.

In the rest of the chapter, we will assume that each agent is associated with a distortion risk measure.

3.4 General relations

Before delving into the optimal risk-sharing mechanisms for risk-averse and risk-seeking agents, we first provide an overview of the relationships between three types of inf-convolutions: $\square_{i=1}^n \rho_{h_i}$, $\boxminus_{i=1}^n \rho_{h_i}$ and $\boxplus_{i=1}^n \rho_{h_i}$. In the homogeneous case, these relationships are straightforward to characterize, and they are outlined in Ghossoub et al. (2026a, Theorem 1). However, the situation becomes more complex in the heterogeneous case where agents have different risk preferences. Among these three inf-convolutions, the unconstrained one is always the smallest, since it works on the largest allocation set, but the relationship between the comonotonic and counter-monotonic inf-convolutions is not always the same. Below, we present several special cases in which the relationship between these three variations can be explicitly determined.

In the following examples, we explore the cases of Value-at-Risk (VaR) and Expected Shortfall (ES) using the results of Embrechts et al. (2018).

Example 3.1 (Inf-convolution of VaRs)**.** By Embrechts et al. (2018, Corollary 2), for $\alpha_1, \dots, \alpha_n \geqslant 0$ and an integrable random variable X, if $\sum_{i=1}^n \alpha_i < 1$ then

$$\square_{i=1}^n \mathrm{VaR}_{\alpha_i}(X) = \mathrm{VaR}_{\sum_{i=1}^n \alpha_i}(X),$$

and by Embrechts et al. (2018, Theorem 2), there exists a counter-monotonic optimal allocation of X. This gives $\boxminus_{i=1}^n \mathrm{VaR}_{\alpha_i} \leqslant \square_{i=1}^n \mathrm{VaR}_{\alpha_i}$, thereby implying that $\boxminus_{i=1}^n \mathrm{VaR}_{\alpha_i} = \square_{i=1}^n \mathrm{VaR}_{\alpha_i}$, since $\boxminus_{i=1}^n \mathrm{VaR}_{\alpha_i} \geqslant \square_{i=1}^n \mathrm{VaR}_{\alpha_i}$ always holds. Since, in addition, $\boxplus_{i=1}^n \mathrm{VaR}_{\alpha_i} =$

$\mathrm{VaR}_{\bigvee_{i=1}^n \alpha_i}$, we obtain

$$\boxminus_{i=1}^{n} \mathrm{VaR}_{\alpha_i} = \square_{i=1}^{n} \mathrm{VaR}_{\alpha_i} = \mathrm{VaR}_{\sum_{i=1}^n \alpha_i} \leqslant \boxplus_{i=1}^{n} \mathrm{VaR}_{\alpha_i} = \mathrm{VaR}_{\bigvee_{i=1}^n \alpha_i}\,.$$

Note, in particular, that the inequality above is generally not an equality.

Example 3.2 (Inf-convolution of ESs)**.** For any $\beta_1, \dots, \beta_n \in [0,1)$, it holds that

$$\boxminus_{i=1}^{n} \mathrm{ES}_{\beta_i} = \square_{i=1}^{n} \mathrm{ES}_{\beta_i} = \boxplus_{i=1}^{n} \mathrm{ES}_{\beta_i} = \mathrm{ES}_{\bigvee_{i=1}^n \beta_i}\,.$$

Using the subadditivity of ES, we can easily verify that, if j is the agent with $\beta_j = \bigvee_{i=1}^n \beta_i$, then the allocation given by $X_j = X$ and $X_i = 0$ for $i \neq j$ is optimal (this is also implied by Embrechts et al. (2018, Theorem 2)). Since this allocation is both comonotonic and counter-monotonic, the three inf-convolutions have the same value here. This is a special case of Corollary 3.1 below.

Within the case of VaR, it follows that $\boxminus_{i=1}^n \rho_{h_i} \leqslant \boxplus_{i=1}^n \rho_{h_i}$ from Example 3.1. However, this is not the only possible relationship. The following theorem provides conditions under which $\boxminus_{i=1}^n \rho_{h_i} \geqslant \boxplus_{i=1}^n \rho_{h_i}$, and it identifies sufficient conditions for which the inequality becomes equality. First, recall from Ghossoub et al. (2026a) that a function $h \in \mathcal{H}$ is *dually subadditive* if h is subadditive (i.e., $h(x+y) \leqslant h(x)+h(y)$ for $x, y \in [0,1]$ with $x+y \leqslant 1$) and its dual $\tilde{h}$ is superadditive (i.e., $\tilde{h}(x+y) \geqslant \tilde{h}(x)+\tilde{h}(y)$ for $x, y \in [0,1]$ with $x+y \leqslant 1$).

Theorem 3.2. *Let $h_i \in \mathcal{H}$ for $i \in [n]$ and $h = \bigwedge_{i=1}^n h_i$.*

(i) If h is dually subadditive, then

$$\boxminus_{i=1}^{n} \rho_{h_i} \geqslant \boxplus_{i=1}^{n} \rho_{h_i} = \rho_h \geqslant \square_{i=1}^{n} \rho_{h_i}. \tag{3.5}$$

(ii) If h is dually subadditive and $h_j = h$ for some $j \in [n]$, then

$$\boxminus_{i=1}^{n} \rho_{h_i} = \boxplus_{i=1}^{n} \rho_{h_i} = \rho_h \geqslant \square_{i=1}^{n} \rho_{h_i}. \tag{3.6}$$

(iii) The function h is concave if and only if

$$\boxminus_{i=1}^{n} \rho_{h_i} \geqslant \boxplus_{i=1}^{n} \rho_{h_i} = \rho_h = \square_{i=1}^{n} \rho_{h_i}. \tag{3.7}$$

(iv) If (3.6) holds then h is dually subadditive.

Proof. (i) It is trivial to see that $\square_{i=1}^{n} \rho_{h_i} \leqslant \boxminus_{i=1}^{n} \rho_{h_i}$ and $\square_{i=1}^{n} \rho_{h_i} \leqslant \boxplus_{i=1}^{n} \rho_{h_i}$. The first equality in (3.5) directly follows from Embrechts et al. (2018, Proposition 5). For any distortion functions h and g, we have $\rho_h \leqslant \rho_g$ if $h \leqslant g$ (see Wang et al. (2020b, Lemma 1)). Therefore, for any $h_i \in \mathcal{H}$ such that h is dually subadditive, we have

$$\boxminus_{i=1}^{n} \rho_{h_i} \geqslant \boxminus_{i=1}^{n} \rho_h = \rho_h,$$

which directly follows from the dual subadditivity of h and Ghossoub et al. (2026a, Theorem 3).

(ii) Take $(X_1, \dots, X_n) \in \mathbb{A}_n(X)$ such that $X_j = X$ and $X_i = 0$ for $i \in [n] \setminus \{j\}$. It is straightforward to verify that this allocation is counter-monotonic. Then it follows that

$$\square_{i=1}^{n} \rho_{h_i} \leqslant \boxminus_{i=1}^{n} \rho_{h_i} \leqslant \rho_h = \boxplus_{i=1}^{n} \rho_{h_i}.$$

By the equality (3.5), we can obtain the desired result.

(iii) We first show the "if " part. Letting $h_i = h$ for $i \in [n]$, we have $\square_{i=1}^{n} \rho_h = \rho_h$, which implies that h is concave since ρ_h is subadditive (see Wang et al. (2020b, Theorem 3)). Next, we show the "only if " part. By Wang et al. (2020b, Lemma 1), we have

$$\square_{i=1}^{n} \rho_{h_i} \geqslant \square_{i=1}^{n} \rho_h = \rho_h = \boxplus_{i=1}^{n} \rho_{h_i}.$$

The first equality follows from the comonotonic improvement theorem of Landsberger and Meilijson (1994); see also Ghossoub et al. (2026a, Theorem 1). Moreover, it always holds that $\square_{i=1}^{n} \rho_{h_i} \leqslant \boxplus_{i=1}^{n} \rho_{h_i}$. Thus, $\square_{i=1}^{n} \rho_{h_i} = \boxplus_{i=1}^{n} \rho_{h_i}$.

(iv) The inequality in (3.6) implies that $\boxminus_{i=1}^{n} \rho_h = \rho_h$ by taking $h_i = h$ for $i \in [n]$. Thus, we obtain that h is dually subadditive by Ghossoub et al. (2026a, Theorem 2). $\square$

We immediately obtain the following corollary:

Corollary 3.1. *Let $h_i \in \mathcal{H}$ for $i \in [n]$ and $h = \bigwedge_{i=1}^{n} h_i$. If h_i is concave for $i \in [n]$ and $h_j = h$ for some j, then*

$$\boxminus_{i=1}^{n} \rho_{h_i} = \boxplus_{i=1}^{n} \rho_{h_i} = \rho_h = \square_{i=1}^{n} \rho_{h_i}. \tag{3.8}$$

The proof is straightforward since the concavity of h_i for $i \in [n]$ guarantees the concavity of h , thus implying the dual subadditivity of h. In this case, the total risk will be absorbed by the least risk-averse agent if such an agent exists within the group.

Example 3.3. If h is concave, (3.8) holds when X is a constant $x \in \mathbb{R}$, without assuming the condition $h_j = h$ for some j. Indeed, any constant vector $(x_1, \dots, x_n)$ in $\mathbb{A}_n(x)$ gives $\sum_{i=1}^{n} \rho_{h_i}(x_i) = \rho_h(x) = x$, and the rest follows from Theorem 3.2 (iii).

Example 3.4. Suppose that all agents have the same risk preferences, i.e., $h_1 = \cdots = h_n = h$. If h is dually subadditive, then it holds that

$$\boxminus_{i=1}^{n} \rho_h = \boxplus_{i=1}^{n} \rho_h = \rho_h.$$

The converse statement also holds true by using (iv) of Theorem 3.2; this also follows from Ghossoub et al. (2026a, Theorem 3).

Based on the results above for risk-averse agents, if there is an agent j with the lowest level of risk aversion (that is, h_j is the smallest among $h_1, \dots, h_n$), then the optimal way to allocate the total risk X is to assign it entirely to this most risk-tolerant agent. The risk preferences of the other agents are irrelevant in this case. However, when heterogeneous

risk-seeking agents are involved, the situation is dramatically different; all agents will participate in the gambling and contribute to the aggregate risk. The details will be discussed in Section 3.5.

3.5 Risk-seeking agents

This section contains the main technical contributions of this chapter, which characterize optimal allocations for the unconstrained and counter-monotonic risk sharing problem in the setting of risk-seeking agents.

3.5.1 Main results

We first note that, when the agents are risk-seeking, we need to constrain the set of allocations to be bounded from below or above, as discussed by Lauzier et al. (2026a). The next result shows that the inf-convolution of $\rho_{h_1}, \dots, \rho_{h_n}$ for risk-seeking agents is typically negative infinity if the set of payoffs $\mathcal{X}$ is taken to be L^∞.

Proposition 3.2. *Let $\mathcal{X} = L^\infty$. If each $h_i \in \mathcal{H}$, $i \in [n]$ is convex and is not the identity function, then*

$$\boxminus_{i=1}^{n} \rho_{h_i}(X) = \boxplus_{i=1}^{n} \rho_{h_i}(X) = -\infty \quad \text{for all } X \in \mathcal{X}.$$

Proof. Let $h = \bigvee_{i=1}^n h_i$. It is clear that $\boxminus_{i=1}^n \rho_{h_i}(X) \leqslant \boxminus_{i=1}^n \rho_h(X)$, for all X. We will show $\boxminus_{i=1}^n \rho_h(0) = -\infty$. For a convex distortion function h_i, we have $h_i(1/n) < 1/n$ if and only if h_i is not the identity function. Hence, $h(1/n) = \bigvee_{i=1}^n h_i(1/n) < 1/n$, and h is not the identity function. Let $\theta = nh(1/n) < 1$. Take $(A_1, \dots, A_n) \in \Pi_n$ with $\mathbb{P}(A_i) = 1/n$ for $i \in [n]$ and $m > 0$. Define $X_i = nm\mathbb{1}_{A_i} - m$ for $i \in [n]$. Clearly, $(X_1, \dots, X_n) \in \mathbb{A}_n^-(0)$. It follows that $\rho_h(X_i) = nmh(1/n) - m = (\theta - 1)m$. Therefore,

$$\boxplus_{i=1}^{n} \rho_h(0) \leqslant \sum_{i=1}^{n} \rho_h(X_i) = n(\theta - 1)m.$$

Letting $m \to \infty$ shows that $\boxminus_{i=1}^{n} \rho_h(0) = -\infty$. For any $X \in \mathcal{X}$, using translation invariance and monotonicity of ρ_h, and the fact that $X \leqslant \text{ess-sup}X$, we obtain

$$\boxplus_{i=1}^{n} \rho_{h_i}(X) \leqslant \boxminus_{i=1}^{n} \rho_{h_i}(X) \leqslant \boxminus_{i=1}^{n} \rho_h(X) \leqslant \boxminus_{i=1}^{n} \rho_h(\text{ess-sup}X) = \boxminus_{i=1}^{n} \rho_h(0) + \text{ess-sup}X = -\infty,$$

which completes the proof. □

Due to Proposition 3.2, we will take $\mathcal{X}$ to be L^+ or L^- in the remainder of the section. These choices correspond to the natural constraint of no short-selling in a financial market. For instance, if $\mathcal{X} = L^+$, then the total loss is nonnegative, and every agent cannot receive a negative loss (that is, a gain) from the allocation. If $\mathcal{X} = L^-$, then the total risk is a gain, and every agent cannot take a loss by sharing the gain.

Before we proceed, we introduce some additional terminology and notation that will be useful in our further analysis. Given n distortion functions $h_i \in \mathcal{H}$ for $i = 1, \dots, n$, their inf-convolution $\square_{i=1}^{n} h_i(x) : [0,1] \mapsto \mathbb{R}$ is defined as

$$\boxplus_{i=1}^{n} h_i(x) = \inf \left\{ \sum_{i=1}^{n} h_i(x_i) : x_i \in [0,1] \text{ for } i \in [n];\ \sum_{i=1}^{n} x_i = x \right\}.$$

Similarly, the sup-convolution can be defined as

$$\boxminus_{i=1}^{n} h_i(x) = \sup \left\{ \sum_{i=1}^{n} h_i(x_i) : x_i \in [0,1] \text{ for } i \in [n];\ \sum_{i=1}^{n} x_i = x \right\}.$$

Here we also introduce the sup-convolution since it will be useful in our analysis and will simplify our result. Note that the inf-convolution of real functions $h_1, \dots, h_n$ is similar to the inf-convolution of risk measures, but the domain is restricted to real numbers in $[0,1]$.

Theorem 3.3. *Suppose that $h_i \in \mathcal{H}$ is continuous and convex for $i \in [n]$, and that $X \in \mathcal{X}^{\perp}$ with $\mathcal{X} = L^+$ or $\mathcal{X} = L^-$. Then*

$$\boxplus_{i=1}^{n} \rho_{h_i}(X) = \boxminus_{i=1}^{n} \rho_{h_i}(X) = \rho_g(X), \tag{3.9}$$

	X	$\mathcal{X}$	$\boxplus_{i=1}^{2} \rho_{h_i}(X)$	$\boxminus_{i=1}^{2} \rho_{h_i}(X) = \square_{i=1}^{2} \rho_{h_i}(X)$
$Y \sim \text{Uniform}(0,1)$	Y	L^+	0.3692	0.2074
	$-Y$	L^-	-0.7667	-1.0435
$Y \sim \text{Pareto}(3,2)$	Y	L^+	2.7291	1.3828
	$-Y$	L^-	-9.4262	-11.0881
$Y \sim \text{logN}(0,1)$	Y	L^+	1.0825	0.5849
	$-Y$	L^-	-5.5828	-6.4773

Table 3.1: Comparison of the three inf-convolutions.

where g is such that (i) $g = \square_{i=1}^{n} h_i$ if $\mathcal{X} = L^+$; (ii) $\tilde{g} = \diamondsuit_{i=1}^{n} \tilde{h}_i$ if $\mathcal{X} = L^-$.

The proof of Theorem 3.3 involves three lemmas and is presented in Section 3.5.2. For the case where all agents are risk averse, the relationship among the three types of inf-convolutions can be established using Theorem 3.2(iii), implying that agents always prefer comonotonic allocations. In contrast, when all agents are risk seeking, Theorem 3.3 indicates that the relationship is determined by a preference for counter-monotonic allocations. Specifically, for $\mathcal{X} = L^+$ (resp. $\mathcal{X} = L^-$) and $X \in \mathcal{X}^{\perp}$, we have

$$\boxplus_{i=1}^{n} \rho_{h_i}(X) \geqslant \square_{i=1}^{n} \rho_{h_i}(X) = \boxminus_{i=1}^{n} \rho_{h_i}(X). \tag{3.10}$$

Notably, the comonotonic inf-convolution is always a distortion risk measure, whereas the inf-convolution of convex distortion risk measures is no longer a distortion risk measure, but rather a monotone distortion riskmetric, as stated in Theorem 3.3. To further verify (3.10) for risk-seeking agents, we provide several numerical experiments, reported in Table 3.1. In these examples, we consider two agents with $h_1(x) = 1-(1-x)^{0.3}$ and $h_2(x) = 1-(1-x)^{0.6}$, respectively. Clearly, both agents are risk seeking.

Example 3.5. When all agents have the same risk preference, that is, $h_1 = \cdots = h_n = h$, and when h is convex and continuous on $[0,1]$, Theorem 3.3 reduces to the homogeneous

case, examined in Ghossoub et al. (2026a, Theorem 3). For example, when $\mathcal{X} = L^+$, the function g becomes

$$g(x) = \square_{i=1}^{n} h(x) = nh(x/n), \quad x \in [0,1],$$

which follows from the convexity of h. Similarly, when $\mathcal{X} = L^-$, we have

$$g(x) = \tilde{l}(x), \text{ and } l(x) = \diamondsuit_{i=1}^{n} \tilde{h}(x) = n\tilde{h}(x/n), \quad x \in [0,1],$$

and it is straightforward to verify that $g(x) = nh(1-(1-x)/n) - nh(1-1/n)$.

3.5.2 Three technical lemmas and the proof of Theorem 3.3

We first present a technical lemma.

Lemma 3.1. *For a given $x > 0$, any random variable X, and $h \in \mathcal{H}$, we have*

$$\inf\left\{\sum_{i=1}^{n} h(\mathbb{P}(X\mathbb{1}_{A_i} \geqslant x)) : (A_1,\dots,A_n) \in \Pi_n\right\} = \square_{i=1}^{n} h(\mathbb{P}(X \geqslant x)).$$

Proof. Let $p = \mathbb{P}(X \geqslant x)$. If $p = 0$, then there is nothing to show. Suppose $p > 0$ in what follows. Since the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is atomless, for any $(p_1,\dots,p_n) \in \mathbb{R}_+^n$ with $p_1 + \cdots + p_n = p$, there exists a composition $(A_1,\dots,A_n)$ of Ω such that $\mathbb{P}(A_i \cap \{X \geqslant x\}) = p_i$ for $i \in [n]$. Therefore,

$$\inf\left\{\sum_{i=1}^{n} h(\mathbb{P}(X\mathbb{1}_{A_i} \geqslant x)) : (A_1,\dots,A_n) \in \Pi_n\right\} \geqslant \square_{i=1}^{n} h(\mathbb{P}(X \geqslant x)).$$

The converse direction is straightforward since $\sum_{i=1}^{n} \mathbb{P}(X\mathbb{1}_{A_i} \geqslant x) = \mathbb{P}(X \geqslant x) = p$. □

Lemma 3.2. *Let $h_1,\dots,h_n : [0,1] \to \mathbb{R}$ be continuous and strictly convex. Then there exist increasing functions $f_i : [0,1] \to [0,1]$ such that for $x \in [0,1]$, $\square_{i=1}^{n} h_i(x) = \sum_{i=1}^{n} h_i(f_i(x))$, where $\sum_{i=1}^{n} f_i(t) = t$, $t \in [0,1]$.*

Proof. Fix $x \in [0,1]$. Set $\mathcal{S}(x) = \{(y_1, \ldots, y_n) \in [0,1]^n : \sum_i y_i = x\}$ and $F(y) = \sum_i h_i(y_i)$, where $y = (y_1, \ldots, y_n) \in \mathcal{S}(x)$. Then $\mathcal{S}(x)$ is a nonempty compact convex set. Since each h_i is continuous, F is continuous, therefore F attains its minimum on $\mathcal{S}(x)$. F is strictly convex on the convex set $\mathcal{S}(x)$. Hence, F has a unique minimizer $f(x) = (f_1(x), \ldots, f_n(x))$. By definition, we have $\sum_i f_i(x) = x$ and $\square_i h_i(x) = F(f(x))$. It remains to show each f_i is increasing.

By strict convexity of h, we have

$$h(u+\varepsilon) - h(u) \;<\; h(v+\varepsilon) - h(v), \tag{3.11}$$

where $u < v$, $\varepsilon > 0$ with $u, u+\varepsilon, v, v+\varepsilon \in [0,1]$. Take $x < y$. Write $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$, where $a_i = f_i(x)$ and $b_i = f_i(y)$. Suppose, for contradiction, that $a_j > b_j$ for some j. As $\sum_i (b_i - a_i) = y - x > 0$, there exists $k \in [n]$ such that $b_k > a_k$ ($k \neq j$). Take $\varepsilon \in \big(0, \min\{a_j - b_j,\ b_k - a_k\}\big)$. Then $a_j - \varepsilon > b_j \geqslant 0$ and $a_k + \varepsilon < b_k \leqslant 1$. Define $\tilde{a} = (\tilde{a}_1, \ldots, \tilde{a}_n)$, where $\tilde{a}_j = a_j - \varepsilon$, $\tilde{a}_k = a_k + \varepsilon$ and $\tilde{a}_i = a_i$ for $i \neq j, k$. Similarly, define $\tilde{b} = (\tilde{b}_1, \ldots, \tilde{b}_n)$, where $\tilde{b}_k = a_k - \varepsilon$, $\tilde{b}_j = b_j + \varepsilon$ and $\tilde{b}_i = b_i$ for $i \neq j, k$. Optimality of a and b therefore gives $F(a) \leqslant F(\tilde{a})$ and $F(b) \leqslant F(\tilde{b})$, which leads to

$$h_j(a_j) - h_j(a_j - \varepsilon) \leqslant h_k(a_k + \varepsilon) - h_k(a_k), \qquad h_k(b_k) - h_k(b_k - \varepsilon) \leqslant h_j(b_j + \varepsilon) - h_j(b_j).$$

By (3.11) and the fact of $a_k < b_k - \varepsilon$ and $b_j < a_j - \varepsilon$, we have

$$h_k(a_k + \varepsilon) - h_k(a_k) < h_k(b_k) - h_k(b_k - \varepsilon), \qquad h_j(b_j + \varepsilon) - h_j(b_j) < h_j(a_j) - h_j(a_j - \varepsilon).$$

Combining the four inequalities, it follows that

$$\begin{aligned} h_j(a_j) - h_j(a_j - \varepsilon) \;&\leqslant\; h_k(a_k + \varepsilon) - h_k(a_k) \;<\; h_k(b_k) - h_k(b_k - \varepsilon) \\ &\leqslant\; h_j(b_j + \varepsilon) - h_j(b_j) \;<\; h_j(a_j) - h_j(a_j - \varepsilon), \end{aligned}$$

a contradiction. Hence, $a_i \leqslant b_i$ for all $i \in [n]$ and each f_i is increasing on $[0,1]$. □

In what follows, $C^2[0,1]$ is the set of all continuous functions on $[0,1]$ with continuous second-order derivatives on $(0,1)$.

Lemma 3.3. *Suppose that $h_i \in \mathcal{H} \cap C^2[0,1]$ is strictly convex for $i \in [n]$, and that $X \in \mathcal{X}^{\perp}$.*

(i) If $\mathcal{X} = L^+$, then $\square_{i=1}^n \rho_{h_i}(X) = \boxminus_{i=1}^n \rho_{h_i}(X) = \rho_g(X)$, where $g = \square_{i=1}^n h_i$.

(ii) If $\mathcal{X} = L^-$, then $\square_{i=1}^n \rho_{h_i}(X) = \boxminus_{i=1}^n \rho_{h_i}(X) = \rho_g(X)$, where g is such that $\tilde{g} = \diamondsuit_{i=1}^n \tilde{h}_i$.

Proof. (i) Suppose that $\mathcal{X} = L^+$ and $X \in \mathcal{X}^{\perp}$. By Theorem 3.1, for any allocation $(X_1, \dots, X_n) \in \mathbb{A}_n(X)$, there exists a jackpot allocation $(X\mathbb{1}_{A_1}, \dots, X\mathbb{1}_{A_n})$, $(A_1, \dots, A_n) \in \Pi_n$ such that $X\mathbb{1}_{A_i} \leqslant_{\mathrm{cv}} X_i$ for each $i \in [n]$. Since $h_i \in \mathcal{H}$ for $i \in [n]$ are convex, there exists $(A_1, \dots, A_n) \in \Pi_n$ such that

$$\sum_{i=1}^n \rho_{h_i}(X\mathbb{1}_{A_i}) = \sum_{i=1}^n \int_0^{\infty} h_i(\mathbb{P}(X\mathbb{1}_{A_i} \geqslant x))\mathrm{d}x \leqslant \sum_{i=1}^n \rho_{h_i}(X_i)$$

holds for $(X_1, \dots, X_n) \in \mathbb{A}_n(X)$. Taking the infimum on both sides yields

$$\inf\left\{ \int_0^{\infty} \sum_{i=1}^n h_i(\mathbb{P}(X\mathbb{1}_{A_i} \geqslant x))\mathrm{d}x : (A_1, \dots, A_n) \in \Pi_n \right\} \leqslant \underset{i=1}{\overset{n}{\square}} \rho_{h_i}(X) \leqslant \underset{i=1}{\overset{n}{\boxminus}} \rho_{h_i}(X),$$

and the above inequalities are in fact equalities since $(X\mathbb{1}_{A_1}, \dots, X\mathbb{1}_{A_n}) \in \mathbb{A}_n^-(X)$. Now, let $g = \square_{i=1}^n h_i$. Using Lemma 3.1, the above inequalities imply that

$$\rho_g(X) = \int_0^{\infty} \inf\left\{ \sum_{i=1}^n h_i(\mathbb{P}(X\mathbb{1}_{A_i} \geqslant x)) : (A_1, \dots, A_n) \in \Pi_n \right\} \mathrm{d}x \leqslant \underset{i=1}{\overset{n}{\square}} \rho_{h_i}(X) \leqslant \underset{i=1}{\overset{n}{\boxminus}} \rho_{h_i}(X).$$

Thus, we obtain that $\rho_g(X) \leqslant \boxminus_{i=1}^n \rho_{h_i}(X)$.

Next, we show the converse direction, that is, $\rho_g(X)$ is attainable by some allocation. Since h_i is strictly convex and differentiable for each $i \in [n]$, there exist increasing functions

f_i, $i \in [n]$, such that $\square_{i=1}^n h_i(x) = \sum_{i=1}^n h_i(f_i(x))$ and $\sum_{i=1}^n f_i(x) = x$, $x \in [0,1]$, as stated in Lemma 3.2. Define the events $A_1, \dots, A_n$ by

$$A_i = \left\{ 1 - \sum_{j=1}^{i} f_j'(1-U_X) < U \leqslant 1 - \sum_{j=1}^{i-1} f_j'(1-U_X) \right\}, \quad i \in [n-1],$$
$$A_n = \left\{ U \leqslant 1 - \sum_{j=1}^{n-1} f_j'(1-U_X) \right\}.$$

Note that the strict convexity of $h_i \in C^2[0,1]$ guarantees the existence of f_i' on $(0,1)$. By the above construction, it is straightforward to verify that $(A_1, \dots, A_n)$ is a composition of Ω, since $\sum_{i=1}^n f_i'(x) = 1$ for $x \in (0,1)$. Moreover, for $i \in [n]$ and $x > 0$, $\mathbb{P}(A_i | X \geqslant x) = \mathbb{E}[f_i'(1-U_X) | X \geqslant x]$, due to independence between U and X. Consider the allocation $(X_1, \dots, X_n) = (X\mathbb{1}_{A_1}, \dots, X\mathbb{1}_{A_n})$, which is in $\mathbb{A}_n^-(X)$. Note that $f_i(y) \to 0$ for $y \downarrow 0$ because $f_i(x) \leqslant x$ for $x \in (0,1)$. Then

$$\begin{aligned}
\sum_{i=1}^n \rho_{h_i}(X\mathbb{1}_{A_i}) &= \sum_{i=1}^n \int_0^\infty h_i(\mathbb{P}(X\mathbb{1}_{A_i} \geqslant x))\mathrm{d}x \\
&= \sum_{i=1}^n \int_0^\infty h_i(\mathbb{P}(A_i | X \geqslant x)\mathbb{P}(X \geqslant x))\mathrm{d}x \\
&= \sum_{i=1}^n \int_0^\infty h_i\left(\mathbb{E}[f_i'(1-U_X) | X \geqslant x]\mathbb{P}(X \geqslant x)\right)\mathrm{d}x \\
&= \sum_{i=1}^n \int_0^\infty h_i\left(\mathbb{E}\left[f_i'(1-U_X)\mathbb{1}_{\left\{F_X^{-1}(U_X) \geqslant x\right\}}\right]\right)\mathrm{d}x \\
&= \sum_{i=1}^n \int_0^\infty h_i\left(\int_{\mathbb{P}(X \leqslant x)}^1 f_i'(1-t)\mathrm{d}t\right)\mathrm{d}x \qquad (3.12) \\
&= \sum_{i=1}^n \int_0^\infty h_i\left(f_i(S_X(x))\right)\mathrm{d}x \\
&= \int_0^\infty \square_{i=1}^n h_i(S_X(x))\mathrm{d}x = \rho_g(X).
\end{aligned}$$

The equality (3.12) holds due to the equivalence of $F_X^{-1}(U_X) \geqslant x$ and $U_X \geqslant \mathbb{P}(X \leqslant x)$ (see Guan et al. (2024, Lemma 1)). Hence, the result implies that $\boxminus_{i=1}^n \rho_{h_i}(X) \leqslant \rho_g(X)$. Combining the above, we obtain $\boxminus_{i=1}^n \rho_{h_i}(X) = \rho_g(X)$.

(ii) This part follows by symmetric arguments to part (i). Denote by $\ell_i(x) = -\tilde{h}_i(x)$ and $h^*(x) = \square_{i=1}^n \ell_i(x)$. By straightforward calculations, we have

$$\begin{aligned}
\overset{n}{\underset{i=1}{\square}} \rho_{h_i}(X) &= \inf\left\{\sum_{i=1}^n \rho_{h_i}(X_i), (X_1,\dots,X_n) \in \mathbb{A}_n(X)\right\} \\
&= \inf\left\{\sum_{i=1}^n \rho_{\ell_i}(-X_i), (-X_1,\dots,-X_n) \in \mathbb{A}_n(-X)\right\} \\
&= \rho_{h^*}(-X) = \rho_g(X),
\end{aligned}$$

where $g(x) = h^*(1-x) - h^*(1) = \square_{i=1}^n \ell_i(1-x) - \square_{i=1}^n \ell_i(1) = \Diamond_{i=1}^n \tilde{h}_i(1) - \Diamond_{i=1}^n \tilde{h}_i(1-x)$. The last but one equality follows from the case of $\mathcal{X} = L^+$. □

Proof of Theorem 3.3. The main idea of the proof is to approximate a general function h_i by its Bernstein polynomial B_k^i for $k \geqslant 1$, which is twice differentiable, and then to apply Lemma 3.3. We only prove case (i). Fix $i \in [n]$ for now. For the continuous function h_i on $[0,1]$, and for $\varepsilon > 0$, there exists an integer $N_i \geqslant 2$ such that for $k > N_i$,

$$\sup_{x\in[0,1]} |h_i(x) - B_k^i(x)| < \varepsilon, \text{ where } B_k^i(x) = \sum_{r=0}^k h_i\left(\frac{r}{k}\right)\binom{k}{r} x^r (1-x)^{k-r}, \ x \in [0,1]; \quad (3.13)$$

see Phillips (2003, Theorem 7.1.5). Clearly, for all $k \geqslant 1$, $B_k^i(0) = h_i(0) = 0$ and $B_k^i(1) = h_i(1) = 1$. Also, since h_i is increasing and convex, B_k^i is also increasing and convex (see Phillips (2003, Theorem 7.1.4)). Let $\tilde{B}_k^i(x) = (1-\varepsilon)B_k^i(x) + \varepsilon x^2$ for $x \in [0,1]$ and $k \geqslant 1$. It is clear that $\tilde{B}_k^i$ is a strictly convex and increasing function such that $|\tilde{B}_k^i - B_k^i| \leqslant \varepsilon$, $\tilde{B}_k^i(0) = 0$ and $\tilde{B}_k^i(1) = 1$.

