This is a post-peer-review, pre-copyedit version of an article published in Semigroup Forum. The final authenticated version is available online at: https://doi.org/10.1007/s00233-015-9772-7

RANGES OF VECTOR STATES ON IRREDUCIBLE OPERATOR SEMIGROUPS

L.W. MARCOUX[1], M. OMLADIČ[2], A.I. POPOV[3], H. RADJAVI[1], AND B. YAHAGHI[4]

ABSTRACT. Let φ be a linear functional of rank one acting on an irreducible semigroup $\mathcal{S}$ of operators on a finite- or infinite-dimensional Hilbert space. It is a well-known and simple fact that the range of φ cannot be a singleton. We start a study of possible finite ranges for such functionals. In particular, we prove that in certain cases, the existence of a single such functional φ with a two-element range yields valuable information on the structure of $\mathcal{S}$.

1. INTRODUCTION

1.1. In the last decade or so there has been increasing interest in questions dealing with the so-called "local-to-global" properties of matrix semigroups: let $\mathcal{S}$ be a (multiplicative) semigroup of matrices in $\mathbb{M}_n(\mathbb{C})$, and assume further that $\mathcal{S}$ is irreducible, i.e. the members of $\mathcal{S}$ have no common invariant subspaces other than the trivial ones $\{0\}$ and $\mathbb{C}^n$. Vaguely speaking, we know that some property, say some "smallness" property such as finiteness or boundedness holds locally, and we are asking whether it holds globally. One specific kind of problem with which we are dealing in this paper is the following. Assume that φ is a non-zero linear functional acting on $\mathbb{M}_n(\mathbb{C})$. Does mere knowledge about the set

$$\varphi(\mathcal{S}) := \{\varphi(S) : S \in \mathcal{S}\}$$

yield any knowledge about $\mathcal{S}$ itself? Or, raising hopes even further, does enough knowledge about $\varphi(\mathcal{S})$ characterize $\mathcal{S}$?

In [4] it was shown that if $\varphi(\mathcal{S})$ is finite, then so is $\mathcal{S}$. (An upper bound on the size of $\mathcal{S}$ was also given: $|\mathcal{S}| \leqslant |\varphi(\mathcal{S})|^{n^2}$.) The same statement was shown to be true also when "bounded" replaced "finite".

We start by recalling that $\varphi(\mathcal{S})$ cannot be a singleton for an irreducible semigroup $\mathcal{S}$. This is a special case of the result that when φ is permutable on $\mathcal{S}$, i.e. if $\varphi(S_1S_2S_3) = \varphi(S_2S_1S_3)$ for every $S_1, S_2, S_3 \in \mathcal{S}$, then $\mathcal{S}$ is reducible. (E.g. see [3], Lemma 2.1.4.) It follows that the first interesting case for consideration for an irreducible semigroup $\mathcal{S} \subseteq \mathbb{M}_n(\mathbb{C})$ is the existence of a linear functional φ on $\mathbb{M}_n(\mathbb{C})$ for which $\varphi(\mathcal{S})$ consists of precisely two elements.

We shall consider a linear functional of rank one, and concentrate on the particular case where φ is given in the inner-product form

$$\varphi(T) = \langle T\xi, \eta \rangle, \quad T \in \mathbb{M}_n(\mathbb{C}),$$

where ξ and η are vectors with $\langle \xi, \eta \rangle = 1$. Particular attention will be paid to the case where $\xi = \eta$, i.e., where φ is a state on $\mathbb{M}_n(\mathbb{C})$. One of our definitive results is obtained when $\mathcal{S}$ consists of unitary matrices, and φ is a state for which $\varphi(\mathcal{S})$ is a two-element set. As we have seen, this forces $\mathcal{S}$ to be finite, and hence a group. Hence $\varphi(\mathcal{S}) = \{1, \omega_2\}$

[1] Research supported in part by NSERC (Canada).
March 27, 2015.

for some $1 \neq \omega_2 \in \mathbb{C}$. In this case, we show that $\omega_2 = -\frac{1}{n}$ and $\mathcal{S}$ is contained, up to simultaneous unitary similarity, in the well-known irreducible group

$$\mathcal{P}_{n+1}|_{\mathbb{1}^\perp},$$

where $\mathcal{P}_k$ denotes the group of all permutation matrices in $\mathbb{M}_k(\mathbb{C})$, and where $\mathbb{1} = (1, 1, \ldots, 1)^t$ is the common fixed column vector for all such permutations (so that $\mathcal{S}$ is the restriction of $\mathcal{P}_{n+1}$ to the common n-dimensional invariant subspace orthogonal to the span of $\mathbb{1}$).

Self-adjoint semigroups, i.e., those $\mathcal{S}$ satisfying

$$\mathcal{S}^* := \{S^* : S \in \mathcal{S}\} = \mathcal{S},$$

are easier to handle, but we also obtain results about which two-elements sets are possible in the general case and, in particular, in the case of semigroups consisting of matrices of rank at most one.

The questions raised in this paper make sense of course in the infinite-dimensional setting of operators on a Hilbert space. Some examples are mentioned in this connection.

1.2. Throughout this paper, we shall use $\mathcal{H}$ to denote a complex, separable Hilbert space and $\mathcal{B}(\mathcal{H})$ to denote the algebra of bounded linear operators acting on $\mathcal{H}$. If $\dim \mathcal{H} = n < \infty$, we identify $\mathcal{H}$ with $\mathbb{C}^n$ and $\mathcal{B}(\mathcal{H})$ with the algebra $\mathbb{M}_n(\mathbb{C})$ of complex $n \times n$ matrices. The standard basis for $\mathbb{C}^n$ is denoted by $\{e_i : 1 \leqslant i \leqslant n\}$, and we fix an orthonormal basis $\{e_i : i \geqslant 1\}$ for $\mathcal{H}$ when $\dim \mathcal{H} = \infty$.

1.3. Definition. *A semigroup $\mathcal{S}$ of bounded linear operators acting on a complex Hilbert space $\mathcal{H}$ is said to be **irreducible** if for each $0 \neq x \in \mathcal{H}$,* span $\{Sx : S \in \mathcal{S}\}$ *is norm-dense in $\mathcal{H}$.*

Of course, when $\dim \mathcal{H} < \infty$, this just says that for all $0 \neq x \in \mathcal{H}$, $\operatorname{span} \{Sx : S \in \mathcal{S}\} = \mathcal{H}$, or equivalently (thanks to Burnside's Theorem) that $\operatorname{span} \mathcal{S} = \mathbb{M}_n(\mathbb{C})$.

1.4. Notation. If $\mathcal{A} \subseteq \mathbb{M}_n(\mathbb{C})$ is a non-empty subset and $\varphi \in \mathbb{M}_n(\mathbb{C})^*$ is a linear functional on $\mathbb{M}_n(\mathbb{C})$, we write $\varphi(\mathcal{A})$ to denote the set $\{\varphi(A) : A \in \mathcal{A}\}$.

Given $1 \leqslant k \leqslant n$, we write $\operatorname{rank} \mathcal{A} \leqslant k$ as shorthand for the statement that $\operatorname{rank}(A) \leqslant k$ for all $A \in \mathcal{A}$, and similarly $\operatorname{rank} \mathcal{A} = k$ to mean that $\operatorname{rank}(A) = k$ for all $A \in \mathcal{A}$.

If $0 \neq x, y \in \mathcal{H}$, we denote by $x \otimes y^*$ the rank-one operator $x \otimes y^*(z) = \langle z, y \rangle\, x$, for all $z \in \mathcal{H}$.

We shall also have occasion to deal with the constant vector in $\mathbb{C}^n$, which we shall denote by $\mathbb{1} = (1, 1, \ldots, 1)^*$. The length of the vector should be clear from the context, but we shall also use the notation $\mathbb{1}_n$ should the need arise.

Given $n \geqslant 2$, we denote by $\mathcal{P}_n \subseteq \mathbb{M}_n(\mathbb{C})$ the unitary group of **permutation matrices**: that is, relative to the standard orthonormal basis $\{e_k\}_{k=1}^n$ for $\mathbb{C}^n$, $P \in \mathcal{P}_n(\mathbb{C})$ implies that there exists a permutation σ of $\{1, 2, \ldots, n\}$ so that $Pe_k = e_{\sigma(k)}$ for all $1 \leqslant k \leqslant n$. Since $\mathbb{1}$ is a fixed point for each element of $\mathcal{P}_n$, it is clear that $\mathbb{C}\mathbb{1}$ is a non-trivial invariant (in fact, orthogonally reducing) subspace for $\mathcal{P}_n$, and so the latter is reducible. It trivially follows that $\mathcal{N} := \mathbb{1}^\perp \subseteq \mathbb{C}^n$ is orthogonally reducing for $\mathcal{P}_n$ and as we shall see, $\mathcal{P}_n|_{\mathcal{N}}$ is an irreducible group of unitary operators.

We begin with a few standard results to which we shall repeatedly refer throughout the paper. In the next result and throughout the paper, the cardinality of a set X will be denoted by $|X|$.

1.5. Lemma. *Let $n \geqslant 2$ be an integer.*

(a) *If $\mathcal{S}$ is an irreducible semigroup and $0 \neq \varphi \in \mathbb{M}_n(\mathbb{C})^*$ is a non-zero linear functional, then*

$$|\varphi(\mathcal{S})| \geqslant 2.$$

Moreover, $|\varphi(\mathcal{S})| \leqslant \infty$ if and only if $|\mathcal{S}| < \infty$.

(b) *If $\mathcal{S} \subseteq \mathbb{M}_n(\mathbb{C})$ is an irreducible semigroup and A, B are two non-zero elements of $\mathbb{M}_n(\mathbb{C})$, then $B\mathcal{S}A \neq \{0\}$.*

(c) *If $\mathcal{S} \subseteq \mathbb{M}_n(\mathbb{C})$ is an irreducible semigroup which does not contain any non-zero nilpotents, then $\mathcal{S} \setminus \{0\}$ is again an irreducible semigroup.*

Proof.

(a) The first assertion is a simple reformulation of Corollary 2.1.6 of [3]. As for the second, note that if $|\mathcal{S}| < \infty$, then trivially $|\varphi(\mathcal{S})| < \infty$. The converse is Theorem 1 of [4].

(b) Choose $x \in \mathbb{C}^n$ so that $Ax \neq 0$. Then $\operatorname{span} \mathcal{S}(Ax) = \mathbb{C}^n$. Thus $B \neq 0$ implies that $B\mathcal{S}Ax \neq 0$, i.e. $B\mathcal{S}A \neq 0$.

(c) Suppose that R, T are two non-zero elements of $\mathcal{S}$ for which $RT = 0$. By part (b), $T\mathcal{S}R \neq 0$, and so we can find $S \in \mathcal{S}$ so that $TSR \neq 0$. Since $(TSR)^2 = 0$, $\mathcal{S}$ admits a non-zero nilpotent element, a contradiction.

Thus $0 \neq A, B \in \mathcal{S}$ implies that $AB \neq 0$; i.e. $\mathcal{S} \setminus \{0\}$ is a semigroup.

□

2. Notation and preliminary results

2.1. Definition. *Let $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H})$ be a non-empty semigroup of operators acting on a complex Hilbert space $\mathcal{H}$. We say that $\Omega \subseteq \mathbb{C}$ is an **admissible set** for $\mathcal{S}$ if there exists a unit vector $\xi \in \mathcal{H}$ so that*

$$\Omega = \{\langle S\xi, \xi \rangle : S \in \mathcal{S}\}.$$

That is to say, Ω is the image of $\mathcal{S}$ under the vector state $\eta_\xi \in \mathcal{B}(\mathcal{H})^*$ defined by $\eta_\xi(T) = \langle T\xi, \xi \rangle$.

*We refer to ξ as an **admissible vector** corresponding to Ω.*

Alternatively, with $\mathcal{S}$ as above, given a norm-one vector $\xi \in \mathcal{H}$, we define the corresponding admissible set $\Omega_\xi := \{\langle S\xi, \xi \rangle : S \in \mathcal{S}\}$.

2.2. If $\xi \in \mathcal{H}$ is a norm-one vector, then we can always extend $\{\xi\}$ to an orthonormal basis - say $\{\xi, e_2, e_3, \ldots, \}$ for $\mathcal{H}$. Writing the matrix $T = [t_{i,j}]$ with respect to this ordered basis (having ξ as the first basis vector), we see that $\Omega_\xi = \{t_{1,1} : T = [t_{i,j}] \in \mathcal{S}\}$.

It follows easily from Lemma 1.5 (a) that if $\mathcal{S}$ is an irreducible semigroup of operators on $\mathcal{H}$, then any admissible set Ω of $\mathcal{S}$ must contain at least two elements. Our goal is to obtain as much information about the sets $\Omega = \{\omega_1, \omega_2\} \subseteq \mathbb{C}$ of cardinality two which can occur as an admissible set for an irreducible semigroup $\mathcal{S} \subseteq \mathcal{B}(\mathbb{C}^n) \simeq \mathbb{M}_n(\mathbb{C})$ of linear maps acting on $\mathbb{C}^n$, where $n \geqslant 2$ is an integer. It is again an immediate consequence of Lemma 1.5 (a) that if an irreducible semigroup $\mathcal{S} \subseteq \mathbb{M}_n(\mathbb{C})$ possesses an admissible set Ω with finitely many elements, then $\mathcal{S}$ is itself a finite set. For this reason, we shall restrict our attention to these.

It is worth pointing out that the existence of an admissible set of cardinality two for an irreducible semigroup of operators is a geometric, as opposed to an algebraic property. By this we mean that if $\mathcal{S}$ is an irreducible semigroup of operators for which there exists an admissible set Ω of cardinality two, and if $\mathcal{T}$ is unitarily equivalent to $\mathcal{S}$, then Ω is again admissible for $\mathcal{T}$ (albeit with a potentially different admissible vector), whereas if $\mathcal{T}$ is only similar to $\mathcal{S}$, then $\mathcal{T}$ need not possess an admissible set of cardinality two at all. That this is the case is made clear by the following example.

2.3. Example. Let $\mathcal{S} = \{E_{i,j} : 1 \leqslant i,j \leqslant 2\} \cup \{0\}$, where $E_{i,j}$ denotes the (i,j) matrix unit. Clearly $\mathcal{S}$ is an irreducible semigroup in $\mathbb{M}_2(\mathbb{C})$. In particular, setting $\xi = e_1$, the first standard orthonormal basis vector, shows that $\Omega = \{0,1\}$ is an admissible set for $\mathcal{S}$.

Let $P = \begin{bmatrix} 1 & \sqrt{2} \\ 0 & 1 \end{bmatrix}$, so that P is invertible. Then

$$\mathcal{T} := P^{-1}\mathcal{S}P = \{P^{-1}SP : S \in \mathcal{S}\}$$

is again an irreducible semigroup of operators of rank at most one.

Now

$$\mathcal{T} = \left\{ \begin{bmatrix} 1 & \sqrt{2} \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -\sqrt{2} & -2 \\ 1 & \sqrt{2} \end{bmatrix}, \begin{bmatrix} 0 & -\sqrt{2} \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}.$$

We shall show that if $\xi = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \in \mathbb{C}^2$ is an arbitrary unit vector, then Ω_ξ has at least three elements. Indeed, a simple but tedious calculation shows that $\Omega_\xi = \{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5\}$, where

$$\begin{aligned}
\beta_1 &= |\xi_1|^2 + \sqrt{2}\,\overline{\xi_1}\xi_2 \\
\beta_2 &= \overline{\xi_1}\xi_2 \\
\beta_3 &= -\sqrt{2}|\xi_1|^2 - 2\overline{\xi_1}\xi_2 + \xi_1\overline{\xi_2} + \sqrt{2}|\xi_2|^2 \\
\beta_4 &= -\sqrt{2}\,\overline{\xi_1}\xi_2 + |\xi_2|^2 \\
\beta_5 &= 0.
\end{aligned}$$

CASE ONE: $\xi_1 = 0$. In this case, $|\xi_2| = 1$. Then $\beta_3 = \sqrt{2}$, $\beta_4 = 1$, and $\beta_1 = \beta_2 = \beta_5 = 0$, so $|\Omega_\xi| = 3$.

CASE TWO: $\xi_2 = 0$. In this case, $|\xi_1| = 1$. Then $\beta_2 = \beta_4 = \beta_5 = 0$, $\beta_1 = 1$ and $\beta_3 = -\sqrt{2}$, so $|\Omega_\xi| = 3$.

CASE THREE: $\xi_1 \neq 0 \neq \xi_2$. Then $\beta_2 = \overline{\xi_1}\xi_2 \neq 0$, and moreover, $0 < |\beta_2| < 1$. Also, $\beta_5 = 0$.

Suppose that $|\Omega_\xi| = 2$. Then $\beta_1 \in \{0, \beta_2\}$.

- If $\beta_1 = 0$, then $\overline{\xi_1}\xi_2 = -\frac{1}{\sqrt{2}}|\xi_1|^2$. From this it follows that $\beta_4 = |\xi_1|^2 + |\xi_2|^2 = 1$, so $|\Omega_\xi| \geqslant |\{0, \beta_2, \beta_4\}| = 3$.
- If $\beta_1 = \beta_2$, then $\overline{\xi_1}\xi_2 = \frac{1}{1-\sqrt{2}}|\xi_1|^2$, and so $\xi_2 = \frac{1}{1-\sqrt{2}}\xi_1$. But then $\beta_1 = \beta_2 = \frac{1}{1-\sqrt{2}}|\xi_1|^2$, while $\beta_3 = \beta_4 = \frac{3-\sqrt{2}}{(1-\sqrt{2})^2}|\xi_1|^2$ and $\beta_5 = 0$, so that $|\Omega_\xi| = 3$.

3. SEMIGROUPS OF SMALL RANK

3.1. In this section we shall prove the existence and examine the possible values of admissible sets of cardinality two for semigroups $\mathcal{S}$ of operators in $\mathbb{M}_n(\mathbb{C})$ of rank at most one.

We begin with a simple existence result whose converse we shall soon establish.

3.2. Lemma. *Let $n \geqslant 2$ and $R = [r_{i,j}] \in \mathbb{M}_n(\mathbb{C})$ be invertible. Suppose that $r_{i,j} \in \{0,1\}$ for all $1 \leqslant i, j \leqslant n$, and set*

$$\mathcal{S} := \{E_{i,j}R : 1 \leqslant i, j \leqslant n\} \cup \{0\}.$$

Then $\mathcal{S}$ is an irreducible semigroup in $\mathbb{M}_n(\mathbb{C})$.

Proof. Clearly $E_{i,j}RE_{k,l}R = r_{j,k}E_{i,l}R \in \mathcal{S}$ for all $1 \leqslant i, j, k, l \leqslant n$, so that $\mathcal{S}$ is a semigroup. Moreover, $\mathcal{S}$ is irreducible since it is clear that $\mathcal{S} \setminus \{0\} \subseteq \mathbb{M}_n(\mathbb{C})$ is linearly independent, and contains n^2 elements.

□

We remark that if $T, R \in \mathbb{M}_n(\mathbb{C})$ are invertible, then clearly $\mathcal{T} = T^{-1}\mathcal{S}T := \{T^{-1}E_{i,j}RT : 1 \leqslant i, j \leqslant n\} \cup \{0\}$ is an irreducible semigroup in $\mathbb{M}_n(\mathbb{C})$, with rank $\mathcal{T} \leqslant 1$.

