

Black hole perturbations beyond the leading order

by

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Author's Declaration

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis.

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Statement of Contributions

The chapters included in this thesis are based on collaborative work.

Chapter 2 is based on the work in [151] with Sizheng Ma, Justin Ripley, and William East. Chapter 3 is based on the work in [150], co-authored with William East and Nils Siemonsen. Chapter 4 is based on unpublished work in collaboration with William East.

Abstract

This thesis employs numerical methods to study black hole perturbations beyond the leading order. This includes vacuum gravitational perturbations as well as superradiant scalar, vector, and tensor boson perturbations. Understanding and quantifying beyond leading order effects enables more accurate tests of General Relativity and of physics beyond the Standard Model.

First, we explore the nonlinear behavior of gravitational perturbations on a Kerr black hole. The ringdown gravitational wave signal, during the final stage of binary black hole mergers, contains important information about the properties of the remnant black hole, and can be used to perform clean tests of general relativity. However, interpreting the loudest portion of the ringdown signal requires understanding the role of nonlinearities and their potential impact on modeling this phase using quasinormal modes. Here, we focus on a particular nonlinear effect arising from the change in the black hole’s mass and spin due to the partial absorption of a quasinormal perturbation. We estimate the size and characteristics of this third-order effect using numerical techniques. Quantifying these effects, we find that they may be relevant in analyzing the ringdown in black hole mergers.

Next, we discuss self-gravity corrections to beyond the standard model particle dynamics around black holes. Specifically, for scalar and vector bosons forming superradiant clouds. Oscillating clouds of ultralight bosons can grow around spinning black holes through superradiance, extracting energy and angular momentum, and eventually dissipating through gravitational radiation. This makes gravitational wave detectors powerful probes of ultralight bosonic fields. Here, we use fully general-relativistic solutions of the black hole-boson cloud systems to study the self-gravity effects of scalar and vector boson clouds. We calculate the self-gravity shift in the cloud oscillation frequency, which determines the frequency evolution of the gravitational wave signal, improving the accuracy of gravitational wave searches for physics beyond the Standard Model. We also perform an analysis of the spacetime geometry of these systems, We compute how the presence of the cloud changes the innermost stable circular orbit and light ring.

Lastly, we investigate the nonlinear phenomenology of the spin-2 boson’s superradiant instability. Models that result in a massive spin-2 boson at low energies have been proposed as solutions to the dark matter problem or as modifications to general relativity. The existence of ultralight scalar and vector bosons is constrained using measurements of black hole spins, due to the mechanism of black hole superradiance, and attempts have been made to place constraints on the existence of spin-2 bosons using the same approach. However, those constraints so far have relied on the assumption that the spin-2 superradiant behavior

matches that of lower-spin fields. Here we consider a particular nonlinear theory, quadratic gravity, to study the behavior of spin-2 particles that undergo superradiance. We find that the phenomenology is different from that of spin zero or one bosons, increasing the spin of the central black hole and resulting in an extremal black hole horizon.

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Chapter 1

Introduction

Einstein formulated General Relativity (GR) in 1915 [97], introducing a geometric theory of gravity in which massive objects warp spacetime. GR has some philosophical differences from Newton’s gravity [164, 106], and it was not immediately embraced as a more natural theory. However, experimental evidence soon showed a strong preference for GR over Newtonian gravity. One of the most convincing early tests was the prediction of the deflection of starlight by the Sun, as was measured by Eddington in 1919 [88]. GR also explained the anomalous perihelion precession of Mercury’s orbit, which Newtonian physics could not account for, as well as the gravitational redshifting of light emitted from distant stars. These tests solidified GR’s position as the dominant theory of gravity. It has been our best description of gravity over the last century. Newtonian gravity, which had been our best theory of gravity for over two hundred years, was replaced through the careful application of experimental evidence.

In addition to accurately describing weak-field regimes, such as planetary motion, GR predicts the existence of black holes [194] and gravitational waves [98]. The first detection of gravitational waves in 2015 [4] was another huge success for GR, confirming their existence as predicted. Gravitational wave experiments are predominantly sensitive to merging compact objects, like compact stars or black holes, but are also sensitive to longer duration background signals, like misshapen spinning objects (for example neutron stars with mountains) or a stochastic background. Since their detection, gravitational waves have been used to better understand the nature of compact objects, as well as their cosmological and astrophysical environment.

Just as Newtonian gravity was tested and new theories explored, we must continue to test and seek potential extensions of GR. The advent of gravitational wave experiments

[2, 11, 15] offers new testing possibilities, allowing us to probe the regime where the gravitational field is at its strongest. Likewise, black hole imaging telescopes [85, 13] yield clues as to the bending of light around supermassive black holes. Gravitational wave and astrophysical telescopes together open the door to many possible tests of gravity in the regimes where it is at its strongest, as well as of the black hole paradigm.

Black holes are the most compact objects in GR. They consist of a spacetime singularity hidden behind an event horizon – this event horizon is what determines their blackness, preventing any light from escaping the interior. The black hole paradigm describes a series of assumptions about our universe and the objects in it. First, it assumes that black holes exist in our universe and are described by the Kerr spinning black hole solution to the equations of GR. It also assumes that the majority of the gravitational waves we detect are generated by the merging of binary black holes into a single remnant black hole. Tests of the black hole paradigm often overlap with those of GR, as deviations from the Kerr solution can indicate either new matter content or modifications to gravity. In order to carry out these tests it is necessary to accurately model gravitational waves and black hole physics in GR as well as in alternative scenarios. Those models can then be applied to analyze gravitational wave and astrophysical data to test their consistency.

While GR has proven to be robust in many of its predictions, it is not a perfect theory, and there are some reasons to look beyond it to other theories of gravity. By predicting spacetime singularities, GR suggests its own breakdown in extreme regimes. This singularity seems like an unphysical, unnatural artifact of applying the mathematics of GR in a region where they no longer apply. It is theorized that there should be some other higher energy theory that would describe the physics at the center of black holes. This theory should at some point be the theory of quantum gravity, but, at scales above the Planck length, such a theory might be well-described by a classical effective field theory with higher-order curvature terms supplementing the Einstein-Hilbert action. Since gravitational curvature is at its largest in the vicinity of small black holes, some of the best tests of these theories are tests of black hole dynamics using gravitational waves.

Tests of GR using gravitational waves are a core motivation for gravitational wave experiments [221, 7]. Black hole binary mergers are usually described by three distinct phases. The first is the inspiral phase, when the two constituent black holes are relatively far apart, and their dynamics and radiation can be well described using perturbation theory. Usually the inspiral dynamics are solved using a perturbative expansion in the velocity of the orbiting black holes. The second is the merger phase, where higher order nonlinear effects become increasingly important and the dynamics of the two merging black hole horizons are less well described by perturbation theory. Finally, once a common horizon is formed, the ringdown phase is the phase where physics is well described by a single remnant

black hole with perturbations in gravitational radiation. Tests of GR using gravitational waves often focus on either the inspiral or the ringdown phases, where there are analytic perturbative models and the deviations from GR can be parametrized in a way that is easy to understand. The ringdown phase is particularly interesting due to the simplicity of the physical model, and the promise of direct tests of the black hole paradigm [96, 100, 87, 38]. In order to correctly interpret the results of these tests, it is essential to understand the perturbative models in each regime.

As the system transitions from the nonlinear and highly dynamical merger phase to the ringdown phase, it eventually becomes well approximated by linear gravitational perturbations of a Kerr black hole, characterized by a superposition of damped sinusoids called quasinormal modes [208, 135, 37]. The frequencies and decay rates of these quasinormal modes are completely determined by the properties of the remnant black hole [62], and are independent of the form of the initial perturbation. Thus, measuring this behavior in a gravitational wave signal would be a strong test of consistency with general relativity [41, 61, 87], as well as an independent measurement of the mass and spin of the black hole remnant. They can also be used to probe the environment for matter [29]. Measuring these modes accurately from gravitational wave observations is an active area of research [77, 127, 6, 56, 126, 63, 99].

Understanding the form and magnitude of nonlinear effects is crucial for accurately modeling the ringdown gravitational wave signal. Linear analyses based on quasinormal modes typically assume that such nonlinearities are negligible. To correctly interpret these analyses, it is important to identify the regime in which nonlinear effects become significant. Accurately modeling these effects may enable more refined analyses, potentially allowing the detection of nonlinear features in the early stages of ringdown.

Modern and precise tests of GR in the strong field regime are predominantly carried out using gravitational waves, however other experiments and observations are also used to constrain physics beyond GR. Astrophysical observations such as black hole masses and spins, or black hole imaging, may be used to constrain possible theories of gravity. Those experiments can also constrain regions of parameter space relevant to physics beyond the Standard Model.