Since X is bounded, we assume that $X \leqslant M$, for some constant $M \geqslant 0$. Fix $k >$

$\max\{N_1, \dots, N_n\}$. It holds that for all Y supported on $[0, M]$,

$$|\rho_{h_i}(Y) - \rho_{B_k^i}(Y)| \leqslant \int_0^M \left|h_i(\mathbb{P}(Y \geqslant x)) - B_k^i(\mathbb{P}(Y \geqslant x))\right| \mathrm{d}x < M\varepsilon,$$

for all $i \in [n]$, and hence,

$$|\rho_{h_i}(Y) - \rho_{\tilde{B}_k^i}(Y)| \leqslant 2M\varepsilon \quad \text{and} \quad \left|\boxminus_{i=1}^n \rho_{h_i}(Y) - \boxminus_{i=1}^n \rho_{\tilde{B}_k^i}(Y)\right| < 2nM\varepsilon.$$

Note that $\tilde{B}_k^i \in \mathcal{H} \cap C^2[0,1]$ for $i \in [n]$. It follows that $\boxminus_{i=1}^n \rho_{\tilde{B}_k^i}(X) = \rho_{g_k}(X)$, where $g_k = \square_{i=1}^n \tilde{B}_k^i$, by Lemma 3.3. Using the inequality (3.13) again, and writing $h = \square_{i=1}^n h_i$, we have

$$\|h - g_k\| = \left\| \square_{i=1}^n h_i - \square_{i=1}^n \tilde{B}_k^i \right\| < 2n\varepsilon,$$

where $\|\cdot\|$ is the supremum norm for continuous functions on $[0,1]$. Because $\mathcal{X} = L^+$ and $X \in \mathcal{X}^\perp$, it follows that

$$\begin{aligned}\left|\boxminus_{i=1}^n \rho_{h_i}(X) - \rho_h(X)\right| &= \left|\boxminus_{i=1}^n \rho_{h_i}(X) - \rho_{g_k}(X) + \rho_{g_k}(X) - \rho_h(X)\right| \\ &\leqslant \left|\boxminus_{i=1}^n \rho_{h_i}(X) - \rho_{g_k}(X)\right| + |\rho_{g_k}(X) - \rho_h(X)| \leqslant 4nM\varepsilon.\end{aligned}$$

Thus, (i) holds true. The proof of (ii) follows from noticing that $-X$ satisfies the assumptions of (i). □

3.5.3 Sharing a constant payoff

We take a closer look at the case $\mathcal{X} = L^+$ and when the aggregate risk X is a constant. By (3.9) and the positive homogeneity of distortion risk metrics, we set $X = 1$ without loss of generality. In this case, Theorem 3.3 implies that a class of Pareto-optimal allocations takes the form of $X_i = \mathbb{1}_{A_i}$ for $i \in [n]$, where $(A_1, \dots, A_n) \in \Pi_n$ satisfies $\square_{i=1}^n h_i(1) =$

$\sum_{i=1}^n h_i(\mathbb{P}(A_i))$. As we can see, the optimal partition depends entirely on the agents' distortion functions h_i, which capture their individual risk preferences. Thus, in a setting with no aggregate uncertainty, the optimal allocation assigns to each risk-seeking agent i a probability $\mathbb{P}(A_i)$ of bearing the total loss. This highlights a key economic feature of risk sharing among risk-seeking agents: optimal allocations concentrate the loss rather than diversify it, and each agent receives a "probability share" of the full loss according to the structure of the solution. Essentially, the problem boils down to a "probability sharing" problem among agents with different attitudes toward risk. It is therefore natural to conjecture that an efficient sharing rule tends to assign higher probabilities of bearing the loss to agents with stronger risk–seeking preferences. However, as shown in Proposition 3.3, such monotonicity can fail in the case of two agents. The following result focuses on agents whose preferences are represented by convex power distortions, and it illustrates how the agents' attitudes toward risk shape the structure of an optimal allocation within this particular class.

Proposition 3.3. *Let $n \in \mathbb{N}$, $\alpha_1 \leqslant \ldots \leqslant \alpha_n \in (1, \infty)$, and $h_i(x) = x^{\alpha_i}$ for $x \in [0,1]$. Suppose that $(x_1, \ldots, x_n) \in [0,1]^n$ satisfies $\sum_{i=1}^n x_i = 1$ and $\sum_{i=1}^n h_i(x_i) = \square_{i=1}^n h_i(1)$.*

(i) In case $n = 2$, we have $x_1 \leqslant x_2$ if and only if $\alpha_2 2^{-\alpha_2} \leqslant \alpha_1 2^{-\alpha_1}$.

(ii) In case $n \geqslant 3$, we have $x_1 \leqslant \ldots \leqslant x_n$.

Proof. Our goal is to find $x_1, \ldots, x_n$ such that $\sum_{i=1}^n x_i = 1$ and $\square_{i=1}^n h_i(1) = \sum_{i=1}^n h_i(x_i)$; that is, to solve

$$\underset{w_1,\ldots,w_n}{\arg\min} \left\{ \sum_{i=1}^n (w_i)^{\alpha_i} : w_1, \ldots, w_n \geqslant 0 \text{ and } \sum_{i=1}^n w_i = 1 \right\}. \tag{3.14}$$

First, we consider the case of $n = 2$. The optimal share $(x_1, x_2) \in [0,1]^2$ can be solved by the first-order condition of $\alpha_1 {x_1}^{\alpha_1 - 1} - \alpha_2 (1 - x_1)^{\alpha_2 - 1} = 0$. Let $m = \frac{x_1}{1 - x_1}$. The condition can be written as:

$$m^{\alpha_1 - 1}(1 + m)^{\alpha_2 - \alpha_1} = \frac{\alpha_2}{\alpha_1}.$$

Clearly, $F(y) := y^{\alpha_1-1}(1+y)^{\alpha_2-\alpha_1}$ is non-decreasing in y and $F(1) = 2^{\alpha_2-\alpha_1}$. Therefore, the condition of $\alpha_2/\alpha_1 \leqslant F(1)$ is equivalent with $m \leqslant 1$, implying that $x_1 \leqslant 1/2$ and $x_1 \leqslant x_2$.

Next, we consider the case of $n \geqslant 3$. To solve the problem, we define the Lagrangian as

$$\mathcal{L}(w_1, \ldots, w_n; \lambda) = \sum_{i=1}^{n} w_i^{\alpha_i} + \lambda \left(1 - \sum_{i=1}^{n} w_i\right), \ \lambda \in \mathbb{R}. \tag{3.15}$$

The first-order condition for (3.15) is given by

$$\alpha_1 {x_1}^{\alpha_1-1} = \cdots = \alpha_n {x_n}^{\alpha_n-1} = \lambda, \text{ and } \sum_{i=1}^{n} x_i = 1. \tag{3.16}$$

Thus, it follows that

$$f(\lambda) := \sum_{i=1}^{n} \left(\frac{\lambda}{\alpha_i}\right)^{\frac{1}{\alpha_i-1}} = 1.$$

Let

$$y_i = \left(1 + \frac{1}{\beta_i}\right)^{-\beta_i} \text{ and } \beta_i = \frac{1}{\alpha_i - 1}.$$

Following the fact that $g(x) := (1+1/x)^x$ is increasing on $(0, \infty)$ with range $(1, e)$, we obtain that y_i is decreasing in β_i and satisfies $y_i > 1/e$. Therefore,

$$\sum_{i=1}^{n} y_i = f(1) = \sum_{i=1}^{n} \left(\frac{1}{\alpha_i}\right)^{\frac{1}{\alpha_i-1}} > \frac{n}{e} \geqslant \frac{3}{e} > 1.$$

Since $f(\lambda)$ is strictly increasing with λ, the solution of $f(\lambda) = 1$ satisfies $\lambda < 1$. Consequently, it follows that $x_i = \lambda^{\beta_i} y_i$ is decreasing in β_i, and hence increasing in α_i. □

Proposition 3.3 shows that when $n \geqslant 3$, optimal allocations always assign higher probabilities of bearing the loss to agents with more risk-seeking distortion functions. The

ordering of probabilities then aligns naturally with the ordering of α_i. However, this monotonicity does not necessarily hold in the two-agent case. When $n = 2$, the feasibility constraint $x_1 + x_2 = 1$ restricts the set of admissible allocations to be a one-dimensional curve. The optimal allocation is therefore determined by comparing the agents' marginal distortion penalties at the symmetric point $x_1 = x_2 = 1/2$. As a result, the ordering of (x_1, x_2) is governed not directly by the risk-seeking parameters (α_1, α_2), but by the transformed quantities $(\alpha_1 2^{-\alpha_1}, \alpha_2 2^{-\alpha_2})$. Since the mapping $x \mapsto x2^{-x}$ is not monotone on $(1, \infty)$, it is possible for the efficient allocation to assign a larger probability of bearing the loss to the less risk-seeking agent. For example, when $\alpha_2 2^{-\alpha_2} > \alpha_1 2^{-\alpha_1}$, as in the case $h_1(x) = x^{1.2}$ and $h_2(x) = x^{1.4}$, the efficient solution assigns a larger probability of loss to the less risk-seeking agent, as shown in the Table 3.2. In contrast, when the ordering reverses, such as the pair of $h_1(x) = x^{1.2}$ and $h_2(x) = x^5$, the efficient allocation assigns a larger probability to the more risk-seeking agent. This counterintuitive outcome arises from a structural feature of the two-agent setting: with only one degree of freedom, increasing one agent's share automatically decreases the other's. Consequently, the exact curvature of the distortion functions near the symmetric split becomes more influential than the overall level of risk seeking, and the less risk-seeking agent may end up bearing more of the loss.

h_1	h_2	$\mathbb{P}(A_1)$	$\mathbb{P}(A_2)$	$\rho_{h_1} \square \rho_{h_2}(1)$
$x^{1.2}$	$x^{1.4}$	0.5129	0.4871	0.8141
$x^{1.2}$	x^5	0.3371	0.6629	0.3992

Table 3.2: Comparison of the inf-convolution and the optimal allocation with different h_1 and h_2.

3.6 Conclusion

The counter-monotonic improvement theorem (reported as Theorem 3.1 herein) not only provides insights into solving risk sharing problem among non-risk-averse agents, but also serves as the foundation for studying a counter-monotonic risk exchange mechanism, in which only counter-monotonic risk allocations are allowed. In this chapter, we investigate the counter-monotonic risk sharing problem for heterogeneous underlying risk measures $\rho_{h_1}, \dots, \rho_{h_n}$.

When all agents are risk averse, meaning that h_i is concave for each $i \in [n]$, the following relationship holds (Theorem 3.2 and Corollary 3.1):

$$\boxminus_{i=1}^{n} \rho_{h_i} \geqslant \boxplus_{i=1}^{n} \rho_{h_i} = \rho_h = \boxdot_{i=1}^{n} \rho_{h_i}, \text{ where } h = \bigwedge_{i=1}^{n} h_i.$$

In this case, comonotonic allocations are always preferred, implying that the counter-monotonic inf-convolution generally yields a larger value than the other two. As a result, finding a closed-form characterization of the counter-monotonic inf-convolution becomes challenging, and it is not our focus in this chapter. However, we provide a sufficient condition under which $\boxminus_{i=1}^{n} \rho_{h_i}$ is equal to the other two inf-convolutions.

When h_i is convex and continuous on $[0, 1]$ for each $i \in [n]$, indicating that agents are risk seeking, the inf-convolution for such distortion risk measures admits explicit formulas for $X \in \mathcal{X}^{\perp}$:

$$\boxdot_{i=1}^{n} \rho_{h_i}(X) = \boxminus_{i=1}^{n} \rho_{h_i}(X) = \rho_g(X),$$

where g is such that (i) $g = \square_{i=1}^{n} h_i$ if $\mathcal{X} = L^+$; and (ii) $\tilde{g} = \diamondsuit_{i=1}^{n} \tilde{h}_i$ if $\mathcal{X} = L^-$. In this setting, these results show that a representative agent (defined by the inf-convolution as its reference) of several agents with distortion risk measures is no longer an agent with a distortion risk measure, but rather with a distortion riskmetric.

3.7 Application to a portfolio manager's problem

The main body of this chapter focuses on the structural characterization of comonotonic risk sharing under heterogeneous distortion risk measures. The portfolio manager's problem below is included as a supplementary application rather than as part of the main theoretical development. While its economic interpretation requires additional modeling choices beyond the main risk sharing problem, it provides useful insight into how heterogeneity in agents' distortion functions can affect the optimal risky investment level. We therefore present it in Section 3.7 as an illustrative use of the explicit formulas derived above.

In this section, we analyze the portfolio optimization problem described in Section 6 of Ghossoub et al. (2026a), with the key difference that we consider agents with different risk preferences instead of identical ones. Consider a financial market scenario where a portfolio manager is responsible for managing the investments of a group of agents who collectively hold a fixed initial aggregate endowment of W. The portfolio manager, in this case, acts as the group representative or the casino strategist, deciding how much of the total pool to bet on the risky games and how much to hold back. The manager considers investing in a risky asset that yields a payoff at time 1, modeled by a non-negative random variable X. The first task for the manager is to decide the proportion λ of the total investment to allocate to the risky asset in a way that aligns with the agents' risk preferences and minimizes the overall risk exposure. This investment incurs a cost $c(\lambda)$, which is assumed to be increasing and convex. Once the investment decision is made, the subsequent task is to find the optimal way to redistribute the total wealth at time 1, given by $W + \lambda X - c(\lambda)$, among the agents. Therefore, the manager's goal is the following:

$$\begin{aligned} \text{to minimize} \quad & \sum_{i=1}^{n} \rho_{h_i}(-X_i) \\ \text{subject to} \quad & \lambda \in [0,1], \quad c(\lambda) \leqslant W; \\ & X_1 + \cdots + X_n = W + \lambda X - c(\lambda); \quad X_1, \ldots, X_n \geqslant 0. \end{aligned} \tag{3.17}$$

In this model, the constraint $X_1, \dots, X_n \geqslant 0$ ensures that the manager does not impose additional loss to the participants at time 1. To guarantee feasibility, we also assume that $c(\lambda) \leqslant W$ to ensure that the terminal wealth remains nonnegative.

Ghossoub et al. (2026a) showed that when the agents involved in the pool have the same risk-seeking or risk-averse preferences, the optimal proportion of risky investments for this group, denoted by λ^*, can be determined by the representative preferences of all agents. Moreover, the manager tends to invest more in risky assets when acting on behalf of a risk-seeking group compared with a risk-averse group. We extend this result to cases where agents exhibit varying levels of risk aversion or risk seeking. In such heterogeneous settings, the optimal allocation of risky investments can be explicitly determined, as outlined in Proposition 3.4. Specifically, for risk-averse groups, λ^* arises from the interplay between individual risk preferences and a minimization mechanism, reflecting the cautious nature of these agents. Conversely, for risk-seeking groups, the sup-convolution mechanism plays a critical role, capturing the group's collective risk preference.

Proposition 3.4. *Suppose that $\mathcal{X} = L^+$ and $h_i \in \mathcal{H}$ for $i \in [n]$. For $X \in \mathcal{X}^\perp$, the following hold.*

(i) If h_i is concave for each $i \in [n]$, the optimal value λ^* is

$$\lambda^* = \min\left\{c'^{-1}\left(\rho_\ell(X)\right), c^{-1}(W)\right\}, \quad \text{where } \tilde{\ell}(x) = \bigwedge_{i=1}^{n} h_i(x),\ x \in [0,1]. \tag{3.18}$$

(ii) If h_i is convex and continuous for each $i \in [n]$, the optimal value λ^* is

$$\lambda^* = \min\left\{c'^{-1}\left(\rho_g(X)\right), c^{-1}(W)\right\}, \quad \text{where } g(x) = \frac{\Diamond_{i=1}^{n} \tilde{h}_i(x)}{\Diamond_{i=1}^{n} \tilde{h}_i(1)},\ x \in [0,1]. \tag{3.19}$$

Proof. (i) Let $g = \tilde{h}$. Since h_i are concave, we have $\square_{i=1}^n \rho_{h_i} = \boxplus_{i=1}^n \rho_{h_i} = \rho_h$, by Theorem 3.2. Problem (3.17) is equivalent to solving the following:

$$\square_{i=1}^{n} \rho_{h_i}(-W - \lambda X + c(\lambda)) = \rho_h(-W - \lambda X + c(\lambda))$$

$$= -\rho_g(W + \lambda X - c(\lambda))$$
$$= -\lambda\rho_g(X) + c(\lambda) - W.$$

By differentiating with respect to λ, the optimal λ^* is determined by $c'(\lambda^*) = \rho_g(X)$. Given the constraint that $c(\lambda) \leqslant W$, it follows that

$$\lambda^* = \min\left\{c'^{-1}\left(\rho_g(X)\right), c^{-1}(W)\right\}.$$

(ii) This follows from Theorem 3.3 and similar arguments to the proof of (i). □

By Proposition 3.4, both ρ_ℓ and ρ_g are distortion risk measures because both $g(x)$ and $\ell(x)$ are increasing, with $\ell(1) = g(1) = 1$ and $\ell(0) = g(0) = 0$. Furthermore, it is straightforward to verify that $\ell(x)$ is convex and $g(x)$ is concave, implying $\ell(x) \leqslant g(x)$ for $x \in [0, 1]$. Consequently, $\rho_\ell \leqslant \rho_g$ holds. The results indicate that the risk-seeking group would prefer to take on more risk than the risk-averse group, which is consistent with the findings in Ghossoub et al. (2026a). A natural question to ask is whether one can make a comparison between the optimal λ^* across groups with varying levels of risk aversion or risk seeking. Intuitively, one might expect that the more risk-seeking a group is, the higher the proportion of investment in risky assets. However, this is not always the case, as we discuss below.

Let us consider a problem of comparing risk-averse or risk-seeking behavior at the group level, with varying risk preferences. Initially, each agent in the group has risk preferences described by the distortion function h_i, for $i \in [n]$. Over time, their risk preferences may vary due to external factors, individual experiences, or economic conditions, resulting in updated risk preferences described by g_i, for each $i \in [n]$. In these two settings, the optimal proportions of total investment allocated to risky assets are denoted by λ_h and λ_g, corresponding to $(h_i)_{i\in[n]}$ and $(g_i)_{i\in[n]}$, respectively. Our focus is primarily on special cases where, despite changes in the levels of risk aversion or risk seeking, agents remain within their original classification – either risk averse or risk seeking. The following proposition

examines the scenario in which all agents are risk averse. It demonstrates that if all agents become more risk averse than before (i.e., $h_i \leqslant g_i$ for each i), then the group collectively would adopt a more conservative attitude, allocating a smaller proportion of their investment to risky assets.

Proposition 3.5. *Suppose that $h_i, g_i \in \mathcal{H}$ are concave, for $i \in [n]$. If $h_i \leqslant g_i$ for each $i \in [n]$, then $\lambda_h \geqslant \lambda_g$.*

Proof. Let $h = \bigwedge_{i=1}^{n} h_i$ and $g = \bigwedge_{i=1}^{n} g_i$. By using expression (3.18) in Proposition 3.4, the optimal λ_h and λ_g are given by

$$\lambda_h = \min\left\{c'^{-1}\left(\rho_{\tilde{h}}(X)\right), c^{-1}(W)\right\} \text{ and } \lambda_g = \min\left\{c'^{-1}\left(\rho_{\tilde{g}}(X)\right), c^{-1}(W)\right\}.$$

Since $h_i \leqslant g_i$ for $i \in [n]$, we have $h \leqslant g$, and thus $\tilde{h} \geqslant \tilde{g}$. Consequently, from the convexity of c, it follows that $\lambda_h \geqslant \lambda_g$. □

We present a numerical example to illustrate how varying levels of risk aversion among risk-averse agents affect the optimal proportion of investments allocated to risky assets within a group. In the following examples, we consider a quadratic cost function of the form $c(\lambda) = \lambda^2/2$.

Example 3.6. Consider a group of two risk-averse agents with $g_1 = 1 - (1-x)^{2\alpha}$ and $g_2 = 1-(1-x)^{3\alpha}$, respectively. To ensure the concavity of g_1 and g_2, we take $\alpha \in [1/2, \infty)$. As α increases, both agents become increasingly risk-averse. Thus, it is expected that a higher value of α would lead to a lower optimal λ^*. We assume that $X \sim \mathrm{U}(0,1)$. The values of the optimal proportion λ^* are displayed in Figure 3.1, illustrating that the group's overall risk aversion increases as α grows.

Next, we analyze the case of risk-seeking agents, where h_i and g_i are assumed to be convex for $i \in [n]$. Intuitively, it may be reasonable to conjecture that, similarly to the risk-averse case, the group would allocate a greater proportion of their investments to risky

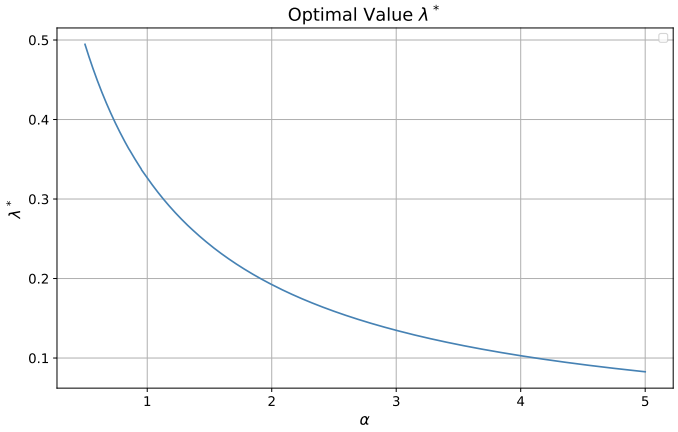


Figure 3.1: Optimal value of λ^* with $g_1(x) = 1 - (1-x)^{2\alpha}$ and $g_2(x) = 1 - (1-x)^{3\alpha}$

assets if each agent becomes more risk seeking. Specifically, we aim to determine whether the inequality $\lambda_h \leqslant \lambda_g$ holds when every agent in the group becomes more risk seeking; that is, $h_i \geqslant g_i$ for each $i \in [n]$. By Proposition 3.4, the problem reduces to verifying whether the following relationship holds for $X \in \mathcal{X}^{\perp}$:

$$\rho_{f_h}(X) \leqslant \rho_{f_g}(X), \text{ where } f_h(x) = \frac{\Diamond_{i=1}^n \tilde{h}_i(x)}{\Diamond_{i=1}^n \tilde{h}_i(1)} \text{ and } f_g(x) = \frac{\Diamond_{i=1}^n \tilde{g}_i(x)}{\Diamond_{i=1}^n \tilde{g}_i(1)}. \tag{3.20}$$

Problem (3.20) is equivalent to determining whether $f_h \leqslant f_g$ holds on $[0,1]$ since f_h and f_g are distortion functions; see e.g., Wang et al. (2020b, Lemma 1). Note that $h_i \geqslant g_i$ implies $\tilde{h}_i \leqslant \tilde{g}_i$. Also, both $\tilde{h}_i$ and $\tilde{h}_i$ are concave as h_i and g_i are convex. Although $\tilde{h}_i \leqslant \tilde{g}_i$ leads to $\Diamond_{i=1}^n \tilde{h}_i \leqslant \Diamond_{i=1}^n \tilde{g}_i$, the inequality $f_h \leqslant f_g$ does not necessarily hold because f_h and f_g are defined as ratios. To illustrate this point, we present two numerical examples. The numerical results in Example 3.7 align with our intuition, whereas Example 3.8 provides a counterexample, showing that (3.20) is not always true.

Example 3.7. We consider a group of two risk-seeking agents and assume that their risk preferences are modeled by $\ell_1(x,\alpha) = 1 - (1-x)^{0.2\alpha}$ and $\ell_2(x,\alpha) = 1 - (1-x)^{0.3\alpha}$, respectively. We take $\alpha \in [0,3]$ to ensure the convexity of ℓ_1 and ℓ_2. As α decreases, both

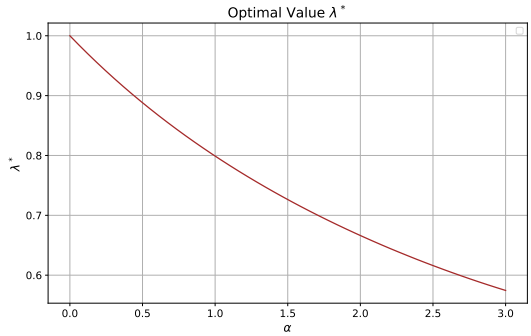
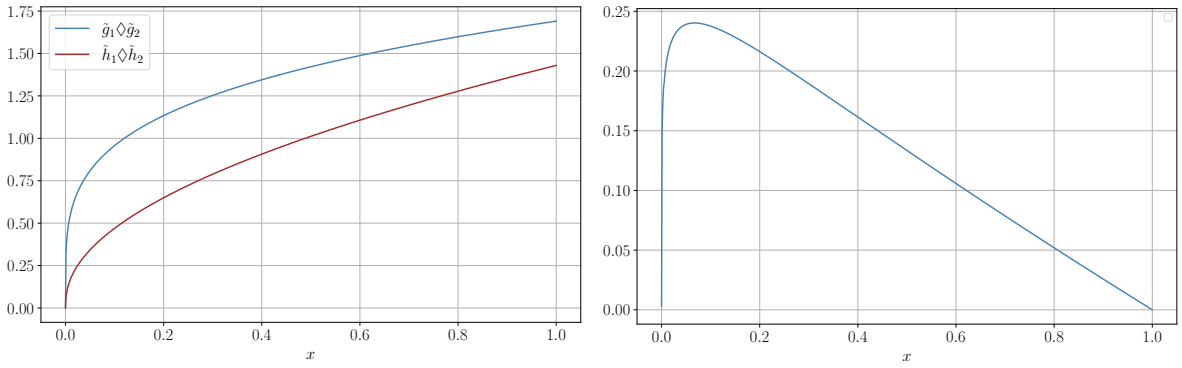


Figure 3.2: Optimal value of λ^* with $g_1(x) = 1 - (1-x)^{0.2\alpha}$ and $g_2(x) = 1 - (1-x)^{0.3\alpha}$



(a) Comparison of $\tilde{h}_1 \Diamond \tilde{h}_2$ and $\tilde{g}_1 \Diamond \tilde{g}_2$

(b) The difference $f_g - f_h$

Figure 3.3: Comparison of sup-convolutions and the normalized ones in Example 3.7

agents exhibit increasingly risk-seeking behavior, thus a larger optimal λ^* is expected. We also assume $X \sim \mathrm{U}(0,1)$. A numerical simulation of the optimal λ^* with varying values of α is presented in Figure 3.2.

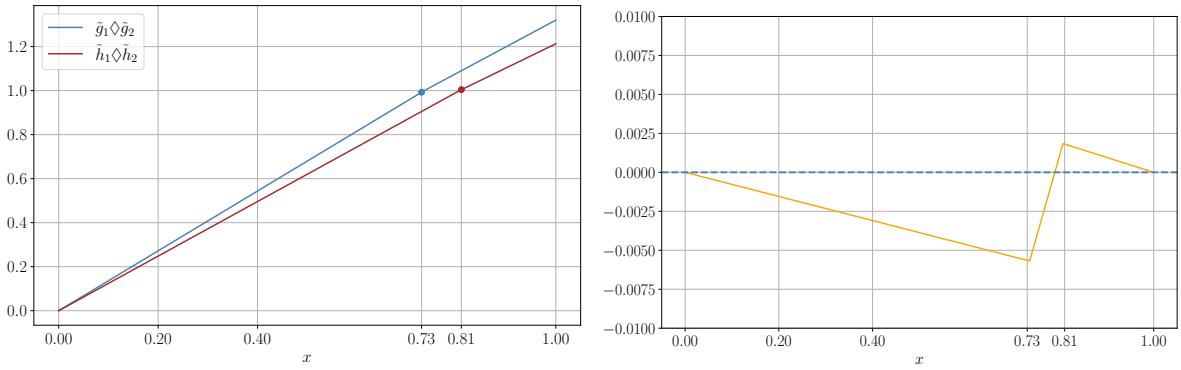
In (3.17), taking $h_1(x) = \ell_1(x,2)$, $h_2(x) = \ell_2(x,2)$, $g_1(x) = \ell_1(x,1)$, and $g_2(x) = \ell_2(x,1)$, we have $h_i \geqslant g_i$ for $i = 1,2$, and thus $\tilde{h}_1 \Diamond \tilde{h}_2 \leqslant \tilde{g}_1 \Diamond \tilde{g}_2$, as shown in Figure 3.3a. In this case, it holds that $f_h \leqslant f_g$, as illustrated in Figure 3.3b.

The following counter-example shows that $f_h(x) > f_g(x)$ may occur for some $x \in (0,1)$, implying that (3.20) does not always hold true for a given X.

Example 3.8. Take $h_1(x) = (1.24x - 0.24) \vee 0$, $h_2(x) = (1.10x - 0.1) \vee 0$ and $g_1(x) = (1.36x - 0.36) \vee 0$, $g_2(x) = (1.21x - 0.21) \vee 0$. Thus, it is straightforward to verify $h_i \geqslant g_i$ for $i = 1, 2$. Also, we can obtain that $\tilde{h}_1(x) = (1.24x) \wedge 1$, $\tilde{h}_2(x) = (1.1x) \wedge 1$, and $\tilde{g}_1(x) = (1.36x) \wedge 1$ and $\tilde{g}_2(x) = (1.21x) \wedge 1$. Then we can show $\tilde{h}_1 \Diamond \tilde{h}_2$ and $\tilde{g}_1 \Diamond \tilde{g}_2$ have the following explicit forms

$$\tilde{h}_1 \Diamond \tilde{h}_2(x) = \begin{cases} 1.24x & x \in [0, 0.81] \\ 1.1x + a & x \in [0.81, 1]; \end{cases} \quad \tilde{g}_1 \Diamond \tilde{g}_2(x) = \begin{cases} 1.36x & x \in [0, 0.73] \\ 1.21x + b & x \in [0.73, 1]. \end{cases}$$

where $a = 0.14/1.24$ and $b = 0.15/1.36$. Figure 3.4a compares two sup-convolutions, showing that $\tilde{h}_1 \Diamond \tilde{h}_2 \leqslant \tilde{g}_1 \Diamond \tilde{g}_2$ holds over $[0, 1]$. However, with normalization at 1, $f_h \leqslant f_g$ no longer holds over $[0, 1]$. As shown in Figure 3.4b, the difference between f_h and f_g crosses the zero line, with f_h exceeding f_g beyond a certain point near 0.8.



(a) Comparison of $\tilde{h}_1 \Diamond \tilde{h}_2$ and $\tilde{g}_1 \Diamond \tilde{g}_2$ (b) The difference $f_g - f_h$

Figure 3.4: Comparison of sup-convolutions and the normalized ones in Example 3.8.

One possible explanation of this phenomenon is that, for a group of risk-seeking agents, the overall risk they encounter stems from two sources: external randomness, introduced by a risky asset, and internal randomness, generated through counter-monotonic risk-sharing within the group, thus gambling among themselves. As agents become increasingly risk-seeking, their preference for internal randomness, arising from gambling within the group,

may dominate their interest in external randomness. Recognizing this, the manager might allocate less to the external risky asset because the group is already sufficiently satisfied from their internal gambling. This strategy not only aligns with the agents' preferences but also achieves the goal of minimizing risk exposure as a group. To explain this plainly, one may think that for a group of investors who see the stock market as a casino, they may directly go to the casino without putting much money in the stock market.

Chapter 4

Optimal Allocations with Distortion Risk Measures and Mixed Risk Attitudes

4.1 Introduction

In classical risk sharing problems with risk-averse agents, Pareto-optimal allocations are comonotonic under mild conditions, as a result of the classical comonotonic improvement theorem of Landsberger and Meilijson (1994). On the other hand, when only risk-seeking agents are involved, inter-agent gambling arises, thereby leading to counter-monotonic allocations, as shown in Lauzier et al. (2023, 2026a). This contrast naturally leads to the question of how heterogeneous preferences shape optimal allocations in markets that comprise both risk-averse and risk-seeking participants, a more realistic assumption in practice. A simple conjecture is that the optimal allocation structure is determined by the relative strength of risk aversion versus risk seeking behavior.