Let us first state a part of Lemma 4.2.4 of [3]:

3.3. Lemma. *Let $\mathcal{S}$ be an irreducible semigroup of $\mathbb{M}_n(\mathbb{C})$ whose members have rank no greater than 1. Then there exist two bases $\{u_i\}$ and $\{v_j\}$ of column vectors for $\mathbb{C}^n$ such that the basis*

$$\{u_i \otimes v_j^* \,;\, i, j = 1, 2, \ldots, n\}$$

of $\mathbb{M}_n(\mathbb{C})$ is contained in $\mathcal{S}$.

3.4. Notation. Let $n \geqslant 1$ and let $\mathcal{S} \subseteq \mathbb{M}_n(\mathbb{C})$ be an irreducible semigroup. We shall denote by $\Gamma(= \Gamma_{\mathcal{S}})$ the set of all $\gamma \in \mathbb{C}$ for which there exist $0 \neq S \in \mathcal{S}$ with $S, \gamma S \in \mathcal{S}$.

3.5. Proposition. *Let $n \geqslant 2$ be an integer and $\mathcal{S} \subseteq \mathbb{M}_n(\mathbb{C})$ be an irreducible semigroup satisfying* rank $\mathcal{S} \leqslant 1$. *Then:*

(a) *There exist invertible matrices R and T such that*

$$\mathcal{B} = \{T^{-1}E_{ij}RT \,;\, i, j = 1, 2, \ldots, n\},$$

where $E_{ij} = e_i \otimes e_j^*$, $1 \leqslant i, j \leqslant n$ are the standard matrix units, forms a basis of $\mathbb{M}_n(\mathbb{C})$ contained in $\mathcal{S}$.

(b) *The entries of the matrix R relative to the standard basis belong to Γ.*

(c) *Γ is a semigroup and its nonzero elements form a group.*

(d) *If Γ is finite, then it consists of $\rho, \rho^2, \ldots, \rho^{k-1}, 1$ for some $k \geqslant 1$, where ρ is a primitive k-th root of 1, and possibly zero.*

Proof.

(a) Let $\{u_i\}_i$ and $\{v_j\}_j$ be the bases for $\mathbb{C}^n$ for which $u_i \otimes v_j^* \in \mathcal{S}$ for all $1 \leqslant i, j \leqslant n$, as provided by Lemma 3.3. Choose invertible matrices T and then $R \in \mathbb{M}_n(\mathbb{C})$ so that

(i) $Tu_i = e_i$, $1 \leqslant i \leqslant n$, and

(ii) $R^*e_j = (T^*)^{-1}v_j$, $1 \leqslant j \leqslant n$.

For all $1 \leqslant i, j \leqslant n$ we then have

$$T^{-1}E_{i,j}RT = T^{-1}(e_i \otimes e_j^*)RT = (T^{-1}e_i) \otimes ((RT)^*e_j)^* = u_i \otimes v_j^* \in \mathcal{S}.$$

(b) Writing $B_{ij} = T^{-1}E_{ij}RT \in \mathcal{S}$ for all $1 \leqslant i, j \leqslant n$, we get

$$r_{jk}B_{il} = (T^{-1}E_{ij}RT)\,(T^{-1}E_{kl}RT) = B_{ij}B_{kl} \in \mathcal{S},$$

for all $i, j, k, l \in \{1, 2, \ldots, n\}$. As $B_{il} \in \mathcal{S}$, it follows that $r_{jk} \in \Gamma$.

(c) If $\gamma, \gamma' \in \Gamma$, there exist $S_1, S_2, S_1', S_2' \in \mathcal{S}$ such that $S_1 = \gamma S_2$ and $S_1' = \gamma' S_2'$. This implies that $S_1 S_1' = \gamma\gamma' S_2 S_2'$, so that $\gamma\gamma' \in \mathcal{S}$. Now if $0 \neq \gamma \in \Gamma$ and $S_1, S_2 \in \mathcal{S}$ are such that $S_1 = \gamma S_2$, then $S_2 = \gamma^{-1} S_1$.

(d) The form of finite complex groups is well-known.

□

We have seen that an admissible set for an irreducible semigroup cannot be a singleton set. On the other hand, we will show that when $n \geqslant 2$, there are irreducible semigroups $\mathcal{S} \subseteq \mathbb{M}_n(\mathbb{C})$ with rank at most one which possess admissible sets of cardinality two. In view of Proposition 3.5(e), it is clear that we have to consider two substantially different cases depending upon whether or not the admissible set contains zero. We begin with the case where 0 does not lie in our admissible set.

3.6. Proposition. *Suppose that $n \geqslant 2$ is an integer and that $\mathcal{S} \subseteq \mathbb{M}_n(\mathbb{C})$ is an irreducible semigroup satisfying* $\operatorname{rank} \mathcal{S} \leqslant 1$. *Suppose also that Ω is an admissible set for $\mathcal{S}$ consisting of two elements. If $0 \notin \Omega$, then our coefficient set $\Gamma = \{-1, 1\}$ and*

(a) *there exist an invertible matrix R whose entries belong to Γ and vectors f and g in $\mathbb{C}^n$ such that $\langle f, g\rangle = 1$ and $\langle E_{ij} R f, g\rangle \in \Omega$ for all $1 \leqslant i, j \leqslant n$. It follows that $\Omega = \{-\omega, \omega\} = \omega\Gamma$ for some $\omega \in \mathbb{C}$.*

(b) *The condition that $\langle E_{ij} R f, g\rangle \in \Omega$ for all i, j can be replaced by $\langle R E_{ij} f, g\rangle \in \Omega$ for all i, j to get an equivalent statement.*

(c) *All admissible sets of cardinality 2 not containing zero can be obtained in this way, that is: suppose that $0 \neq \omega \in \mathbb{C}$ and that $\Omega = \{-\omega, \omega\}$ (from part (a), this is the only possible form of an admissible set of cardinality two not containing zero). If there exist $f, g \in \mathbb{C}^n$ with $\langle f, g\rangle = 1$ and R an invertible matrix with entries in $\Gamma = \{-1, 1\}$ satisfying $\langle E_{ij} R f, g\rangle \in \Omega$ for all $1 \leqslant i, j \leqslant n$, then there exists an irreducible semigroup $\mathcal{S}$ for which Ω is an admissible set.*

Proof.

(a) Let Ω be admissible and satisfy the assumptions of the proposition. Let Γ, R and T be defined as in Proposition 3.5. Denote by ξ a corresponding norm-one admissible vector for $\mathcal{S}$.

We first claim that $\Gamma \subseteq \{-1, 1\}$. Indeed, suppose that $\gamma \in \Gamma$ but that $\gamma \notin \{-1, 1\}$.

Since $0 \notin \Omega$ by hypothesis, $\mathcal{S}$ can not contain any nilpotents (0 or otherwise). Thus every element of $\mathcal{S}$ is a non-zero multiple of an idempotent. Fix $S \in \mathcal{S}$ so that $S, \gamma S \in \mathcal{S}$ (such an S exists by definition of Γ). Fix $\beta \in \mathbb{C} \setminus \{0\}$ and $E = E^2 \in \mathbb{M}_n(\mathbb{C})$ so that $S = \beta E$.

Denoting Ω by $\{\omega_1, \omega_2\}$, we may assume without loss of generality that $\omega_1 = \langle S\xi, \xi\rangle$. Now $\gamma S \in \mathcal{S}$ and $\gamma \neq 1$ implies that

$$\gamma\omega_1 = \langle \gamma S\xi, \xi\rangle \in \Omega,$$

whence $\omega_2 = \gamma\,\omega_1$.

Note that $S^2 = \beta^2 E^2 = \beta^2 E = \beta S$, and more generally, $S^k = \beta^{k-1} S \in \mathcal{S}$ for all $k \geqslant 2$. Similarly, $(\gamma S)^k = \gamma^k \beta^{k-1} S \in \mathcal{S}$ for all $k \geqslant 1$. It easily follows from this that

$$\omega_1, \beta^{k-1}\omega_1, \gamma^k \beta^{k-1}\omega_1 \in \Omega \quad \text{for all } k \geqslant 1.$$

In particular,

$$\omega_1, \beta\,\omega_1, \gamma^2 \beta\,\omega_1 \in \Omega.$$

If $\beta = 1$, then $\omega_1, \gamma\omega_1, \gamma^2\omega$ are three distinct elements of Ω, a contradiction of our hypotheses.

Thus $\beta \neq 1$, so that $\omega_2 = \beta\omega_1 = \gamma\omega_1$. But $\omega_1 \neq 0$ implies that $\gamma = \beta$. But then $\beta = \gamma \notin \{-1, 1\}$ and $\omega_1,\ \beta\omega_1,\ \beta^2\omega_1 \in \Omega$ implies that $|\Omega| \geqslant 3$, again a contradiction.

Since $\Gamma \subseteq \{-1, 1\}$ is a group, there are only two possibilities, namely: $\Gamma = \{1\}$ and $\Gamma = \{-1, 1\}$. By Proposition 3.5 (b) the first possibility is ruled out by the fact that R is invertible, and so $\Gamma = \{-1, 1\}$. Define f and g by

$$f = T\xi \quad \text{and} \quad g = (T^{-1})^*\xi \tag{1}$$

to get the desired result.

Note that $\Gamma = \{-1, 1\}$ implies that there exists an element $S \in \mathcal{S}$ so that $-S \in \mathcal{S}$, and hence $\langle S\xi, \xi\rangle, \langle -S\xi, \xi\rangle \in \Omega$. Since Ω has two elements and $0 \notin \Omega$ by hypothesis, $\Omega = \{-\omega, \omega\}$ for some $\omega \in \mathbb{C}$.

(b) This follows by considering the irreducible semigroup $\mathcal{S}^* := \{S^* : S \in \mathcal{S}\} \subseteq \mathbb{M}_n(\mathbb{C})$, and applying part (a) of the proposition.

(c) Let us write $R = [r_{ij}]$ and note that $r_{ij} \in \{-1, 1\}$ for all $1 \leq i, j \leq n$. It is a simple exercise in linear algebra to observe that for any two vectors f and g satisfying $\langle f, g\rangle = 1$ there exist a unit vector $\eta \in \mathbb{C}^n$ and an invertible matrix T such that the relations

$$f = T\eta \quad \text{and} \quad g = (T^{-1})^*\eta$$

are satisfied. Let $\mathcal{S} = \{\pm T^{-1}E_{ij}RT \mid 1 \leqslant i, j \leqslant n\}$. Clearly $\mathcal{S}$ contains a linearly independent set of cardinality n^2 (namely $\{T^{-1}E_{ij}RT : 1 \leq i, j \leq n\}$), and thus $\operatorname{span}\mathcal{S} = \mathbb{M}_n(\mathbb{C})$. Moreover, for $1 \leq i, j, k, l \leq n$,

$$T^{-1}E_{ij}RT\ T^{-1}E_{kl}RT = r_{jk}T^{-1}E_{il}RT.$$

But $r_{jk} \in \{-1, 1\}$ and thus $\mathcal{S}$ is an irreducible semigroup in $\mathbb{M}_n(\mathbb{C})$. For all $S = \pm T^{-1}E_{ij}RT \in \mathcal{S}$, we have

$$\langle S\eta, \eta\rangle = \langle \pm T^{-1}E_{ij}RT\eta, \eta\rangle = \pm\langle E_{ij}Rf, g\rangle \in \Omega.$$

It is clear that indeed $\{\langle S\eta, \eta\rangle : S \in \mathcal{S}\} = \Omega$.

□

3.7. Theorem. *Suppose that* $0 \notin \Omega \subseteq \mathbb{C}$, $|\Omega| = 2$ *and* $\Gamma = \{-1, 1\}$. *Then the following are equivalent:*

(a) Ω *is an admissible set.*

(b) *There exists an invertible matrix* W *with entries from* Γ *such that* $\langle W^{-1}\mathbb{1}, \mathbb{1}\rangle = \omega^{-1}$ *and* $\Omega = \omega\Gamma$.

Proof.

(a) implies (b).

Assume that Ω is an admissible set and let R, f and g be defined as in Proposition 3.6 (b). Denote by r_j the j^{th} column of R, so that $R = [r_1, r_2, \ldots, r_n]$; apply Proposition 3.6 (b) to get $\{\langle RE_{ij}f, g\rangle : 1 \leqslant i, j \leqslant n\} \subseteq \Omega$. Note, however, that for each $1 \leqslant i, j \leqslant n$ we get $\langle RE_{ij}f, g\rangle = f_j\langle r_i, g\rangle$. We may assume with no loss of generality that one of the entries of f equals 1 (after multiplying f by a nonzero constant and dividing g by the same constant). Since $\langle r_i, g\rangle \neq 0$ (otherwise $0 \in \Omega$), it then follows from Proposition 3.6 that $\{f_j\}_{j=1}^n \subseteq \Gamma = \{-1, 1\}$ and that $\{\langle r_i, g\rangle\}_{i=1}^n \subseteq \Omega$.

We can choose $\omega \in \Omega$, and then multiply the columns of R by appropriate members of Γ if necessary to obtain a new matrix T with entries in Γ so that $\langle t_i, g\rangle = \omega$ for all $i = 1, 2, \ldots, n$ (where t_i denotes the i^{th} column of T). Note that T is still invertible since it is simply R multiplied by a diagonal matrix with entries in $\Gamma = \{-1, 1\}$.

Consequently, we have $g^* T = \omega \mathbb{1}^*$, so that $g^* = \omega \mathbb{1}^* T^{-1}$ and recalling that $\langle f, g\rangle = 1$, we get that

$$1 = \langle f, g\rangle = \omega\langle f, (T^{-1})^* \mathbb{1}\rangle = \omega\langle T^{-1} f, \mathbb{1}\rangle.$$

Finally, the vector f has entries in $\{-1, 1\}$ and so we can multiply it by a diagonal matrix D with entries in $\{-1, 1\} = \Gamma$ so that $Df = \mathbb{1}$. Then

$$1 = \omega\langle T^{-1} D^{-1} \mathbb{1}, \mathbb{1}\rangle.$$

Let $W = DT$. The entries of W are still in Γ and W is invertible since each of D and T are.

This completes the proof.

(b) implies (a). To get the converse we use Proposition 3.6 (c), namely: set $f = W^{-1}\mathbb{1}$, $g = \overline{\omega}\mathbb{1}$ and observe that $\langle f, g\rangle = 1$, while $\langle E_{ij} W f, g\rangle = \omega \in \Omega$ for all $1 \leqslant i, j \leqslant n$.

□

Having dealt with the case where our admissible set does not contain zero, we now turn our attention to the case where it does.

3.8. Theorem. *Let $n \geqslant 2$ and $\mathcal{S} \subseteq \mathbb{M}_n(\mathbb{C})$ be an irreducible semigroup for which* $\operatorname{rank} \mathcal{S} \leqslant 1$. *Suppose that $\Omega = \{\omega_1, \omega_2\}$ is an admissible set of cardinality two with corresponding admissible vector ξ. As always, we may assume without loss of generality that $\omega_2 \neq 0$. Then either*

(a) *there does not exist a non-zero nilpotent element in $\mathcal{S}$, in which case*

$$\operatorname{tr}(\mathcal{S}) := \{\operatorname{tr}(T) : T \in \mathcal{S}\} = \{-1, 1\}$$

and $\omega_1 = -\omega_2$; or

(b) *there exists a non-zero nilpotent element in $\mathcal{S}$ in which case $0 \in \Omega$ and* $\operatorname{tr}(\mathcal{S}) = \{0, 1\}$.

Proof. Let us begin with some general comments. Since $\mathcal{S}$ is irreducible, it can not consist solely of nilpotent matrices, by Levitzki's Theorem (see, e.g. Theorem 2.1.7 of [3]).

Let $B = x \otimes y^* \in \mathcal{S}$ be any element which is not nilpotent. Since $\operatorname{rank} B = 1$, it follows that $\operatorname{tr}(B) = \langle x, y\rangle \neq 0$. Also, for all $k \geqslant 1$, $0 \neq B^k = \langle x, y\rangle^{k-1} x \otimes y^* = \langle x, y\rangle^{k-1} B \in \mathcal{S}$.

Next, observe that $\mathcal{T} := \mathcal{S}B\mathcal{S} = \{RBT : R, T \in \mathcal{S}\}$ is a semigroup ideal in the irreducible semigroup $\mathcal{S}$, and therefore is itself irreducible, by Lemma 2.1.10 of [3]. As such, by Lemma 1.5, there exist $R_0, T_0 \in \mathcal{S}$ so that

$$\langle R_0 B T_0 \xi, \xi\rangle = \omega_2.$$

But then for all $k \geqslant 1$, $R_0 B^k T_0 = \langle x, y\rangle^{k-1} R_0 B T_0 \in \mathcal{S}$, so that

$$\operatorname{tr}(B)^{k-1}\omega_2 = \langle x, y\rangle^{k-1}\omega_2 = \langle R_0 B^k T_0 \xi, \xi\rangle \in \Omega.$$

Since $\omega_2 \neq 0$ and $\langle x, y\rangle \neq 0$, it follows from the fact that $|\Omega| = 2$ that $\operatorname{tr}(B) = \langle x, y\rangle \in \{-1, 1\}$. Moreover, if $\operatorname{tr}(B) = -1$, then from above

$$\operatorname{tr}(B)\omega_2 = -\omega_2 \in \Omega.$$

That is, $\omega_1 = -\omega_2$.

(a) Suppose that $\mathcal{S}$ does not contain a non-zero nilpotent element. Then $\mathcal{S} \setminus \{0\}$ is an irreducible semigroup of rank at most one, and from above, $\operatorname{tr}(\mathcal{S} \setminus \{0\}) \subseteq \{-1, 1\}$. Since $|\operatorname{tr}(\mathcal{S} \setminus \{0\})| \geqslant 2$ by Lemma 1.5, it follows that $\operatorname{tr}(\mathcal{S} \setminus \{0\}) = \{-1, 1\}$. Hence there exists $B \in \mathcal{S}$ with $\operatorname{tr}(B) = -1$, and so from the above argument $\Omega = \{-\omega_2, \omega_2\}$.

Note that this implies that $0 \notin \mathcal{S}$, for otherwise $0 = \langle 0\xi, \xi \rangle \in \Omega$, implying that $|\Omega| \geqslant 3$, a contradiction.

(b) Now suppose that $\mathcal{S}$ admits a non-zero nilpotent element N. Then $N^n = 0 \in \mathcal{S}$, and thus $0 = \langle 0\xi, \xi \rangle \in \Omega$; i.e. $\omega_1 = 0$. Furthermore, $\operatorname{tr}(N) = 0$ as N is nilpotent.

From above, if $B \in \mathcal{S}$ is not nilpotent, then $0 \neq \operatorname{tr}(B)^{k-1}\omega_2 \in \Omega$ for all $k \geqslant 1$. Since $|\Omega| = 2$, and $0 \in \Omega$, we conclude that $\operatorname{tr}(B) = 1$.

□

3.9. Remark. If, in the above result, the admissible set Ω is only assumed to be finite (as opposed to having cardinality two), the a similar proof can be used to show that either $\mathcal{S}$ does not admit a non-zero nilpotent, in which case $\operatorname{tr}(\mathcal{S}) \in \Omega_k$, or there exists a non-zero nilpotent in $\mathcal{S}$, in which case $\operatorname{tr}(\mathcal{S}) \in \{0\} \cup \Omega_k$, where $k \leq |\Omega|$ and Ω_k stands for the set of the kth roots of unity.

3.10. There remains the question of which subsets $\Omega = \{\omega_1, \omega_2\} \subseteq \mathbb{C}$ actually occur as the admissible set of cardinality two for an irreducible semigroup of rank less than or equal to one. This is the question we now address.

