Exotic constructions on covers branched over hyperplane arrangements

by

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Author's Declaration

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of Contributions

I am the sole author of Chapters 1 and 2.

Chapter 3 is background material and takes inspiration from [34] (co-authored by Robert Harris, Amey Joshi, Doug Park and Mainak Poddar), [33] (co-authored by Robert Harris and Doug Park) and [35] (co-authored by Robert Harris, Patrick Naylor and Doug Park).

In Chapter 4, Sections 4.1-4.3 is background material and takes inspiration from [34] (co-authored by Robert Harris, et-al) and [33] (co-authored by Robert Harris and Doug Park). Sections 4.4 and 4.5 are original research and I was the sole author

Chapter 5 is original research and is based on the articles [34] (co-authored by Robert Harris, et-al) and [33] (co-authored by Robert Harris and Doug Park). I am the sole author of the main result of this chapter, Theorem 5.2.11.

Chapter 6 is original research and Sections 6.1-6.3 is based the articles [34] (co-authored by Robert Harris, et-al) and [33] (co-authored by Robert Harris and Doug Park). I am the sole author of Section 6.4.

Abstract

The purpose of this thesis is to study and construct geometric structures on 4-manifolds from a starting point of combinatorial objects.

In part as a consequence of embedded surfaces and codimension two submanifolds coinciding in dimension four, many of the tools that are used in higher dimensions fail or are underwhelming when applied to 4-manifolds. For this reason, the development and advancement of techniques that are applicable to 4-manifolds are of particular interest and importance to low dimensional topologists. The general techniques of interest are those that either construct a 4-manifold in a novel way or find interesting embedded submanifolds.

Consequently, we shall contribute to the study of 4-manifolds by investigating ways to construct 4-manifolds with positive signature in addition to describing construction methods that can guarantee the existence of algebraically interesting embedded symplectic submanifolds.

In this thesis we investigate how the combinatorial data of line arrangements and the algebraic data of their complements in rational surfaces can be utilized to construct symplectic 4-manifolds with arbitrarily large signatures through the method of branched coverings. In general, we not only show that these line arrangements can be used to provide asymptotic bounds for the existence of symplectic 4-manifolds but we also show that for any line arrangement, there exist symplectic branched covers with sufficiently nice geometric and topological properties. Namely, we show they contain embedded symplectic surfaces which carry their fundamental group.

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Dedication

This is dedicated to Grandma and Grandpa Gillmaier, Grandpa Harris, and George.

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Chapter 1

Introduction

There are two primary questions in the field of low dimensional topology. One asks about the topological behaviour of a manifold and one asks about the geometric behaviour (more specifically the behaviour of the geometric structures) of a manifold. When considering 4-dimensional manifolds that are closed (compact and boundaryless) and orientable (hereafter simply referred to as a 4-manifold), this first question has been considered in various cases. For instance, it was essentially completely answered in the simply connected setting by Freedman [21] when he published his landmark result that classified simply connected 4-manifolds.

Theorem 1.0.1. (§1 of [21]) *Suppose Q is a unimodular symmetric bilinear form. Then there is exactly one homeomorphism type of simply connected 4-manifolds X having intersection form $Q_X = Q$ if Q is even and exactly two homeomorphism types if Q is odd. Furthermore, for any such Q, at most one corresponding homeomorphism type can carry a smooth structure.*

On the other hand, when considering 4-manifolds with non-trivial fundamental groups, there has been mixed success, examples include the homeomorphism classifications for finite cyclic fundamental groups by Hambleton and Kreck [28], [29], in addition to the various homotopy classifications provided by the likes of Hillman, Kasprowski, Powell, Ray and Ruppik [36], [40].

Thanks to these classifications in addition to many others, in many cases we have the tools to distinguish the topology of a given 4-manifold. Consequently, we can then start

to consider geometric questions. Specifically, we can ask how the (known) topological type of a 4-manifold can dictate/restrict the behaviour of the geometric structures that they can admit (if any). In general, this geometric problem can be separated further into the following two questions.

1. Does a given topological 4-manifold X admit at least one smooth structure? If so, how many?

2. Does a given smooth 4-manifold admit other geometric structures on top of its smooth structure? If so, which ones?

The goal of this thesis is to discuss a specific formulation of question 2. However, we will still provide some motivation and insight into both questions to better set the stage.

If we first consider question 1, the question as to whether or not a topological 4-manifold *cannot* carry a smooth structure has brought about many interesting results. For instance, both Rohlin's Theorem [57] and Donaldson's Theorem [16] can be used to provide examples of non-smoothable 4-manifolds. In contrast, regarding a count of how many can exist, gauge theoretic tools like the Seiberg-Witten invariants [69], [17] or Heegaard Floer Homology [51] have been used to great effect to exhibit infinitely many non-diffeomorphic smooth structures on the same underlying topological 4-manifold (these are referred to as exotic copies of the 4-manifold). For question 1, it is this problem on the uniqueness of smooth structures that sees the most study. The culmination of which is the largest open problem in the field.

Conjecture 1.0.2. (Smooth 4-dimensional Poincaré conjecture) *Every 4-manifold which is homeomorphic to S^4 must be diffeomorphic to S^4.*

Regarding question 2 above, there are many other geometric structures that geometers (and topologists) find interesting. While the most well known additional structure is a complex (or holomorphic) structure, there is significant interest in considering only almost complex structures. For example, as motivated by physics and their relations to Riemannian metrics, symplectic structures in addition to other differential forms have also been deeply studied. Going the other way and asking about multiple compatible structures one often arrives at the study of Kähler manifolds. For a holistic overview of many of the open geometric problems in the field we refer the reader to Kirby's problem list [41], whereas a

more focused discussion on questions surrounding symplectic and Kähler 4-manifolds can be found in [9].

This thesis is primarily interested in the following specific formulation of question 2: which topological 4-manifolds admit symplectic structures? Such a question (on whether a topological manifold admits a specific type of smooth structure) is often referred to as a *geography problem* with the name coming from Persson in his investigations into the classification of complex surfaces of general type [55]. Further investigations and techniques for the case of algebraic surfaces were also provided by Sommese [61]. Finally, the problem this thesis considers, the *symplectic* geography problem was first asked by McCarthy and Wolfsen [47]. For an outline on both of these specific problems mentioned and some insight into the other types of geography problems we refer the reader to [62].

We also highlight that since many of the constructions considered in this thesis are generalizations of the constructions of Hirzebruch, the results have connections to some important results in combinatorics. For example, we note the similarities between our equation (6.15) and Hirzebruch's inequality and its proof.

Theorem 1.0.3. (cf. §3 of [39] or Theorem 1.5 in [56]) *Let $\mathcal{L}$ be an arrangement of $n \geq 4$ lines that is neither a pencil nor a near pencil. If t_k denotes the number of intersection points of $\mathcal{L}$ with multiplicity k, then we must have*

$$t_2 + t_3 \geq n + \sum_{r \geq 5} (r-4)t_r.$$

That being said, many of the results of this thesis lie on the side of geometry, as the aim of this thesis is to build 4-manifolds which admit symplectic structures (in fact we will build lots of Kähler manifolds) whereas many of the ideas that form the basis for this thesis come from the combinatorics of hyperplane arrangements. This results in many proofs that rely on combinatorial and algebraic arguments, yet eventually lead to geometric conclusions.

The structure of this thesis is as follows.

In Chapter 3, we define and discuss the primary target of this thesis, the *symplectic geography problem* for simply connected 4-manifolds. Specifically, we discuss in Section 3.1 the fact that there is a function $\lambda(\sigma)$ which completely determines the answer to the geography problem in the currently unsolved case of non-negative signature.

We will close out Chapter 3 by also providing an overview of the primary geometric invariant that is relevant to our later constructions, namely the Seiberg-Witten invariants. We will also provide examples as to how those invariants are used when considering the symplectic geography problem. We further highlight that two of these examples will provide us with one half of the building blocks that are required for our constructions in Chapter 6.

In Chapter 4, we discuss and review the main construction method used in this thesis, namely the concept of branched covering spaces. We also provide many useful results pertaining to branched covers that will be used repetitively throughout this thesis. We shall also provide some new insights into less studied cases of branched covers.

While branched covers form the basis for the 4-manifolds considered, our specific construction breaks apart into two pieces. First, in Chapter 5 we discuss the main building blocks of this thesis; namely, we construct smooth symplectic 4-manifolds with arbitrarily large signature by taking covers of rational surfaces branched over embedded arrangements of hyperplanes.

We also prove that almost all so-called "supersolvable" line arrangements can be used to construct positive signature 4-manifolds. Specifically, we classify the behaviour of covers branched over these arrangements in the following theorem:

Theorem. *Suppose $\mathcal{L}$ is a supersolvable arrangement of n lines in $\mathbb{CP}^2$. Let $\rho : X \to \mathbb{CP}^2$ be a $(\mathbb{Z}/p\mathbb{Z})^k$ cover branched over $\mathcal{L}$. The behaviour of the signature of X is determined by the number of intersection points with multiplicity greater than 2. In particular, we know that:*

1. *If $\sum_{i=3}^n t_i = 0$, then $n = 3$, and the $(\mathbb{Z}/p\mathbb{Z})^2$ cover branched over $\mathcal{L}$ is $\mathbb{CP}^2$. In particular, $\sigma(X) = 1$.*

2. *If $\sum_{i=3}^n t_i = 1$, then $\mathcal{L}$ is either a pencil ($t_n = 1$) or a near pencil ($t_{n-1} = 1$). If $\mathcal{L}$ is a near pencil, then any $(\mathbb{Z}/p\mathbb{Z})^k$ cover branched over $\mathcal{L}$ satisfies $\sigma(X) = 0$.*

3. *If $\sum_{i=3}^n t_i = 2$, then $\mathcal{L}$ is constructed from two pencils by adding a line through both intersection points (which we call a connected double pencil). Furthermore any $(\mathbb{Z}/p\mathbb{Z})^k$ cover branched over $\mathcal{L}$ satisfies $\sigma(X) = -p^{k-2}$. In particular, if X is a $(\mathbb{Z}/p\mathbb{Z})^2$ branched cover, then $\sigma(X) = -1$.*

4. *If $\sum_{i=3}^n t_i \geq 3$, then $\sum_{i=3}^n t_i \geq 4$ and the leading coefficient of $\sigma(X)$ satisfies*

$$\left(3 - n + \sum_{i=3}^{n}(i-2)t_i\right) > 0, \tag{1.1}$$

and specifically, $\sigma(X) \to \infty$ *as* $p \to \infty$.

Finally, in Chapter 6 we start by providing a condition under which embedded submanifolds carry sufficient data on the fundamental group of their ambient space. We then utilize this condition, in conjunction with the symplectic 4-manifolds constructed in the previous chapter as well as a building block reviewed in Section 3.2 to obtain symplectic 4-manifolds that are applicable to the symplectic geography problem.

This results in one of the main results of this thesis. Specifically we show that the following holds.

Theorem. *Let* $\lambda(\sigma)$ *denote the function in Definition 3.1.8 that determines a solution to the geography problem for non-negative signature. Then the asymptotic behaviour of* $\lambda(\sigma)$ *is bounded by*

$$\limsup_{\sigma\to\infty} \frac{\lambda(\sigma)}{\sigma} \leq \frac{11}{8} = 1.375.$$

As of writing, this result is the current best known asymptotic limit of the ratio $\lambda(\sigma)/\sigma$. This is Corollary 22 of [33] and the proof can be found under Corollary 6.3.3.

At the end of this chapter we also prove the existence of smooth covers branched over a line arrangement with a Galois group of the form $G = (\mathbb{Z}/p\mathbb{Z})^2$. Specifically, we prove the following.

Theorem. *Let* $\mathcal{L}$ *be any (non-pencil) arrangement of* $n+1$ *lines in* $\mathbb{CP}^2$. *Then for all primes*

$$p \geq 2n \sum_{i=0}^{n-1} 2^i = 2n(2^n - 1),$$

there exists a smooth (and symplectic) $(\mathbb{Z}/p\mathbb{Z})^2$ *covering* $X \to \widetilde{\mathbb{CP}^2}$ *with branch locus* B *and a birational map* $\beta : \widetilde{\mathbb{CP}^2} \to \mathbb{CP}^2$ *satisfying* $B = \beta^{-1}(\mathcal{L})$.

Finally, we conclude this thesis by showing that for each smooth cover guaranteed to exist by the above theorem, there also exists an embedded symplectic submanifold of (real) dimension 2 for which the fundamental group of the submanifold surjects onto the fundamental group of the ambient symplectic 4-manifold.

Chapter 2

Notation

Before we can follow this structure as outlined in the introduction, we must first commit to the notations and conventions that will be followed in this thesis.

All manifolds in this thesis are assumed to be smooth, closed, connected and orientable unless otherwise specified. Furthermore, a 4-manifold we will mean either a 4-dimensional *real* manifold or a 2-dimensional *complex* manifold (complex surface) that satisfies the above conditions. While we may occasionally call a 4-manifold a complex surface, surface will typically be used to denote a 2-dimensional *real* surface (or a complex curve).

All homology and cohomology groups of a manifold will be taken to have coefficients in $\mathbb{Z}$ and so we will simply write $H_i(X)$ or $H^i(X)$, respectively. Furthermore, any group that is neither a fundamental group, a homology group, nor a cohomology group will be assumed to be finite.

By a symplectic 4-manifold, we mean a pair (X, ω) of a 4-manifold X which admits a symplectic structure ω that is a closed non-degenerate smooth 2-form. However, as at no time in this thesis will the specific structure ω be used, we will simply drop the pair notation. That said, while the majority of 4-manifolds discussed will be symplectic, we will still make it clear by stating when a manifold is symplectic or not.

While we will not delve into the language of algebraic geometry too often, we still make it clear that any variety discussed will be over the complex numbers and is assumed to be irreducible.

Chapter 3

Geometric structures on 4-manifolds

The goal of this thesis is to construct 4-manifolds on which there exists various geometric structures. Specifically we will be concerned with manifolds that admit symplectic structures (a closed non-degenerate 2-form) or exotic smooth structures (non-diffeomorphic smooth structures on the same topological manifold). Therefore, to fully contextualize the results in this thesis, this chapter will be dedicated to providing a brief overview of what is known regarding various classifications of these structures.

While the definition and some facts regarding symplectic structures will be assumed, an in-depth understanding of such structures will not be required to follow this thesis as the construction methods are more algebraic and combinatorial than strictly geometric. That said, we will also assume familiarity with some fundamental results on 4-manifolds. Namely, the definitions that appear in addition to some consequences of Freedman's theorem on the classification of simply connected topological 4-manifolds.

In this chapter, we will start by providing an overview of the *symplectic geography problem* for 4-manifolds. The second half of this chapter will then be dedicated to a review of a standard method (the Fintushel-Stern knot surgery) for which the symplectic (or more generally the smooth) structure on a 4-manifold can be modified under certain circumstances.

3.1 The geography of symplectic 4-manifolds

To help set the stage, we shall start by recalling that for *smooth* 4-manifolds, Freedman's Theorem can be simplifed.

Theorem 3.1.1. (§1 of [21]) *Two smooth simply connected 4-manifolds are homeomorphic if and only if they have isomorphic intersection forms.*

As a consequence of Freedman's theorem, the homeomorphism type of a simply connected (smooth) 4-manifold is completely and uniquely determined by its Euler characteristic, signature and the parity of its intersection form. Furthermore, if a 4-manifold X has no 2-torsion (i.e., $H_1(X)$ has no 2-torsion), it is known that we can replace the parity of the intersection form by whether or not the 4-manifold is spin.

Proposition 3.1.2. (cf. p. 164 of [60]) *Any 4-manifold without 2-torsion admits spin structures if and only if its intersection form is even.*

Therefore, if we restrict ourselves to only consider spin (or in our case nonspin) 4-manifolds, the homeomorphism type of a *simply connected, smooth 4-manifold* X is completely determined by only its Euler characteristic $e(X)$ and its signature $\sigma(X)$.

In the above context, a geography problem asks for which possible Euler characteristics and signatures (i.e., for which topological 4-manifolds) can certain geometric structures exist. For this thesis we are primarily interested in simply connected nonspin irreducible symplectic 4-manifolds. Correspondingly, we consider the following *specific* geography problem, which is a special case of a more general symplectic geography problem first asked by McCarthy and Wolfsen [47].

Definition 3.1.3. The *symplectic geography problem* asks: For which pairs $(e, \sigma) \in \mathbb{Z} \times \mathbb{Z}$ does there exist a simply connected nonspin irreducible symplectic 4-manifold X with $e(X) = e$ and $\sigma(X) = \sigma$?

Remark 3.1.4. In general there are other well known geography problems [62]. The most notorious of which is the complex geography problem (also referred to as the classification of complex surfaces) in which the above definition replaces simply connected *nonspin irreducible symplectic 4-manifold* with simply connected *minimal complex surface*. For information on the complex geography problem we refer the reader to [25].

Alternatively, we could drop the nonspin condition and in doing so one would arrive at the form of the symplectic geography problem that McCarthy and Wolfsen first asked. However, unless stated otherwise, we will use symplectic geography problem to refer to the nonspin formulation, for reasons that we will see below.

In the case that X is a symplectic 4-manifold (or more generally any manifold admitting an almost complex structure), it follows from the work of Taubes [65] and Hirzebruch [38] that the holomorphic Euler characteristic and first Chern class satisfy the equations

$$\chi_h(X) = \frac{1}{4}(e(X) + \sigma(X)), \tag{3.1}$$

$$c_1^2(X) = 2e(X) + 3\sigma(X), \tag{3.2}$$

respectively. Furthermore, as both $\chi_h(X)$ and $c_1^2(X)$ are non-negative integers (in fact $\chi_h \geq 1$), we can rephrase the symplectic geometry problem to ask for which pairs (a, b) with $a \geq 1$ and $b \geq 0$ does there exist a simply connected nonspin irreducible symplectic 4-manifold X satisfying $\chi_h(X) = a$ and $c_1^2(X) = b$.

Remark 3.1.5. When considering the relationships among $\chi_h(X)$, $c_1^2(X)$, $e(X)$ and $\sigma(X)$, we can observe that two necessary condition on the pairs (e, σ) is that $e + \sigma \equiv 0 \pmod 4$ and $2e + 3\sigma \geq 0$. It turns out, that in the case of $\sigma < 0$ these are also sufficient conditions.

Theorem 3.1.6. (can be deduced from Theorem 2 in [5]) *If $(e, \sigma) \in \mathbb{Z} \times \mathbb{Z}$ and $\sigma < 0$, then there exists a simply connected nonspin irreducible symplectic 4-manifold X with $e(X) = e$ and $\sigma(X) = \sigma$ if and only if*

- $2e + 3\sigma \geq 0$,
- $e + \sigma \equiv 0 \pmod 4$.

As a slight digression, we further note a result of Rohlin which states that any simply connected *spin* 4-manifold X satisfies $\sigma(X) \equiv 0 \pmod{16}$ [57]. In particular, we highlight an earlier result of Park and Szábo [53] in which they completed an answer to the geography problem in the spin case, building off the foundational work of Gompf [24]. The statement of which is the same as the above theorem with only this additional restriction from Rohlin.

Returning to the nonspin case, A complete answer to the geography problem in the case of $\sigma \geq 0$ is still an open problem. Furthermore, the condition $2e + 3\sigma \geq 0$ is comparatively

weak in this case. Therefore, to fully contextualize the progress on the geography problem in the $\sigma \geq 0$ case we require the following definition.

Definition 3.1.7. We say that a 4-manifold X has ∞^2-*property* if there exist infinitely many pairwise nondiffeomorphic irreducible symplectic 4-manifolds and infinitely many pairwise nondiffeomorphic irreducible nonsymplectic 4-manifolds, all of which are homeomorphic to X.

In the context of the above definition, we define the condition that replaces of the non-negativity condition of $2e + 3\sigma$ in Theorem 3.1.6.

Definition 3.1.8. (cf. Definition 13 in [3]) Given an integer $\sigma \geq 0$, let $\lambda(\sigma)$ be the smallest positive integer with the following two properties.

(i) $\lambda(\sigma) \geq \lceil (\sigma + 1)/2 \rceil$.

(ii) Every integral point (a, b) on the line $b = 8a + \sigma$ satisfying $a \geq \lambda(\sigma)$ is realized as $(\chi_h(X), c_1^2(X))$, where X is a compact simply connected nonspin 4-manifold without boundary that has ∞^2-property.

The well-defined-ness of $\lambda(\sigma)$ was first recognized in [54] and further investigated in [4]. Furthermore, if X is simply connected (and so $b_1 = b_3 = 0$) we have $e(X) \geq 2 + \sigma(X)$ and so we see that the strongest possible solution to the geography problem would be to show that $\lambda(\sigma) = \lceil (\sigma + 1)/2 \rceil$. The hopeful claim that λ takes this form was first conjectured in [3]. This problem is currently unsolved and the difference between the regimes of "small" and "large" values of σ suggests that geometers likely lack the tools to completely handle this problem at the current time. It is therefore expected that this problem will remain open for some time as it is comparatively more difficult to construct manifolds with positive signature as opposed to manifolds with negative signature.

This is perhaps most apparent if we were to drop the condition of irreducibility. In this case, if one has (for instance) a Kähler manifold X and takes the blow up at a point, then the blown-up 4-manifold $\widehat{X} \cong X \# \overline{\mathbb{CP}}^2$ will be a symplectic 4-manifold with $\sigma(\widehat{X}) = \sigma(X) - 1$. That is, it is 'trivial' (straightforward) to make non-irreducible symplectic 4-manifolds with arbitrarily negative signature. On the other hand, the vanishing theorem (cf. Theorem 3.2.2 below) implies that we cannot perform the same operation but in the other direction.

That is, we cannot simply 'add on' summands of $\mathbb{CP}^2$ to obtain large signature symplectic manifolds.

In this context, this thesis seeks to discuss the behaviour of $\lambda(\sigma)$ for the case of "large" σ by investigating construction techniques that result in 4-manifolds with arbitrarily large signature. We shall also provide conditions under which certain constructions are guaranteed to yield manifolds with positive signature.

3.2 Modifying smooth and symplectic structures

In general, the geography problem is a question of existence. However, as suggested by Definition 3.1.7, there is also a separate question that is often asked regarding the uniqueness of structures on 4-manifolds (sometimes referred to as botany problem). This is perhaps most apparent when considering strictly smooth structures (as opposed to symplectic structures) as there is also what one can call the *exotic geography problem*, which asks which topological 4-manifolds admit exotic smooth structures.

In fact, when we consider the definition of $\lambda(\sigma)$, it is required that we construct 4-manifolds with not only a symplectic structure but also infinitely many symplectic structures as well as infinitely many nonsymplectic smooth structures. For this reason, this section will review the knot surgery of Fintushel and Stern [19] and how it can be used modify the smooth (symplectic) structures on a 4-manifold.

Definition 3.2.1. Let X be a 4-manifold containing an embedded torus T with trivial normal bundle, and suppose that K is a knot in S^3. The result of *knot surgery* along T is a 4-manifold of the form

$$X_K := (X - \nu(T)) \cup_\varphi (S^1 \times (S^3 - \nu(K)),$$

where $\nu(T) \cong T \times D^2$ is a tubular neighborhood of T, and the gluing map φ between the boundary 3-tori sends the homology class of the meridian $\mu(T) = \{\text{point}\} \times \partial D^2 \subset \partial(\nu(T))$ of T to that of the longitude of K.

In this thesis, we will often refer to a torus T as in the above definition as a near-cusp embedded torus (cf. page 533 of [60]) as T will be contained in a tubular neighbourhood of a *twice pinched* embedded torus (cusp) in X.

We also highlight that the above definition does not completely account for the framing of the knot and so the isotopy class of the map φ is not completely determined. However, for our purposes this does not make a difference. Specifically, if one has

$$\pi_1(X) = \pi_1(X - T) = 1,$$

then X_K is also simply connected and has the same intersection form as X (a related example that serves as a sketch of the fact that X_K is simply connected can be found in the proof of Theorem 6.2.5). Consequently, an application of Freedman's Theorem then implies that X and X_K are homeomorphic. We refer the reader to [19] for further details on the specifics of this construction.

The most important reason as to why we consider Definition 3.2.1 is the fact that a knot surgery *can* (but not necessarily always) change the smooth (symplectic) structure on a 4-manifold without changing the homeomorphism type. However, to fully appreciate the consequences of this construction, a cursory understanding of some gauge theory is required. Therefore, while we shall omit some (many) details, we provide a brief overview of the Seiberg-Witten invariants in preparation for a survey of the results that are pertinent to our case.

Suppose that X is a 4-manifold with $b_2^+(X) > 1$. We shall omit the strict definition and simply recall from Lecture 2 in [20] that the Seiberg-Witten invariant of X can be expressed as an integer-valued function

$$SW_X : \mathrm{Spin}^c(X) \longrightarrow \mathbb{Z},$$

where $\mathrm{Spin}^c(X)$ is the set of all $\mathrm{Spin}^c(4)$ structures on X. It is a known fact to (cf. Proposition 2.4.16 of [25]) that if $H^2(X)$ has no 2-torsion, then the set of all $\mathrm{Spin}^c(4)$ on X is in 1-1 correspondence with a subset of $H^2(X)$. Namely,

$$\mathrm{Spin}^c(X) \longleftrightarrow \mathcal{C}_X = \{K \in H^2(X) \mid K \equiv w_2(X) \pmod 2\},$$

where $w_2(X)$ is the second Stiefel–Whitney class of X (or more specifically of the tangent bundle TX). Given a $\mathrm{Spin}^c(4)$ structure $\mathfrak{s}$, the exact correspondence is given by $\mathfrak{s} \mapsto c_1(\mathfrak{s})$. We also note that even if $H^2(X)$ has 2-torsion, we can still choose to work with $H^2(X)$ instead of $\mathrm{Spin}^c(X)$. In particular, given an element $L \in H^2(X)$, we can set

$$I_L = \{\mathfrak{s} \in \mathrm{Spin}^c(X) \mid c_1(\mathfrak{s}) = L\}.$$

Then we can define the Seiberg-Witten invariant evaluated on L by summing over all $\mathrm{Spin}^c(4)$ structures in I_L. That is,

$$SW_X(L) = \sum_{\mathfrak{s} \in I_L} SW_X(\mathfrak{s}).$$

Note in the line above, we set $SW_X(L) = 0$ if I_L is the empty set. This results in a function

$$SW_X : H^2(X) \longrightarrow \mathbb{Z}.$$

A cohomology class $K \in H^2(X)$ for which $SW_X(K) \neq 0$ is called a Seiberg-Witten *basic* class of X and the set of all such basic classes will be denoted by Bas_X. It can be shown (for instance by the generalized adjunction formula in [44], [50]) that there are at most finitely many Seiberg-Witten basic classes for a given 4-manifold.

The most important fact regarding the Seiberg-Witten invariant of a 4-manifold X is that it *is* an invariant. Specifically, if $h : X_1 \to X_2$ is a diffeomorphism, then

$$SW_{X_1}(h^*(L)) = \pm SW_{X_2}(L)$$

for all $L \in H^2(X_2)$. Finally, if we write $H = H^2(X)$, then the Seiberg-Witten invariant of X can also be expressed as an element

$$\overline{SW}_X = \sum_{L \in H} SW_X(L)\, L \in \mathbb{Z}[H],$$

in the integer group ring of H.

We note that in general the Seiberg-Witten invariants are not a complete diffeomorphism invariant of 4-manifolds. In particular, there are many distinct 4-manifolds for which their Seiberg-Witten invariants vanish (i.e., $SW_X \equiv 0$).

Theorem 3.2.2. (cf. Theorem 2.4.6 (1) in [25]) *Suppose X is a simply connected 4-manifold with with $b_2^+(X) > 1$ and odd. Then if $X = X_1 \# X_2$ and $b_2^+(X_i) > 0$ $(i = 1, 2)$, then $SW_X \equiv 0$.*

Before we return to our discussion on the modification of symplectic 4-manifolds, we state how the Fintushel and Stern knot surgery interacts with the Seiberg-Witten invariant.