An intriguing possibility of recent focus is the prospect of using observations of black holes and gravitational waves to probe the existence of ultralight bosons. There are several theoretical motivations for considering scalar and vector particles that are weakly coupled to the Standard Model. The quantum chromodynamics (QCD) axion is a (pseudo) scalar example [170, 213], whose mass is observationally constrained to $m_b \leq 10^{-3}$ eV [177], and for some models is theoretically expected to be $m_b \sim 10^{-10}$ eV [133, 21, 125]. Axions,

along with dark photons [124, 169, 109, 12], have been considered as potential dark matter candidates; string theory also features ultralight axionlike particles [20], as well as vector bosons [108] in the mass range 10^{-33} to 10^{-10} eV.

The graviton of GR is the only known spin-2 field in existence, and is predicted to be massless. Models including a massive spin-2 degree of freedom have been proposed as potential modifications of GR [74], there has been recent interest in these models in the context of the Dark Matter problem [16, 26, 148, 107, 129, 147, 136]. Those models include theories with quadratic curvature terms added to the action, like Einstein-Weyl gravity and quadratic gravity [203, 204], as well as theories which include a mass term for the metric in the action, like nonlinear massive gravity [80, 79], bimetric gravity [114, 113], or other multi-metric theories.

Black holes can produce ultralight bosons through the superradiant mechanism. A scattering field can grow in the vicinity of a black hole if it satisfies the superradiance condition, $\omega < m\Omega_H$, where ω is the angular frequency of the field, m is its azimuthal mode number, and Ω_H is the frequency of the black hole horizon [222, 154, 201]. A field in a bound state that satisfies the superradiance condition will be continuously amplified through this process, and grows exponentially with time. This process is possible in principle for a scalar, vector, or tensor bosonic field with any mass, though the instability rate is only astrophysically relevant if the Compton wavelength of the field is roughly comparable to the black hole's radius. This is equivalent to $\alpha \equiv Gm_b M_{\text{BH}}/(c^3 \hbar) \sim 1$, where m_b is the boson's mass and M_{BH} the black hole's mass. What happens next depends on the backreaction of the cloud on the spacetime. In the scalar and vector boson cases, and in the absence of other couplings, the resulting boson cloud eventually spins down the black hole sufficiently to saturate the superradiance condition; after this, there is minimal flux through the black hole horizon, and the cloud will dissipate through gravitational radiation.

The signatures of black hole superradiance are used to constrain the existence of ultralight bosons. Those signatures are determined by a combination of perturbation theory and full numerical solutions. It is important to fully understand the theoretical model, and its level of accuracy, in the regimes of interest in order to test the existence of these fields accurately.

Black hole perturbation theory is a particularly useful approach in the modeling of physics near black holes and the associated gravitational wave signals. Black hole perturbation theory can be used to describe the physics of a stationary black hole and a field which can be understood as small. The form of equations that determine the dynamics of that small field can be difficult to solve. In 1973, Teukolsky showed that, in some scenarios, those equations become more straightforward when studied using the Newman-Penrose

formalism, where the physical system is described using spin coefficients and Weyl scalars, defined using a null tetrad [208]. Using Teukolsky’s approach, massless scalar, vector, and tensor black hole perturbations can be described at leading order by separable equations. These equations can be solved analytically in the frequency domain or numerically in the time domain. These techniques have been used to show the linear stability of Kerr to small amounts of gravitational radiation and matter falling in from the spacetime [214, 78]. They have also been used to model the ringdown phase of black hole mergers in GR, as well as in modified gravity theories [39].

Black hole perturbation theory can also be used to model the gravitational radiation of astrophysical sources, and to determine the leading order behavior of extra matter fields in the presence of a black hole [48]. In the case of massive boson fields, the behavior of superradiant perturbations can be solved linearly in the time domain, using carefully chosen ansätze and nonrelativistic approximations.

Often a linear analysis alone does not describe the full phenomenology of the physical case. Black hole perturbation theory is invaluable for understanding the growth or decay of fields, and the associated physical timescales. However, nonlinear effects may render a linearly stable system unstable, or quench a linear instability. To fully understand these systems, full numerical solutions, or nonlinear analyses, are often necessary. These analyses are important in the context of constraining deviations from GR, the presence of fields beyond the Standard Model, or in determining the phenomenology of a black hole system with a linear instability.

In this thesis, we develop our physical models of the behavior of perturbations on black holes to include beyond leading-order effects. These improved models can be used to more accurately test physics beyond GR and beyond the Standard Model. We use numerical relativity to go beyond leading-order perturbation theory calculations, and to determine nonlinear components to black hole ringdown, gravitational waves sourced by scalar and vector ultralight bosons, and the phenomenology of superradiant spin-2 bosons.

1.1 Outline

In Chapter 2 we will discuss non-linear effects in black hole ringdown. The ringdown gravitational wave signal arising, e.g. in the final stage of a black hole binary merger, contains important information about the properties of the remnant, and can potentially be used to perform clean tests of general relativity. However, interpreting the ringdown signal, in particular when it is at its loudest, requires understanding the role of nonlinearities and

their potential impact on modeling this phase using quasinormal modes. Here, we focus on a particular nonlinear effect arising from the change in the black hole’s mass and spin due to the partial absorption of a quasinormal perturbation. We isolate and systematically study this third-order, secular effect by evolving the equations governing linear metric perturbations on the background of a spinning black hole, but allowing the properties of the background to evolve in a prescribed way. We find that this leads to the excitation of quasinormal modes with higher polar angular number, retrograde modes (counterrotating with respect to the black hole), and overtones, as well as giving rise to a component of the signal at early times that cannot be fully described using quasinormal modes. Quantifying these effects, we find that they may be relevant in analyzing the ringdown phase of black hole mergers.

In Chapter 3, we discuss self gravity effects in superradiant clouds. Oscillating clouds of ultralight bosons can grow around spinning black holes through superradiance, extracting energy and angular momentum, and eventually dissipating through gravitational radiation. Gravitational wave detectors like LIGO, Virgo, KAGRA, and LISA can thus probe the existence of ultralight bosons. In this study, we use fully general-relativistic solutions of the black hole-boson cloud systems to study the self-gravity effects of scalar and vector boson clouds, making only the simplifying assumption that the spacetime is axisymmetric (essentially corresponding to taking an oscillation average). We calculate the self-gravity shift in the cloud oscillation frequency, which determines the frequency evolution of the gravitational wave signal, finding that this effect can be up to twice as large in the relativistic regime compared to nonrelativistic estimates. We use this to improve the **superrad** waveform model, and estimate that this reduces the theoretical phase error significantly for the louder vector boson signals. We also perform an analysis of the spacetime geometry of these systems, calculating how the cloud changes the innermost stable circular orbit and light ring around the black hole.

In Chapter 4, we investigate the phenomenology of spin-2 superradiance. Massive spin-2 fields, which arise in proposed extensions of the standard model of particle physics and general relativity, give rise to a superradiant instability around spinning black holes. We study the nonlinear evolution of this instability in quadratic gravity. We find that as the massive spin-2 field grows, the black hole’s spin increases, eventually leading to a superextremal horizon. This sharply contrasts with the spin-down that occurs due to the backreaction of the superradiant instability of a minimally coupled massive spin-0 or spin-1 field. We discuss the implications for attempts to constrain the existence of spin-2 fields using black hole observations.

Throughout this thesis, we use units and conventions as indicated individually in each chapter.

Chapter 2

Nonlinear Effect of Absorption on the Ringdown of a Spinning Black Hole

2.1 Introduction

An important aspect both of modeling and physically interpreting the ringdown gravitational wave signal is determining the form and magnitude of nonlinear effects. Understanding in what regime such effects are significant determines how accurate a purely linear analysis in terms of quasinormal modes will be at different stages of the ringdown; it also has the potential to unlock new aspects of the ringdown signal that might be observed. A number of recent works have addressed different aspects of this problem using tools from numerical relativity and beyond linear-order black hole perturbation theory.

At second-order in the perturbation amplitude, linear quasinormal modes combine to generate quadratic quasinormal modes. Analyses of numerical relativity simulations of binary black hole mergers have found evidence for the presence of these modes [71, 155, 143, 70], while second-order perturbation calculations have sought to determine the relation between the amplitude of the quadratic and linear modes [158, 178, 227, 146, 53, 44]. Including these modes in a fit to the ringdown can improve the extraction of the linear modes [70, 104].

Attendant in the decay of a quasinormal mode is a flux of energy and angular momentum through the black hole horizon. At third order, the resulting change in the black hole's spin and mass will affect the original perturbation, including by exciting additional modes through the background dynamics—an effect sometimes referred to as absorption induced

mode excitation (AIME) [193, 179, 226]. While this effect is subdominant to quadratic effects in a naïve order counting, it is cumulative with time and has been shown to contribute significantly in some scenarios. In Ref. [193], fully nonlinear calculations of scalar quasinormal modes of spherical black holes in asymptotically anti-de Sitter spacetimes showed this to be the dominant nonlinear effect in that context, and motivated analytic estimates of AIME for overtones in asymptotically flat, spherically symmetric spacetimes, suggesting this effect would play a role in black hole mergers. In Ref. [179], the effect of a changing black hole mass in spherical symmetry was modelled with null dust, and a model to capture the resulting changing frequency and amplitude of a ringdown signal was proposed. Recently, Ref. [226] studied how a spinning black hole was perturbed by an incoming gravitational wave packet as a function of its amplitude using nonlinear simulations. They quantified the effect of the change in the black hole’s mass and spin resulting from the incoming gravitational wave packet on the amplitude and frequency of the ensuing ringdown signal.