3.11. Corollary. *Let $n \geqslant 2$ be an integer, and let $\mathcal{S} \subseteq \mathbb{M}_n(\mathbb{C})$ be an irreducible semigroup with* $\operatorname{rank} \mathcal{S} \leqslant 1$. *Suppose that there exists an admissible set $\Omega = \{0, \omega_2\}$ for $\mathcal{S}$, where $\omega_2 \neq 0$.*

If $\Gamma, \mathcal{B}$ are defined as in paragraph 3.4 and Proposition 3.5, then $\Gamma \subseteq \{0, 1\}$ and $\mathcal{S}_0 := \mathcal{B} \cup \{0\}$ is a minimal irreducible subsemigroup which also has Ω as an admissible set.

Proof. We argue by contradiction. Choose an admissible, norm-one vector $\xi \in \mathbb{C}^n$ corresponding to $\Omega = \{0, \omega_2\}$. Suppose that $\gamma \in \Gamma$ and that $\gamma \notin \{0, 1\}$. Fix $0 \neq S \in \mathcal{S}$ so that $S, \gamma S \in \mathcal{S}$. Since $\mathcal{S}$ acts irreducibly on $\mathbb{C}^n$ and $\xi \neq 0$, there exists $T_1 \in \mathcal{S}$ so that $ST_1\xi \neq 0$.

Since $ST_1\xi \neq 0$ and $\mathcal{S}$ acts irreducibly on $\mathbb{C}^n$, there exists $T_2 \in \mathcal{S}$ so that

$$\langle T_2 S T_1 \xi, \xi \rangle \neq 0.$$

Thus $\langle T_2 S T_1 \xi, \xi \rangle = \omega_2$. But $\gamma S \in \mathcal{S}$ and so

$$\langle T_2(\gamma S) T_1 \xi, \xi \rangle = \gamma \langle T_2 S T_1 \xi, \xi \rangle = \gamma \omega_2 \in \Omega = \{0, \omega_2\}.$$

But $\gamma \notin \{0, 1\}$, and $\omega_2 \neq 0$, providing the desired contradiction.

Finally, the fact that $\mathcal{S}_0$ is an irreducible semigroup in $\mathbb{M}_n(\mathbb{C})$ stems from Lemma 3.3, and the minimality of $\mathcal{S}_0$ is a consequence of the fact that any irreducible semigroup in $\mathbb{M}_n(\mathbb{C})$ must contain at least n^2 non-zero elements, since it must span $\mathbb{M}_n(\mathbb{C})$.

□

3.12. Theorem. *Let $\mathcal{S} \subseteq \mathbb{M}_n(\mathbb{C})$ be an irreducible semigroup and suppose that there exists $S_0 \in \mathcal{S}$ with* $\operatorname{rank} S_0 = 1$. *If $\Omega = \{\omega_1, \omega_2\}$ is an admissible set of cardinality two for $\mathcal{S}$, then $\Omega \subseteq \mathbb{Q}$.*

Proof. As always, we denote by ξ the norm-one admissible vector corresponding to Ω. We note that by passing to the irreducible semigroup ideal generated by S_0, we may assume without loss of generality that $\operatorname{rank} \mathcal{S} \leqslant 1$. We continue under this assumption.

It follows from Theorem 3.8 above that $\operatorname{tr}(\mathcal{S}) \in \mathbb{Q}$ (whether or not $\mathcal{S}$ contains a non-zero nilpotent).

Since $\operatorname{rank}(\mathcal{S}) \leqslant 1$, we can use Theorem 2.6 of [5] to conclude that there exists an invertible matrix $W \in \mathbb{M}_n(\mathbb{C})$ so that for all $S \in \mathcal{S}$,

$$W^{-1}SW \in \mathbb{M}_n(\mathbb{Q}).$$

Next, by the second remark after the above cited theorem, we see that there exist $r_1, r_2, \dots, r_k \in \mathbb{Q}$ and $S_1, S_2, \dots, S_k \in \mathcal{S}$ so that

$$I = r_1\, W^{-1}S_1W + r_2\, W^{-1}S_2W + \cdots + r_k\, W^{-1}S_kW.$$

But then $I = \sum_{i=1}^k r_i\, S_i$.

Recall from Theorem 3.8 that there exist two cases, namely:

(a) $\omega_1 = 0$, i.e. $\Omega = \{0, \omega_2\}$, or
(b) $\omega_1 = -\omega_2$, i.e. $\Omega = \{-\omega_2, \omega_2\}$.

In either case, the equation

$$1 = \langle I\xi, \xi\rangle = \langle \sum_{i=1}^k r_i\, S_i\xi, \xi\rangle = \sum_{i=1}^k r_i\, \langle S_i\xi, \xi\rangle,$$

with $r_i \in \mathbb{Q}$ and $\langle S_i\xi, \xi\rangle \in \{0, -\omega_2, \omega_2\}$ for all $1 \leqslant i \leqslant k$, implies that $1 = r\omega_2$ for some $r \in \mathbb{Q}$, whence $\omega_2 \in \mathbb{Q}$, and therefore $\Omega \subseteq \mathbb{Q}$.

$\square$

3.13. Remark. We point out that irreducibility of the semigroup $\mathcal{S}$ is crucial to the above result. For example, if $\lambda \in \mathbb{C}$, then with $E = \begin{bmatrix} \lambda & 1 \\ \lambda - \lambda^2 & 1 - \lambda \end{bmatrix} \in \mathbb{M}_2(\mathbb{C})$, $\mathcal{E} := \{E, 0\}$ is a (reducible) semigroup with $\operatorname{rank} \mathcal{E} \leqslant 1$, and setting $\xi = e_1$ yields $\Omega_\xi = \{0, \lambda\}$.

Having established that a two-element admissible set for an irreducible semigroup of rank at most one in $\mathbb{M}_n(\mathbb{C})$ must consist of rational numbers, we now seek to identify which pairs of rational numbers can appear as such admissible sets.

3.14. Lemma. *Let $A \in \mathbb{M}_n(\mathbb{C})$ be invertible and $x, y \in \mathbb{C}^n$. Let $B = \begin{bmatrix} A & x \\ y^* & 0 \end{bmatrix} \in \mathbb{M}_{n+1}(\mathbb{C})$. The following are equivalent:*

(a) *B is invertible in $\mathbb{M}_{n+1}(\mathbb{C})$.*
(b) *$y^*A^{-1}x \neq 0$.*

Proof. It is easy to see that the matrices $V = \begin{bmatrix} I_n & 0 \\ -y^* & 1 \end{bmatrix}$ and $W = \begin{bmatrix} A^{-1} & 0 \\ 0 & 1 \end{bmatrix}$ are invertible and that

$$VWB = \begin{bmatrix} I_n & A^{-1}x \\ 0 & -y^*A^{-1}x \end{bmatrix}.$$

But B is invertible if and only if $\det B \neq 0$ if and only if $\det VWB \neq 0$, which clearly happens if and only if $y^*A^{-1}x \neq 0$.

$\square$

3.15. Lemma. *For each $n \geqslant 3$ and for each integer $1 \leqslant m \leqslant n-2$, there exists $R = [r_{i,j}] \in \mathbb{M}_n(\mathbb{C})$ so that*

(a) *$r_{i,j} \in \{0,1\}$ for all $1 \leqslant i, j \leqslant n$;*
(b) *R is invertible; and*
(c) *for all $1 \leqslant i \leqslant n$, $\sum_{j=1}^{n} r_{i,j} = m$.*

In other words, R is an invertible matrix with $\{0,1\}$ entries, each of whose rows contains exactly m non-zero entries, each equal to 1.

Proof. The proof is by induction on n.

For $n = 3$, we must have $m = 1$, and hence it suffices to consider $R = I_3$, the identity matrix in $\mathbb{M}_3(\mathbb{C})$.

Now suppose that the result holds for $n = 3, 4, 5, \ldots, n_0$ and for all $1 \leqslant m \leqslant n-2$. We shall prove that it holds for $n = n_0 + 1$ and all $1 \leqslant m \leqslant n_0 - 1$.

Again, if $m = 1$, it suffices to consider the identity matrix $R = I_{n_0+1}$. Suppose therefore that $m \geqslant 2$.

Now $2 \leqslant m \leqslant n_0$ implies that $1 \leqslant m - 1 \leqslant n_0 - 1$, and so by our induction hypothesis, there exists $R_0 = [r_{i,j}] \in \mathbb{M}_{n_0}(\mathbb{C})$ invertible, so that $r_{i,j} \in \{0,1\}$ for all $1 \leqslant i, j \leqslant n_0$ and $\sum_{j=1}^{n_0} r_{i,j} = m - 1$ for all $1 \leqslant i \leqslant n_0$.

Let $\mathbb{1} = (1, 1, \ldots, 1)^t \in \mathbb{C}^{n_0}$. We shall prove the existence of a vector $y = (y_1, y_2, \ldots, y_{n_0})^* \in \mathbb{C}^{n_0}$ so that

(i) $y_i \in \{0,1\}$ for all $1 \leqslant i \leqslant n_0$,
(ii) $\sum_{i=1}^{n_0} y_i = m$, and
(iii) $R := \begin{bmatrix} R_0 & \mathbb{1} \\ y^* & 0 \end{bmatrix} \in \mathbb{M}_{n_0+1}(\mathbb{C})$ is invertible.

This will complete the induction step, thereby proving the Lemma.

By the above Lemma, R will be invertible if and only if $y^* R_0^{-1} \mathbb{1} \neq 0$. Suppose to the contrary that $y^* R_0^{-1} \mathbb{1} = 0$ for all choices of y with exactly m non-zero entries, all of these equal to 1. If we denote the vector $R_0^{-1} \mathbb{1}$ by $(z_1, z_2, \ldots, z_{n_0})^t$, then it follows that for each $1 \leqslant k \leqslant m$,

$$
\begin{aligned}
0 &= z_1 + z_2 + \cdots + z_m \\
0 &= z_1 + z_2 + \cdots + z_{k-1} + z_{k+1} + \cdots + z_{m+1}
\end{aligned}
$$

from which we deduce that $z_{m+1} - z_k = 0$, i.e. $z_1 = z_2 = \cdots = z_{m+1}$, and for all $m+1 \leqslant k \leqslant n_0$,

$$
\begin{aligned}
0 &= z_1 + z_2 + \cdots + z_m \\
0 &= \quad\; z_2 + z_3 + \cdots + z_m + z_k
\end{aligned}
$$

from which we deduce that $z_1 - z_k = 0$, i.e. $z_1 = z_2 = \cdots = z_{n_0}$. In other words, $z = z_1 \mathbb{1}$.

But then $0 = z_1 + z_2 + \cdots + z_m$ implies that $0 = m z_1$ and hence that $z = 0$. But $z = R_0^{-1} \mathbb{1}$. Since R_0 is invertible and $\mathbb{1} \neq 0$, this is a contradiction.

This proves the existence of such a vector y as we have described, which completes the proof.

$\square$

3.16. Proposition. *Let $n \geqslant 3$, and let p, q be integers satisfying $1 \leqslant p \leqslant q-2 \leqslant n-2$. Then there exists an irreducible semigroup $\mathcal{S} \subseteq \mathbb{M}_n(\mathbb{C})$ with* $\operatorname{rank} \mathcal{S} \leqslant 1$ *and a unit vector $\xi \in \mathbb{C}^n$ so that*

$$\Omega_\xi = \{0, \frac{p}{q}\}.$$

Proof. By Lemma 3.15, there exists an invertible matrix $R_0 = [r_{i,j}] \in \mathbb{M}_q(\mathbb{C})$ so that $r_{i,j} \in \{0,1\}$ for all $1 \leqslant i,j \leqslant q$ and $\sum_{j=1}^{q} r_{i,j} = p$, $1 \leqslant i \leqslant q$. Let $R = R_0 \oplus I_{n-q} \in \mathbb{M}_n(\mathbb{C})$ and observe that R is still invertible, and all of its entries are either 0 or 1.

By Lemma 3.2, the semigroup $\mathcal{S} = \{E_{i,j}R : 1 \leqslant i,j \leqslant n\} \cup \{0\}$ is irreducible.

Let $\xi = \frac{1}{\sqrt{q}}(1,1,\ldots,1,0,0,\ldots,0)^t$, where the entry 1 occurs exactly q times.

Obviously $\langle 0\xi, \xi\rangle = 0 \in \{0, \frac{p}{q}\}$, while for $0 \neq S \in \mathcal{S}$,

$$\begin{aligned}
\langle S\xi, \xi\rangle &= \langle E_{i,j}R\xi, \xi\rangle \\
&= \langle R\xi, E_{j,i}\xi\rangle \\
&= \langle p\xi, E_{j,i}\xi\rangle \in \{0, \frac{p}{q}\}.
\end{aligned}$$

□

3.17. Example. Let $n \geqslant 2$. For each $1 \leqslant k \leqslant n-1$, there exists an irreducible semigroup $\mathcal{S} \subseteq \mathbb{M}_n(\mathbb{C})$ so that $\operatorname{rank} \mathcal{S} \leqslant 1$ and $\mathcal{S}$ possesses an admissible set of the form $\Omega = \{0, -k\}$.

Indeed, as always, we denote the standard orthonormal basis for $\mathbb{C}^n$ by $\{e_j : 1 \leqslant j \leqslant n\}$. Let $\mathcal{X} = \{x_1, e_2, e_3, \ldots, e_n\}$ and $\mathcal{Y} = \{y_1, e_2, e_3, \ldots, e_n\}$, where

$$\begin{aligned}
x_1 &= -ke_1 + e_2 + e_3 + \cdots + e_{k+1} \quad \text{and} \\
y_1 &= e_1 + e_2 + \cdots + e_{k+1}.
\end{aligned}$$

Observe that for all $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ we have that $\langle x, y\rangle \in \{0,1\}$ and so

$$\mathcal{S} := \{x \otimes y^* : x \in \mathcal{X}, y \in \mathcal{Y}\} \cup \{0\}$$

is a semigroup with $\operatorname{rank} \mathcal{S} \leqslant 1$. Since $\mathcal{X}$ and $\mathcal{Y}$ each form bases for $\mathbb{C}^n$, it follows that $\mathcal{S}$ is irreducible.

Letting $\xi := e_1$ shows that ξ is an admissible vector for $\mathcal{S}$ with $\Omega_\xi = \{0, -k\}$.

Of particular interest to us is the fact that by setting $n = 2$ and $k = 1$, we deduce that $\mathcal{S}_{-1} := \{\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\} \cup \{0\}$ is an irreducible semigroup in $\mathbb{M}_2(\mathbb{C})$ consisting of operators of rank at most one for which $\xi = e_1$ is an admissible vector with $\Omega_\xi = \{0, -1\}$.

3.18. Proposition. *For each $0 \neq q \in \mathbb{Q}$, there exist $n \geqslant 2$ an integer and $\mathcal{S} \subseteq \mathbb{M}_n(\mathbb{C})$ an irreducible semigroup satisfying* $\operatorname{rank} \mathcal{S} \leqslant 1$ *such that $\mathcal{S}$ possesses an admissible set of the form $\Omega = \{0, q\}$.*

Proof. Suppose first that $q < 0$, and write $q = -\frac{a}{b}$, where $a, b \in \mathbb{N}$. Choose a positive integer k so that $0 < \frac{q}{-k} = \frac{a}{bk} < \frac{1}{3}$. It follows from Proposition 3.16 that there exists an irreducible semigroup $\mathcal{S}_1 \subseteq \mathbb{M}_{bk}(\mathbb{C})$ with $\operatorname{rank} \mathcal{S}_1 \leqslant 1$ and an admissible vector - say ξ_1 for $\mathcal{S}_1$ - so that $\Omega_{\xi_1} = \{0, \frac{a}{bk}\} = \{0, -\frac{q}{k}\}$. (Note: the fact that $\frac{a}{bk} < \frac{1}{3}$ forces $a \leqslant (bk) - 2$.)

As we saw in Example 3.17, there exists an irreducible semigroup $\mathcal{S}_2 \subseteq \mathbb{M}_{k+1}(\mathbb{C})$ with rank $\mathcal{S}_2 \leqslant 1$ and an admissible vector - say ξ_2 for $\mathcal{S}_2$ - so that $\Omega_{\xi_2} = \{0, -k\}$.

It is a standard fact and routine to verify that the tensor product of two irreducible semigroups of matrices is again an irreducible semigroup, and the fact that rank $\mathcal{S}_i \leqslant 1$, $i = 1, 2$ implies that $\mathcal{S} := \mathcal{S}_1 \otimes \mathcal{S}_2 \subseteq \mathbb{M}_{bk}(\mathbb{C}) \otimes \mathbb{M}_{k+1}(\mathbb{C}) \simeq \mathbb{M}_{bk(k+1)}(\mathbb{C})$ satisfies rank $\mathcal{S} \leqslant 1$.

Finally, letting $\xi := \xi_1 \otimes \xi_2$ shows that ξ is an admissible vector for $\mathcal{S}$ and $\Omega_\xi = \{0, (-k)\frac{q}{-k}\} = \{0, q\}$. Let us denote this semigroup by $\mathcal{S}_q$.

If $q > 0$, then let $\mathcal{T} = \mathcal{S}_{-1} \otimes \mathcal{S}$ be the irreducible semigroup of operators of rank at most one obtained by tensoring the semigroup $\mathcal{S}_{-1}$ constructed at the end of Example 3.17 whose admissible set relative to e_1 is $\{0, -1\}$ with the example $\mathcal{S}_q \subseteq \mathbb{M}_{bk(k+1)}(\mathbb{C})$ constructed immediately above and whose admissible set is $\{0, -q\}$. Again, the tensor product of irreducible semigroups is irreducible, the tensor product of two semigroups of operators of rank at most one still consists of operators of rank at most one, and the vector $\eta := e_1 \otimes \xi$ is easily seen to be an admissible vector for $\mathcal{T}$ with $\Omega_\eta = \{0, q\}$.

$\square$

3.19. Next we turn our attention to the case of admissible sets of irreducible semigroups of rank-one operators. Note that such semigroups cannot contain any nilpotents, for otherwise they would contain 0, which is clearly not of rank one. We have seen that if $\Omega = \{\omega_1, \omega_2\}$ is such a set, then $\omega_1 = -\omega_2 \in \mathbb{Q}$.

3.20. Example. For each $n \geqslant 2$, there exists an irreducible semigroup $\mathcal{S} \subseteq \mathbb{M}_n(\mathbb{C})$ of rank-one operators for which $\Omega = \{-1, 1\}$ is an admissible set.

Proof. For each $1 \leqslant i \leqslant n$, set $x_i = e_1 + e_2 + \cdots + e_i$. For each $1 \leqslant j \leqslant n$, set $y_j = -e_1 + 2e_j$. Then $\mathcal{X} = \{x_i : 1 \leqslant i \leqslant n\}$ and $\mathcal{Y} = \{y_j : 1 \leqslant j \leqslant n\}$ are easily seen to be bases for $\mathbb{C}^n$, and $\langle x_i, y_j \rangle \in \{-1, 1\}$ for all $1 \leqslant i, j \leqslant n$, from which we deduce that $\mathcal{S} = \{\pm\, x_i \otimes y_j^* : 1 \leqslant i, j \leqslant n\}$ is an irreducible semigroup in $\mathbb{M}_n(\mathbb{C})$.

Let $\xi := e_1$. Clearly $\langle S\xi, \xi \rangle = \pm \langle x_i, e_1 \rangle \langle e_1, y_j \rangle \in \{-1, 1\}$ for all $S \in \mathcal{S}$, i.e. $\Omega_\xi = \{-1, 1\}$.