Lemma 3.2.3. (§1 of Lecture 3 in [20]) *Let X be a 4-manifold with $b_2^+(X) > 1$. Let T be a homologically nontrivial near-cusp embedded torus in X sith self-intersection $[T]^2 = 0$. Further assume that $\pi_1(X) = \pi_1(X - T) = 1$. Let X_K denote the result of performing a knot surgery along T using the knot K. Then X_K is homeomorphic to X and via the isomorphism of the group rings induced by the homeomorphism, we further have*

$$\overline{SW}_{X_K} = \overline{SW}_X \cdot \Delta_K(PD(2[T])),$$

where $PD : H_2(X) \to H^2(X)$ denotes the Poincaré duality homomorphism and the product on the right-hand side is the product inside the group ring $\mathbb{Z}[H^2(X_K)]$. Furthermore, $\Delta_K(t)$ is the symmetrized Alexander polynomial of K.

Example 3.2.4. Let X be an elliptic K3 surface for which the elliptic fibration $f : X \to \mathbb{CP}^1$ contains a cusp fiber. It is known that $SW_X(0) = 1$ and $SW_X(L) = 0$ for all $L \neq 0 \in H^2(X)$. If T denotes any smooth torus fiber of the elliptic fibration, then T is a near-cusp embedded torus of self-intersection 0. Furthermore, we can conclude $\pi_1(X - T) = 1$ since any meridian circle of T will correspond to a boundary of a punctured sphere section, in addition to $\pi_1(X) = 1$. Suppose K is a trefoil knot with symmetrized Alexander polynomial equal to

$$\Delta_K(t) = t - 1 + t^{-1}.$$

Performing knot surgery along T with the trefoil K, we compute using Lemma 3.2.3 the Seiberg-Witten invariant of X_K to be

$$\overline{SW}_{X_K} = \Delta_K(PD(2[T])) = PD(2[T]) - [0] + PD(-2[T]),$$

where $[0] \in H^2(X)$ denotes the trivial class.

The above example shows how the Seiberg-Witten invariant of a 4-manifold undergoing a knot surgery is computed in practice (a practice we will eventually return to). It also provides us with an example of two smooth 4-manifolds that form an exotic pair. We know from Lemma 3.2.3 that X and X_K are homeomorphic to each other. However, it is clear that their Seiberg-Witten invariants are different by counting basic classes. It follows that the smooth structures on X and X_K are distinct. In fact, we will see (Example 3.2.8 below) that both X and X_K admit non-equivalent symplectic structures as well.

However, before this aforementioned example we discuss an interesting fact regarding the Seiberg-Witten invariant of a symplectic 4-manifold first discovered by Taubes.

Theorem 3.2.5. (cf. Main Theorem in [64]) *If X is a symplectic 4-manifold with $b_2^1(X) > 1$, then the classes $\pm c_1(X)$ are Seiberg-Witten basic classes. In particular, we have*

$$SW_X(c_1(X)) = \pm 1.$$

The primary consequence of the above theorem is that 4-manifolds for which the Seiberg-Witten invariants vanish cannot admit any symplectic structure.

Example 3.2.6. If we consider the 4-manifold $Y = 3\mathbb{CP}^2 \# 19\overline{\mathbb{CP}}^2$. By Theorem 3.2.2, we see that $SW_Y \equiv 0$ and so Y does not admit symplectic structures. At the same time, we compute $e(Y) = 24$ and $\sigma(Y) = 16$. That is, Y carries the same the Euler characteristic and signature as both complex surfaces in Example 3.2.4 whereas Y is the only 4-manifold of the three not admitting symplectic structures. This helps to showcase why care must be taken when studying the symplectic geography problem, as there is some nuance to finding the appropriate geometric structures.

Theorem 3.2.5 can also be used to provide examples of 4-manifolds whose Seiberg-Witten invariants don't vanish and yet also do not admit symplectic structures.

Theorem 3.2.7. (cf. Corollary 1.3 in [19]) *Assume that T is a near-cusp embedded torus in a (possibly symplectic) 4-manifold X, and that K is a knot whose Alexander polynomial is not monic. Then X_K admits no symplectic structures.*

As a consequence to Theorem 3.2.7, we obtain many more examples of topological 4-manifolds that don't carry symplectic structures. Returning to the context of Example 3.2.4, instead of the trefoil knot we may consider the alternating twist knot with $2m-1$ half twists, which we will denote by K_m. Then for each positive integer $m > 0$ we have

$$\Delta_{K_m}(t) = mt - (2m-1) + mt^{-1}.$$

In particular, if $m \neq 1$ (i.e., K_m is not the trefoil) then since $\Delta_{K_m}(t)$ is not monic, the smooth manifolds X_{K_m} that are each homeomorphic to a K3 surface (by Lemma 3.2.3), will not only all be pairwise non-diffeomorphic but each will also not admit any symplectic structure. We contrast this infinite family of non-symplectic 4-manifolds to the following example.

Example 3.2.8. Suppose K is a fibered knot in the 3-sphere S^3 with genus $g(K)$ and X is a symplectic 4-manifold such that a near-cusp embedded torus $T \subset X$ is a symplectic submanifold. It is known that the 4-manifold X_K obtained from the knot surgery defined above inherits a symplectic structure (cf. Corollary 1.2 in [19]).

As a torus knot, the trefoil is a fibered knot. Therefore we can conclude that both 4-manifolds in Example 3.2.4 carry a symplectic structure. Moreover, since the knot sum of fibered knots is a fibered knot (cf. page 326 of [58]), the sum of n trefoil knots K'_n each has distinct Alexander polynomial

$$\Delta_{K'_n}(t) = (t - 1 + t^{-1})^n.$$

Therefore, performing various knot surgeries along the elliptic K3 surface from above we find infinitely many symplectic 4-manifolds $X_{K'_n}$, for which they are all pairwise non-diffeomorphic by considering their Seiberg-Witten invariants. In fact, this also implies that the infinitely many symplectic structures are also all distinct.

Now, as we are interested in 4-manifolds that cannot be decomposed into a non-trivial connected sum, we shall review some useful facts that can be used to conclude when a symplectic 4-manifold is irreducible. For that we will start by recalling the definition of minimality.

Definition 3.2.9. A (symplectic) 4-manifold X is said to be *minimal* if it does not contain an embedded (symplectic) sphere of self-intersection -1.

We note that while the above definition is often stated for symplectic 4-manifolds, it has been observed that even in the symplectic case, this is equivalent to X not admitting any embedded *smooth* sphere with self-intersection -1 [68].

In general, a 4-manifold being minimal does not say anything regarding the irreducibility of the 4-manifold. However, in the case of symplectic manifolds, there are occasions for which they are equivalent.

Lemma 3.2.10. (cf. Corollary 1.4 in [30] and Corollary 5.5 in [43]) *Minimal symplectic 4–manifolds with $b_2^+ > 1$ and with residually finite fundamental groups are irreducible.*

Of particular interest to us is that the above lemma works for simply connected 4-manifolds. Not only that, by considering the Seiberg-Witten invariant of a symplectic 4-manifold, it can also be possible to conclude minimality (and hence irreducibility).

Lemma 3.2.11. (cf. Exercise 10.1.20 (b) in [25]) *A symplectic 4-manifold X with $b_2^+(X) > 1$ is minimal if and only if there is no Seiberg-Witten basic class $K \in Bas_X$ with $(c_1(X) - K)^2 = -4$.*

We briefly return to the infinitely many symplectic (and non-symplectic) 4-manifolds above that are all homeomorphic to a K3 surface. A computation of the Seiberg-Witten invariants can show using Lemma 3.2.11 that the symplectic 4-manifolds $X_{K'_n}$ are all irreducible (for an example of this computation, see the proof of Theorem 6.2.5 below). It can also be shown that the non-symplectic 4-manifolds X_{K_m} are also all irreducible. In particular, referring back to Definition 3.1.7 we see that the elliptic K3 surface provides a concrete example of 4-manifolds that have the ∞^2-property. Albeit, they do not have positive signature.

Chapter 4

Abelian branched covers

The main construction method of this thesis is that of branched coverings of complex surfaces. For this reason we provide a brief review and overview of the important facts that are known surrounding branched coverings for which the covering group is abelian. For a more in depth description of abelian branched covers, we refer the reader to [37] for a relatively topological perspective or [52] for a primarily algebraic geometric perspective.

Definition 4.0.1. Let Y be a (smooth) variety. A Galois (or regular) branched covering is a finite surjective map $\rho : X \to Y$ together with a group G acting faithfully on X and transitively on the fibers of $\rho : X \to Y$. In this case Y can be exhibited as the quotient X/G. The group G is referred to as the Galois group of the branched covering. We say this covering is smooth if X is smooth.

While we will be primarily considering the case in which Y is a compact complex surface, we may occasionally state results in the context of varieties similar to the definition above.

In general we will let $\rho^u : X^u \to Y - B$ denote the unbranched covering corresponding to ρ. It is well known that unbranched coverings of $Y - B$ with Galois group G are in one-to-one correspondence with surjective group homomorphisms $\psi : \pi_1(Y - B) \to G$. Furthermore, for such a covering, there is a unique way to complete X^u to X by the following theorem of Grauert and Remmert [26].

Theorem 4.0.2. *Let Y be a variety and $B \subset Y$ a closed subvariety such that $Y - B$ is dense in Y. If $\rho^u : X^u \to Y - B$ is a finite unbranched cover, then there is a unique*

irreducible normal variety X containing a homeomorphic copy of X^u and a unique covering map $\rho : X \to Y$ such that $\rho|_{X^u} = \rho^u$.

With this result it follows that Galois coverings of a compact complex surface Y branched along a divisor B are also in correspondence with surjective group homomorphisms $\psi : \pi_1(Y - B) \to G$. It is also known that the covering space X is a smooth complex surface except possibly at points in the fiber of a singular point of B. Consequently, by understanding the algebraic properties of homomorphisms between $\pi_1(Y - B)$ and G in addition to investigating neighbourhoods of points in $\mathrm{Sing}(B)$ (the singular points of B), it is possible to construct smooth covers of Y exhibiting B as a branching set.

Remark 4.0.3. If G is abelian, then we may factor $\psi : \pi_1(Y - B) \to G$ uniquely through $H_1(Y - B)$ via a surjective homomorphism $\phi : H_1(Y - B) \to G$ so that $\psi = \phi \circ h$ with $h : \pi_1(Y - B) \to H_1(Y - B)$ denoting the Hurwicz map. In general, when G is abelian we typically choose to work with ϕ instead of ψ.

4.1 Algebraic/geometric data of branched coverings

Throughout this section $\rho : X \to Y$ will denote a branched covering of Y with branch locus B and Galois group G. Furthermore, $\phi : H_1(Y - B) \to G$ will be the *defining map* for ρ (i.e., the surjective group homomorphism corresponding to ρ) if G is abelian.

As ρ can also be considered as a transitive group action on X, to fully understand branched covers we must consider the subsets of X on which G does not act freely. For this we follow [37] and begin by considering two classes of subgroups of G that carry information of the branch (ramification) locus of ρ.

Definition 4.1.1. For $W \subset X$, the inertia and stabilizer subgroups of W are

$$I_W = \{g \in G \mid g(x) = x \text{ for all } x \in W\},$$
$$H_W = \{g \in G \mid g(x) \in W \text{ for all } x \in W\},$$

respectively. Furthermore, given a curve C in Y, I_C and H_C are defined to be the inertia and stabilizer subgroups of any individual connected component C' of $\rho^{-1}(C)$.

Remark 4.1.2. In general, for $C \subset Y$ these subgroups are not well defined as for two different connected components C_1' and C_2' of $\rho^{-1}(C)$, $I_{C_1'}$ and $I_{C_2'}$ (or their stabilizer subgroups) are in general only subgroups conjugate to each other. However, in the case that G is abelian, or more generally a group in which all subgroups are normal, these subgroups will be well defined. For this reason, we will assume that all groups G discussed for the remainder of the thesis, with the exception of those in Section 4.5, are abelian.

The act of constructing specific branched covers of a complex surface in general comes from fixing a desired branch locus and then defining the surjective homomorphism from $H_1(Y - B)$ to G. The following lemma is proved in [37] and demonstrates how the distinguished subgroups of Definition 4.1.1 are determined by the defining map ϕ.

Lemma 4.1.3. (cf. Propositions I.3.8 and I.3.13 in [37]) *Let C be a curve in the branch locus B. Let $\mu_C \in H_1(Y - B)$ be the homology class of the meridian circle of C. Then*

(i) I_C is generated by $\phi(\mu_C)$.

(ii) H_C contains $\phi(\mu_C)$ and the elements $(C \cdot V)\phi(\mu_V)$ for all irreducible components $V \subset B$, where $C \cdot V$ is the intersection number. If C is smooth and has genus 0, then these elements generate H_C.

This next lemma is also proved in [37] and in conjunction with the above result yields a complete description on the smoothness of the covering space X in terms of the map ϕ.

Lemma 4.1.4. (cf. Propositions I.4.1 in [37]) *The covering surface X is smooth if and only if whenever two curves C and D in the branch locus intersect, the inertia subgroups I_C and I_D intersect only in the identity element.*

As we are primarily interested in the construction of symplectic (and therefore smooth) 4-manifolds, we will repeatedly return to Lemma 4.1.4 to provide us with an easily verifiable criterion as to when a cover is smooth. For this reason, we will refer throughout this thesis to the condition of trivial intersection of inertia subgroups simply as the *smoothness criterion*.

We will also see later in Chapter 6 that being able to find some distinguished subvarieties of X will be of interest. For that reason, we end this section by cherry picking three final

results from [37] that show how important information on curves in X can be computed with the inertia and stabilizer subgroups.

Let T be a union of curves in Y so that all the intersections in $T \cup B$ are transversal. Let $C, D \subset T$ be irreducible curves.

Lemma 4.1.5. (cf. p. 23 of [37]) *The number of curves in the lift $\rho^{-1}(C)$ is given by $\frac{|G|}{|H_C|}$.*

Lemma 4.1.6. (cf. Lemma II.3.3 in [37]) *The self-intersection number of a component C' in $\rho^{-1}(C)$ is given by $(C')^2 = \frac{|H_C|}{|I_C|^2} C^2$.*

Lemma 4.1.7. (cf. Lemma II.3.7 in [37]) *Let $p \in C \cap D$ and let C' and D' be two curves in $\rho^{-1}(C)$ and $\rho^{-1}(D)$ respectively so that C' and D' meet at a point above p. Then the number of points where C' and D' meet in the fiber $\rho^{-1}(p)$ equals*

$$\frac{|H_C \cap H_D|}{|I_C||I_D|}. \tag{4.1}$$

4.2 Branched covers of line arrangements

The starting point of all branched cover constructions considered in this thesis is the complex projective plane $\mathbb{CP}^2$. We also restrict to the case when the branch locus comes in the following form of a line arrangement.

Definition 4.2.1. A (complex) *line arrangement* $\mathcal{L}$ is a set of n distinct complex lines in $\mathbb{CP}^2$. Specifically, if $f_1, \ldots, f_n$ are distinct homogeneous polynomials defining linear hyperplanes $L_1, \ldots, L_n \subset \mathbb{CP}^2$, then $\mathcal{L} = L_1 \cup L_2 \cup \cdots \cup L_n$ is the reducible curve given by the equation $f_1 f_2 \cdots f_n = 0$.

If all coefficients of the f_i's can be chosen to be real, then $\mathcal{L}$ can be further called a real line arrangement, as it can also correspond to an arrangement of real lines in $\mathbb{RP}^2$. In the case of real line arrangements, their (complex) counterparts can be expressed visually as an $\mathbb{R}^2$ slice of $\mathbb{C}^2$, see Figure 4.1. We note however, that in general it must be taken into consideration that there may be intersections occurring at infinity, corresponding to the case that there are parallel lines in the representation of the arrangement.

If an arrangement is not real, we often still represent them in the plane with the understanding that there will be some 'non-physical' aspects. These typically include lines

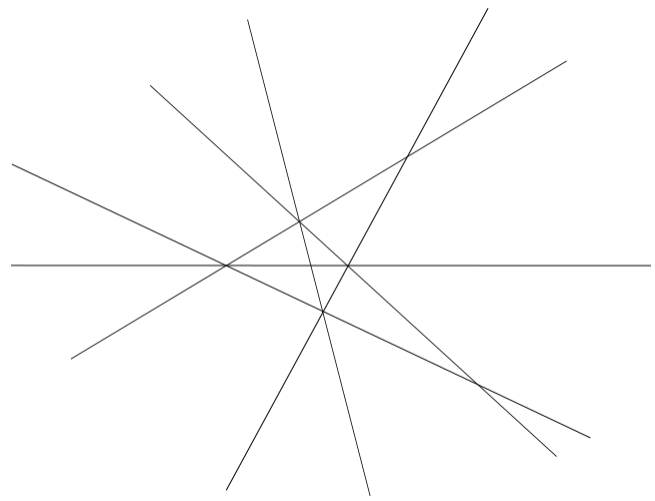

Figure 4.1: A (real) line arrangement of six lines

appearing to cross at points in which they don't or the addition of requiring lines to be viewed visually as generic curves and not hyperplanes (i.e., not straight). In general we will omit this distinction between the types of line arrangement (in part due to it being an open problem on exactly which arrangements can be made to be real). Occasionally we will highlight whether an arrangements is real or complex when it is known and 'non-obvious'.

As evidenced by Lemma 4.2.2 below, any cover branched over a line arrangement of 2 or less lines will have a cyclic Galois group. This thesis is interested in the case of non-cyclic Galois coverings and so all line arrangements considered will always consist of $n \geq 3$ lines. Furthermore, we will also assume that not all the lines pass through a single shared point of intersection. Such line arrangements are called *pencils*. A partial justification as to the decision to disregard pencils can be found in the paragraph preceding Remark 4.2.7.

For now, let us suppose we have a surjective homomorphism $\phi : H_1(\mathbb{CP}^2 - \mathcal{L}) \to G$. We let μ_i denote the homology class of a meridian circle around L_i in $H_1(\mathbb{CP}^2 - \mathcal{L})$, where we have chosen the orientation to be compatible with that of the normal plane in a tubular neighbourhood of L_i. In this case, we have from Lemma 4.1.3 that $I_{L_i} = \langle \phi(\mu_i) \rangle$ and so a covering map $\rho : X \to Y = \mathbb{CP}^2$ branched over a line arrangement $B = \mathcal{L}$ is in general determined by associating to each line L_i a non-identity element of G (as a curve C is contained in the branch locus of a covering if and only if I_C is non-trivial). In fact, the

order of $\phi(\mu_i)$ in G corresponds to the ramification degree of L_i under the mapping ρ.

In the case of a line arrangement in $\mathbb{CP}^2$, $H_1(\mathbb{CP}^2 - \mathcal{L})$ comes with the following explicit description.

Lemma 4.2.2. (cf. Lemma III.1.3 in [37]) *There is an isomorphism $H_1(\mathbb{CP}^2 - \mathcal{L}) \cong \mathbb{Z}^{n-1}$. Moreover, $H_1(\mathbb{CP}^2 - \mathcal{L})$ is generated by $\mu_1, \mu_2, \ldots, \mu_n$, and these generators satisfy only one relation $\mu_1 + \mu_2 + \cdots + \mu_n = 0$.* □

As an (immediate) consequence of Lemma 4.2.2, we have:

Corollary 4.2.3. *Let $g_1, \ldots, g_n$ be non-identity elements of G containing a generating set for G such that $g_1 + \cdots + g_n = 0$. Then $\phi : H_1(\mathbb{CP}^2 - \mathcal{L}) \to G$ given by $\phi(\mu_i) = g_i$ defines a topological covering of $\mathbb{CP}^2$ branched exactly over $\mathcal{L}$.*

Thanks to Lemma 4.2.2, we also note that in some specific cases, information on certain inertia or stabilizer subgroups of various embedded curves in the total space X of a covering branched over a line arrangement can be determined without any knowledge of the specific defining map ϕ.

Corollary 4.2.4. *Let $\mathcal{L} = L_1 \cup L_2 \cup \cdots \cup L_n$ be a line arrangement and suppose that $\phi : H_1(\mathbb{CP}^2 - \mathcal{L}) \to G$ defines a cover $\rho : X \to \mathbb{CP}^2$ branched over $\mathcal{L}$. Further assume that $\mathcal{L}$ contains a line in general position. That is, assume for some i, the line $L = L_i$ intersects $\bigcup_{j \neq i} L_j$ at exactly $n-1$ points. Then $H_L = G$ and so there is a unique curve in the lift $\rho^{-1}(L)$.*

Proof. Suppose L is a line in $\mathcal{L}$ in general position. Since L intersects $\bigcup_{j \neq i} L_j$ at exactly $n-1$ points, it follows that L intersects all lines in $\mathcal{L}$ except itself exactly once (this comes from the fact that lines in a line arrangement correspond to *linear* forms, in addition to Bezout's theorem). In particular, since L is a smooth genus 0 curve, Lemma 4.1.3 implies that H_L is generated by $\phi(\mu_{L_i})$ for all lines L_i in $\mathcal{L}$. Now by Lemma 4.2.2, the μ_i's generate $H_1(\mathbb{CP}^2 - \mathcal{L})$ and since ϕ is necessarily surjective, we find

$$H_L = \langle \phi(\mu_{L_1}), \phi(\mu_{L_2}), \ldots, \phi(\mu_{L_n}) \rangle = G.$$

We then apply Lemma 4.1.5 to obtain the irreducibility of $\rho^{-1}(L)$. □

In general, a line arrangement $\mathcal{L}$ may not contain a line in general position (for examples, see Section 5.3.1 and Section 5.3.2). However, we nonetheless consider the following example (which will come up multiple times in this thesis) for which there is a distinguished line in general position.

Example 4.2.5. We will consider an arrangement $\mathcal{L}_n$ of $n \geq 4$ lines for which all but one line passes through a common point. Such a line arrangement is called the *near pencil* of n lines and an example is given in Figure 4.2 for $n = 6$ lines. We highlight that it is clear from the definition and the figure below that the line which does not pass through the $(n-1)$-fold intersection point is a line in general position.

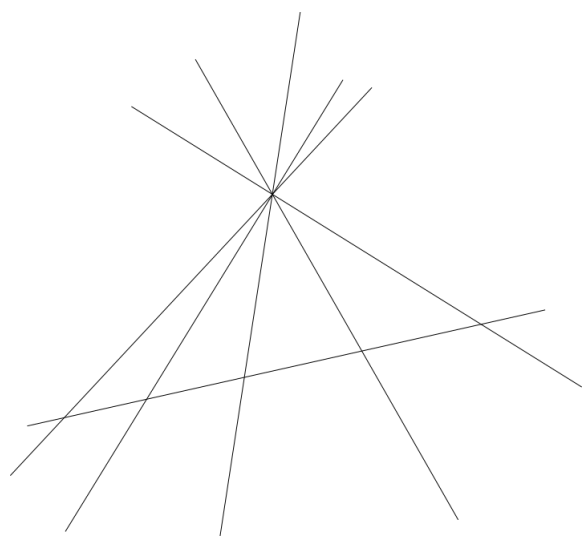

Figure 4.2: A near pencil arrangement

Remark 4.2.6. If X denotes the total space of a branched cover corresponding to a map ϕ as in Corollary 4.2.3, then it has been observed in Theorem I.2.15 in [37] and the start of Section 1 in [45] that the space X is a complex algebraic surface (which is not necessarily smooth). In particular, X is a projective surface, which is Kähler if it is smooth. Most importantly, this means that if we can construct *smooth* 4-manifolds using this construction, then we know they will also carry a symplectic structure. Finally, we also know that since Y is compact and boundaryless, so is X by Corollary I.2.3 in [37].

However, in general a branched cover as described in Corollary 4.2.3 will not be smooth.

As previously noted, the singular points of the cover X can occur only above singular points of the branch locus. As a line arrangement consists solely of linear hyperplanes, $\mathrm{Sing}(\mathcal{L})$ is exactly the set of points in $\mathbb{CP}^2$ that are contained in two or more lines. For such a point with multiplicity r (contained in r distinct lines), we will fix the notation

$$p_{i_1,i_2,\ldots,i_r} = L_{i_1} \cap L_{i_2} \cap \cdots \cap L_{i_r} \tag{4.2}$$

Now, combining Lemma 4.1.4 and Lemma 4.1.3, we see that smoothness of X above double points of $\mathcal{L}$ (those with multiplicity two) carries a simple and easy to check algebraic condition.

A bit more work is required to check when X is smooth at points contained in a fiber above a point with higher multiplicity. Suppose $p = p_{i_1,i_2,\ldots,i_r}$ is a singular point of $\mathcal{L}$ with multiplicity $r \geq 3$. We perform a blow up $\beta_p : \widetilde{Y}_p \to \mathbb{CP}^2$ at p so that the intersections of the exceptional sphere $E_p = \beta^{-1}(p)$ with the lines L_{i_j}, $1 \leq j \leq r$ are transverse. We can then consider the smoothness criterion (Lemma 4.1.4) for the intersection points of $\mathcal{L} \cup E_p$ in $\widetilde{Y}_p \cong \mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$. Therefore, by blowing up all points of higher multiplicity than 2, we can determine when X will be smooth.

Suppose $\widetilde{\mathbb{CP}^2}$ is the rational surface obtained by blowing up all points $p_{i_1,i_2,\ldots,i_r}$ where $r \geq 3$. Let t_r denote the number of intersection points of $\mathcal{L}$ with multiplicity r. Then it follows that there is a diffeomorphism $\widetilde{\mathbb{CP}^2} \cong \mathbb{CP}^2 \# N\overline{\mathbb{CP}}^2$ where $N = \sum_{r=3}^{n} t_r$.

For $k \in \{1, \ldots, N\}$, let p_k denote one of the higher multiplicity intersection points. Then if the blow-down map is given by $\beta : \widetilde{\mathbb{CP}^2} \to \mathbb{CP}^2$, we let $E_k := \beta^{-1}(p_k)$ be the exceptional sphere corresponding to p_k. If $p_k = p_{i_1,i_2,\ldots,i_r}$, we will use $\varepsilon_k = \varepsilon_{i_1,i_2,\ldots,i_r}$ to represent the homology class of a meridian around E_k in $\widetilde{\mathbb{CP}^2} - \beta^{-1}(\mathcal{L})$. It is known (for instance, Lemma 1.3 in [45]) that

$$\varepsilon_{i_1,i_2,\ldots,i_r} = \beta_*^{-1}(\mu_{i_1} + \mu_{i_2} + \cdots + \mu_{i_r}). \tag{4.3}$$

We should note that we still get a well defined branched cover after blowing up $\mathbb{CP}^2$ finitely many times. In fact, let $\widehat{L}_i$ denote the proper transform of L_i under β for each $1 \leq i \leq n$. Then for $\widehat{\mathcal{L}} = \widehat{L}_1 \cup \cdots \cup \widehat{L}_n$ and $\mathcal{E} = E_1 \cup \cdots \cup E_N$, we have the total transform of $\mathcal{L}$, which is $\beta^{-1}(\mathcal{L}) = \widehat{\mathcal{L}} \cup \mathcal{E}$. Moreover, the induced map

$$\beta_* : H_1(\widetilde{\mathbb{CP}^2} - \beta^{-1}(\mathcal{L})) \longrightarrow H_1(\mathbb{CP}^2 - \mathcal{L})$$

is an isomorphism as β is a diffeomorphism on $\widetilde{\mathbb{CP}^2} - \beta^{-1}(\mathcal{L})$. It follows that for a surjective homomorphism $\phi : H_1(\mathbb{CP}^2 - \mathcal{L})$ defining a branched cover, the composition

$$\phi \circ \beta_* \circ h : \pi_1(\widetilde{\mathbb{CP}^2} - \beta^{-1}(\mathcal{L})) \to G$$

will also define a branched cover. We will abuse notation and also refer to the cover $\rho : X \to \widetilde{\mathbb{CP}^2}$ as the cover of $\mathbb{CP}^2$ branched along $\mathcal{L}$ due to the aforementioned diffeomorphism between the unbranched bases $\widetilde{\mathbb{CP}^2} - \beta^{-1}(\mathcal{L})$ and $\mathbb{CP}^2 - \mathcal{L}$.