In this work, we also consider the nonlinear effect of a changing black hole mass and spin, but focus directly on the backreaction that the least-damped quasinormal mode of a spinning black hole will have on itself through this channel. We calculate this effect in isolation by evolving the linearized Einstein equations in terms of the Weyl tensor—i.e., the Teukolsky equation—but on the background of a black hole whose mass and spin changes according to the expected flux of energy and angular momentum through the horizon. Aided by the choice of hyperboloidal slicing, we use initial data corresponding exactly to a single quasinormal mode and quantify the resulting change in the gravitational wave signal due to its partial absorption by the black hole. While this analysis neglects other nonlinear effects, it allows us to identify and quantify the impact of one particular effect in isolation and determine in what regime it will impact the consistency of a purely linear treatment. We find that, in addition to changing the amplitude and phase of the original least-damped quasinormal mode, this excites overtones, higher angular number, and retrograde modes. The amplitudes found for these excited modes are comparable, in some cases, to those found in the ringdown of black hole mergers, as well as to quadratic quasinormal modes.

The remainder of this Chapter is organized as follows. In Sec. 2.2, we calculate, to leading order in perturbation theory, the change in mass and spin of a perturbed black hole due to the partial absorption of a gravitational perturbation, including for the specific case of a pure quasinormal mode. In Sec. 2.3, we describe our methods for constructing quasinormal mode initial data, numerically evolving linear gravitational perturbations on a changing black hole background, and identifying and quantifying quasinormal modes in the resulting gravitational waves. In Sec. 2.4, we present our main results, including identifying which new modes are excited and their relative amplitudes. We also find evidence for a

component in the ringdown signal at earlier times that cannot be well described by a quasinormal mode, which we examine and compare to a changing frequency model in Sec. 2.5. In Sec. 2.6, we discuss our results, including their relevance to black hole mergers, and conclude. In the appendixes, we provide details on the analytic calculation of the change in black hole mass and spin due to a solution of the Teukolsky equation (Appendix A.1), discuss some the subtleties in connect the gravitational wave signal with a gravitational perturbation near a black hole (Appendixes A.2 and A.3), compare different choices for quasinormal mode initial data (Appendix A.4), and provide details on numerical errors and convergence (Appendix A.5).

2.2 Analytic quasinormal mode absorption

Our notation is as follows. We set $G = c = 1$. We use superscripts to refer to the order in black hole perturbation theory (e.g. $f^{(2)}$ denotes a second-order perturbation of f). As is common in applications of the Newman-Penrose (NP) formalism, we use the $+- --$ metric signature. The black hole mass is M and its dimensionful spin is a (such that a/M is dimensionless). We use $\Re/\Im$ to denote the real/imaginary parts of complex numbers e.g., $\Re(1 + 2i) = 1$.

2.2.1 Flux for a frequency-space perturbation

Here we briefly review how a black hole's mass and spin change due to a linear gravitational perturbation [115, 207], as calculated using the NP formalism [159]. We provide more details of the calculation, as well as a review of the literature, in Appendix A.1.

We describe linearized gravitational waves via the linearized NP scalars $\Psi_4^{(1)}$ and $\Psi_0^{(1)}$. Linear gravitational perturbations can be absorbed by the black hole, which changes its mass and spin. To find a quantitative relation between the linearized gravitational field and the black hole mass, we follow an approach described in Ref. [115]. First, from the NP equations, we relate $\Psi_0^{(1)}$ to the *second* order perturbation of the NP scalar $\rho^{(2)}$. The surface integral of this quantity on the black hole horizon then gives the change in the black hole area A . In ingoing Kerr-Schild coordinates, the relation is

$$\frac{dA^{(2)}}{dv} = -2 \int d\varphi d\vartheta \sqrt{s} \rho^{(2)}(v, \vartheta, \varphi), \quad (2.1)$$

where s is the induced metric on the black hole horizon. The integral over the azimuthal angle φ and polar angle ϑ is evaluated at the horizon radius $r_+ = M + \sqrt{M^2 - a^2}$. Following

Ref. [207], we perform a Fourier decomposition of all perturbed quantities according to their frequency ω and azimuthal number m , e.g.,

$$\Psi_0^{(1)}(v, r, \vartheta, \varphi) = \int \frac{d\omega}{2\pi} e^{-i\omega v} \sum_m e^{im\varphi} \tilde{\Psi}_0^{(1)}(\omega, r, \vartheta, m). \quad (2.2)$$

For a given (fixed) ω and m , we can calculate $dA^{(2)}/dv$ (see Appendix A.1.6). In addition, the first law of black hole thermodynamics relates the change in black hole area to the change in the mass and spin of the black hole (see Appendix A.1.5). We then have [207]

$$\frac{d^2 M}{dv d\Omega} = \frac{\sqrt{M^2 - a^2} |\omega|^2}{8\pi (2Mr_+ |\omega|^2 - am\Re\omega)} \frac{d^2 A}{dv d\Omega}, \quad (2.3)$$

and

$$\frac{d^2 a}{dv d\Omega} = \frac{\sqrt{M^2 - a^2} (a|\omega|^2 + m\Re\omega)}{8\pi M (2Mr_+ |\omega|^2 - am\Re\omega)} \frac{d^2 A}{dv d\Omega}. \quad (2.4)$$

In short, for a given quasinormal mode solution for $\Psi_0^{(1)}$, we can find the corresponding differential change in the black hole mass and spin.

Next, we summarize the change in mass and spin of a black hole due to the presence of a single quasinormal mode. We consider a quasinormal mode solution to Ψ_0 :

$$\Psi_0^{(1)}(v, r, \vartheta, \varphi) = \mathcal{A} \times e^{im\varphi - i\omega v} R(r) S(\vartheta), \quad (2.5)$$

where $R(r)$ is normalized to unity on the horizon, $R(r_+) = 1$. Using Eq. (2.1) and the relevant NP equations (see Appendix A.1.6 for details), we find that

$$\frac{dA^{(2)}}{dv} = 4\pi |\mathcal{A}|^2 \frac{M \exp[2(\Im\omega)v] r_+}{\left(\epsilon_{(HH)}^{(0)} - \Im\omega\right) \left(\Re\omega^2 + (2\epsilon_{(HH)}^{(0)} - \Im\omega)^2\right)}, \quad (2.6)$$

where $\epsilon_{(HH)}^{(0)}$ is the NP scalar ϵ in the Hawking-Hartle tetrad on the horizon [see Eq. (A.39)]. This result matches others in the literature [Eqs. (5.14) and (5.22) in Ref. [172], as well as [207] and [193]] for a normal mode (i.e. $\Im\omega = 0$) up to coordinate and tetrad changes.¹

¹Note that our result disagrees with Ref. [193] slightly for the complex mode case. The factor of $\left(\epsilon_{(HH)}^{(0)} - \Im\omega\right)$ in the denominator of Eq. (2.6) is replaced by $\left(\epsilon_{(HH)}^{(0)}\right)$ in Ref. [193].

2.2.2 Mass and spin change due the absorption of the least-damped mode

In order to connect a quasinormal mode, as measured through a gravitational wave signal, to an induced change in the black hole's mass and spin, we need to relate the amplitude of the quasinormal mode at null infinity to the amplitude of the gravitational perturbation on the horizon. This will then allow us to apply Eqs. (2.3), (2.4), and (2.6). In Appendix A.2, we relate the amplitude for the Weyl scalar $\Psi_4^{(1)}$ extrapolated to future null infinity to the Weyl scalar $\Psi_0^{(1)}$ evaluated at the black hole horizon via the NP equations. This gives

$$|\mathcal{A}| = |\Psi_0^{(HH)}(r_+)| \quad (2.7)$$

$$= C_{\text{lookback}} \left| \frac{C_{\text{to-horizon}}}{16M^2 r_+^3} \sqrt{\left| \frac{4D'}{\mathfrak{C}'} \right|^2 + \left| \frac{48\omega M}{\mathfrak{C}'} \right|^2} \left(\frac{|h_{lmn}\omega^2|}{2} \right) \right|.$$

In the above, D' and $\mathfrak{C}'$ are Starobinsky constants defined by Eqs. (2.25) and (2.28) in Ref. [35], and $C_{\text{to-horizon}}$ [defined in Eq. (A.53)] relates the solution at null infinity to one at the horizon. The quantity C_{lookback} refers to the time coordinate shift due to a slicing choice Eq. (A.66). The uncertainty in this value relates to an uncertainty in the distance of the quasinormal mode peak from the black hole horizon. We have also introduced the spheroidal components of the gravitational wave strain h (times extraction radius r) of a quasinormal mode, indexed by an overtone number n , angular number ℓ , and azimuthal angular number m ,

$$h_{\ell mn} = \int d\Omega \frac{h}{M} r {}_{-2}S_{\ell mn}^*, \quad (2.8)$$

where the spheroidal harmonics are normalized with $\int_{-1}^1 SS^* d\cos\theta = 1$.