$\square$

3.21. We will now give an inductive construction of admissible sets of cardinality 2. Let W be as in Theorem 3.7, i.e. an invertible square matrix of order n, whose entries belong to $\Gamma = \{-1, 1\}$. From that Theorem, we see that if $\omega = \langle W^{-1}\mathbb{1}, \mathbb{1} \rangle^{-1}$, then there exists an irreducible semigroup $\mathcal{S}$ of rank-one operators in $\mathbb{M}_n(\mathbb{C})$ for which $\Omega = \{-\omega, \omega\}$ is an admissible set for $\mathcal{S}$.

Let $\gamma \in \mathbb{C}^n$ be a column whose entries belong to Γ as well. We will write $\gamma \in \Gamma^n$ and denote by D_γ the diagonal matrix whose diagonal entries are precisely those of γ (as an ordered set). Note that $D_\gamma W$ is again an invertible matrix whose entries belong to Γ, so that for $\omega_\gamma^{-1} := \langle (D_\gamma W)^{-1}\mathbb{1}, \mathbb{1} \rangle$, we have that again $\Omega_\gamma = \{-\omega_\gamma, \omega_\gamma\}$ is admissible for some irreducible semigroup $\mathcal{S}_\gamma$ of rank-one operators. Since $D_\gamma = D_\gamma^{-1} = D_\gamma^*$, we have $\langle (D_\gamma W)^{-1}\mathbb{1}, \mathbb{1} \rangle = \langle W^{-1}\mathbb{1}, (D_\gamma^{-1})^*\mathbb{1} \rangle = \langle W^{-1}\mathbb{1}, \gamma \rangle$. Similar conclusions are valid if we multiply W on either the right or on both the left and the right by a diagonal matrix with diagonal entries from $\{-1, 1\}$.

We introduce some notation. It turns out that we need to introduce four types of admissible points. However, to simplify the notation, we will introduce the inverses of these points. For columns $\gamma, \delta \in \Gamma^n$ we define the *neutral*, respectively the *left*, respectively the *right*, respectively the *two-sided inverse-admissible point*:

$$\begin{aligned}
\omega(W) &= \langle W^{-1}\mathbb{1}, \mathbb{1}\rangle, \\
\omega(\gamma, W) &= \langle W^{-1}\gamma, \mathbb{1}\rangle, \\
\omega(W, \delta) &= \langle W^{-1}\mathbb{1}, \delta\rangle, \\
\omega(\gamma, W, \delta) &= \langle W^{-1}\gamma, \delta\rangle.
\end{aligned}$$

Note that there is only a formal distinction between different inverse-admissible points. Namely, if we let $\widehat{W} = D_\gamma W D_\delta$, then $\omega(\widehat{W}) = \omega(\gamma, W, \delta)$. Next, write the matrix W in the block partition with respect to the first $n-1$ rows, respectively columns, and the last row, respectively column, as

$$W = \begin{pmatrix} S & \gamma \\ \delta^* & \kappa \end{pmatrix}. \tag{2}$$

Using standard linear algebra computations we see that

$$W^{-1} = \begin{pmatrix} S^{-1} + (\kappa - \omega(\gamma, S, \delta))^{-1} S^{-1}\gamma\delta^* S^{-1} & -(\kappa - \omega(\gamma, S, \delta))^{-1} S^{-1}\gamma \\ -(\kappa - \omega(\gamma, S, \delta))^{-1}\delta^* S^{-1} & (\kappa - \omega(\gamma, S, \delta))^{-1} \end{pmatrix}, \tag{3}$$

provided that S is invertible. Observe that this assumption may be fulfilled in the following way. Since W is invertible, its submatrix made of the first $n-1$ columns is of full rank. Consequently, there are $n-1$ rows of W such that the corresponding submatrix is of full rank. Now we can apply a permutation matrix on W from the left-hand side to get S in partition (2) invertible. Note that it is also possible to make S invertible by applying a permutation matrix on W from the right-hand side. In either case the neutral admissible point $\omega(W)$ does not change, while for the other points we have to apply the same permutation on the corresponding diagonal.

Choose now $\eta, \zeta \in \Gamma^{n-1}$ and define for $r, t \in \Gamma$ the n-tuples $\hat{\eta}_r = (\eta, r) \in \Gamma^n$ and $\hat{\zeta}_t = (\zeta, t) \in \Gamma^n$. From Formula (3) we get easily:

$$\begin{aligned}
\omega(W) &= \omega(S) + (\kappa - \omega(\gamma, S, \delta))^{-1}(\omega(\gamma, S) - 1)(\omega(S, \delta) - 1), \\
\omega(\hat{\eta}_r, W) &= \omega(\eta, S) + (\kappa - \omega(\gamma, S, \delta))^{-1}(\omega(\gamma, S) - 1)(\omega(\eta, S, \delta) - r), \\
\omega(W, \hat{\zeta}_t) &= \omega(S, \zeta) + (\kappa - \omega(\gamma, S, \delta))^{-1}(\omega(\gamma, S, \zeta) - t)(\omega(S, \delta) - 1), \\
\omega(\hat{\eta}_r, W, \hat{\zeta}_t) &= \omega(\eta, S, \zeta) + (\kappa - \omega(\gamma, S, \delta))^{-1}(\omega(\gamma, S, \zeta) - t)(\omega(\eta, S, \delta) - rt).
\end{aligned} \tag{4}$$

3.22. Proposition. *Equations* (4) *give recursive formulas for obtaining inverses of admissible points dimension by dimension and all the admissible points can be obtained in this way.*
Proof. Clear.

□

3.23. If we limit ourselves to more concrete cases of matrices W, we can get even better insight into what admissible points may be at the expense of generality of the result. In the following proposition we consider matrices W to be of a special kind that is called in the literature **circulant**.

Let C be the *basic circulant matrix* sending standard basis vectors e_i to e_{i-1} for $i = 2, 3, \ldots, n$ and e_1 to e_n. For any polynomial $\phi(x) \in \mathbb{C}[x]$ we can define $V_\phi = \phi(C)$. The matrices so obtained are called *circulant matrices*. Since $C^n = I$ we may view these polynomials as members of $\mathbb{C}[x]/(x^n - 1)$. Note that the entries of $R = V_\phi$ belong to $\Gamma = \{-1, 1\}$ if and only if the polynomial ϕ (which we may assume, without loss of generality, to be of degree no more than $n-1$) has coefficients from Γ. The question of whether W is invertible is clearly equivalent to the question of whether ϕ is invertible in the ring $\mathbb{C}[x]/(x^n-1)$. Now, using elementary ring theory we know that this is equivalent to the question of whether or not ϕ is coprime to x^n-1. If this is so, the inverse of W is obtained as V_ψ, where ψ is the unique element of the quotient $\mathbb{C}[x]/(x^n-1)$ such that $\psi(x)\phi(x) \equiv 1\ (x^n-1)$.

3.24. Proposition. *For any integers j,k such that $0 \leqslant j < \frac{n}{2}$ and $0 < k < \frac{n}{2}$ is coprime to n, it holds that*

$$\Omega = \{\pm\frac{n-2k}{n-2j}\}$$

is an admissible set for some semigroup $\mathcal{S}$ of rank-one operators in $\mathbb{M}_n(\mathbb{C})$.
Proof. Let j,k be as above and define

$$\phi(x) = -(1+x+\cdots+x^{k-1}) + x^k(1+x+\cdots+x^{n-k-1}).$$

We want to show that ϕ is coprime to x^n-1. Now, $\phi(1) = n-2k \neq 0$ so that it suffices to show that ϕ is coprime to $\theta_0(x) = 1+x+\cdots+x^{n-1}$. Assume not. Then, they have a nontrivial common factor with their sum and consequently a nontrivial common factor with $\theta_1(x) = 1+x+\cdots+x^{n-k-1}$. So, this factor divides the polynomial $\theta_2(x) = 1+x+\cdots+x^{k-1}$ as well. Let ρ be a zero of this factor. It is clear that it is both a k^{th} root of unity and an n^{th} root of unity. So, its nontrivial order divides both k and n contradicting the assumption that they are coprime. These considerations imply that no n^{th} root of unity is a zero of ϕ, so that there exists a ψ in $\mathbb{C}[x]/(x^n-1)$ such that $\psi(x)\phi(x) \equiv 1\ (x^n-1)$. Finally, $W^{-1} = V_\psi$ and a short computation reveals that

$$\langle W^{-1}\mathbb{1}, \mathbb{1}\rangle = n\psi(1) = n\phi(1)^{-1} = \frac{n}{n-2k}.$$

Here we used the fact that the leftmost expression above equals the sum of all entries of W^{-1}. We can compute this sum cycle by cycle, each of them being equal to n times the corresponding coefficient of the polynomial ψ and the desired result follows. Now, if we replace one of the two columns $\mathbb{1}$ with a column γ containing j entries equal to -1 and $n-j$ entries equal to 1, we get n in this formula replaced by $n-2j$, as was to be shown.

$\square$

3.25. Example.

(a) For each of the sets $\Omega_1 = \{-3,3\}, \Omega_2 = \{-1,1\}, \Omega_3 = \{-\frac{3}{5},\frac{3}{5}\}, \Omega_4 = \{-\frac{1}{3},\frac{1}{3}\}$ and $\Omega_5 = \{-\frac{1}{5},\frac{1}{5}\}$, there exists an irreducible semigroup $\mathcal{S}_k$ of rank-one operators in $\mathbb{M}_5(\mathbb{C})$ for which Ω_k is an admissible set, $1 \leqslant k \leqslant 5$.
(b) With Ω_k as above, $1 \leqslant k \leqslant 5$, set $\Omega_6 = \{-2,2\}$, $\Omega_7 = \{-\frac{2}{3},\frac{2}{3}\}$. Then for each $1 \leqslant k \leqslant 7$, there exists an irreducible semigroup $\mathcal{T}_k \subseteq \mathbb{M}_6(\mathbb{C})$ of rank-one operators for which Ω_k is an admissible set.

Proof.

(a) This follows directly from Proposition 3.24 by letting $n=5, k=1,2$ and $j=0,1,2$.
(b) In the case where the dimension of the underlying space is 6, the index $k=2$ of Proposition 3.24 is not coprime to $n=6$. So, while we can get Ω_6, Ω_2 and Ω_7 in this way, the sets $\{-\frac{2}{6},\frac{2}{6}\}, \{-\frac{2}{4},\frac{2}{4}\}$, and $\{-\frac{2}{2},\frac{2}{2}\}$ cannot be obtained via circulant matrices. Indeed, the circulant matrix containing 4 consecutive cycles of 1's and 2

cycles of -1's has zero determinant because -1 is a zero of the polynomial $\phi(x) = 1 + x + x^2 + x^3 - x^4 - x^5$. So, we apply Proposition 3.22 instead.

Let S_k denote the circulant matrix corresponding to the polynomial ϕ as in the proof of Proposition 3.24, for $k = 1, 2$, and let γ_j be the 5-tuple made of j entries equal to -1 and $5 - j$ entries equal to 1 for $j = 0, 1, 2$. Define W using Formula (2) with $S = \gamma_j S_k$, $\gamma = D_{\gamma_j}\gamma_k$, $\delta = \mathbb{1}$ (for $k = 1, 2$ and $j = 0, 1, 2$) and let $\kappa \in \Gamma$ be chosen to be different from $\omega(\gamma, S, \delta)$ so that W is invertible. Using the first one of the Formulas (4) we get that all points of item (a) are admissible in dimension $n = 6$ as well.

□

3.26. Theorem. *For each $0 \neq q \in \mathbb{Q}$ there exists $n \in \mathbb{N}$ and an irreducible semigroup $\mathcal{S} \subseteq \mathbb{M}_n(\mathbb{C})$ of rank-one operators for which $\Omega = \{-q, q\}$ is an admissible set.*

Proof.

- First let $m_1 \geqslant 1$ be an integer and set $n_1 = 2(m_1 + 1)$. Let $k_1 = 1$ (clearly $gcd(k_1, n_1) = 1$) and $j_1 = m_1$, so that $0 \leqslant j_1 \leqslant \frac{n_1}{2}$. Observe that
$$\frac{n_1 - 2k_1}{n_1 - 2j_1} = m_1.$$
By Proposition 3.24, there exists an irreducible semigroup $\mathcal{S}_1 \subseteq \mathbb{M}_{n_1}(\mathbb{C})$ of rank-one operators for which $\Omega_1 = \{-m_1, m_1\}$ is an admissible set. Let $\xi_1 \in \mathbb{C}^{n_1}$ be a corresponding admissible unit vector.
- Let $m_2 \in \mathbb{N}$ be an even integer and $n_2 = 2m_2$. Set $k_2 = m_2 - 1$ and $j_2 = 0$. Observe that $gcd(k_2, n_2) = gcd(m_2 - 1, 2m_2) = 1$, and that
$$\frac{n_2 - 2k_2}{n_2 - 2j_2} = \frac{1}{m_2},$$
so that we may once again apply Proposition 3.24 to conclude that there exists an irreducible semigroup $\mathcal{S}_2 \subseteq \mathbb{M}_{n_2}(\mathbb{C})$ of rank-one operators for which $\Omega_2 = \{-\frac{1}{m_2}, \frac{1}{m_2}\}$ is an admissible set. Let $\xi_2 \in \mathbb{C}^{n_2}$ be a corresponding admissible vector.

Consider $\mathcal{S} = \{S_1 \otimes S_2 : S_1 \in \mathcal{S}_1, S_2 \in \mathcal{S}_2\}$, so that $\mathcal{S}$ is an irreducible semigroup of rank-one operators acting on $\mathcal{H} := \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2}$. Let $\xi = \xi_1 \otimes \xi_2$, so that ξ is a unit vector in $\mathcal{H}$. It is routine to verify that for each $S_1 \otimes S_2 \in \mathcal{S}$ we have
$$\langle (S_1 \otimes S_2)(\xi_1 \otimes \xi_2), (\xi_1 \otimes \xi_2) \rangle = \langle S_1\xi_1, \xi_1 \rangle \, \langle S_2\xi_2, \xi_2 \rangle \in \{-\frac{m_1}{m_2}, \frac{m_1}{m_2}\}.$$
Hence $\Omega = \{-\frac{m_1}{m_2}, \frac{m_1}{m_2}\}$ is an admissible set for $\mathcal{S}$.

Finally, given any $0 < q \in \mathbb{Q}$, it is clear that we may write $q = \frac{m_1}{m_2}$ for some $m_1 \geqslant 1$ and $m_2 \geqslant 1$ with m_2 even, which completes the proof.

□

3.27. Remark. We remark that some results in this section remain valid for sets Ω of cardinality p, where p is a prime number. The proofs of Proposition 3.6, Theorem 3.7, remark 3.21 and Proposition 3.22 work, with little change, in this more general context. The set Γ in these statements becomes $\{\alpha, \alpha^2, \alpha^3, \ldots, \alpha^p = 1\}$, where α is a primitive p-th root of unity, and the set Ω is still of the form $\omega\Gamma$, for some $\omega \in \mathbb{C}$.

Proposition 3.24 is stated slightly differently for an admissible set Ω of cardinality p. The polynomial ϕ in the proof of Proposition 3.24 can be replaced with the polynomial

$$\varphi(x) = \alpha(1 + x + \cdots + x^{k-1}) + x^k(1 + x + \cdots + x^{n-k-1}),$$

which produces the admissible set $\Omega = \omega\Gamma$, where $\Gamma = \{\alpha, \alpha^2, \alpha^3, \ldots, \alpha^p = 1\}$ and $\omega = \frac{n+k(\alpha-1)}{n+j(\alpha-1)}$.

The analogue of Theorem 3.12 is no longer true if $|\Omega| > 2$. This follows from the generalized Proposition 3.24: one can produce n, k, j in the proposition such that $\omega = \frac{n+k(\alpha-1)}{n+j(\alpha-1)}$ is not rational.

4. Semigroups of invertible operators

4.1. In this section, we look at irreducible semigroups of $n \times n$ matrices which consist of invertible operators. Since our interest lies in irreducible semigroups $\mathcal{S}$ which possess an admissible semigroup of cardinality two, it follows from Lemma 1.5 (a) that such an $\mathcal{S}$ must be finite. But a finite semigroup consisting of invertible operators is easily seen to be a group. Indeed, if $S \in \mathcal{S}$ and the latter is finite, then there exist $k_1 < k_2$ so that $S^{k_1} = S^{k_2}$. But then $S^{k_1}(S^{k_2-k_1} - I) = 0$, and since S^{k_1} is invertible, we have that $S^{k_2-k_1} = I \in \mathcal{S}$. Furthermore, $S^{-1} = S^{k_2-k_1-1} \in \mathcal{S}$.

It is well-known that every finite group of invertible matrices in $\mathbb{M}_n(\mathbb{C})$ is simultaneously similar to a group of unitaries. We shall begin our analysis of groups of invertible matrices possessing two-element admissible sets by considering groups of unitaries.

4.2. By an **open half-space** of $\mathbb{C}$ we shall mean a set of the form $H_\alpha = \{z \in \mathbb{C} : \mathrm{Re}\,(\alpha z) > 0\}$, where $0 \neq \alpha \in \mathbb{C}$ is a constant. Recall that every locally compact group $\mathcal{G}$ admits a (positive, left-translation invariant) **Haar measure**, which we shall denote by ν. This measure is unique up to scaling by a positive real number.

Note that if $n \geqslant 2$ is an integer and $\mathcal{G} \subseteq \mathbb{M}_n(\mathbb{C})$ is a finite irreducible group, then $\sum_{G\in\mathcal{G}} G = 0$. Indeed, if $T = \sum_{G\in\mathcal{G}} G \neq 0$, then by choosing $x \in \mathbb{C}^n$ so that $Tx \neq 0$, we see that $GT = T$ and hence $GTx = Tx$ for all $G \in \mathcal{G}$, implying that $\mathbb{C}(Tx)$ is a non-trivial invariant subspace for $\mathcal{G}$, a contradiction.

4.3. Proposition. *Let $\mathcal{S} \subseteq \mathbb{M}_n(\mathbb{C})$ be a compact group of unitary matrices, and let Ω be an admissible set for $\mathcal{S}$ with a corresponding admissible norm-one vector ξ. If there exists an open half-space $H \subseteq \mathbb{C}$ so that $\Omega \subseteq H \cup \{0\}$, then $\mathcal{S}$ is reducible.*

Proof. Let ν denote normalized Haar measure on $\mathcal{S}$, (which is positive and whose support is $\mathcal{S}$), and let

$$T := \int_{\mathcal{S}} S d\nu.$$

We claim that $T \neq 0$. Indeed, suppose otherwise and consider the non-zero linear functional $\varphi : \mathbb{M}_n(\mathbb{C}) \to \mathbb{C}$ defined by $\varphi(X) = \langle X\xi, \xi\rangle$. Then

$$\begin{aligned} 0 &= \varphi(T) \\ &= \langle T\xi, \xi\rangle \\ &= \int_{\mathcal{S}} \langle S\xi, \xi\rangle d\nu. \end{aligned}$$

Fix $0 \neq \alpha \in \mathbb{C}$ so that $H = H_\alpha = \{z \in \mathbb{C} : \mathrm{Re}\,\alpha z > 0\}$. Now $\langle S\xi, \xi\rangle \in H \cup \{0\}$ for all $S \in \mathcal{S}$, and so $\mathrm{Re}\,\alpha\varphi(S) = \mathrm{Re}\,\alpha\langle S\xi, \xi\rangle \geqslant 0$ for all $S \in \mathcal{S}$. In particular, $I \in \mathcal{S}$ (as $\mathcal{S}$ is a group), and thus $1 = \varphi(I_n) = \langle I_n\xi, \xi\rangle \in \Omega \subseteq H$; i.e. $\operatorname{Re}\alpha > 0$. Since φ is continuous, there exists an open subset $\mathcal{T} \subseteq \mathcal{S}$ so that $I \in \mathcal{T}$ and $\operatorname{Re}\alpha\varphi(S) > \frac{1}{2}\operatorname{Re}\alpha$ for all $S \in \mathcal{T}$. But then $\nu(\mathcal{T}) > 0$ and so

$$\operatorname{Re}\alpha\langle T\xi, \xi\rangle = \operatorname{Re}\alpha\langle \left(\int_{\mathcal{S}} S d\nu\right)\xi, \xi\rangle = \int_{\mathcal{S}} \operatorname{Re}\alpha\langle S\xi, \xi\rangle d\nu > 0,$$

contradicting the fact that $T = 0$.