The exchange of risk between agents with mixed risk attitudes remains theoretically underdeveloped. In the literature, risk sharing has been studied primarily in markets with

homogeneous preference classes, e.g., all risk-averse, all risk-seeking, or all quantile-type agents. For risk-averse agents, optimal allocations have been characterized in various frameworks, from the seminal work by Borch (1962) within the context of risk-averse Expected-Utility (EU) agents, to the case of convex risk measures in Barrieu and El Karoui (2005) and Jouini et al. (2008), all showing comonotonicity of optimal allocations. The results can also be extended to non-monotone risk measures (Acciaio (2007) and Filipović and Svindland (2008)) and quasi-convex risk measures (Mastrogiacomo and Rosazza Gianin (2015)). For more developments on efficient risk sharing with monetary and convex risk measures or concave utility functionals, see Heath and Ku (2004), Tsanakas (2009), and Ghossoub and Zhu (2026), for instance. Although risk aversion serves as a foundational and tractable assumption in theory, empirical observations frequently reveal risk-seeking attitudes in practice. Building upon the counter-monotonic improvement theorem of Lauzier et al. (2026a), the recent work of Ghossoub et al. (2026b,a) provided a systematic study of risk sharing among risk-seeking agents with distortion risk measures. They showed that counter-monotonic allocations are optimal in this setting, and they provided some insight into computing the inf-convolution (infimal convolution) of concave risk measures. Beyond convexity or concavity, risk-sharing problems have also been examined in quantile-based and more general distortion risk-metrics frameworks; see Embrechts et al. (2018), Weber (2018) and Lauzier et al. (2026b).

The relationship between heterogeneous risk attitudes and the existence of Pareto optima or competitive equilibria remains an open and challenging question. One should remark that the presence of non-convex preferences can cause technical problems since the existence of optima can fail in this setting. The market that combines risk-averse agents and risk-seeking agents is of particular interest; see Araujo et al. (2017, 2018), Herings and Yang (2022), and Beißner and Werner (2023). In a finite-state exchange economy, Araujo et al. (2017) proved the existence of individually rational Pareto optima even in the presence of non-convex preferences, and they suggested that risk-averse agents exhibit comonotonic sharing, while risk-seeking agents bet on the allocation of risk. Within the framework of distortion risk measures, we obtain analogous results. Moreover, rather than

providing only qualitative statements, we deliver quantitative characterizations via the inf-convolution. Beißner et al. (2024) studied a pure-exchange economy without aggregate uncertainty, considering two agents who maximize their rank-dependent utility (RDU), with concave utility functions and arbitrary (possibly nonconvex) probability distortion functions. They derived a closed-form characterization of Pareto-optimal allocations in this general setting. This chapter also touches on a related risk-sharing problem with a constant aggregate. Although distortion risk measures can be viewed as a subclass of RDU with linear utility, the results of Beißner et al. (2024) cannot be directly extended to our framework, as they require strict concavity of the utility functions.

To overcome the challenges arising from mixed risk attitudes in the market, we study a finite-agent economy with heterogeneous risk attitudes, where the agents' preferences are represented by distortion risk measures. For such preferences, risk attitudes are characterized by their probability distortion functions. We analyze Pareto-optimal allocations through the tool of inf-convolution. Our first step is to reduce the multi-agent problem into a 2-agent problem, where one agent acts as the representative agent of the risk-averse participants, and the other acts as the representative agent of the risk-seeking participants (Theorem 4.1). Zooming in on the reduced 2-agent problem, we provide necessary and sufficient conditions for the existence of optimal allocations. Since unbounded optimal allocations may not exist for risk-seeking agents due to excessive gambling, we consider separately the case where the allocations are possibly signed and the case of nonnegative allocations; the latter corresponds to the economically relevant assumption that no agents can profit from an aggregate pure loss. We offer several explicit characterizations of the inf-convolution and optimal allocations under some assumptions on the distortion functions. As another main contribution, we show that, indeed, the relative strength of risk aversion and risk seeking of the representative agents determines the structure of the optimal allocations for the original market, formalizing the aforementioned simple conjecture.

The rest of this chapter is structured as follows. Section 4.2 formulates the risk-sharing problem that we consider in this chapter. Section 4.3 presents an n-agent setting with mixed attitudes (risk-averse and risk-seeking agents), and shows how the problem reduces

to a two-agent formulation, which motivates the analysis in Sections 4.4 and 4.5. Regarding constraints on the feasible allocation set, Section 4.4 considers general allocation sets and establishes our main existence result for optimal allocations, while Section 4.5 focuses on positive allocation sets. Several results extend beyond the specific risk-averse/risk-seeking setting. Section 4.6 integrates these results to resolve the n-agent problem in certain settings. Finally, Section 4.7 concludes.

4.2 Problem formulation

Let $\mathcal{X}$ be a convex cone of random variables on an atomless probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Consider a setting with $n \in \mathbb{N}$ agents who share a random variable $X \in \mathcal{X}$, representing the aggregate risk, without central authority involvement. Write $[n] = \{1, \dots, n\}$. Positive and negative values of random variables represent losses and surpluses, respectively. We denote by $F_X : \mathbb{R} \mapsto [0,1]$ the cumulative distribution function of X, and by $S_X : \mathbb{R} \mapsto [0,1]$ the survival function of X. Let U_X be a uniform random variable on $[0,1]$, such that $F_X^{-1}(U_X) = X$ almost surely. The existence of U_X for a general X is guaranteed; see Lemma A.32 of Föllmer and Schied (2016). We consider a one-period economy, where an aggregate risk X is redistributed among the agents at the end of the period. For a given $X \in \mathcal{X}$ and $n \in \mathbb{N}$, the set of allocations of X is

$$\mathbb{A}_n(X) = \left\{ (X_1, \dots, X_n) \in \mathcal{X}^n : \sum_{i=1}^{n} X_i = X \right\}.$$

That is, the set of allocations of X includes all possible divisions of X among the n agents. Note that the choice of $\mathcal{X}$ may impose constraints on the admissible allocations. In this chapter, the space $\mathcal{X}$ can be chosen as the space L^∞ of bounded random variables or the space L^+ of nonnegative random variables. The choice will be specified in each context.

We next impose some dependence structures on admissible allocations. A random vector (X, Y) is said to be comonotonic (resp. counter-monotonic) if $(X(\omega) - X(\omega'))(Y(\omega) -$

$Y(\omega')) \geqslant 0$ (resp. $\leqslant 0$) for all $\omega, \omega' \in \Omega$. An n-tuple $(X_1, \dots, X_n)$ is called comonotonic (resp. counter-monotonic) if each pair of its components is comonotonic (resp. counter-monotonic). For results on and applications of these concepts in actuarial science, see Dhaene et al. (2002) on comonotonicity and Dhaene and Denuit (1999) and Lauzier et al. (2023) on counter-monotonicity.

The preference of each agent is represented by a *distortion risk measure* $\rho_h : \mathcal{X} \mapsto \mathbb{R}$, defined as

$$\rho_h(X) = \int X \mathrm{d}h \circ \mathbb{P} = \int_0^\infty h(\mathbb{P}(X > x))\,\mathrm{d}x + \int_{-\infty}^0 (h_i(\mathbb{P}(X > x)) - h(1))\,\mathrm{d}x, \tag{4.1}$$

where h is in the set $\mathcal{H}$ of all normalized distortion functions, i.e.,

$$\mathcal{H} = \{h : [0,1] \to [0,1] \mid h \text{ is increasing, } h(0) = 0 \text{ and } h(1) = 1\}.$$

If $h \in \mathcal{H}$, then ρ_h is also called a dual utility of Yaari (1987). In the dual theory, the distortion function provides a full characterization of risk aversion: a Yarri agent is (strongly) risk-averse if and only if h is concave (Yaari (1987)). Similarly, risk seeking corresponds to convexity of h. We omit "strong" in risk aversion and risk seeking, which is the only sense that we consider in this chapter (weak risk aversion is different from strong risk aversion for distortion risk measures).

If one drops the normalization requirement of $h(1) = 1$ and monotonicity of h, we obtain the more general class of *distortion riskmetrics* (see Wang et al. (2020b)) ρ_h with $h \in \mathcal{H}^{\mathrm{BV}}$, where

$$\mathcal{H}^{\mathrm{BV}} = \{h : [0,1] \to \mathbb{R} \mid h \text{ is bounded variation and } h(0) = 0\}.$$

This broader class is introduced as it will be useful in the subsequent analysis.

Our goal is to study Pareto-optimal allocations among the n agents in this market. An allocation $(X_1, \dots, X_n) \in \mathbb{A}_n(X)$ is said to be Pareto optimal in $\mathbb{A}_n(X)$ if for any

$(Y_1, \dots, Y_n) \in \mathbb{A}_n(X)$ satisfying $\rho_{h_i}(Y_i) \leqslant \rho_{h_i}(X_i)$ for all $i \in [n]$, we have $\rho_i(Y_i) = \rho_i(X_i)$, for all $i \in [n]$. That is, no agent can strictly improve their position without worsening that of another.

An alternative method of finding Pareto optima is via the tool of infimal convolution. The inf-convolution $\square_{i=1}^n \rho_{h_i}$ of n risk measures $\rho_{h_1}, \dots, \rho_{h_n}$ is defined as

$$\square_{i=1}^n \rho_{h_i}(X) = \inf \left\{ \sum_{i=1}^n \rho_i(X_i) : (X_1, \dots, X_n) \in \mathbb{A}_n(X) \right\}, \quad X \in \mathcal{X}. \tag{4.2}$$

That is, the inf-convolution of n risk measures is the infimum over aggregate risk values for all possible allocations. An allocation $(X_1, \dots, X_n)$ is said to be optimal in $\mathbb{A}_n(X)$ if it attains the infimum, i.e., $\square_{i=1}^n \rho_{h_i}(X) = \sum_{i=1}^n \rho_{h_i}(X_i)$. It is immediate that every optimal allocation is Pareto optimal. Moreover, when $\mathcal{X} = L^\infty$, the converse also holds and Pareto optimality coincides with optimality through inf-convolution, because of translation invariance of the risk measures (see Proposition 1 of Embrechts et al. (2018)). By contrast, the converse may not hold if $\mathcal{X}$ is chosen as L^+. Both cases will be considered in the subsequent analysis; see Section 4.4 and Section 4.5.

A particularly useful property of distortion risk measures is the following duality. For any $X \in \mathcal{X}$, we have $\rho_h(-X) = -\rho_{\widetilde{h}}(X)$, where $\widetilde{h}(t) = 1 - h(1-t)$, $t \in [0,1]$, is the dual function of h. Consequently, an optimal allocation $(X_1^*, \dots, X_n^*)$ that attains the infimum in (4.2) also solves the utility maximization problem:

$$\sup \left\{ \sum_{i=1}^n \rho_{\widetilde{h}_i}(-X_i) : (-X_1, \dots, -X_n) \in \mathbb{A}_n(-X) \right\}, \quad X \in \mathcal{X},$$

and here $\rho_{\widetilde{h}_i}$ can be interpreted as agent i's utility functional in the theory of Yaari (1987). That is, a risk sharing minimizer is simultaneously a utility maximizer under the dual distortions. In this chapter, we adopt the risk sharing perspective and focus on minimizing the aggregate risk across all agents.

In a manner similar to the inf-convolution of risk measures, we define the inf-convolution

of real functions, adopting the same notation. Given n distortion functions $h_i \in \mathcal{H}$ for $i \in [n]$, their inf-convolution $\square_{i=1}^n h_i(x) : [0,1] \mapsto \mathbb{R}$ is defined as

$$\square_{i=1}^{n} h_i(x) = \inf \left\{ \sum_{i=1}^{n} h_i(x_i) : x_i \in [0,1] \text{ for } i \in [n];\ \sum_{i=1}^{n} x_i = x \right\}.$$

Note that the $\square_{i=1}^n h_i$ is not necessarily a distortion function since $\square_{i=1}^n h_i(1) = 1$ does not always hold. Throughout the chapter, we write $\bigvee_{i=1}^n x_i = \max\{x_1, \dots, x_n\}$ and $\bigwedge_{i=1}^n x_i = \min\{x_1, \dots, x_n\}$ for any $x_1, \dots, x_n \in \mathbb{R}$, and these operation are naturally extended to functions.

4.3 Problem reduction

Our goal is to understand how the coexistence of risk-averse and risk-seeking behaviors shapes Pareto-optimal allocations. Specifically, suppose that n agents share a total risk X. The first m are risk averse, with concave distortion functions $h_i, i \in [m]$, and the remaining $n - m$ are risk seeking with convex distortion functions $h_i, i \in [n] \setminus [m]$. The following theorem shows that under some mild conditions, this n-agent risk-sharing problem can always be reduced to an equivalent two-agent problem. Specifically, let $g_1 = \bigwedge_{i=1}^m h_i$ and $g_2 = \square_{j=m+1}^n h_j$. These two are indeed the representative risk preferences for the risk-averse and risk-seeking groups, as shown in Theorem 4.1. The key insight of this reduction is that the collective behavior of each class can be represented by a single "aggregate" distortion, so the optimal allocation can be studied through a two-distortion formulation. Before moving to the result, we first introduce the following set:

$$\mathcal{X}^{\perp} = \{X \in \mathcal{X} : \text{there exists a uniform random variable } U \text{ independent of } X\}.$$

For more discussions on $\mathcal{X}^{\perp}$, in particular its relation to $\mathcal{X}$, we refer to Section 5 of Liu et al. (2020).

Theorem 4.1. *Let $\mathcal{X} = L^+$. Assume that $h_i \in \mathcal{H}$ are continuous for all $i \in [n]$, with h_i concave for $i \in [m]$ and h_j convex for $j \in [n] \setminus [m]$. For $X \in \mathcal{X}^\perp$, let $(X_1^*, \dots, X_n^*) \in \mathbb{A}_n(X)$ be an optimal allocation. Then the following holds:*

(i) If h_i is strictly concave for $i \in [m]$ and h_j is strictly convex for $j \in [n] \setminus [m]$, then the vector $(X_1^*, \dots, X_m^*)$ is comonotonic and the vector $(X_{m+1}^*, \dots, X_n^*)$ is counter-monotonic.

(ii) It holds that

$$\boxed{\sum_{i=1}^{n} \rho_{h_i}(X) = \rho_{g_1} \boxdot \rho_{g_2}(X),} \tag{4.3}$$

where $g_1 = \bigwedge_{i=1}^{m} h_i$ and $g_2 = \boxdot_{j=m+1}^{n} h_j$.

Proof. (i) The comonotonicity of $(X_1^*, \dots, X_m^*)$ follows directly from the comonotonic improvement theorem of Landsberger and Meilijson (1994), while the counter-monotonicity of $(X_{m+1}^*, \dots, X_n^*)$ follows from the counter-monotonic improvement theorem of Lauzier et al. (2026a).

(ii) For $X \in \mathcal{X}^\perp$,

$$\begin{aligned}
\sum_{i=1}^{n} \rho_{h_i}(X) &= \inf\left\{ \sum_{i=1}^{m} \rho_{h_i}(X_i) + \sum_{i=m+1}^{n} \rho_{h_i}(X_i) : (X_1, \dots, X_n) \in \mathbb{A}_n(X) \right\} \\
&= \inf\left\{ \sum_{i=1}^{m} \rho_{h_i}(Y) + \sum_{i=m+1}^{n} \rho_{h_i}(Z) : (Y, Z) \in \mathbb{A}_2(X) \right\} \qquad (4.4) \\
&= \inf\left\{ \sum_{i=1}^{m} \rho_{h_i}(Y') + \sum_{i=m+1}^{n} \rho_{h_i}(Z') : (Y', Z') \in \mathbb{A}_2(X) \text{ and } Z' \in \mathcal{X}^\perp \right\} \qquad (4.5) \\
&= \inf\{\rho_{g_1}(Y) + \rho_{g_2}(Z) : (Y, Z) \in \mathbb{A}_2(X)\} \qquad (4.6) \\
&= \rho_{g_1} \boxdot \rho_{g_2}(X).
\end{aligned}$$

Applying Theorem 2 of Ghossoub et al. (2026b) to $\boxdot_{i=m+1}^{n} \rho_{h_i}(Z)$ in (4.4) requires $Z \in \mathcal{X}^\perp$. We now show that it suffices to assume $X \in \mathcal{X}^\perp$. Indeed, for any $(Y, Z) \in \mathbb{A}_2(X)$, there

exists $(Y', Z') \in \mathbb{A}_2(X)$ and a standard uniform random variable U such that $(X, Y, Z) \overset{d}{=} (X, Y', Z')$ and U is independent of (X, Y', Z'); see Lemma 2 of Lauzier et al. (2026a). The allocation $(Y', Z') \in \mathbb{A}_2(X)$ can be constructed as follows. Since $X \in \mathcal{X}^{\perp}$, there exists a standard uniform random variable U independent of X, then we can generate i.i.d. standard uniform random variables U_1, U_2 independent of X. Let

$$Y' = F^{-1}_{Y|X}\left(U_1 \mid X_1\right) \text{ and } Z' = F^{-1}_{Z|X,Y}\left(U_2 \mid X, Y'\right),$$

where $F^{-1}_{Y|X}(\cdot \mid x)$ is the conditional quantile function of Y given $X = x$ and $F^{-1}_{Z|X,Y}(\cdot \mid x, y)$ is the quantile function of Z given $(X, Y) = (x, y) \in \mathbb{R}^2_+$. By construction, we have $(X, Y, Z) \overset{\mathrm{d}}{=} (X, Y', Z')$ and $Y' + Z' = Y + Z = X$. By Proposition 1 of Liu et al. (2020) and uniformly continuity of ρ_h, the inf-convolutions $\square_{i=1}^{m} \rho_{h_i}$ and $\square_{i=m+1}^{n} \rho_{h_i}$ are law-invariant; thus (4.5) follows. Also, Theorem 3 of Ghossoub et al. (2026b) identifies the inner inf-convolution in (4.6):

$$\square_{i=1}^{m} \rho_{h_i}(X) = \rho_{g_1}(X) \text{ and } \square_{i=m+1}^{n} \rho_{h_i}(X) = \rho_{g_2}(X) \text{ for } X \in \mathcal{X}^{\perp},$$

with concave distortion functions h_i, $i \in [m]$ and convex distortion functions h_i, $i \in [n] \backslash [m]$. Thus, the result is obtained. □

The first part of Theorem 4.1 also holds for the expected utility model; see Lauzier et al. (2026a). This can be generalized to other decision models accounting for risk-averse and risk-seeking behaviors. In Theorem 4.1, g_1 is concave and g_2 is convex. Equation (4.3) shows that, under appropriate assumptions, the n-agent problem reduces to risk sharing between a representative risk-averse agent with distortion function g_1 and a representative risk-seeking agent with distortion function g_2. The intuition behind this result is that agents with similar risk attitudes can be treated collectively: among the risk-averse agents, comonotonic allocations are optimal, while among the risk-seeking agents, countermonotonic allocations are optimal. Hence, the only remaining complexity lies in how the total risk is divided between these two representative agents.

It is important to note that g_2, being the inf-convolution of several convex distortion functions, is not necessarily a distortion function. In fact, one typically has $g_2(1) < 1$ unless all underlying $h_i, i \in [n]$ are the identity. Let $\hat{g}_2 = g_2/g_2(1)$ the normalization of g_2. Clearly, such normalization would not change the preferences of the agent and a Pareto-optimal allocation for agents using ρ_{g_1} and ρ_{g_2} is also Pareto optimal for agents using ρ_{g_1} and $\rho_{\hat{g}_2}$. This observation motivates the next step of our analysis, where we examine the detailed two-agent model in detail, showing how the form of distortion functions drives the optimal risk sharing.

4.4 Allocations in the whole space

In this section, we consider an economy with two agents who may have different risk preferences. As shown by Example 4.1 below, a Pareto-optimal allocation may fail to exist in the presence of risk-seeking behavior; this is a known phenomenon of risk-seeking agents. One possible resolution is to impose constraints on admissible allocations. Our first step, however, is to identify conditions on the distortion functions under which a Pareto-optimal allocation exists even without any external constraints. Throughout this section, we will work on the space $\mathcal{X} = L^\infty$. In Section 4.5, we turn to the setting with boundedness constraints on each individual allocation, where the existence of the inf-convolution is guaranteed.

Example 4.1. Consider two agents sharing a sure amount $X = 1$. Agent 1 is extremely risk-seeking with $h_1(x) = \mathbb{1}_{\{x=1\}}$, and Agent 2 is risk-averse with distortion $h_2(x) = \sqrt{x}$. Fix any event $A \in \mathcal{F}$ with $0 < \mathbb{P}(A) < 1$ and an arbitrary scalar $a > 0$, consider the allocation $(X_1, X_2) = (a\,\mathbb{1}_A,\ 1 - a\,\mathbb{1}_A)$. A direct calculation yields

$$\rho_{h_1}(a\mathbb{1}_A) + \rho_{h_2}(1 - a\mathbb{1}_A) = 1 - a\,\widetilde{h}_2(\mathbb{P}(A)). \tag{4.7}$$

Since $\widetilde{h}_2\big(\mathbb{P}(A)\big) > 0$, the right-hand side of (4.7) tends to $-\infty$ as $a \to \infty$. Thus, by taking unbounded bets on A, the total measured risk can be driven arbitrarily negative.

Intuitively, any gamble of the form $a\,\mathbb{1}_A$ with $0 < \mathbb{P}(A) < 1$ is perceived by Agent 1 as perfectly safe, no matter how large a is. The balancing position $1 - a\,\mathbb{1}_A$ falls to Agent 2, whose concave distortion function evaluates this exposure ever more negatively. Together, this makes the pair drive the total risk down without bound. Hence, a Pareto-optimal allocation fails to exist because one side can always "game" the allocation by scaling up risky bets.

4.4.1 Two general agents

We now present our first main result, which establishes necessary and sufficient conditions for the existence of a well-defined optimal risk sharing value between two agents with heterogeneous distortion preferences.

Theorem 4.2. *Suppose that $\mathcal{X} = L^\infty$. For any h_1, $h_2 \in \mathcal{H}$, the following are equivalent.*

(i) $h_1 \,\square\, h_2(1) = 1$*; equivalently,* $h_1 \geqslant \tilde{h}_2$.

(ii) $\rho_{h_1} \,\square\, \rho_{h_2}(c) = c$ *for any* $c \in \mathbb{R}$.

(iii) $\rho_{h_1} \,\square\, \rho_{h_2}(X) > -\infty$ *holds for any* $X \in \mathcal{X}$.

(iv) $\rho_{h_1} \,\square\, \rho_{h_2}(X) > -\infty$ *holds for some* $X \in \mathcal{X}$.

Proof. (i) $\to$ (ii): Note that the condition $h_1 \,\square\, h_2(1) = 1$ is equivalent to $h_1 \geqslant \widetilde{h}_2$ on $[0, 1]$. This implies that $\rho_{h_1}(Y) \geqslant \rho_{\widetilde{h}_2}(Y)$ for any $Y \in L^\infty$. Using translation invariance of ρ_{h_2}, we obtain

$$\rho_{h_1} \,\square\, \rho_{h_2}(c) = \inf_{Y \in \mathcal{X}} \{\rho_{h_1}(Y) + \rho_{h_2}(c - Y)\} = \inf_{Y \in \mathcal{X}} \left\{c + \rho_{h_1}(Y) - \rho_{\widetilde{h}_2}(Y)\right\} \geqslant c.$$

On the other hand, it is always the case that $\rho_{h_1} \,\square\, \rho_{h_2}(c) \leqslant c$, for $c \in \mathbb{R}$. Hence, the equality follows.

(ii) $\to$ (iii): Take $X \in \mathcal{X}$ and $Z = X - \text{ess-inf}X \geqslant 0$. By monotonicity of ρ_{h_2}, we have

$$\begin{aligned}
\rho_{h_1} \square \rho_{h_2}(X) &= \rho_{h_1} \square \rho_{h_2}(Z + \text{ess-inf}X) \\
&= \inf_{Y \in \mathcal{X}} \{\rho_{h_1}(Y) + \rho_{h_2}(Z + \text{ess-inf}X - Y)\} \\
&\geqslant \inf_{Y \in \mathcal{X}} \{\rho_{h_1}(Y) + \rho_{h_2}(\text{ess-inf}X - Y)\} \\
&= \rho_{h_1} \square \rho_{h_2}(\text{ess-inf}X) = \text{ess-inf}X > -\infty.
\end{aligned}$$

(iii) $\to$ (iv): Immediate.

(iv) $\to$ (i): We argue by contradiction. Assume that $h_1 \square h_2(1) < 1$, i.e., there exists $m \in [0,1]$ such that $h_1(m) < \widetilde{h}_2(m)$. Construct a zero-sum allocation $(Y, -Y)$ with $Y = 2a\mathbb{1}_A - a$, $a > 0$ and $\mathbb{P}(A) = m$. Then

$$\begin{aligned}
\rho_{h_1} \square \rho_{h_2}(0) &\leqslant \rho_{h_1}(2a\mathbb{1}_A - a) + \rho_{h_2}(2a\mathbb{1}_{A^c} - a) \\
&= 2ah_1(\mathbb{P}(A)) + 2ah_2(1 - \mathbb{P}(A)) - 2a \\
&= 2a(h_1(m) - \widetilde{h}_2(m)).
\end{aligned}$$

As $a \to \infty$, $\rho_{h_1} \square \rho_{h_2}(0) \to -\infty$. For any $X \in \mathcal{X}$, by the fact of $X \leqslant \text{ess-sup}X$, we have

$$\rho_{h_1} \square \rho_{h_2}(X) \leqslant \rho_{h_1} \square \rho_{h_2}(0) + \text{ess-sup}X = -\infty,$$

which leads to a contradiction. Therefore, we conclude $h_1(x) \geqslant \widetilde{h}_2(x)$ for $x \in [0,1]$, which implies that $h_1 \square h_2(1) = 1$. □

In fact, condition (i) in Theorem 4.2 captures a dominance of caution over speculation in the perception of risk, when both risk-averse and risk-seeking behaviors coexist. This condition requires that the cautious agent's loss weights are strong enough to offset the other's optimism at each quantile level, in the sense that h_1 never falls below the dual $\tilde{h}_2$. As a result, this condition prevents the "gambling" agent from driving the total risk value

to $-\infty$. Without this no-free-gambling condition, the risk-sharing mechanism may fail, allowing one agent to absorb all risk without sufficient penalty.

Remark 4.1. Theorem 4.2 yields a criterion for when the inf-convolution of two distortion risk measures cannot admit a finite and attainable minimum. If $h_1 \square h_2(1) < 1$, then $\rho_{h_1} \square \rho_{h_2}(X) = -\infty$ holds for any $X \in \mathcal{X}$, implying that the inf-convolution cannot be exact at X.

We now highlight several well-known instances that Theorem 4.2 recovers.

Example 4.2 (Two risk-averse agents)**.** If both h_1 and h_2 are concave, then condition (i) in Theorem 4.2 is satisfied; i.e., $h_1 \geqslant \widetilde{h}_2$ since $\widetilde{h}_2$ is convex. In particular, for a constant total risk, any risk-free split is Pareto optimal.

Example 4.3 (Two risk-seeking agents)**.** If both h_1 and h_2 are convex and neither is the identity, then $h_1 < \widetilde{h}_2$ since $\widetilde{h}_2$ is concave. By Theorem 4.2, it follows that

$$\rho_{h_1} \square \rho_{h_2}(X) = -\infty, \text{ for } X \in \mathcal{X},$$

which is consistent with the result established in Proposition 2 of Ghossoub et al. (2026b).

Example 4.4 (Inf-convolution of VaRs)**.** By Corollary 2 of Embrechts et al. (2018), for $\alpha_1, \alpha_2 \geqslant 0$, we have

$$\mathrm{VaR}_{\alpha_1} \square \mathrm{VaR}_{\alpha_2}(X) = \mathrm{VaR}_{\alpha_1+\alpha_2}(X), \text{ for } X \in \mathcal{X}.$$

Note that for $\alpha \geqslant 1$, $\mathrm{VaR}_\alpha(X) = -\infty$ for $X \in \mathcal{X}$. Therefore, if $\alpha_1 + \alpha_2 \geqslant 1$, it holds that

$$\mathrm{VaR}_{\alpha_1} \square \mathrm{VaR}_{\alpha_2}(X) = -\infty, \text{ for } X \in \mathcal{X}. \tag{4.8}$$

This conclusion also follows directly from Theorem 4.2. In this case, the two agents are associated with distortion functions $h_1(x) = \mathbb{1}_{\{x>\alpha_1\}}$ and $h_2(x) = \mathbb{1}_{\{x>\alpha_2\}}$, respectively. Indeed, $h_1 \square h_2(1) = 0 \neq 1$ with $\alpha_1 + \alpha_2 \geqslant 1$, and Theorem 4.2 then yields (4.8).

As a direct application of Theorem 4.2, we now investigate the special case where one agent's distortion function is the dual of the other's. This setting is of particular interest because the pair $(h, \tilde{h})$ acts as a "mirror" on the same risk. Valuing a loss with h equals (up to sign) valuing a gain with the dual $\tilde{h}$. The following proposition shows that, under this mirror setting, the inf-convolution is always well-defined, and it even reduces to a particularly simple form when one agent is risk averse (the mirror agent is necessarily risk seeking).

Proposition 4.1. *Suppose that $\mathcal{X} = L^\infty$. For any $h \in \mathcal{H}$ and $X \in \mathcal{X}$, the following hold.*

(i) $|\rho_h \square \rho_{\widetilde{h}}(X)| \leqslant \|X\|_\infty$.

(ii) $\rho_h \square \rho_{\widetilde{h}}(X) > -\infty$.

(iii) $\rho_h \square \rho_{\widetilde{h}}(c) = c$ *for any constant* $c \in \mathbb{R}$.

(iv) If h or $\widetilde{h}$ is concave, then $\rho_h \square \rho_{\widetilde{h}} = \rho_{h \square \widetilde{h}} = \rho_{h \wedge \widetilde{h}}$.

Proof. (i). For any $h \in \mathcal{H}$, it follows that

$$\rho_h \square \rho_{\widetilde{h}}(X) = \inf_{Y \in \mathcal{X}} \left\{ \rho_h(Y) + \rho_{\widetilde{h}}(X - Y) \right\} = \inf_{Y \in \mathcal{X}} \left\{ \rho_h(Y) - \rho_h(Y - X) \right\} \leqslant \|X\|_\infty.$$

Moreover,

$$\inf_{Y \in \mathcal{X}} \left\{ \rho_h(Y) - \rho_h(Y - X) \right\} \geqslant -\|X\|_\infty$$

The two inequalities hold since ρ_h is lipschitz-continuous with respect to L^∞ -norm.

(ii) and (iii) follow directly from Theorem 4.2, since h is the dual function of $\tilde{h}$.

(iv). Without loss of generality, we assume that h is concave. Then

$$\rho_h \square \rho_{\widetilde{h}}(X) = \inf_{Y \in \mathcal{X}} \left\{ \rho_h(Y) + \rho_{\widetilde{h}}(X - Y) \right\}$$

$$\geqslant \inf_{Y\in\mathcal{X}} \left\{\rho_h(Y)+\rho_{\widetilde{h}}(X)+\rho_{\widetilde{h}}(-Y)\right\} = \rho_{\widetilde{h}}(X).$$

The inequality follows from the superadditivity of $\rho_{\widetilde{h}}$ and the fact of $\rho_h(-Y) = -\rho_{\widetilde{h}}(Y)$. Also, $\rho_h \,\square\, \rho_{\widetilde{h}} \leqslant \rho_{\widetilde{h}}$ holds. Thus, we obtain $\rho_h \,\square\, \rho_{\widetilde{h}} = \rho_{\widetilde{h}}$. A symmetric argument applies if $\widetilde{h}$ is concave. Consequently, we obtain $\rho_h \,\square\, \rho_{\widetilde{h}} = \rho_{h\wedge\widetilde{h}}$.