Here we calculate the leading-order change in mass and spin due to the absorption of the $\ell = 2$, $m = 2$, $n = 0$ quasinormal mode, which we consider since this mode has longest decay time of all quasinormal modes, and it has been the most robustly measured of all ringdown modes [5, 103]. To estimate the magnitude of such a quasinormal mode that is relevant for the aftermath of a comparable mass binary black hole merger, we consider the fits to ringdown modes that were computed from numerical relativity simulations in Ref. [105]. In that reference, the authors analyzed a quasicircular binary black hole merger with mass ratio 1.22, which led to a remnant that had dimensionless spin $a/M = 0.692$, and found a strain amplitude of $|h_{220}| \approx 0.41$ (converting to our conventions) when fitting from the peak of the strain onward. Using Eq. (2.7), this translates to $|\mathcal{A}| \approx 0.013(C_{\text{lookback}}/1.86)$,

where we estimate that the dimensionless constant $C_{\text{lookback}} = 1.86 \pm 0.9$ based on the light crossing time from the black hole light ring, with an uncertainty of $\delta r = {}^{+M}_{-M/5}$ in Boyer-Lindquist radius (see Appendix A.2 for further discussion).

Integrating Eq. (2.6) from $v_1 = 0$ to $v_2 = \infty$ to find the change in the black hole’s area, and applying Eqs. (A.46) and (A.47), we can approximate the changes induced on this example black hole due to the partial absorption of its fundamental mode. We find, for remnant spin $a/M = 0.692$ and perturbing frequency $\omega_{220}(a/M)$,

$$\Delta M/M = (0.92\%) \times \left(\frac{C_{\text{lookback}}}{1.86} \right)^2 \left(\frac{h_{220}}{0.408} \right)^2, \quad (2.9)$$

$$\Delta a/a = (6.34\%) \times (100 \times \Delta M/M). \quad (2.10)$$

The change in the black hole parameters scales with the square of the perturbing amplitude, and there is a factor of 2 uncertainty in C_{lookback} corresponding to a factor of 4 uncertainty in ΔM and Δa . This change in background corresponds to a change in the quasinormal mode frequency of $\Delta\omega_{(220)}/\omega \approx 2.6\%$.

In Fig. 2.1, we show the result of this analytic calculation for $M(v)$ and $a(v)$. The timescale of the change in mass and spin is determined by the imaginary part of the perturbing mode. We compare this result to a full general relativity calculation for the same event that motivated our amplitude choice [132]. (Recall, this black hole binary has a quasicircular inspiral with mass ratio 1.22 and remnant dimensionless spin $a/M = 0.692$.) The data in [132] include calculations of the apparent horizon (Christodoulou) mass and spin at each time step. These quantities are calculated using an approximate azimuthal Killing vector and are not expected to be exact. We plot the simulation time coordinates divided by the remnant mass, which will not correspond exactly to the time coordinate v of the analytic calculation. In Ref. [105], the fit to quasinormal modes also found overtones with significant amplitude at early times, which could also contribute to the flux through the horizon; however, we only include the fundamental mode in our analytic calculation. Because of these caveats, we only expect an approximate relation between the calculations from numerical relativity and our analytic estimate, even at late times. However, we find it reassuring that the plotted quantities are comparable.

It is informative to compare our total mass change calculation to the numerical relativity calculation postmerger. From the numerical relativity result, the black hole’s mass changes by $\approx 6\%$ after forming a single horizon. The change in spin is $\approx 17\%$ over the same period. Using our analytic calculation [considering only the $(\ell mn) = (220)$ mode], we found a mass change of $\approx 1\%$, with a corresponding spin change of $\approx 6\%$. We can also look at the time dependence in the numerical relativity mass approximation. For time

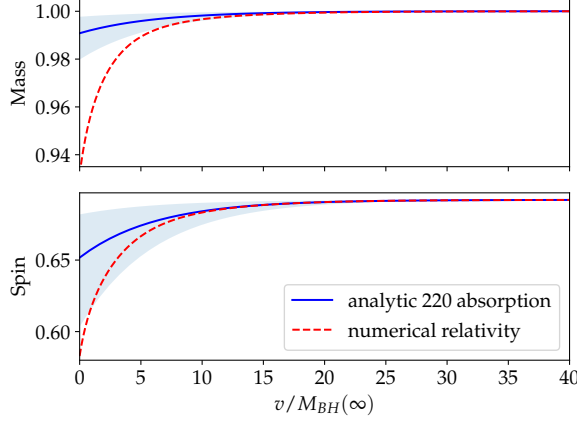


Figure 2.1: Mass and spin of a binary black hole merger remnant according to numerical relativity calculations [132]. Units are of the remnant mass, with “spin” corresponding to $[J(v)/M(v)] M(\infty)$. For the simulation, v refers to the coordinate time and $v = 0$ is when a common horizon is formed [the strain peaks at infinity at $v \approx 7M(\infty)$]. We compare these numerical relativity results with our analytic calculation of the change in black hole parameters due to the absorption of a quasinormal mode with $\omega_{\ell mn} = \omega_{220}$ using our estimate of its amplitude. We include our estimated error on the change in mass and spin due to the absorption of that mode. The high uncertainty in this calculation is due to an uncertainty in the lookback time (these error bars correspond to assuming a peak at the light ring with uncertainty of $1M$ farther from the black hole and $M/5$ closer to the black hole). Note that the time coordinate is different for the numerical relativity and analytic cases.

$t = (10 - 100)M$ (the last half a percent in mass change), the mass change is well described by $M = M(\infty)(1 - \exp Cv)$ with real constant C . Fitting the exponent C , we find that it matches the value expected from the absorption of the (220) mode associated with the final black hole ($C = 2\Im\omega_{220}$) to within $\sim 0.5\%$.

2.3 Methods

Here we describe how we generate numerical waveforms using a modified Teukolsky evolution code that allows the black hole mass and spin to change with time in a prescribed way. We then detail how we identify quasinormal modes in the gravitational wave signal

using a quasinormal mode filter [143, 145, 144]. To check the robustness of our results, we then outline another approach to fitting for quasinormal modes in the time domain using a least-squares fitting procedure. Once the mode content has been determined, mode amplitudes can be fit in the time domain. We describe our fitting template and how we apply it in this case.

We use the symbols $M_{\text{BH}}(t)$ and $a_{\text{BH}}(t)$ to denote the time-dependent black hole quantities, with $M_{\text{BH}}(\infty) = M$ and $a_{\text{BH}}(\infty) = a$. We use $\mathcal{A}$ to denote the amplitude of Ψ_4 at null infinity, such that the perturbing quasinormal mode has the form

$$\Psi_4 = \mathcal{A} \times e^{im\varphi - i\omega t} \frac{R(r)}{R(\infty)} S(\vartheta). \quad (2.11)$$

2.3.1 Numerical evolution setup

We use a Teukolsky evolution code (henceforth TEC) based on the horizon-penetrating, hyperboloidally compactified (HPHC) coordinates described in Ref.s [228, 184] (for more discussion on these coordinates, see also Refs. [186, 185]). Using TEC, we evolve the linearized Weyl scalar $\Psi_4^{(1)}$ by solving the Teukolsky equation. As TEC evolves the Teukolsky equation in HPHC coordinates, the black hole horizon and future null infinity are both accessible in a finite amount of “time” from the interior. We construct quasinormal mode initial data using the methods of Ref. [185].

For this study, we have made several minor modifications to the version of TEC described in Ref. [228]. We have adapted the code to recalculate evolution operators at each time substep using a parametrized black hole mass and spin. We also made minor changes to both the initial data and evolution codes to allow for arbitrary grid choices (instead of always using a grid corresponding to the fixed physical background), which we use to construct initial data and evolve on a grid where the inner boundary is always set to $r = M$ in Boyer-Lindquist coordinates. Using a fixed grid makes it more straightforward to compare the results of simulations with different backgrounds. In what follows, t corresponds to the hyperboloidal time coordinate defined in Ref. [228].

To analyze the results of our numerical simulations, we extract $\Psi_4(t, \theta, \varphi)$ from a sphere at future null infinity, and project out its spin-weighted spherical harmonic components $\Psi_4^{\ell m}(t)$. We note that each component $\Psi_4^{\ell m}(t)$ does not generally describe an individual quasinormal mode, as for nonspinning black holes the angular dependence of the quasinormal modes are described by “spin-weighted spheroidal harmonics” [208], not spin-weighted

spherical harmonics,

$${}_sS_{\ell m}(\theta, \phi; a\omega) = \sum_{\ell'=\ell_{\min}(s,m)}^{\ell_{\max}} c_{\ell'\ell m} {}_sY_{\ell'm}(\theta, \phi) \quad (2.12)$$

$$c_{\ell'\ell m}(a > 0) \neq 0. \quad (2.13)$$

We calculate these factors of $c_{\ell'\ell m}$ using Ref. [202] and use them to weight amplitudes extracted from a particular spherical harmonic projection of the data.