Hence $T \neq 0$. Choose $x \in \mathbb{C}^n$ so that $Tx \neq 0$. Observe that by left-invariance of Haar measure, $ST = T$ for all $S \in \mathcal{S}$, and thus $STx = Tx \neq 0$ is a fixed point of $\mathcal{S}$. In particular, $\mathbb{C}x$ is an invariant subspace for $\mathcal{S}$, and so $\mathcal{S}$ is reducible.

□

The above Proposition admits the following simple Corollary. We shall improve upon it below. Note that for any group $\mathcal{G} \subseteq \mathbb{M}_n(\mathbb{C})$ and for any admissible set Ω for $\mathcal{G}$, the fact that $I_n \in \mathcal{G}$ implies that $1 \in \Omega$. Also, a closed subset $\Omega \subseteq \mathbb{C}$ is an admissible set for $\mathcal{G}$ if and only if Ω is an admissible set for $\overline{\mathcal{G}}$.

4.4. Corollary. *Let $n \geqslant 1$ be an integer, and let $\mathcal{G} \subseteq \mathbb{M}_n(\mathbb{C})$ be an irreducible group of unitaries. If $\Omega = \{1, \omega_2\}$ is an admissible set for $\mathcal{G}$, then $\omega_2 < 0$.*

Proof. Since $\overline{\mathcal{G}}$ is an irreducible group of unitaries, it follows from Proposition 4.3 that Ω - being an admissible set for $\overline{\mathcal{G}}$ - can not be contained in $H \cup \{0\}$ for any open half-space H. Given that $\Omega = \{1, \omega_2\}$, the only way to avoid this is if $\omega_2 < 0$.

□

We will now demonstrate that for each integer $n \geqslant 2$, there exists an irreducible group $\mathcal{G} \subseteq \mathbb{M}_n(\mathbb{C})$ of unitary matrices which possesses an admissible set Ω of cardinality two. As we shall see, this condition is rather rigid, and to a large extent it determines the structure of the group $\mathcal{G}$.

If $n \geq 2$ and $\mathcal{A} \subseteq \mathbb{M}_n(\mathbb{C})$, we denote by $\langle A\rangle$ the group generated by $\mathcal{A}$.

4.5. Example. Let $n = 2$ and consider the group

$$\mathcal{E}_2 = \langle \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} \rho & 0 \\ 0 & \rho^2 \end{bmatrix} \rangle \subseteq \mathbb{M}_2(\mathbb{C}),$$

where $\rho = e^{\frac{2\pi i}{3}}$ is the cube root of unity in $\mathbb{C}$. Then

$$\mathcal{E}_2 = \{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \rho & 0 \\ 0 & \rho^2 \end{bmatrix}, \begin{bmatrix} \rho^2 & 0 \\ 0 & \rho \end{bmatrix}, \begin{bmatrix} 0 & \rho \\ \rho^2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \rho^2 \\ \rho & 0 \end{bmatrix} \}.$$

Then relative to the distinguished vector $z = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$, we find that

$$\{\langle Gz, z\rangle : G \in \mathcal{E}_2\} = \{-\frac{1}{2}, 1\}.$$

This can be seen by expressing each element of the group with respect to the orthonormal basis $\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \}$, with respect to which we find that

$$\mathcal{E}_2 = \{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \}.$$

4.6. Example. Let $n = 2$, and recall that for $k \geqslant 2$ we use $\mathcal{P}_k$ to denote the permutation unitary matrices in $\mathbb{M}_k(\mathbb{C})$. Then the matrix $C_3 := \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ lies in $\mathcal{P}_3$, and C_3 is a normal matrix with eigenvalues $\{1, \rho, \rho^2\}$, where ρ is the third root of unity, $e^{\frac{2\pi i}{3}}$.

The norm-one eigenvector corresponding to the eigenvalue 1 (for every permutation matrix $P \in \mathcal{P}_3$ is $y = \frac{1}{\sqrt{3}} \mathbb{1}_3$. Let $x_1 = \frac{1}{\sqrt{3}}(1, \rho^2, \rho)^t$ and $x_2 = \frac{1}{\sqrt{3}}(1, \rho, \rho^2)^t$, so that x_1 and x_2 are two norm-one eigenvectors for C_3 corresponding to the eigenvalues ρ and ρ^2 respectively.

Relative to the orthonormal basis $\{\mathbb{1}_3, x_1, x_2\}$, we see that $C_3 = \operatorname{diag}(1, \rho, \rho^2)$ and that every $P \in \mathcal{P}_3$ may be written as

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & p_{22} & p_{23} \\ 0 & p_{32} & p_{33} \end{bmatrix}.$$

Next, observe that if $Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, then $Q \in \mathcal{P}_3$, and $Qx_1 = x_2$, $Qx_2 = x_1$, from which we find that relative to the orthonormal basis $\{y, x_1, x_2\}$, we *still have*

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

For each $P \in \mathcal{P}_3$, let G_P denote the compression of P to $\mathbb{1}^\perp$. Thus $\mathcal{G}_2 = \{G_P : P \in \mathcal{P}_3\}$ contains $\begin{bmatrix} \rho & 0 \\ 0 & \rho^2 \end{bmatrix}$ as well as $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Since $\mathcal{G}_2$ is a group, we see that it must contain the group $\mathcal{E}_2$ defined in Example 4.5 above. But $|\mathcal{G}_2| = |\mathcal{P}_3| = 3! = 6 = |\mathcal{E}_2|$, whence $\mathcal{G}_2 = \mathcal{E}_2$.

This proves that there exists a unique irreducible group of unitary matrices in $\mathbb{M}_2(\mathbb{C})$ which possesses an admissible set of cardinality two. This is not entirely surprising. There are only two groups of order 6, and one of them is abelian. Since an abelian group in $\mathbb{M}_n(\mathbb{C})$ is never irreducible, it is clear that if we can find an irreducible subgroup of $\mathbb{M}_2(\mathbb{C})$ of order 6, it must be isomorphic to any other such group, and so $\mathcal{G}_2$ and $\mathcal{E}_2$ are isomorphic.

4.7. Example.

(a) Let $n = 3$ and

$$\mathcal{E}_3^0 = \langle \begin{bmatrix} -1 & & \\ & -1 & \\ & & 1 \end{bmatrix}, \mathcal{P}_3 \rangle,$$

where $\mathcal{P}_3$ denotes the set of all 3×3 permutation matrices. Then $\mathcal{E}_3^0$ consists of all weighted permutations, where the weights lie in $\{-1, 1\}$, and for which there is an even number of -1's. Of these there are $|\mathcal{P}_3| = 6$ where all of the weights are 1, and also $|\mathcal{P}_3| = 6$ weighted permutations where two of the three weights are -1 and the other is 1. Thus $|\mathcal{E}_3^0| = 12$.

Let $\xi = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ be our distinguished unit vector. Then

$$\{\langle G\xi, \xi\rangle : G \in \mathcal{E}_3^0\} = \{-\frac{1}{3}, 1\}.$$

(b) Let $n = 3$. Let $\mathcal{P}_4 \subseteq \mathbb{M}_4(\mathbb{C})$ denote the group of permutation matrices described in the introduction. Recall that $\mathcal{N} := \mathbb{1}^\perp \subseteq \mathbb{C}^4$ is an orthogonally reducing subspace for $\mathcal{P}_4$. Let $\mathcal{E}_3 := \mathcal{P}_4|_{\mathcal{N}}$. Since each element $P \in \mathcal{P}_4$ admits the matrix decomposition $\begin{bmatrix} 1 & 0 \\ 0 & P_{\mathcal{N}} \end{bmatrix}$ relative to the decomposition $\mathbb{C}^4 = \mathbb{C}\mathbb{1} \oplus \mathcal{N}$, it follows that $\mathcal{E}_3 := \{P_{\mathcal{N}} : P \in \mathcal{P}_4\}$ is a group of unitary matrices.

It is a routine exercise to check that $\mathcal{E}_3 \subseteq \mathcal{B}(\mathcal{N}) \simeq \mathbb{M}_3(\mathbb{C})$ is irreducible. Let $\xi := \frac{1}{\sqrt{12}} \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. For $P \in \mathcal{P}_4$,

$$P\xi \in \{\frac{1}{\sqrt{12}} \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{12}} \begin{bmatrix} 1 \\ -3 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{12}} \begin{bmatrix} 1 \\ 1 \\ -3 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{12}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ -3 \end{bmatrix}\}.$$

Hence $\langle P\xi, \xi\rangle \in \Omega := \{1, -\frac{1}{3}\}$.

It follows that there are at least two non-isomorphic, irreducible groups $\mathcal{E}_3^0$ and $\mathcal{E}_3$ of unitary matrices in $\mathbb{M}_3(\mathbb{C})$ for which $\Omega = \{1, -\frac{1}{3}\}$ is an admissible set.

The group $\mathcal{E}_3$ constructed above is actually prototypical of a class of groups of unitaries which possess an admissible set of cardinality two, as we now demonstrate.

4.8. Proposition. *Let $n \geqslant 1$ and consider the inner product space $(\mathbb{K}^n, \langle\cdot,\cdot\rangle)$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $u_1, u_2, \ldots, u_m \in \mathbb{K}^n$ be distinct norm-one vectors, and suppose that there exists $r \in \mathbb{K}$ so that*

$$\langle u_i, u_j\rangle = r \qquad \textit{for all} \qquad 1 \leqslant i \neq j \leqslant m.$$

Then $m \leqslant n+1$ and if $m = n+1$, then $r = -\frac{1}{n}$.

Proof. First note that since the norm of each u_k is one, and since the vectors are distinct, we have that $r \neq 1$ unless $m = 1$, a trivial case. For the rest of the argument, therefore, we shall assume that $m \geqslant 2$ and hence $r \neq 1$.

Fix $s > 1$, and suppose that $\alpha_1, \alpha_2, \ldots, \alpha_s \in \mathbb{K}$ are chosen so that $\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_s u_s = 0$. Then for each $1 \leqslant k \leqslant s$, we have

$$0 = \langle 0, u_k\rangle = \langle \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_s u_s, u_k\rangle,$$

which gives rise to a system of s equations, expressed as the single matrix equation

$$\begin{bmatrix} 1 & r & r & \cdots & r \\ r & 1 & r & \cdots & r \\ \vdots & \vdots & & \cdots & \vdots \\ r & r & \cdots & r & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Let R_s denote the $s \times s$ matrix on the left-hand side of the equation, and observe that

$$\det R_s = (1-r)^{s-1}(1+(s-1)r).$$

Now suppose that $m > n$. By applying the above analysis with $s = m$, we see that the set $\{u_1, u_2, \ldots, u_m\}$ is obviously linearly dependent in the n-dimensional space $\mathbb{K}^n$, and thus there must exist a non-zero solution to the equation

$$R_m \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

This in turn implies that $\det R_m = 0$, which, when coupled with the fact that $r \neq 1$ from above, implies that $1 + (m-1)r = 0$, i.e. $r = -\frac{1}{m-1}$. Since $m > n$, we have that $r \neq -\frac{1}{n-1}$.

From this it follows that $\{u_1, u_2, \ldots, u_n\}$ is linearly independent in $\mathbb{K}^n$, because

$$\det R_n = (1-r)^{n-1}(1+(n-1)r) \neq 0,$$

by virtue of the fact that $r \notin \{-\frac{1}{n-1}, 1\}$.

Next suppose that $n < t \leqslant m$ and note that $\langle u_k, u_t \rangle = r$ for all $1 \leqslant k \leqslant n$. Since $\{u_1, u_2, \ldots, u_n\}$ is a basis for $\mathbb{K}^n$, we may choose $\beta_1, \beta_2, \ldots, \beta_{n+1} \in \mathbb{K}$ not all equal to zero so that

$$\beta_1 u_1 + \beta_2 u_2 + \cdots + \beta_n u_n + \beta_{n+1} u_t = 0.$$

Arguing as before, by considering the inner product of this with each vector u_k, $1 \leqslant k \leqslant n$ and then with u_t, we obtain a system of $n+1$ equations which we express as a single matrix equation:

$$R_{n+1} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{n+1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

If we subtract the first equation from the k^{th} equation, $2 \leqslant k \leqslant n+1$, we get an equation of the form:

$$(r-1)\beta_1 + 0\beta_2 + \cdots + 0\beta_{k-1} + (1-r)\beta_k + 0\beta_{k+1} + \cdots + 0\beta_{n+1} = 0,$$

from which we deduce (recall that $r \neq 1$) that $\beta_k = \beta_1$. Note that $\beta_1 \neq 0$, since otherwise $\beta_k = 0$ for all $1 \leqslant k \leqslant n+1$, contradicting our choice of β_k's.

This implies that $u_t = -\sum_{k=1}^{n} u_k$ is uniquely determined; i.e. $u_{n+1} = u_{n+2} = \cdots = u_m$. Since we originally specified that all vectors u_k, $1 \leqslant k \leqslant m$ were distinct, this can only happen if $m = n+1$ in which case $r = -\frac{1}{m-1} = -\frac{1}{n}$.

$\square$

4.9. Proposition. *Let $n \geqslant 2$ be an integer, and $\mathbb{1} = \mathbb{1}_{n+1} \in \mathbb{C}^{n+1}$. Then $\mathcal{R}_n := \mathcal{P}_{n+1}|_{\mathbb{1}^\perp}$ is an irreducible group of unitaries in $\mathcal{B}(\mathbb{1}^\perp) \simeq \mathbb{M}_n(\mathbb{C})$ for which $\Omega = \{1, -\frac{1}{n}\}$ is an admissible set.*

Proof. Let $\{e_k\}_{k=1}^{n+1}$ denote the standard orthonormal basis for $\mathbb{C}^{n+1}$. It is clear that $P\mathbb{1} = \mathbb{1}$ for all $P \in \mathcal{P}_{n+1}$, and hence the fact that $\mathcal{P}_{n+1}$ is self-adjoint shows that $\mathbb{C}\mathbb{1}$ is an orthogonally reducing subspace for $\mathcal{P}_{n+1}$. Indeed, relative to the decomposition $\mathbb{C}^{n+1} = \mathbb{C}\mathbb{1} \oplus \mathbb{1}^\perp$, $P \in \mathcal{P}_{n+1}$ implies that P has the form

$$P = \begin{bmatrix} 1 & 0 \\ 0 & R_P \end{bmatrix}.$$

By definition, we have set $\mathcal{R}_n = \{R_P : P \in \mathcal{P}_{n+1}\} \subseteq \mathcal{B}(\mathbb{1}^\perp) \simeq \mathbb{M}_n(\mathbb{C})$.

Since $\mathcal{P}_{n+1}$ and hence $\mathcal{R}_n$ is self-adjoint, any invariant subspace of $\mathcal{R}_n$ must be orthogonally reducing, and if Q_0 is an orthogonal projection in $\mathcal{B}(\mathbb{1}^\perp)$ which commutes with every element of $\mathcal{R}_n$, then

$$Q := \begin{bmatrix} 0 & 0 \\ 0 & Q_0 \end{bmatrix}$$

is an orthogonal projection which commutes with every element of $\mathcal{P}_{n+1}$. Proving that $\mathcal{R}_n$ is irreducible is equivalent to proving that $Q_0 \in \{0, I\} \subseteq \mathcal{B}(\mathbb{1}^\perp)$.

Let $Z \in \mathbb{M}_{n+1}(\mathbb{C})$ be an orthogonal projection which commutes with every element of $\mathcal{P}_{n+1}$, and for $1 \leqslant i \neq j \leqslant n+1$, let $P_{i,j}$ denote the permutation matrix in $\mathbb{M}_{n+1}(\mathbb{C})$ induced by the transposition of e_i and e_j. The computation $ZP_{i,j} = P_{i,j}Z$ for all $1 \leqslant i \neq j \leqslant n+1$ implies that

$$Z = \alpha I_{n+1} + \beta Z_1,$$

where Z_1 is the rank-one projection all of whose entries are equal. Note that the range of Z_1 is therefore $\mathbb{C}\mathbb{1}$, and hence, relative to the decomposition $\mathbb{C}^{n+1} = \mathbb{C}\mathbb{1} \oplus \mathbb{1}^\perp$, Z is given by the diagonal matrix $Z = \operatorname{diag}(\alpha + \beta, \alpha I)$. Since $Z = Z^2$, we either have $\alpha = 0$ and $\beta \in \{0, 1\}$, or $\alpha = 1$ and $\beta \in \{0, -1\}$.

Either way, for Q as above, we see that this implies that $Q_0 \in \{0, I\}$, completing the proof that $\mathcal{R}_n$ is irreducible.

Next, let $\xi = \frac{1}{\sqrt{n^2+n}}(-n, 1, 1, ..., 1)^* \in \mathbb{C}^{n+1}$. Note that $\xi \in \mathbb{1}^\perp$ and $\|\xi\| = 1$. Furthermore, thinking of $R_P \in \mathcal{B}(\mathbb{1}^\perp)$, the action of R_P upon $\xi \in \mathbb{1}^\perp$ is just the action of P on $\xi \in \mathbb{C}^{n+1}$. Thus

$$\langle R_P \xi, \xi \rangle = \langle P\xi, \xi \rangle \in \{1, -\frac{1}{n}\}$$

for all $P \in \mathcal{P}_{n+1}$, proving that $\Omega = \{1, -\frac{1}{n}\}$ is an admissible set for $\mathcal{R}_n$.

□

It is clear that if $\mathcal{G} \subseteq \mathcal{R}_n$ is an irreducible group, then $\Omega = \{1, -\frac{1}{n}\}$ is also admissible for $\mathcal{G}$. Our present goal is to prove a converse to this statement. (Note that $I \in \mathcal{G}$ implies that $1 \in \Omega$ whenever Ω is admissible for $\mathcal{G}$.)

4.10. Theorem. *Let $n \geqslant 2$ be an integer. Suppose that $\mathcal{G} \subseteq \mathbb{M}_n(\mathbb{C})$ is an irreducible group of unitary matrices and that $\Omega = \{1, \omega_2\}$ is an admissible set of cardinality two for $\mathcal{G}$. It follows that $\omega_2 = -\frac{1}{n}$, and that $\mathcal{G}$ is unitarily equivalent to a subgroup of the group $\mathcal{R}_n = \mathcal{P}_{n+1}|_{\mathbb{1}^\perp}$ defined in Proposition 4.9.*

In particular, $\mathcal{G}$ has at most $(n+1)!$ elements.

Proof. We may assume with no loss of generality that $\mathcal{G}$ is *maximal* with respect to the condition that it be an irreducible group and that it possess an admissible set $\Omega = \{1, \omega_2\}$ of cardinality two. From Corollary 4.4, we see that $\omega_2 < 0$.