One of the consequences of the above discussion on the exceptional spheres is that $\phi(\varepsilon_{i_1,i_2,\ldots,i_r}) = \phi(\mu_{i_1}) + \phi(\mu_{i_2}) + \cdots + \phi(\mu_{i_r})$ and so the inertia subgroup

$$I_{E_k} = \langle \phi(\varepsilon_{i_1,i_2,\ldots,i_r}) \rangle \tag{4.4}$$

depends only on the the meridians around lines intersecting the point p_k. Moreover an analysis on the smoothness of X at points contained in $\rho^{-1}(E_k)$ is determined by a comparison between subgroups of G of the form $\langle \phi(\mu_{i_j}) \rangle$ and $\langle \phi(\mu_{i_1}) + \phi(\mu_{i_2}) + \cdots + \phi(\mu_{i_r}) \rangle$.

We note that in general we may also need to perform blow ups at double points. That is, if $\langle \phi(\mu_i) \rangle \cap \langle \phi(\mu_j) \rangle \neq \{id_G\}$ for lines L_i and L_j intersecting at a double point p, then $\phi(\mu_i) + \phi(\mu_j)$ may generate a subgroup of G that intersects I_{L_i} and I_{L_j} trivially (for instance, when the sum is the identity). However, we will see that for the case in which we are interested, this will not occur.

Remark 4.2.7. From the above we see that there are in general two different ways for the cover X to be smooth at a point in the pre-image of an intersection point $E_k \cap \widehat{L_i}$. If I_{E_k} is trivial, then the smoothness criterion is automatically satisfied. This corresponds to E_k not being contained in the branch locus of the cover and so the intersection point of E_k and L_i is not a singular point of B after the blow up. Alternatively, if I_{E_k} is non-trivial but still has trivial intersection with I_{L_i}, then the cover will still be smooth. In general we will focus on the case in which $B = \beta^{-1}(\mathcal{L})$ so that the branch locus is necessarily a connected subvariety of $\widetilde{\mathbb{CP}^2}$ (it is possible for the branch locus to be connected while not satisfying this criterion but we will not consider those cases here). For examples in addition to a discussion on the case when the cover may no longer be branched over E_k, we refer the reader to [45].

We also make note that if it is possible for a line arrangement $\mathcal{L}$ to satisfy the smoothness criteria in the way described in Remark 4.2.7 ($B = \beta^{-1}(\mathcal{L})$ is the branch locus) then it

cannot be a pencil of lines. As a pencil of lines is an arrangement where the only intersection point is a single shared intersection point amongst all of the lines, Lemma 4.2.2 implies that after blowing up the unique singular point of $B = \mathcal{L}$, the cover will not be branched over this corresponding exceptional sphere and so the total corresponding branch locus will be disconnected. For this reason, we remain steadfast when following the examples set by the pioneers of line arrangements and shall continue to only consider non-pencil line arrangements.

Remark 4.2.8. For a fixed line arrangement $\mathcal{L}$ and a fixed group G (even possibly cyclic), the above desingularization method is in general insufficient to find a smooth resolution of a topological branched covering of a line arrangement (i.e., one defined as in Corollary 4.2.3), even if you allow for a sequence of blow ups of intersection points on exceptional spheres. For an example, we refer the reader to [32] in which the authors provided an example of such a procedure failing to desingularize certain $\mathbb{Z}/5\mathbb{Z}$ covers.

4.3 Branched covers with elementary abelian Galois group

For now we will consider the case when the Galois group G is an elementary abelian group. In this case we must have that $G = (\mathbb{Z}/p\mathbb{Z})^k$ for some prime p and some $k \in \mathbb{N}$. In fact, we shall also assume that $k \geq 2$ to remain distinct from the deeply studied case of cyclic branched covers of 4-manifolds.

It is not hard to see that if $\mathcal{L}$ is a line arrangement consisting of n lines, then we must have $k \leq n - 1$ since

$$\phi : H_1(\mathbb{CP}^2 - \mathcal{L}) \cong \mathbb{Z}^{n-1} \to G = (\mathbb{Z}/p\mathbb{Z})^k \tag{4.5}$$

must be surjective. For any $2 \leq k \leq n - 1$ there is a multitude of ways to define a map as in Corollary 4.2.3, and so we can see that the nuance of this case comes from the question on the existence of defining maps for which the resulting branched cover is smooth.

We remark that in the case of G being an elementary abelian group, by applying Lemma 4.1.4, a condition for the total branched covering space to be smooth while also satisfying the branching condition of Remark 4.2.7 takes on the following form.

Lemma 4.3.1. (cf. Lemmas 1.2 and 1.4 in [45]) *Let $G = (\mathbb{Z}/p\mathbb{Z})^k$ and $\rho : X \to \widetilde{\mathbb{CP}^2}$ be as above. Then the complex surface X is smooth if the following two conditions hold.*

(i) For each double point $p_{i_1,i_2} = L_{i_1} \cup L_{i_2}$, we have $\langle \phi(\mu_{i_1}), \phi(\mu_{i_2}) \rangle \cong (\mathbb{Z}/p\mathbb{Z})^2$.

(ii) For each r-fold intersection point $p_{i_1,i_2,\ldots,i_r} = L_{i_1} \cap L_{i_2} \cap \cdots \cap L_{i_r}$ with $r \geq 3$, we have $\langle \phi(\varepsilon_{i_1,i_2,\ldots,i_r}), \phi(\mu_{i_j}) \rangle \cong (\mathbb{Z}/p\mathbb{Z})^2$ for all $j \in \{1, 2, \ldots, r\}$. □

One can observe that Lemma 4.3.1 is equivalent to Lemma 4.1.4 for the case of $G = (\mathbb{Z}/p\mathbb{Z})^k$ by viewing $\langle \phi(\mu_{i_1}), \phi(\mu_{i_2}) \rangle$ as the internal direct product of $I_{L_{i_1}}$ and $I_{L_{i_2}}$.

Remark 4.3.2. As stated above, we are interested in the case when the branch locus is $B = \beta^{-1}(\mathcal{L})$. However, we can observe that if we have a smooth cover in the sense of Lemma 4.3.1, then all irreducible curves $C \subset \beta^{-1}(\mathcal{L})$ are spheres. Therefore, Lemma 4.1.3 implies that H_C contains a subgroup isomorphic to $(\mathbb{Z}/p\mathbb{Z})^2$ for any such curve C.

For a given line arrangement $\mathcal{L}$ and any prime p, the group $G = (\mathbb{Z}/p\mathbb{Z})^k$ will admit a map defining a smooth branched cover for k large enough. In particular, if $k = n-1$, then there is a "standard choice" for the defining map.

Example 4.3.3. (cf. page 968 in [45]) Let $\mathcal{L}$ be any line arrangement consisting of n complex lines and consider the surjective homomorphism

$$\phi : H_1(\mathbb{CP}^2 - \mathcal{L}) \cong \mathbb{Z}^{n-1} \to (\mathbb{Z}/p\mathbb{Z})^{n-1}$$

given by

$$\phi(\mu_i) = \begin{cases} \mathbf{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0) & i \in \{1, 2, \ldots, n-1\}, \\ -\sum_{i<n} \mathbf{e}_i = (p-1, \ldots, p-1) & i = n, \end{cases} \tag{4.6}$$

where $\mathbf{e}_i$ is the ith standard basis vector that has 1 as the ith component and 0 elsewhere. It is easy to see that at any intersection the smoothness criterion will be satisfied and so the corresponding branched covering has a smooth total space X. We shall refer to such a homomorphism and its corresponding smooth branched cover as the canonical branched cover associated to a line arrangement.

The above example suggests the following interesting question. For a line arrangement $\mathcal{L}$ and prime p. What is the smallest k for which $(\mathbb{Z}/p\mathbb{Z})^k$ is the Galois group of a smooth branched over $\mathcal{L}$? Examples for which $p = k = 2$ are known to exist. We refer the reader to the discussion on Burniat surfaces in Section 4.2 of [45]. We will later return to this question and provide some introductory results towards an answer in Section 6.4.

Remark 4.3.4. We note that the condition that p be prime in Example 4.3.3 is not required. That is, for any abelian group of the form $(\mathbb{Z}/m\mathbb{Z})^{n-1}$ with $m \in \mathbb{N}$, a map $\phi : H_1(\mathbb{CP}^2 - \mathcal{L}) \cong \mathbb{Z}^{n-1} \to (\mathbb{Z}/m\mathbb{Z})^{n-1}$ defined in the form of equation (4.6) will define a smooth branched covering. The branched coverings from these maps were first studied by [39] whereas relaxing the condition to allow for cases including $k \leq n-1$ was first utilized by [45]. However, we further highlight that even though m may be composite, one can compute the inertia subgroups for such a cover. Furthermore we observe that for the canonical defining map (4.6), all inertia subgroups corresponding to any line of a line arrangement will have the same order.

We now conclude this section by highlighting the fact that various invariants of interest for the total covering space X can be easily computed in the case of branched coverings with Galois groups of the form $(\mathbb{Z}/p\mathbb{Z})^k$ (or $(\mathbb{Z}/m\mathbb{Z})^{n-1}$ referring to the above remark).

Lemma 4.3.5. (cf. Lemma 9 in [34]) *Given an arrangement $\mathcal{L}$ of n complex lines in $\mathbb{CP}^2$, let t_i denote the number of intersection points of $\mathcal{L}$ with multiplicity i. Assume that we blow up at all intersection points of $\mathcal{L}$ with multiplicities greater than 2. Let X be the (smooth) total space of the branched covering map corresponding to a surjective homomorphism (4.5) whose branch locus is equal to the total transform of $\mathcal{L}$ (hence all exceptional spheres are in the resulting branch locus). Let K_X^2, $e(X)$ and $\sigma(X)$ denote the self-intersection number of the canonical class K_X, the Euler characteristic and the signature of X, respectively. Suppose that p and k are positive integers satisfying one of the following two conditions:*

(i) $p \geq 2$ *and* $k = n-1$,

(ii) p *is a prime number and* $1 \leq k \leq n-1$.

Then we have

$$K_X^2 = p^{k-2}\Big[\Big((n-3)p-n\Big)^2 - \sum_{i=3}^{n} t_i\Big((2p-1)-(p-1)i\Big)^2\Big],$$

$$e(X) = p^{k-2}\Big[\Big(3-2n+\sum_{i=2}^{n}(i-1)t_i\Big)p^2 + 2\Big(n+\sum_{i=2}^{n}(1-i)t_i\Big)p + t_2 + \sum_{i=3}^{n} it_i\Big],$$

$$\sigma(X) = \frac{p^{k-2}}{3}\Big[\Big(3-n+\sum_{i=3}^{n}(i-2)t_i\Big)p^2 + n - \sum_{i=3}^{n}(i+1)t_i\Big]. \qquad \square$$

For case (i), the formulas for $e(X)$ and K_X^2 are found in equations (5) and (6) in Subsection 2.2 of [39], respectively. For case (ii), the formulas for K_X^2 and $e(X)$ are found in Theorems 2.1 and 2.3 in [45], respectively. (Note that we have $t_i' = 0$ and $t_i'' = t_i$ in the notation of [45], where $t_i' = 0$ follows from our assumption on the branch locus in Remark 4.2.7). The second formula also easily follows from the inclusion-exclusion principle for the Euler characteristic. The formula for $\sigma(X)$ can be derived easily from Equation (3.2). To express the formulas in their final forms, we also need to use the well-known identity:

$$\binom{n}{2} = \sum_{i=2}^{n}\binom{i}{2}t_i, \tag{4.7}$$

which is equation (2.1) in [31].

4.4 Non-elementary abelian Galois groups.

The most useful part when considering elementary abelian groups as the Galois groups of a branched cover is that all subgroups have the same order and in particular, since all cyclic subgroups are of prime order, all inertia subgroups are isomorphic. We will see in all of the constructions considered in the later chapters of this thesis that this plays a pivotal role when verifying the smoothness criterion (Lemma 4.1.4) for a branched cover. An important question to then ask in this context is what can occur when we drop the assumption that our Galois group takes on the form $(\mathbb{Z}/p\mathbb{Z})^k$.

Let p and q be distinct primes. We shall start by considering covers with Galois group $G \cong (\mathbb{Z}/p\mathbb{Z})^k \times (\mathbb{Z}/q\mathbb{Z})^{n-k}$ and provide an analogue to the canonical smooth branched cover of Example 4.3.3, albeit restricted to the case in which all lines are generic.

Proposition 4.4.1. *Let $\mathcal{L}$ be a line arrangement consisting of $n+1>4$ lines in general position. For all $2 \leq k \leq n-2$, there is exists a smooth Galois cover $\rho : X \to \mathbb{CP}^2$ branched along $\mathcal{L}$ with Galois group $(\mathbb{Z}/p\mathbb{Z})^k \times (\mathbb{Z}/q\mathbb{Z})^{n-k}$.*

Proof. Let $\mu_i \in H_1(\mathbb{CP}^2 - \mathcal{L})$ denote the loop around the line L_i. We consider the map

$$\phi : H_1(\mathbb{CP}^2 - \mathcal{L}) \cong \mathbb{Z}^n \to (\mathbb{Z}/p\mathbb{Z})^k \times (\mathbb{Z}/q\mathbb{Z})^{n-k}$$

given by

$$\phi(\mu_i) = \begin{cases} \mathbf{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0) & i \in \{1, 2, \ldots, n\}, \\ -\sum_{i<n} \mathbf{e}_i = (p-1, \ldots, p-1, q-1, \ldots, q-1) & i = n+1, \end{cases} \tag{4.8}$$

Since this homomorphism satisfies the relation in Lemma 4.2.2, this map is a well defined surjection and so ϕ corresponds to an abelian covering $\rho : X \to \mathbb{CP}^2$.

We now consider the smoothness criterion. Computing inertia subgroups we find

$$I_{L_i} \cong \begin{cases} \mathbb{Z}/p\mathbb{Z} & i \in \{1, 2, \ldots, k\}, \\ \mathbb{Z}/q\mathbb{Z} & i \in k+1, k+1, \ldots, n\,. \\ \mathbb{Z}/pq\mathbb{Z} & i = n+1. \end{cases} \tag{4.9}$$

If $1 \leq i < j \leq n$, then it is clear that $I_{L_i} \cap I_{L_j} = \{0\}$.

On the other hand, if $1 \leq i \leq n$ and $j = n+1$, then all non-identity elements of I_{L_i} generate I_{L_i} and so if $I_{L_i} \cap I_{L_{n+1}}$ is non-empty, then I_{L_i} is a subgroup of $I_{L_{n+1}}$. Moreover, we must have that $\phi(\mu_i)$ generates the unique subgroup of $\mathbb{Z}/pq\mathbb{Z}$ of order $|\phi(\mu_i)|$. However, the unique subgroup of $\mathbb{Z}/pq\mathbb{Z}$ of order p consists of elements of the form

$$(m(p-1), \ldots, m(p-1), 0, \ldots, 0),$$

whereas the subgroup of order q consists of elements of the form

$$(0, \ldots, 0, m'(q-1), \ldots, m'(q-1)).$$

Specifically, since $2 \leq k \leq n-2$, every non-identity element of $\mathbb{Z}/pq\mathbb{Z}$ contains at least two coordinates with non-zero representatives. As any power of $\phi(\mu_i)$ consists of at most one such coordinate, it cannot be an element of $I_{L_{n+1}}$. It follows that the cover defined by ϕ is smooth. $\square$

Example 4.4.2. Since the group $G = (\mathbb{Z}/p\mathbb{Z})^2 \times (\mathbb{Z}/q\mathbb{Z})^3$ is not elementary abelian, The smooth cover defined by applying Proposition 4.4.1 to an arrangement of 6 lines and group G will in general be different from any such cover previously studied (for instance by Hirzebruch [39]).

We also notice that if n is even and $k = n/2$, then we will have $G \cong (\mathbb{Z}/pq\mathbb{Z})^k$. However, in contrast to Remark 4.3.4 in which all inertia subgroups of the canonical cover with (non-elementary) abelian group $(\mathbb{Z}/m\mathbb{Z})^n$ had the same order, there are three different orders for the inertia subgroups of lines coming from Proposition 4.4.1.

We can observe that the above construction is not restricted to the case when only 2 distinct primes divide the order of the group G. In particular, for a set of primes $\{p_m\}$, provided each summand $\mathbb{Z}/p_m\mathbb{Z}$ appears at least twice in the decomposition of G, we will have a similar result. Namely:

Corollary 4.4.3. *For a fixed integer $k \geq 2$ and distinct primes $p_1, p_2, \ldots, p_k$, let $\mathcal{L}$ be a line arrangement consisting of $n + 1 > 2k$ lines in general position. Further assume there are integers $n_1, n_2, \ldots, n_k \geq 2$, for which $\sum_{i=1}^{k} n_i = n$. Then there exists a smooth Galois cover $\rho : X \to \mathbb{CP}^2$ branched along $\mathcal{L}$ with Galois group $(\mathbb{Z}/p_1\mathbb{Z})^{n_1} \times (\mathbb{Z}/p_2\mathbb{Z})^{n_2} \times \cdots \times (\mathbb{Z}/p_k\mathbb{Z})^{n_k}$.*

We also note that the ability to construct smooth branched covers with non-elementary abelian Galois group is not limited to arrangements of lines in general position.

Proposition 4.4.4. *Let $\mathcal{L}_n$ be a near pencil line arrangement with $n > 4$ lines (Example 4.2.5). Then there exists a smooth Galois cover $\rho : X \to \mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ branched along $\mathcal{L}_n$ with Galois group $(\mathbb{Z}/p\mathbb{Z}) \times (\mathbb{Z}/q\mathbb{Z})^{n-2}$*

Proof. Let L_n denote the line which does not pass through the $(n-1)$-fold intersection point. We define the surjective group homomorphism

$$\phi : H_1(\mathbb{CP}^2 - \mathcal{L}_n) \cong \mathbb{Z}^{n-1} \to (\mathbb{Z}/p\mathbb{Z}) \times (\mathbb{Z}/q\mathbb{Z})^{n-2}$$

as follows. First, let $\phi(\mu_1) = (1, 0, \ldots, 0)$ and $\phi(\mu_2) = (p-1, 0, \ldots, 0)$. Then for $3 \leq i \leq n-1$ we send μ_i to $(0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 in the $(i-1)$-th place and sending μ_n to $(0, q-1, \ldots, q-1)$. An example of the labeling of the near pencil $\mathcal{L}_n$ is provided in Figure 4.3.

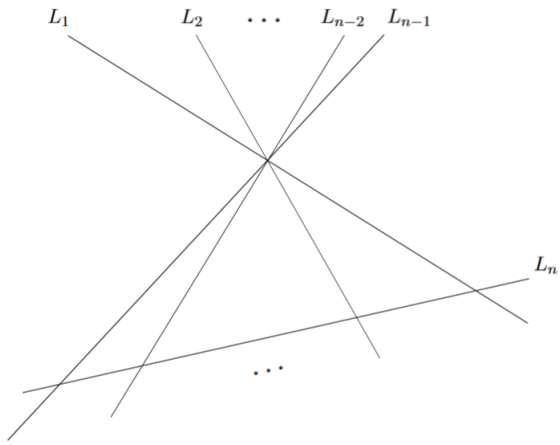

Figure 4.3: A labeled near pencil arrangement

It is easy to see that this map is surjective. To check smoothness, we again must verify the smoothness criterion after we blow up the $(n-1)$-fold intersection point. For the case of the intersection of a line L_i with the line L_n it follows from a similar argument to above. However, the other intersection condition is now to check the subgroup generated by $g_i = \phi(\mu_i)$ intersects trivially with the group generated by $\varepsilon = \sum_{i=1}^{n-1} \phi(\mu_i)$.

In this case we find $\varepsilon = (0, 1, \ldots, 1)$ which generates a group with elements of the form $(0, m, \ldots, m)$. Then again, for all L_i's which pass through the $(n-1)$-fold intersection point, the non-zero elements of their inertia groups contain only a single non-zero coordinate and therefore they do not intersect. $\square$

Similar to the previous section, one may be interested in computing the topological invariants for the covers considered in this section. To both provide a comparison to Lemma 4.3.5 above, in addition to providing insight into the difficulty of finding enlightening examples in this section, we provide these computations for the covers guaranteed by Proposition 4.4.1.

Example 4.4.5. Let $\mathcal{L}$ be a line arrangement consisting of $n+1 > 4$ lines in general position. Suppose $\rho : X \to \mathbb{CP}^2$ is the smooth $(\mathbb{Z}/p\mathbb{Z})^k \times (\mathbb{Z}/q\mathbb{Z})^{n-k}$ cover branched over $B = \mathcal{L}$ as defined by equation (4.8). Then if H denotes the class of a line in $\mathbb{CP}^2$ so that

$K_{\mathbb{CP}^2} = -3H$. By the Hurwitz formula (cf. page 54 of [12]), the canonical class of X is given by

$$K_X = \rho^*(K_{\mathbb{CP}^2}) + \sum_j (e_j - 1)R_j \tag{4.10}$$

where R_j is an irreducible component of the ramification locus $R = f^{-1}(B)$ and e_j is the branching order of R_j. Specifically we have $R_j = \rho^{-1}(L_j)$ by Lemma 4.2.4. Now, maintaining the same indexing for lines L_j from Proposition 4.4.1, we know

$$e_j = \begin{cases} p & \text{if } j \in \{1, 2, \ldots, k\}, \\ q & \text{if } j \in \{k+1, k+2, \ldots, n\}, \\ pq & \text{if } j = n+1 \end{cases}$$

by considering the inertia subgroups given by equation (4.9). Furthermore, denoting $P_i = R_i$ for $i \leq k$, $Q_j = R_j$ for $k+1 \leq j \leq n$, and $N = R_{n+1}$, we have

$$K_X = \rho^*(K_{\mathbb{CP}^2}) + \sum_{i=1}^{k} (p-1)P_i + \sum_{j=k+1}^{n} (q-1)Q_j + (pq-1)N.$$

Applying Lemma 4.1.7 and Lemma 4.1.6 we compute the following intersection (and self-intersection) numbers to be

$$\begin{aligned}
(\rho^*(K_{\mathbb{CP}^2}))^2 &= (f^*(-3H))^2 = 9p^k q^{n-k}, \\
P_i^2 &= p^{k-2}q^{n-k} \\
Q_j^2 &= p^k q^{n-k-2} \\
N^2 &= p^{k-2}q^{n-k-2} \\
\rho^*(K_{\mathbb{CP}^2}) \cdot P_i &= p^{k-1}q^{n-k} \\
\rho^*(K_{\mathbb{CP}^2}) \cdot Q_j &= p^k q^{n-k-1} \\
\rho^*(K_{\mathbb{CP}^2}) \cdot N &= p^{k-1}q^{n-k-1} \\
P_i \cdot Q_j &= p^{k-1}q^{n-k-1} \\
P_i \cdot N &= p^{k-2}q^{n-k-1} \\
Q_j \cdot N &= p^{k-1}q^{n-k-2}.
\end{aligned}$$

Consequently, we can compute

$$\begin{aligned}K_X^2 &= 9p^kq^{n-k} + k^2(p-1)^2p^{k-2}q^{n-k} + (n-k)^2(q-1)^2p^kq^{n-k-2} + (pq-1)^2p^{k-2}q^{n-k-2}\\ &\quad - 6\Big[k(p-1)p^{k-1}q^{n-k} + (n-k)(q-1)p^kq^{n-k-1} + (pq-1)p^{k-1}q^{n-k-1}\Big]\\ &\quad + 2\Big[k(n-k)(p-1)(q-1)p^{k-1}q^{n-k-1} + k(pq-1)(p-1)p^{k-2}q^{n-k-1}\\ &\quad + (n-k)(pq-1)(q-1)p^{k-1}q^{n-k-2}\Big].\end{aligned}$$

The Euler characteristic of X can also be computed using the inclusion-exclusion principle and we get

$$\begin{aligned}e(X) &= p^kq^{n-k}(e(\mathbb{CP}^2) - e(B))\\ &\quad + \left[kp^{k-1}q^{n-k} + (n-k)p^kq^{n-k-1} + p^{k-1}q^{n-k-1}\right](e(\mathbb{CP}^1) - e(\mathrm{Sing(L)})\\ &\quad + \frac{k(k-1)}{2}p^{k-2}q^{n-k} + \frac{(n-k)(n-k-1)}{2}p^kq^{n-k-2} + k(n-k)p^{k-1}q^{n-k-1}\\ &\quad + kp^{k-2}q^{n-k-1} + (n-k)p^{k-1}q^{n-k-2},\end{aligned}$$

where $e(\mathrm{Sing}(L))$ is the number of intersection points on a line L (in this case we have $e(\mathrm{Sing}(L)) = n$ since all lines are in general position). We also compute $e(B)$ to be

$$e(B) = 2(n+1) - \frac{n(n+1)}{2} = (2 - \frac{n}{2})(n+1).$$

It then follows that $e(X)$ can be expressed as

$$\begin{aligned}e(X) &= \frac{n^2-3n+2}{2}\cdot p^kq^{n-k}\\ &\quad + (2-n)\left[kp^{k-1}q^{n-k} + (n-k)p^kq^{n-k-1} + p^{k-1}q^{n-k-1}\right]\\ &\quad + \frac{k(k-1)}{2}p^{k-2}q^{n-k} + \frac{(n-k)(n-k-1)}{2}p^kq^{n-k-2} + k(n-k)p^{k-1}q^{n-k-1}\\ &\quad + kp^{k-2}q^{n-k-1} + (n-k)p^{k-1}q^{n-k-2}.\end{aligned}$$

We note that even if we proceed similarly to Lemma 4.3.5 and factor out $p^{k-2}q^{n-k-2}$ for both K_X^2 and $e(X)$, the expressions are significantly more complex. Especially since we are considering lines in general position and so in the expressions of Lemma 4.3.5, all terms consisting of t_i, $i \geq 3$, would vanish.

Example 4.4.6. While the above invariants are particularly unwieldy, we can still consider 'small' examples. For instance, for the group $G = (\mathbb{Z}/2\mathbb{Z})^2 \times (\mathbb{Z}/3\mathbb{Z})^2$ and the arrangement of 5 lines in general position we compute

$$\begin{aligned} K_X^2 &= 1 \\ e(X) &= 23 \\ \sigma(X) &= -15. \end{aligned}$$

We compare this result to other known results on covers branched over an arrangement of lines in general position. Specifically, Hirzebruch observed (cf. p. 125 of [39]) that any such cover with Galois group $(\mathbb{Z}/m\mathbb{Z})^{n-1}$ is a complete intersection of Fermat surfaces of fixed degree $m \geq 2$. That is, surfaces in $\mathbb{CP}^N$ defined by the equations

$$\sum_{i=0}^{N} a_i z_i^m.$$

However, it is also known that the self-intersection of the canonical class of any such complete intersection surface is

$$\left(\sum_{k=1}^{N-2} d - (N+1)\right)^2 \left(\prod_{k=1}^{N-2} d\right)$$

which is never 1. Therefore, the surface X is completely distinct from any of Hirzebruch's covers that are branched over lines in general position.