When we begin a numerical evolution, we expect the first few M of data in the time domain to be dominated by the initial data in the wave zone, and to be only weakly affected by the background. We want to choose a reference time where the data at null infinity start to be strongly dependent on the background. To estimate this, we calculate the propagation time from the black hole light ring to null infinity and to the horizon—see Appendix A.3. For a background black hole with spin $a/M = 0.7$, we calculate a delay time of $9M$. For this case, we analyze data with $t > 9M$, where we are sure that the signal at null infinity is from initial data that were evolved from the near black hole region, and not from the wave zone. Choosing a later reference time is more conservative, giving a smaller measurement for excited amplitudes relative to the initial perturbation. This is because the chosen perturbation is longer lived than the excited modes, and so their ratio decreases with time. The uncertainty in lookback time δt introduces an uncertainty in any amplitude fits, with $\delta(A_{\ell mn}/\mathcal{A}) = e^{(\mathcal{I}\omega_{\ell mn} - \mathcal{I}\omega_{220})\delta t}$.

2.3.2 Fitting quasinormal modes

We adopt three steps for quasinormal mode fitting. To summarize, first we use quasinormal mode filters [143, 145, 144] to select candidate modes. This filtering method enables the identification and subtraction of modes without requiring corresponding amplitude information, thereby eliminating the need for fitting and avoiding the risk of overfitting. Second, we implement free frequency fits for the candidate modes with varied starting times. We identify a mode when the fit converges to the targeted quasinormal frequency. In some cases, we perform the fit after filtering out other candidate modes, since it reduces the number of fitting parameters and improves fit performance. Once all the quasinormal modes are identified, we finally fit for their complex amplitudes while keeping frequencies fixed. Below, we provide more details about our procedure.

In Ref. [143], the authors introduced a frequency-domain filter for a quasinormal mode

$\omega_{\ell mn},$

$$F_\omega = \frac{\omega - \omega_{\ell mn}}{\omega - \omega_{\ell mn}^*}. \quad (2.14)$$

After filtering out the dominant mode, they demonstrated the presence of oscillations induced by subdominant modes in time-domain waveforms, thereby aiding in the identification of these modes. Here, we extend this analysis by examining the Fourier transformation of filtered waveforms. In Fig. 2.2, we show how to use the Fourier spectrum to identify modes using an example consisting of a fixed background and remnant spin $a/M = 0.7$, but where the initial data consist of a fundamental quasinormal mode with respect to a black hole with slightly different parameters. After applying a discrete Fourier transform (DFT), several filters (listed in the figure) are applied to the data, resulting in the blue curve. There is a prominent peak at the real frequency of the $(220)_R$ retrograde mode (the negative center frequency indicates it is a retrograde mode). Retrograde modes are modes that travel in the opposite direction around the black hole relative to its spin. Here, we use a subscript R to indicate that a mode is retrograde. They have frequency

$$\omega_{\ell mn R} = -\Re \omega_{\ell - mn} + i \Im \omega_{\ell - mn}. \quad (2.15)$$

The orange curve shows the result of the DFT after applying another filter to remove the $(220)_R$ mode. Clearly, this filter removes the peak at $(220)_R$. It also reveals a smaller new bump in the negative frequency band, corresponding to $(\ell, m, n) = (320)_R$. The green curve shows the result after filtering out $(\ell, m, n) = (320)_R$, removing the peak at $(\ell, m, n) = (320)_R$. In this example, we have identified two modes potentially present in the signal by examining DFTs of the filtered waveform. We repeat this procedure iteratively to determine all the candidate modes.

Once we have identified candidate modes using the filter, we use a free frequency fit to confirm the presence of that mode. Our free frequency fits use a model where a single free complex exponential is fit to data in different time windows,

$$\psi^{\text{free freq}} = A \exp[-i(\omega_r + i\omega_i)t], \quad (2.16)$$

where the free parameters are $A \in \mathbb{C}$ and $\omega_r, \omega_i \in \mathbb{R}$. These fits are useful for confirming the presence of a mode in data after its real frequency is identified using the filter. In Fig. 2.3, a free frequency fit is performed on the data after filtering out all of the modes in the list of modes already identified, except for $(\ell, m, n) = (320)_R$. We see a clustering of points around the value of the remnant $(320)_R$ mode. We conclude that this mode is physically present.

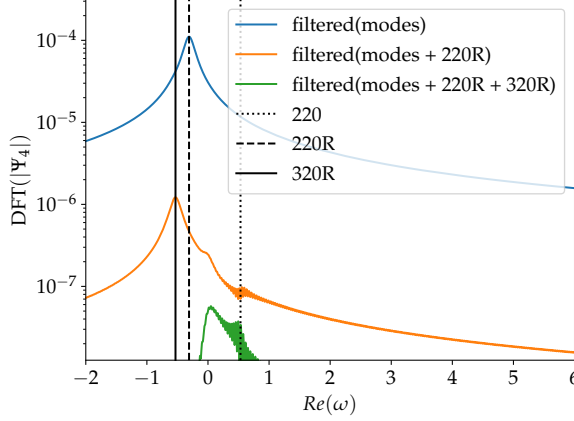


Figure 2.2: Fourier transform of a gravitational wave signal on a fixed background and remnant spin $a/M = 0.7$. Here we see that there is some evidence for both the 220_R and the 320_R frequency in the data. Here “modes” corresponds to: $(22n)$ with $n = 0, \dots, 7$; (320) ; $(221)_R$; and $(222)_R$.

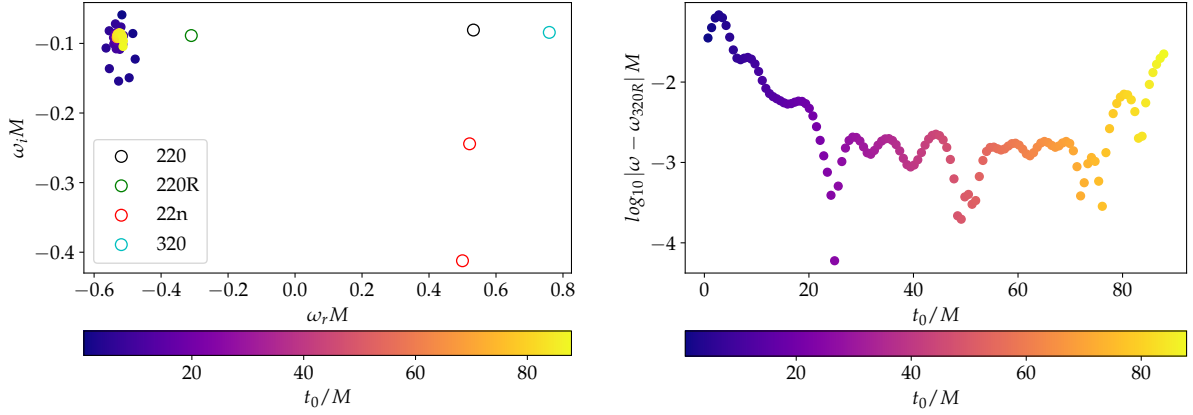
After determining the mode content, we perform fixed-frequency fits. We use the model

$$\psi^{\text{fixed-freq}} = \sum_1^N A_N \exp[-i\omega_N t], \quad (2.17)$$

where the free parameters are $A_N \in \mathbb{C}$, and ω_N are the fixed frequencies of quasinormal modes calculated using linear perturbation theory. So for N modes, this model has $2N$ real free parameters. In order to determine the physical amplitude and extract it, we vary the time window of data considered in the fit, as in Ref. [28]. We introduce an arbitrary reference time to describe the fitted amplitudes in a time-independent way. The time-independent amplitude is defined relative to the amplitude fit at $t = t_0$,

$$|A_N^{t_{\text{ref}}}| = |A_N^{\text{fit}}(t_0)| \exp(-\mathfrak{I}\omega_N(t_0 - t_{\text{ref}})). \quad (2.18)$$

Then a region where the fitted amplitude has the expected decay rate corresponds to a region where the time-independent fitted amplitude is constant. We vary the start of the time window and check if the time-independent fitted amplitude is constant in some region. This region would correspond to a time window where the mode is resolved, and where the signal is well described by a superposition of quasinormal modes with fixed amplitudes.