Next, we show that $\rho_{h\wedge\widetilde{h}} = \rho_{h\square\widetilde{h}}$. Note that $h\square\widetilde{h}(1) = 1$, and hence it suffices to show that $h \wedge \widetilde{h} = h\square\widetilde{h}$. Let $g(y) = h(y) - h(1-x+y)$, for each $x \in [0,1]$. Then $g(y)$ is non-decreasing in $y \in [0,x]$ due to concavity of h. Therefore, $g(y)$ attains its infimum at the endpoints $y = 0$. Then it follows that

$$h\square\widetilde{h}(x) = \inf_{0\leqslant y\leqslant x} \Big\{h(y)+\widetilde{h}(x-y)\Big\} = 1 + \inf_{0\leqslant y\leqslant x} \{h(y)-h(1-x+y)\} = \widetilde{h}(x).$$

Similarly, if $\widetilde{h}$ is concave, we have $h\square\widetilde{h}(x) = h(x)$. □

4.4.2 One risk-averse agent and one risk-seeking agent

To approach a solution to (4.3), we now analyze the two-agent setting in detail. The following theorem identifies structural conditions on the distortion functions under which the inf-convolution admits a simple representation, including cases in which one agent is risk averse or risk seeking.

Theorem 4.3. *Suppose that $\mathcal{X} = L^\infty$ and $h_1, h_2 \in \mathcal{H}$. The following hold.*

(i) If $h_1 \wedge h_2$ is concave, then $\rho_{h_1} \,\square\, \rho_{h_2} = \rho_{h_1 \,\square\, h_2} = \rho_{h_1\wedge h_2}$.

(ii) If h_2 is convex, then $\rho_{h_1} \,\square\, \rho_{h_2} = \rho_{h_2}$ if and only if $h_1 \,\square\, h_2(1) = 1$.

Proof. (i) By concavity of $h_1 \wedge h_2$, we have

$$h_1 \,\square\, h_2 \geqslant (h_1 \wedge h_2) \,\square (h_1 \wedge h_2) = h_1 \wedge h_2.$$

On the other hand, it is clear that $h_1 \,\square\, h_2 \leqslant h_1 \wedge h_2$. Hence, we conclude $h_1 \,\square\, h_2 = h_1 \wedge h_2$. Then it suffices to show $\rho_{h_1} \,\square\, \rho_{h_2} = \rho_{h_1 \wedge h_2}$. For $X \in \mathcal{X}$ and $(X_1, X_2) \in \mathbb{A}_2(X)$, we have

$$\rho_{h_1}(X_1) + \rho_{h_2}(X_2) \geqslant \rho_{h_1 \wedge h_2}(X_1) + \rho_{h_1 \wedge h_2}(X_2) \geqslant \rho_{h_1 \wedge h_2}(X).$$

This implies that $\rho_{h_1} \,\square\, \rho_{h_2} \geqslant \rho_{h_1 \wedge h_2}$. The last inequality follows from the equivalence between concavity of $h_1 \wedge h_2$ and subadditivity of $\rho_{h_1 \wedge h_2}$; see Theorem 3 of Wang et al. (2020b). Furthermore, $\rho_{h_1} \,\square\, \rho_{h_2} \leqslant \rho_{h_1 \wedge h_2}$ holds. Therefore, the desired result is obtained.

(ii) "If": Note that $h_1 \,\square\, h_2(1) = 1$ is equivalent to $h_1 \geqslant \widetilde{h}_2$. For any $X, Y \in \mathcal{X}$, it follows that

$$\rho_{h_1}(Y) + \rho_{h_2}(X - Y) \geqslant \rho_{h_1}(Y) + \rho_{h_2}(X) - \rho_{\widetilde{h}_2}(Y) \geqslant \rho_{h_2}(X).$$

The inequality holds due to the convexity of h_2. The above result implies that $\rho_{h_1} \,\square\, \rho_{h_2} \geqslant \rho_{h_2}$. Also, it is immediate that $\rho_{h_1} \,\square\, \rho_{h_2} \leqslant \rho_{h_2}$. Hence, the equality holds.

"Only if": Let $X = \mathbb{1}_A$ and $x = \mathbb{P}(A)$. Take an allocation (X_1, X_2) of X as: $X_1 = \mathbb{1}_B$ and $X_2 = \mathbb{1}_{A \setminus B}$, where $B \subseteq A$. Write $y = \mathbb{P}(B)$. Then it follows that

$$\rho_{h_1}(X_1) + \rho_{h_2}(X_2) = h_1(y) + h_2(x - y) \text{ and } \rho_{h_2}(X) = h_2(x).$$

By the condition $\rho_{h_1} \,\square\, \rho_{h_2} = \rho_{h_2}$, we have $h_1(y) + h_2(x - y) \geqslant h_2(x)$ for any $x \in [0, 1]$. Consequently, $h_1(y) + h_2(1 - y) \geqslant 1$ for any $y \in [0, 1]$, implying that $h_1 \,\square\, h_2(1) = 1$. □

Theorem 4.3 provides some insights for the mixed two-agent case (one risk-averse and one risk-seeking). In particular, if the existence condition $h_1 \,\square\, h_2(1) = 1$ holds, i.e., the level of risk aversion dominates the level of risk seeking, then the allocation $(X_1, X_2) = (0, X)$ is optimal. Intuitively, the "force" of caution of the risk-averse agent exceeds the "force" of gambling of the risk-seeking agent in this setting. As a result, the efficient arrangement is that the risk-seeking agent absorbs the entire uncertain part of the risk and the risk-averse agent holds the safe position.

It is important to note that in both cases in Theorem 4.3 and throughout Example 4.2–4.4 (with existence conditions), the identity

$$\rho_{h_1} \boxdot \rho_{h_2} = \rho_{h_1 \boxdot h_2} \tag{4.9}$$

always holds. A natural question to ask is whether this relationship still holds in more general settings, especially when two agents' are neither risk-averse nor risk-seeking. In fact, the equality is not universal. Example 4.5 shows that if both agents have subadditive distortion functions, then equality (4.9) may fail once concavity is lost.

Example 4.5 ($\rho_{h_1} \boxdot \rho_{h_2} = \rho_{h_1 \boxdot h_2}$ does not always hold.)**.** Suppose that two agents share the same risk preference $h \in \mathcal{H}$, where h is continuous and subadditive but not concave on $[0,1]$. Then there exists $x, y \in [0,1]$ such that

$$2h(x/2 + y/2) < h(x) + h(y). \tag{4.10}$$

Consider the total risk $X = \mathbb{1}_A + \mathbb{1}_B$ with $\mathbb{P}(A) = \mathbb{P}(B) = x/2 + y/2$ and $\mathbb{P}(A \cap B) = y$. Define an allocation (Y, Z) of X with $Y = \mathbb{1}_A$ and $Z = \mathbb{1}_B$. By (4.10), we can obtain that

$$\rho_h(Y) + \rho_h(Z) < \rho_h(X) = h(\mathbb{P}(A \cup B)) + h(\mathbb{P}(A \cap B)).$$

Therefore, for such X, we have $\rho_h \boxdot \rho_h(X) < \rho_h(X) = \rho_{h \boxdot h}(X)$. The equality holds due to subadditivity of h, implying $h \boxdot h = h$. Below we give a concrete example. Let $h(x) = \max\{\sqrt{x}, 2x - x^2\}$ for $x \in [0,1]$. Clearly, both $\sqrt{x}$ and $2x - x^2$ are subadditive, hence $h(x)$ is subadditive. Consider $X = \mathbb{1}_A + \mathbb{1}_B$ with $\mathbb{P}(A) = \mathbb{P}(B) = 0.38$ and $\mathbb{P}(A \cap B) = 0.26$. Then we can calculate

$$\rho_h(Y) + \rho_h(Z) = 2h(0.38) = 1.233 < \rho_h(X) = h(0.5) + h(0.26) = 1.260.$$

Whereas the results in this section are presented for generality on the space $\mathcal{X} = L^\infty$, the subsequent sections will focus on $\mathcal{X} = L^+$, the set of nonnegative random variables, where

only nonnegative allocations are admissible to ensure well-defined risk-sharing problems.

4.5 Nonnegative allocations

Without any constraints on the allocation set, an optimal allocation may fail to exist under certain conditions, for instance, when $h_1(x) < \widetilde{h}_2(x)$ on $[0,1]$, as shown in Theorem 4.2. In this section, we consider the setting $\mathcal{X} = L^+$; that is, both the aggregate risk and the admissible allocations are required to be nonnegative. This setting is economically intuitive: it means that for each agent, there cannot be any profit from an aggregate pure loss. It is a natural assumption in many applications, such as peer-to-peer insurance.

Even with the nonnegativity constraint, solving $\rho_{h_1} \square \rho_{h_2}$ with heterogeneous preferences can be challenging. In this section, we proceed in two directions: (i) we study the inf-convolution for specific structural classes of distortion functions h_1 and h_2; (ii) we evaluate $\rho_{h_1} \square \rho_{h_2}(X)$ for tractable classes of risks X (e.g., Bernoulli-type random variables) without specifying the distortion type. These studies also include, as notable special cases, settings with one risk-averse and one risk-seeking agent.

4.5.1 Two agents with special distortion functions

In analyzing risk sharing with specific preference classes, we focus on the relationship between the constrained inf-convolution $\rho_{h_1} \square \rho_{h_2}$ and the benchmark $\rho_{h_1 \square h_2}$. We identify when this benchmark provides an upper bound for $\rho_{h_1} \square \rho_{h_2}$ and when this bound is attained. Using $\rho_{h_1 \square h_2}$ as the benchmark is natural since, in all known tractable settings such as risk-averse/risk-seeking pairs and quantile-based specifications, one has the identity (4.9).

An assumption of monotonicity of optimizers for $h_1 \square h_2$ would be important in the subsequent analysis.

Assumption COIN. Let $h_1, h_2 \in \mathcal{H}$. There exists an increasing function f with $x - f(x)$ increasing such that $h_1(f(x)) + h_2(x - f(x)) = h_1 \square h_2(x)$.

Clearly, Assumption COIN ensures that the optimal split of x moves monotonically as x increases; equivalently, there exists an increasing selector f with a co-increasing residual $x \mapsto x - f(x)$. In particular, when both h_1 and h_2 are strictly convex, such selector exists; see Lemma 2 of Ghossoub et al. (2026b).

Remark 4.2. The importance of Assumption COIN arises precisely when one attempts to achieve the benchmark value $\rho_{h_1 \square h_2}(X)$ through an explicit allocation construction. To attain $\rho_{h_1 \square h_2}(X)$, one would naturally construct an allocation (X_1, X_2) of X so that the tail of each component corresponds to the optimal split of the tail of X. This requires that the splitting function f, which determines how the tail mass of X is divided between X_1 and X_2, moves monotonically with the tail level. Assumption COIN ensures that the benchmark allocation is well-defined.

The following theorem shows that under Assumption COIN, the constrained inf-convolution $\rho_{h_1} \square \rho_{h_2}$ can always be bounded by $\rho_{h_1 \square h_2}$.

Theorem 4.4. *Suppose that Assumption COIN holds and $\mathcal{X} = L^+$. For $X \in \mathcal{X}$, it holds that*

$$\rho_{h_1 \square h_2} \square \rho_{h_1 \square h_2}(X) \leqslant \rho_{h_1} \square \rho_{h_2}(X) \leqslant \rho_{h_1 \square h_2}(X). \tag{4.11}$$

Proof. Since $h_1 \square h_2 \leqslant \min\{h_1, h_2\}$, the monotonicity of distortion risk measures yields the first inequality. It remains to show that $\rho_{h_1} \square \rho_{h_2}(X) \leqslant \rho_{h_1 \square h_2}(X)$. For $X \geqslant 0$ and $n \in \mathbb{N}$, define

$$X_n = \sum_{i=0}^{n2^n - 1} \frac{i}{2^n} \mathbb{1}_{\left\{\frac{i}{2^n} < X \leqslant \frac{i+1}{2^n}\right\}} + n \mathbb{1}_{\{X \geqslant n\}}, \tag{4.12}$$

so that $X_n \uparrow X$. Equivalently, (4.12) can be formulated as:

$$X_n = \frac{1}{2^n}\sum_{k=1}^{n2^n} \mathbb{1}_{A_n^k}, \text{ where } A_n^k = \left\{X \geqslant \frac{k}{2^n}\right\}, k \in [2^n n].$$

Clearly, $A_n^k \subseteq A_n^{k-1}$ for $k \in [2^n n] \setminus [1]$. Hence, $(\mathbb{1}_{A_n^1}, \ldots, \mathbb{1}_{A_n^{2^n n}})$ is comonotonic. Write $p_n^k = \mathbb{P}(A_n^k)$ and $\mathcal{S}_n^k = \arg\min_{0\leqslant t\leqslant p_n^k}\left\{h_1(t) + h_2(p_n^k - t)\right\}$ for $k \in [2^n n]$. Let $u_n^k \in \mathcal{S}_n^k$ be the selector within Assumption COIN such that u_n^k and $p_n^k - x_n^k$ are non-increasing in k. By comonotonic additivity of $\rho_{h_1 \square h_2}$, it follows that

$$\begin{aligned}\rho_{h_1\square h_2}(X_n) &= \frac{1}{2^n}\sum_{k=1}^{n2^n}\rho_{h_1\square h_2}(\mathbb{1}_{A_n^k}) = \frac{1}{2^n}\sum_{k=1}^{n2^n} h_1 \square h_2(p_n^k)\\ &= \frac{1}{2^n}\sum_{k=1}^{n2^n}(h_1(u_n^k) + h_2(p_n^k - u_n^k)).\end{aligned}$$

Our goal is to construct an allocation $(Y_n, Z_n) \in A_2(X_n)$ such that $\rho_{h_1}(Y_n) + \rho_{h_1}(Z_n) = \rho_{h_1\square h_2}(X_n)$ for each $n \in \mathbb{N}$. Define the sets

$$\begin{aligned}B_n^k &= B_n^{k+1} \cup \left\{F_{X_n}\left(\frac{k}{2^n}\right) \leqslant U_{X_n} < F_{X_n}\left(\frac{k}{2^n}\right) + (u_n^k - u_n^{k+1})\right\},\ k = 1, \ldots, 2^n n - 1,\\ B_n^{2^n n} &= \left\{F_{X_n}(n) \leqslant U_{X_n} < F_{X_n}(n) + u_n^{2^n n}\right\}.\end{aligned}$$

Then for all $k \in [2^n n]$, we have $\mathbb{P}(B_n^k) = u_n^k$. By construction, it follows that $B_n^k \subseteq B_n^{k-1}$ and $B_n^{k-1}\backslash B_n^k \subseteq A_n^{k-1}\backslash A_n^k$, $k = 2, \ldots, 2^n n$. Since $B_n^{2^n n} \subseteq A_n^{2^n n}$, backward induction gives $B_n^k \subseteq A_n^k$ for $k \in [2^n n]$. Therefore, we also have $A_n^k\backslash B_n^k \subseteq A_n^{k-1}\backslash B_n^{k-1}$ for $k \in [2^n n] \setminus [1]$. Define the allocation

$$Y_n = \frac{1}{2^n}\sum_{k=1}^{n2^n}\mathbb{1}_{B_n^k} \text{ and } Z_n = \frac{1}{2^n}\sum_{k=1}^{n2^n}\mathbb{1}_{A_n^k\backslash B_n^k}. \tag{4.13}$$

Clearly, $Y_n + Z_n = X_n$ for each n. Moreover, comonotonic additivity leads to

$$\begin{aligned}\rho_{h_1}(Y_n) + \rho_{h_2}(Z_n) &= \frac{1}{2^n}\sum_{k=1}^{n2^n}\left(h_1(\mathbb{P}(B_n^k)) + h_2(\mathbb{P}(A_n^k) - \mathbb{P}(B_n^k))\right) \\ &= \frac{1}{2^n}\sum_{k=1}^{n2^n}\left(h_1(u_n^k) + h_2(p_n^k - u_n^k)\right) \\ &= \frac{1}{2^n}\sum_{k=1}^{n2^n} h_1 \,\square\, h_2(p_n^k) = \rho_{h_1 \,\square\, h_2}(X_n). \end{aligned} \tag{4.14}$$

Consequently, it follows that

$$\begin{aligned}\rho_{h_1} \,\square\, \rho_{h_2}(X) &= \lim_{n\to\infty} \rho_{h_1} \,\square\, \rho_{h_2}(X_n) \\ &\leqslant \lim_{n\to\infty} \left(\rho_{h_1}(Y_n) + \rho_{h_2}(Z_n)\right) = \lim_{n\to\infty} \rho_{h_1 \,\square\, h_2}(X_n) = \rho_{h_1 \,\square\, h_2}(X).\end{aligned}$$

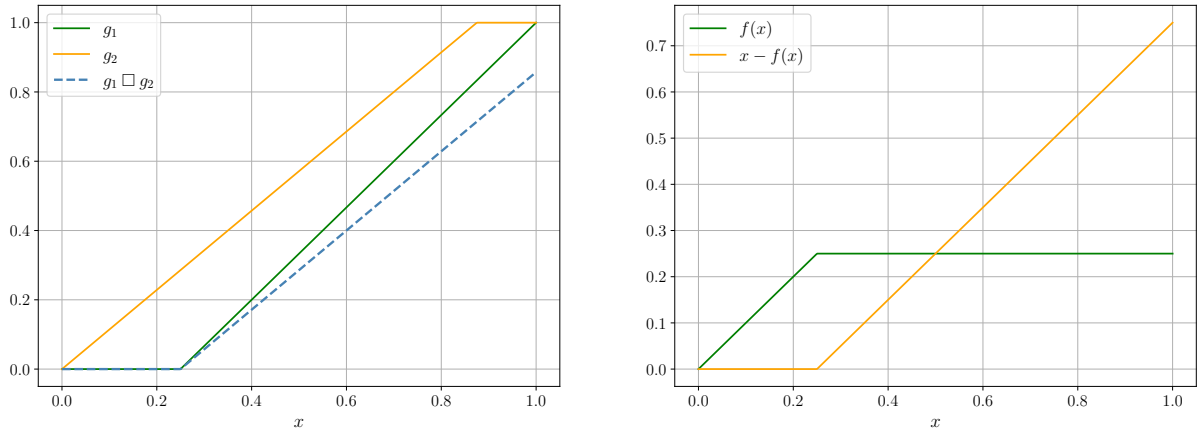
The first equality holds since $\rho_{h_1} \,\square\, \rho_{h_2}$ is continuous from below; see Lemma 4.2 in Section 4.8 for more details. Therefore, the result implies that $\rho_{h_1} \,\square\, \rho_{h_2}(X) \leqslant \rho_{h_1 \,\square\, h_2}(X)$. This completes the proof. □

As we can see, Theorem 4.4 is non-constructive and it does not specify an allocation that attains the upper bound $\rho_{h_1 \,\square\, h_2}(X)$. On an atomless space, however, when there exists a uniform random variable independent of X, the attainable allocation is available; see Lemma 3 of Lauzier et al. (2026a).

As noted earlier, Assumption COIN holds whenever both h_1 and h_2 are convex; however, convexity is not required. There exists nonconvex pairs (h_1, h_2) that still admits such an increasing minimizer, as demonstrated in Example 4.6 involving a risk-averse agent and a risk-seeking agent. Therefore, Theorem 4.4 applies strictly beyond the convex setting.

Example 4.6. Let $g_1(x) = \max\{0, 4/3(x - 1/4)\}$ and $g_2(x) = \min\{8/7x, 1\}$ for $x \in [0, 1]$. Clearly, g_1 is convex and g_2 is concave with $g_2 \leqslant \widetilde{g}_1$. Then we can calculate $g_1 \,\square\, g_2(x) = \max\{0, 8/7(x - 1/4)\}$. Define $f(x) = \min\{x, 1/4\}$, thus $x - f(x) = \max\{0, x - 1/4\}$.

Both $f(x)$ and $x - f(x)$ are non-decreasing; see Figure 4.1 for a detailed illustration. It can be verified that $g_1(f(x)) + g_2(x - f(x)) = g_1 \square g_2(x)$ for $x \in [0, 1]$.



(a) g_1, g_2 and $g_1 \square g_2$ (b) Optimizer for $g_1 \square g_2$

Figure 4.1: An illustration of $g_1 \square g_2$ with $a = 1/4$ and $b = 7/8$

In fact, the second inequality in (4.11) becomes an equality when the two agents' distortion functions are of the type described in Example 4.6. Before stating that result, we provide a technical lemma that will be used in the sequel, which analyzes risk sharing between a VaR agent and a distortion agent. The result also generalizes Theorem 5.3 of Wang and Wei (2020). To make the statement more precise, we introduce the following notation. For $h \in \mathcal{H}$, let $\alpha(h) = \sup\{t \in [0, 1] : h(t) = 0\}$. The function

$$\hat{h}(t) = h((t + \alpha(h)) \wedge 1), \quad t \in [0, 1]$$

is called the active part of h. Additionally, let $h^a(t) = h((t-a)_+)$ for $t \in [0, 1]$ and $a \in [0, 1]$.

Lemma 4.1. *Suppose that $h \in \mathcal{H}$ and $\mathcal{X} = L^+$. For any $\alpha \in [0, 1]$ and $X \in \mathcal{X}$, we have*

$$\mathrm{VaR}_\alpha \square \rho_h(X) = \rho_{h^\alpha}(X). \tag{4.15}$$

Proof. We first show that $\mathrm{VaR}_\alpha \Box \rho_h(X) \geqslant \rho_{h^\alpha}(X)$. For any allocation $(Y, X-Y)$ of X with $0 \leqslant Y \leqslant X$, it follows that

$$\begin{aligned}
\mathrm{VaR}_\alpha(Y) + \rho_h(X-Y) &= \mathrm{VaR}_\alpha(Y) + \int_0^1 \mathrm{VaR}_\beta(X-Y)\mathrm{d}h(\beta) \\
&= \int_0^{1-\alpha} \mathrm{VaR}_\alpha(Y) + \mathrm{VaR}_\beta(X-Y)\mathrm{d}h(\beta) \\
&\quad + \int_{1-\alpha}^1 \mathrm{VaR}_\alpha(Y) + \mathrm{VaR}_\beta(X-Y)\mathrm{d}h(\beta) \\
&\geqslant \int_0^{1-\alpha} \mathrm{VaR}_{\alpha+\beta}(X)\mathrm{d}h(\beta) \\
&= \int_0^1 \mathrm{VaR}_{\beta'}(X)\mathrm{d}h((\beta'-\alpha)_+) = \rho_{h^\alpha}(X).
\end{aligned}$$

The inequality is due to $0 \leqslant Y \leqslant X$ and $\mathrm{VaR}_{\alpha+\beta}(X_1+X_2) \leqslant \mathrm{VaR}_\alpha(X_1) + \mathrm{VaR}_\beta(X_2)$ for $X_1, X_2 \in \mathcal{X}$; see Corollary 1 of Embrechts et al. (2018).

Next we show that $\mathrm{VaR}_\alpha \Box \rho_h(X) \leqslant \rho_{h_\alpha}(X)$. Note that $\mathrm{VaR}_\alpha\left(X\mathbb{1}_{\{U_X>1-\alpha\}}\right) = 0$. By straightforward calculation, we have

$$\begin{aligned}
\mathrm{VaR}_\alpha \Box \rho_h(X) &\leqslant \mathrm{VaR}_\alpha\left(X\mathbb{1}_{\{U_X>1-\alpha\}}\right) + \rho_h\left(X\mathbb{1}_{\{U_X\leqslant 1-\alpha\}}\right) \\
&= \int_0^\infty h\left(\mathbb{P}\left(X\mathbb{1}_{\{U_X\leqslant 1-\alpha\}} > t\right)\right)\mathrm{d}t \\
&= \int_0^\infty h\left(\mathbb{P}\left(\{X>t\}\cap\{U_X\leqslant 1-\alpha\}\right)\right)\mathrm{d}t \\
&= \int_0^\infty h\left((\mathbb{P}(X>t)-\alpha)_+\right)\mathrm{d}t = \rho_{h^\alpha}(X).
\end{aligned}$$

Therefore, the desired result is obtained. □

Remark 4.3. For any $\alpha \in [0,1]$, we denote by g^α the distortion function of VaR_α, i.e., $g^\alpha(x) = \mathbb{1}_{\{x>\alpha\}}$. Clearly, $g^\alpha \Box h(x) = h^\alpha(x)$ for $x \in [0,1]$. Therefore, (4.15) in fact states that $\rho_{g^\alpha} \Box \rho_h(X) = \rho_{g^\alpha \Box h}(X)$ for $X \in \mathcal{X}$.

We now investigate the structure of optimal allocations in a two-agent risk sharing

setting involving one risk-averse and one risk-seeking participant, where both agents are characterized by piecewise linear distortion functions, as generalized from Example 4.6. In the next proposition, the distortion functions h_1 and h_2 correspond to a left-tail Expected Shortfall (see Embrechts et al. (2015)) and an Expected Shortfall, respectively.

Proposition 4.2. *Suppose that* $\mathcal{X} = L^+$. *Let* $h_1(x) = \max\{0, (x-a)/(1-a)\}$ *and* $h_2(x) = \min\{x/b, 1\}$ *for* $x \in [0,1]$, *where* $a, b \in (0,1)$ *and* $a + b \geqslant 1$. *Then*

$$\rho_{h_1} \square \rho_{h_2}(X) = \rho_{h_1 \square h_2}(X) = \rho_h(X) \quad \textit{for all } X \in \mathcal{X},$$

where

$$h(x) = \max\{0, (x-a)/b\}, \quad x \in [0,1].$$

Moreover, an optimal allocation (X_1, X_2) *of* X *is given by*

$$X_1 = X\mathbb{1}_{\{U_X \geqslant 1-a\}}, \text{ and } X_2 = X\mathbb{1}_{\{U_X < 1-a\}}.$$

Proof. Clearly, $\alpha(h_1) = a$ and $\hat{h}_1(x) = \min\{x/(1-a), 1\}$. We can calculate

$$h_1 \square h_2(x) = \max\{0, (x-a)/b\} = h_2^a(x) = h(x), \quad x \in [0,1]. \tag{4.16}$$

By Lemma 4.1, it follows that

$$\mathrm{VaR}_a \square \rho_{\hat{h}_1}(X) = \rho_{h_1}(X).$$

Note that by using Lemma 2 of Liu et al. (2020), we can see that inf-convolutions are associative. Therefore,

$$\begin{aligned}\rho_{h_1} \square \rho_{h_2}(X) &= (\mathrm{VaR}_a \square \rho_{\hat{h}_1}) \square \rho_{h_2}(X) \\ &= \mathrm{VaR}_a \square (\rho_{\hat{h}_1} \square \rho_{h_2})(X) = \mathrm{VaR}_a \square \rho_{h_2}(X) = \rho_{h_1 \square h_2}(X),\end{aligned}$$

where the last equality follows from Lemma 4.1 and (4.16).

Next, we find the optimal allocation. Let $f(x) = \min\{x, a\}$, so that $x - f(x) = \max\{0, x - a\}$. Both $f(x)$ and $x - f(x)$ are non-decreasing. It can be verified that $h_1(f(x)) + h_2(x - f(x)) = h_1 \,\square\, h_2(x)$ for $x \in [0, 1]$. Then it follows that

$$\begin{aligned}
\rho_{h_1}(X_1) &= \int_0^\infty h_1(\mathbb{P}(X\mathbb{1}_{\{U_X \geqslant 1-a\}} > t))\mathrm{d}t \\
&= \int_0^\infty h_1(\mathbb{P}(\{X > t\} \cap \{U_X \geqslant 1 - a\}))\mathrm{d}t \\
&= \int_0^\infty h_1(\mathbb{P}(U_X \geqslant \max\{F_X(t), 1 - a\}))\mathrm{d}t \\
&= \int_0^\infty h_1(\min\{a, S_X(t)\})\mathrm{d}t = \int_0^\infty h_1(f(S_X(t)))\mathrm{d}t.
\end{aligned}$$

Similarly, we have $\rho_{h_2}(X_2) = \int_0^\infty h_1(S_X(t) - f(S_X(t)))\mathrm{d}t$. This implies that $\rho_{h_1}(X_1) + \rho_{h_2}(X_2) = \rho_{h_1 \,\square\, h_2}(X)$. □

Although Proposition 4.2 covers only a subset of the mixed (one risk-averse and one risk-seeking) cases, it extends part (ii) of Theorem 4.3 by removing the constraint of $h_1 \square h_2(1) = 1$. The proposition shows that, in this setting, an optimal allocation can be constructed with a counter-monotonic structure when the parameters do not satisfy $(a + b = 1)$. This indicates that counter-monotonicity can arise naturally in the mixed risk-attitude case. The following example further illustrates that, when the risk-seeking effect is sufficiently strong, a counter-comonotonic allocation may strictly outperform all comonotonic allocations.

Example 4.7. Set $h_1(x) = 1 - (1 - x)^2$ and $h_2(x) = x^3$ for $x \in [0, 1]$ and let $X \sim$ Bernoulli(p) for $p \in [0, 1]$. For any comonotonic allocation (X_1, X_2) of X, it holds that

$$\rho_{h_1}(X_1) + \rho_{h_2}(X_2) \geqslant \rho_{h_1 \wedge h_2}(X) = h_2(p).$$

By straightforward calculation, we have

$$h_1 \,\square\, h_2(x) = h_2(x)\mathbb{1}_{\left\{x \leqslant \sqrt{\frac{2}{3}}\right\}} + (h_1(f(x)) + h_2(x - f(x)))\mathbb{1}_{\left\{x > \sqrt{\frac{2}{3}}\right\}},$$

where $f(x) = (3x - 1 - \sqrt{7 - 6x})/3$. It can be seen from Figure 4.2 that $h_1 \square h_2(x) < h_2(x)$ for $x > \sqrt{2/3}$. Take $X_1' = \mathbb{1}_A$ and $X_2' = \mathbb{1}_B$, where A and B are disjoint, $\mathbb{P}(A \cup B) = p$ and satisfies $h_1(\mathbb{P}(A)) + h_2(\mathbb{P}(B)) = h_1 \square h_2(p)$. Clearly, (X_1', X_2') is a counter-monotonic allocation of X. For $p > \sqrt{2/3}$, we can derive that

$$\rho_{h_1}(X_1') + \rho_{h_2}(X_2') = h_1 \square h_2(p) < h_2(p).$$

Hence, a comonotonic allocation (X_1, X_2) is never optimal.

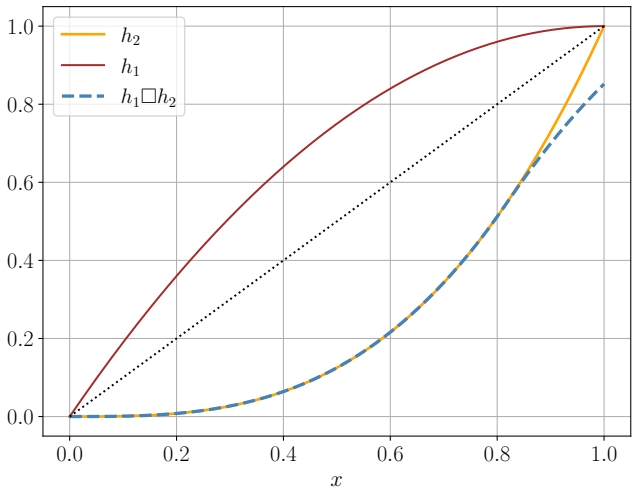


Figure 4.2: An illustration of Example 4.7.

4.5.2 Two agents sharing a Bernoulli-type risk

In this subsection, we consider a simple setting of two agents sharing a Bernoulli random risk with unrestricted risk preferences. In this binary setting, the inf-convolution and optimal allocations admit closed-form solutions. Furthermore, we explore how these insights extend to richer risk distributions.

Theorem 4.5. *Suppose that $h_1, h_2 \in \mathcal{H}$ and $\mathcal{X} = L^+$. For any $A \in \mathcal{F}$ and $a \in \mathbb{R}_+$, it*

holds that

$$\rho_{h_1} \Box \rho_{h_2}(a\mathbb{1}_A) = \rho_{h_1 \Box h_2}(a\mathbb{1}_A) = ah_1 \Box h_2(\mathbb{P}(A)). \tag{4.17}$$

Moreover, a Pareto-optimal allocation is given by $X_1 = a\mathbb{1}_B$ *and* $X_2 = a\mathbb{1}_{A\setminus B}$ *satisfying* $h_1(\mathbb{P}(B)) + h_2(\mathbb{P}(A \setminus B)) = h_1 \Box h_2(\mathbb{P}(A))$.