Remark 4.4.7. We end this section by highlighting that while non-elementary abelian topological branched covers can be found in a relatively straightforward manner (for instance, there is an analogous statement to Corollary 4.2.3). Essentially nothing is known in general about when a *smooth* cover will exist for an arbitrary line arrangement. Specifically, since there can be cyclic inertia subgroups not of prime order, checking the smoothness criterion is comparatively difficult. Moreover, as suggested by Corollary 4.4.3, line arrangements admitting such smooth covers will likely be comparatively large. Furthermore, since the general behaviour of 'large' line arrangements is relatively unknown in many cases, an overall algorithm for finding such smooth covers escapes us at this time.

4.5 Hamiltonian branched covers

In this section we will take a slight tangent and move away from the abelian case. To start, let us return briefly to the definition of the inertia and stabilizer subgroups (Definition 4.1.1).

Let $\rho : X \to Y$ be a branched covering with branched locus B and Galois group G. We recall from Section 4.1 that for a curve $C \subset Y$ the inertia and stabilizer subgroups for different components C' of $\rho^{-1}(C)$ are conjugate subgroups of G. Consequently, I_C (or H_C) is well defined exactly when one (and hence all) inertia (stabilizer) subgroup $I_{C'}$ ($H_{C'}$) is normal. While this means that for a generic non-abelian group we cannot immediately look to extend the notion of inertia and stabilizer subgroups to all subvarieties of Y, if a non-abelian group carries the property that all subgroups are normal, then some meaningful definition can be made. Groups with the property that all subgroups are normal are called Dedekind groups and a non-abelian Dedekind group is called a Hamiltonian group.

Example 4.5.1. The most well known example of a Hamiltonian group is the order 8 group of quaternions Q_8.

The structure of a Hamiltonian group is completely known. Specifically Dedekind [15] and Baer [10] have shown that a Hamiltonian group has the following form.

Proposition 4.5.2. *Suppose G is a Hamiltonian group. The $G = Q_8 \times B \times D$ where Q_8 denotes the group of Quaternions, B is an elementary abelian 2-group and D is a periodic abelian group with no 2-torsion.*

We note that a group G not being Hamiltonian does not prevent it from being the Galois group of a branched cover as noted at the start of Chapter 4, since only a surjective homomorphism

$$\psi : \pi_1(Y - B) \to G$$

is required. However, as demonstrated by Sections 4.1–4.4, in depth knowledge of the subgroups of G is necessary to determine the topology of the covering space.

For now, let us consider the following example. If $\rho : X \to \mathbb{CP}^2$ is a cover branched along a (potentially reducible) curve $B \subset \mathbb{CP}^2$ with Galois group Q_8, then by the corresponding surjective homomorphism ψ, we know that $\pi_1(\mathbb{CP}^2 - B)$ is non-abelian. In the case in which

B is a line arrangement, we obtain restrictions on how the lines can intersect. In particular, it is known [70] that the complement of lines in general position has abelian fundamental group and so any line arrangement must have at least one intersection point of multiplicity at least 3. Therefore, let us consider the simplest (non-pencil) line arrangement that has this property, the near pencil consisting of 4 lines.

Example 4.5.3. If $\mathcal{L}_4$ in $\mathbb{CP}^2$ is a near pencil of 4 lines, then we may assume that the general line is the sphere at infinity. Consequently we may instead consider the case of

$$\mathbb{C}^2 - B \cong \mathbb{CP}^2 - \mathcal{L}_4$$

where B consists of 3 lines in $\mathbb{C}^2$ passing through the origin. Up to homotopy, B is the curve $zw(z-w) = 0$ in $\mathbb{C}^2$.

Proposition 4.5.4. *Let $B = \{(z,w) \in \mathbb{C}^2 \mid zw(z-w) = 0\} \subset \mathbb{C}^2$. Then $\pi_1(\mathbb{C}^2 - B)$ is non-abelian. In particular, there is a surjective map $\phi : \pi_1(\mathbb{C}^2 - B) \to Q_8$.*

Proof. Let $B = L_1 \cup L_2 \cup L_3$, where

$$\begin{aligned} L_1 &= \{(z,w) \in \mathbb{C}^2 \mid z = 0\}, \\ L_2 &= \{(z,w) \in \mathbb{C}^2 \mid w = 0\}, \\ L_3 &= \{(z,w) \in \mathbb{C}^2 \mid z - w = 0\}. \end{aligned}$$

Restricting the standard deformation retraction $r : \mathbb{C}^2 - \{(0,0)\} \to S^3$ to $\mathbb{C}^2 - B$, it follows that

$$\pi_1(\mathbb{C}^2 - B) \cong \pi_1(S^3 - (S^3 \cap B)).$$

Furthermore, we have that $S^3 \cap L_i \cong S^1$ for $i = 1, 2, 3$. That is, $S^3 \cap B$ is a link with three components. Up to ambient isotopy this link is 6^3_3 in Rolfsen's table of links and is shown in Figure 4.4.

Adding meridian circles as in Figure 4.5, we can compute the Wirtinger presentation of $\pi_1(S^3 - (S^3 \cap B))$ to be

$$\langle a, b, c, d, e, f \mid b = a^{-1}ea, b = c^{-1}ec, a = c^{-1}dc, a = ede^{-1}, c = e^{-1}fe, c = a^{-1}fa \rangle$$

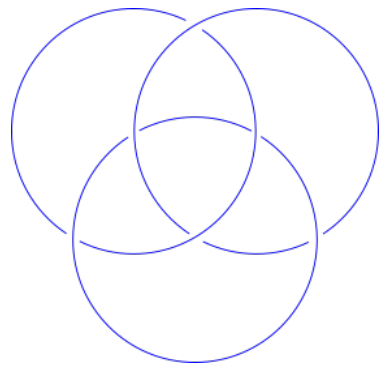

Figure 4.4: The link 6^3_3

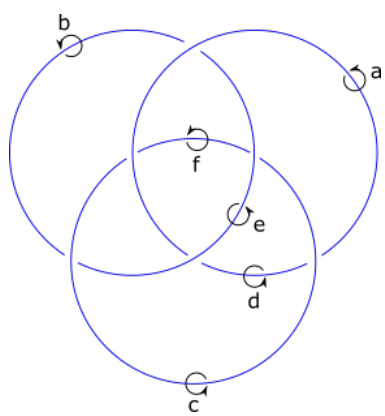

Figure 4.5: Generating loops of $\pi_1(S^3 - (S^3 \cap B))$

and repeated applications of Tietze transformations yields

$$\begin{aligned}
&\langle a, b, c, d, e, f \mid b = a^{-1}ea, b = c^{-1}ec, a = c^{-1}dc, a = ede^{-1}, c = e^{-1}fe, c = a^{-1}fa\rangle \\
&= \langle a, b, c, d, f \mid b = c^{-1}aba^{-1}c, a = c^{-1}dc, d = ab^{-1}aba^{-1}, c = ab^{-1}a^{-1}faba^{-1}, c = a^{-1}fa\rangle \\
&= \langle a, b, c, d \mid b = c^{-1}aba^{-1}c, a = c^{-1}dc, d = ab^{-1}aba^{-1}, c = ab^{-1}cba^{-1}\rangle \\
&= \langle a, b, c \mid b = c^{-1}aba^{-1}c, a = c^{-1}ab^{-1}aba^{-1}c, c = ab^{-1}cba^{-1}\rangle \\
&= \langle a, b, c \mid b = c^{-1}aba^{-1}c, a = b^{-1}c^{-1}acb, c = ab^{-1}cba^{-1}\rangle.
\end{aligned}$$

We now define $\phi : \pi_1(S^3 - (S^3 \cap B)) \to Q_8$ by

$$\phi(a) = i, \phi(b) = j, \phi(c) = k.$$

It is easy to see that ϕ is surjective. Furthermore it is a well defined group homomorphism since

$$\begin{aligned}
\phi(a) &= i = -i^3 = (-j)(-k)(i)(k)(j) = \phi(b^{-1}c^{-1}acb),\\
\phi(b) &= j = -j^3 = (-k)(i)(j)(-i)(k) = \phi(c^{-1}aba^{-1}c),\\
\phi(c) &= k = -k^3 = (i)(-j)(k)(j)(-i) = \phi(ab^{-1}cba^{-1}).
\end{aligned}$$

It follows that $\pi_1(\mathbb{C}^2 - B) \cong \pi_1(S^3 - (S^3 \cap B))$ is non-abelian. □

Remark 4.5.5. As observed in Lemma 4.5.4, there is a surjective map $\phi : \pi_1(\mathbb{C}^2 - B) \to Q_8$ and so there is a corresponding (non-compact) topological cover $\rho : X \to \mathbb{C}^2$ branched along B with Galois group Q_8. Furthermore, as the only singular point of B is the origin, X is smooth except possibly at the points in $f^{-1}((0,0))$.

Also, passing through the isomorphism $\pi_1(\mathbb{C}^2 - B) \cong \pi_1(\mathbb{CP}^2 - \mathcal{L}_4)$, ϕ also defines a (compact) topological cover $\rho_c : X_c \to \mathbb{CP}^2$ branched over $\mathcal{L}_4$ with Galois group Q_8. However in this case, there are also 3 additional singular points of $\mathcal{L}_4$ occurring on the line at infinity. In both cases, we would be interested in providing a systematic way to determine the minimal resolution of X (or X_c).

One might optimistically hope that an analogous statement to Lemma 4.1.4 could be given for quaternionic (or Hamiltonian) branched covers of line arrangements. Unfortunately, we can easily see that this is not going to be the case in general.

Remark 4.5.6. Suppose $\phi : \pi_1(\mathbb{CP}^2 - \mathcal{L}) \to Q_8$ is a surjective map defining a branched cover. If two curves C and D are in the branch locus, then we known both I_C and I_D are non-trivial. However, the element -1 in Q_8 is a non-identity element contained in all subgroups of Q_8. Therefore, it is impossible for the same algebraic condition as Lemma 4.1.4 to distinguish smoothness.

Remark 4.5.7. The above issue is contrasted to the case of non-elementary abelian branched covers In that case, while there were subgroups of the Galois group with different order (similar to how there are order 2 and 4 subgroups of Q_8), the various inertia subgroups that arose in Section 4.4 were able to be chosen to intersect trivially. However, even in the previous section, this came at the cost of requiring more summands in the abelian decomposition of the group.

This suggests that while there may not be an algebraic condition equivalent to smoothness for Q_8, when considering larger Hamiltonian groups (i.e., those with abelian summands as provided by Propostion 4.5.2), it may be possible to determine at least a one directional implication of smoothness. We believe that further investigations into not only this case of 'large' Hamiltonian groups but also the case in the previous section for non-elementary abelian groups could provide greater insight into how the geometric and topological aspects of a branched cover are determined by the algebraic structure of the Galois group.

Chapter 5

Building manifolds with non-negative signature

In this chapter we will construct symplectic 4-manifolds with non-negative signatures by using branched coverings with elementary abelian Galois groups as described in Section 4.3. To accomplish this goal, we first return to Lemma 4.3.5 to gain insight into how we can obtain 4-manifolds with non-negative signature.

From Lemma 4.3.5, we see that as p grows large, the positivity (or negativity) of the signature will depend on the sign of the coefficient of the p^2 term

$$\Big(3 - n + \sum_{i=3}^{n}(i-2)t_i\Big). \tag{5.1}$$

Consequently, we shall begin by investigating this coefficient further.

5.1 Zero leading coefficient

Now, while we will eventually construct branched covers for which the leading coefficient is positive, we first provide some brief observations for the case on the boundary,

$$\Big(3 - n + \sum_{i=3}^{n}(i-2)t_i\Big) = 0. \tag{5.2}$$

Substituting equation 5.2 into the signature formula from Lemma 4.3.5, we find that the coefficient of the $p^{k-2}/3$ term is

$$\begin{aligned} n - \sum_{i=3}^{n}(i+1)t_i &= 3 + \sum_{i=3}^{n}(i-2)t_i - \sum_{i=3}^{n}(i+1)t_i \\ &= 3 + \sum_{i=3}^{n}(-3)t_i \\ &= 3\Big(1 - \sum_{i=3}^{n} t_i\Big). \end{aligned}$$

It then follows that in this case $\sigma(X)$ will be positive exactly when there are no higher order intersection points, zero when there exists exactly one point of multiplicity ≥ 3 and negative otherwise.

Example 5.1.1. First, in the case that $t_i = 0$ for all $i \geq 3$, equation 5.2 implies $n = 3$. That is, the line arrangement $\mathcal{L}$ consists of exactly 3 lines in general position (Figure 5.1).

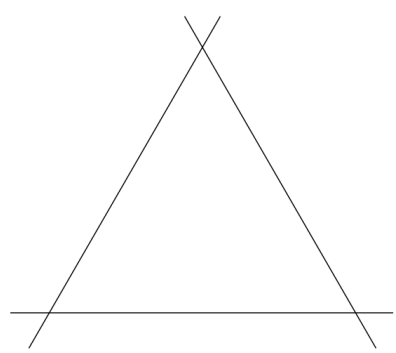

Figure 5.1: Three lines in general position

Furthermore, it was observed in [39] and [45] (and can also be checked by considering invariants) that the cover of $\mathbb{CP}^2$ branched over this 'triangle' of 3 lines with Galois group $(\mathbb{Z}/p\mathbb{Z})^2$ is again $\mathbb{CP}^2$.

Now, if there is exactly one intersection point of multiplicity greater than 2, we will let j denote the multiplicity of this point. Then equation 5.2 simplifies to $n = 3 + (j - 2)$ and so the higher multiplicity point is a point in which $n - 1$ points intersect. We can further use equation (4.7) to find that in this case we have $t_2 = n-1$. That is, the line arrangement must be a near pencil. We refer to section 5.2.1 below for a detailed exposition of this case.

A partial discussion on the case of more than one higher order intersection point occurring in the line arrangement can also be found in the next section in Theorem 5.2.11.

5.2 Supersolvable arrangements

In this section we consider a class of line arrangements that all satisfy equation (5.1) being non-negative. Specifically, they are line arrangements for which there is a contained pencil of lines 'seeing' enough of the line arrangement. However, to better motivate the results at the end of the section, we shall first investigate the (non-pencil) line arrangements that are the closest to being a pencil.

5.2.1 An example: The near pencil

We shall return to the setting of Example 4.2.5 and take $\mathcal{L}_n$ to be a near pencil line arrangement consisting of $n \geq 4$ lines. For a labeling of the lines in the near pencil, we refer back to Figure 4.3.

For a fixed odd prime p, we can exhibit $\mathcal{L}_n$ as the branch locus of a $(\mathbb{Z}/p\mathbb{Z})^2$ cover of $\mathbb{CP}^2$ via the following defining map $\phi : H_1(\mathbb{CP}^2 - \mathcal{L}_n) \to (\mathbb{Z}/p\mathbb{Z})^2$, where

$$\begin{aligned}
\phi(\mu_1) &= \begin{cases} (2,0) & \text{if } n \equiv 2 \pmod{p}, \\ (1,0) & \text{if } n \not\equiv 2 \pmod{p}, \end{cases} \\
\phi(\mu_i) &= (1,0), \quad i \in \{2,3,\ldots,n-2\}, \\
\phi(\mu_{n-1}) &= (0,1), \\
\phi(\mu_n) &= \begin{cases} (p-1,p-1) & \text{if } n \equiv 2 \pmod{p}, \\ (2-n,p-1) & \text{if } n \not\equiv 2 \pmod{p}. \end{cases}
\end{aligned}$$

Then since neither $(*,0)$ nor $(0,*)$ can generate an element with two non-zero components $(*,*)$, the branch cover defined by ϕ will be smooth above all double points of $\mathcal{L}_n$. Furthermore,

$$\sum_{i=1}^{n-1} \phi(\mu_i) = \begin{cases} (n-1,1) & \text{if } n \equiv 2 \pmod p, \\ (n-2,1) & \text{if } n \not\equiv 2 \pmod p \end{cases}$$
$$\neq (0,1)$$

which again cannot be generated by elements of the form $(*,0)$ or $(0,*)$. It follows that X is smooth above the $(n-1)$-fold point of $\mathcal{L}_n$ and hence, X is a smooth complex surface.

Let $\rho : X_{n,p} \to \mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ denote the branch cover corresponding to ϕ and $\beta : \mathbb{CP}^2 \# \overline{\mathbb{CP}}^2 \to \mathbb{CP}^2$ be the blow-down map. Then the branch locus of ρ is

$$B = \beta^{-1}(\mathcal{L}_n) = \widehat{L}_1 \cup \cdots \cup \widehat{L}_{n-1} \cup \widehat{L}_n \cup E = \widehat{L}_1 \cup \cdots \cup \widehat{L}_{n-1} \cup \beta^{-1}(L_n) \cup E$$

where E is the exceptional sphere coming from the blow-up and $\beta^{-1}(L_n) = \widehat{L_n}$ is the proper transform of L_n. Applying Lemma 4.3.5, we have

$$e(X_{n,p}) = 2(3-n)p + 2(n-1)$$
$$\sigma(X_{n,p}) = 0.$$

We also note that since $\beta^{-1}(L_n) = \widehat{L}_n$ does not intersect E, $\widehat{L}_n$ is a sphere with $(\widehat{L}_n)^2 = 1$. In particular, since $\widehat{L}_n$ is a curve of genus 0, Lemma 4.1.3 implies that

$$I_{\widehat{L}_n} \cong \mathbb{Z}/p\mathbb{Z},$$
$$H_{\widehat{L}_n} = G.$$

Specifically, we then see that there is a unique component of $L'_n = \rho^{-1}(\widehat{L}_n)$ by Lemma 4.1.5 and that the self intersection of this curve is $(L'_n)^2 = (\widehat{L}_n)^2 = 1$ by Lemma 4.1.6. Furthermore, since all the intersection points on L_n are the $t_2 = n-1$ double points of the arrangement, the Euler characteristic of L'_n is computed to be

$$\begin{aligned} e(L'_n) &= \frac{|G|}{|I_{\widehat{L}_n}|} e(\widehat{L}_n) - t_2 \frac{|G|}{|I_{\widehat{L}_n}|} e(\{pt\}) + t_2 e(\{pt\}) \\ &= 2p - (n-1)p + (n-1) \\ &= (3-n)p + (n-1). \end{aligned} \tag{5.3}$$

Here we are using the fact that $|G|/|I_{\widehat{L}_n}|$ is the ramification degree of $\widehat{L}_n$ and then applying inclusion-exclusion principle.

An identical analysis shows that $E' = \rho^{-1}(E)$ also has a unique component and that this curve satisfies $(E')^2 = (E)^2 = -1$ and $e(E') = (3-n)p + (n-1)$.

Remark 5.2.1. Starting with $e(X_{4,3}) = 0$ and observing that $\frac{\partial}{\partial p} e(X_{n,p}) = 6 - 2n < 0$ and $\frac{\partial}{\partial n} e(X_{n,p}) = 2 - 2p < 0$ for $n \geq 4$ and $p \geq 3$ respectively, it follows that $e(X_{n,p}) \leq 0$ for all $n \geq 4$ and odd primes p. In particular, we observe that the first betti number $b_1(X_{n,p}) \neq 0$ and so $X_{n,p}$ is not simply connected. We shall return to this point later (Chapter 6) but for now we simply remark that in general, the Kähler manifolds constructed via branched cover constructions are not applicable to the geography problem without applying further geometric techniques to handle this issue regarding the fundamental group.

Remark 5.2.2. Instead of fixing $k = 2$, we can set $k = n - 1$ (or in fact any k satisfying $2 \leq k \leq n - 1$ by a similar construction). In this case we will have $G = (\mathbb{Z}/p\mathbb{Z})^{n-1}$, and so we can use the canonical homomorphism ϕ in Example 4.3.3. Furthermore, from Remark 4.3.4 we can let $p \geq 2$ be any (possibly non-prime) integer. It was observed in Subsection 2.4 of [39] that the covering spaces $X_{n-1,p}$ in this case are birationally equivalent to ruled surfaces.

5.2.2 Supersolvability

Having seen the above example, we are prepared to discuss a generalization of the near pencil. This will be line arrangements for which there is an intersection point for which the lines passing through it 'see' the entirety of the line arrangement.

Definition 5.2.3. A point x of a line arrangement $\mathcal{L}$ is called a *modular point* for $\mathcal{L}$ if x is an intersection point and for any other intersection point q of the line arrangement, the line passing through x and q is in $\mathcal{L}$. A line arrangement $\mathcal{L}$ is said to be *supersolvable* if it has a modular point.

Example 5.2.4. Let x be the unique higher order intersection point of multiplicity $n - 1$ in a near pencil line arrangement. Then x is a modular point and so all near pencil line arrangements are supersolvable.

Example 5.2.5. It is also straight forward to construct a supersolvable arrangement with exactly two intersection points of multiplicity not equal to 2. Specifically, If $\mathcal{L}_1$ and $\mathcal{L}_2$ are pencils consisting of u and v many lines respectively, then in general, their union arrangement $\mathcal{L}_1 \cup \mathcal{L}_2$ will not be supersolvable as there may not be a line connecting these two higher multiplicity intersection points (see Figure 5.2).

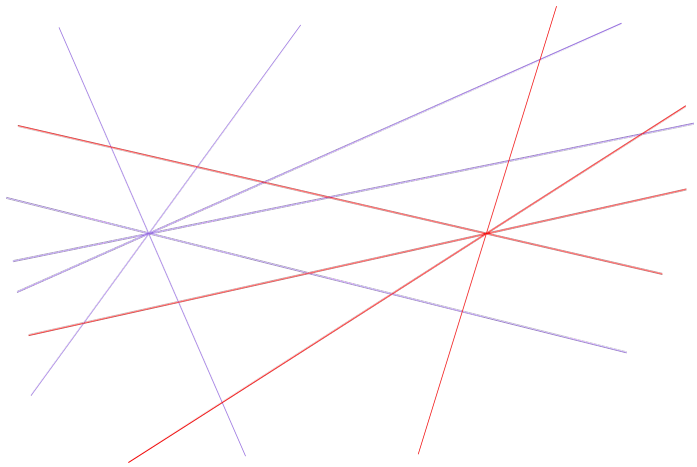

Figure 5.2: From a (dis)connected double pencil ...

However, if we modify the arrangement by adding in the line (call it L) passing through the two singular points of the pencils, then the arrangement $\mathcal{L}_1 \cup \mathcal{L}_2 \cup L$ will be supersolvable and an example is shown in Figure 5.3. Here, the black line L is being added to the pencils $\mathcal{L}_1$ (the purple lines) and $\mathcal{L}_2$ (the red lines). We shall refer to this arrangement as a connected double pencil. It is straight forward to compute that for such an arrangement (of $n = u + v + 1$ lines), we have $t_{u+1} = t_{v+1} = 1$ (unless $u = v$, then $t_{u+1} = t_{v+1} = 2$), $t_2 = uv$ and $t_r = 0$ for all other r.

Example 5.2.6. For a fixed integer $m \geq 3$, we follow [27] (as well as [39]) and will denote $A_1(2m)$ to be the arrangement of $2m$ lines in $\mathbb{CP}^2$ defined as follows: Let the first m lines be determined by placing them on the edges of a regular m-gon. Then place the other m lines so that they make up the lines of symmetry of that m-gon. The m lines of symmetry all pass through the intersection point of $A_1(2m)$ that is the centroid of the m-gon. Furthermore, every other intersection point of $A_1(2m)$ lays on exactly one of the lines of symmetry. It follows that the centroid is a modular point and so $A_1(2m)$ is supersolvable. We note that

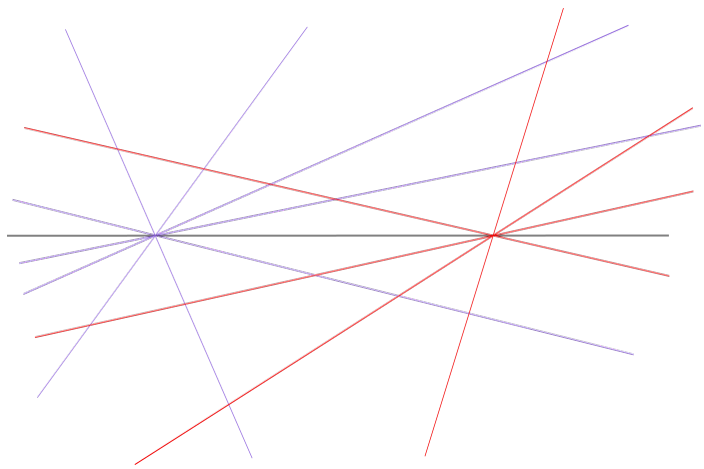

Figure 5.3: ... to a connected double pencil

a branched covering corresponding to the $A_1(6)$ arrangement (Figure 5.4) was used in [7] to obtain an upper bound $\lambda(0) \leq 12$. It is easy to see from Figure 5.4 that it is supersolvable.

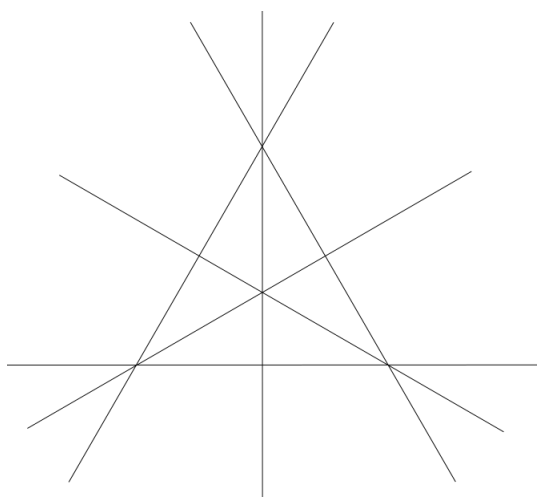

Figure 5.4: The $A_1(6)$ arrangement.

We refer the reader to [31] for a broader description of supersolvable line arrangements. We now highlight the following result from [31] in preparation for Theorems 5.2.10 and

5.2.11.

Lemma 5.2.7. (cf. Lemma 2 in [31]) *Let $\mathcal{L}$ be a complex supersolvable line arrangement with a modular point x with multiplicity m. If q is an intersection point of $\mathcal{L}$ with multiplicity $r \geq m$, then q is also a modular point for $\mathcal{L}$.* □

With Lemma 5.2.7 we are prepared to discuss supersolvable line arrangements in the context of branched cover constructions of 4-manifolds with non-negative signature. First, we state and prove a useful fact about building supersolvable line arrangements.

Lemma 5.2.8. *Let $\mathcal{L}$ be a supersolvable line arrangement consisting of n lines. There is a sequence of supersolvable line arrangements $\mathcal{L}_3, \mathcal{L}_4, \ldots, \mathcal{L}_n = \mathcal{L}$ and a set of lines $L_4, L_5, \ldots, L_n$ such that $\mathcal{L}_i = \mathcal{L}_{i-1} \cup L_i$ for all $i \in \{4, 5, \ldots, n\}$.*

Proof. Let $\mathcal{L}$ be a supersolvable line arrangement with n lines and suppose x is an intersection point with the highest multiplicity, say multiplicity m. Index the lines in $\mathcal{L}$ so that the lines $L_1, L_2, \ldots, L_m$ are the lines intersecting at x. Choose $\mathcal{L}_i$ to be the line arrangement consisting of lines $L_1, L_2, \ldots, L_i$. Then for $i \leq m$, $\mathcal{L}_i$ is a pencil of i lines and so is supersolvable. Furthermore, by Lemma 5.2.7 we know that x is a modular point for $\mathcal{L}$ and so any intersection point of $\mathcal{L}$ lies on one of the lines of index $i \leq m$. In particular, if $k > m$, then the intersection points obtained by adding L_k to $\mathcal{L}_{k-1}$ all lie on lines already in $\mathcal{L}_{k-1}$ and so $\mathcal{L}_k$ will also be supersolvable. The result follows. □

Remark 5.2.9. We note that in the proof of Lemma 5.2.8, since for all $i \geq m$ the arrangement $\mathcal{L}_i$ contained all the lines of $\mathcal{L}$ intersecting at x, the order in which the lines $L_k, k > m$ were added does not matter. That is, for any choice of ordering of $L_k, k > m$, the arrangements defined inductively by $\mathcal{L}_i = \mathcal{L}_{i-1} \cup L_i$ will still all be supersolvable.