Recall from Lemma 1.5 (a) that $\mathcal{G}$ must be finite, say $\mathcal{G} = \{G_k\}_{k=1}^m$. Let ξ be a unit vector for $\mathcal{G}$ corresponding to Ω, and for $1 \leqslant k \leqslant m$, set $u_k = G_k\xi$. Note that $\|u_k\| = 1$ for all $1 \leqslant k \leqslant m$. (At this stage it is very possible that $u_i = u_j$ even if $i \neq j$.) Clearly $\{u_1, u_2, \ldots, u_m\}$ is nothing more than the orbit of ξ under $\mathcal{G}$. Since $\mathcal{G}$ is irreducible, span $\{u_1, u_2, \ldots, u_m\} = \mathbb{C}^n$. In particular, $m \geqslant n$, and $\{u_1, u_2, \ldots, u_m\}$ contains a basis for $\mathbb{C}^n$. By reindexing if necessary, we may assume that $\{u_1, u_2, \ldots, u_n\}$ is such a basis.

We claim that there exists $n < k \leqslant m$ so that $u_k \notin \{u_1, u_2, \ldots, u_n\}$. For suppose otherwise; i.e., suppose that $G_k\xi \in \{u_1, u_2, ..., u_n\}$ for all $1 \leqslant k \leqslant m$. From paragraph 4.2, we know that $\sum_{k=1}^m G_k = 0$, and thus

$$0 = \left(\sum_{k=1}^m G_k\right)\xi = \sum_{k=1}^m G_k\xi = \sum_{k=1}^m u_k.$$

For each $1 \leqslant i \leqslant n$, let $\delta_i = |\{j : 1 \leqslant j \leqslant m \text{ and } G_j\xi = u_i\}|$. Then $1 \leqslant \delta_i$ is an integer for all i and

$$0 = \sum_{k=1}^m u_k = \sum_{i=1}^n \delta_i u_i,$$

contradicting the linear independence of $\{u_1, u_2, \ldots, u_n\}$. Thus $\{u_1, u_2, \ldots, u_m\}$ contains at least $n+1$ distinct vectors.

However, $u_i \neq u_j$ implies that

$$\langle u_i, u_j\rangle = \langle G_i\xi, G_j\xi\rangle = \langle G_j^* G_i\xi, \xi\rangle \in \Omega = \{1, \omega_2\}.$$

But $\|u_k\| = 1$ for all $k \geqslant 1$, and so $u_i \neq u_j$ implies that $\langle u_i, u_j\rangle = \omega_2$. By Proposition 4.8 above, there can be *at most* $n+1$ distinct vectors in $\mathbb{C}^n$ with the property that the angle between any two of them is a single fixed value – in our case, ω_2. Hence $\{u_1, u_2, \ldots, u_m\}$ chains at most (and therefore exactly) $n+1$ distinct vectors. After reindexing (if necessary) we may assume that $\{u_1, u_2, \ldots, u_m\} = \{u_1, u_2, \ldots, u_{n+1}\}$, and from Proposition 4.8, we also know that $\omega_2 = -\frac{1}{n}$.

Next, identify $\mathbb{C}^n$ with an n-dimensional subspace of $\mathbb{C}^{n+1}$, and let $u_0 \in \mathbb{C}^n$ be a norm-one vector perpendicular to $\mathrm{Orb}_{\mathcal{G}}$. Define

$$z_k = (1 + 1/n)^{-\frac{1}{2}} \left(u_k + \frac{1}{\sqrt{n}} u_0\right), \quad \text{for} \quad k = 1, 2, \ldots, n+1.$$

Note that if $1 \leqslant i \neq j \leqslant n+1$, then

$$\langle z_i, z_j\rangle = (1 + 1/n)\left(\langle u_i, u_j\rangle + \frac{1}{n}\langle u_0, u_0\rangle\right) = 0,$$

while

$$\|z_i\|^2 = \left(1 + \frac{1}{n}\right)\left(\|u_i\|^2 + \frac{1}{n}\|u_0\|^2\right) = 1.$$

We have shown that $\mathcal{B} := \{z_1, z_2, \ldots, z_{n+1}\}$ is an orthonormal basis for $\mathbb{C}^{n+1}$.

Consider the group $\widehat{\mathcal{P}}_{n+1}$ of permutation matrices relative to the orthonormal basis $\mathcal{B}$: that is, an operator $\widehat{P} \in \mathbb{M}_{n+1}(\mathbb{C})$ lies in $\widehat{\mathcal{P}}_{n+1}$ if and only if there exists a permutation σ of the set $\{1, 2, \ldots, n+1\}$ so that $\widehat{P}(z_k) = z_{\sigma(k)}$ for all $1 \leqslant k \leqslant n+1$. Obviously $\widehat{\mathcal{P}}_{n+1}$

is unitarily equivalent to $\mathcal{P}_{n+1}$ via a unitary that takes the standard orthonormal basis for $\mathbb{C}^{n+1}$ to $\mathcal{B}$.

Let $\eta = \sum_{k=1}^{n+1} z_k$, so that η plays the same role relative to the basis $\mathcal{B}$ that the vector $\mathbb{1}$ plays relative to the standard orthonormal basis for $\mathbb{C}^{n+1}$. Then $\widehat{P}\eta = \eta$ for all $\widehat{P} \in \widehat{\mathcal{P}}_{n+1}$ and $\mathbb{C}\,\eta \subseteq \mathbb{C}^{n+1}$ is the unique one-dimensional reducing subspace for $\widehat{\mathcal{P}}_{n+1}$. But an easy calculation shows that

$$\begin{aligned}\eta &= \sum_{k=1}^{n+1} z_k \\ &= \left(1 + \frac{1}{n}\right)^{-\frac{1}{2}} \left(\sum_{k=1}^{n+1} (u_k + \frac{1}{\sqrt{n}} u_0)\right) \\ &= \left(1 + \frac{1}{n}\right)^{-\frac{1}{2}} \left(0 + \frac{n+1}{\sqrt{n}} u_0\right) \\ &= \sqrt{n+1} u_0.\end{aligned}$$

Thus $\widehat{P} \in \widehat{\mathcal{P}}_{n+1}$ implies that $\widehat{P}u_0 = u_0$, and hence if σ is a permutation of $\{1, 2, \ldots, n+1\}$ so that $\widehat{P}z_k = z_{\sigma(k)}$ for all $1 \leqslant k \leqslant n+1$, then $\widehat{P}u_k = u_{\sigma(k)}$ for all $1 \leqslant k \leqslant n+1$. Moreover, there exists such an operator $\widehat{P} \in \widehat{\mathcal{P}}_{n+1}$ for each such permutation σ.

Since $\operatorname{span}\{u_1, u_2, \ldots, u_{n+1}\} = u_0^\perp \subseteq \mathbb{C}^{n+1}$, we see that $\widehat{\mathcal{P}}_{n+1}|_{u_0^\perp}$ consists of all possible operators on $\operatorname{span}\{u_1, u_2, \ldots, u_{n+1}\}$ which permute the u_k's. Since each $G \in \mathcal{G}$ permutes the u_k's, $1 \leqslant k \leqslant n+1$ and is totally determined by its action on this set, $\mathcal{G} \subseteq \widehat{\mathcal{P}}_{n+1}|_{u_0^\perp} \simeq \mathcal{P}_{n+1}|_{\mathbb{1}^\perp}$.

By Proposition 4.9, Ω is an admissible set for $\widehat{\mathcal{P}}_{n+1}|_{u_0^\perp}$. By the maximality of $\mathcal{G}$, we conclude that $\mathcal{G} = \widehat{\mathcal{P}}_{n+1}|_{u_0^\perp}$.

The final statement is simply the observation that $\mathcal{P}_{n+1}$ has $(n+1)!$ elements.

□

4.11. Proposition. *Let $n \geqslant 2$ be an integer and $\mathbb{1} = \mathbb{1}_{n+1} \in \mathbb{C}^{n+1}$. Let $\mathcal{R}_n := \mathcal{P}_{n+1}|_{\mathbb{1}^\perp}$. Then $\mathcal{R}_n$ contains a unitarily equivalent copy of $\mathcal{P}_n$. That is, there exists a unitary operator $V : \mathbb{C}^n \to \mathbb{1}^\perp$ so that $V\mathcal{P}_n V^* \subseteq \mathcal{R}_n$.*

Proof. Fix $n \geqslant 2$ as above. Let $\mathcal{T} = \{P \in \mathcal{P}_{n+1} : Pe_1 = e_1\}$, so that

(i) $\mathcal{T}$ is isomorphic to $\mathcal{P}_n$ as a group, and in fact

(ii) $\mathcal{T}|_{e_1^\perp} := \{P|_{e_1^\perp} : P \in \mathcal{T}\}$ is unitarily equivalent to $\mathcal{P}_n$.

Let $\mathcal{M} := \operatorname{span}\{e_1, \mathbb{1}\}$, and extend $\{e_1\}$ to an orthonormal basis $\{e_1, f\}$ for $\mathcal{M}$. Relative to the decomposition $\mathbb{C}^{n+1} = \mathbb{C}e_1 \oplus \mathbb{C}f \oplus \mathcal{M}^\perp$, $P \in \mathcal{T}$ is of the form

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & T_P \end{bmatrix},$$

where $T_P = P|_{\mathcal{M}^\perp}$. In particular, $P|_{e_1^\perp} = \begin{bmatrix} 1 & 0 \\ 0 & T_P \end{bmatrix}$. Hence

$$\mathcal{P}_n \simeq \mathcal{T}|_{e_1^\perp} \simeq \{\begin{bmatrix} 1 & 0 \\ 0 & T_P \end{bmatrix} : P \in \mathcal{P}_{n+1}\},$$

with $T_P = P|_{\mathcal{M}^\perp}$.

Now extend $\{\mathbb{1}\}$ to an orthonormal basis $\{\mathbb{1}, g\}$ for $\mathcal{M}$. Relative to the decomposition $\mathbb{C}^{n+1} = \mathbb{C}\mathbb{1} \oplus \mathbb{C}g \oplus \mathcal{M}^\perp$, $P \in \mathcal{T}$ admits the same matrix form

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & T_P \end{bmatrix},$$

where $T_P = P|_{\mathcal{M}^\perp}$.

Clearly $P \in \mathcal{T}$ implies that

$$P|_{\mathbb{1}^\perp} \simeq \begin{bmatrix} 1 & 0 \\ 0 & T_P \end{bmatrix} \simeq P|_{e_1^\perp}.$$

Hence the map $W : \text{span}\,\{f, \mathcal{M}^\perp\} \to \text{span}\,\{g, \mathcal{M}^\perp\}$ determined by $Wf = g$ and $Wx = x$ for all $x \in \mathcal{M}^\perp$ is unitary and

$$W^* P|_{\mathbb{1}^\perp} W = P|_{e_1^\perp} \quad \text{for all} \quad P \in \mathcal{T},$$

proving that $W\mathcal{T}|_{e_1^\perp} W^* \subseteq \mathcal{R}_n$ is a unitarily equivalent copy of $\mathcal{P}_n$ in $\mathcal{R}_n$.

□

Before turning our attention to irreducible, finite groups of invertible matrices which do not necessarily consist of unitary operators (but which are necessarily simultaneously similar to a group of unitaries), we make the following simple observation.

4.12. Remark. Let $n \geqslant 2$. We remark that no subset $\Lambda \subseteq \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ can serve as an admissible set for an irreducible group $\mathcal{G} \subseteq \mathbb{M}_n(\mathbb{C})$ of unitary matrices.

Indeed, suppose otherwise. Let $\mathcal{G}$ be such a group and let ξ be a corresponding admissible vector. Let $G \in \mathcal{G}$ and note that $\|G\| = 1$. Then $\langle G\xi, \xi\rangle \in \Lambda$ implies that $|\langle G\xi, \xi\rangle| = 1$, and hence (by the Cauchy-Schwarz inequality), $G\xi \in \mathbb{T}\xi$. It follows that $\mathbb{C}\xi$ is a non-trivial invariant subspace for $\mathcal{G}$, contradicting the irreducibility of $\mathcal{G}$.

4.13. Lemma. *Let $n \geqslant 1$ be an integer and $f, g \in \mathbb{C}^n$. Suppose that $\langle f, g\rangle = 1$. Then there exists an invertible, positive definite operator $T \in \mathbb{M}_n(\mathbb{C})$ so that $T^*g = Tg = T^{-1}f$, and $\|T^*g\| = 1$.*

Proof. First observe that it suffices to prove this in the case where $\|g\| = 1$, otherwise we can scale g by $r := \frac{1}{\|g\|}$ and f by $\|g\|$.

Under the assumption that $\|g\| = 1$, let $\mathcal{M} = \text{span}\{f, g\}$ and consider an orthonormal basis $\{g, h\}$ for $\mathcal{M}$, which we may extend to an orthonormal basis $\{g, h, h_3, \ldots, h_n\}$ for $\mathbb{C}^n$.

Writing $f = \alpha_1 g + \alpha_2 h$ and noting that $\langle f, g\rangle = 1$ implies that $\alpha_1 = 1$, define

$$K = \begin{bmatrix} 1 & \overline{\alpha_2} \\ \alpha_2 & \beta \end{bmatrix} \oplus I_{n-2} \in \mathbb{M}_n(\mathbb{C}),$$

where $\beta > 0$ is chosen so that $\beta > |\alpha_2|^2$. It follows that K is positive definite and invertible, and that $Kg = f$.

Since $K > 0$, we can find a positive square root T for K, and clearly with respect to the above decomposition $\mathbb{C}^n = \mathcal{M} \oplus \mathcal{M}^\perp$, we obtain

$$T = \begin{bmatrix} t_1 & \overline{t_2} \\ t_2 & t_4 \end{bmatrix} \oplus I_{n-2}.$$

Then $T^2 = T^*T = K$ implies that $|t_1|^2 + |t_2|^2 = 1$ and $T^2 g = Kg = f$.

But then

$$T^*g = Tg = T^{-1}f,$$

and $\|T^*g\| = \|Tg\| = \sqrt{|t_1|^2 + |t_2|^2} = 1$.

$\square$

4.14. Example. Let $n \geqslant 1$ and set $\mathcal{N} = \mathbb{1}^\perp \subseteq \mathbb{C}^{n+1}$. Then there exists an irreducible group $\mathcal{G} \subseteq \mathcal{B}(\mathcal{N}) \simeq \mathbb{M}_n(\mathbb{C})$ and a unit vector $\xi \in \mathcal{N}$ so that $\Omega := \{\langle U\xi, \xi\rangle : U \in \mathcal{G}\} = \{1, -n\}$. In particular, there exists an admissible set of cardinality two for $\mathcal{G}$.

Now $\mathcal{N} = \mathbb{1}^\perp = \{y = (y_k)_{k=1}^{n+1} \in \mathbb{C}^{n+1} : \sum_{k=1}^{n+1} y_k = 0\}$. Consider $\mathcal{G}_0 = \mathcal{P}_{n+1}|_{\mathcal{N}}$. Set

$$f = \frac{\sqrt{n}}{\sqrt{n+1}}(-n, 1, 1, \dots, 1)^t,$$
$$g = \frac{1}{\sqrt{n^2+n}}(-1, n, -1, -1, \dots, -1)^t,$$

so that $f, g \in \mathcal{N}$, $\|g\| = 1$ and $\langle f, g\rangle = 1$.

By Lemma 4.13, we can find a positive invertible matrix $T \in \mathcal{B}(\mathcal{N}) \simeq \mathbb{M}_n(\mathbb{C})$ so that $T^*g = Tg = T^{-1}f$ and $\|T^*g\| = 1$. Let $\xi = T^*g = T^{-1}f$.

Set $\mathcal{G} = T^{-1}\mathcal{G}_0 T$. Then with $U \in \mathcal{G}$, say $U = T^{-1}PT$, where $P \in \mathcal{G}_0$, we find that $\langle U\xi, \xi\rangle = \langle PT\xi, (T^{-1})^*\xi\rangle = \langle Pf, g\rangle$.

Finally, an easy computation shows that for any permutation matrix $P \in \mathcal{P}_{n+1}$,

$$\langle Pf, g\rangle \in \frac{\sqrt{n}}{\sqrt{n+1}}\frac{1}{\sqrt{n^2+n}}\{2n - (n-1), -n^2 - n\} = \{1, -n\}.$$

The irreducibility of $\mathcal{G}_0$ implies that the non-zero functional $P \mapsto \langle Pf, g\rangle$ takes on at least two values (see Lemma 1.5 (a)), so that

$$\Omega = \{\langle U\xi, \xi\rangle : U \in \mathcal{G}\} = \{1, -n\}.$$

4.15. Lemma. *Let $f = (\alpha_1, \alpha_2, \alpha_3)^t$ and $g = (\beta_1, \beta_2, \beta_3)^t \in \mathbb{C}^3$ and suppose that*

$$\Lambda := \{\langle Pf, g\rangle \in \Omega \quad \textit{for all} \quad P \in \mathcal{P}_3\}$$

has cardinality at most 2. If $|\{\alpha_1, \alpha_2, \alpha_3\}| \geqslant 2$ and $|\{\beta_1, \beta_2, \beta_3\}| \geqslant 2$, then

$$|\{\alpha_1, \alpha_2, \alpha_3\}| = 2 = |\{\beta_1, \beta_2, \beta_3\}|.$$

Proof. By symmetry, it suffices to prove that $|\{\alpha_1, \alpha_2, \alpha_3\}| = 2$. Assume to the country that all of the α_i's are distinct.

Denote the elements of $\mathcal{P}_3$ by I, S, S^2, J_1, J_2 and J_3, where

$$S = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \ J_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \ J_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \ \text{and } J_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We can assume without loss of generality that $\beta_1 \neq \beta_2 \neq \beta_3$ (which leaves room for the possibility that $\beta_1 = \beta_3$).

Set $\lambda_1 := \langle f, g\rangle \in \Lambda$. Then

$$\langle (I - J_3)f, g\rangle = (\alpha_1 - \alpha_2)(\overline{\beta}_1 - \overline{\beta}_2) \neq 0,$$

and thus $\lambda_2 := \langle J_3 f, g\rangle \neq \lambda_1$, implying that $|\Lambda| = 2$. Similarly,

$$\langle (I - J_1) f, g\rangle = (\alpha_1 - \alpha_2)(\overline{\beta}_2 - \overline{\beta}_3) \neq 0$$

and

$$\langle (S^2 - J_3) f, g\rangle = -(\alpha_1 - \alpha_3)(\overline{\beta}_2 - \overline{\beta}_3) \neq 0,$$

implying that $\langle J_1 f, g\rangle = \lambda_2$ and $\langle S^2 f, g\rangle = \lambda_1$. Finally,

$$\langle (S^2 - J_2) f, g\rangle = (\alpha_2 - \alpha_3)(\overline{\beta}_1 - \overline{\beta}_2) \neq 0$$

and

$$\langle (S - J_1) f, g\rangle = (\alpha_3 - \alpha_1)(\overline{\beta}_1 - \overline{\beta}_2) \neq 0,$$

which allows us to complete the list, namely:

$$\begin{aligned} \langle I f, g\rangle &= \langle S f, g\rangle = \langle S^2 f, g\rangle = \lambda_1 \\ \langle J_1 f, g\rangle &= \langle J_2 f, g\rangle = \langle J_3 f, g\rangle = \lambda_2. \end{aligned}$$

From these six equations we obtain

$$(\alpha_2 + \alpha_3 - 2\alpha_1)(\overline{\beta}_2 - \overline{\beta}_1) = \langle (I - S + J_1 - J_3) f, g\rangle = 0,$$

and

$$(\alpha_1 + \alpha_2 - 2\alpha_3)(\overline{\beta}_2 - \overline{\beta}_1) = \langle (S - S^2 + J_2 - J_1) f, g\rangle = 0.$$

Hence

$$\alpha_2 + \alpha_3 - 2\alpha_1 = 0 = \alpha_1 + \alpha_2 - 2\alpha_3,$$

which implies that $\alpha_1 = \alpha_3$, a contradiction.