Proof. By the positive homogeneity of $\rho_{h_1 \Box h_2}$, it is trivial to verify the second equality in (4.17). To show the first equality, we first state that $\rho_{h_1} \Box \rho_{h_2}(a\mathbb{1}_A) \geqslant \rho_{h_1 \Box h_2}(a\mathbb{1}_A)$. With $0 \leqslant Y \leqslant a\mathbb{1}_A$, it follows that

$$\begin{aligned}
\rho_{h_1}(Y) + \rho_{h_2}(a\mathbb{1}_A - Y) &= \int_0^\infty h_1(\mathbb{P}(Y > t))\mathrm{d}t + \int_0^\infty h_2(\mathbb{P}(a\mathbb{1}_A - Y > t))\mathrm{d}t \\
&= \int_0^a h_1(\mathbb{P}(Y \geqslant t))\mathrm{d}t + \int_0^a h_2(\mathbb{P}(a - Y > t|A)\mathbb{P}(A))\mathrm{d}t \\
&= \int_0^a h_1(\mathbb{P}(Y \geqslant t))\mathrm{d}t + \int_0^a h_2(\mathbb{P}(Y < t|A)\mathbb{P}(A))\mathrm{d}t \\
&= \int_0^a h_1(\mathbb{P}(Y \geqslant t))\mathrm{d}t + \int_0^a h_2(\mathbb{P}(\{Y < t\} \cap A))\mathrm{d}t \\
&\geqslant \int_0^a h_1 \Box h_2(\mathbb{P}(A))\mathrm{d}t = ah_1 \Box h_2(\mathbb{P}(A)),
\end{aligned}$$

where we use $\mathbb{P}(Y \geqslant t) \geqslant \mathbb{P}(\{Y \geqslant t\} \cap A)$. Thus, we have $\rho_{h_1} \Box \rho_{h_2}(\mathbb{1}_A) \geqslant ah_1 \Box h_2(\mathbb{P}(A))$.

Next, we show the converse direction. Take an allocation (X_1, X_2) of X as $X_1 = a\mathbb{1}_B$ and $X_2 = a\mathbb{1}_C$, where $B \cup C = A$, $B \cap C = \varnothing$ and $h_1(\mathbb{P}(B)) + h_2(\mathbb{P}(C)) = ah_1 \Box h_2(\mathbb{P}(A))$, we have $\rho_{h_1}(X_1) + \rho_{h_2}(X_2) = ah_1 \Box h_2(\mathbb{P}(A))$. Consequently, $\rho_{h_1} \Box \rho_{h_2}(a\mathbb{1}_A) \leqslant \rho_{h_1 \Box h_2}(a\mathbb{1}_A)$ holds for any $a \in \mathbb{R}_+$. □

Notably, Theorem 4.5 does not rely on any concavity or convexity assumptions about the distortion functions. Moreover, it characterizes how the total probability mass $\mathbb{P}(A)$ is divided between the two agents so as to minimize the total risk value, with the splitting probabilities determined by $h_1 \Box h_2$. To gain intuition, we provide an example to see how

the optimal split changes when one agent is risk averse and the other is risk seeking, particularly in cases where the dominance condition $h_1 \geqslant \widetilde{h}_2$ fails. Example 4.8 illustrates this behavior explicitly, showing how the share of risk held by each agent varies with the total probability level $\mathbb{P}(A)$.

Example 4.8. Suppose that $h_1(x) = 1-(1-x)^\alpha$ and $h_2(x) = x^\beta$ with $1 < \alpha < \beta$. Clearly, h_1 is concave, h_2 is convex, and $h_1 \leqslant \tilde{h}_2$. Let $p_0 = (\alpha/\beta)^{\frac{1}{\beta-1}}$ and $(\mathbb{1}_B, \mathbb{1}_{A\setminus B})$ be an optimal allocation of $\mathbb{1}_A$. We can show that

(i) If $\mathbb{P}(A) \leqslant p_0$, then $\mathbb{P}(B) = 0$ and $\mathbb{P}(A\setminus B) = \mathbb{P}(A)$, so the risk-seeking agent bear the entire risk.

(ii) If $\mathbb{P}(A) > p_0$, then as $\mathbb{P}(A)$ increases, $\mathbb{P}(B)$ increases strictly, while $\mathbb{P}(A\setminus B)$ falls strictly. In this case, the risk-averse agent begins to take on an increasing share of the risk, while the risk-seeker's share correspondingly shrinks.

The proof is non-trivial and the details are provided in Proposition 4.7 of Section 4.8. For the special case $\alpha = 2$ and $\beta = 3$, Figure 4.3 illustrates the trend of $\mathbb{P}(B)$ and $\mathbb{P}(A \setminus B)$ as $\mathbb{P}(A)$ varies.

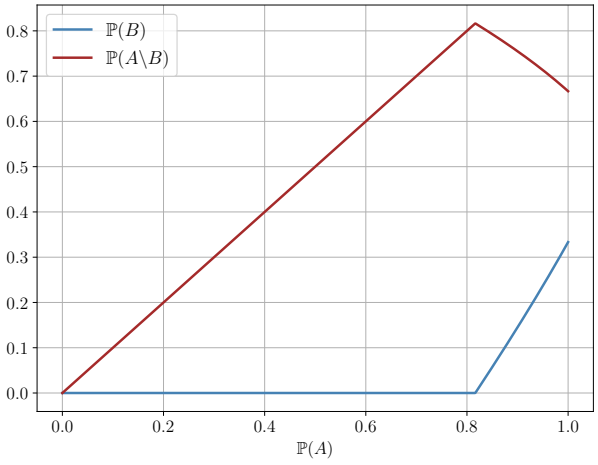


Figure 4.3: An illustration of Example 4.8.

It is clear that examining an economy without aggregate uncertainty, where the total endowment is constant, is a special case of Theorem 4.5; as shown in the following corollary.

Corollary 4.1. *Suppose that $\mathcal{X} = L^+$. For any $a \in \mathbb{R}_+$, it holds that*

$$\rho_{h_1} \,\square\, \rho_{h_2}(a) = a h_1 \,\square\, h_2(1). \tag{4.18}$$

Moreover, a Pareto-optimal allocation is given by $X_1 = \mathbb{1}_B$ and $X_2 = \mathbb{1}_{A\setminus B}$ with $h_1(\mathbb{P}(B)) + h_2(\mathbb{P}(A \setminus B)) = h_1 \,\square\, h_2(1)$.

The result directly follows from Theorem 4.5 by taking $A = \Omega$. If additionally $h_1 \geqslant \widetilde{h}_2$, implying that $h_1 \,\square\, h_2(1) = 1$, then $\rho_{h_1} \,\square\, \rho_{h_2}(a) = a$ for any positive constant a. In this case, any constant split $(b, a-b)$ with $0 \leqslant b \leqslant a$ is Pareto optimal.

In fact, the equality (4.18) relies on the richness of the probability space to ensure that $\mathbb{P}(B)$ can achieve the infimum. On non-atomless space (e.g., finite probability space), the required probability level may be unattainable and the equality can fail; see more details in Example 4.9.

Example 4.9 (Counter-example in a finite probability space)**.** Define a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega = \{\omega_1, \omega_2\}$, and $\mathbb{P}$ is such that $\mathbb{P}(\{\omega_1\}) = 1/6$, $\mathbb{P}(\{\omega_2\}) = 5/6$. Suppose that two agents have distortion functions given by

$$h_1(x) = \begin{cases} 2x & x \in [0, 0.5] \\ 1 & x \in [0.5, 1]; \end{cases} \qquad h_2(x) = \begin{cases} 0 & x \in [0, 2/3] \\ 3x - 2 & x \in [2/3, 1]. \end{cases}$$

We can show that

$$h_1 \,\square\, h_2(x) = \begin{cases} 0 & x \in [0, 2/3] \\ 2x - 4/3 & x \in [2/3, 1]. \end{cases}$$

Assume that two agents are sharing a constant 1. The only possible allocation (X_1, X_2) of 1 is in the form of $(a\mathbb{1}_{\{\omega_1\}} + b\mathbb{1}_{\{\omega_2\}}, (1-a)\mathbb{1}_{\{\omega_1\}} + (1-b)\mathbb{1}_{\{\omega_2\}})$, where $0 \leqslant a, b \leqslant 1$. If $a \leqslant b$,

then $\rho_{h_1}(X_1) + \rho_{h_2}(X_2) = a + (b-a)h_2(5/6) + (1-b) + (b-a)h_1(1/6) = 1$. If $a \geqslant b$, then $\rho_{h_1}(X_1)+\rho_{h_2}(X_2) = b+(a-b)h_1(1/6)+(1-a)+(a-b)h_2(5/6) = 1-1/6(a-b) \geqslant 5/6$. Also, it is straightforward to verify that $\rho_{h_1 \,\square\, h_2}(1) = 2/3$. This shows that $\rho_{h_1} \,\square\, \rho_{h_2}(1) > \rho_{h_1 \,\square\, h_2}(1)$.

Having established the equality (4.17) for indicator risks, we now turn to more complex distributions. When the total risk involves multiple components rather than a single Bernoulli variable, the exact equality no longer necessarily holds. Even so, this broader setting yields useful insights, particularly regarding whether and when the inf-convolution can still attain the benchmark value. Proposition 4.3 demonstrates that such attainability persists in a structured class of random risks. Specifically, when the total risk is composed of disjoint indicator components, there always exists a feasible allocation achieving the value of $\rho_{h_1 \,\square\, h_2}$.

Proposition 4.3. *Suppose that $h_1, h_2 \in \mathcal{H}$ and $\mathcal{X} = L^+$. Let $X = a\mathbb{1}_A + b\mathbb{1}_B$, where $A, B \in \mathcal{F}$ are disjoint and $a, b \in \mathbb{R}_+$ are constants. Then it holds that*

$$\rho_{h_1} \,\square\, \rho_{h_2}(X) \leqslant \rho_{h_1 \,\square\, h_2}(X). \tag{4.19}$$

Proof. Without loss of generality, we assume that $a \leqslant b$. Then X can be reformulated as $X = a\mathbb{1}_{A'} + (b-a)\mathbb{1}_{B'}$, where $A' = A \cup B$ and $B' = B$. clearly, $B' \subseteq A'$, which implies that $(a\mathbb{1}_{A'}, (b-a)\mathbb{1}_{B'})$ is comonotonic. Let $p_1 = \mathbb{P}(A')$ and $p_2 = \mathbb{P}(B')$. By comonotonic additivity of $\rho_{h_1 \,\square\, h_2}$, we have

$$\rho_{h_1 \,\square\, h_2}(X) = ah_1 \,\square\, h_2(p_1) + (b-a)h_1 \,\square\, h_2(p_2).$$

Define the set

$$S_i = \{t \in [0, p_i] : h_1(t) + h_2(p_i - t) = h_1 \,\square\, h_2(p_i)\}.$$

Let $u_i \in S_i$ for each $i \in [2]$. Our aim is to construct an allocation (Y, Z) of X such that $\rho_{h_1}(Y) + \rho_{h_2}(Z) = ah_1 \,\square\, h_2(p_1) + (b-a)h_1 \,\square\, h_2(p_2)$. We construct an allocation of X as

follows:

$$Y = a\mathbb{1}_C + b\mathbb{1}_D \quad \text{and} \quad Z = a\mathbb{1}_{A'\setminus C} + b\mathbb{1}_{B'\setminus D} \quad \text{with} \quad \mathbb{P}(C) = u_1 \quad \text{and} \quad \mathbb{P}(D) = u_2.$$

In fact, the construction of C and D vary with the magnitude relationship between u_1 and u_2, as well as $p_1 - u_1$ and $p_2 - u_2$. Next, we will show the details about how to construct C and D in different cases. We consider the following three cases.

Case 1: $u_1 \leqslant u_2$ and $p_1 - u_1 \leqslant p_2 - u_2$. This case cannot happen unless $\mathbb{P}(A) = 0$, then it reduces to Theorem 4.5. Therefore, (4.19) holds trivially.

Case 2: $u_1 \leqslant u_2$ and $p_1 - u_1 \geqslant p_2 - u_2$. It implies that $u_1 \leqslant u_2$ since $p_1 \geqslant p_2$. Take $C \subseteq B$ with $\mathbb{P}(C) = u_1$ and $D = C \cup E$ with $E \subseteq B'\setminus C$ and $\mathbb{P}(E) = u_2 - u_1$. Hence, $\mathbb{P}(D) = u_2$. Also, we have $B'\setminus D \subseteq A'\setminus C$.

Case 3: $u_1 \geqslant u_2$ and $p_1 - u_1 \geqslant p_2 - u_2$. Thus, $u_1 \geqslant u_2 \geqslant p_2 - p_1 + u_1$. Let $D \subseteq B'$ with $\mathbb{P}(D) = u_2$. The take $C = D \cup E$ and $E \subseteq A'\setminus B'$ with $\mathbb{P}(E) = u_1 - u_2$. Hence, $\mathbb{P}(C) = u_1$. In this case, we have $B'\setminus D \subseteq A'\setminus C$.

Case 4: $u_1 \geqslant u_2$ and $p_1 - u_1 \leqslant p_2 - u_2$. It implies that $u_1 \geqslant p_1 - p_2 + u_2$. Let $D \subseteq B'$ with $\mathbb{P}(D) = u_2$. Take $C = D \cup (A'\setminus B') \cup E$ and $E \subseteq B'\setminus D$ with $\mathbb{P}(E) = u_1 - u_2 - (p_1 - p_2)$. Thus, we have $\mathbb{P}(C) = u_1$ and $A'\setminus C \subseteq B'\setminus D$.

By above constructions, both $(a\mathbb{1}_C, (b-a)\mathbb{1}_D)$ and $(a\mathbb{1}_{A'\setminus C}, (b-a)\mathbb{1}_{B'\setminus D})$ are comonotonic since the underlying events are nested. Then it follows that

$$\begin{aligned}\rho_{h_1}(Y) + \rho_{h_2}(Z) &= a\big(h_1(u_1) + h_1(u_2)\big) + (b-a)\big(h_2(p_1 - u_1) + h_2(p_2 - u_2)\big) \\ &= ah_1 \square h_2(p_1) + (b-a)h_1 \square h_2(p_2) = \rho_{h_1 \square h_2}(X).\end{aligned}$$

Consequently, $\rho_{h_1} \square \rho_{h_2}(X) \leqslant \rho_{h_1 \square h_2}(X)$. □

By translation invariance, one typically has $\rho_{h_1} \square \rho_{h_2}(\mathbb{1}_A + c) \leqslant \rho_{h_1 \square h_2}(\mathbb{1}_A) + c$ for any constant c, where the bound comes from handling the constant and the indicator

separately. Using Theorem 4.3, a tighter bound can be obtained by treating the constant as an additional layer over a disjoint set; see Corollary 4.2.

Corollary 4.2. *Suppose that $h_1, h_2 \in \mathcal{H}$ and $c \in \mathbb{R}_+$. For any $A \in \mathcal{F}$, it holds that*

$$\max\{h_1 \,\square\, h_2(\mathbb{P}(A)), c\, h_1 \,\square\, h_2(1)\} \leqslant \rho_{h_1} \,\square\, \rho_{h_2}(\mathbb{1}_A + c) \leqslant \rho_{h_1 \,\square\, h_2}(\mathbb{1}_A + c).$$

Proof. The first inequality directly follows from Theorem 4.5. It is immediate to show the second inequality by applying Theorem 4.3 and replacing $B = A^c$ and $a = 1+c = 1+b$. □

The following proposition provides some results for random variables X satisfying $\mathbb{P}(X > 0) \leqslant a$ for some $a \in [0, 1]$. We first introduce some notations for convenient. For any $h_1, h_2 \in \mathcal{H}$, let $x_c = \sup\{x \in [0, 1] : h_1 \,\square\, h_2(x) = h_2(x)\}$ and $x_d = \sup\left\{x > 0 : h_{2+}'(x) \leqslant 1\right\}$.

Proposition 4.4. *Suppose that $\mathcal{X} = L^+$ and $h_1, h_2 \in \mathcal{H}$. For $X \in \mathcal{X}$ with $\mathbb{P}(X > 0) \leqslant \alpha$, $\alpha \in [0, 1)$, the following hold.*

(i) If $h_1 \,\square\, h_2$ is convex and $\alpha \leqslant 1/2$, then $\rho_{h_1} \,\square\, \rho_{h_2}(X) \geqslant \rho_{h_1 \,\square\, h_2}(X)$.

(ii) If h_2 is convex and $\alpha \leqslant x_c/2$, then $\rho_{h_1} \,\square\, \rho_{h_2}(X) = \rho_{h_2}(X)$.

(iii) If h_1 is concave, h_2 is convex and $\alpha \leqslant \max\{x_c/2, x_d\}$, then $\rho_{h_1} \,\square\, \rho_{h_2}(X) = \rho_{h_2}(X)$.

Proof. (i) For any allocation $(X_1, X_2) \in \mathbb{A}_2(X)$ and $a \in (0, \infty)$, we have $\mathbb{E}(X_2 \wedge a) + \mathbb{E}(X_1 \wedge a) \geqslant \mathbb{E}(X \wedge a)$, implying that

$$\int_0^a (\mathbb{P}(X_1 > t) + \mathbb{P}(X_2 > t))\mathrm{d}t \geqslant \int_0^a \mathbb{P}(X > t)\mathrm{d}t.$$

By integral majorization theorem (Pečarić and Tong, 1992, Theorem 12.15) and convexity of $h_1 \,\square\, h_2$, it holds that

$$\int_0^a h_1 \,\square\, h_2(\mathbb{P}(X_1 > t) + \mathbb{P}(X_2 > t))\mathrm{d}t \geqslant \int_0^a h_1 \,\square\, h_2(\mathbb{P}(X > t))\mathrm{d}t. \tag{4.20}$$

Note that $\mathbb{P}(X > 0) \leqslant 1/2$ ensures that $h_1 \,\square\, h_2$ is well-defined. Then it follows that

$$\begin{aligned}\rho_{h_1}(X_1) + \rho_{h_2}(X_2) &= \int_0^\infty h_1(\mathbb{P}(X_1 > t)) + h_2(\mathbb{P}(X_2 > t))\mathrm{d}t \\ &\geqslant \int_0^\infty h_1 \,\square\, h_2(\mathbb{P}(X_1 > t) + \mathbb{P}(X_2 > t))\mathrm{d}t \geqslant \rho_{h_1 \,\square\, h_2}(X),\end{aligned}$$

which implies that $\rho_{h_1} \,\square\, \rho_{h_2}(X) \geqslant \rho_{h_1 \,\square\, h_2}(X)$.

(ii) Clearly, $\rho_{h_1} \,\square\, \rho_{h_2}(X) \leqslant \rho_{h_2}(X)$. It suffices to show the converse direction. We first show that for any $x \leqslant x_0$, it holds that $h_1 \square h_2(x) = h_2(x)$. By definition of x_0, it follows that $f_{x_0}(y) := h_1(y) + h_2(x_0 - y) - h_2(x_0) \geqslant 0$ for $y \leqslant x_0$. Then for $x \leqslant x_c$, we have

$$h_1(y) + h_2(x - y) - h_2(x) = f_{x_0}(y) - h_2(x_0 - y) + h_2(x - y) + h_2(x_0) - h_2(x) \geqslant 0. \tag{4.21}$$

The inequality (4.21) holds due to convexity of h_2, implying that $h_2(x_0) - h_2(x) \geqslant h_2(x_0 - y) - h_2(x - y)$. For any allocation $(X_1, X_2) \in \mathbb{A}_2(X)$, it holds that

$$\begin{aligned}\rho_{h_1}(X_1) + \rho_{h_2}(X_2) &\geqslant \int_0^\infty h_1 \,\square\, h_2(\mathbb{P}(X_1 > t) + \mathbb{P}(X_2 > t))\mathrm{d}t \\ &= \int_0^\infty h_2(\mathbb{P}(X_1 > t) + \mathbb{P}(X_2 > t))\mathrm{d}t \\ &\geqslant \int_0^\infty h_2(\mathbb{P}(X > t))\mathrm{d}t = \rho_{h_2}(X).\end{aligned}$$

The equality holds due to $\mathbb{P}(X > 0) \leqslant x_0/2$, which implies that $h_1 \,\square\, h_2(g_1(t) + g_2(t)) = h_2(g_1(t) + g_2(t))$ for $t \geqslant 0$. The second inequality follows from (4.20) and convexity of h_2. Thus, it follows that $\rho_{h_1} \,\square\, \rho_{h_2}(X) \geqslant \rho_{h_2}(X)$ for any $X \in \mathcal{X}$ with $\mathbb{P}(X > 0) \leqslant x_0/2$. Therefore, the desired result is obtained.

(iii) Following from (ii), it suffices to show that the result holds for $\alpha = x_d$. For any allocation (X_1, X_2) of $X \in \mathcal{X}$, it holds that

$$\int_0^\infty h_1(\mathbb{P}(X_1 > t))\mathrm{d}t + \int_0^\infty h_2(\mathbb{P}(X_2 > t))\mathrm{d}t$$

$$\geqslant \int_0^\infty \big(\mathbb{P}(X_1 > t) + h_2(\mathbb{P}(X > t)) + h_{2+}'(\mathbb{P}(X > t))(\mathbb{P}(X_2 > t) - \mathbb{P}(X > t))\big)\,\mathrm{d}t. \quad (4.22)$$

The inequality holds due to concavity of h_1 and convexity of h_2, implying that $h_1(x) \geqslant x$ and $h_2(y) \geqslant h_2(x) + h_{2+}'(x)(y-x)$ for any $x, y \in [0,1]$; see Theorem 25.1 of Rockafellar (1970). Next we show it always holds that $h_{2+}'(\mathbb{P}(X > t)) \leqslant 1$. If $x_d = 0$, then h_2 is the identity function and $h_{2+}'(x) = 1$. If $x_d > 0$, then $h_{2+}'(\mathbb{P}(X > t)) \leqslant 1$ since $\mathbb{P}(X > 0) \leqslant x_d$. By the fact of $\mathbb{P}(X_2 > t) \leqslant \mathbb{P}(X > t)$ for $t > 0$ and (4.22), we have

$$\begin{aligned}\rho_{h_1}(X_1) + \rho_{h_2}(X_2) &\geqslant \int_0^\infty (h_2(\mathbb{P}(X > t)) + \mathbb{P}(X_1 > t) + \mathbb{P}(X_2 > t) - \mathbb{P}(X > t))\,\mathrm{d}t \\ &= \int_0^\infty h_2(\mathbb{P}(X > t))\mathrm{d}t = \rho_{h_2}(X).\end{aligned}$$

The above result implies that $\rho_{h_1} \square \rho_{h_2}(X) = \rho_{h_2}(X)$ with $\mathbb{P}(X > 0) \leqslant x_d$. Consequently, the desired result is obtained. □

Proposition 4.4 shows that when the loss probability is sufficiently small, the efficient arrangement assigns the entire risky slice to the risk-seeking agent. Intuitively, small-probability losses are nearly "free" under a convex distortion function, so letting the risk-seeker absorb them minimizes the total risk value. Moreover, if the condition $h_1 \geqslant \tilde{h}_2$ holds, the relative strength of risk aversion over risk seeking is strong enough that the optimal arrangement assigns the entire risk (not just the small layer) to the risk-seeking agent; see Theorem 4.2.

4.6 Applications to the original problem

In this section, we return to the n-agent risk-sharing problem and demonstrate how the results established in Sections 4.4 and 4.5 can be applied to characterize optimal allocations in mixed economies, including both risk-averse and risk-seeking agents. Recall that $g_1 = \bigwedge_{i=1}^m h_i$ and $g_2 = \square_{j=m+1}^n h_j$ for $m \leqslant n$.

Proposition 4.5. *Let $\mathcal{X} = L^+$. Assume that $h_i \in \mathcal{H}$ are continuous for all $i \in [n]$, with h_i concave for $i \in [m]$ and h_j convex for $j \in [n] \setminus [m]$, where $m \leqslant n$. If $g_1 \geqslant \widetilde{h}_i$ for some $i \in [n] \setminus [m]$, then for any $X \in \mathcal{X}^\perp$, we have*

$$\square_{i=1}^{n} \rho_{h_i}(X) = \rho_{g_2}(X).$$

Proof. We first note that the inf-convolution $\square_{i=1}^{n} \rho_{h_i}(X)$ is associative; see Lemma 2 of Liu et al. (2020). Without loss of generality, we assume $g_1 \geqslant \widetilde{h}_{m+1}$. By Theorem 4.3, we have $\rho_{g_1} \square \rho_{h_{m+1}}(X) = \rho_{h_{m+1}}(X)$. Thus,

$$\square_{i=1}^{n} \rho_{h_i}(X) = \rho_{g_1} \square \rho_{h_{m+1}} \square \ldots \square \rho_{h_n}(X) = \rho_{g_2}(X).$$

Therefore, the desired result is obtained. □

Proposition 4.5 states that if the least risk-averse agent in the cautious group is more conservative than at least one risk-seeking agent is adventurous (formally, if $g_1 \geqslant \widetilde{h}_i$ for some convex h_i), then the efficient split assigns all the randomness to the risk-seeking side. Intuitively, once there exists a single risk-seeker willing to absorb the entire uncertain part when they are sharing the risk with the most tolerant risk-averse agent, the problem effectively turns into a "betting game" among the risk-seeking agents. In this case, the optimal allocation is counter-monotonic, that is, risk-averse agents bear nothing and risk-seekers bet on who takes the total risk.

When such dominance condition fails and the risk seeking dominates risk aversion, the optimal risk sharing would change accordingly. As we know, a full analysis with arbitrary distortion functions is challenging, so we focus on a tractable subclass of piecewise linear distortions. Building on the two-agent analysis in Section 4.5, the following proposition provides explicit formulas for the n-agent inf-convolution across different settings.

Proposition 4.6. *Suppose that $\mathcal{X} = L^+$ and $m, n \in \mathbb{N}$. Let $h_i(x) = \min\{x/b_i, 1\}$ for $i \in [m]$ and $h_i(x) = \max\{0, (x - a_i)/(1 - a_i)\}$ for $i \in [n] \setminus [m]$ over $x \in [0, 1]$, where*

$a_i, b_i \in (0,1)$ *with* $\sum_{i=m+1}^{n} a_i \leqslant 1$. *Denote by* $b = \bigvee_{i=1}^{m} b_i$. *Then the following hold for* $X \in \mathcal{X}^{\perp}$.

(i) If $a_i + b \leqslant 1$ *for some* $i \in [n]\backslash[m]$, *then*

$$\mathop{\square}_{i=1}^{n} \rho_{h_i}(X) = \rho_g(X), \text{ where } g(t) = \max\left\{0, \frac{t - \sum_{i=m+1}^{n} a_i}{1 - \bigvee_{i=m+1}^{n} a_i}\right\}, \ t \in [0,1].$$

(ii) If $\bigwedge_{i=m+1}^{n} a_i + b > 1$, *then*

$$\mathop{\square}_{i=1}^{n} \rho_{h_i}(X) = \rho_g(X), \text{ where } g(t) = \max\left\{0, \frac{t - \sum_{i=m+1}^{n} a_i}{b}\right\}, \ t \in [0,1].$$

Proof. Let $\ell(t) = \max\left\{0, (t - \sum_{i=m+1}^{n} a_i)/(1 - \bigvee_{i=m+1}^{n} a_i)\right\}$. It is straightforward to verity that $\square_{i=m+1}^{n} h_i(x) = \ell(x)$.

(i) The result directly follows from Proposition 4.5.

(ii) To make presentations more precise, we first introduce some notations, define

$$\ell_1(x) = \max\left\{0, \frac{x - \sum_{i=m+1}^{n} a_i}{1 - \sum_{i=m+1}^{n} a_i}\right\}, \text{ and } \ell_2(x) = \min\left\{\frac{x}{1 - \bigvee_{i=m+1}^{n} a_i}, 1\right\} \ t \in [0,1].$$

Then it can be verified that $\ell_1 \square \ell_2(x) = \ell(x)$ for $x \in [0,1]$. Note that $\bigwedge_{i=m+1}^{n} a_i + b > 1$ implies that $\sum_{i=m+1}^{n} a_i + b > 1$. By Proposition 4.2, we have

$$\rho_{\ell_1} \square \rho_{\ell_2}(X) = \rho_{\ell}(X). \tag{4.23}$$

Let $h = \bigwedge_{i=1}^{m} h_i$. It follows that

$$\begin{aligned}\mathop{\square}_{i=1}^{n} \rho_{h_i}(X) &= \rho_h \square \left(\mathop{\square}_{i=m+1}^{n} \rho_{h_i}\right)(X) = \rho_h \square \rho_{\ell}(X) = \rho_h \square (\rho_{\ell_1} \square \rho_{\ell_2})(X) \\ &= \rho_h \square \rho_{\ell_1}(X) = \rho_{\ell}(X),\end{aligned}$$

where $\ell(t) = h \,\square\, \ell_1(t) = \max\left\{0, (t - \sum_{i=m+1}^{n} a_i)/b\right\}$. The first equality follows from Theorem 4.2. By applying (4.23) and Proposition 4.2, we obtain the last three equalities. $\square$

As shown in Proposition 4.6, when risk aversion dominates sufficiently, the risk-seeking group absorbs all randomness, consistent with Proposition 4.5. In contrast, when risk seeking dominates risk aversion, as described by condition (ii), the pattern of optimal risk sharing changes substantially. The risk-averse participants continue to share risk comonotonically within their own group, but they would collectively bet with the risk-seeking group. The resulting allocation is counter-monotonic across groups, reflecting a betting structure in which the risk-seeking side and the risk-averse side effectively compete over who absorbs the uncertainty.

4.7 Conclusion

In this chapter, we study risk sharing in economies where agents need not be all risk-averse or all risk-seeking. This mixed setting is economically important with capturing the coexistence of cautious and speculative participants, and mathematically challenging because the underlying distortion risk measures are neither all convex nor all concave.

With mixed attitudes in place (some are risk-averse and others are risk-seeking), we establish a two-agent reduction by aggregating risk attitudes on each side into representative distortions; see Theorem 4.1. The reduced problem can be solved via the inf-convolution $\rho_{h_1} \,\square\, \rho_{h_2}$. One of our main results (Theorem 4.2) provides necessary and sufficient conditions for existence of optimal allocations. When the conditions hold, the concave-convex case admits explicit solutions, in which the risk-seeking side bears all the risk, as shown in Theorem 4.3.

We also consider the case of restricted allocations that are nonnegative. This setting ensure a well-defined risk sharing problem and corresponds to the natural assumption of no profit from pure losses. We compare the constrained value $\rho_{h_1} \,\square\, \rho_{h_2}(X)$ with the

benchmark $\rho_{h_1 \square h_2}(X)$. Our analysis yields constructive bounds and equality conditions for specific distortion families (Theorems 4.4 and Proposition 4.2) and for canonical risks (Theorems 4.5 and Proposition 4.3). Finally, we apply these results to resolve the original multi-agent problem.

4.8 Omitted details in Example 4.8

The details in Example 4.8 are guaranteed by the following proposition.

Proposition 4.7. *Suppose that* $h_1(x) = 1 - (1-x)^\alpha$ *and* $h_2(x) = x^\beta$ *for* $x \in [0,1]$ *with* $1 < \alpha < \beta$. *Let* $p_0 = (\alpha/\beta)^{\frac{1}{\beta-1}}$. *Then there exists* $y^*(x)$ *such that* $h_1(y^*(x)) + h_2(x - y^*(x)) = h_1 \square h_2(x)$. *Moreover,*

(i) If $x \leqslant p_0$, $y^*(x) = 0$ *and* $x - y^*(x) = x$.

(ii) If $x > p_0$, $y^*(x)$ *satisfies that* $\alpha(1-y^*(x))^{\alpha-1} = \beta(x-y^*(x))^{\beta-1}$. *Moreover,* $y^*(x)$ *is increasing and* $x - y^*(x)$ *is decreasing.*

Proof. Let $g(y) = h_1(y) + h_2(x-y)$ for $x \in [0,1]$ and $y \in [0,x]$. Then

$$g'(y) = h_1'(y) - h_2'(x-y) = \alpha(1-y)^{\alpha-1} - \beta(x-y)^{\beta-1}.$$

Let $R(y) = \frac{\alpha(1-y)^{\alpha-1}}{\beta(x-y)^{\beta-1}}$. By straightforward calculation, we have

$$\ln R(y) = \ln\alpha - \ln\beta + (\alpha-1)\ln(1-y) - (\beta-1)\ln(x-y) \text{ and } (\ln R(y))' = \frac{\beta-1}{x-y} - \frac{\alpha-1}{1-y}.$$

Since $\alpha < \beta$, $(\ln R(y))' > 0$, thus $\ln R(y)$ is increasing. Also, $\ln R(0) = \ln\alpha - \ln\beta - (\beta - 1)\ln x$. Next, we consider two cases:

(i) If $x \leqslant p_0$, then $\ln R(y) \geqslant \ln R(0) \geqslant 0$ for $y \in [0,x]$. This implies that $R(y) \geqslant 1$ and $g'(y) \geqslant 0$, thus $g(y)$ is increasing. Hence, the infimum of $g(y)$ over $y \in [0,x]$ attains at $y^* = 0$.