We now prove the following result stating a connection between the leading coefficient of signature and supersolvability.

Theorem 5.2.10. (cf. Lemma 13 in [34]) *If $\mathcal{L}$ is a supersolvable arrangement of n lines in $\mathbb{CP}^2$, then $\mathcal{L}$ satisfies*

$$\left(3 - n + \sum_{i=3}^{n} (i-2)t_i\right) \geq 0. \tag{5.4}$$

Proof. We will prove the theorem by using induction on the number of lines n. For $n = 3$ case, there are only two possible line arrangements: a pencil (all lines passing through one point) and a near pencil . It is easy to see that pencils and near pencils are supersolvable and always satisfy (5.4) (even for a greater number of lines).

Now let us assume that all the supersolvable line arrangements with n lines satisfy (5.4). Let $\mathcal{L}$ be a supersolvable line arrangement with $n+1$ lines which is neither a pencil nor a near pencil. By Lemma 5.2.8, there is a supersolvable line arrangement $\mathcal{L}'$ such that $\mathcal{L} = \mathcal{L}' \cup L$ for some line L. From the proof of Lemma 5.2.8, we may assume that L does not pass through at least one point of highest intersection multiplicity (call this point x). On the other hand, since $\mathcal{L}$ is not a near pencil, we may assume that L contains at least one other intersection point of multiplicity ≥ 3. If L does not, then L is a line in general position and since $\mathcal{L}$ is not a near pencil, there is another line L' in $\mathcal{L}$ that is distinct from L and also does not pass through x. In this case, the intersection point $L \cap L'$ is a double point and so there is no line in $\mathcal{L}$ passing through both x and $L \cap L'$. However, this contradicts the supersolvability of $\mathcal{L}$ since Lemma 5.2.7 implies x is a modular point.

Now, by induction $\mathcal{L}'$ satisfies (5.4) (note we will let t_i and t'_i denote the total number of multiplicity i intersection points on $\mathcal{L}$ and $\mathcal{L}'$, respectively). Now, $\mathcal{L}$ is constructed from $\mathcal{L}'$ by adding a line, which decreases the $3 - n$ term by 1. On the other hand, if $x_j \in L$ is an intersection point of $\mathcal{L}$ of multiplicity m_j, then $x_j \in \mathcal{L}'$ has multiplicity $m_j - 1$ (where $m_j - 1 = 1$ simply means that x_j is a smooth point of $\mathcal{L}'$). In particular, letting n_i denote the number of intersection points of $\mathcal{L}$ of multiplicity i which lie on L, then we must have

$$t_i = t'_i + n_i - n_{i+1}.$$

However, we then see that

$$\begin{aligned}
\sum_{i=3}^{n}(i-2)t_i &= \sum_{i=3}^{n}(i-2)(t'_i + n_i - n_{i+1}) \\
&= \sum_{i=3}^{n}(i-2)t'_i + \sum_{i=3}^{n}(i-2)n_i - \sum_{i=3}^{n}(i-2)n_{i+1} \\
&= \sum_{i=3}^{n}(i-2)t'_i + \sum_{i=3}^{n}(i-2)n_i - \sum_{i=3}^{n}(i-3)n_i \\
&= \sum_{i=3}^{n}(i-2)t'_i + \sum_{i=3}^{n} n_i.
\end{aligned}$$

Furthermore, as we observed above, L contains at least one point of intersection with multiplicity ≥ 3 and so $\sum_{i=3}^{n} n_i \geq 1$. It then follows that adding the line L to $\mathcal{L}'$ does not decrease the sum on the left side of (5.4) and so the coefficient remains nonnegative. $\square$

In fact, we observe that the trick of inductively constructing any supersolvable arrangement from another can be used to provide a stronger overview of the covers branched over such an arrangement. Specifically, the total number of higher order intersection points characterize the behaviour of the signature of a cover branched along a supersolvable line arrangement.

Theorem 5.2.11. *Suppose $\mathcal{L}$ is a supersolvable arrangement of n lines in $\mathbb{CP}^2$. Let $\rho : X \to \mathbb{CP}^2$ be a (smooth) $(\mathbb{Z}/p\mathbb{Z})^k$ cover branched over $\mathcal{L}$. The behaviour of the signature of X is determined by the number of intersection points with multiplicity greater than 2.*

(1) *If $\sum_{i=3}^{n} t_i = 0$ then $n = 3$, and the $(\mathbb{Z}/p\mathbb{Z})^2$ cover branched over $\mathcal{L}$ is $\mathbb{CP}^2$. In particular, $\sigma(X) = 1$.*

(2) *If $\sum_{i=3}^{n} t_i = 1$, then $\mathcal{L}$ is either a pencil ($t_n = 1$) or a near pencil ($t_{n-1} = 1$). If $\mathcal{L}$ is a near pencil, then any $(\mathbb{Z}/p\mathbb{Z})^k$ cover branched over $\mathcal{L}$ satisfies $\sigma(X) = 0$.*

(3) *If $\sum_{i=3}^{n} t_i = 2$, then $\mathcal{L}$ is constructed from two pencils by adding a line through both intersection points. Furthermore any $(\mathbb{Z}/p\mathbb{Z})^k$ cover branched over $\mathcal{L}$ satisfies $\sigma(X) = -p^{k-2}$. In particular, if X is a $(\mathbb{Z}/p\mathbb{Z})^2$ branched cover, then $\sigma(X) = -1$.*

(4) *If $\sum_{i=3}^{n} t_i \geq 3$, then $\sum_{i=3}^{n} t_i \geq 4$ and the leading coefficient of $\sigma(X)$ satisfies*

$$\left(3 - n + \sum_{i=3}^{n} (i-2)t_i\right) > 0.$$

Proof. (1) If $\sum_{i=3}^{n} t_i = 0$, then $\mathcal{L}$ consists of n lines in general position and by Lemma 5.2.7 any intersection point is a modular point. However, if $n \geq 4$, then for two lines L_1, L_2 intersecting at a modular point x, we have that there are exactly $2(n-1) - 1 = 2n - 3$ intersection points of $\mathcal{L}$ on $L_1 \cup L_2$. However, an arrangement of lines in general position have $\frac{n(n-1)}{2}$ total intersection points. Therefore, when $n \geq 4$, there will be at least one intersection point that does not lie on $L_1 \cup L_2$ contradicting that x is a modular point.

Hence $n = 3$ and as discussed in Example 5.1.1, any smooth $(\mathbb{Z}/p\mathbb{Z})^2$ cover branched over $\mathcal{L}$ is $\mathbb{CP}^2$.

(2) If $\sum_{i=3}^{n} t_i = 1$, then there is a unique intersection point x of multiplicity $m \geq 3$. We again have that x is a modular point by Lemma 5.2.7. It is well known that the only line arrangements with $t_n = 1$ are pencils. If $t_{n-1} = 1$ then the only possibility if that the line not passing through x lies in general position, hence $\mathcal{L}$ is a near pencil. Finally, if $t_n = t_{n-1} = 0$, then there are at least two lines in general position (not passing through x) which intersect at a two-fold point. As this two-fold point does not lie on a line in $\mathcal{L}$ passing through x, we arrive at a contradiction that x is a modular point. Hence any supersolvable line arrangement with exactly one intersection point of higher multiplicity is either a pencil or a near pencil. As shown in Section 5.2.1, if $\mathcal{L}$ is a near pencil then the branched cover X has $\sigma(X) = 0$.

(3) Suppose $\sum_{i=3}^{n} t_i = 2$, and let x_u and x_v denote the intersection points of multiplicities $u \geq 3$ and $v \geq 3$ respectively. As at least one of x_u or x_v is a modular point, it follows that there is a line in $\mathcal{L}$ on which points x_u and x_v lie. We further observe that we must have $n = u + v - 1$. If not, i.e., if there is a line L that passes through neither x_u nor x_v, then L must be a line in general position as otherwise there is another intersection point of multiplicity ≥ 3, contradicting the assumption $\sum_{i=3}^{n} t_i = 2$. However, if L is in general position, then for the point x_u (or x_v), the intersection of L and a line passing through x_v (or x_u) is a double point for which no line passing through x_u (or x_v) passes through. Therefore x_u (and x_v) cannot be a modular point for $\mathcal{L}$, a contradiction. Now, using $n = u + v - 1$, we can compute t_2 by manipulating equation (4.7). Specifically, starting with

$$\frac{n(n-1)}{2} = t_2 + \frac{u(u-1)}{2} + \frac{v(v-1)}{2},$$

we find

$$\begin{aligned} 2t_2 &= n^2 - n - u^2 + u - v^2 + v \\ &= (u + v - 1)^2 - u^2 - v^2 - (n - u - v) \\ &= (u^2 + v^2 + 1 - 2u - 2v + 2uv) - u^2 - v^2 + 1 \\ &= 2 - 2u - 2v + 2uv \\ &= 2(1 - u)(1 - v). \end{aligned}$$

That is, $t_2 = (u - 1)(v - 1)$ and so $\mathcal{L}$ is necessarily the connected double pencil from Example 5.2.5. Applying Lemma 4.3.5 we find $\sigma(X) = -p^{k-2}$ and so the result follows.

(4) Suppose $\sum_{i=3}^{n} t_i \geq 3$ and let $\mathcal{L}_3, \mathcal{L}_4, \ldots, \mathcal{L}_n = \mathcal{L}$ be the sequence of supersolvable arrangements from Lemma 5.2.8. Let x_1 be an intersection point of maximal multiplicity m_1 and suppose x_2 is another intersection point with multiplicity $m_2 \geq 3$. By Remark 5.2.9, we may assume that L_{m_1} is a pencil of m_1 lines intersecting at x and $\mathcal{L}_{m_1+m_2-1}$ is a connected double pencil of $m_1 + m_2 - 1$ lines. We may also suppose that L_{m_1} is the line that passes through both x_1 and x_2 so that the lines $L_i, i < m_1$ pass through only x_1 (and not x_2) and $L_i, i > m_1$ pass through only x_2 (and not x_1). It is easy to check that for a connected double pencil

$$3-(m_1+m_2-1)+\sum_{i=3}^{m_1+m_2-1}(i-2)t_i = 3-(m_1+m_2-1)+(m_1-2)+(m_2-2) = 0. \quad (5.5)$$

Now, since there are at least three intersection points of multiplicity at least three, $\mathcal{L} \neq \mathcal{L}_{m_1+m_2-1}$. That is, if there is another point of multiplicity $m_3 \geq 3$ in $\mathcal{L}$, then there is at least one line in $\mathcal{L}$ that does not pass through either of x_1 or x_2, call it L. We now consider $\mathcal{L}_{m_1+m_2-1} \cup L$. Since L must intersect L_{m_1+1} and the point of intersection between L and L_{m_1+1} must lie on a line passing through x_1 (since x_1 is a modular point), there is some $L_i, i < m_1$ such that $L \cap L_i \cap L_{m_1+1} \neq \emptyset$; call this point x_3. Furthermore, since $m_2 \geq 3$, there is a line L_{m_1+2} that also passes through x_2 and not x_1 that is different from L_{m_1+1}. A similar argument shows that the intersection point of L and L_{m_1+2} must also lie on some $L_j, j < m_1$ and in fact, $i \neq j$ as otherwise we would have $L = L_j = L_i$. Therefore $L \cap L_j \cap L_{m_1+2} \neq \emptyset$ is an intersection point different than x_3 that also has multiplicity three. That is, after adding in the remaining lines, there is at least four intersection points with multiplicity at least three, hence $\sum_{i=3}^{n} t_i \geq 4$. We also see that by adding the line L, the $3-n$ term of Equation 5.5 decreases by one where as the t_3 term increases by at least two. Furthermore, as L did not pass through any higher multiplicity intersection points of $\mathcal{L}_{m_1+m_2-1}$, it follows that

$$3-(m_1+m_2)+\sum_{i=3}^{m_1+m_2}(i-2)t_i \geq 1. \quad (5.6)$$

Finally, as observed in the proof of Theorem 5.2.10, any other line added to a supersolvable arrangement that maintains supersolvability and is not a near pencil cannot decrease the leading coefficient expression. It follows that since the coefficient in equation 5.6 is stricitly

positive, for the total line arrangement $\mathcal{L}$ we must have

$$\left(3 - n + \sum_{i=3}^{n}(i-2)t_i\right) > 0. \tag{5.7}$$

This concludes our proof. □

We note as a consequence of Theorem 5.2.11 that the supersolvable line arrangements of Examples 5.2.6 will all correspond to positive leading coefficients and hence positive signature for corresponding covers as $p \to \infty$. Furthermore, we also see that supersolvable line arrangements that yield negative signature are comparatively rare to those that give non-negative or positive signatures. However, while supersolvable arrangements provide an abundance of covers with positive signature, they do not fully encapsulate the line arrangements that are 'applicable' to the geography problem. For instance, the converse of Theorem 5.2.10 is not true. For a counterexample, we refer the reader to Remark 5.3.1 below.

5.3 Non-supersolvable arrangements

As mentioned above, 4-manifolds with positive signature can be constructed using branched covers of line arrangements that are not supersolvable. We will focus on two such line arrangements, the Wiman configuration (more specifically, we will be interested in the Wiman arrangement with an extra line added) and the Dual Hesse arrangement. We will later consider both of these arrangements in the context of the geography problem and contrast them to motivate the results of Section 6.4.

5.3.1 Wiman arrangement plus one line

The Wiman arrangement (which we will denote by $\mathcal{W}$) is a line arrangement consisting of 45 lines with $t_3 = 120, t_4 = 45, t_5 = 36$, and $t_i = 0$ for all other multiplicities. In general we refer the reader to Subsection 4.2 in [14] for further reading on the Wiman configuration. However, we do note that since $t_2 = 0$, the Sylvester-Gallai theorem (proved by Melchior in [48]) implies that $\mathcal{W}$ is not a real line arrangement and so does not have a corresponding picture in $\mathbb{R}^2$ without curved lines and 'phantom' intersections.

Now, let L be any generic line (i.e., a line distinct from the 45 lines in the Wiman configuration and which does not intersect $\mathcal{W}$ at any intersection point of W). We will let $\mathcal{L} = \mathcal{W} \cup L$ denote the line arrangement consisting of 46 lines and satisfying $t_2 = 45, t_3 = 120, t_4 = 45, t_5 = 36$ and $t_i = 0, i \geq 6$. For any $p \geq 2$, we consider the smooth branched cover of $\mathbb{CP}^2$ defined by equation (4.6) with Galois group $G = (\mathbb{Z}/p\mathbb{Z})^{45}$ and resolved by first blowing up all 201 intersection points of multiplicity at least three before taking the cover. In this case, we will let $\beta : \mathbb{CP}^2 \# 201 \overline{\mathbb{CP}}^2 \to \mathbb{CP}^2$ denote the blow-down map and $\rho : X_p \to \mathbb{CP}^2 \# 201 \overline{\mathbb{CP}}^2$ denote the covering map. The branch locus of ρ is $B = \beta^{-1}(\mathcal{L})$ which is the union of proper transforms of the 45 lines of $\mathcal{W}$, the proper transform $\widehat{L}$ of the generic line L, and the 201 exceptional spheres. Applying Lemma 4.3.5 we have

$$e(X_p) = p^{43}(475p^2 - 1036p + 765), \tag{5.8}$$

$$\sigma(X_p) = \frac{p^{43}}{3}\left(275p^2 - 875\right). \tag{5.9}$$

We highlight the fact that for all $p \geq 2$ we have $\sigma(X_p) > 0$. Now, $\widehat{L}$ is a sphere with self-intersection $(\widehat{L})^2 = 1$ and Lemma 4.1.3 implies

$$I_{\widehat{L}} \cong \mathbb{Z}/p\mathbb{Z} \tag{5.10}$$

$$H_{\widehat{L}} = G. \tag{5.11}$$

Now, on any line in the Wiman arrangement, it is known that there are exactly 8 intersection points with multiplicity 3, 4 intersection points with multiplicity 4 and 4 intersection points with multiplicity 5. Therefore, in $\mathcal{L}$ every line C that is different from the general line L contains exactly 17 intersection points. It then follows from Lemma 4.1.3 that $|H_C| \leq p^{17+1}$. That is, $\widehat{L}$ is the only curve $C' \subset \beta^{-1}(\mathcal{L})$ for which $\rho^{-1}(C')$ has a unique component, say $L' = \rho^{-1}(\widehat{L})$ for which $(L')^2 = p^{43}(\widehat{L})^2 = p^{43}$. Now, a computation similar to equation (5.3) (or an application of Lemma V.3.1 in [37]) shows that the Euler characteristic of L' is

$$e(L') = (2 - 45)p^{44} + 45p^{43}. \tag{5.12}$$

That is, L' is a surface of genus $g(L') = \frac{1}{2}(43p^{44} - 45p^{43}) + 1$.

Alternatively, if we consider an exceptional sphere E that corresponds to the blow-up of any triple point, Then applying Lemma 4.1.3 we find

$$I_E \cong \mathbb{Z}/p\mathbb{Z} \tag{5.13}$$

$$H_E \cong (\mathbb{Z}/p\mathbb{Z})^3. \tag{5.14}$$

In contrast to all above examples, we see that Lemma 4.1.5 implies that $\rho^{-1}(E)$ consists of p^{42} components. If we consider one of these components $E' \subset \rho^{-1}(E)$, then we find that $(E')^2 = p \cdot E^2 = -p$ and $e(E') = -p^2 + 3p$. We note that when $p = 2$, E' is a sphere and for $p = 3$, E' is a torus.

Remark 5.3.1. Both the Wiman arrangement $\mathcal{W}$ and the Wiman arrangement with a line added, $\mathcal{L}$, are examples of arrangements that are not supersolvable but also satisfy equation (5.4). For instance, it was proved in [1] that a supersolvable line arrangement that is not a pencil must have at least one double point. As it is easily seen that the Wiman arrangement is not a pencil, it is not supersolvable. The 'line added' Wiman arrangement also cannot be supersolvable. If it was, then all the intersection points of multiplicity 5 would be modular points by Lemma 5.2.7. Let us consider one such modular point x. As there are only 5 lines passing through x whereas there are 45 double points of the arrangement, at least one of the lines passing through x contains more than one double point. In this case, as L is the only line in $\mathcal{L}$ containing more than one double point, it follows that L passes through x. However, L was added to $\mathcal{W}$ in a generic way and does not intersect any other intersection point of $\mathcal{W}$, a contradiction.

5.3.2 The dual Hesse arrangement

We now focus on another arrangement that is not supersolvable while also resulting in a family of smooth Kähler manifolds with positive signature. Giving $\mathbb{CP}^2$ the homogeneous coordinates $(x : y : z)$, the linear hyperplanes of the arrangement are given by the 9 linear factors of

$$(x^3 - y^3)(y^3 - z^3)(z^3 - x^3).$$

There are exactly 12 intersection points of the arrangement and they are all triple points.

This arrangement carries two common names. First, it is a special case of line arrangements defined by the equation

$$(x^n - y^n)(y^n - z^n)(z^n - x^n)$$

and in [67] these are referred to as the CEVA(n) arrangements. Alternatively, this arrangement can also be called the dual Hesse arrangement. A partial intuition behind this name is as follows: if one starts with the Hesse arrangement, then one can consider the 'dual' of

the arrangement in a certain sense. That is, take the intersection points to be lines and assume that they intersect in accordance to the lines that pass through them originally. In doing as described above, one can obtain an arrangement that is the same as the CEVA(3). From here on out we shall refer to this arrangement as the dual Hesse. For a discussion on the application of the Hesse arrangement in addition to a diagram of its intersection pattern, we refer the reader to [8].

We note that similar to the Wiman configuration, the dual Hesse configuration also has $t_2 = 0$. In particular, this implies that it is not supersolvable and that it is not a real arrangement (these follow similarly to above by [1] and [48], respectively). Nonetheless, as the number of lines in the dual Hesse arrangement is comparatively small, we provide a diagram that represents the intersection properties of the lines in Figure 5.5. A similar diagram showing an unlabeled dual Hesse arrangement can be found in [18].

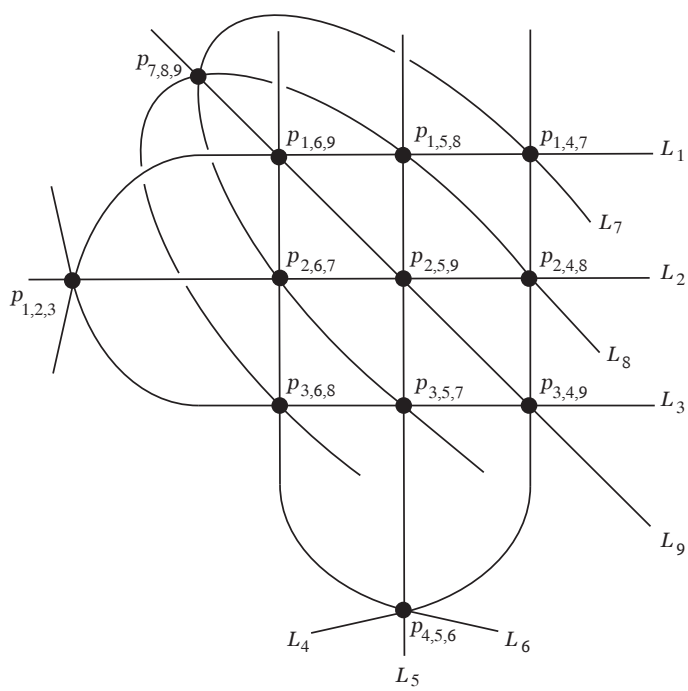

Figure 5.5: Dual Hesse arrangement $\mathcal{H}$

Letting $\mathcal{H}$ denote the dual Hesse arrangement, we define a $(\mathbb{Z}/p\mathbb{Z})^2$ covering of $\mathbb{CP}^2$

branched over $\mathcal{H}$ for a fixed odd prime p by considering the map

$$\phi : H_1(\mathbb{CP}^2 - \mathcal{H}) \cong \mathbb{Z}^8 \longrightarrow (\mathbb{Z}/p\mathbb{Z})^2$$

given by,

$$\begin{aligned}
\phi(\mu_1) &= (1,0), \quad \phi(\mu_2) = (1,4), \quad \phi(\mu_3) = (-6,-8),\\
\phi(\mu_4) &= (0,1), \quad \phi(\mu_5) = (1,0), \quad \phi(\mu_6) = (1,0),\\
\phi(\mu_7) &= (0,1), \quad \phi(\mu_8) = (1,1), \quad \phi(\mu_9) = (1,1).
\end{aligned}$$

We blow up $\mathbb{CP}^2$ at the twelve intersection points denoted by

$$p_{i_1,i_2,i_3} = L_{i_1} \cap L_{i_2} \cap L_{i_3}.$$

In preparation for Example 6.1.7 in the next chapter, a schematic providing an example of what this does to the line L_1 is given in Figure 5.6, where as always $\beta : \mathbb{CP}^2\#12\overline{\mathbb{CP}}^2 \to \mathbb{CP}^2$ denotes the blow-down map.

Denoting E_{i_1,i_2,i_3} as the exceptional sphere of the blow-up at p_{i_1,i_2,i_3}, it follows that the meridian class of E_{i_1,i_2,i_3} is given by

$$\varepsilon_{i_1,i_2,i_3} = \beta_*^{-1}(\mu_{i_1} + \mu_{i_2} + \mu_{i_3})$$

as in equation (4.3). We observe that ϕ represents a well defined branched cover which we will denote by $\rho : X_p \to \mathbb{CP}^2\#12\overline{\mathbb{CP}}^2$. Moreover, applying Lemma 4.3.5, we compute the following topological invariants of X_p to be

$$\begin{aligned}
e(X_p) &= 9p^2 - 30p + 36,\\
\sigma(X_p) &= 2p^2 - 13.
\end{aligned}$$

Unlike Examples 5.2.1 and 5.3.1, many of the properties of X_p are not immediate. For this reason, we will go through each of them separately.

Lemma 5.3.2. (cf. Lemma 7 in [33]) *For every prime number $p \geq 11$, X_p is smooth.*

Proof. As the singular points of $\mathcal{H}$ are the 12 triple points, we must verify Lemma 4.1.4 for the three intersection points on each exceptional sphere. For instance, if we consider the

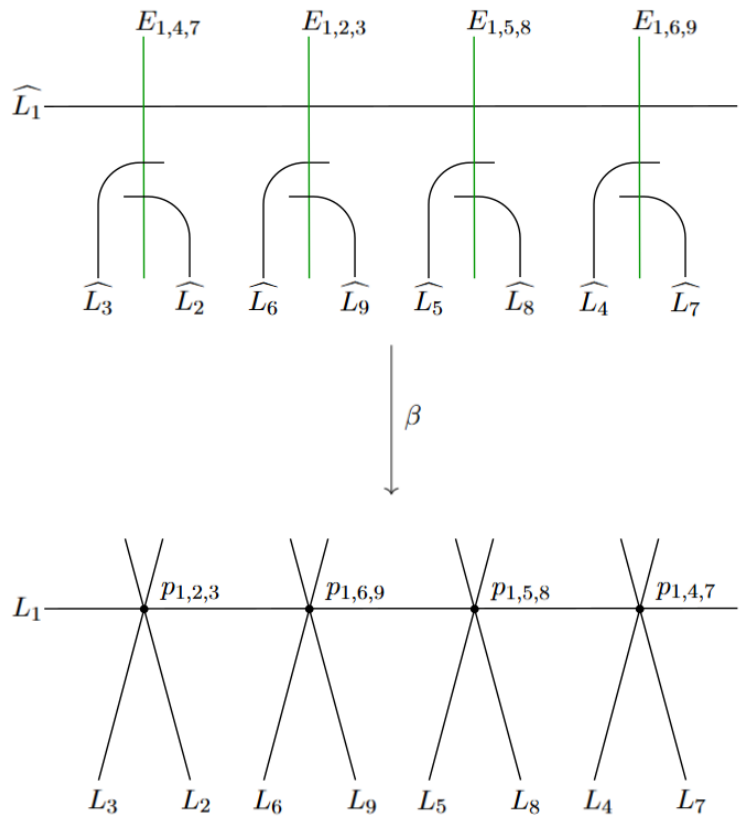

Figure 5.6: Blowing up the intersection points on L_1

sphere $E_{1,4,7} = \beta^{-1}(p_{1,4,7})$, then by Lemma 4.1.3 we compute the relevant inertia subgroups to be

$$\begin{aligned}
I_{\widehat{L_1}} &= \langle \phi(\mu_1) \rangle = \langle (1,0) \rangle, \\
I_{\widehat{L_4}} &= \langle \phi(\mu_4) \rangle = \langle (0,1) \rangle, \\
I_{\widehat{L_7}} &= \langle \phi(\mu_7) \rangle = \langle (0,1) \rangle, \\
I_{E_{1,4,7}} &= \langle \phi(\mu_1 + \mu_4 + \mu_7) \rangle = \langle (1,2) \rangle.
\end{aligned}$$

Since G is elementary abelian, all subgroups of G are cyclic of prime order p, and so two subgroups of G intersect non-trivially if and only if they are equal. Alternatively, in the

language of Lemma 4.3.1 for $G = (\mathbb{Z}/p\mathbb{Z})^2$, if we can exhibit a (non-identity) element lying exclusively in either $I_{\widehat{L}_i}$ or $I_{E_{1,4,7}}$, then $\langle \phi(\mu_i), \phi(\varepsilon_{i_1,i_2,i_3}) \rangle = G \cong (\mathbb{Z}/p\mathbb{Z})^2$ and smoothness above the point $p_{1,4,7}$ would follow. For this intersection point, similar to smoothness in Example 5.2.1, since $(1,2)$ is not of the form $(*,0)$ nor $(0,*)$, it is not an element of any $I_{\widehat{L}_i}$ for $i = 1,4,7$.