□

4.16. Lemma. *Let $0 \neq f, g \in \mathbb{C}^{n+1}$ and suppose that*

(a) $f, g \in \mathbb{1}^{\perp}$ *(where $\mathbb{1} = \mathbb{1}_{n+1} \in \mathbb{C}^{n+1}$); and*
(b) $|\{\langle P f, g\rangle : P \in \mathcal{P}_{n+1}\}| = 2$.

Writing $f = (\alpha_1, \alpha_2, \ldots, \alpha_{n+1})^t$ and $g = (\beta_1, \beta_2, \ldots, \beta_{n+1})^t$, we have that

$$|\{\alpha_k\}_{k=1}^{n+1}| = 2 = |\{\beta_k\}_{k=1}^{n+1}|.$$

Proof. Since $f, g \in \mathbb{1}^{\perp}$ in turn implies that $\sum_{k=1}^{n+1} \alpha_k = 0 = \sum_{k=1}^{n+1} \beta_k$, it then follows that $|\{\alpha_k\}_{k=1}^{n+1}| \geqslant 2$, and similarly, $|\{\beta_k\}_{k=1}^{n+1}| \geqslant 2$.

It suffices to prove that for any integers $1 \leqslant p_1 < p_2 < p_3 \leqslant n+1$ and $1 \leqslant q_1 < q_2 < q_3 \leqslant n+1$, considering the vectors $\varphi = (\alpha_{p_1}, \alpha_{p_2}, \alpha_{p_3})^t$ and $\psi = (\beta_{q_1}, \beta_{q_2}, \beta_{q_3})^t$ satisfying $|\{\alpha_{p_1}, \alpha_{p_2}, \alpha_{p_3}\}| \geqslant 2$ and $|\{\beta_{q_1}, \beta_{q_2}, \beta_{q_3}\}| \geqslant 2$, we necessarily have

$$|\{\alpha_{p_1}, \alpha_{p_2}, \alpha_{p_3}\}| = 2 = |\{\beta_{q_1}, \beta_{q_2}, \beta_{q_3}\}|.$$

After applying fixed permutations P_0 to f and Q_0 to g we may assume that $p_i = q_i = i$, $1 \leqslant i \leqslant 3$.

Let $\widehat{\mathcal{P}}_3$ be the set of all permutation unitary matrices which fix all basis vectors e_j with $j > 3$. Also, let $f_0 = (\alpha_1, \alpha_2, \alpha_3)^t$ and $g_0 = (\beta_1, \beta_2, \beta_3)^t$. Then for all $P \in \mathcal{P}_3$, we get

$$\langle P f_0, g_0\rangle = \langle \widehat{P} f, g\rangle - \kappa,$$

where $\kappa = \sum_{i=4}^{n+1} \alpha_i \overline{\beta}_i$ is constant and where $\widehat{P} \in \widehat{\mathcal{P}}_3$ is the permutation unitary matrix which acts like P on span $\{e_1, e_2, e_3\}$ and which fixes e_k, $4 \leqslant k \leqslant n$.

If follows from the hypotheses that

$$|\{\langle Pf_0, g_0\rangle : P \in \mathcal{P}_3\}| = 2.$$

The proof now follows by applying the preceding lemma.

$\square$

4.17. Proposition. *Let $n \geqslant 4$, $f, g \in \mathbb{C}^{n+1}$ and suppose that*

(a) *$f, g \in \mathbb{1}^{\perp}$, where $\mathbb{1} = \mathbb{1}_{n+1} \in \mathbb{C}^{n+1}$;*
(b) *$\langle f, g\rangle = 1$; and*
(c) *$|\{\langle Pf, g\rangle : P \in \mathcal{P}_{n+1}\}| = 2$.*

Write $f = (\alpha_1, \alpha_2, \ldots, \alpha_{n+1})^t$ and $g = (\beta_1, \beta_2, \ldots, \beta_{n+1})^t$. Then either exactly n of the α_i's are the same, or exactly n of the β_i's are the same.

Proof. It follows from Lemma 4.16 that

$$|\{\alpha_k\}_{k=1}^{n+1}| = 2 = |\{\beta_k\}_{k=1}^{n+1}|.$$

We argue by contradiction. Condition (c) above then implies that we may reindex the sequences $\{\alpha_k\}_{k=1}^{n+1}$ and $\{\beta_k\}_{k=1}^{n+1}$ so that $\alpha_2 = \alpha_1$, $\alpha_3 = \alpha_4 \neq \alpha_1$, $\beta_2 = \beta_1$, $\beta_3 = \beta_4 \neq \beta_1$.

Write $\Omega = \{\langle Pf, g\rangle : P \in \mathcal{P}_{n+1}\} = \{\langle f, Pg\rangle : P \in \mathcal{S}_{n+1}\}$. Consider the permutations corresponding to $\sigma_1 = (1)$, $\sigma_2 = (23)$ and $\sigma_3 = (13)(24)$, and let $\gamma = \sum_{k=5}^{n+1} \alpha_k \overline{\beta_k}$. Then with

$$\delta_1 := \alpha_1\overline{\beta_1} + \alpha_1\overline{\beta_1} + \alpha_3\overline{\beta_3} + \alpha_3\overline{\beta_3} \tag{5}$$

$$\delta_2 := \alpha_1\overline{\beta_1} + \alpha_3\overline{\beta_1} + \alpha_1\overline{\beta_3} + \alpha_3\overline{\beta_3} \tag{6}$$

$$\delta_3 := \alpha_3\overline{\beta_1} + \alpha_3\overline{\beta_1} + \alpha_1\overline{\beta_3} + \alpha_1\overline{\beta_3}, \tag{7}$$

we get that $\{\delta_1 + \gamma, \delta_2 + \gamma, \delta_3 + \gamma\} \subseteq \Omega$. Since $|\Omega| = 2$, it follows that $\delta_i = \delta_j$ for some $1 \leqslant i \neq j \leqslant 3$.

If $\delta_1 = \delta_2$, then considering Eqn(5) - Eqn(6) implies

$$(\alpha_1 - \alpha_3)(\overline{\beta_1 - \beta_3}) = 0,$$

which in turn implies that $\beta_1 = \beta_3$, a contradiction. Thus $\delta_1 \neq \delta_2$.

Similarly, by considering Eqn(6) - Eqn(7), we obtain that $\delta_2 \neq \delta_3$ and by Eqn(5) - Eqn(7), we obtain that $\delta_1 \neq \delta_3$. But this contradicts the fact that $\delta_i = \delta_j$ for some $1 \leqslant i \neq j \leqslant 3$.

This completes the proof.

$\square$

4.18. Proposition. *Let $n \geqslant 4$, $f, g \in \mathbb{C}^{n+1}$ and suppose that*

(a) *$f, g \in \mathbb{1}^{\perp}$, where $\mathbb{1} = \mathbb{1}_{n+1} \in \mathbb{C}^{n+1}$;*
(b) *$\langle f, g\rangle = 1$; and*
(c) *$|\{\langle Pf, g\rangle : P \in \mathcal{P}_{n+1}\}| = 2$.*

Then $\Omega(= \Omega(n, f, g)) := \{\langle Pf, g\rangle : P \in \mathcal{P}_{n+1}\} = \{\langle f, Pg\rangle : P \in \mathcal{P}_{n+1}\} = \{1, \omega_2\}$, where

$$\omega_2 \in \left\{\frac{-r}{n+1-r} : 1 \leqslant r \leqslant n\right\}.$$

Also, for each value of $\omega_2 \in \{\frac{-r}{n+1-r} : 1 \leqslant r \leqslant n\}$, there exist f and g satisfying conditions (a), (b) and (c) so that $\Omega(n, f, g) = \{1, \omega_2\}$.

Thus,

$$\cup_{n=4}^{\infty} \Omega(n, f, g) = \{1\} \cup \{-\frac{m}{n} : 1 \leqslant m, n\} = \{1\} \cup (\mathbb{Q} \cap (-\infty, 0)).$$

Proof. We saw in Proposition 4.17 that if we write $f = (\alpha_1, \alpha_2, \ldots, \alpha_{n+1})^t$ and $g = (\beta_1, \beta_2, \ldots, \beta_{n+1})^t$, then either exactly n of the α_i's are the same or exactly n of the β_i's are the same.

Since condition (c) effectively allows us to permute the entries of f and g as we wish, and since moving $P \in \mathcal{P}_{n+1}$ from the left to the right of the inner product means that the situation is symmetric in f and g, we may assume without loss of generality that the second condition holds, in which case - by further stipulating that $\|g\| = 1$ and keeping in mind that $g \in \mathbb{1}^\perp$ - we may also assume that

$$g = \frac{1}{\sqrt{n^2+n}}(-n, 1, 1, \ldots, 1)^t,$$

and for some choice of $1 \leqslant r \leqslant n$,

$$f = \kappa\,(-(n+1-r), -(n+1-r), \ldots, -(n+1-r), r, r, \ldots, r)^t,$$

where κ is chosen to ensure that $\langle f, g\rangle = 1$, as required in (b). (That is, the fact that $f \in \mathbb{1}^\perp$ forces this ratio on α_1 vs α_2.)

Solving for κ, we have that

$$\begin{aligned} 1 &= \langle f, g\rangle \\ &= \frac{\kappa}{\sqrt{n^2+n}}\left[n(n+1-r) - (r-1)(n+1-r) + (n+1-r)r\right] \\ &= \kappa\frac{\sqrt{n+1}}{\sqrt{n}}(n+1-r). \end{aligned}$$

Hence $\kappa = \frac{\sqrt{n}}{\sqrt{n+1}}\frac{1}{n+1-r}$.

For this choice of r, $\Omega = \{1, \omega_2\}$, where

$$\begin{aligned} \omega_2 &= \frac{\kappa}{\sqrt{n^2+n}}\left[-nr + r(-(n+1-r)) + (n-r)r\right] \\ &= \frac{-\kappa r}{\sqrt{n^2+n}}(n+1) \\ &= \frac{-r}{n+1-r}. \end{aligned}$$

From this, we see that $\Omega(n, f, g) = \{1, \omega_2\}$ with $\omega_2 \in \{\frac{-r}{n+1-r} : 1 \leqslant r \leqslant n\}$. The last statement is a simple verification.

□

4.19. Theorem. *Let $n \geqslant 4$, $\mathcal{N} = \mathbb{1}^\perp \subseteq \mathbb{C}^{n+1}$ and suppose that $\mathcal{G}_0 = \mathcal{P}_{n+1}|_{\mathcal{N}}$. If $T \in \mathcal{B}(\mathcal{N}) \simeq \mathbb{M}_n(\mathbb{C})$ is invertible and $\mathcal{G} = T\mathcal{G}_0T^{-1}$ admits a distinguished unit vector $\xi \in \mathcal{N}$ so that $\Omega := \{\langle G\xi, \xi\rangle : G \in \mathcal{G}\}$ has cardinality two, then $\Omega = \{1, \omega_2\}$, where $\omega_2 \in \{\frac{-r}{n+1-r} : 1 \leqslant r \leqslant n\}$. Furthermore, any Ω of this form is possible.*

Proof. Set $f = T^{-1}\xi$ and $g = T^*\xi$. It is clear that $\langle f, g\rangle = \|\xi\|^2 = 1$, and that $f, g \in \mathcal{N}$ since $\xi \in \mathcal{N}$ and $T \in \mathcal{B}(\mathcal{N})$. Furthermore,

$$\Omega = \{\langle TUT^{-1}\xi, \xi\rangle : U \in \mathcal{G}_0\} = \{\langle Pf, g\rangle : P \in \mathcal{P}_{n+1}\}.$$

By Proposition 4.18, Ω is of the stated form.

□

5. SEMIGROUPS OF INTERMEDIATE RANK

5.1. Our goal in this section is to show that for each $n \geqslant 4$ and each $1 < k < n$, there exists an irreducible semigroup $\mathcal{S} \subseteq \mathbb{M}_n(\mathbb{C})$ so that

(a) $S \in \mathcal{S}$ implies that $\operatorname{rank} S = k$; and
(b) $\mathcal{S}$ has an admissible set of cardinality two.

It will follow from Theorem 6.3 below that such a semigroup cannot be selfadjoint.

5.2. Example. Fix $n \geqslant 4$ and $1 < k < n$. Let $\mathbb{1} = \mathbb{1}_{k+1} \in \mathbb{C}^{k+1}$. Recall from Proposition 4.9 that $\mathcal{R}_k = \mathcal{P}_{k+1}|_{\mathbb{1}^\perp}$ is an irreducible group of unitary operators in $\mathcal{B}(\mathbb{1}^\perp) \simeq \mathbb{M}_k(\mathbb{C})$, and that $\Omega = \{1, -\frac{1}{k}\}$ is admissible for $\mathcal{R}_k$.

By Proposition 4.11, $\mathcal{R}_k \subseteq \mathbb{M}_k(\mathbb{C})$ contains a subsemigroup unitarily equivalent to $\mathcal{P}_k$; that is, there exists an orthonormal basis $\mathcal{B} = \{f_1, f_2, ..., f_k\}$ of $\mathbb{1}^\perp \simeq \mathbb{C}^k$ so that if $\tilde{\mathcal{P}}_k$ denotes the group of permutation unitaries in $\mathcal{B}(\mathbb{1}^\perp) \simeq \mathbb{M}_k(\mathbb{C})$ which permute the elements of $\mathcal{B}$, then $\tilde{\mathcal{P}}_k \subseteq \mathcal{R}_k$.

Let $G \in \tilde{\mathcal{P}}_k$ be the transposition unitary determined by: $Gf_1 = f_2$, $Gf_2 = f_1$ and $Gf_j = f_j$ for $3 \leqslant j \leqslant k$. A routine calculation shows that G is unitarily equivalent to the matrix $-1 \oplus I_{k-1}$, and as such G is a rank-one perturbation of the identity $I \in \tilde{\mathcal{P}}_k \subseteq \mathcal{R}_k$.

Let $x_0, y_0 \in \mathbb{1}^\perp \simeq \mathbb{C}^k$ be chosen so that $\|x_0\| = 1$ and $G = I + x_0 \otimes y_0^*$; that is, $x_0 \otimes y_0^* = G - I \simeq -2 \oplus 0_{k-1}$. (It follows that $y_0 = -2x_0$.)

For each $1 \leqslant j \leqslant n-k$, define $X_j \in \mathbb{M}_{k\times(n-k)}(\mathbb{C})$ to be the matrix all of whose columns are zero except for the j^{th} column, which is the vector x_0. Thus

$$X_j = \begin{bmatrix} 0 & 0 & \cdots & 0 & x_0 & 0 & \cdots & 0 \end{bmatrix}.$$

Similarly, let $Y_j \in \mathbb{M}_{k\times(n-k)}(\mathbb{C})$ to be the matrix all of whose columns are zero except for the j^{th} column, which is the vector y_0. Thus

$$Y_j = \begin{bmatrix} 0 & 0 & \cdots & 0 & y_0 & 0 & \cdots & 0 \end{bmatrix}.$$

Set $\mathcal{X} = \{X_0 = 0, X_1, X_2, \ldots, X_{n-k}\}$ and $\mathcal{Y} = \{Y_0 = 0, Y_1, Y_2, \ldots, Y_{n-k}\}$. We define

$$\mathcal{S} = \left\{ \begin{bmatrix} U & UX \\ Y^*U & Y^*UX \end{bmatrix} : U \in \mathcal{R}_k,\ X \in \mathcal{X}, Y \in \mathcal{Y} \right\}.$$

If $S_1 = \begin{bmatrix} U_1 & U_1X_j \\ Y_r^*U_1 & Y_r^*U_1X_j \end{bmatrix}$ and $S_2 = \begin{bmatrix} U_2 & U_2X_t \\ Y_s^*U_2 & Y_s^*U_2X_t \end{bmatrix}$ belong to $\mathcal{S}$, then a routine calculation shows that

$$S_1S_2 = \begin{bmatrix} V & VX_t \\ Y_r^*V & Y_r^*VX_t \end{bmatrix},$$

where $V = U_1U_2 + U_1X_jY_x^*U_2 = U_1(I + X_jY_s^*)U_2$.

But $X_jY_s^* = 0$ if $j \neq s$, if $X_j = 0$ or $Y_j = 0$; otherwise $X_jY_s^* = x_0 \otimes y_0^*$. Thus $(I + X_jY_s^*) = G \in \mathcal{R}_k$, and so $V \in \mathcal{R}_k$.

This shows that $\mathcal{S}$ is a semigroup. Next, we verify that $\mathcal{S}$ is irreducible.

(i) By choosing $X = 0 = Y$, we see that $S = \begin{bmatrix} U & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{S}$ for all $U \in \mathcal{R}_k$. Thus $\operatorname{span} \mathcal{S} \supseteq \mathbb{M}_k(\mathbb{C}) \oplus 0$.

(ii) Let $1 \leqslant j \leqslant n-k$, $X = X_j$ and $Y = 0$. Then for any $U \in \mathcal{R}_k$, $S_1 = \begin{bmatrix} U & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{S}$ and $S_2 = \begin{bmatrix} I & X_j \\ 0 & 0 \end{bmatrix} \in \mathcal{S}$, so that $S_1 S_2 = \begin{bmatrix} U & UX_j \\ 0 & 0 \end{bmatrix} \in \mathcal{S}$. Combining this with the result from (i) shows that

$$\begin{bmatrix} 0 & UX_j \\ 0 & 0 \end{bmatrix} \in \operatorname{span} \mathcal{S}$$

for all $1 \leqslant j \leqslant n-k$.

But $\mathcal{R}_k$ acts irreducibly on $\mathbb{C}^k$, and $x_0 \neq 0$, so that $\operatorname{span}\{UX_j : U \in \mathcal{R}_k\} = \begin{bmatrix} 0 & 0 & \cdots & 0 & \mathbb{C}^k & 0 & \cdots & 0 \end{bmatrix}$.

Hence $\operatorname{span} \mathcal{S} \supseteq \begin{bmatrix} 0 & \mathbb{M}_{k \times (n-k)}(\mathbb{C}) \\ 0 & 0 \end{bmatrix}$.

(iii) A similar argument applied to the $(2,1)$ corner of $\mathcal{S}$ shows that

$$\operatorname{span} \mathcal{S} \supseteq \begin{bmatrix} 0 & 0 \\ \mathbb{M}_{(n-k) \times k}(\mathbb{C}) & 0 \end{bmatrix}.$$

But $\operatorname{span} \mathcal{S}$ is an algebra which contains

$$\left\{ \begin{bmatrix} A & B \\ C^* & 0 \end{bmatrix} : A \in \mathbb{M}_k(\mathbb{C}), B, C \in \mathbb{M}_{k \times (n-k)}(\mathbb{C}) \right\},$$

and thus $\operatorname{span} \mathcal{S} = \mathbb{M}_n(\mathbb{C})$, proving that $\mathcal{S}$ is irreducible.

Note also that $S = \begin{bmatrix} U & UX \\ Y^*U & Y^*UX \end{bmatrix} \in \mathcal{S}$ implies that

$$k = \operatorname{rank} U \geqslant \operatorname{rank} \begin{bmatrix} I \\ Y^* \end{bmatrix} U \begin{bmatrix} I & X \end{bmatrix} = \operatorname{rank} S \geqslant \operatorname{rank} U = k,$$

so that $\operatorname{rank} S = k$.