(ii) If $x > p_0$, then $\ln R(0) < 0$ and $\ln R(y) \to +\infty$ as $y \downarrow x$. Thus, $g'(y)$ would be first less than zero and then cross zero. Then first order condition gives:

$$g'(y^*) = h_1'(y^*) - h_2'(x - y^*) = \alpha(1 - y^*)^{\alpha-1} - \beta(x - y^*)^{\beta-1} = 0. \tag{4.24}$$

Write $F(x, y^*) = \alpha(1 - y^*)^{\alpha-1} - \beta(x - y^*)^{\beta-1}$. Then it follows that

$$\frac{\mathrm{d}y^*}{\mathrm{d}x} = -\frac{\partial F/\partial x}{\partial F\ \partial y} = \frac{\beta(\beta-1)(x-y^*)^{\beta-2}}{\beta(\beta-1)(x-y^*)^{\beta-2} - \alpha(\alpha-1)(1-y^*)^{\alpha-2}} = \frac{1}{1 - \frac{\alpha-1}{\beta-1}\frac{x-y^*}{1-y^*}} > 0.$$

The last equality can be obtained from (4.24). By straightforward calculation, we have

$$\frac{\mathrm{d}(x-y^*)}{\mathrm{d}x} = 1 - \frac{\mathrm{d}y^*}{\mathrm{d}x} = \frac{-(\alpha-1)(x-y^*)}{(\beta-1)(1-y^*) - (\alpha-1)(x-y^*)} < 0.$$

Consequently, the desired result is obtained. □

4.9 Basic properties of the constrained inf-convolution

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be an atomless probability space and $\mathcal{X} = L^+$ be the set of nonnegative random variables in this space. For risk functionals $\rho_1, \rho_2 : \mathcal{X} \mapsto [0, \infty)$, define the constrained inf-convolution

$$\rho_1 \,\square\, \rho_2(X) = \inf\left\{\rho_1(Y) + \rho_2(X - Y), 0 \leqslant Y \leqslant X\right\}, \quad X \in \mathcal{X}.$$

The constraint $0 \leqslant Y \leqslant X$ ensures $Y, X - Y \in \mathcal{X}$. Here we record several properties that can pass from ρ_1 and ρ_2 to their constrained inf-convolution $\rho_1 \,\square\, \rho_2$. Our focus in this section is the constrained formulation. The corresponding properties in the unconstrained case have been well studied in Liu et al. (2020).

Lemma 4.2. *Suppose that $\mathcal{X} = L^+$ and $X \in \mathcal{X}$. Let ρ_1, ρ_2 be two risk functionals.*

(i) *If ρ_1 and ρ_2 are monotone, then $\rho_1 \square \rho_2$ is monotone.*

(ii) *If ρ_1 and ρ_2 are uniformly continuous, then $\rho_1 \square \rho_2$ is uniformly continuous.*

(iii) *If ρ_1 and ρ_2 are monotone and one of them is continuous from above, then $\rho_1 \square \rho_2$ is continuous from above.*

Proof. (i) Suppose that $X, Y \in \mathcal{X}$ with $X \leqslant Y$. For any $\varepsilon > 0$, there exists $Z_Y \in \mathcal{X}$ such that $0 \leqslant Z_Y \leqslant Y$ and

$$\rho_1(Z_Y) + \rho_2(Y - Z_Y) \leqslant \rho_1 \square \rho_2(Y) + \varepsilon.$$

Let $\widetilde{Z} = Z_Y \wedge X$ (so $0 \leqslant \widetilde{Z} \leqslant X$). Thus, $\widetilde{Z} \leqslant Z_Y$ and $X - \widetilde{Z} \leqslant Y - Z_Y$. Then

$$\rho_1 \square \rho_2(X) \leqslant \rho_1(\widetilde{Z}) + \rho_2(X - \widetilde{Z}) \leqslant \rho_1(Z_Y) + \rho_2(Y - Z_Y) \leqslant \rho_1 \square \rho_2(Y) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we obtain $\rho_1 \square \rho_2(X) \leqslant \rho_1 \square \rho_2(Y)$.

(ii) Since ρ_1 and ρ_2 are uniformly continuous, for any $\varepsilon > 0$, there exists δ_i for $i = 1, 2$ such that for all $X, Y \in \mathcal{X}$, $\|X - Y\| \leqslant \delta_i$ implies $|\rho_i(X) - \rho_i(Y)| \leqslant \varepsilon/3$. By definition of $\rho_1 \square \rho_2$, there exists $Z_X \in [0, X]$ such that $\rho_1(Z_X) + \rho_2(X - Z_X) \leqslant \rho_1 \square \rho_2(X) + \varepsilon/3$. Let $Z' = Z_X \wedge Y$ and $\delta = \min\{\delta_1, \delta_2/2\}$. For any $X, Y \in \mathcal{X}$ with $\|X - Y\| \leqslant \delta_1$, we have

$$\|Z' - Z_X\| \leqslant \|X - Y\| \leqslant \delta \text{ and } \|(X - Z_X) - (Y - Z')\| \leqslant \|X - Y\| + \|Z' - Z_X\| \leqslant \delta_2.$$

Then it follows that

$$\begin{aligned}
&|\rho_1 \square \rho_2(X) - \rho_1 \square \rho_2(Y)| \\
&\leqslant |\rho_1(Z_X) + \rho_2(X - Z_X) - \rho_1(Z') - \rho_2(Y - Z')| + \varepsilon/3 \\
&\leqslant |\rho_1(Z_X) - \rho_1(Z')| + |\rho_2(X - Z_X) - \rho_2(Y - Z')| + \varepsilon/3 \leqslant \varepsilon.
\end{aligned}$$

Hence, $\rho_1 \square \rho_2$ is uniformly continuous.

(iii) Without loss of generality, we assume ρ_2 is continuous from above. By part (i), we have $\rho_1 \square \rho_2$ is monotone. Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{X}$ such that $X_n \downarrow X$ as $n \to \infty$. For any $\varepsilon > 0$ and choose $Z \in \mathcal{X}$ with $0 \leqslant Z \leqslant X$ such that

$$\rho_1(Z) + \rho_2(X - Z) \leqslant \rho_1 \square \rho_2(X) + \varepsilon.$$

Since $X_n \downarrow X$ and $Z \leqslant X$, we have $Z \leqslant X_n$ for all n. Hence,

$$\limsup_{n\to\infty} \rho_1 \square \rho_2(X_n) \leqslant \rho_1(Z) + \limsup_{n\to\infty} \rho_2(X_n - Z) = \rho_1(Z) + \rho_2(X - Z).$$

Thus, $\limsup_{n\to\infty} \rho_1 \square \rho_2(X_n) \leqslant \rho_1 \square \rho_2(X)$. On the other hand, since $\rho_1 \square \rho_2$ is monotone, we have $\rho_1 \square \rho_2(X_n) \geqslant \rho_1 \square \rho_2(X)$. This implies

$$\lim_{n\to\infty} \rho_1 \square \rho_2(X_n) = \rho_1 \square \rho_2(X).$$

Hence, $\rho_1 \square \rho_2$ is continuous from above.

The desired result is obtained. □

Lemma 4.2 shows that monotonicity of $\rho_1 \square \rho_2$ under nonnegative allocations requires both ρ_1 and ρ_2 to be monotone. For general allocation sets, one monotone component is sufficient, as shown in Lemma 1 of Liu et al. (2020). The example below demonstrates that this sufficiency fails when allocations are constrained to be nonnegative.

Example 4.10. Take $(\Omega, \mathcal{F}, \mathbb{P}) = ([0,1], \mathcal{B}([0,1]), \lambda)$ where λ is the Lebesgue measure. Let $B \subset [0,1]$ satisfy $\lambda(B) = \frac{1}{2}$. Define two risk functionals $\rho_1(X) = 1 - \mathbb{1}_{\{X \sim \text{Bernoulli}(\frac{1}{2})\}}$ and $\rho_2(X) = \mathbb{E}[X]$. Set

$$X = c\mathbb{1}_B \quad \text{with } c \in (0,1) \quad \text{and} \quad Y = \mathbb{1}_B.$$

Clearly $0 \leqslant X \leqslant Y$. Any feasible $0 \leqslant Z \leqslant X$ satisfies $Z = 0$ on B^c and $0 \leqslant Z \leqslant c < 1$ on

B. Hence $\rho_1(Z) = 1$ for all $0 \leqslant Z \leqslant X$. Therefore,

$$\rho_1 \,\square\, \rho_2(X) = \rho_1(X) + \rho_2(0) = 1 > \rho_1(Y) + \rho_2(0) = 0 \geqslant \rho_1 \,\square\, \rho_2(Y).$$

In this case, $\rho_1 \,\square\, \rho_2$ is not monotone.

Chapter 5

Pareto optimality for distortion risk measures beyond risk aversion

5.1 Introduction

Risk sharing is a central problem in actuarial science, mathematical finance and decision theory. A group of agents facing an aggregate loss seeks to divide it into individual positions so that no agent can be made better off without making another agent worse off. This leads naturally to the notion of Pareto optimality. The set of attainable risk vectors is called the risk possibility set, and its efficient boundary is the Pareto frontier. A standard way to obtain Pareto-optimal allocations is to minimize a weighted sum of the agents' individual risk values, where the weights represent the relative importance assigned to the agents in the social objective. When all weights are strictly positive, any optimizer of such a weighted-sum problem is Pareto optimal. The converse direction is more complicated. Since smaller risk values are preferred, the relevant geometric object is the risk possibility set together with all componentwise worse risk vectors. If this upper set is convex, supporting-hyperplane arguments imply that the full Pareto frontier can be recovered by varying the weights. The main goal of this chapter is to investigate whether this convexity

property holds in constrained risk sharing problems under distortion risk measures, since this determines whether weighted-sum methods can recover the full constrained Pareto frontier.

Weighted-sum methods are standard tools for studying Pareto optimality. This idea underlies much of the classical risk-sharing literature, beginning with the expected-utility results of Borch (1962) and Wilson (1968). In insurance design, related Pareto and weighted-objective methods appear in the classical deductible results of Arrow (1963), Mossin (1968), and Raviv (1979). More recent work studies Pareto efficiency in centralized and decentralized insurance markets, including centralized flood-risk insurance markets, peer-to-peer insurance, and robust distortion-risk-measure models (Abdikerimova and Feng, 2022; Denuit, 2020; Feng et al., 2023; Boonen et al., 2024; Ghossoub et al., 2025).

For monetary utilities and convex risk measures, this approach is closely connected to inf-convolution. The inf-convolution corresponds to the equal-weight case and computes the smallest aggregate risk value attainable by decomposing the total risk among the agents. This approach has been developed in the risk-sharing literature by Gerber (1978), Barrieu and El Karoui (2005), Jouini et al. (2008), Filipović and Svindland (2008), and Acciaio (2007). Further existence and structural results have been obtained for law-invariant monetary utilities, non-monotone monetary functionals, quasi-convex or quasi-concave risk measures, and SSD-preserving utility functionals; see, for instance, Heath and Ku (2004), Tsanakas (2009), Mastrogiacomo and Rosazza Gianin (2015), Ravanelli and Svindland (2014), Mao and Wang (2020), Liebrich (2024), and Ghossoub and Zhu (2026). In risk-averse settings, these optimization methods are closely connected to comonotonicity, following the comonotonic improvement theorem of Landsberger and Meilijson (1994) and its extensions by Dana et al. (2003), Ludkovski and Rüschendorf (2008), Carlier et al. (2012), and Denuit et al. (2025).

Weighted-sum and inf-convolution methods also play an important role beyond the convex risk-measure setting. For quantile-based and tail-risk measures, Pareto-optimal risk sharing has been studied by Embrechts et al. (2018), Embrechts et al. (2020), Liu et al.

(2022), and Weber (2018). In contrast to the risk-averse case, optimal allocations in such models may exhibit counter-monotonic or non-comonotonic structures. Recent work by Lauzier et al. (2023) and Lauzier et al. (2026a) develop a counter-monotonic improvement theorem for risk-seeking agents, while Ghossoub et al. (2026a) and Ghossoub et al. (2026b) study counter-monotonic risk sharing under homogeneous and heterogeneous distortion risk measures. These results show that beyond universal risk aversion, the geometry of the attainable risk set and the structure of optimal allocations can differ substantially from the classical comonotonic case.

In the unconstrained setting on L^∞, the weighted-sum characterization simplifies further for monetary risk measures. As shown by Gerber (1978) and developed in Barrieu and El Karoui (2005), Jouini et al. (2008), Filipović and Svindland (2008), Acciaio (2007), and Embrechts et al. (2018), translation invariance gives a particularly simple Pareto structure. Deterministic side payments can be exchanged among agents without changing the aggregate loss, so once the minimum total risk value is attained, this value can be redistributed through constants. The Pareto frontier is therefore flat, and Pareto optimality is characterized by the inf-convolution of the agents' risk measures.

The setting of this chapter is different in two respects. First, we work with distortion risk measures generated by general distortion functions, without imposing concavity or convexity. Thus, agents need not be risk averse or risk seeking, and the usual convexity arguments may not apply. Second, we impose feasibility constraints on allocations. A typical requirement is that a positive aggregate loss should be shared through nonnegative individual allocations. Such a constraint is natural in insurance and reinsurance, where a risk-transfer contract should not generate gains from realized losses or assign more than the total loss to one party. Moreover, as shown in Chapter 4, the unconstrained inf-convolution may be unbounded even when the aggregate risk is bounded, which provides an additional motivation for studying constrained allocation problems. These constraints change the geometry of the Pareto frontier. In the unconstrained L^∞ setting, translation invariance allows deterministic side payments to move allocations along a flat frontier. Under nonnegativity constraints, however, such side payments may violate the bounds,

and the flat-frontier simplification no longer applies. The constrained risk possibility set therefore becomes the central object of study.

The main contributions of the chapter are summarized as follows. First, we identify structural conditions under which Pareto optimality can still be characterized by weighted sum optimality. Beyond the classical risk-averse setting, Theorem 5.1 shows that such characterization also holds for a group of risk-seeking agents under mild conditions. Second, we show that Pareto-optimal allocations can be found through a one-parameter family of constrained problems, as shown in Theorem 5.2. The method remains valid beyond the risk-averse setting and provides a direct way to study Pareto optimality when weighted-sum method may fail to capture the full frontier. The third contribution is an explicit analysis of Bernoulli and two-point aggregate risks. We characterize both the Pareto frontier and the structure of the allocations that attain it. For Bernoulli risks, Theorem 5.3 shows that the frontier is the lower convex envelope of an explicit one-dimensional function, and that every frontier point admits a three-atom optimizer. For two-point aggregate risks, Theorem 5.4 and Proposition 5.1 provide finite-atom representations of Pareto-optimal allocations. Although a complete convex-envelope formula is not obtained, finite-atom representations provide a tractable structural description of efficient allocations and a step toward a fuller characterization of the constrained frontier.

5.2 Pareto optimality and distortion risk measures

All random variables are defined on an atomless probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{X}$ be a convex cone of random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. We write L^0 for the set of all real-valued random variables, L^∞ for the bounded ones and L^+ for the nonnegative bounded ones. Let $\mathcal{X}^\perp$ be the set of all random variables $X \in \mathcal{X}$ such that there exists a standard uniform random variable U independent of X. All equalities and inequalities between vectors are understood componentwise. Let $n \in \mathbb{N}$ be a positive integer and $[n] = \{1, \dots, n\}$.

Let $\mathcal{H}$ be the set of all distortion functions, that is,

$$\mathcal{H} = \{h : [0,1] \to [0,1] \mid h \text{ is increasing, } h(0) = 0,\ h(1) = 1\}.$$

For $h \in \mathcal{H}$, the corresponding distortion risk measure $\rho_h : \mathcal{X} \to \mathbb{R}$ is the Choquet integral

$$\rho_h(X) = \int X \mathrm{d}h \circ \mathbb{P} = \int_0^\infty h(\mathbb{P}(X > x))\mathrm{d}x + \int_{-\infty}^0 (h(\mathbb{P}(X > x)) - h(1))\mathrm{d}x.$$

We recall several notions used throughout the chapter. For a risk measure $\rho : L^\infty \to \mathbb{R}$, we say that ρ is convex if for all $\lambda \in [0,1]$ and $X, Y \in L^\infty$, $\rho(\lambda X + (1-\lambda)Y) \leqslant \lambda\rho(X) + (1-\lambda)\rho(Y)$; ρ is law-invariant if for $X, Y \in L^\infty$ that have the same distribution under $\mathbb{P}$, $\rho(X) = \rho(Y)$. For a law-invariant ρ, define $\tilde{\rho} : \mathcal{M} \to \mathbb{R}$ by $\tilde{\rho}(F) = \rho(X)$, where $\mathcal{M}$ is the set of distribution functions of $X \in L^\infty$ and $X \sim F$, we say that ρ is concave on mixtures if $\tilde{\rho}$ is concave. The following result connects the shape of the distortion function with convexity properties of the associated distortion risk measure. Further equivalent conditions can be found in Wang et al. (2020b).

Lemma 5.1 (Wang et al. (2020b, Theorem 3))**.** *For $h \in \mathcal{H}$, the following are equivalent: (i) h is concave; (ii) ρ_h is convex; (iii) ρ_h is concave on mixtures.*

We next introduce the constrained allocation set. Let $\mathcal{V} \subseteq (L^0)^n$ be a set of random vectors. For an aggregate risk X, define

$$\mathbb{A}_n^{\mathcal{V}}(X) = \left\{ (X_1, \dots, X_n) \in \mathcal{V} : \sum_{i=1}^n X_i = X \right\}.$$

Thus, $\mathbb{A}_n^{\mathcal{V}}(X)$ is the set of feasible allocations of X within the constraint set $\mathcal{V}$. We also write $\overline{\mathcal{V}}_n = \{\sum_{i=1}^n X_i : (X_1, \dots, X_n) \in \mathcal{V}\}$ and $\overline{\mathcal{V}}_n^{\perp} = \overline{\mathcal{V}}_n \cap \mathcal{X}^{\perp}$. Note that $\overline{\mathcal{V}}_n$ is the set of random variables X such that $\mathbb{A}^{\mathcal{V}}(X)$ is nonempty. We say that $\mathcal{V}$ is *law invariant* if $\mathbf{X} \in \mathcal{V}$ implies $\mathbf{Y} \in \mathcal{V}$ for all $\mathbf{Y} \in (L^0)^n$ with $\mathbf{Y} \overset{\mathrm{d}}{=} \mathbf{X}$; $\mathcal{V}$ is *mixture-closed* if it is law invariant and

for $\mathbf{X}, \mathbf{Y} \in \mathcal{V}$ and $A \in \mathcal{F}$ independent of $(\mathbf{X}, \mathbf{Y})$, we have

$$\mathbb{1}_A \mathbf{X} + \mathbb{1}_{A^c} \mathbf{Y} \in \mathcal{V}.$$

In all commonly used models in risk sharing, $\mathcal{V}$ is convex and mixture-closed. For instance, for $p \in [0, \infty]$, the sets $(L^p)^n$ and $(L^p_+)^n$ are convex and mixture-closed. Moreover, the set of all random vectors taking values in $[a, b]^n$ for some $a, b \in \mathbb{R}$ is convex and mixture-closed. An allocation $(X_1, \dots, X_n) \in \mathbb{A}^{\mathcal{V}}(X)$ is *Pareto optimal within* $\mathcal{V}$ if for any $(Y_1, \dots, Y_n) \in \mathbb{A}^{\mathcal{V}}(X)$,

$$\rho_{h_i}(Y_i) \leqslant \rho_{h_i}(X_i) \text{ for all } i \in [n] \implies \rho_{h_i}(Y_i) = \rho_{h_i}(X_i) \text{ for all } i \in [n].$$

That is, no feasible allocation can reduce every agent's risk, with at least one strict reduction. Let Δ_n be the unit simplex in $\mathbb{R}^n$, that is, $\Delta_n = \{(w_1, \dots, w_n) \in [0,1]^n : \sum_{i=1}^n w_i = 1\}$. For $\mathbf{w} = (w_1, \dots, w_n) \in \Delta_n$ and $X \in \mathcal{X}$, an allocation $(X_1, \dots, X_n) \in \mathbb{A}^{\mathcal{V}}(X)$ is $\mathbf{w}$-optimal if $\sum_{i=1}^n w_i \rho_{h_i}(X_i) \leqslant \sum_{i=1}^n w_i \rho_{h_i}(Y_i)$ for all $(Y_1, \dots, Y_n) \in \mathbb{A}^{\mathcal{V}}(X)$. Throughout the chapter, when the constraint set $\mathcal{V}$ is fixed, we simply say Pareto optimal and $\mathbf{w}$-optimal, omitting the phrase "within $\mathcal{V}$". Clearly, if $\mathbf{w} \in \Delta_n$ has all components strictly positive, then every $\mathbf{w}$-optimal allocation is Pareto optimal. However, in nonconvex or constrained settings, a Pareto-optimal allocation may not be supported by any weight vector. One of our main goals is to identify when such a supporting weight exists. In the unconstrained setting $\mathcal{V} = (L^\infty)^n$ with monetary risk measures, however, a stronger result is well known. Every Pareto-optimal allocation is $\mathbf{w}$-optimal for the equal weight vector $\mathbf{w} = (1/n, \dots, 1/n)$, as shown in Embrechts et al. (2018).

For $X \in \mathcal{X}$ and $\mathcal{V} \subseteq (L^0)^n$, the *risk possibility set* is defined by

$$\mathrm{RPS}(X) = \left\{ \left(\rho_{h_1}(X_1), \dots, \rho_{h_n}(X_n)\right) : (X_1, \dots, X_n) \in \mathbb{A}_n^{\mathcal{V}}(X) \right\},$$

and its upper risk set is

$$\mathrm{RPS}^{\diamond}(X) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \geqslant \mathbf{y} \text{ for some } \mathbf{y} \in \mathrm{RPS}(X)\}.$$

The *risk possibility frontier*, denote by $\mathrm{RPF}(X)$, is the lower boundary of $\mathrm{RPS}^{\diamond}(X)$, i.e., the set of risk vectors attained by Pareto-optimal allocations. We also refer to $\mathrm{RPF}(X)$ as the Pareto frontier. If $\mathrm{RPS}^{\diamond}(X)$ is convex and closed, then every point on the risk possibility frontier admits a supporting hyperplane. Consequently, every Pareto-optimal allocation whose risk vector lies on this frontier is $\mathbf{w}$-optimal for some $\mathbf{w} \in \Delta_n$.

5.3 Risk-averse agents and risk-seeking agents

In this section, we study when Pareto-optimal allocations can be supported by weighted-sum problems. The key geometric issue is whether the upper risk set is convex. For risk-averse agents, this follows from the standard convex risk measure argument and is therefore included mainly as a benchmark. The key observation is that an analogous supporting-weight result also holds for risk-seeking agents under some mild conditions. The two cases are linked by Lemma 5.1, which gives the convexity property appropriate to each type of distortion function. The following theorem gives the corresponding supporting-weight characterization.

Theorem 5.1. *Let $h_1, \dots, h_n \in \mathcal{H}$ be continuous and $X \in \overline{\mathcal{V}}_n^{\perp}$. Suppose one of the two cases:*

(i) $\mathcal{V}$ is convex and $h_1, \dots, h_n$ are all concave.

(ii) $\mathcal{V}$ is mixture-closed and $h_1, \dots, h_n$ are all convex.

If an allocation $(X_1, \dots, X_n) \in \mathbb{A}^{\mathcal{V}}(X)$ is Pareto optimal within $\mathcal{V}$, then it is $\mathbf{w}$-optimal for some $\mathbf{w} \in \Delta_n$. The converse holds if, in addition, $h_1, \dots, h_n$ are strictly increasing.

Proof. For $\mathbf{X} = (X_1, \dots, X_n) \in \mathbb{A}^{\mathcal{V}}(X)$, let $\mathbf{r}(\mathbf{X}) = (\rho_{h_1}(X_1), \dots, \rho_{h_n}(X_n))$. Write

$$\mathrm{RPS}(X) = \{\mathbf{r}(\mathbf{X}) : \mathbf{X} \in \mathbb{A}^{\mathcal{V}}(X)\}; \quad \mathrm{RPS}^{\diamond}(X) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \geqslant \mathbf{y} \text{ for some } \mathbf{y} \in \mathrm{RPS}(X)\}.$$

It suffices to show that $\mathrm{RPS}^{\diamond}(X)$ is convex and closed, and the desired statement follows by applying the Hahn–Banach Theorem.

To show the convexity of $\mathrm{RPS}^{\diamond}(X)$, we will use two well-known properties: (a) if h is concave, then ρ_h is convex; (b) h is convex, then ρ_h is mixture-convex. For these two statements, see Wang et al. (2020b, Theorem 3). Take $\mathbf{X}, \mathbf{Y} \in \mathbb{A}^{\mathcal{V}}(X)$, a standard uniform random variable U independent of $\mathbf{X}, \mathbf{Y}$, and $\lambda \in [0, 1]$. It suffices to consider allocations $\mathbf{X}$ and $\mathbf{Y}$ such that $(\mathbf{X}, \mathbf{Y})$ and U are independent; see Lauzier et al. (2026a, Lemma 2).

If $h_1, \dots, h_n$ are concave, then due to the convexity of ρ_{h_i} for $i \in [n]$, we have

$$\mathbf{r}(\lambda\mathbf{X} + (1-\lambda)\mathbf{Y}) \leqslant \lambda\mathbf{r}(\mathbf{X}) + (1-\lambda)\mathbf{r}(\mathbf{Y}).$$

If $h_1, \dots, h_n$ are convex, then due to the mixture-convexity of ρ_{h_i} for $i \in [n]$,

$$\mathbf{r}\left(\mathbb{1}_{\{U \leqslant \lambda\}}\mathbf{X} + \mathbb{1}_{\{U > \lambda\}}\mathbf{Y}\right) \leqslant \lambda\mathbf{r}(\mathbf{X}) + (1-\lambda)\mathbf{r}(\mathbf{Y}).$$

Since both $\lambda\mathbf{X} + (1-\lambda)\mathbf{Y}$ and $\mathbb{1}_{\{U \leqslant \lambda\}}\mathbf{X} + \mathbb{1}_{\{U > \lambda\}}\mathbf{Y}$ are in $\mathbb{A}^{\mathcal{V}}(X)$, we conclude that in both cases, $\mathrm{RPS}^{\diamond}(X)$ is convex. The closedness of $\mathrm{RPS}^{\diamond}(X)$ follows from Lauzier et al. (2026a, Lemma 3), which uses continuity of $h_1, \dots, h_n$.

The converse statement is straightforward. □

Theorem 5.1 shows that the supporting-weight representation is not determined only by whether the agents are risk averse or risk seeking. What matters is whether the feasible set remains feasible after applying the operation used to convexify the attainable risk region. In the risk-averse case, convexity of the distortion risk measures is combined with deterministic averaging, so ordinary convexity of $\mathcal{V}$ is sufficient. In the risk-seeking case, the relevant convexification is probabilistic mixing, which requires an external randomization

device. Therefore, mixture-closedness of $\mathcal{V}$ is the natural condition. However, this method relies on structural assumptions on the agents' risk preferences. The supporting-weight argument applies in the two benchmark cases above, where the distortion functions share the same convexity structure. It does not directly apply to mixed-attitude markets, such as the case of one risk-averse agent and one risk-seeking agent. This motivates the direct frontier approach developed in the next section, where Pareto optimality is characterized through a family of constrained optimization problems.

5.4 Equivalent formulations in the case of two agents

In this section, we restrict attention to the two-agent case with $\mathcal{V} = (L^+)^2$ and a non-negative aggregate risk $X \geqslant 0$. Admissible allocations are therefore of the form $(Y, X-Y)$ with $0 \leqslant Y \leqslant X$. The following theorem provides an equivalent formulation of Pareto optimality in the two-agent case, reducing the problem to a one-parameter family of constrained optimization problems.

Theorem 5.2. *Suppose that $h_1, h_2 \in \mathcal{H}$ and h_2 is strictly increasing. The following are equivalent.*

i) $(Y^*, X-Y^*)$ is Pareto optimal in $(L^+)^2$, i.e., $0 \leqslant Y^* \leqslant X$.

ii) There exists some $\lambda \geqslant 0$ such that Y^* solves the problem

$$(\mathcal{P}_\lambda) \quad \inf_{0 \leqslant Y \leqslant X} \{\rho_{h_2}(X-Y) : \rho_{h_1}(Y) \leqslant \lambda\}.$$

iii) There exists some $\lambda \geqslant 0$ such that Y^* solves the problem

$$(\tilde{\mathcal{P}}_\lambda) \quad \inf_{0 \leqslant Y \leqslant X} \{\rho_{h_2}(X-Y) : \rho_{h_1}(Y) = \lambda\}.$$

Proof. *i)* $\Rightarrow$ *ii)*. Suppose that $(Y^*, X-Y^*)$ is Pareto optimal and let $\lambda^* = \rho_{h_1}(Y^*)$. We show that Y^* solves $(\mathcal{P}_{\lambda^*})$. Assume, by contradiction, that Y^* is not optimal for $(\mathcal{P}_{\lambda^*})$.