We compare the intersection point above with the case for $p_{2,5,9}$. We compute

$$\begin{aligned}
I_{\widehat{L}_2} &= \langle \phi(\mu_2) \rangle = \langle (1,4) \rangle, \\
I_{\widehat{L}_5} &= \langle \phi(\mu_5) \rangle = \langle (1,0) \rangle, \\
I_{\widehat{L}_9} &= \langle \phi(\mu_9) \rangle = \langle (1,1) \rangle, \\
I_{E_{2,5,9}} &= \langle \phi(\mu_2 + \mu_5 + \mu_9) \rangle = \langle (3,5) \rangle.
\end{aligned}$$

As above, we must show at least one (and hence all) non-identity elements of $I_{E_{2,5,9}}$ are not contained in $I_{\widehat{L}_i}$ for $i = 2,5,9$. The $i = 5$ case is identical to the previous case. Since all elements of $\langle (1,1) \rangle$ are of the form (k,k) for some $k \in \mathbb{Z}$, and combining with the fact that $3 \not\equiv 5 \pmod{p}$ (when $p \geq 11$), it follows that $(3,5) \notin I_{\widehat{L}_9}$.

When $i = 2$, supposing for contradiction that $(3,5) \in I_{\widehat{L}_2}$, then

$$(0,7) = 3(1,4) - (3,5) \in I_{\widehat{L}_2}.$$

In particular, if s is the multiplicative inverse of 7 in $\mathbb{Z}/p\mathbb{Z}$, which exists for primes $p \geq 11$, then

$$(0,1) = 3s(1,4) - s(3,5) \in I_{\widehat{L}_2}.$$

That is, $(0,1) \in I_{\widehat{L}_2}$ and since $I_{\widehat{L}_2}$ is cyclic of prime order, we must have $I_{\widehat{L}_2} = \langle (0,1) \rangle$. However, as with all previous cases mentioned, it is then observed that we cannot have $(1,4) \notin I_{\widehat{L}_2}$, a contradiction. It follows that X_p will be smooth at any point above $p_{2,5,9}$.

In general, the smoothness criterion can be checked above all points of the form p_{i_1,i_2,i_3} using an argument similar to one of those found in the above two examples in addition to those provided at other times throughout this thesis. For this reason we shall omit the arguments for the remaining 10 cases. □

Remark 5.3.3. We can allow p to be in a certain set of composite numbers but it would be more difficult to check that the corresponding branched covers X_p are smooth. Specifically, we again highlight the fact that since all cyclic subgroups of elementary abelian groups

have the same (prime) order, it makes checking the intersection condition more efficient due to only needing to show any two subgroups are not exactly equal. However, it is enough for our purposes to only work with prime p and so we shall continue to do so.

As we are primarily interested in the case for which X_p is smooth (so that it is Kähler (Remark 4.2.6), we will assume that $p \geq 11$ unless stated otherwise.

In the case of $G = (\mathbb{Z}/p\mathbb{Z})^{n-1}$ where n is the number of lines in an arrangement, [39] provides a description of the minimality of the branched cover. To argue that each X_p constructed in this section is minimal, more work is required. To that end, we first state and prove the following result regarding the exceptional spheres found in the branch locus.

Lemma 5.3.4. (cf. Lemma 10 in [33]) *$H_{E_{i_1,i_2,i_3}} = (\mathbb{Z}/p\mathbb{Z})^2 = G$ for every prime $p \geq 11$ and every exceptional sphere E_{i_1,i_2,i_3}. Specifically, this result holds for all primes p for which X_p is smooth.*

Proof. An exceptional sphere E_{i_1,i_2,i_3} intersects only the curves $\widehat{L}_{i_1}$, $\widehat{L}_{i_2}$ and $\widehat{L}_{i_3}$ in the branch locus $\beta^{-1}(\mathcal{H})$. Therefore, applying Lemma 4.1.3, we find that $\phi(\mu_{i_1})$, $\phi(\mu_{i_2})$, $\phi(\mu_{i_3})$ and $\phi(\varepsilon_{i_1,i_2,i_3})$ generate $H_{E_{i_1,i_2,i_3}}$. Furthermore, from Lemma 5.3.2 we know X_p is smooth, and so Lemma 4.3.1 immediately implies that

$$\langle \phi(\varepsilon_{i_1,i_2,i_3}), \phi(\mu_{i_j}) \rangle = (\mathbb{Z}/p\mathbb{Z})^2 = G$$

for all $j = 1, 2, 3$. It then immediately follows that

$$H_{E_{i_1,i_2,i_3}} = \langle \phi(\mu_{i_1}), \phi(\mu_{i_2}), \phi(\mu_{i_3}), \phi(\varepsilon_{i_1,i_2,i_3}) \rangle = G. \qquad \square$$

We are now prepared to consider the minimality of X_p.

Lemma 5.3.5. (cf. Lemma 11 in [33]) *For every prime number $p \geq 11$, X_p is minimal.*

Proof. We suppose for contradiction that X_p is not minimal. Then there is an irreducible curve C' contained in X_p with genus 0 and self-intersection -1. Since C' is irreducible, we find the curve $C = \rho(C')$ is also irreducible. Furthermore, the Riemann-Hurwitz formula states

$$2g(C') - 2 = |G|(2g(C) - 2) + \sum_{x \in B \cap C} (r_x - 1) \tag{5.15}$$

where g denotes the genus and r_x is the ramification degree of $\rho^{-1}(x)$. If $g(C) > 0$, then

$$\begin{aligned} -2 &= p^2(2g(C) - 2) + \sum_{x \in B \cap C} (r_x - 1) \\ &\geq 0 + \sum_{x \in B \cap C} (r_x - 1) \\ &\geq 0. \end{aligned}$$

Therefore, C is also a curve of genus 0.

Now, suppose C is a curve not fully contained in $B = \beta^{-1}(\mathcal{H})$, then in $H_2(\mathbb{CP}^2 \# 12\overline{\mathbb{CP}}^2)$, C and B intersect in a finite set of points, say $r = [C] \cdot [B]$. We will let s denote the number of intersection points for which B and C intersect along a single component of B and t denote the number points in C intersecting B at a pre-exisisting intersection point of B (a double point of $\beta^{-1}(\mathcal{H})$). Then $r = s + t$ and applying an inclusion-exclusion argument similar to that done in equation (5.3) we find

$$2 = e(C') = p^2(2 - r) + ps + t. \tag{5.16}$$

Since $p \geq 11$, we observe that the only solution to equation (5.16) is $r = t = 2$ and $s = 0$. So C intersects exactly two double points of B. More specifically, we have that C intersects the proper transform of at least one line of $\mathcal{H}$ and at least one exceptional sphere.

Next we consider a generic line L in $\mathbb{CP}^2$ so that its proper transform $\widehat{L}$ does not intersect any exceptional sphere of B. Then $H_2(\mathbb{CP}^2 \# 12\overline{\mathbb{CP}}^2)$ is generated by $[\widehat{L}]$ and the exceptional spheres $[E_{i_1,i_2,i_3}]$ that correspond to the twelve intersection points of $\mathcal{H}$. Expressing $[C]$ in this basis we have

$$[C] = a_L[\widehat{L}] + \sum_{p_{i_1,i_2,i_3} \in \mathrm{Sing}(B)} a_{i_1,i_2,i_3}[E_{i_1,i_2,i_3}].$$

Since C is not one of the exceptional spheres (as it is not fully contained in B), C must intersect L and so $a = a[\widehat{L}]^2 = [C] \cdot [\widehat{L}] \neq 0$. Moreover, since both C and $\widehat{L}$ are distinct holomorphic curves, it also follows that $a = [C] \cdot [\widehat{L}] > 0$. Now, $\beta(C)$ is an irreducible genus 0 curve in $\mathbb{CP}^2$. In particular, it must then correspond to an embedding of degree 1 or 2. That is, by Bezout's theorem, we can conclude that $a = [C] \cdot [\widehat{L}] \in \{1, 2\}$

On the other hand, since $t = 2$, there are two distinct exceptional spheres E_{i_1,i_2,i_3} and E_{j_1,j_2,j_3} for which

$$[C] \cdot [E_{i_1,i_2,i_3}] = [C] \cdot [E_{j_1,j_2,j_3}] = 1.$$

It follows that the homology class of C must be of the form

$$[C] = [L] - [E_{i_1,i_2,i_3}] - [E_{j_1,j_2,j_3}]$$

or

$$[C] = 2[L] - [E_{i_1,i_2,i_3}] - [E_{j_1,j_2,j_3}].$$

Now we note that there are at least three mutually disjoint lines $\widehat{L}_{k_1}$, $\widehat{L}_{k_2}$, $\widehat{L}_{k_3}$ in B that are also disjoint from E_{i_1,i_2,i_3} and E_{j_1,j_2,j_3}. Specifically, by considering indices, we take

$$\{k_1, k_2, k_3\} \subseteq \{1, 2, \ldots, 9\} - \big(\{i_1, i_2, i_3\} \cup \{j_1, j_2, j_3\}\big).$$

However, we must then have either

$$[C] \cdot [\,\widehat{L}_{k_1}] = [C] \cdot [\,\widehat{L}_{k_2}] = [C] \cdot [\,\widehat{L}_{k_3}] = 1,$$

or

$$[C] \cdot [\,\widehat{L}_{k_1}] = [C] \cdot [\,\widehat{L}_{k_2}] = [C] \cdot [\,\widehat{L}_{k_3}] = 2.$$

Therefore, C and B intersect in at least three or six points, respectively. But this contradicts our previous observation that $r = 2$.

Finally, suppose that C is contained in B. Since C is irreducible and has negative self-intersection by Lemma 4.1.6, C must equal one of the exceptional spheres, say E_{l_1,l_2,l_3}. By combining Lemma 4.1.5 and Lemma 5.3.4, we conclude that C' is the only curve in the lift $\rho^{-1}(C)$. As observed in the proof of Lemma 5.3.4, $C = E_{l_1,l_2,l_3}$ contains three double points of B. From the inclusion-exclusion principle, we conclude that the lift $C' = \rho^{-1}(C)$ has Euler characteristic equal to $p(2-3) + 3 = 3 - p$. It follows that the genus of C' is $(p-1)/2 \geq 5$, which contradicts our assumption that C' has genus 0. □

Remark 5.3.6. An alternative proof of the minimality of X_p can be given using Assertion 2.2 in [45]. The minimality of X_p is equivalent to the following divisor

$$D_K = (6p-9)L - \sum_{i_1,i_2,i_3} (p-2)E_{i_1,i_2,i_3}$$

being numerically effective (nef) on $Y = \mathbb{CP}^2 \# 12\overline{\mathbb{CP}}^2$. Here, the last sum is taken over all twelve exceptional spheres (In the notation of [45], $q = p$, $n = 9$, T'_r is the empty set for

all $r \geq 2$, $|T_3''| = 12$, and T_r'' is the empty set for all $r \geq 4$.). Hence we must show that $D_K \cdot C \geq 0$ for every irreducible curve C in Y. Since we have

$$3[D_K] = (9p-9)[L] + (p-2)\sum_{i=1}^{9}[\widehat{L}_i]$$

inside $H_2(Y)$, it is not too hard to show that $(3D_K) \cdot C \geq 0$ for every irreducible curve C in Y.

Chapter 6

Building simply connected symplectic manifolds

In the previous chapter, we saw how to construct symplectic 4-manifolds with invariants applicable to the geography problem. Unfortunately, the 4-manifolds constructed do not immediately apply. For instance, while the 4-manifolds constructed above have non-negative signature, they are in general not simply connected. For instance, computations by [37] of b_1 for 'small' line arrangements suggests that the majority of covers branched over line arrangements will have infinite fundamental group. That said, in [39], [37] and [59] there are interesting examples of covers with $b_1 = 0$. Unfortunately, even for most of these examples, it is still not known as to what exactly is the fundamental group of the cover. For this reason, in this chapter we shall start by investigating under which conditions useful information on the fundamental group of a 4-manifold will be known.

6.1 Fundamental groups of reducible divisors

The general structure of this section is as follows. First we will find divisors on complex surfaces that contain a generating set of the the fundamental group of the ambient 4-manifold. In doing so, we will then be able to use this data when performing surgeries on the 4-manifolds along these divisors to construct simply connected symplectic 4-manifolds.

We start by recalling that given a divisor D on a variety X, one can define a rational

morphism $\psi_D : X \to \mathbb{CP}^N$ (where $N = \dim |D|$). Specifically, the maps ψ_D is defined by considering a basis of global sections of the line bundle corresponding to D and taking the coordinates of the map ψ_D to be evaluation on those sections. There is quite a bit of literature surrounding these maps, especially in consideration to their relation with the divisor K_X and the Kodaira dimension of a complex surface (cf. Section IV.1 of [12]). In general, for our purposes, we will only require the existence of this map. Specifically, we consider the following definition from Nori [49].

Definition 6.1.1. (cf. Definition 2.2 in [49]) An effective divisor D on a complex surface X is *not composed of a pencil* if for some positive integer m the rational morphism $\psi_m : X \to \mathbb{CP}^N$ induced by the complete linear system $|mD|$ satisfies $\dim_{\mathbb{C}}(\psi_m(X)) = 2$.

This definition in general picks out divisors admitting enough sections to construct a *surface* in $\mathbb{CP}^N$ that is birational to X. Alternatively, one can think of a divisor being composed of a pencil if the rational morphism ψ_D can be factored, where you first project X onto a complex curve (this is a Stein factorization) and then embed it into $\mathbb{CP}^N$ and so X can be viewed as a pencil of curves.

We also see that a sufficient condition for a divisor to be not composed of a pencil is for the divisor to be ample, as in this case there is some $m > 0$ for which the morphism ψ_m is an embedding.

We now state Corollary 2.3 and 2.4 in [49] (with R equal to the empty set) as lemmas to motivate our interests in divisors that are not composed of a pencil and their relation to fundamental groups.

Lemma 6.1.2. (cf. Corollary 2.3 in [49]) *Let D be an effective divisor on a smooth complete complex surface that is not composed of a pencil. If U denotes a tubular neighborhood of the support of D, then the inclusion-induced homomorphism $\pi_1(U) \to \pi_1(X)$ is surjective.*

Lemma 6.1.3. (cf. Corollary 2.4 in [49]) *If C is an irreducible curve on a surface X with $C^2 > 0$ and if U denotes a tubular neighborhood of the support of D, then $\pi_1(U) \to \pi_1(X)$ is surjective.*

The above lemmas show that for a suitably chosen embedded surface, some information on the fundamental group $\pi_1(X)$ of the ambient 4-manifold X can be obtained from the topology of the surface. In this context, we say that the divisor D (or the curve C) *carries* the fundamental group of X.

Example 6.1.4. We return briefly to example 5.3.1. The curve $L' = \rho^{-1}(\widehat{L})$ is irreducible and $(L')^2 = p^{43} > 0$. Therefore, Lemma 6.1.3 implies that the curve L' carries the fundamental group of the covering space branched over the Wiman arrangement with an added line.

In general, if a line arrangement $\mathcal{L}$ contains at least one generic line, then Lemma 4.2.4 implies the preimage of the line is an irreducible curve. Moreover, as such a line contains no intersection points that have higher multiplicity greater than 2, it will follow that this unique lift has positive self intersection (in the context of the covers discussed in this thesis). Therefore it will carry the fundamental group. However, a line arrangement in general may not include a generic line (for instance, the dual Hesse). In this case, any irreducible curve C in $\mathcal{L}$ contains at least one higher order intersection point and so after blowing up, we will have $\widehat{C}^2 \leq 0$. In particular, any irreducible curve in the lift of any (reducible or irreducible) curve in $\mathcal{L}$ will have non-positive self intersection. Consequently, we wish to provide a generalization of Lemma 6.1.3 in the case of reducible curves with positive self intersection. That way, we will be able to 'stitch' curves together in the cover in the hopes of finding positive self intersection.

In preparation to prove this generalization, we state the following fact from [12] as a lemma.

Lemma 6.1.5. (cf. Proposition IV.7.4 in [12]) *Let D be a divisor on a complex surface X. If $D^2 > 0$, then $D \cdot H \neq 0$ for any ample divisor H. Moreover, if $D \cdot H > 0$, then some positive multiple of D is effective, and $h^0(\mathcal{O}_X(mD)) = \dim H^0(X, \mathcal{O}_X(mD))$ grows quadratically in m.*

Theorem 6.1.6. (cf. Theorem 16 in [33]) *Let D be a (possibly reducible) effective divisor on a compact projective Kähler surface X with $D^2 > 0$. If $\pi_1(D)$ denotes the fundamental group of the support of D, then the inclusion-induced homomorphism $\pi_1(D) \to \pi_1(X)$ is surjective.*

Proof. Let D satisfy the hypotheses of the theorem. Since D is effective, we can use Nakai's criterion (Corollary IV.6.4 in [12]) to conclude that $D \cdot H > 0$ for any ample divisor H. Thus D satisfies the hypotheses of Lemma 6.1.5 and so $h^0(\mathcal{O}_X(mD))$ grows quadratically in m.

By Theorem 8.1 in [66], if $h^0(\mathcal{O}_X(mD))$ grows quadratically in m, then the Iitaka dimension of D,

$$\kappa(D, X) = \max\{\dim_{\mathbb{C}}(\psi_m(X)) \mid m \geq 1\},$$

must be 2. In particular, we must have $\dim_{\mathbb{C}}(\psi_m(X)) = 2$ for at least one positive integer m. It follows that D is not composed of a pencil.

Let U denote a tubular neighborhood of the support of D in X. Since a projective surface is complete, we can now use Lemma 6.1.2 to conclude that the inclusion-induced homomorphism $\pi_1(U) \to \pi_1(X)$ is surjective. Since U and the support of D are homotopy equivalent, $\pi_1(D) \to \pi_1(X)$ must be surjective as well. □

Example 6.1.7. For the remainder of this section, we return to the context of example 5.3.2 where X_p is a smooth cover branched over the dual Hesse arrangement. Specifically, we will apply Theorem 6.1.6 to a suitable divisor on X_p.

We consider the effective divisor

$$D = \widehat{L}_1 + E_{1,2,3} + E_{1,6,9} + E_{1,5,8} + E_{1,4,7} \tag{6.1}$$

on $Y = \mathbb{CP}^2 \# 12\overline{\mathbb{CP}}^2$. Lemma 4.1.3 then implies that

$$\begin{aligned}
I_{\widehat{L}_1} &= \langle \phi(\mu_1) \rangle = \langle (1,0) \rangle \cong \mathbb{Z}/p\mathbb{Z}, \\
H_{\widehat{L}_1} &= \langle \phi(\mu_1), \phi(\beta_*(\varepsilon_{1,2,3})), \phi(\beta_*(\varepsilon_{1,6,9})), \phi(\beta_*(\varepsilon_{1,5,8})), \phi(\beta_*(\varepsilon_{1,4,7})) \rangle \\
&= \langle \phi(\mu_1), \phi(\mu_1 + \mu_2 + \mu_3), \phi(\mu_1 + \mu_6 + \mu_9), \phi(\mu_1 + \mu_5 + \mu_8), \phi(\mu_1 + \mu_4 + \mu_7) \rangle \\
&= \langle (1,0), (-4,-4), (3,1), (3,1), (1,2) \rangle = (\mathbb{Z}/p\mathbb{Z})^2 = G.
\end{aligned}$$

Similarly, we have $I_{E_{1,2,3}} \cong I_{E_{1,6,9}} \cong I_{E_{1,5,8}} \cong I_{E_{1,4,7}} \cong \mathbb{Z}/p\mathbb{Z}$. By Lemma 5.3.4, we also have $H_{E_{1,2,3}} = H_{E_{1,6,9}} = H_{E_{1,5,8}} = H_{E_{1,4,7}} = (\mathbb{Z}/p\mathbb{Z})^2 = G$.

Let C denote one of the irreducible curves in the above effective decomposition of D (that is, let C be one of $\widehat{L}_1$, $E_{1,2,3}$, $E_{1,6,9}$, $E_{1,5,8}$, or $E_{1,4,7}$). Applying Lemma 4.1.5 we find that there is a unique irreducible curve C' in $\rho^{-1}(C)$. Furthermore, since all stabilizer subgroups are the same and are equal to G, we find $(C')^2 = C^2$ by Lemma 4.1.6.

Since the lift of C is a unique curve, we find the effective divisor

$$D' = \widehat{L}'_1 + E'_{1,2,3} + E'_{1,6,9} + E'_{1,5,8} + E'_{1,4,7} \tag{6.2}$$

on X_p is exactly the lift of the divisor expressed in equation (6.1). Moreover we also find that $(E'_{1,2,3})^2 = (E'_{1,6,9})^2 = (E'_{1,5,8})^2 = (E'_{1,4,7})^2 = -1$. Then by expressing the homology class of $\widehat{L}_1$ as

$$[\widehat{L}_1] = [L] - [E_{1,2,3}] - [E_{1,6,9}] - [E_{1,5,8}] - [E_{1,4,7}],$$

where $[L]$ is the homology class of a line in $\mathbb{CP}^2$, we also have $(\widehat{L}'_1)^2 = (\widehat{L}_1)^2 = 1 - 4 = -3$. We then compute the self intersection of D' to be

$$\begin{aligned}(D')^2 = \ & (\widehat{L}'_1)^2 + (E'_{1,2,3})^2 + (E'_{1,6,9})^2 + (E'_{1,5,8})^2 + (E'_{1,4,7})^2 \\ & + 2\,(\widehat{L}'_1) \cdot (E'_{1,2,3} + E'_{1,6,9} + E'_{1,5,8} + E'_{1,4,7}) \\ = \ & -3 - 4 + 2 \cdot 4 = 1 > 0.\end{aligned}$$

A schematic diagram showing the intersection pattern of D' is given in Figure 6.1.

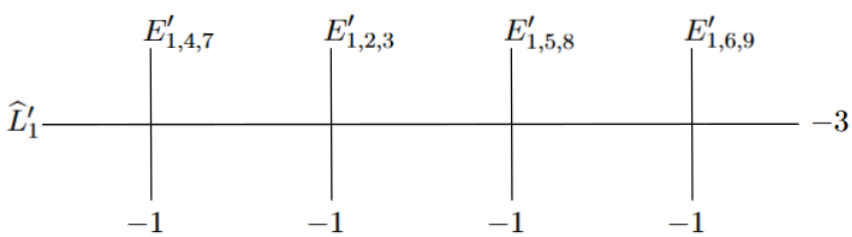

Figure 6.1: The divisor D'

Finally, we may apply Theorem 6.1.6 to the divisor D' on X_p, we obtain the following result regarding the fundamental group of X_p.

Corollary 6.1.8. (cf. Corollary 17 in [33]) *Let $p \geq 11$ be a prime number. If $\pi_1(D')$ denotes the fundamental group of the support of the divisor D' on X_p given by (6.2), then the inclusion-induced homomorphism $\pi_1(D') \to \pi_1(X_p)$ is surjective.*

Remark 6.1.9. From the inclusion-exclusion principle, we can deduce that the Euler characteristic of $\widehat{L}'_1$ is $(2-4)p+4 = 4-2p$, and so the genus of $\widehat{L}'_1$ is $p-1$. We also noticed in the proof of Lemma 5.3.5 that the Euler characteristic of the lift of an exceptional sphere is $(2-3)p+3 = 3-p$, and so in general we find that the genera of the lifts $E'_{1,2,3}$, $E'_{1,6,9}$, $E'_{1,5,8}$, and $E'_{1,4,7}$ are all equal to $(p-1)/2$.

Viewing X_p as a symplectic 4-manifold, we can sympletically resolve the four intersection points on D' to obtain a connected smooth symplectic submanifold S inside X_p with self-intersection $S^2 = (D')^2 = 1$ and genus $g(S) = 3(p-1)$.

Specifically, referring back to Figure 6.1 and Remark 6.1.9, D' is composed of 4 embedded genus $(p-1)/2$ curves intersecting a common genus $p-1$ curve transversely. Considering a small open 4-ball $B(p)$ around an intersection point p, $D' \cap B(p)$ can be modeled as the coordinate axes. The symplectic resolution of D' around B is a surface S obtained by perturbing these axes. This is equivalent to replacing $D' \cap B(p)$ with an annulus (cf. [25]). It follows that $[S] = [D']$ in $H_2(X_p)$ and so $S^2 = (D')^2$. Topologically, we also see that

$$S \simeq \widehat{L}'_1 \# E'_{1,2,3} \# E'_{1,6,9} \# E'_{1,5,8} \# E'_{1,4,7}$$

and so by inclusion-exclusion principle we find that

$$\begin{aligned} e(S) &= 4(3-p) + (4-2p) - 4(2) \\ &= 8 - 6p. \end{aligned}$$

It then follows that $g(S) = 1 - (4 - 3p) = 3p - 3$. Lastly, we show that this resolved surface also carries the fundamental group of X.

Lemma 6.1.10. (cf. Lemma 18 in [33]) *For every prime number $p \geq 11$, the inclusion-induced homomorphism $\pi_1(S) \to \pi_1(X_p)$ is surjective.*

Proof. We note that S and the support of D' only differ inside small open 4-balls centered at the intersection points of D' wherein the symplectic resolutions occur. Let U' denote a tubular neighborhood of the support of D'. Let $\imath : S \to U'$ denote the inclusion map, and let $\jmath : U' \to D'$ denote a homotopy equivalence. Near each intersection point of D', the composition $\jmath \circ \imath$ collapses an annulus on S onto a pair of transversely intersecting disks on D' (cf. p. 38 of [25]). The effect on the fundamental groups is that we obtain a group presentation for $\pi_1(D')$ by introducing two new relations to the group presentation for $\pi_1(S)$ at each of the four intersection points of D'. Hence we obtain a surjective homomorphism $(\jmath \circ \imath)_* : \pi_1(S) \to \pi_1(D')$. If $k : D' \to X_p$ denotes the inclusion map of the support of D' into X_p, then the composition $k \circ \jmath \circ \imath : S \to X_p$ is homotopy equivalent to the inclusion map of S into X_p. By Corollary 6.1.8, $k_* : \pi_1(D') \to \pi_1(X_p)$ is surjective, and our result follows. $\square$

6.2 Symplectic normal connected sums

In this section, we will construct an infinite family of simply connected nonspin irreducible symplectic 4-manifolds by taking the family of smooth Kähler surfaces X_p as the starting point. To accomplish this we will perform symplectic normal connected sums (cf. [24], [47]).

Definition 6.2.1. Let X_1 and X_2 be symplectic 4-manifolds containing embedded surfaces Σ_1 and Σ_2 of genus g and with normal bundle ν_1 and ν_2, respectively. Suppose further that $e(\nu_1) = -e(\nu_2)$ and that there exists an orientation-reversing diffeomorphism between their respective tubular neighbourhoods $\varphi : \nu(\Sigma_1) \to \nu(\Sigma_2)$. Then the *(symplectic) normal connected sum* along Σ_1 and Σ_2 is the (symplectic) smooth 4-manifold

$$X_1 \#_\varphi X_2 = (X_1 - \nu(\Sigma_1)) \cup_\varphi (X_2 - \nu(\Sigma_2)). \tag{6.3}$$

Remark 6.2.2. In general, it is known (cf. [24]) that after performing this operations, the topological invariants pertinent to our case are given by

$$\begin{aligned} e(X_1 \#_\varphi X_2) &= e(X_1) + e(X_2) - 2e(\Sigma), \\ \sigma(X_1 \#_\varphi X_2) &= \sigma(X_1) + \sigma(X_2). \end{aligned}$$

where $e(\Sigma) = e(\Sigma_i)$.