(a) Free frequency fits in complex plane

(b) $|\omega_{\text{fit}} - \omega_{320R}| M$

Figure 2.3: We perform a free frequency fit on Ψ_4 after filtering out previously identified modes except for $(\ell, m, n) = (320)_R$. Fits are performed for data in the window $t \in (t_0, 100M)$. In the complex plane, the free frequency fits are clustered around the physical value of ω_{320R} . In (b), we plot the absolute difference between $\omega_{\text{fit}} = \omega_r + i\omega_i$ and the physical ω_{320R} . From $|\omega_{\text{fit}} - \omega_{320R}| M$, we can see that in the first $\sim 50M$ the fit frequencies converge to the value of ω_{320R} . At later times, the frequency fits diverge from this as the fitting window becomes small. From the combination of these results and Fig. 2.2, we conclude that $(320)_R$ is present in the data.

We fit mode amplitudes in the time domain using a least-squares algorithm² to minimize the difference between the fit model and the signal.

In general, we apply filters to the data before fitting mode amplitudes. Note that applying the filter to a mode will affect the amplitude of that mode. Applying a filter F_ω to a mode with frequency ω' gives

$$\begin{aligned} & F_\omega \cdot \exp(i\omega't + \phi) \\ &= \begin{cases} 0 & \text{if } \omega = \omega', \\ A^{\text{filter}}(\omega, \omega') \exp(i\omega't + \phi') & \text{otherwise.} \end{cases} \end{aligned} \quad (2.19)$$

This amplitude change has the form

$$A^{\text{filter}}(\omega, \omega') = \frac{\omega' - \omega}{\omega' - \omega^*}, \quad (2.20)$$

where ω^* is the complex conjugate. Applying multiple filters results in an amplitude change given by the product of these suppression factors. We use Ref. [202] to calculate quasinormal mode frequencies to high precision as input to $A^{\text{filter}}(\omega, \omega')$. Then we divide any amplitudes fit to filtered data by this suppression factor to recover the amplitude in the unfiltered signal. We test our numerical computation of the filter factor by comparing the amplitude change of a filtered pure mode to the product of $O(10)$ suppression factors. We find agreement to the level of 0.01%. We weight extracted amplitudes both by their suppression by the filter and by the spherical-spheroidal mixing component, so the final amplitude results are with respect to the spheroidal basis.

2.4 Results

In Sec. 2.2, we determined to leading order how we expect the mass and spin of the black hole to change with time in the presence of a particular quasinormal mode. Here we present the results of our numerical experiments, probing how a changing black hole background affects a single perturbing quasinormal mode. The change in background is second-order in perturbing amplitude, so the change in the perturbation is expected to be third order,

$$\Psi_4 = \Psi_4^{\text{initial}}[O(\mathcal{A})] + \Psi_4^{\text{excited}}[O(\mathcal{A}^3)]. \quad (2.21)$$

²We use the Levenberg-Marquardt method as implemented in `scipy.optimize.least_squares`. However, since the amplitude is a linear parameter, it can be solved exactly, without optimization, by inverting a matrix representation of the Fourier transform of the model.

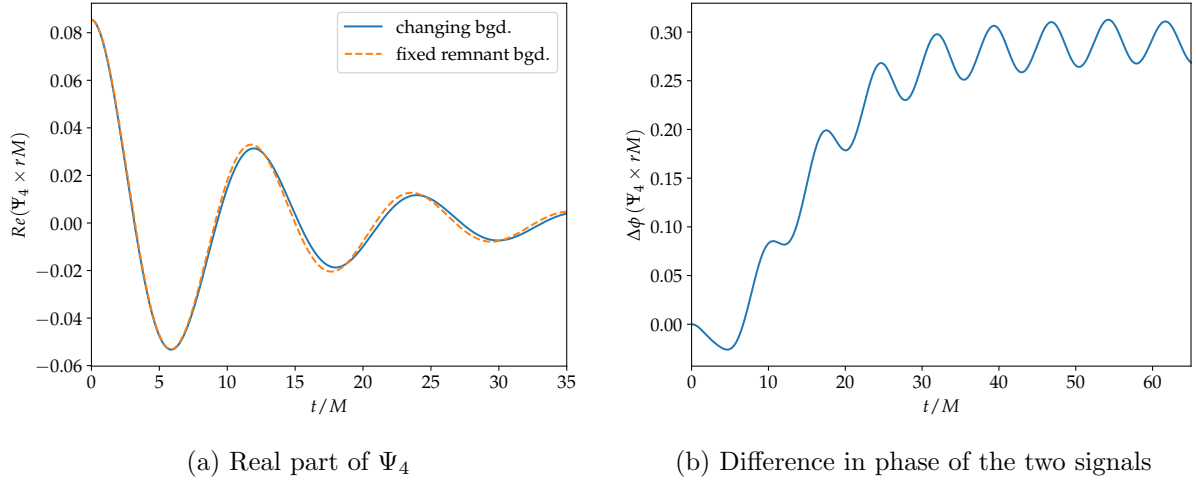


Figure 2.4: (a) The real part of Ψ_4 at null infinity for a changing background case with $\Delta M/M = 4.7\%$ compared to a fixed black hole background. Both cases have identical initial perturbation and remnant black hole parameters $M_{\text{BH}}(\infty) = M$, $a_{\text{BH}}(\infty) = a = 0.7M$. The change in the signal is on the level of a few percent. (b) The phase difference between the two signals. The phase difference asymptotes to a constant at late times.

In this section, we study the change in a pure quasinormal mode perturbation due to the change it induces on the black hole. Given an initial perturbation, we can calculate the resulting background change. Using our flux calculation, we determine an initial black hole that, in the presence of the fundamental ($\ell = 2, m = 2, n = 0$) quasinormal mode of that remnant, will change mass and spin to have exactly the properties of that remnant. We use the remnant quasinormal mode as initial data on a background that changes by ΔM and Δa from the determined initial black hole to the remnant according to our flux calculation. So we set the background to have mass and spin:

$$M_{\text{BH}}(t) = M - \Delta M \exp[2\mathfrak{I}\omega_{220}t], \quad (2.22)$$

$$a_{\text{BH}}(t) = a - \Delta a \exp[2\mathfrak{I}\omega_{220}t]. \quad (2.23)$$

We compare this case with the same initial data dissipating on the fixed remnant background with mass M and spin a , i.e. a pure quasinormal mode, in Fig. 2.4. As we will detail below, the perturbation to the quasinormal mode from the changing background can be characterized by both a change to the amplitude and phase of the original, fundamental quasinormal mode, as well as the excitation of additional quasinormal modes.

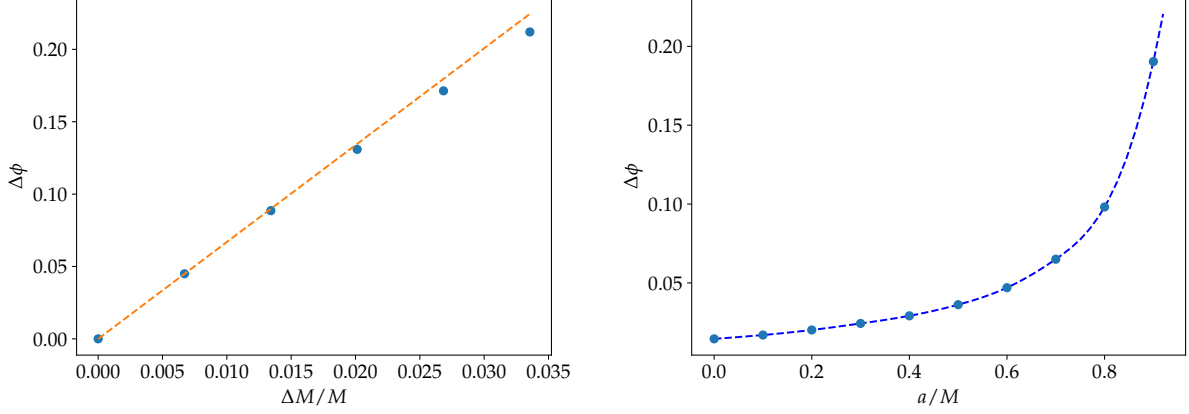
We evolve the linear Teukolsky equations for Ψ_4 on a black hole background evolving according to Eqs. (2.22) and (2.23), with some small ΔM and $\Delta a \propto \Delta M$. When computing the resulting shift in Ψ_4 with respect to the fixed background case (i.e. a pure quasinormal mode), to leading order we will get a linear-order dependence on ΔM , and an additional linear dependence on $\mathcal{A}$ (the initial amplitude of Ψ_4), which combine to give the cubic dependence in Eq. (2.21). This means that our results can be rescaled to different values of ΔM , e.g., $\Delta A_{\ell mn}/\mathcal{A} \propto \Delta M$ for an excited mode amplitude, provided this quantity is sufficiently small.

Technically, the coefficients in the Teukolsky equation depend nonlinearly on the black hole mass and spin, but we demonstrate that we recover the expected scaling for the values within the range of interest by simulating several systems with varying ΔM and showing the resulting change in gravitational wave signal in Fig. 2.5a. Here we use the accumulated phase offset $\Delta\phi$ as a relatively clean measure of the change in signal. $\Delta\phi$ is the phase of $\Delta\Psi_4$, $\Delta\phi = \phi(\Delta M) - \phi(\Delta M = 0)$, which asymptotes to a constant at late times. As shown, for these choices of $\mathcal{A}$, the leading-order term dominates, and we have $\Delta\phi \propto \Delta M$ within 6% for $\Delta M/M \lesssim 3.5\%$.