Then there exists $0 \leqslant Y \leqslant X$ such that $\rho_{h_2}(X-Y) < \rho_{h_2}(X-Y^*)$ and $\rho_{h_1}(Y) \leqslant \rho_{h_1}(Y^*)$, contradicting Pareto optimality of $(Y^*, X-Y^*)$.

ii) $\Rightarrow$ *i)*. Suppose that X^* solves $(\mathcal{P}_\lambda)$ for some $\lambda \geqslant 0$. Then we show that $(Y^*, X-Y^*)$ is Pareto optimal. Assume by contradiction that $(Y^*, X-Y^*)$ is not Pareto optimal, then there exists $0 \leqslant Y \leqslant X$ such that $\rho_{h_2}(X-Y) \leqslant \rho_{h_2}(X-Y^*)$ and $\rho_{h_1}(Y) \leqslant \rho_{h_1}(Y^*)$ with at least one strict inequality. If the first inequality is strict, then it immediately contradicts with optimality of Y^* for $(\mathcal{P}_{\lambda^*})$. Thus, $\rho_{h_2}(X-Y) = \rho_{h_2}(X-Y^*)$ and $\rho_{h_1}(Y) < \rho_{h_1}(Y^*)$. Let

$$M = \|X-Y\|_\infty \text{ and } \varepsilon \in \left(0, \min\left\{1, \frac{\rho_{h_1}(Y^*) - \rho_{h_1}(Y)}{2M}\right\}\right).$$

Note that $M > 0$ since $X \neq Y$ a.s., otherwise by strictly increasing of h_2, we have $X = Y = Y^*$ and $\rho_{h_1}(Y) = \rho_{h_1}(Y^*) = \rho_{h_1}(X)$, contradicting with $\rho_{h_1}(Y) < \rho_{h_1}(Y^*)$. Define $\tilde{Y} = Y + \varepsilon(X-Y)$. Clearly, $0 \leqslant \tilde{Y} \leqslant X$. By Lipschitz continuity of ρ_{h_2}, we have

$$\rho_{h_1}(\tilde{Y}) \leqslant \rho_{h_1}(Y) + \varepsilon M \leqslant \rho_{h_1}(Y) + \frac{\rho_{h_1}(Y^*) - \rho_{h_1}(Y)}{2} < \rho_{h_1}(Y^*) \leqslant \lambda,$$

which implies that $\tilde{Y}$ is feasible for Problem $(\mathcal{P}_\lambda)$. Also, $\rho_{h_2}(X-\tilde{Y}) = (1-\varepsilon)\rho_{h_2}(X-Y) < \rho_{h_2}(X-Y)$, contradicting that Y^* solves $(\mathcal{P}_\lambda)$.

ii) $\Rightarrow$ *iii)*. Suppose that Y^* solves $(\mathcal{P}_{\lambda_0})$ for some $\lambda_0 \geqslant 0$. Set $\lambda^* = \rho_{h_1}(Y^*) \leqslant \lambda_0$. Then Y^* is feasible for $(\tilde{\mathcal{P}}_{\lambda^*})$. Assume by contradiction that Y^* is not optimal for $(\tilde{\mathcal{P}}_{\lambda^*})$. Then there exists $0 \leqslant Y \leqslant X$ such that $\rho_{h_2}(X-Y) < \rho_{h_2}(X-Y^*)$, contradicting Y^* solves $(\mathcal{P}_{\lambda_0})$.

iii) $\Rightarrow$ *ii)* Suppose that Y^* solves $(\tilde{\mathcal{P}}_{\lambda^*})$. Then we show that Y^* also solves $(\mathcal{P}_{\lambda^*})$. Assume, by contradiction, that Y^* is not optimal $(\mathcal{P}_{\lambda^*})$, then there exists $0 \leqslant Y \leqslant X$ such that $\rho_{h_1}(Y) \leqslant \lambda^*$ and $\rho_{h_2}(X-Y) < \rho_{h_2}(X-Y^*)$. If $\rho_{h_1}(Y) = \lambda^*$, this contradicts with optimality of Y^* for $(\tilde{\mathcal{P}}_{\lambda^*})$. Hence, $\rho_{h_1}(Y) < \lambda^*$. Define $\tilde{Y}_t = Y + t(X-Y)$ with $t \in [0,1]$. It is trivial to see that $0 \leqslant \tilde{Y}_t \leqslant X$ and $\rho_{h_1}(\tilde{Y}_t)$ is non-decreasing and continuous. Since $0 \leqslant Y^* \leqslant X$, we have $\rho_{h_1}(X) \geqslant \rho_{h_1}(Y^*) = \lambda^*$. Then there exists $t^* > 0$ such that

$\rho_{h_1}(\tilde{Y}_{t^*}) = \lambda^*$. Define $\tilde{Y} = \tilde{Y}_{t^*}$. Then we have

$$X - \tilde{Y} = X - (Y + t^*(X - Y)) = (1 - t^*)(X - Y).$$

Since $t^* \in (0, 1]$ and $X - Y \geqslant 0$ a.s., we have $0 \leqslant X - \tilde{Y} \leqslant X - Y$ a.s., and in fact $X - \tilde{Y} < X - Y$ on $\{X > Y\}$. If $\mathbb{P}(X > Y) > 0$, strict monotonicity of ρ_{h_2} yields $\rho_{h_2}(X - \tilde{Y}) < \rho_{h_2}(X - Y)$. Combining with $\rho_{h_2}(X - Y) < \rho_{h_2}(X - Y^*)$ gives $\rho_{h_2}(X - \tilde{Y}) < \rho_{h_2}(X - Y^*)$, contradicting optimality of Y^* for $\left(\tilde{\mathcal{P}}_{\lambda^*}\right)$. If instead $\mathbb{P}(X > Y) = 0$, then $Y = X$ a.s., hence $\rho_{h_1}(Y) = \rho_{h_1}(X) \geqslant \rho_{h_1}(Y^*)$, contradicting with $\rho_{h_1}(Y) < \rho_{h_1}(Y^*)$. □

Theorem 5.2 shows that, in the two-agent case, Pareto optimality can be studied through the value function

$$\phi(\lambda) = \inf\{\rho_{h_2}(X - Y) : 0 \leqslant Y \leqslant X,\ \rho_{h_1}(Y) = \lambda\},\ \lambda \in [0, \rho_{h_1}(X)]. \tag{5.1}$$

Indeed, the theorem states that every Pareto-optimal allocation appears as a solution to this problem for some value of λ, and conversely every solution of the constrained problem is Pareto optimal. Thus, in two-agent problems, the graph $\{(\lambda, \phi(\lambda)) : \lambda \in \Lambda(X)\}$ is the lower risk possibility frontier, where $\Lambda(X) = \{\rho_{h_1}(Y) : 0 \leqslant Y \leqslant X\}$. This frontier function ϕ is the main object in the next two sections. For Bernoulli aggregate risks, we compute ϕ explicitly as a convex envelope. For two-point aggregate risks, we use this formulation to prove that Pareto-optimal allocations admit finite-atom representations.

5.5 Sharing a Bernoulli total risk

In this section, we study the case where the aggregate risk follows Bernoulli distribution, $X = \mathbb{1}_A$ for $A \in \mathcal{F}$ with $p = \mathbb{P}(A)$ under the constraint set $\mathcal{V} = (L^+)^2$. By Theorem 5.2, finding the Pareto-optimal allocation reduces to solving the following optimization problem (5.1). Since $0 \leqslant Y \leqslant X = \mathbb{1}_A$, the allocation Y vanishes on A^c and takes values in $[0, 1]$ on

A, so the problem reduces to deciding how the event A is split between the two agents. The following theorem characterizes the Pareto frontier ϕ as the convex envelope of an explicit one-dimensional function, and exhibits an optimal allocation supported on at most three atoms.

Theorem 5.3. *Let $\mathcal{V} = (L^+)^2$ and $X = \mathbb{1}_A$ with $p = \mathbb{P}(A)$. Then for every $\lambda \in [0, h_1(p)]$, we have $\phi(\lambda) = \delta(\lambda)$, where δ is the convex envelope on $[0, h_1(p)]$ of the function*

$$\psi(t) = h_2(p - h_1^{-1}(t)), \qquad t \in [0, h_1(p)]. \tag{5.2}$$

Moreover, for every $\lambda \in [0, h_1(p)]$, there exists an optimal $Y^* \in \mathcal{V}$ with $\rho_{h_1}(Y^*) = \lambda$ and $\rho_{h_2}(X - Y^*) = \delta(\lambda)$ which has the form

$$Y^* = \mathbb{1}_{B_2} + \theta\, \mathbb{1}_{B_1 \setminus B_2}, \tag{5.3}$$

for some measurable sets $B_2 \subseteq B_1 \subseteq A$ and some $\theta \in [0, 1]$.

Proof. Since $0 \leqslant Y \leqslant X = \mathbb{1}_A$, we have $0 \leqslant Y \leqslant 1$ and $Y = 0$ on A^c. Hence,

$$0 \leqslant \rho_{h_1}(Y) \leqslant \rho_{h_1}(\mathbb{1}_A) = h_1(p).$$

Fix $\lambda \in [0, h_1(p)]$. For Y with $0 \leqslant Y \leqslant \mathbb{1}_A$, define $q(t) := \mathbb{P}(Y > t),\ t \in [0, 1]$. Then q is nonincreasing, right-continuous, and $0 \leqslant q(t) \leqslant p,\ t \in [0, 1]$. Take $x(t) = h_1(q(t))$. Then we have $\rho_{h_1}(Y) = \int_0^1 x(t)\, \mathrm{d}t$. Also,

$$\begin{aligned}
\rho_{h_2}(X - Y) &= \int_0^1 h_2(\mathbb{P}(\mathbb{1}_A - Y \geqslant t))\, \mathrm{d}t = \int_0^1 h_2(\mathbb{P}(A \cap \{Y \leqslant 1 - t\}))\, \mathrm{d}t \\
&= \int_0^1 h_2(p - \mathbb{P}(Y > 1 - t))\, \mathrm{d}t = \int_0^1 h_2(p - \mathbb{P}(Y > s))\, \mathrm{d}s = \int_0^1 \psi(x(s))\, \mathrm{d}s
\end{aligned}$$

The third equality holds since $\{Y > 1-t\} \subset A$. Therefore,

$$\phi(\lambda) = \inf\left\{\int_0^1 \psi(x(s))\,ds : \int_0^1 x(s)\,ds = \lambda,\ x : [0,1] \to [0, h_1(p)] \text{ nonincreasing}\right\}.$$

Let δ be the convex envelope of ψ on $[0, h_1(p)]$. Since $\delta \leqslant \psi$, for every admissible x,

$$\int_0^1 \psi(x(s))\,\mathrm{d}s \geqslant \int_0^1 \delta(x(s))\,\mathrm{d}s \geqslant \delta\left(\int_0^1 x(s)\,\mathrm{d}s\right) = \delta(\lambda),$$

where the second inequality follows Jensen inequality. Hence $\phi(\lambda) \geqslant \delta(\lambda)$.

Then we show the opposite direction. Fix $\lambda \in [0, h_1(p)]$. Since δ is the convex envelope of ψ, there exist $x_1, x_2 \in [0, h_1(p)]$ with $x_2 \geqslant x_1$ and $\theta \in [0,1]$ such that

$$\lambda = \theta x_2 + (1-\theta)x_1, \qquad \delta(\lambda) = \theta\psi(x_2) + (1-\theta)\psi(x_1).$$

Let $q_i := h_1^{-1}(x_i), i = 1, 2$. Then $0 \leqslant q_1 \leqslant q_2 \leqslant p$. Choose measurable sets $B_2 \subseteq B_1 \subseteq A$ such that $\mathbb{P}(B_1) = q_2$ and $\mathbb{P}(B_2) = q_1$. Define

$$Y^* = \mathbb{1}_{B_2} + \theta\,\mathbb{1}_{B_1 \setminus B_2}.$$

Then $0 \leqslant Y^* \leqslant \mathbb{1}_A$, so $Y^* \in \mathcal{V}$. Moreover, for $t \in [0,1]$,

$$\mathbb{P}(Y^* > t) = \begin{cases} q_2, & 0 \leqslant t < \theta, \\ q_1, & \theta \leqslant t < 1. \end{cases} \quad \text{and} \quad x(t) = h_1(\mathbb{P}(Y^* > t)) = \begin{cases} x_2, & 0 \leqslant t < \theta, \\ x_1, & \theta \leqslant t < 1. \end{cases}$$

Hence,

$$\rho_{h_1}(Y^*) = \int_0^1 x(t)\,\mathrm{d}t = \theta x_2 + (1-\theta)x_1 = \lambda$$
$$\text{and } \rho_{h_1}(X - Y^*) = \int_0^1 \psi(x(t))\,\mathrm{d}t = \theta\psi(x_2) + (1-\theta)\psi(x_1) = \delta(\lambda),$$

which implies that $\phi(\lambda) \leqslant \delta(\lambda)$. Therefore, we conclude that $\phi = \delta$. □

While Theorem 5.3 characterizes ϕ as the lower frontier of the achievable region in the Bernoulli setting, the same reduction also describes the upper frontier. Define

$$\hat{\phi}(\lambda) = \sup\left\{\rho_{h_2}(X - Y) : 0 \leqslant Y \leqslant X,\ \rho_{h_1}(Y) = \lambda\right\},\ \lambda \in [0, \rho_{h_1}(X)]. \tag{5.4}$$

Equivalently, $\hat{\phi}$ represents the utility frontier under the dual utility interpretation of Yaari (1987).

Corollary 5.1. *Under the assumptions of Theorem 5.3, it holds that* $\hat{\phi}(\lambda) = \hat{\delta}(\lambda)$ *for* $\lambda \in [0, h_1(p)]$*, where* $\hat{\delta}$ *is the concave envelope of the function* ψ *defined in* (5.2)*.*

Proof. The argument of Theorem 5.3 applies immediately after replacing inf by sup, the convex envelope δ by the concave envelope $\hat{\delta}$. □

Theorem 5.3 studies the sharing of a Bernoulli aggregate risk under the constraint that both agents receive nonnegative allocations. It shows that Pareto frontier in this case is convex, which is equivalent to convexity of the corresponding upper risk possibility set $\mathrm{RPS}^{\diamond}(X)$. Consequently, each Pareto-optimal Y^* admits some weight vector $\mathbf{w} \in \Delta_2$ for which Y^* is $\mathbf{w}$-optimal; that is, Y^* minimizes

$$\alpha\,\rho_{h_1}(Y) + (1-\alpha)\,\rho_{h_2}(\mathbb{1}_A - Y), \qquad 0 \leqslant Y \leqslant \mathbb{1}_A,$$

for some weight vector $\mathbf{w} = (\alpha, 1-\alpha) \in \Delta_2$. The minimum value of this problem is the inf-convolution $(\alpha\,\rho_{h_1} \,\square\, (1-\alpha)\,\rho_{h_2})(\mathbb{1}_A)$. Recall that the inf-convolution of two functions $h_1, h_2 : [0,1] \to \mathbb{R}$ is

$$(h_1 \,\square\, h_2)(p) = \inf_{q \in [0,p]} \big\{h_1(q) + h_2(p-q)\big\}, \qquad p \in [0,1].$$

The following corollary provides the explicit form of $(\alpha\,\rho_{h_1} \,\square\, (1-\alpha)\,\rho_{h_2})(\mathbb{1}_A)$.

Corollary 5.2. *Under the assumptions of Theorem 5.3, for every $\alpha \in [0,1]$,*

$$(\alpha\, \rho_{h_1} \,\square\, (1-\alpha)\, \rho_{h_2})(\mathbb{1}_A) = (\alpha\, h_1 \,\square\, (1-\alpha)\, h_2)(p). \tag{5.5}$$

Proof. By definition of the inf-convolution and the constraint $0 \leqslant Y \leqslant \mathbb{1}_A$,

$$\begin{aligned}(\alpha\, \rho_{h_1} \,\square\, (1-\alpha)\, \rho_{h_2})(\mathbb{1}_A) &= \inf_{0 \leqslant Y \leqslant \mathbb{1}_A} \big\{\alpha\, \rho_{h_1}(Y) + (1-\alpha)\, \rho_{h_2}(\mathbb{1}_A - Y)\big\} \\ &= \inf_{\lambda \in [0, h_1(p)]} \big\{\alpha\lambda + (1-\alpha)\, \phi(\lambda)\big\}.\end{aligned}$$

By Theorem 5.3, $\phi = \delta$ on $[0, h_1(p)]$. It is straightforward to show that

$$\inf_{\lambda \in [0, h_1(p)]} \big\{\alpha\lambda + (1-\alpha)\, \delta(\lambda)\big\} = \inf_{\lambda \in [0, h_1(p)]} \big\{\alpha\lambda + (1-\alpha)\, \psi(\lambda)\big\}.$$

Substituting $q = h_1^{-1}(\lambda) \in [0,p]$ and $\psi(\lambda) = h_2(p-q)$,

$$\inf_{\lambda \in [0, h_1(p)]} \big\{\alpha\lambda + (1-\alpha)\, \psi(\lambda)\big\} = \inf_{q \in [0,p]} \big\{\alpha\, h_1(q) + (1-\alpha)\, h_2(p-q)\big\} = (\alpha\, h_1 \,\square\, (1-\alpha)\, h_2)(p).$$

This completes the proof. □

Theorem 5.3 and Corollary 5.2 describe the same Pareto frontier from two dual perspectives. Theorem 5.3 gives the frontier point by point, each risk level λ identifies a frontier point $(\lambda, \phi(\lambda))$ together with an explicit three-atom optimizer. The corollary gives the frontier through its supporting lines: each weight α identifies the supporting line of slope $-\alpha/(1-\alpha)$, with minimum value $(\alpha\, h_1 \,\square\, (1-\alpha)\, h_2)(p)$. Since ϕ is convex, the two views are equivalent, varying α over $[0,1]$ traces out the entire frontier. Furthermore, for every $\lambda \in [0, h_1(p)]$, there exists an optimal Y^* supported on at most three atoms $\{0, \theta, 1\}$ with $\theta \in [0,1]$ and this optimizer can be realized explicitly in the form given in (5.3). This implies that each point on the lower convex envelope can be obtained by splitting the event A into regions on which the one of the agents receives none, part or all of the Bernoulli

loss. This observation motivates the following finite-dimensional numerical problem:

$$\xi(\lambda) = \inf\left\{\rho_{h_2}(X-Y) : 0 \leqslant Y \leqslant X, Y \text{ takes at most three atoms, } \rho_{h_1}(Y) = \lambda\right\}. \quad (5.6)$$

By Theorem 5.3, we expect $\xi(\lambda) = \phi(\lambda)$ for all $\lambda \in [0, h_1(p)]$. The following example numerically illustrates these conclusions and shows the corresponding atom structure of the optimal allocation.

Example 5.1. Let $T(p) = \exp\{-(-\ln p)^\gamma\}$, $\gamma > 0$, denote the Prelec probability weighting function Prelec (1998), which is inverse S-shaped for $\gamma \in (0,1)$, linear for $\gamma = 1$ and S-shaped for $\gamma > 1$. We consider the Bernoulli aggregate risk $X = \mathbb{1}_A$ with $p = \mathbb{P}(A) = 0.35$, and take h_1, h_2 to be Prelec distortion functions with parameters $\gamma_1 = 1.8$ and $\gamma_2 = 0.7$. Figure 5.1 illustrates Theorem 5.3 and Corollary 5.1. It plots the explicit function ψ,

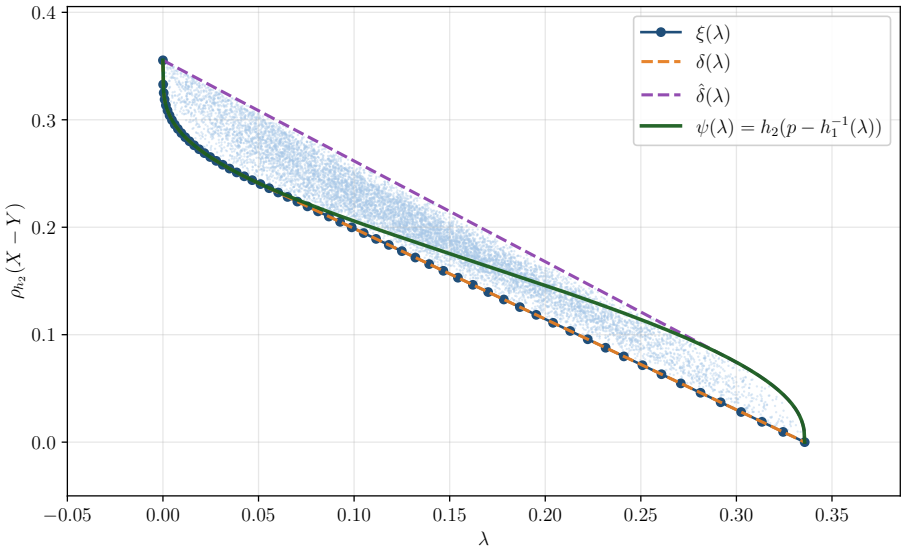


Figure 5.1: Achievable region (blue cloud), the explicit function ψ (green), its convex envelope δ (orange dashed) and concave envelope $\hat{\delta}$ (purple dashed), and the numerical curve ξ (blue) for the Bernoulli risk $X = \mathbb{1}_A$ under the Prelec example with $\gamma_1 = 1.8$ and $\gamma_2 = 0.7$.

together with its theoretical convex envelope δ and concave envelope $\hat{\delta}$. The figure also

displays the feasible region of risk pairs $(\rho_{h_1}(Y), \rho_{h_2}(X-Y))$, generated by randomly sampling admissible allocations satisfying $0 \leqslant Y \leqslant X$. The numerical curve ξ, obtained from the three-atom optimization problem in (5.6), agrees with the convex envelope δ, which also validates the three-atom reduction in this Bernoulli example. Figure 5.2 visualizes

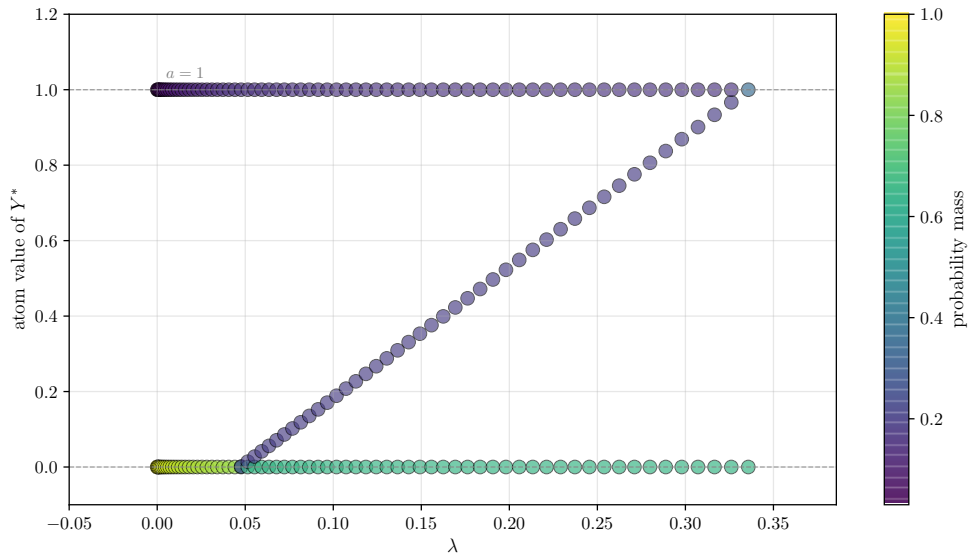


Figure 5.2: Three-atom structure of the optimizer Y_λ^* for the Bernoulli risk $X = \mathbb{1}_A$. Color indicates the probability mass assigned to each atom

the atom structure of the optimizer along the frontier. For each value of λ, the optimal allocation takes at most three values. As λ increases, this intermediate atom moves upward from 0 toward 1, representing a gradual transfer of the Bernoulli loss to Agent 1 with h_1.

5.6 Finite-atom allocations with two-point aggregate risk

In this section, we move beyond the Bernoulli aggregate risk by adding a deterministic layer. Let $X = a\mathbb{1}_A + b$ with $a, b > 0$ and $\mathbb{P}(A) = p \in (0, 1)$. Compared with the Bernoulli

case, the aggregate risk now contains a base layer b, which is present in all states, and an excess layer $a\mathbb{1}_A$, which appears only on the event A. This additional base layer changes the structure of feasible allocations. In the Bernoulli case $X = \mathbb{1}_A$, the constraint $0 \leqslant Y \leqslant \mathbb{1}_A$ forces Y to vanish on A^c, so the allocation is determined entirely by how the event A is split between the two agents. This leads to the convex-envelope formula in Section 5.5. For $X = a\mathbb{1}_A + b$, the allocation Y may split both the base layer and the excess layer. These two parts are coupled through the distortion risk measures, and the constrained optimization problem no longer reduces to the one-dimensional Bernoulli structure. The main result of this section shows that, despite this additional complexity, efficient allocations still admit a finite-layer representation. The structure, however, depends on the relative size of a and b. When $a \geqslant b$, the support interval can be decomposed in a clean ordered way, leading to a sharp seven-atom representation. When $a < b$, the common layer is longer than the excess layer, so the clean separation between the common layer and the event-dependent layer is lost. This makes the reduction less direct and leads to a looser finite-atom bound, which we discuss separately in Section 5.9.

The proof of Theorem 5.4 relies on the following reduction lemma. It states that a componentwise nonincreasing vector-valued function can be replaced by a two-step function with ordered values, while preserving one integral functional and weakly improving another. This is the key reduction that converts the infinite-dimensional optimization into a finite-atom construction. In this sense, the lemma plays the role of a local sweeping principle. It concentrates the variation of a monotone curve onto two ordered values, without worsening the objective.

Lemma 5.2. *Let $L > 0$, and let $e : [0, L] \to [0, 1]^d$ be componentwise nonincreasing. Let $F, G : [0, 1]^d \to \mathbb{R}$ be continuous functions. Then there exists a componentwise nonincreasing lower semicontinuous step function $\widetilde{e} : [0, L] \to [0, 1]^d$ taking at most two vector values such that*

$$\int_0^L F(\widetilde{e}(u))\,\mathrm{d}u = \int_0^L F(e(u))\,\mathrm{d}u, \text{ and } \int_0^L G(\widetilde{e}(u))\,\mathrm{d}u \leqslant \int_0^L G(e(u))\,\mathrm{d}u.$$

Proof. Define $\pi(z) : [0,1]^d \to \mathbb{R}^2$ by $\pi(z) = (F(z), G(z))$. Then we consider the set $\mathcal{E} = \{(\pi(e(u)) : u \in [0,L]\} \subseteq \mathbb{R}^2$ and $S = \overline{\mathcal{E}}$, where $\overline{\mathcal{E}}$ is the closure of $\mathcal{E}$. Clearly, S is compact and closed. Hence, the convex hull of S, denoted by $\operatorname{conv}(S)$, is also compact. Then for a given $e : [0,L] \to [0,1]^d$, we have

$$\left(\frac{1}{L}\int_0^L F(e(u))\,\mathrm{d}u, \frac{1}{L}\int_0^L G(e(u))\,\mathrm{d}u\right) \in \operatorname{conv}(S).$$

Let $m = M/L$ and $\bar{G} = \frac{1}{L}\int_0^L G(e(u))\,\mathrm{d}u$. Consider the vertical slice $C_t = \{y : (t,y) \in \operatorname{conv}(S)\}$ and $g_0 = \min\{y : y \in C_m\}$. Clearly, the set C_m is nonempty since $(m, \bar{G}) \in \operatorname{conv}(S)$. Thus, g_0 is attained and $g_0 \leqslant \bar{G}$. Since the point (m, g_0) lies on the lower boundary of $\operatorname{conv}(S)$, there exist $s^{(1)}, s^{(2)} \in S$ and $\theta \in [0,1]$ such that

$$(m, g_0) = \theta s^{(1)} + (1-\theta)s^{(2)}. \tag{5.7}$$

If $s^{(1)} = s^{(2)}$, then $(m, g_0) = s^{(1)} \in S$.

For each $s^{(i)} \in S$, $i \in [2]$, there exist a sequence $u_n^{(i)} \in [0,L]$ such that $\lim_{n\to\infty} \pi(e(u_n^{(i)})) = s^{(i)}$. Since $[0,L]$ is totally ordered, for each n either $u_n^{(1)} \leqslant u_n^{(2)}$ or $u_n^{(2)} \leqslant u_n^{(1)}$. Without loss of generality, assume that $u_n^{(1)} \leqslant u_n^{(2)}$ for each n. By compactness of $[0,1]^d$, there exists a subsequence $u_{n_k}^{(i)}$ and $z^{(i)} \in [0,1]^d$ such that $e(u_{n_k}^{(i)}) \to z^{(i)}$ for $k \to \infty$. Since both F and G are continuous, π is also continuous, which gives $\pi(z^{(i)}) = s^{(i)}$. Componentwise monotonicity of e leads to $e(u_n^{(1)}) \geqslant e(u_n^{(2)})$, implying that $z^{(1)} \geqslant z^{(2)}$ componentwise. Now define

$$\widetilde{e}(u) = \begin{cases} z^{(1)}, & 0 \leqslant u < \theta L, \\ z^{(2)}, & \theta L \leqslant u \leqslant L. \end{cases}$$

Because $z^{(1)} \geqslant z^{(2)}$ componentwise, the function $\widetilde{e}$ is componentwise nonincreasing. It is

lower semicontinuous by construction. By (5.7), we compute

$$\int_0^L F(\widetilde{e}(u))\,\mathrm{d}u = L\left[\theta F(z^{(1)}) + (1-\theta)F(z^{(2)})\right] = Lm = M.$$

Similarly,

$$\int_0^L G(\widetilde{e}(u))\,\mathrm{d}u = L\left[\theta G(z^{(1)}) + (1-\theta)G(z^{(2)})\right] = Lg_0 \leqslant L\bar{G} = \int_0^L G(e(u))\,\mathrm{d}u.$$

This completes the proof. □

Theorem 5.4. *Let $X = a\mathbb{1}_A + b$ with $a \geqslant b > 0$, where $A \in \mathcal{F}$ and denote by $p = \mathbb{P}(A)$. Let $h_1, h_2 \in \mathcal{H}$ be continuous. For $\lambda \in [0, \rho_{h_1}(X)]$, the infimum $\phi(\lambda)$ is attained by some $Y^* \in Y_\lambda$ taking at most seven distinct values. More precisely, Y^* may be chosen so that its possible atom locations are contained in*

$$\{0, \tau_0, b, \tau_1, a, a+\tau_0, a+b\},$$

where $\tau_0 \in [0, b]$ and $\tau_1 \in [b, a]$.

Proof sketch. The full proof is given in Section 5.8. We outline the main ideas here. First, any feasible allocation $0 \leqslant Y \leqslant X$ can be described by its conditional survival functions on A and A^c. Write $\mu(t) = \mathbb{P}(Y > t \mid A^c)$, $0 \leqslant t \leqslant b$ and $\nu(t) = \mathbb{P}(Y > t \mid A)$, $0 \leqslant t \leqslant a+b$. Thus, the quantities $\rho_{h_1}(Y)$ and $\rho_{h_2}(X-Y)$ can be written in terms of (μ, ν). For $a \geqslant b$, the support of Y naturally splits into three layers, $[0, b]$, $[b, a]$ and $[a, a+b]$. The terms over the lower interval $[0, b]$ and the upper interval $[a, a+b]$ are coupled through the shift by a, while the middle interval $[b, a]$ can be treated separately. Lemma 5.2 is first applied to the coupled lower and upper layers. This reduces them to a two-step structure and produces the paired breakpoints τ_0 and $a+\tau_0$. Applying the same lemma to the middle layer produces one additional breakpoint τ_1. This yields an optimizer whose atom locations are contained in $\{0, \tau_0, b, \tau_1, a, a+\tau_0, a+b\}$.

Theorem 5.4 therefore shows that, when $a \geqslant b > 0$, every frontier point can be attained by an allocation with at most seven distinct values. This support has a natural layered interpretation. The point $\tau_0 \in [0, b]$ represents a split in the base layer, and the shifted point $a+\tau_0$ represents the corresponding split in the upper layer on A. The point $\tau_1 \in [b, a]$ represents one additional split in the middle layer. When $b = 0$, the aggregate risk reduces to the scaled Bernoulli risk $a\mathbb{1}_A$. In this case, $\tau_0 = 0$, and the seven-point support collapses to $\{0, \tau_1, a\}$, which agrees with the three-atom structure in Theorem 5.3.

Theorem 5.4 focuses on the case $a \geqslant b$, where the loss layers have the clean decomposition $[0, b]$, $[b, a]$, and $[a, a + b]$. This ordering allows the reduction argument to separate the relevant terms and obtain the seven-atom structure. When $a < b$, the base layer extends beyond the excess layer, so the same decomposition no longer separates the terms as cleanly. The reduction idea still yields a finite-atom bound, but the resulting bound is less sharp. For completeness, Proposition 5.1 in Section 5.9 discusses the case $a < b$. The bound obtained there should be viewed as a preliminary finite-atom bound rather than an optimal one. A more refined analysis of the overlap structure may further reduce the number of atoms.

This finite-atom result is useful for both theory and computation. Theoretically, it shows that the infinite-dimensional constrained optimization problem admits a representative optimizer with finitely many layers. Computationally, it provides a finite-dimensional framework for approximating the constrained Pareto frontier. This framework also motivates the future-work discussion on two-point aggregate risks, where inverse S-shaped distortions raise further questions about the geometry of the constrained frontier and the possible limitations of weighted-sum methods.

5.7 Conclusion

This chapter studies Pareto-optimal risk sharing under distortion risk measures without imposing universal risk aversion. We work in a constrained setting, where admissible

allocations may be subject to natural feasibility requirements such as nonnegativity. This changes the geometry of the problem. Translation invariance no longer reduces the Pareto frontier to a flat inf-convolution identity, and weighted-sum methods may fail to recover the full frontier. The first main result identifies when the weighted-sum approach remains valid. Theorem 5.1 shows that every Pareto-optimal allocation can be supported by weights in two benchmark cases, one with risk-averse agents and one with risk-seeking agents. Thus the familiar weighted-sum method extends beyond the classical risk-averse setting, provided the appropriate notion of convexity is available. Our second result provides a complementary approach for two agents that does not rely on convexity. Theorem 5.2 reduces the Pareto problem to a one-parameter family of constrained optimization problem. This leads to the function ϕ in the form of (5.1) whose graph describes the Pareto frontier. Our third result studies this frontier explicitly for Bernoulli and two-point aggregate risks. For a Bernoulli aggregate risk, the frontier is the lower convex envelope of a one-dimensional function, and every frontier point can be attained by a three-atom allocation, as shown in Theorem 5.3. For a two-point aggregate risk, Theorem 5.4 shows that every frontier point admits an optimizer with at most seven distinct payment levels. This finite-atom structure gives a layered representation of efficient allocations and provides a practical basis for numerical computation.

Several questions remain open. First, although the seven-atom theorem gives a finite-dimensional structure, it does not provide a closed-form expression for the Pareto frontier in the two-point case. A sharper characterization of how the optimal atoms and probability masses vary with λ would be valuable. Second, the finite-atom bound for the case $a < b$ appears less sharp, and a more refined analysis may reduce the number of required atoms. Third, extending the direct frontier approach to more than two agents would require a higher-dimensional analogue of the function ϕ, which remains an important direction for future work.