Since for any smooth orientation-reversing diffeomorphism $\varphi : \nu(\Sigma_1) \to \nu(\Sigma_2)$ the resulting 4-manifold will inherit a symplectic structure (cf. [24], [47]), we wish to find another symplectic 4-manifold (in addition to X_p) with which we can perform this operation. Note however, that as our goal is to construct a *simply connected* 4-manifold by modifying X_p, it is insufficient to simply 'double' X_p and consider $X_p \#_\varphi \overline{X_p}$. In fact, because the two submanifolds along which the summing is done are required to be compatible, we will see that it will also be convenient to modify the manifolds X_p.

Before that however, we will state one final result regarding the symplectic normal connected sum. Specifically, we borrow a fact from Usher on how the minimality of the manifolds X_i relate to the minimality of $X_1 \#_\varphi X_2$ and state it as a lemma.

Lemma 6.2.3. (cf. Theorem 1.1 in [68]) *Let $Z = X_1 \#_\varphi X_2$ denote the symplectic normal sum along surfaces Σ_1 and Σ_2 that have positive genus $g(\Sigma_i)$. Then:*

(1) *If either $X_1 - \Sigma_1$ or $X_2 - \Sigma_2$ contains an embedded symplectic sphere of self-intersection -1, then Z is not minimal.*

(2) *If one of the summands X_i (for definiteness, say X_1) admits the structure of an S^2-bundle over a surface of genus g such that Σ_1 is a section of this fiber bundle, then Z is minimal if and only if X_2 is minimal.*

(3) *In all other cases, Z is minimal.*

Remark 6.2.4. For our purposes, we can simplify the statement of the above lemma to the following. If at least one of X_1 or X_2 is minimal and neither of $X_1 - \Sigma_1$ nor $X_2 - \Sigma_2$ contain a sphere of self intersection -1, then Z is minimal.

For one of the components for the symplectic normal sum, we shall start by considering the elliptic surfaces. For $n \in \mathbb{Z}$ with $n > 0$, we let $E(n)$ denote the simply connected complex elliptic surface. Topologically we have $e(E(n)) = 12n$ and $\sigma(E(n)) = -8n$ and we may also assume that the elliptic fibration $f : E(n) \to \mathbb{CP}^1$ contains a singular cusp fiber.

For $n = 2$, then $E(2)$ is diffeomorphic to a K3 surface which is both spin and minimal. We let $E(2)_K$ denotes the result of performing a knot surgery (Definition 3.2.1) along a near-cusp fiber T of $E(2)$. Specifically, if K is a fibered knot of genus $g(K)$ then as observed in Example 3.2.8, $E(2)_K$ is a simply connected symplectic 4-manifold that is homeomorphic to $E(2)$. Furthermore, if s is a section of the fibration f, then it corresponds to a punctured sphere in $E(2)_K$ that intersects a Seifert surface Σ_K for the knot K exactly along a longitude of K (a circle c_m parallel to K lying on the boundary of $S^3 - \nu(K)$). Gluing these surfaces together we obtain a genus $g(K)$ symplectic submanifold Σ'_K with self-intersection -2 inside $E(2)_K$. Moreover, any smooth torus fiber of the restriction $f|_{E(2)_K - \nu(T)}$ intersects Σ'_K at exactly one transverse intersection point. We symplectically resolve this intersection as done above to obtain a surface F^1_K. The surface F^1_K is therefore a symplectic submanifold of $E(2)_K$ with genus $g(K) + 1$ and self-intersection 0. The pair $E(2)_K$ and F^1_K will be the second component of our symplectic normal sum. A schematic showing how the components of F^1_K are embedded into $E(2)_K$ can be found in Figure 6.2.

For the other component of the normal connected sum, we proceed as follows: Fix some $p \geq 11$, we again let S denote the symplectic submanifold of genus $3(p-1)$ that carries the fundamental group of X_p. Since we require an embedded submanifold that is compatible

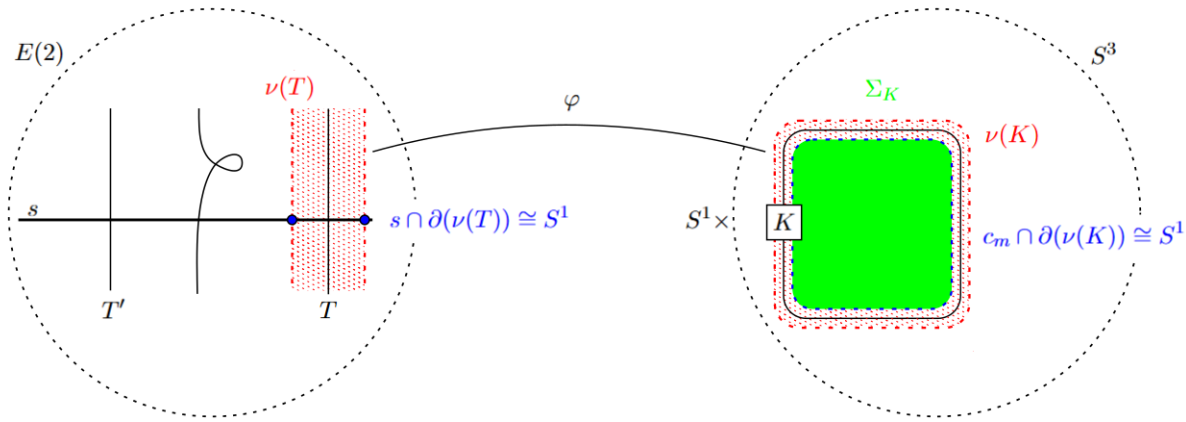

Figure 6.2: Before resolving, F_K^1 is $T' \cup \left[(s - \nu(T)) \cup_\varphi (\Sigma_K - \nu(K))\right]$

to the surface F_K^1 (for instance, a submanifold with self-intersection 0), we perform a blow up in X_p at a point on S. Then in $X_p \# \overline{\mathbb{CP}}^2$, the proper transform $\widehat{S}$ of S is a symplectic submanifold with $(\widehat{S})^2 = 0$ and genus $3(p-1)$. We will let this $X_p \# \overline{\mathbb{CP}}^2$ and its embedded surface $\widehat{S}$ be the other ingredients for our construction.

We will perform the symplectic normal connected sum of the symplectic 4-manifolds $X_p \# \overline{\mathbb{CP}}^2$ and $E(2)_K$ along the symplectic submanifolds $\widehat{S}$ and F_K^1 to obtain a symplectic 4-manifold which we will denote as Z_p. We note that, since we must have

$$g(K) = g(F_K^1) - 1 = g(\widehat{S}) - 1 = 3(p-1) - 1,$$

we could have taken K to be the knot sum of $3(p-1) - 1$ trefoil knots. Now, as per Remark 6.2.2 we have

$$\begin{aligned} e(Z_p) &= e(X_p \# \overline{\mathbb{CP}}^2) + e(E(2)_K) - 2e(\widehat{S}) = e(X_p) + 1 + 24 + 4g(\widehat{S}) - 4 \\ &= 9p^2 - 18p + 45, \\ \sigma(Z_p) &= \sigma(X_p \# \overline{\mathbb{CP}}^2) + \sigma(E(2)_K) = \sigma(X_p) - 1 - 16 = 2p^2 - 30. \end{aligned}$$

In preparation to apply Z_p to the symplectic geography problem, we now state and verify some of the pertinent properties of Z_p.

Theorem 6.2.5. (cf. Theorem 19 in [33]) *Let p be a prime number satisfying $p \geq 11$. Let Z_p be the closed symplectic 4-manifold that we constructed above. Then Z_p is simply*

connected, nonspin and irreducible. Moreover, Z_p contains two disjoint symplectic tori T_1 and T_2, both with self-intersection 0, such that $\pi_1(Z_p - (T_1 \cup T_2)) = 1$.

Proof. First we prove that Z_p is simply connected. We have

$$Z_p = ((X_p \# \overline{\mathbb{CP}}^2) - \nu(\widehat{S})) \cup_\varphi (E(2)_K - \nu(F_K^1)), \tag{6.4}$$

where $\psi : \partial(\nu(\widehat{S})) \to \partial(\nu(F_K^1))$ is an orientation-reversing diffeomorphism on the boundaries. Let $\imath_1 : \partial(\nu(\widehat{S})) \to (X_p \# \overline{\mathbb{CP}}^2) - \nu(\widehat{S})$ and $\imath_2 : \partial(\nu(F_K^1)) \to (E(2)_K - \nu(F_K^1))$ denote the inclusion maps.

By Seifert-Van Kampen theorem, we have

$$\pi_1(Z_p) \cong \frac{\pi_1((X_p \# \overline{\mathbb{CP}}^2) - \nu(\widehat{S})) * \pi_1(E(2)_K - \nu(F_K^1))}{N},$$

where N is the normal subgroup of the free product generated by elements of the form $(\imath_1)_*(\xi)^{-1}(\imath_2 \circ \varphi)_*(\xi)$, where $\xi \in \pi_1(\partial(\nu(\widehat{S})))$, and $(\imath_1)_*$ and $(\imath_2 \circ \varphi)_*$ are the induced homomorphisms on the fundamental groups.

A singular cusp fiber of $E(2)$ corresponds to two vanishing cycles in $H_1(T)$ where T is the near-cusp fiber along which the knot surgery was performed. Since in homology, one of these vanishing cycles was glued to a meridian of F_K^1, it follows that such meridian is homotopically trivial. Furthermore, since all meridians of F_K^1 are conjugate, we may conclude that

$$\pi_1(E(2)_K - \nu(F_K^1)) \cong \pi_1(E(2)) = 1.$$

It then follows that

$$\pi_1(Z_p) \cong \frac{\pi_1((X_p \# \overline{\mathbb{CP}}^2) - \nu(\widehat{S}))}{(\imath_1)_*(\pi_1(\partial(\nu(\widehat{S}))))}.$$

Furthermore, the class of a meridian of $\widehat{S}$ is trivial in the group above as they necessarily correspond to elements of $\pi_1(\partial(\nu(\widehat{S})))$. Therefore we can 'fill' back in the tubular neighbourhood to get

$$\pi_1(Z_p) \cong \frac{\pi_1(X_p \# \overline{\mathbb{CP}}^2)}{\jmath_*(\pi_1(\widehat{S}))}.$$

where $\jmath_*$ is the inclusion homomorphism induced by $\jmath : \widehat{S} \to X_p \# \overline{\mathbb{CP}}^2$. Finally, since the blow-up operation does not modify $\pi_1(X)$, we have

$$\pi_1(Z_p) \cong \frac{\pi_1(X_p)}{k_*(\pi_1(S))},$$

where k_* is the inclusion homomorphism induced by $k : S \to X_p$. It now follows that since S carries the fundamental group of X_p (Lemma 6.1.10) we must have $\pi_1(Z_p) = 1$.

Now, since Z_p is simply connected, it has no 2-torsion. In particular, by Proposition 3.1.2, to show that Z_p is nonspin, it is sufficient to show that the intersection form of Z_p is odd. That is, we shall find an embedded surface in Z_p with odd self-intersection.

The exceptional sphere of the blow-up $X_p\#\overline{\mathbb{CP}}^2$ corresponds to a once-punctured sphere in the complement $(X_p\#\overline{\mathbb{CP}}^2) - \nu(\widehat{S})$. This punctured sphere can be glued to a once-punctured torus fiber in the complement $E(2)_K - \nu(F_K^1)$ which results in a smooth embedded torus in Z_p with self intersection $-1 + 0 = -1$.

We now consider the irreducibility of Z_p. First we know that X_p is minimal from Lemma 5.3.5. Therefore, the only candidate for an embedded sphere with self-intersection -1 is the exceptional sphere of the blow-up. However, as noted previously, this sphere is punctured in the complement $(X_p\#\overline{\mathbb{CP}}^2) - \nu(\widehat{S})$ and so $(X_p\#\overline{\mathbb{CP}}^2) - \nu(\widehat{S})$ cannot contain a -1 sphere.

On the other hand, it is not hard to see that $E(2)_K$ is a minimal symplectic 4-manifold by looking at its Seiberg-Witten invariants. Indeed, by Lemma 3.2.3, we compute the Seiberg-Witten invariants of $E(2)_K$ to be

$$\overline{SW}_{E(2)_K} = \Delta_K(PD(2[T])).$$

where $\Delta_K(t)$ is the (monic) symmetrized Alexander polynomial of the fibered knot K. If the degree of $\Delta_K(t)$ is m, then it is well known that the Seiberg-Witten basic class associated to $c_1(E(2)_K)$ must be $\pm PD(2m[T])$. Moreover, as the other Seiberg-Witten basic classes of $E(2)_K$ must be of the form $\pm PD(2r[T])$ for some $0 \leq r \leq m$, we compute.

$$\begin{aligned}
(c_1(E(2)_K) - PD(2r[T]))^2 &= (PD(2m[T]) - PD(2r[T]))^2 \\
&= (2m[T] - 2r[T])^2 \\
&= (2(m-r))^2[T]^2 \\
&= 0 \neq -4.
\end{aligned}$$

Where the last line follows from the fact that the torus T is a near-cusp fiber of $E(2)_K$ and necessarily has zero self intersection. Consequently, Lemma 3.2.11 implies that $E(2)_K$ is minimal (recalling that $E(2)_K$ is homeomorphic to $E(2)$ and so $b_2^+(E(2)_K) = 3 > 1$). We can now use Lemma 6.2.3 to conclude that the symplectic normal connected sum

Z_p is minimal. Finally, recalling Lemma 3.2.10 (simply connected minimal symplectic 4-manifolds are irreducible), we find that Z_p is irreducible.

Now, Lemma 2.3 in [6] tells us that the $E(2)_K - \nu(F_K^1)$ half of Z_p contains two disjoint symplectic tori T_1 and T_2 of self-intersection 0 such that

$$\pi_1\left((E(2)_K - \nu(F_K^1)) - (T_1 \cup T_2)\right) = 1.$$

An application of the Seifert-Van Kampen theorem in a similar manner as above but for the decomposition

$$Z_p - (T_1 \cup T_2) = ((X_p \# \overline{\mathbb{CP}}^2) - \nu(\widehat{S})) \cup_\varphi \left((E(2)_K - \nu(F_K^1)) - (T_1 \cup T_2)\right)$$

instead of (6.4) allows us to conclude that

$$\pi_1(Z_p - (T_1 \cup T_2)) = 1.$$

This concludes our proof. □

6.3 Applications to the geography problem

We are now prepared to consider how the smooth simply connected symplectic manifolds Z_p constructed in the previous section can be applied to the symplectic geography problem. We will also later apply the same treatment to an analogous construction that can be performed on the 4-manifolds constructed from the Wiman arrangement (Section 5.3.1) in preparation and as motivation for Section 6.4.

6.3.1 Asymptotic upper bound for $\lambda(\sigma)$

In this section, we provide the smallest known asymptotic upper bound for $\lambda(\sigma)$. The following theorem had been proved in [3] and serves as the basis for our application.

Theorem 6.3.1. (cf. Corollary 17 in [3]) *Let X be a closed simply connected nonspin minimal symplectic 4-manifold with $b_2^+(X) > 1$ and $\sigma(X) \geq 0$. Assume that X contains two disjoint symplectic tori T_1 and T_2 with self-intersection 0 such that $\pi_1(X - (T_1 \cup T_2)) =$*

1. *Suppose σ is a fixed integer satisfying $0 \leq \sigma \leq \sigma(X)$. If $\lceil x \rceil = \min\{k \in \mathbb{Z} \mid k \geq x\}$ and if we define*

$$\ell(\sigma) = \left\lceil \frac{\sigma(X) - \sigma}{8} - 1 \right\rceil,$$

then we have

$$\lambda(\sigma) \leq \chi_h(X) + \ell(\sigma) + 1. \tag{6.5}$$

□

Now we apply Theorem 6.3.1 to the 4-manifolds Z_p in the previous section that are indexed by prime numbers $p \geq 11$. However, because the signature and the holomorphic Euler characteristic of Z_p are functions of prime p and not simply the integers, we need to estimate the lengths of gaps between consecutive prime numbers in order to discuss the asymptotic relations between the above invariants. We will make use of the following theorem from [11].

Theorem 6.3.2. (cf. Theorem 1 in [11]) *Let p_n denote the nth prime number so that $p_1 = 2$, $p_2 = 3$, etc. If $\theta = 0.525$, then there is a positive integer N such that $p_{n+1} - p_n \leq p_{n+1}^{\theta}$ for all $n \geq N$.*

Combining the last two theorems, we obtain the following.

Corollary 6.3.3. (cf. Corollary 22 in [33]) *Let $\lambda(\sigma)$ be as in Definition 3.1.8. If $\theta = 0.525$, then*

$$\lambda(\sigma) \leq \frac{11}{8}\sigma + O\big(\sigma^{\frac{1+\theta}{2}}\big) \tag{6.6}$$

as $\sigma \to \infty$.

Proof. For each prime $p \geq 11$, let Z_p be the closed simply connected nonspin irreducible symplectic 4-manifold in the previous section. We recall that

$$\begin{aligned} e(Z_p) &= 9p^2 - 18p + 45, \\ \sigma(Z_p) &= 2p^2 - 30, \\ \chi_h(Z_p) &= \frac{1}{4}(e(Z_p) + \sigma(Z_p)) = \frac{1}{4}(11p^2 - 18p + 15). \end{aligned}$$

It is easy to check that Z_p satisfies the hypothesis of Theorem 6.3.1. Specifically, we refer the reader back to Theorem 6.2.5. Given any integer $\sigma \geq 0$, let p_σ be the smallest prime number such that $p_\sigma \geq 11$ and $\sigma \leq \sigma(Z_{p_\sigma})$.

Let p_n denote the nth prime number as in Theorem 6.3.2. If $p_\sigma = p_{n+1}$, then we have $\sigma > \sigma(Z_{p_n})$, provided that Z_{p_n} is defined. Let $d_n = p_{n+1} - p_n$. By Theorem 6.3.2, there is a positive integer N such that $d_n \leq p_{n+1}^\theta = p_\sigma^\theta$ for all $n \geq N$. Thus for all $n \geq N$, we can deduce that $p_n = p_{n+1} - d_n \geq p_\sigma - p_\sigma^\theta$. It follows that for all σ large enough,

$$p_\sigma^2 < 2(p_\sigma - p_\sigma^\theta)^2 - 30 \leq \sigma(Z_{p_n}) < \sigma \leq \sigma(Z_{p_\sigma}) = 2p_\sigma^2 - 30.$$

Hence we conclude that $p_\sigma = O(\sigma^{1/2})$ as $\sigma \to \infty$, and

$$\sigma = 2p_\sigma^2 + O\left(p_\sigma^{1+\theta}\right) = 2p_\sigma^2 + O\big(\sigma^{\frac{1+\theta}{2}}\big)$$

as $p_\sigma, \sigma \to \infty$. It follows that

$$\begin{aligned}
\chi_h(Z_{p_\sigma}) &= \frac{11}{4}p_\sigma^2 + O\left(p_\sigma\right) = \frac{11}{8}\sigma + O\big(\sigma^{\frac{1+\theta}{2}}\big) + O(\sigma^{1/2}) \\
&= \frac{11}{8}\sigma + O\big(\sigma^{\frac{1+\theta}{2}}\big), \\
\ell(\sigma) &= \frac{1}{8}(\sigma(Z_{p_\sigma}) - \sigma) + O(1) \\
&\leq \frac{1}{8}\left(\sigma(Z_{p_\sigma}) - (2(p_\sigma - p_\sigma^\theta)^2 - 30)\right) + O(1) \\
&= O\left(p_\sigma^{1+\theta}\right) = O\big(\sigma^{\frac{1+\theta}{2}}\big).
\end{aligned}$$

Substituting these expressions into (6.5), we obtain (6.6). □

We note in particular that as a consequence of Corollary 6.3.3 we have

$$\limsup_{\sigma\to\infty} \frac{\lambda(\sigma)}{\sigma} \leq \frac{11}{8} = 1.375, \tag{6.7}$$

which at the time of writing this thesis is the best known asymptotic bound for this ratio. We again highlight that very little is known for the case of small symplectic 4-manifolds with non-negative signature.

This fact is perhaps most evident for covers branched over line arrangements when considering the signature formula we computed in Lemma 4.3.5. Namely, if we do not consider the asymptotic behavior, to determine when such a branched cover results in positive signature will in general require an understanding of the zeroes of the polynomial

$\sigma(X)$ in addition to possibly the zeroes of $\frac{\partial}{\partial p}\sigma(X)$. Consequently we see that determining which line arrangements can result in positive signature for 'small' values of p can be comparatively difficult.

To further accentuate this observation and to serve as greater motivation for the construction discussed in Section 6.4, we review the precursor to the bound found in Corollary 6.3.3 which we found in [34].

6.3.2 An alternate bound from the Wiman configuration

We begin by noting that the construction method of the manifold Z_p above is not limited to the covers branched over the dual Hesse arrangement. In particular:

Example 6.3.4. We return to the covers branched over the 'Wiman+1' arrangement. That is, for each integer $p \geq 2$, let $X = X_p$ and $S = L'$ be as in Subsection 5.3.1. Then as noted in Example 6.1.4, L' carries the fundamental group of X_p. Furthermore, it is a symplectic submanifold of X_p with genus

$$g(L') = \frac{1}{2}(43p^{44} - 45p^{43}) + 1.$$

That is, if K is a fibered knot of genus $g(K) = \frac{1}{2}(43p^{44} - 45p^{43}) + 1$, then we can construct a minimal symplectic 4-manifold $E(p^{43})_K$ that contains a symplectic submanifold F'_K of genus

$$g(F'_K) = g(K) = g(L')$$

and with self-intersection

$$(F'_K)^2 = -p^{43} = -(L')^2.$$

We note that unlike the construction for the dual Hesse, since $(L')^2 > 1$, we do not need to perform a blow up to find a 4-manifold with $b_2^+ > 1$ that contains a self-intersection $-(L')^2$ surface. It follows that we can perform an analogous symplectic normal connected sum between X_p and $E(p^{43})_K$ which yields

$$W_p = (X_p - \nu(L')\,) \cup_\varphi (E(p^{43})_K - \nu(F'_K)), \tag{6.8}$$

a simply connected minimal (and hence irreducible) symplectic 4-manifold that satisfies the same conclusions as Theorem 6.2.5. For instance, W_p contains a submanifold E' in the $X_p - \nu(L')$ half with self-intersection $-p$. Therefore, if p is an odd integer, then we can conclude that W_p is nonspin since the intersection form of Z_p is not even (Lemma 3.1.2).

Now we apply Theorem 6.3.1 to the 4-manifolds W_p in Example 6.3.4 corresponding to odd integers $p \geq 3$ and obtain the following.

Corollary 6.3.5. (cf. Corollary 21 in [34]) *Let $\lambda(\sigma)$ be as in Definition 3.1.8. Then we have*

$$\lambda(\sigma) \leq \frac{17}{11}\sigma + O\big(\sigma^{\frac{44}{45}}\big) \tag{6.9}$$

as $\sigma \to \infty$.

Proof. For each odd integer p, let W_p be the closed simply connected nonspin irreducible symplectic 4-manifold in Example 6.3.4. As done previously, we can similarly compute

$$\begin{aligned} e(W_p) &= p^{43}\left(475p^2 - 950p + 687\right), \\ \sigma(W_p) &= \frac{p^{43}}{3}\left(275p^2 - 899\right), \\ \chi_h(W_p) &= p^{43}\left(\frac{425}{3}p^2 - \frac{475}{2}p + \frac{581}{6}\right). \end{aligned}$$

As suggested in Example 6.3.4 and fully argued in [34], each W_p satisfies the hypothesis of Theorem 6.3.1. Given any integer $\sigma \geq 0$, let p be the smallest odd integer such that $\sigma \leq \sigma(W_p)$.

It follows that $\sigma > \sigma(W_{p-2})$, and for all σ large enough, we have

$$p^{45} < \frac{(p-2)^{43}}{3}\left(275(p-2)^2 - 899\right) < \sigma \leq \frac{p^{43}}{3}\left(275p^2 - 899\right).$$

Hence we conclude that $p = O(\sigma^{1/45})$ as $\sigma \to \infty$, and

$$\sigma = \frac{275}{3}p^{45} + O\left(p^{44}\right) = \frac{275}{3}p^{45} + O\big(\sigma^{\frac{44}{45}}\big)$$

as $p, \sigma \to \infty$. It follows that

$$\begin{aligned} \chi_h(W_p) &= \frac{425}{3}p^{45} + O\left(p^{44}\right) = \frac{17}{11}\sigma + O\big(\sigma^{\frac{44}{45}}\big), \\ \ell(\sigma) &= \frac{1}{8}(\sigma(W_p) - \sigma) + O(1) \\ &\leq \frac{1}{8}\left(\sigma(W_p) - \sigma(Z_{p-2})\right) + O(1) \\ &= O\left(p^{44}\right) = O\big(\sigma^{\frac{44}{45}}\big). \end{aligned}$$

Substituting these expressions into (6.5), we obtain (6.9). □

We included this bound (even in the context of Corollary 6.3.3) for two primary reasons. First, we note that the smaller order terms of $\lambda(\sigma)$ carried the exponent $\frac{44}{45}$ as a consequence of the order of the Galois group of the covering being p^{45}. In comparison, since the cover branched over the dual Hesse arrangement had Galois group of order p^2, in Corollary 6.3.3 the smaller order terms of $\lambda(\sigma)$ carried a comparatively smaller exponent. Specifically we find that while both computations provide an asymptotic bound that converges to a linear slope (as required by Definition 3.1.8), the rate at which these 'families of manifolds converge' to the bound is dependent on the number of summands in the Galois group. Consequently, we observe that it may be useful to determine conditions under which the Galois group of the cover branched over a line arrangement can contain the fewest $\mathbb{Z}/p\mathbb{Z}$ summands.

Secondly, the existence of submanifolds (specifically embedded surfaces) that carry the fundamental group is also critical when it comes constructing simply connected symplectic 4-manifolds that can be applied to the geography problem using the methods of branched covers. However, in the case of the Wiman arrangement, the standard defining map provided by equation (4.6) does not readily produce curves in the cover X_p with positive self-intersection (since each hyperplane in the line arrangement contains multiple intersection points of higher multiplicity). As a result, it was necessary to add the extra line to the Wiman arrangement to guarantee the existence of a positive self-intersection curve. We will see below (in the proof of Corollary 6.4.2) that for branched covers with Galois groups consisting of only two $\mathbb{Z}/p\mathbb{Z}$ summands, there will always exist symplectic submanifolds that carry the fundamental group.

In this context of finding 'small' Galois groups (in terms of summands, not necessarily order), we will return to the Wiman arrangement in Example 6.4.3.

6.4 Asymptotic existence of smooth covers

We consider another motivating factor for one to be interested in Galois coverings with 'few' summands in addition to the two in the previous section. As the true end goal for the geography problem is to 'minimize' $\lambda(\sigma)$, we find from equation (6.5) that it is also important to minimize $\chi_h(X) = \frac{1}{4}(e(X) + \sigma(X))$ and in particular, $e(X)$ for the manifolds that we construct. We then see from Section 6.2 that it is also important to minimize the genus of embedded symplectic surfaces in the branched covering space (or at least for

those surfaces which carry the fundamental group). Specifically, we wish to find embedded surfaces with small $g(\widehat{S})$ (or $g(S)$) along which to perform the symplectic normal connected sum.