In Fig. 2.5b, we show the dependence of the accumulated phase on spin. The phase offset increases significantly for higher spin, which can be explained by the quasinormal mode frequency shifts. In general, for some fixed change in dimensionless spin, the change in quasinormal mode frequency is larger for larger initial spin i.e., $d\omega_{220}/d(a/M)$ increases monotonically with a/M .

This change in gravitational wave signal can be characterized by a change in the perturbing mode amplitude ΔA_{220} , along with the excitation of a superposition of extra modes, with excited amplitudes $A_{\ell mn}$. To determine which modes are excited, we use a combination of frequency-space filtering and free frequency fits in the time domain, according to the methods described in Sec. 2.3.2. Once the relevant modes are determined, we can fit their amplitudes in the time domain.

When analyzing Ψ_4 at null infinity, we find retrograde modes, higher ℓ modes and, potentially, overtones. We recall from the last section that retrograde modes are modes that travel in the opposite direction around the black hole to its spin, see Eq. (2.15). We perform a mode analysis for several spins, fitting amplitudes in each case as outlined in Sec 2.3.2. The results of fitting the $\ell = m = 2$ spherical harmonic are shown in Fig. 2.6. There is some ambiguity in the overtone fit, where the overtone amplitude appears to increase at early times. This may be due to the presence of nonmode content with frequency close to the (221) overtone. This feature will be discussed in detail in Sec. 2.5. Here we provide our best fit for the overtone amplitude with the caveat that the fit is not exact. Apart



(a) Dominance of linear contribution with $a/M = 0.7$ (b) Dependence of $\Delta\phi$ on spin with $\Delta M/M = 1\%$

Figure 2.5: Difference of signal due to the third-order horizon flux effect. Comparing a signal with changing background $\Psi_4(\Delta M)$ to a signal from a fixed background $\Psi_4(0)$. We calculate the accumulated phase offset due to the changing background by calculating $\Delta\phi = \phi(\Delta M) - \phi(\Delta M = 0)$, which asymptotes to a constant at later times ($t \geq 40M$, see Fig. A.1). In (a), the phase offset is calculated for several mass changes. The dashed line is a linear fit to the first two points. We determine that the relative change in signal is dominated by a term linear in the mass and spin change for the range plotted, with a 6% deviation for $\Delta M/M = 3.4\%$. (b) The dependence of the phase offset on spin. In this case, the dashed line is an interpolation of the points.

from the overtone, the amplitude of the other modes can be confidently measured without any significant dependence on the extraction time, as shown below. We also perform a mode analysis on higher-order spherical harmonics, with $\ell \neq 2$. The result of this analysis is shown in Fig. 2.7.

In Appendix A.4, we compare the results above to those obtained using a variation on the initial data where we use a quasinormal mode of the initial black hole, rather than the final black hole, as an initial perturbation. We find that the same modes are excited in both cases, though quantitatively, the excitation of additional modes is larger when the initial data correspond to a quasinormal mode of the initial black hole. This means the results shown here are conservative in terms of estimating the size of this effect.

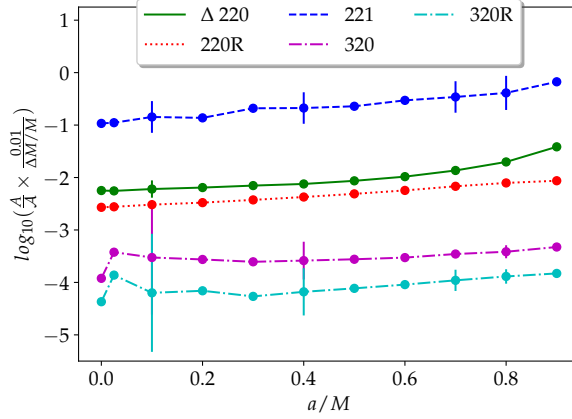


Figure 2.6: Amplitudes of excited modes $A_{\ell mn}$ as function of spin. These amplitudes are fit from the $(\ell m) = (22)$ spherical harmonic projection of the data. Higher ℓ modes are less excited for lower spin and become poorly resolved below $a/M = 0.3$ —in the $a/M = 0$ case the mixing coefficient between the $\ell = 3$ spheroidal harmonic and the $\ell = 2$ spherical harmonic becomes zero. In this case we fit the $\ell = 3$ result from the $\ell m = 32$ spherical projection of the data. These amplitudes are all calculated for a mass change $\Delta M = 1\%$ and a spin change consistent with that. The error bars are calculated for some of the spin cases, they include the numerical error and error due to uncertainty in lookback time. The numerical error is determined by comparing the values extracted from simulations with two resolutions (see Appendix A.5). The error due to an uncertainty in lookback time δt is a factor of $\exp[i(\omega_{\ell mn} - \omega_{220})\delta t]$.

2.5 NonMode Content

Here we discuss the ambiguity in the overtone amplitude extraction and present some evidence for nonmode content in the data. This nonmode content is only present in cases with a background that evolves according to our flux calculation. We discuss the possible effect of a changing frequency in the data and how this kind of nonmode content can manifest.

2.5.1 Ambiguity in overtone extraction

When using a changing background, we find some evidence for an additional component to the gravitational wave signal that is not described by a sum of quasinormal modes with

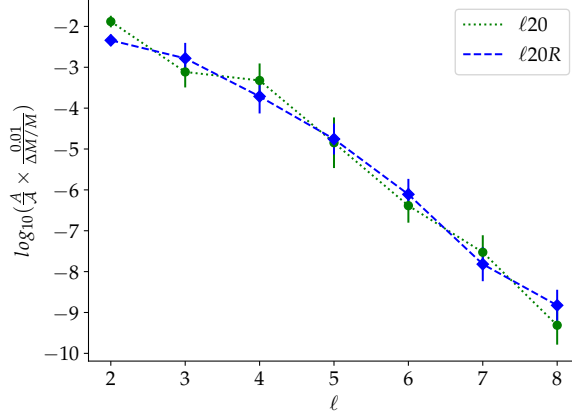


Figure 2.7: Amplitudes of excited modes $A_{\ell 20}$ for a remnant with spin $a/M = 0.7$ and perturbing amplitude such that $\Delta M/M = 1\%$. These modes were fit from spherical modes $\psi_{\ell', m}$ with $\ell = \ell'$, $m = 2$. The error bars are the numerical error, determined by comparing the values extracted from simulations with two resolutions and assuming first order convergence in $A/\mathcal{A}$ (see Appendix A.5).

fixed frequency. We first see this nonmode content in the frequency domain, using the filter analysis described in Sec. 2.3.2. An example Fourier analysis is shown in Fig. 2.8. In contrast to Fig. 2.2 above, where a fixed background was used, the spectrum in Fig. 2.8 indicates a component to the signal with similar real frequency to the (220) mode, even though this mode and seven of its overtones have been removed by filters.

We use free frequency fits to better understand this contribution to the signal. For an example with remnant spin $a/M = 0.7$, all identified modes are filtered from the signal, and a free frequency fit is performed. This fit is shown in Fig. 2.9. We confirm the result from the frequency-domain analysis that there is some nonmode content with real frequency close to the (22n) modes. We see that this part of the signal has a similar decay rate to the first overtone, or 3 times the decay rate of the fundamental mode. At very early times, the fit is close to $\omega = \Re\omega_{220} + i\Im\omega_{221} \approx \Re\omega_{220} + i3\Im\omega_{220}$. Since this frequency is close to the first overtone it will be highly suppressed by the filter relative to any numerical noise or other mode features. This means that this feature is more significant, and better resolved at later times, in data with fewer overtone filters (in particular, without filtering the first overtone). We gain some insight into a possible source for nonmode content with a similar decay rate to the first overtone from a spherically symmetric model below, in Sec. 2.5.2, and conclude that this may be due to a changing frequency contribution.

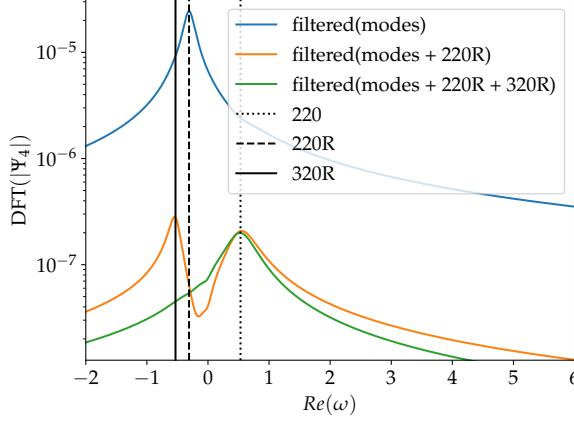


Figure 2.8: The results of performing a DFT on the gravitational wave signal with an evolving background and remnant spin $a/M = 0.7$. We perform the same analysis as in Sec. 2.3.2, Fig. 2.2. Again, we see that there is some evidence for retrograde frequencies in the data. Recall that modes corresponds to the list: (220), (320), (221), (222), (223), (224), (225), (226), (227), (221)_R, (222)_R. Here, even after (220) and seven of its overtones are filtered out, there is a peak at $\Re\omega_{220}$. Filtering out more overtones does not seem to remove this peak.