5.8 Omitted proof of Theorem 5.4

Proof. Let Y be any admissible allocation satisfying $0 \leqslant Y \leqslant X$. Then we have $0 \leqslant Y \leqslant a+b$ on A and $0 \leqslant Y \leqslant b$ on A^c. Define the conditional survival functions $\mu(t) = \mathbb{P}(Y > t \mid A^c)$ for $0 \leqslant t \leqslant b$ and $\nu(t) = \mathbb{P}(Y > t \mid A)$ for $0 \leqslant t \leqslant a+b$. Then μ and ν are nonincreasing and right-continuous. The unconditional survival function of Y is $S_Y(t) := \mathbb{P}(Y > t) = \bar{p}\,\mu(t) + p\nu(t)$. Then we can calculate that

$$\begin{aligned} \rho_{h_1}(Y) &= \int_0^b h_1(\bar{p}\,\mu(t) + p\nu(t))\,\mathrm{d}t + \int_b^{a+b} h_1(p\nu(t))\,\mathrm{d}t \qquad (5.8) \\ &= \int_0^b h_1(\bar{p}\,\mu(t) + p\nu(t))\,\mathrm{d}t + \int_a^{a+b} h_1(p\nu(t))\,\mathrm{d}t + \int_b^a h_1(p\nu(t))\,\mathrm{d}t \end{aligned}$$

Next we compute $\rho_{h_2}(X-Y)$. Since $X - Y = (a+b-Y)\mathbb{1}_A + (b-Y)\mathbb{1}_{A^c}$, we have

$$\begin{aligned} \rho_{h_2}(X-Y) &= \int_0^{a+b} h_2(p\,\mathbb{P}(Y < a+b-t|A) + \bar{p}\,\mathbb{P}(Y < b-t|A^c))\mathrm{d}t \\ &= \int_0^b h_2(p\,(1-\nu(a+b-t)) + \bar{p}\,(1-\mu(b-t)))\mathrm{d}t + \int_b^{a+b} h_2(p\,(1-\nu(a+b-t)))\mathrm{d}t \\ &= \int_0^b h_2(1-\bar{p}\,\mu(u) - p\nu(a+u))\,du + \int_0^a h_2(p(1-\nu(u)))\,\mathrm{d}u \qquad (5.9) \\ &= \int_0^b h_2(1-\bar{p}\,\mu(u) - p\nu(a+u))\,du + \int_0^b h_2(p(1-\nu(u)))\,\mathrm{d}u + \int_b^a h_2(p(1-\nu(u)))\,\mathrm{d}u \end{aligned}$$

Define the set

$$F_{x,y} = \{\text{function from } [0,x] \text{ to } [0,y] \text{ that is nonincreasing and lower semicontinuous}\}$$

and

$$\mathcal{D} = \{(\mu,\nu) : \mu \in F_{b,1},\ \nu \in F_{a+b,1},\ \mu(b) = 0,\ \nu(a+b) = 0\}\,.$$

For $(\mu, \nu) \in \mathcal{D}$, define

$$r_0^{\mu,\nu}(u) = (\nu(u), \mu(u), \nu(a+u)), u \in [0,b] \text{ and } q^\nu(u) = \nu(u),\ u \in [b,a].$$

Define

$$F_0(x,y,z) = h_1(\bar{p}\, y + px) + h_1(pz), \text{ and } G_0(x,y,z) = h_2(1 - \bar{p}\, y - pz) + h_2(p(1-x)).$$

For any measurable $r : [0,b] \mapsto [0,1]^3$, define

$$I_0(r) = \int_0^b F_0(e(u))\mathrm{d}u, \text{ and } J_0(r) = \int_0^b G_0(e(u))\mathrm{d}u.$$

For any measurable $r : [b,a] \mapsto [0,1]$, define

$$I_1(r) = \int_b^a h_1(pe(u))\mathrm{d}u, \text{ and } J_1(r) = \int_0^b h_2(p(1-e(u)))\mathrm{d}u.$$

Then the original optimization problem can be rewritten as

$$\phi(\lambda) = \inf_{(\mu,\nu)\in\mathcal{D}} \left\{ J_0(r_0^{\mu,\nu}) + J_1(q^\nu) : I_0(r_0^{\mu,\nu}) + I_1(q^\nu) = \lambda \right\}.$$

We now prove the existence of an optimizer. Let $(\mu_n, \nu_n)_{n\geqslant 1}$ be a sequence in $\mathcal{D}$. Since (μ_n) are uniformly bounded nonincreasing functions on $[0,b]$, Helly's selection theorem yields a subsequence $(\mu_{n_k})_{k\geqslant 1}$ and a nonincreasing function $\mu : [0,b] \to [0,1]$ such that $\mu_{n_k}(t) \to \mu(t)$ at every continuous point of μ. From this subsequence, apply Helly's selection theorem again to $(\nu_{n_k})_{k\geqslant 1}$ on $[0, a+b]$. Then there exist a further subsequence, still denoted by (n_k) for simplicity, and a nonincreasing function $\nu : [0, a+b] \to [0,1]$ such that $\nu_{n_k}(t) \to \nu(t)$ at every continuity point t of ν. Replacing μ and ν by their right-continuous versions if necessary, we obtain a limit pair $(\mu, \nu) \in \mathcal{D}$. Since monotone functions have at most countably many discontinuities, this convergence holds almost everywhere. Therefore, $\mathcal{D}$ is sequentially compact.

Suppose $(\mu_n, \nu_n) \to (\mu, \nu)$ pointwise at continuity points of the limit. Since monotone functions have at most countably many discontinuities, we have $r_0^{\mu_n,\nu_n}(t) \to r_0^{\mu,\nu}(t)$ for almost every $t \in [0, b]$. Since F_0 is continuous and bounded by $2\|h_1\|_\infty$, the dominated convergence theorem yields $I_0(r_0^{\mu_n,\nu_n}) \to I_0(r_0^{\mu,\nu})$. The same argument applies to I_1, J_0, and J_1. Therefore $I(\mu_n, \nu_n) \to I(\mu, \nu)$, $J(\mu_n, \nu_n) \to J(\mu, \nu)$. Hence $I = I_0 + I_1$ and $J = J_0 + J_1$ are sequentially continuous on $\mathcal{D}$. Fix $\lambda \in [0, \rho_{h_1}(X)]$. The feasible set

$$\mathcal{D}_\lambda := \{(\mu, \nu) \in \mathcal{D} : I(\mu, \nu) = \lambda\}$$

is closed in $\mathcal{D}$, hence sequentially compact. It remains to check that $\mathcal{D}_\lambda$ is nonempty. If $\lambda = 0$, take $Y = 0$. If $\lambda > 0$, take $c = \lambda/\rho_{h_1}(X) \in [0, 1]$. Then $Y = cX$ satisfies $0 \leqslant Y \leqslant X$, and by positive homogeneity of ρ_{h_1}, we have $\rho_{h_1}(Y) = \rho_{h_1}(cX) = \lambda$. Hence $\mathcal{D}_\lambda \neq \varnothing$. Since J is sequentially continuous on the compact nonempty set $\mathcal{D}_\lambda$, J attains its minimum. Thus there exists an optimal pair $(\mu^*, \nu^*) \in \mathcal{D}_\lambda$.

Let Y^* be a random variable realizing this pair (μ^*, ν^*). On $[0, b]$, define

$$r_0(u) = r_0^{\mu^*,\nu^*}(u) = \big(\nu^*(u), \mu^*(u), \nu^*(a + u)\big),\ 0 \leqslant u \leqslant b, \text{ and } R_0 = \{r_0(u) : 0 \leqslant u \leqslant b\}.$$

Since μ^* and ν^* are nonincreasing, r_0 is componentwise nonincreasing. Also, R_0 is a compact subset of $[0, 1]^3$ and a chain under the coordinatewise order. By Lemma 5.2, there exists a componentwise nonincreasing lower semicontinuous function $\widetilde{r}_0(u) = (\widetilde{x}(u), \widetilde{y}(u), \widetilde{z}(u))$ for $0 \leqslant u \leqslant b$, taking at most two vector values from $\overline{R}_0$, such that

$$I_0(\widetilde{r}_0) = I_0(r_0) \text{ and } J_0(\widetilde{r}_0) \leqslant J_0(r_0).$$

Thus there exist $(x_1, y_1, z_1), (x_2, y_2, z_2) \in \overline{R}_0$ and $\tau_0 \in [0, b]$ such that $(x_1, y_1, z_1) \geqslant (x_2, y_2, z_2)$ componentwise, and

$$\widetilde{r}_0(u) = \begin{cases} (x_1, y_1, z_1), & 0 \leqslant u < \tau_0, \\ (x_2, y_2, z_2), & \tau_0 \leqslant u \leqslant b. \end{cases}$$

On $[0, b)$, define

$$\widetilde{\nu}(t) = \begin{cases} x_1, & 0 \leqslant t < \tau_0, \\ x_2, & \tau_0 \leqslant t < b, \end{cases} \text{ and } \widetilde{\mu}(t) = \begin{cases} y_1, & 0 \leqslant t < \tau_0, \\ y_2, & \tau_0 \leqslant t < b. \end{cases}$$

On $[a, a+b)$, define

$$\widetilde{\nu}(t) = \begin{cases} z_1, & a \leqslant t < a + \tau_0, \\ z_2, & a + \tau_0 \leqslant t < a + b. \end{cases}$$

Additionally, take $\widetilde{\nu}(t) = \nu^*(t)$ for $b \leqslant t < a$.

We need to check that $\widetilde{\nu}$ is nonincreasing on the entire interval $[0, a+b]$. The only nontrivial cases are at $t = b$ and $t = a$. By Lemma 5.2's construction, the two values $z^{(1)} = (x_1, y_1, z_1)$ and $z^{(2)} = (x_2, y_2, z_2)$ are taken as limits of sequences in $[0, b]$,

$$z^{(i)} = \lim_{k\to\infty} r_0(u_{n_k}^{(i)}) = \lim_{k\to\infty} \big(\nu^*(u_{n_k}^{(i)}),\, \mu^*(u_{n_k}^{(i)}),\, \nu^*(a + u_{n_k}^{(i)})\big),$$

where $u_{n_k}^{(i)} \in [0, b]$ and $u_{n_k}^{(1)} \leqslant u_{n_k}^{(2)}$ for every n. Since $u_{n_k}^{(2)} \in [0, b]$ and $a + u_{n_k}^{(1)} \in [a, a+b]$, monotonicity of ν^* gives $\nu^*(u_{n_k}^{(2)}) \geqslant \nu^*(b)$ and $\nu^*(a + u_{n_k}^{(1)}) \leqslant \nu^*(a)$. Passing to the limit lead to $x_2 \geqslant \nu^*(b)$ and $z_1 \leqslant \nu^*(a)$.

Apply Lemma 5.2 again on $[b, a]$ to $q(u) = \nu^*(u)$ with $d = 1$, $F(z) = h_1(pz)$, and $G(z) = h_2(p(1-z))$. Then there exist a step function $\widetilde{q}$ on $[b, a]$ with breakpoint $\tau_1 \in [b, a]$ and two values $w_1 \geqslant w_2$, satisfying

$$\int_b^a h_1(p\,\widetilde{q})\,dv = \int_b^a h_1(p\,q)\,dv, \quad \int_b^a h_2(p(1-\widetilde{q}))\,dv \leqslant \int_b^a h_2(p(1-q))\,dv.$$

Each w_i is a limit of ν^*-values of sequences in $[b, a]$, so by monotonicity of ν^*, it follows

that $\nu^*(b) \geqslant w_1 \geqslant w_2 \geqslant \nu^*(a)$. Combining all inequalities, we have

$$x_2 \geqslant \nu^*(b) \geqslant w_1 \geqslant w_2 \geqslant \nu^*(a) \geqslant z_1.$$

Together with $z^{(1)} \geqslant z^{(2)}$ from Lemma 1, this establishes global monotonicity of $\widetilde{\nu}$ on $[0, a+b]$. Set $\widetilde{\mu}(b) = 0$ and $\widetilde{\nu}(a+b) = 0$, changing endpoint values does not affect the integrals. Let $\widetilde{Y}$ be a random variable realizing the pair $(\widetilde{\mu}, \widetilde{\nu})$. By construction,

$$\rho_{h_1}(\widetilde{Y}) = \rho_{h_1}(Y^*) = \lambda, \text{ and, } \rho_{h_2}(X - \widetilde{Y}) \leqslant \rho_{h_2}(X - Y^*).$$

Since Y^* is optimal, equality must hold. Hence $\widetilde{Y}$ is also an optimizer.

It remains to count the number of possible atoms of $\widetilde{Y}$. Let $S_{\widetilde{Y}}(t) = \mathbb{P}(\widetilde{Y} > t)$ be its survival function. On $[0, b]$, both $\widetilde{\mu}$ and $\widetilde{\nu}$ have at most one interior jump on this interval, located at τ_0. On $[b, a]$, the function $\widetilde{\nu}$ has at most one interior jump on this interval, located at τ_1. On $[a, a+b]$, $\widetilde{\nu}(t)$. has at most one interior jump, located at $a + \tau_0$. Combining the three regions, the possible jump locations of $S_{\widetilde{Y}}$ are contained in $\{0, \tau_0, b, \tau_1, a, a+\tau_0, a+b.\}$. Therefore, there exists an optimizer taking at most seven distinct values. □

5.9 Omitted proof of the case $a < b$

Proposition 5.1. *Let $X = a\mathbb{1}_A + b$ with $a < b \leqslant 2a$. Let $h_1, h_2 \in \mathcal{H}$ be continuous. Then for each $\lambda \in [0, \rho_{h_1}(X)]$, there exists an optimizer Y^* of $\phi(\lambda)$ satisfying one of the following.*

(i) If $a < b \leqslant 2a$, then Y^* takes at most 11 distinct values, with possible atom locations contained in

$$\{0,\ \tau_0,\ b-a,\ \tau_1,\ a,\ a+\tau_0,\ b,\ a+\tau_1,\ 2a,\ 2a+\tau_0,\ a+b\},$$

where $\tau_0 \in [0, b-a]$ *and* $\tau_1 \in [b-a, a]$.

(ii) If $b = Ka + r$ *where* $K = \lfloor b/a \rfloor \geqslant 2$ *and* $r \in (0, a)$*, then* Y^* *takes at most* $4K + 7$ *distinct values, with possible atom locations contained in*

$$\bigcup_{j=0}^{K} \{ja,\ ja+\tau_0,\ ja+r,\ ja+\tau_1\} \ \cup\ \{(K+1)a,\ (K+1)a+\tau_0,\ a+b\},$$

where $\tau_0 \in [0, r]$ *and* $\tau_1 \in [r, a]$.

Proof. Similar to the argument in Theorem 5.4, for a given λ, there exists an optimal pair $(\mu^*, \nu^*) \in \mathcal{D}_\lambda$ with corresponding Y^* such that $\rho_{h_2}(X - Y^*) = \phi(\lambda)$.

(i) We consider the case of $a < b \leqslant 2a$. Define

$$\ell_0^*(t) = \big(\mu^*(t),\ \nu^*(t),\ \mu^*(a+t),\ \nu^*(a+t),\ \nu^*(2a+t)\big), \qquad t \in [0, b-a],$$

and

$$r_0^*(t) = \big(\mu^*(t),\ \nu^*(t),\ \nu^*(a+t)\big), \qquad t \in [b-a, a].$$

Both are componentwise nonincreasing on their domains with $\ell_0^* \in [0,1]^5$ and $r_0^* \in [0,1]^3$. Writing $\ell_0 = (P_1, P_2, P_3, P_4, P_5)$ and $r_0 = (Q_1, Q_2, Q_3)$, define

$$\begin{aligned}
F_1(\ell_0) &= h_1(\bar{p}P_1 + pP_2) + h_1(\bar{p}P_3 + pP_4) + h_1(pP_5),\\
G_1(\ell_0) &= h_2(1 - \bar{p}R_1 - pP_4) + h_2(p(1-P_2)) + h_2(1 - \bar{p}P_3 - pP_5),\\
F_2(r_0) &= h_1(\bar{p}Q_1 + pQ_2) + h_1(pQ_3),\\
G_2(r_0) &= h_2(1 - \bar{p}Q_1 - pQ_3) + h_2(p(1-Q_2)).
\end{aligned}$$

By splitting each integral in (5.8) and (5.9) according to $[0, b] = [0, b-a] \cup [b-a, a] \cup$

$[a, b]$, it is straightforward to verify that

$$\rho_{h_1}(Y) = \int_0^{b-a} F_1(\ell_0^*(v))\, dv + \int_{b-a}^{a} F_2(r_0^*(u))\, du, \tag{5.10}$$

$$\rho_{h_2}(X - Y) = \int_0^{b-a} G_1(\ell_0^*(v))\, dv + \int_{b-a}^{a} G_2(r_0^*(u))\, du. \tag{5.11}$$

Apply Lemma 5.2 to ℓ_0 on $[0, b-a]$ with $F = F_1, G = F_2$. The lemma yields a step function $\tilde{\ell} : [0, b-a] \to [0,1]^5$ taking values $z^{(1)} \geqslant z^{(2)}$ componentwise with breakpoint $\tau_0 \in [0, b-a]$,

$$\tilde{\ell}(t) = \begin{cases} z^{(1)}, & 0 \leqslant t < \tau_0, \\ z^{(2)}, & \tau_0 \leqslant t \leqslant b-a, \end{cases}$$

satisfying

$$\int_0^r F^*(\tilde{\ell})\, dv = \int_0^r F^*(\ell_0^*)\, dv, \qquad \int_0^r G^*(\tilde{\ell})\, dv \leqslant \int_0^r G^*(\ell_0^*)\, dv.$$

Similarly, apply Lemma 5.2 to r_0^* on $[b-a, a]$ with $F = F_2, G = G_2$. The lemma yields a step function $\widetilde{r}_0$ taking values $w^{(1)} \geqslant w^{(2)}$ with breakpoint $\tau_1 \in [b-a, a]$. Write the coordinates $z^{(i)} = (z_1^{(i)}, \ldots, z_5^{(i)})$ and $w^{(i)} = (w_1^{(i)}, w_2^{(i)}, w_3^{(i)})$ for $i = 1, 2$. Define $\widetilde{\mu} : [0, b] \to [0,1]$ piecewise by

$$\widetilde{\mu}(t) = \begin{cases} z_1^{(1)}, & 0 \leqslant t < \tau_0, \\ z_1^{(2)}, & \tau_0 \leqslant t < b-a, \\ w_1^{(1)}, & b-a \leqslant t < \tau_1, \\ w_1^{(2)}, & \tau_1 \leqslant t < a, \\ z_3^{(1)}, & a \leqslant t < a + \tau_0, \\ z_3^{(2)}, & a + \tau_0 \leqslant t < b, \end{cases}$$

and $\widetilde{\mu}(b) := 0$. Define $\widetilde{\nu} : [0, a+b] \to [0,1]$ piecewise by

$$\widetilde{\nu}(t) = \begin{cases} z_2^{(1)}, & 0 \leqslant t < \tau_0, \\ z_2^{(2)}, & \tau_0 \leqslant t < b-a, \\ w_2^{(1)}, & b-a \leqslant t < \tau_1, \\ w_2^{(2)}, & \tau_1 \leqslant t < a, \\ z_4^{(1)}, & a \leqslant t < a+\tau_0, \\ z_4^{(2)}, & a+\tau_0 \leqslant t < b, \\ w_3^{(1)}, & b \leqslant t < a+\tau_1, \\ w_3^{(2)}, & a+\tau_1 \leqslant t < 2a, \\ z_5^{(1)}, & 2a \leqslant t < 2a+\tau_0, \\ z_5^{(2)}, & 2a+\tau_0 \leqslant t < a+b, \end{cases}$$

and $\widetilde{\nu}(a+b) := 0$.

The global monotonicity of $\widetilde{\nu}$ and $\widetilde{\mu}$ follows the same argument in Theorem 5.4. Therefore, the jump points of $\widetilde{\mu}$ are contained in $\{0,\, \tau_0,\, b-a,\, \tau_1,\, a,\, a+\tau_0,\, b\}$. Similarly, the jump points of $\widetilde{\nu}$ are contained in $\{0,\, \tau_0,\, b-a,\, \tau_1,\, a,\, a+\tau_0,\, b,\, a+\tau_1,\, 2a,\, 2a+\tau_0,\, a+b\}$. Let $\widetilde{Y}$ be a random variable realizing the pair $(\widetilde{\mu}, \widetilde{\nu})$. Clearly, $\widetilde{Y}$ is also an optimizer. The set of possible atom locations of $\widetilde{Y}$ is the union, namely

$$\mathcal{A} = \{0,\, \tau_0,\, r,\, \tau_1,\, a,\, a+\tau_0,\, b,\, a+\tau_1,\, 2a,\, 2a+\tau_0,\, a+b\},$$

which contains 11 elements. Therefore $\widetilde{Y}$ takes at most 11 distinct values.

(ii) Define the following

$$\ell^*(t) = \big(M_0^*(t),\, \ldots,\, M_K^*(t),\, N_0^*(t),\, \ldots,\, N_{K+1}^*(t)\big) \in [0,1]^{2K+3},\ t \in [0,r]$$

$$r^*(t) = \big(M_0^*(t),\, \ldots,\, M_{K-1}^*(t),\, N_0^*(t),\, \ldots,\, N_K^*(t)\big) \in [0,1]^{2K+1},\ t \in [r, a]$$

where $M_j^*(s) = \mu^*(ja + s)$ and $N_j^*(s) = \nu^*(ja + s)$. The components M_K^* and N_{K+1}^* are well-defined only on $[0, r]$ since μ^* has domain $[0, b] = [0, Ka + r]$ and ν^* has domain $[0, a + b] = [0, (K + 1)a + r]$.

For $\ell = (M_0, \ldots, M_K, N_0, \ldots, N_{K+1}) \in [0,1]^{2K+3}$ and $q = (M_0, \ldots, M_{K-1}, N_0, \ldots, N_K) \in [0,1]^{2K+1}$, define

$$\begin{aligned}
F_1(\ell) &= \sum_{j=0}^{K} h_1(\bar{p}M_j + pN_j) + h_1(pN_{K+1}),\\
G_1(\ell) &= \sum_{j=0}^{K} h_2(1 - \bar{p}M_j - pN_{j+1}) + h_2(p(1 - N_0)),\\
F_2(q) &= \sum_{j=0}^{K-1} h_1(\bar{p}M_j + pN_j) + h_1(pN_K),\\
G_2(q) &= \sum_{j=0}^{K-1} h_2(1 - \bar{p}M_j - pN_{j+1}) + h_2(p(1 - N_0)).
\end{aligned}$$

By splitting $[0, b] = \bigcup_{j=0}^{K-1}[ja, (j + 1)a] \cup [Ka, b]$ and $[b, a + b] = [b, (K + 1)a] \cup [(K + 1)a, a + b]$, it follows from (5.8) and (5.9) that

$$\rho_{h_1}(Y) = \int_0^r F_1(\ell^*(t))\,\mathrm{d}t + \int_r^a F_2(r^*(t))\,\mathrm{d}t, \tag{5.12}$$

$$\rho_{h_2}(X - Y) = \int_0^r G_1(\ell^*(t))\,\mathrm{d}t + \int_r^a G_2(r^*(t))\,\mathrm{d}t. \tag{5.13}$$

Apply Lemma 5.2 to ℓ^* on $[0, r]$ with (F_1, G_1), we obtain a step function $\tilde{\ell} : [0, r] \to [0,1]^5$ taking values $z^{(1)} \geqslant z^{(2)}$ componentwise with breakpoint $\tau_0 \in [0, b - a]$. Similarly, applying Lemma 5.2 to r^* on $[r, a]$ yields a step function with values $w^{(1)} \geqslant w^{(2)}$

and breakpoint $\tau_1 \in [r, a]$. Define $\widetilde{\mu}, \widetilde{\nu}$ by piecing together the components

$$\widetilde{\mu}(ja+u) = \begin{cases} z^{(1)}_{M_j}, & 0 \leqslant u < \tau_0, \\ z^{(2)}_{M_j}, & \tau_0 \leqslant u < r, \\ w^{(1)}_{M_j}, & r \leqslant u < \tau_1, \\ w^{(2)}_{M_j}, & \tau_1 \leqslant u \leqslant a, \end{cases} \qquad j = 0, 1, \ldots, K-1$$

$$\text{and} \quad \widetilde{\mu}(Ka+v) = \begin{cases} z^{(1)}_{M_K}, & 0 \leqslant v < \tau_0, \\ z^{(2)}_{M_K}, & \tau_0 \leqslant v \leqslant r, \end{cases}$$

with $\widetilde{\mu}(b) = 0$. Analogously, for $j = 0, 1, \ldots, K$ and $u \in [0, a]$,

$$\widetilde{\nu}(ja+u) = \begin{cases} z^{(1)}_{N_j}, & 0 \leqslant u < \tau_0, \\ z^{(2)}_{N_j}, & \tau_0 \leqslant u < r, \\ w^{(1)}_{N_j}, & r \leqslant u < \tau_1, \\ w^{(2)}_{N_j}, & \tau_1 \leqslant u \leqslant a, \end{cases} \qquad j = 0, 1, \ldots, K-1$$

$$\text{and} \quad \widetilde{\nu}((K+1)a+v) = \begin{cases} z^{(1)}_{N_{K+1}}, & 0 \leqslant v < \tau_0, \\ z^{(2)}_{N_{K+1}}, & \tau_0 \leqslant v \leqslant r, \end{cases}$$

with $\widetilde{\nu}(a+b) = 0$. Global monotonicity at the points $\{r, a, 2a, \ldots, Ka\}$ follows from componentwise monotonicity of (μ^*, ν^*) via the chain argument, similar as in Theorem 5.4. Therefore, the jump points of $\widetilde{\mu}$ are contained in

$$\bigcup_{j=0}^{K-1} \{ja,\, ja+\tau_0,\, ja+r,\, ja+\tau_1\} \;\cup\; \{Ka,\, Ka+\tau_0\} \;\cup\; \{b\},$$

which has $4K+3$ elements. Similarly, the jump points of $\widetilde{\nu}$ are contained in

$$\bigcup_{j=0}^{K}\{ja,\, ja+\tau_0,\, ja+r,\, ja+\tau_1\} \,\cup\, \{(K+1)a,\, (K+1)a+\tau_0\} \,\cup\, \{a+b\},$$

which has $4K+7$ elements. Since the jump set of $\widetilde{\mu}$ is contained in the jump set of $\widetilde{\nu}$, $\widetilde{Y}$ takes at most $4K+7$ distinct values.

Therefore, the proof is completed. □

Chapter 6

Concluding remarks and future work

6.1 Concluding remarks

This thesis develops a systematic theory of Pareto-optimal risk sharing for agents whose preferences are represented by distortion risk measures (equivalently, Yaari dual utilities). Its unifying concern is to understand how the structure of optimal allocations is shaped by the agents' risk attitudes once universal risk aversion is relaxed, with attention to the contrast between comonotonic and counter-monotonic risk sharing, the role of heterogeneity among agents, and the geometry of the Pareto frontier under feasibility constraints.

The first part of the thesis, Chapters 2 and 3 develop the theory of counter-monotonic risk sharing under distortion risk measures. Chapter 2 treats the homogeneous case (all agents use the same distortion) and gives a comparative analysis of unconstrained, comonotonic, and counter-monotonic inf-convolutions under concave, convex, and inverse S-shaped distortion functions. We obtain explicit formulas for the counter-monotonic inf-convolution in each setting, identify when the counter-monotonic and comonotonic inf-convolutions coincide, and apply the results to a portfolio manager's problem. Chapter 3 extends the analysis to heterogeneous distortions. We establish general relationships among the three inf-convolutions under various combinations of concave and convex distortions, and derive

explicit formulas when all agents are risk-seeking. A notable consequence is that the representative agent of risk-seeking agents with distortion risk measures is in general no longer an agent with a distortion risk measure but rather a distortion riskmetric, and the group's effective risk attitude need not vary monotonically with the agents' individual attitudes.

Chapter 4 studies the more general setting, where the market contains both risk-averse and risk-seeking agents. The key innovation is a reduction collapsing the n-agent problem to a two-agent problem between a representative risk-averse agent and a representative risk-seeking agent. Building on this reduction we establish necessary and sufficient conditions for the existence of optimal allocations, describe the cases where the inf-convolution is unbounded below, and obtain explicit optimal allocations for piecewise linear distortions and Bernoulli aggregate risks. The nonnegative-allocation constraint is also analyzed and the results are applied back to the multi-agent problem, where two regimes emerge. When risk aversion dominates, the risk-seeking group absorbs all randomness, and when risk seeking dominates, the risk-averse agents share risk comonotonically among themselves but bet collectively against the risk-seeking side.

Chapter 5 studies Pareto optimality under feasibility constraints such as nonnegativity, which reshape the geometry of the risk sharing problem. We identify two benchmark situations in which the weighted-sum method remains valid, and develop for two agents a direct approach that does not rely on convexity, reducing the Pareto problem to a one-parameter family of constrained optimization problems whose value function traces the frontier. For Bernoulli aggregate risks, a convex-envelope characterization is obtained, and frontier points are shown to be attainable by three-atom allocations. For two-point aggregate risks, finite-atom representations give a tractable structural description of efficient allocations and provided a step toward understanding the general constrained frontier.

6.2 Potential future work

This section discusses several open questions and natural extensions arising from the results of this thesis. The discussion is organized around four directions: comparative statics under heterogeneous distortion functions, equilibrium analysis in mixed-attitude markets, constrained Pareto optimality and finite-atom structures.

6.2.1 Comparative statics under heterogeneous distortions

One direction concerns the comparative statics of group-level risk attitudes under heterogeneous distortion functions. In Chapter 3, the portfolio manager's problem suggests a counterintuitive phenomenon: when all agents in a group become more risk seeking, it is not necessarily optimal for the manager to allocate a larger proportion of wealth to the risky asset. This reflects the fact that the representative distortion induced by heterogeneous risk-seeking agents is obtained through a normalized aggregation of the individual distortions, as shown in (3.20). Although a pointwise ordering of individual distortions is preserved before normalization, the normalization step may destroy the corresponding ordering at the group level. A natural question is to identify conditions under which stronger individual risk seeking leads to stronger group-level risk seeking, and hence to more aggressive optimal investment decisions.

6.2.2 Equilibrium analysis with mixed risk attitudes

Another direction is to connect the Pareto-optimal allocations obtained in Chapter 4 with competitive equilibrium concepts. The two regimes identified in that chapter, in which either the risk-seeking group absorbs all randomness or the risk-averse and risk-seeking sides bet collectively against each other, are derived from a Pareto-optimality analysis. Whether these regimes also arise as competitive equilibria in an exchange economy with mixed risk attitudes remains an interesting open question. The presence of nonconvex preferences

makes the existence and characterization of equilibria more challenging, as discussed in Araujo et al. (2017) and Beißner and Werner (2023). A complete equilibrium analysis would require identifying appropriate price systems, budget constraints and individual optimality conditions that are compatible with the representative-agent reduction of Chapter 4. Such an analysis would provide a more complete picture of price formation when hedging and speculation coexist.

6.2.3 Constrained Pareto optimality and finite-atom structures

Chapter 5 reduces the two-agent constrained Pareto problem to a one-parameter family of optimization problems and develops finite-atom representations for Bernoulli and two-point aggregate risks. These results suggest two main directions for future work.

The first direction is to complete the analysis of the two-point case. For a two-point aggregate risk in the case $a \geqslant b$, Theorem 5.4 shows that Pareto-optimal allocations can be represented using at most seven atoms. This reduction turns the original infinite-dimensional problem into a finite-dimensional one and makes it possible to study the constrained frontier numerically. A natural next step is to use this finite-dimensional representation to determine whether whether some Pareto-optimal allocations may fail to admit a supporting-weight representation. Establishing such a result analytically would clarify the limitation of weighted-sum methods in the presence of non-convex feasible risk sets and remains an important open problem.

The second direction is to extend the finite-atom reduction beyond two-point aggregate risks. The results of Chapter 5 suggest that a similar layered structure should hold for general finitely supported aggregate risks. For a n-point aggregate risk, one may expect an optimal allocation to be represented by a bounded number of atoms, possibly following a pattern analogous to the seven-atom structure in the two-point case. A general finite-support theorem would provide a tractable finite-dimensional framework for constrained Pareto frontiers. It could also serve as a basis for approximating a continuous aggregate

risks by discrete ones, thereby linking the finite-atom theory to more general aggregate risk models.

Finally, the direct frontier approach of Chapter 5 is developed for two agents. Extending it to more agents would require a higher-dimensional analogue of the value function ϕ and a careful study of multiple risk constraints. Such an extension would clarify how the geometry of constrained Pareto frontiers changes as the number of agents grows.

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