Finally, let $\xi_0 \in \mathbb{C}^k$ be a norm-one vector corresponding to the admissible set $\Omega = \{1, -\frac{1}{k}\}$ for $\mathcal{R}_k$. Set $\xi = \begin{bmatrix} \xi_0 \\ 0 \end{bmatrix} \in \mathbb{C}^n$. For $S = \begin{bmatrix} U & UX \\ Y^*U & Y^*UX \end{bmatrix} \in \mathcal{S}$, we have

$$\langle S\xi, \xi \rangle = \langle U\xi_0, \xi_0 \rangle \in \Omega.$$

Hence $\Omega = \{1, -\frac{1}{k}\}$ is admissible for $\mathcal{S}$.

5.3. Remark. We do not know at this time precisely which subsets $\Omega \subseteq \mathbb{C}$ of cardinality two can appear as the admissible set of an irreducible semigroup $\mathcal{S} \subseteq \mathbb{M}_n(\mathbb{C})$ of rank k, where $1 < k < n$. Nevertheless, we are in a position to make a couple of interesting observations.

5.4. Let $n \geqslant 2$ be an integer and $\mathcal{S} \subseteq \mathbb{M}_n(\mathbb{C})$ be an irreducible group of invertible matrices. Suppose that $\Omega = \{\omega_1, \omega_2\}$ is a two-element admissible set for $\mathcal{S}$ with corresponding norm-one admissible vector ξ. Note that $1 \in \Omega$, say $\omega_1 = 1$, since $I \in \mathcal{S}$.

Recall that the finiteness of Ω implies that of $\mathcal{S}$, and so we may write $\mathcal{S} = \{S_i\}_{i=1}^m$. By reindexing if necessary, we may fix $1 \leqslant k \leqslant m$ so that $\langle S_i\xi, \xi \rangle = 1$, $1 \leqslant i \leqslant k$ and $\langle S_i\xi, \xi \rangle = \omega_2$, $k+1 \leqslant i \leqslant m$.

Recall also from Section 4.2 that $\sum_{i=1}^{m} S_i = 0$. It follows that

$$\begin{aligned}0 &= \langle (\sum_{i=1}^{m} S_i)\xi, \xi\rangle \\ &= \sum_{i=1}^{k}\langle S_i\xi, \xi\rangle + \sum_{i=k+1}^{m}\langle S_i\xi, xi\rangle \\ &= k + (m-k)\omega_2,\end{aligned}$$

and thus $\omega_2 = -\frac{k}{m-k}$ is rational. That is, $\Omega \subseteq \mathbb{Q}$.

More generally we have:

5.5. Theorem. *Let $\mathcal{S} \subseteq \mathbb{M}_n(\mathbb{C})$ be an irreducible semigroup and $\Omega = \{\omega_1, \omega_2\}$ an admissible set for $\mathcal{S}$. If the minimal nonzero rank present in $\mathcal{S}$ is $r > 1$, then Ω is linearly dependent over $\mathbb{N}$, i.e., there are $k_1, k_2 \in \mathbb{N}$ such that $k_1\omega_1 + k_2\omega_2 = 0$, and hence $0 \notin \Omega$. In particular, $0 \notin \mathcal{S}$.*
Proof. Let $\xi \in \mathbb{C}^n$ be a distinguished unit vector for Ω. Note first that the subset of the irreducible semigroup $\mathcal{S} \cup \{0\}$ consisting of all matrices of rank r or zero is a nonzero semigroup ideal of $\mathcal{S} \cup \{0\}$, and hence is irreducible. Thus, we may choose a $P \in \mathcal{S}$ with $\operatorname{rank}(P) = r$ such that $P\xi \neq 0$. Now, note that $\mathcal{G} := P\mathcal{S}|_{P\mathbb{C}^n} \setminus \{0\}$ is an irreducible semigroup of invertible linear operators and moreover

$$\langle PSP\xi, \xi\rangle \in \Omega,$$

for all $S \in \mathcal{S}$. Define the linear functional $\phi : \mathcal{B}(P\mathbb{C}^n) \to \mathbb{C}$ by $\phi(A) = \langle AP\xi, \xi\rangle$. It follows that ϕ is nonzero and that $\phi(\mathcal{G}) = \Omega$ because $\mathcal{G}$ is irreducible. By the second paragraph of Section 4.2, we have $\sum_{G\in\mathcal{G}} G = 0$ because $\mathcal{G}$ is a finite semigroup, and hence a finite group, of invertible operators. This shows that Ω is linearly dependent over $\mathbb{N}$ because $\sum_{G\in\mathcal{G}} \phi(G) = 0$ and $\phi(\mathcal{G}) = \Omega$. That $0 \notin \Omega$ immediately follows from linear dependence of Ω over $\mathbb{N}$. It is now clear that $0 \notin \mathcal{S}$, for otherwise $0 \in \Omega$, which is impossible.

□

5.6. Remark.

(a) Indeed, a proof almost identical to that of this theorem proves the following.
Let $\mathcal{S} \subseteq \mathbb{M}_n(\mathbb{C})$ be an irreducible semigroup and Ω an admissible set for $\mathcal{S}$. If the minimal nonzero rank present in $\mathcal{S}$ is $r > 1$ and either $\mathcal{S}$ or Ω is finite, then $\Omega \setminus \{0\}$ is linearly dependent over $\mathbb{N}$, i.e., there are $m \in \mathbb{N}$, $k_1, \ldots, k_m \in \mathbb{N}$, and $\omega_1, \ldots, \omega_m \in \Omega \setminus \{0\}$ such that $k_1\omega_1 + \cdots + k_m\omega_m = 0$.

(b) If the minimal nonzero rank present in $\mathcal{S}$ is 1 with $\Omega = \{\omega_1, \omega_2\}$, then, by Theorem 3.12, $\Omega \setminus \{0\}$ is linearly dependent over $\mathbb{Q}$ as opposed to being linearly dependent over $\mathbb{N}$.

5.7. Corollary. *Let $\mathcal{S} \subseteq \mathbb{M}_n(\mathbb{C})$ be an irreducible semigroup and $\Omega = \{\omega_1, \omega_2\}$ an admissible set for $\mathcal{S}$. If the minimal nonzero rank present in $\mathcal{S}$ is $r > 1$, then $\Omega = \omega_1\{1, q\}$, where $q \in \mathbb{Q}$ and $q < 0$.*

5.8. Corollary. *Let $\mathcal{S} \subseteq \mathbb{M}_n(\mathbb{C})$ be an irreducible semigroup of invertible matrices and $\Omega = \{\omega_1, \omega_2\}$ an admissible set for $\mathcal{S}$. Then $\Omega = \{1, q\}$, where $q \in \mathbb{Q}$ and $q < 0$.*

Corollary 5.8 immediately follows from Corollary 5.7 because in Corollary 5.8, we have that $1 \in \Omega$.)

5.9. Proposition. *Let $\mathcal{S} \subseteq \mathbb{M}_n(\mathbb{C})$ be an irreducible semigroup and $\Omega = \{\omega_1, \omega_2\}$ an admissible set for $\mathcal{S}$ with the distinguished unit vector $\xi \in \mathbb{C}^n$. If there is a $P \in \mathcal{S}$ with minimal nonzero rank$(P) > 1$ such that $P\xi \neq 0$ and $P^* = P^k$ for some $k \in \mathbb{N}$, then $\Omega \subseteq ||P\xi||^2\mathbb{Q}$.*
Proof. Note first that $P\mathcal{S}|_{P\mathbb{C}^n} \setminus \{0\}$ is an irreducible semigroup of invertible linear operators and moreover

$$\langle PSP\xi, P\xi \rangle = \langle P^{k+1}SP\xi, \xi \rangle \in \Omega,$$

for all $S \in \mathcal{S}$. It is now plain that $||P\xi||^{-2}\,\Omega$ is an admissible set of cardinality two for $P\mathcal{S}|_{P\mathbb{C}^n} \setminus \{0\}$ with corresponding admissible vector $||P\xi||^{-1}\,P\xi$. It thus follows from above that $\Omega \subset ||P\xi||^2\mathbb{Q}$.

□

6. SELFADJOINT SEMIGROUPS

6.1. In this section we examine irreducible selfadjoint semigroups of operators which possess an admissible set consisting of two elements. Our first result does not depend upon the ambient Hilbert space being finite-dimensional.

6.2. Lemma. *A finite, selfadjoint semigroup of bounded linear operators acting on a Hilbert space $\mathcal{H}$ consists of partial isometries.*
Proof. Let $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H})$ be a finite, selfadjoint semigroup, and fix $T \in \mathcal{S}$. Then $S := T^*T \in \mathcal{S}$ is a positive operator. Since $\mathcal{S}$ is finite, there exist $1 \leqslant i < j$ so that $S^i = S^j$. That is, $S^i(I - S^{j-i}) = 0$. It follows from the polynomial functional calculus that if $\alpha \in \sigma(S)$, then $\alpha^i(1 - \alpha^{j-i}) = 0$, so that $\alpha = 0$ or $\alpha^{j-i} = 1$. But $S \geqslant 0$ implies that $\alpha \geqslant 0$, so this in turn implies that $\alpha \in \{0, 1\}$.

That is, $\sigma(S) \subseteq \{0, 1\}$. Since $S \geqslant 0$, this implies that S is an orthogonal projection, and hence that T is a partial isometry.

□

6.3. Theorem. *Let $\mathcal{S} \subseteq \mathbb{M}_n(\mathbb{C})$ be a minimal selfadjoint, irreducible semigroup . Suppose that $\mathcal{S}$ has an admissible set $\Omega = \{\omega_1, \omega_2\}$ consisting of two elements. Either*

(a) *each element of $\mathcal{S}$ is invertible, $\mathcal{S}$ is a group, and $\Omega = \{1, -\frac{1}{n}\}$, or*
(b) *$\mathcal{S} \simeq \{E_{i,j} : 1 \leqslant i, j \leqslant n\} \cup \{0\}$, and there exists an integer $1 \leqslant p \leqslant n$ so that $\Omega = \{0, \frac{1}{\sqrt{p}}\}$.*

Proof.

First note that as always, the fact that there exists a finite admissible set for $\mathcal{S}$ implies that $\mathcal{S}$ itself is finite.

Let $k = \min\{\operatorname{rank} S : 0 \neq S \in \mathcal{S}\}$.

If $k = n$, then every non-zero element of $\mathcal{S}$ is invertible. By minimality of $\mathcal{S}$, $0 \notin \mathcal{S}$ and thus every element of $\mathcal{S}$ is invertible. By Lemma 6.2, each element of $\mathcal{S}$ is a partial isometry and thus by virtue of being invertible, it is a unitary operator. Since $\mathcal{S}$ is finite, it is a group.

Finally, we may appeal to an earlier result, Theorem 4.10, to conclude that $\Omega_{\mathcal{S}} = \{1, -\frac{1}{n}\}$.

If $k < n$, then $\mathcal{K} := \{S \in \mathcal{S} : \operatorname{rank} S \leqslant k\}$ is a semigroup ideal of $\mathcal{S}$ which is selfadjoint and irreducible. By minimality of $\mathcal{S}$, we find that $\mathcal{K} = \mathcal{S}$, and hence $S \in \mathcal{S}$ implies that $S = 0$ or $\operatorname{rank} S = k$. We may now apply the structure Theorem 3.11 of [2], (also [1]) to conclude that (again, by minimality of $\mathcal{S}$), k divides n and there exists an irreducible group $\mathcal{U} \subseteq \mathbb{M}_k(\mathbb{C})$ of unitary matrices so that

$$\mathcal{S} \simeq \{E_{i,j} \otimes U : 1 \leqslant i, j \leqslant \frac{n}{k}, U \in \mathcal{U}\} \cup \{0\}.$$

(Here $\simeq$ denotes simultaneous unitary equivalence.)

Since $0 \in \mathcal{S}$, it immediately follows that $0 \in \Omega$, say $\omega_1 = 0$ and $\Omega = \{0, \omega_2\}$. Denote by ξ a distinguished norm-one vector for $\mathcal{S}$ corresponding to Ω. Let $I_{\mathcal{U}}$ denote the identity element of $\mathcal{U}$, and observe that $F_i := E_{i,i} \otimes I_{\mathcal{U}} \in \mathcal{S}$ for all $1 \leqslant i \leqslant \frac{n}{k}$. Since F_i is positive for all i and since $I = \sum_{i=1}^{n/k} F_i$, it follows that there exists $1 \leqslant i_0 \leqslant \frac{n}{k}$ so that $\langle F_{i_0}\xi, \xi\rangle \neq 0$, and thus

$$\langle F_{i_0}\xi, \xi\rangle = \omega_2.$$

Next, observe that for all $U \in \mathcal{U}$,

$$\langle (E_{i_0,i_0} \otimes U)\xi, \xi\rangle \in \{0, \omega_2\}.$$

If we set $\zeta = F_{i_0}\xi$ and think of this as lying in the subspace $F_{i_0}\mathbb{C}^n$, then $U \in \mathcal{U}$ is an irreducible group acting on $F_{i_0}\mathbb{C}^n$, and

$$\langle U\zeta, \zeta\rangle = \langle (E_{i_0,i_0} \otimes U)\xi, \xi\rangle \in \{0, \omega_2\}.$$

In other words, $\{0, \omega_2\}$ is a two-element admissible set for $\mathcal{U}$, which contradicts Theorem 4.10, *unless* $k = 1$ and $\mathcal{U} = \{1\}$.

We conclude that $\mathcal{S} \simeq \{E_{i,j} : 1 \leqslant i, j \leqslant n\} \cup \{0\}$.

Writing $\xi = (\xi_1, \xi_2, \ldots, \xi_n)^t$, we see that $\langle E_{i,j}\xi, \xi\rangle = \xi_j\overline{\xi_i} \in \{0, \omega_2\}$ for all i, j. From our choice of i_0 above, we know that $\xi_{i_0} \neq 0$. If $1 \leqslant j \leqslant n$ and $\xi_j \neq 0$, then

$$\omega_2 = \xi_j\overline{\xi_{i_0}} = \langle E_{i_0,j}\xi, \xi\rangle = \langle E_{i_0,i_0}\xi, \xi\rangle = \xi_{i_0}\overline{\xi_{i_0}},$$

from which we conclude that $\xi_j = \xi_{i_0}$. That is, there exists a subset $A \subseteq \{1, 2, \ldots, n\}$ so that $\xi_j = \xi_{i_0}$ if $j \in A$ and $\xi_j = 0$ if $j \notin A$.

It is now easy to compute that $\omega_2 = \frac{1}{p}$, where $p = \sqrt{|A|}$.

$\square$

7. Infinite-dimensional results

7.1. Many of the problems that we have formulated and answered above can also be asked in the infinite-dimensional setting. We finish the paper by describing a couple of simple situations where the answers are clear.

If $\mathcal{H}$ is an infinite-dimensional, separable Hilbert space and $\{e_n\}_{n=1}^\infty$ is an orthonormal basis for $\mathcal{H}$, it is easy to see that $\mathcal{S} := \{E_{ij} := e_j \otimes e_i^* : 1 \leq i, j < \infty\} \cup \{0\}$ is an irreducible semigroup of operators of rank at most one for which $\Omega = \{0, 1\}$ is an admissible set with two elements; indeed, one may take $\xi = e_1$ as the corresponding admissible unit vector. It is unclear at this time which two-elements sets Ω can serve as an admissible set for an irreducible semigroup of rank-one operators on $\mathcal{H}$.

7.2. In Section 4 of the paper, we observed that if $\mathcal{G} \subseteq \mathbb{M}_n(\mathbb{C})$ is an irreducible group of unitaries, and if $\Omega = \{\omega_1, \omega_2\}$ is an admissible set for $\mathcal{G}$, then $\Omega = \{1, -\frac{1}{n}\}$.

Our last example is that of an irreducible group $\mathcal{P}$ of unitary operators acting on the infinite-dimensional, separable Hilbert space ℓ_2, for which $\Omega = \{0, 1\}$ is an admissible set. We do not know whether or not this is the only possible admissible set of cardinality two for an irreducible group of unitaries in $\mathcal{B}(\ell_2)$.

7.3. Example. Let $\{e_k\}_{k=1}^{\infty}$ denote the standard orthonormal basis for ℓ_2, and let $\mathcal{P}$ denote the set of all permutation unitaries in $\mathcal{B}(\ell_2)$ relative to this basis, that is: $P \in \mathcal{P}$ if and only if there exists a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ so that $Pe_k = e_{\sigma(k)}$ for all $k \geqslant 1$.

If $\xi = e_1$, then it is obvious that $\langle P\xi, \xi \rangle \in \{0, 1\}$ for all $P \in \mathcal{P}$, and that both 0 and 1 can occur. Hence $\Omega = \{0, 1\}$ is admissible for $\mathcal{P}$.

There remains only to show that $\mathcal{P}$ is irreducible. Note that $\mathcal{P}$ is selfadjoint, so that any invariant subspace is in fact orthogonally reducing. Hence it suffices to show that any orthogonal projection Q commuting with every element of $\mathcal{P}$ is trivial - i.e. $Q \in \{0, I\}$.

For $1 \leqslant i \neq j$, let $P_{i,j}$ denote the permutation unitary given by $P_{i,j}e_i = e_j$, $P_{i,j}e_j = e_i$ and $P_{i,j}e_k = e_k$ if $k \notin \{i, j\}$.

Let $Q = [q_{k,l}]$. Observe that $QP_{i,j}$ interchanges the i^{th} and j^{th} columns of Q, while $P_{i,j}Q$ interchanges the i^{th} and j^{th} rows of Q. The equation $QP_{i,j} = P_{i,j}Q$ therefore implies that $q_{i,k} = q_{j,k}$ if $k \notin \{i, j\}$.)

From this we get that infinitely many entries in the k^{th} column of Q are identical. Given that the norm of Q is at most one, this can only happen if all of those entries are equal to zero. This shows that the off-diagonal entries of Q are all equal to zero, i.e. Q is diagonal.

We also see from this equation that $q_{i,i} = q_{j,j}$, so that Q is scalar.

The only scalar projections are 0 and I, completing the proof that $\mathcal{P}$ is irreducible.

References

[1] A.I. Popov and H. Radjavi. Erratum to: Semigroups of partial isometries. *Semigroup Forum*, 87:679–680, 2013.

[2] A.I. Popov and H. Radjavi. Semigroups of partial isometries. *Semigroup Forum*, 87:663–678, 2013.

[3] H. Radjavi and P. Rosenthal. *Simultaneous Triangularization*. Universitext. Springer Verlag, New York, Berlin, Heidelberg, 2000.

[4] H. Radjavi and P. Rosenthal. Limitations on the size of semigroups of matrices. *Semigroup Forum*, 76:25–31, 2008.

[5] H. Radjavi and B. Yahaghi. On irreducible algebras spanned by triangularizable matrices. *Lin. Alg. and its Appl.*, 436:2001–2007, 2012.

Department of Pure Mathematics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1

E-mail address: `LWMarcoux@uwaterloo.ca`

Department of Mathematics, University of Ljubljana, Ljubljana, Slovenia, Europe

E-mail address: `matjaz@omladic.net`

School of Mathematics and Statistics, Herschel Building, Newcastle University, Newcastle upon Tyne, UK NE1 7RU

E-mail address: `alexey.popov@ncl.ac.uk`

Department of Pure Mathematics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1

E-mail address: `hradjavi@uwaterloo.ca`

Department of Mathematics, Faculty of Sciences, University of Golestan, Gorgan 49138-15759, Iran

E-mail address: `bamdad5@hotmail.com`