To this end, we shall start by recalling Lemma 4.1.7 for two embedded (path-connected) surfaces C' and D' with genus $g(C')$ and $g(D')$, respectively. If they intersect transversely, we consider the symplectic resolution R of their intersection points as described previously. If $\frac{|H_C \cap H_D|}{|I_C||I_D|} = 1$, then R is topologically $C' \# D'$ and so $g(R) = g(C') + g(D')$. On the other hand, if $\frac{|H_C \cap H_D|}{|I_C||I_D|} \neq 1$ then in general we will have $g(R) > g(C') + g(D')$ as there will be at least one additional curve in $\pi_1(R)$ contributing to its genus that is not a curve in either $\pi_1(C')$ or $\pi_1(D')$ (for example, the blue curve in Figure 6.3). More specifically, if C' and D' intersect at k transverse intersection points. Then $e(C' \cup D') = (2 - 2g(C')) + (2 - 2g(D')) - k$. As the symplectic resolution replaces a subset of Euler characteristic 1 (transversely intersecting discs) with a subset of Euler characteristic 0 (an annulus) we compute

$$\begin{aligned} e(R) &= e(C' \cup D') - k(1) + k(0) \\ &= (2 - 2g(C')) + (2 - 2g(D')) - 2k. \end{aligned}$$

It follows that $g(R) = g(C') + g(D') + k - 1$.

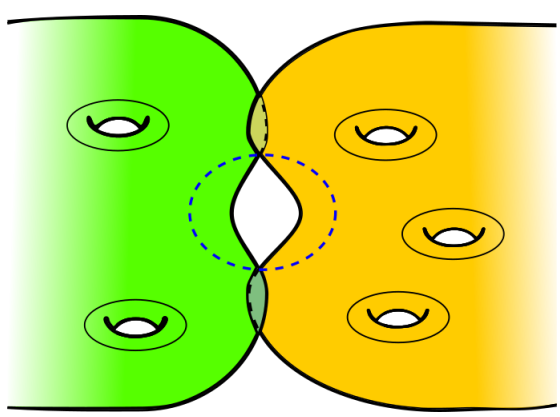

Figure 6.3: Genus grows when there is extra intersection

Hence to minimize the genus of R, we see that of particular interest are line arrangements and branched covers corresponding to the case $|H_C \cap H_D| = |I_C||I_D|$. Now, we saw

previously that for an irreducible curve, the inertia subgroup is in general cyclic and so if G is an elementary abelian group of order p^n, then $|I_C||I_D| = p^2$. On the other hand, we observed in Remark 4.3.2 that H_C and H_D contain a subgroup of order at least p^2 (in fact, it is a shared subgroup that contains both I_C and I_D). We now observe that if the covering group is $G = (\mathbb{Z}/p\mathbb{Z})^2$, then necessarily $H_C = H_D = (\mathbb{Z}/p\mathbb{Z})^2$ and so $|H_C \cap H_D| = |I_C||I_D|$ for any intersecting curves C and D. In fact, this exactly is the case we observed in Section 5.3.2 where there is a unique curve in the lifts of all curves in the branch locus. We therefore investigate under what conditions can we find a defining map

$$\phi : H_1(\mathbb{CP}^2 - \mathcal{L}) \cong \mathbb{Z}^{n-1} \to G = (\mathbb{Z}/p\mathbb{Z})^2 \tag{6.10}$$

that corresponds to a smooth cover.

In general, one can check that for a fixed line arrangement $\mathcal{L}$, there may not be a smooth defining map for any specific prime p. However, if we allow for p to be sufficiently large, then we can conclude that such a map does in fact exist.

Theorem 6.4.1. *Let $\mathcal{L}$ be any (non-pencil) arrangement of $n+1$ lines. Then for all primes*

$$p \geq 2n \sum_{i=0}^{n-1} 2^i = 2n(2^n - 1),$$

there exists a defining map

$$\phi : H_1(\mathbb{CP}^2 - \mathcal{L}) \cong \mathbb{Z}^n \to G = (\mathbb{Z}/p\mathbb{Z})^2$$

for which the resulting branched cover $X \to \widetilde{\mathbb{CP}^2}$ is smooth and the branch locus is $B = \beta^{-1}(\mathcal{L})$ (as always, $\widetilde{\mathbb{CP}^2}$ is the blow up of $\mathbb{CP}^2$ at all points of $\mathcal{L}$ with higher intersection multiplicity).

Proof. Fix a prime satisfying $p \geq 2n \sum_{i=0}^{n-1} 2^i$. We can exhibit $\mathcal{L} = L_0 \cup L_1 \cup \cdots \cup L_n$ as the branch locus of a $(\mathbb{Z}/p\mathbb{Z})^2$ cover of $\mathbb{CP}^2$ via the following defining map

$$\phi : H_1(\mathbb{CP}^2 - \mathcal{L}) \to (\mathbb{Z}/p\mathbb{Z})^2.$$

$$\phi(\mu_k) = \begin{cases} (1, 2^k) & \text{if } 0 \leq k \leq n-2, \\ (-(n-1), 0) & \text{if } k = n-1, \\ (0, -\sum_{i=0}^{n-2} 2^i) = (0, -(2^{n-1} - 1)) & \text{if } k = n. \end{cases} \tag{6.11}$$

Suppose $L_i \cap L_j$ is a double point. If at least one of i or j is $n-1$ or n, then it is clear that

$$\langle \phi(\mu_i) \rangle \cap \langle \phi(\mu_j) \rangle = \{(0,0)\}$$

since no element in $\langle \phi(\mu_i) \rangle$ or $\langle \phi(\mu_j) \rangle$ has two non-zero coordinates. On the other hand, if $i, j \leq n-2$, then since p is prime, a subgroup generated by $(1, *)$ consists of a unique element for which the first coordinate is equal to $n \in \mathbb{Z}/p\mathbb{Z}$. That is, if

$$\langle \phi(\mu_i) \rangle \cap \langle \phi(\mu_j) \rangle \neq \{(0,0)\}$$

then $\phi(\mu_i) = (1, 2^i) = (1, 2^j) = \phi(\mu_j)$. However, since $p > 2^i, 2^j$, $2^i \not\equiv 2^j \pmod{p}$ and so $\phi(\mu_i) \neq \phi(\mu_j)$, a contradiction. Hence, X is smooth in the fiber of any double point.

Now suppose that $L_{i_1} \cap L_{i_2} \cap \cdots \cap L_{i_r}$ with $r \geq 3$ is a higher multiplicity intersection point. Then after blowing up, we know the homology class of the exceptional sphere satisfies

$$\phi(\varepsilon_{i_1, i_2, \ldots, i_r}) = \phi(\mu_{i_1}) + \phi(\mu_{i_2}) + \cdots + \phi(\mu_{i_r}).$$

Letting $I = \{i_1, i_2, \ldots, i_r\}$ denote the index set of lines in the intersection, we consider the following cases: (1) neither $n-1$ nor n is an element of I, (2) exactly one of $n-1$ or n is an element of I, or finally, (3) I contains both $n-1$ and n.

(1) If neither $n-1$ nor n are in I, then it follows that

$$\phi(\varepsilon_{i_1, i_2, \ldots, i_r}) = (r, \sum_{i \in I} 2^i).$$

To check smoothness, we must verify that $(r, \sum_{i \in I} 2^i)$ is not an element of any subgroup of the form $\langle (1, 2^k) \rangle$. Suppose otherwise, then we must have

$$r(1, 2^k) = (r, \sum_{i \in I} 2^i)$$

and so

$$r2^k \equiv \sum_{i \in I} 2^i \pmod{p}.$$

However, since p is strictly larger than both $r2^k$ and $\sum_{i \in I} 2^i$ by assumption, we may reduce to the case of equivalence over the integers.

Now, if $(r)_2$ is the binary representation of r, then it is a well-known fact that the number of non-zero digits of $(r)_2$ is less than $\ln_2(r) < r$. It then follows that the binary representation of $r2^k$ also has fewer than $\ln_2(r) < r$ non-zero digits as multiplication by a power of 2 only performs a left shift on the digits of the representation. On the other hand, by definition, the number of non-zero digits in the binary representation of $\sum_{i\in I} 2^i$ is exactly $|I| = r$. Therefore, we have

$$r2^k \neq \sum_{i\in I} 2^i$$

and so

$$\langle \phi(\mu_k)\rangle \cap \langle \phi(\varepsilon_{i_1,i_2,\ldots,i_r})\rangle = \{(0,0)\}$$

for any $k < n-1$.

(2a) If $n-1 \in I$, then for convenience we re-label the intersection point and the index set I so that $I = \{i_1, i_2, \ldots, i_r, n-1\}$. We then compute

$$\phi(\varepsilon_{i_1,i_2,\ldots,i_r,n-1}) = (r-(n-1), \sum_{i\in I, i\neq n-1} 2^i).$$

Now, if $r-(n-1) = 0$, then $r = n-1$ and so $|I| = n$. That is, we are considering an intersection point of multiplicity n. However, as the only line arrangement with $n+1$ lines and an intersection point of multiplicity n is the near pencil, we may treat this case separately. Specifically, we saw in section 5.2.1 that for any odd prime (and so for any prime $p > 2n\sum_{i=0}^{n-1} 2^i$) there exists a smooth $(\mathbb{Z}/p\mathbb{Z})^2$ branched cover with the desired branch locus. Therefore we may assume that $r-(n-1) \neq 0$. Since $\phi(\varepsilon_{i_1,i_2,\ldots,i_r,n-1})$ consists of two non-zero coordinates,

$$\langle \phi(\mu_{n-1})\rangle \cap \langle \phi(\varepsilon_{i_1,i_2,\ldots,i_r,n-1})\rangle = \{(0,0)\}.$$

Alternatively, suppose for $k \neq n-1$ that we have

$$\langle \phi(\mu_k)\rangle \cap \langle \phi(\varepsilon_{i_1,i_2,\ldots,i_r,n-1})\rangle \neq \{(0,0)\}.$$

Then we must have that

$$-\phi(\varepsilon_{i_1,i_2,\ldots,i_r,n-1}) = (n-1-r, -\sum_{i\in I, i\neq n-1} 2^i) \in \langle (1, 2^k)\rangle,$$

and so

$$(n-1-r)2^k \equiv -\sum_{i\in I, i\neq n-1} 2^i \pmod p.$$

However, again by our bound for p, we have both $(n-1-r)2^k$ and $p-\sum_{i\in I,i\neq n-1} 2^i$ can be compared as representatives. We notice that

$$p-\sum_{i\in I,i\neq n-1} 2^i > p-\sum_{i=0}^{n-1} 2^i > \frac{p}{2},$$

whereas

$$(n-1-r)2^k < n\sum_{i=0}^{n-1} 2^i < \frac{p}{2},$$

and so they are not equal as representatives in $\mathbb{Z}/p\mathbb{Z}$, a contradiction. Therefore,

$$\langle\phi(\mu_k)\rangle \cap \langle\phi(\varepsilon_{i_1,i_2,\ldots,i_r,n-1})\rangle = \{(0,0)\}.$$

(2b) If $n \in I$, then we again re-label the intersection point and the index set I so that $I = \{i_1, i_2, \ldots, i_r, n\}$ and find that

$$\phi(\varepsilon_{i_1,i_2,\ldots,i_r,n}) = (r, -(2^{n-1}-1) + \sum_{i\in I,i\neq n} 2^i).$$

Similarly to the previous case, we consider what happens when $-(2^{n-1}-1)+\sum_{i\in I,i\neq n} 2^i$ is 0. This corresponds to the case when $I-\{n\} = \{0,1,\ldots,n-2\}$ and so $|I| = n$. That is, we are again in the case of a near pencil which has already been considered. Therefore, we may assume $\phi(\varepsilon_{i_1,i_2,\ldots,i_r,i_n})$ has two non-zero coordinates and so by an identical argument to above,

$$\langle\phi(\mu_n)\rangle \cap \langle\phi(\varepsilon_{i_1,i_2,\ldots,i_r,n})\rangle = \{(0,0)\}.$$

Now, suppose for $k \neq n$ that we have

$$\langle\phi(\mu_k)\rangle \cap \langle\phi(\varepsilon_{i_1,i_2,\ldots,i_r,n})\rangle \neq \{(0,0)\}$$

so that

$$\phi(\varepsilon_{i_1,i_2,\ldots,i_r,n}) = (r, -(2^{n-1}-1) + \sum_{i\in I,i\neq n} 2^i) \in \langle(1,2^k)\rangle.$$

Then we must have

$$r2^k \equiv -(2^{n-1}-1) + \sum_{i\in I, i\neq n} 2^i \pmod p.$$

Since $-(2^{n-1}-1) + \sum_{i\in I,i\neq n} 2^i \neq 0$ we must have $-(2^{n-1}-1) + \sum_{i\in I,i\neq n} 2^i < 0$ and so

$$p - (2^{n-1}-1) + \sum_{i\in I,i\neq n} 2^i > p - \sum_{i=0}^{n-2} 2^i > \frac{p}{2},$$

whereas

$$r2^k < n\sum_{i=0}^{n-1} 2^i < \frac{p}{2},$$

and so again we contradict that there is a shared element between the inertia subgroups.

(3) We again re-index so that $I = \{i_1, i_2, \ldots, i_r, n-1, n\}$. Then

$$\phi(\varepsilon_{i_1,i_2,\ldots,i_r,n-1,n}) = (r-(n-1), -(2^{n-1}-1) + \sum_{i\in I, i\neq n-1,n} 2^i).$$

If the first coordinate of $\phi(\varepsilon_{i_1,i_2,\ldots,i_r,n-1,n})$ is zero, then we have $r = n-1$ and so $|I| = n+1$. That is, there are $n+1$ many lines intersecting at a common point, and thus $\mathcal{L}$ is a pencil, a contradiction to the hypothesis. If the second coordinate of $\phi(\varepsilon_{i_1,i_2,\ldots,i_r,n-1,n})$ is zero, then similarly to the above, we can conclude that $|I| = n$ and thus $\mathcal{L}$ is a near pencil. Therefore we may assume that both coordinates of $\phi(\varepsilon_{i_1,i_2,\ldots,i_r,n-1,n})$ are non-zero.

In this final case we can immediately observe that we have both

$$\langle\phi(\mu_{n-1})\rangle \cap \langle\phi(\varepsilon_{i_1,i_2,\ldots,i_r,n-1,n})\rangle = \{(0,0)\},$$

$$\langle\phi(\mu_{n})\rangle \cap \langle\phi(\varepsilon_{i_1,i_2,\ldots,i_r,n-1,n})\rangle = \{(0,0)\}.$$

On the other hand, suppose for $0 \leq k \leq n-2$, that

$$-\phi(\varepsilon_{i_1,i_2,\ldots,i_r,n-1,n}) = (n-1-r,\, 2^{n-1}-1-\sum_{i\in I, i\neq n-1,n} 2^i) \in \langle(1,2^k)\rangle.$$

and so

$$(n-1-r)2^k \equiv 2^{n-1}-1-\sum_{i\in I, i\neq n-1,n} 2^i \pmod p$$

is satisfied. Then the same argument as above shows that there are fewer than $n-1-r$ non-zero digits in the binary representation of $(n-1-r)2^k$. Since

$$2^{n-1} - 1 - \sum_{i\in I, i\neq n-1,n} 2^i = \sum_{i=0}^{n-2} 2^i - \sum_{i\in I, i\neq n-1,n} 2^i = \sum_{\substack{0\leq i\leq n-2,\\ i\notin I}} 2^i,$$

and there are exactly r elements in I which are all less than $n-1$, the binary representation of $2^{n-1}-1-\sum_{i\in I, i\neq n-1,n} 2^i$ must have exactly $n-1-r$ non-zero digits. It follows that the above equivalence cannot be satisfied. Hence, the cover defined by ϕ will be smooth for intersection points of this type as well. Upon combining these cases, our result follows. □

A consequence of Theorem 6.4.1 is that the behaviour and the existence of the embedded surface S in Section 5.3.2 can be generalized to any line arrangement in the asymptotic limit.

Corollary 6.4.2. *Let L be any line in any (non-pencil) fixed line arrangement $\mathcal{L}$. For sufficiently large primes p, denote by X_p the smooth $(\mathbb{Z}/p\mathbb{Z})^2$ branched cover described in Theorem 6.4.1. Furthermore let $\{q_1, q_2, \ldots, q_k\}$ denote the maximal subset of intersection points of $\mathcal{L}$ which have multiplicities m_{q_i} at least 3 and which also lie on L. Then X_p contains a smooth symplectic submanifold of genus*

$$g = \frac{p-1}{2}\left(-k-2+\sum_{i=1}^{k} m_{q_i}\right) \tag{6.12}$$

which carries the fundamental group of X_p

Proof. Let the line L and points $q_1, q_2, \ldots, q_k$ be as in the statement of the corollary. After blowing up $\mathbb{CP}^2$ at all higher order intersection points, we consider the effective divisor

$$A = \widehat{L} + \sum_{i=1}^{k} E_i \tag{6.13}$$

where E_i is the exceptional sphere $\beta^{-1}(q_i)$ in $\widetilde{\mathbb{CP}^2}$. By considering the map ϕ defined in the proof of Theorem 6.4.1 and since the Galois group of the cover is $(\mathbb{Z}/p\mathbb{Z})^2$, it is straightforward to compute that all inertia subgroups are isomorphic to $\mathbb{Z}/p\mathbb{Z}$ and all

stabilizer subgroups are isomorphic to $G = (\mathbb{Z}/p\mathbb{Z})^2$ for all genus 0 components (and hence all components) of A.

Then by Lemma 4.1.5, the lift of A' decomposes as the effective divisor

$$A' = \widehat{L}' + \sum_{i=1}^{k} E_i' \tag{6.14}$$

where $\widehat{L}'$ and E_i' are the unique lifts of the curves $\widehat{L}$ and E_i, respectively. Furthermore, by Lemma 4.1.6 we find that

$$(L')^2 = \widehat{L}^2 = L^2 - k = 1 - k.$$

whereas $(E_i')^2 = -1$ for all i. We can then further compute that

$$\begin{aligned}(A')^2 &= (\widehat{L}' + \sum_{i=1}^{k} E_i')^2 = (\widehat{L}')^2 + 2\sum_{i=1}^{k} \widehat{L}' \cdot E_i' + \sum_{i=1}^{k} (E_i')^2 \\ &= 1 - k + 2(k) - k = 1\end{aligned}$$

and so A' carries the fundamental group of X_p by Theorem 6.1.6. An analogous diagram for the intersection pattern of A' compared to that of Figure 6.1 is given below in Figure 6.4. Symplectically resolving these intersection points yields the desired symplectic submanifold (which following the similarities to the earlier construction we will denote by S). In particular, we note that the curve found for the dual Hesse case is simply an example of this construction.

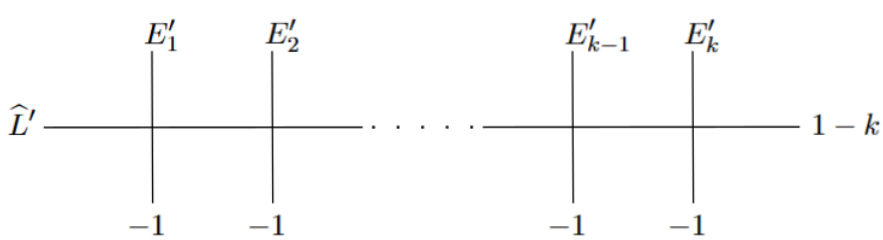

Figure 6.4: The divisor A'

By restricting to the covers of the various irreducible components, we compute that the Euler characteristic of $\widehat{L}'$ is $(2-k)p+k = 2p+k(1-p)$, and so the genus of $\widehat{L}'$ is

$$g(\widehat{L}') = \frac{2-2p-k(1-p)}{2} = \frac{(k-2)(p-1)}{2}.$$

A similar argument also computes $e(E_i') = (2-m_{q_i})p + m_{q_i} = 2p + m_{q_i}(1-p)$, and so in general we find that the genus of E_i' is

$$g(E_i') = \frac{2-2p-m_{q_i}(1-p)}{2} = \frac{(m_{q_i}-2)(p-1)}{2}.$$

Putting this all together we find that the genus of the symplectic resolution of A' is therefore

$$g(S) = \frac{p-1}{2}\left(-k-2+\sum_{i=1}^{k} m_{q_i}\right). \qquad \square$$

We can now describe a general procedure for applying branched covers of line arrangements to the symplectic geography problem. First, we begin with any arrangement $\mathcal{L}$ in $\mathbb{CP}^2$ and blow up all intersection points of multiplicity greater than 2 to obtain $\widehat{\mathcal{L}}$ in $\widetilde{\mathbb{CP}^2}$. For large primes p, we then consider the *smooth* $(\mathbb{Z}/p\mathbb{Z})^2$ cover of $\widetilde{\mathbb{CP}^2}$ branched over $\widehat{\mathcal{L}}$ provided by Theorem 6.4.1. This cover X_p contains some embedded symplectic surface S of genus

$$g(S) = \frac{p-1}{2}\left(-k-2+\sum_{i=1}^{k} m_{q_i}\right)$$

with $S^2 = 1$ carrying the fundamental group.

Letting $E(2)$ be a homotopy $K3$ surface obtained by a knot surgery with a knot K of genus $g(S)-1$ and blowing up X_p at a point on S, we can consider the normal connected sum

$$Z_p = (X_p \# \overline{\mathbb{CP}}^2) \#_\varphi (E(2)_K).$$

This results in a symplectic 4-manifold Z_p with Euler characteristic

$$\begin{aligned} e(Z_p) &= e(X_p) + 1 + 24 + 4g(\widehat{S}) - 4 \\ &= \Big(3 - 2n + \sum_{i=2}^{n}(i-1)t_i\Big)p^2 + 2\Big(n + \sum_{i=2}^{n}(1-i)t_i\Big)p + t_2 + \sum_{i=3}^{n} it_i \\ &\quad + 4\cdot\frac{p-1}{2}\Big(-k-2+\sum_{i=1}^{k} m_{q_i}\Big) + 21, \end{aligned}$$

and signature

$$\begin{aligned}\sigma(Z_p) &= \sigma(X_p) - 17 \\ &= \frac{1}{3}\left(\left(3 - n + \sum_{i=3}^{n}(i-2)t_i\right)p^2 + n - \sum_{i=3}^{n}(i+1)t_i\right) - 17.\end{aligned}$$

Now, performing a similar analysis for Z_p as provided in the proof of Corollary 6.3.3, we find that

$$\begin{aligned}\limsup_{\sigma\to\infty}\frac{\lambda(\sigma)}{\sigma} &\leq \limsup_{p\to\infty}\frac{\chi_h(Z_p)}{\sigma(Z_p)} \\ &= \limsup_{p\to\infty}\frac{e(Z_p)+\sigma(Z_p)}{4\sigma(Z_p)} \\ &= \lim_{p\to\infty}\frac{e(X_p)+\sigma(X_p)+O(p)}{4\sigma(X_p)+O(1)} \\ &= \lim_{p\to\infty}\frac{e(X_p)+\sigma(X_p)}{4\sigma(X_p)},\end{aligned}$$

where the last equality comes from the observation that $e(X_p)$ and $\sigma(X_p)$ are both $O(p^2)$ as $p \to \infty$. That is, we have

$$\limsup_{\sigma\to\infty}\frac{\lambda(\sigma)}{\sigma} \leq \frac{1}{4}\left(1 + 3\cdot\frac{3-2n+\sum_{i=2}^{n}(i-1)t_i}{3-n+\sum_{i=3}^{n}(i-2)t_i}\right). \tag{6.15}$$

We highlight that this asymptotic behaviour is in a sense 'blind' to the fundamental group issues that can arise when considering these branched covers. That is, since the genus of the curves carrying the fundamental group of X_p is linear in p, the contributions from the normal connected sum vanish upon taking large covers.

Example 6.4.3. With this procedure in mind, we compare the result of Theorem 6.4.1 and Corollary 6.4.2 to the case of the Wiman arrangement. If we consider a $(\mathbb{Z}/p\mathbb{Z})^2$ cover branched over the Wiman arrangement (call it X_p), then for sufficiently large primes, the cover will contain a symplectic submanifold that carries the fundamental group without requiring the addition of the generic line that we needed in Section 5.3.1. In this case, we can compute by Lemma 4.3.5 that

$$e(X_p) = 432p^2 - 948p + 720, \tag{6.16}$$

$$\sigma(X_p) = \frac{1}{3}\left(276p^2 - 876\right) = 92p^2 - 292. \tag{6.17}$$

Furthermore, we recall from from Section 5.3.1, that on any line of the arrangement there are 8, 4 and 4 intersection points of multiplicities 3, 4 and 5, respectively. Therefore the surface S constructed in Corollary 6.4.2 will have genus

$$\begin{aligned} g(S) &= \frac{p-1}{2}\left(-k-2+\sum_{i=1}^{k} m_{q_i}\right) \\ &= \frac{p-1}{2}(-16-2+60) \\ &= 21(p-1). \end{aligned}$$

and will carry the fundamental group. Proceeding in a way identical to the above and performing a symplectic normal connected sum on X_p along S we obtain a simply connected irreducible symplectic 4-manifold W'_p with the invariants

$$\begin{aligned} e(W'_p) &= e(X_p\#\overline{\mathbb{CP}}^2) + e(E(2)_K) - 2e(\widehat{S}) \\ &= (432p^2 - 948p + 720 + 1) + 24 + 4(21(p-1)) - 4 \\ &= 432p^2 - 864p + 657, \\ \sigma(W'_p) &= \sigma(X_p\#\overline{\mathbb{CP}}^2) + \sigma(E(2)_K) \\ &= \left(92p^2 - 292 - 1\right) - 16 \\ &= 92p^2 - 309. \\ \chi_h(W'_p) &= \frac{1}{4}(e(W'_p) + \sigma(W'_p)) \\ &= 131p^2 - 216p + 87. \end{aligned}$$

Feeding these invariants into Theorem 6.3.1 we find the asymptotic limit

$$\limsup_{\sigma\to\infty} \frac{\lambda(\sigma)}{\sigma} \leq \frac{131}{92} \approx 1.424 < \frac{17}{11}. \tag{6.18}$$

That is, the asymptotic existence of smooth $(\mathbb{Z}/p\mathbb{Z})^2$ covers branched over line arrangements can be used to find sharper bounds on the asymptotic behaviour of individual line arrangements. The asymptotic bound on $\lambda(\sigma)$ we found in [34] could be improved to that in (6.18) (This also suggests in part why the dual Hesse gives the current best known bound for $\lambda(\sigma)/\sigma$, as it also has covering group $(\mathbb{Z}/p\mathbb{Z})^2$).

Remark 6.4.4. We also contrast the result of Theorem 6.4.1 and Corollary 6.4.2 to that of the dual Hesse. Specifically, the $(\mathbb{Z}/p\mathbb{Z})^2$ covers defined in Section 5.3.2 only require $p \geq 11$, a stark difference compared to the existence bound from Theorem 6.4.1. This suggests that there is is an interesting question to be investigated.

Question 6.4.5. Let $\mathcal{L}$ be a line arrangement and suppose X is a $(\mathbb{Z}/p\mathbb{Z})^2$ cover branched over $B = \beta^{-1}(\mathcal{L})$ where $\beta : \widetilde{\mathbb{CP}^2} \to \mathbb{CP}^2$ is the blow-down map corresponding to the blow-up of each intersection point of $\mathcal{L}$ of multiplicity at least 3. Define $m_{sm}(\mathcal{L})$ to be the minimum $p \geq 2$ for which the cover X can be taken to be smooth. Is there an explicit function for m_{sm} in terms of various invariants of the line arrangement? Specifically, can m_{sm} be determined only from the set $\{t_i\}$ of number of intersection points with specified multiplicities?

Finally, we note that the difficulty in finding an answer to the above question is primarily caused by wanting to guarantee the existence of a $(\mathbb{Z}/p\mathbb{Z})^2$ cover for *any* line arrangement. That is, there are examples of line arrangements consisting of 9 lines (the same amount as the dual Hesse arrangement) that permit a smooth $(\mathbb{Z}/2\mathbb{Z})^2$ branched cover (cf. Section 4.2 of [45]). Therefore, any complete answer to the above question (or likely even a partial answer) will require either powerful techniques that are 'agnostic' to the geometry of the arrangement, or if not, then we expect they will require a deeper understanding than currently exists on how intersection points can be distributed along the lines in an arrangement.

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