Still considering the same example, in Fig. 2.10a we filter the same modes as in Fig. 2.9, except for the $(22n)$ overtones with $n \in \{1, 2, 3\}$, and perform a free frequency fit. We see a shift in the fit decay rate (the imaginary part of the frequency), indicating that this component has some overlap with the overtones or is otherwise affected by the overtone filters. It seems that at early times the free frequency is converging to the first overtone, but that the overtone stops being resolved before the frequency reaches a constant value in time. We see frequencies with similar real part of the first overtone and the fundamental mode, with a decay time slower than the overtone. At $t = 10M$ the free frequency fit gives a result close to the real frequency of the fundamental mode and with decay rate $\approx 10\%$ slower than the first overtone.

For comparison, in Fig. 2.10b we consider a case where the initial data are the (220) quasinormal mode of the initial black hole, instead of the final black hole. In this case we more confidently identify an overtone amplitude and the frequency of the free frequency fit is closer to the remnant overtone at later times. We speculate that the reason this does not happen for the remnant perturbation case is that the overtone amplitude is smaller, and becomes unresolved at earlier times. We also recall that, in the instantaneously changing

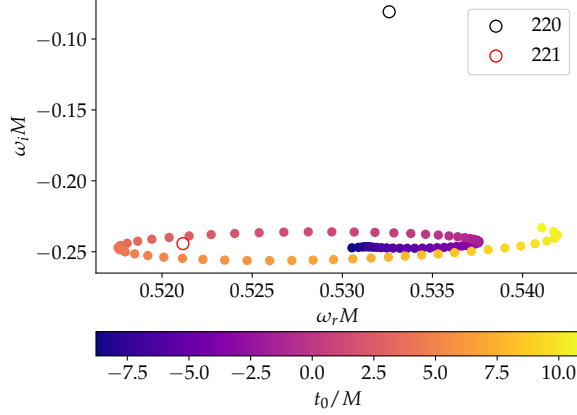
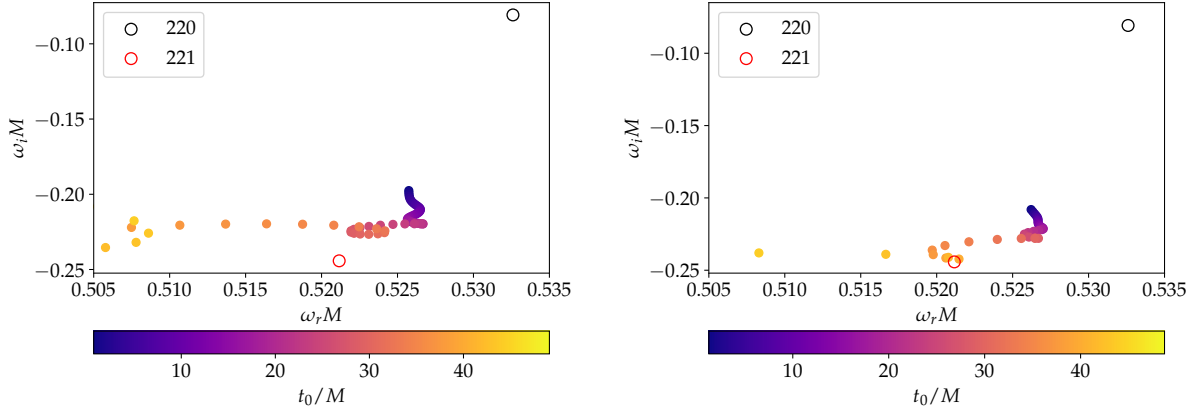


Figure 2.9: A free frequency fit is performed after filtering out all identified modes, including seven overtones of $\ell = 2$, $m = 2$. The remnant is a black hole with dimensionless spin $a/M = 0.7$, and the background evolves according to the absorption of a (220) mode, with $\Delta M/M = 0.67\%$. The x axis here stretches back before the lookback time (defined as $t = 0$, see Sec. 2.3.1) to show the behavior from the start of the simulation. The initial perturbation is the (220) mode of the remnant black hole. Note that this fit identifies a frequency with $\omega \approx \Re\omega_{220} + i\Im\omega_{221} \approx \Re\omega_{220} + i3\Im\omega_{220}$ at early times, where these two complex frequencies are indicated, respectively, by the black and red circles.

background case shown above (Fig. 2.2) there is no evidence for any nonmode content, and the overtone amplitude is extracted cleanly. We further discuss these different initial data and background models in Appendix A.4.

As anticipated above, when fitting explicitly for the (221) overtone we find a time dependence in the measured amplitude not present in the other quasinormal modes. This is shown in Fig. 2.11, where we plot the nominally time-independent amplitudes with fit time (defined relative to an arbitrary reference time by Eq. (2.18)) extracted using the fixed-frequency model in Eq. (2.17). At early times the inferred amplitude of the (221) can be seen to increase by a factor of a few at roughly an exponential rate. (At late times, it increases even more rapidly, but this behavior is expected since this mode decays much faster than the other modes, and eventually becomes buried in the noise.) This means that this residual mode can be interpreted as a (221) mode with a shifted decay rate and indeed, picking a new quasinormal mode frequency from a free frequency fit at $t = 10M$ as our template, we find that the extracted amplitude becomes more flat in fit time (labeled by “new mode” in Fig. 2.11). We find that the shift in decay rate is spin dependent, increasing



(a) Remnant quasinormal mode perturbation

(b) Initial quasinormal mode perturbation

Figure 2.10: A free frequency fit is performed after filtering out modes except for the first three overtones (221), (222), (223). The remnant black hole has dimensionless spin $a/M = 0.7$, and the background evolves according to the absorption of a (220) mode, with $\Delta M/M = 0.67\%$. These two plots correspond to two different initial data cases. (a) An initial perturbation equal to the (220) quasinormal mode of the “remnant” black hole. (b) An initial perturbation equal to the (220) quasinormal mode of the “initial” black hole i.e., the black hole with mass $M - \Delta M$ and spin $a - \Delta a$. In this case, the quickly decaying mode is resolved at later times and is shown to approach the overtone frequency with more confidence. We discuss this initial data case further in Appendix A.4.

for an increasing rate of background change.

Although the measurement is somewhat ambiguous, we choose to extract an overtone amplitude at the time when the amplitude is smoothest with variations in the fitting window (and the free frequency most closely matches the overtone) to obtain the results described in Sec. 2.4, with the caveat that they do not represent a fully resolved overtone. For the example in Figs 2.10a and 2.11, this corresponds to $t = (25 - 30)M$. We find that, when considering higher black hole spins, the region of maximally flat amplitude in fit time moves to later times. This makes sense, since the timescale of the background change is half the timescale of the perturbing mode, which increases with increasing spin. In general, we find that the most flat region for the overtone fit corresponds approximately to the time $t = 2/\mathfrak{I}\omega_{220}$.

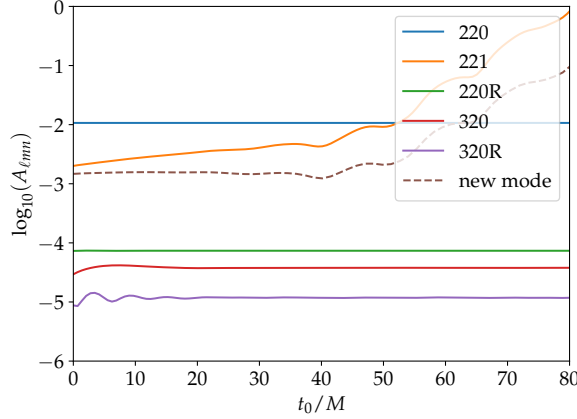


Figure 2.11: Fitted quasinormal mode amplitudes relative to $t = 0$ for a remnant black hole with spin $a/M = 0.7$ and a background changing such that $\Delta M/M = 1\%$. These amplitudes are defined in Eq.s (2.18) and (2.17). We apply filters and do independent fits for fundamental modes [i.e. with $N = 1$ in Eq. (2.17)]. We fit the first overtone together with the $n = 2$ and 3 overtones (setting $N = 3$). We also show a fit with a quasinormal mode with a frequency of $\omega = 0.526 + 0.221i$ (labeled new mode) that comes from a free frequency fit, as opposed to being a quasinormal mode frequency of the remnant black hole.

2.5.2 Changing frequency model

With the goal of understanding the nonmode content to the ringdown signal indicated by the above results, we now consider a changing frequency model, along the lines of the one accounting for the third-order changing background effect [179]. For simplicity, we will focus on an example with zero angular momentum, in particular, considering a nonspinning black hole with an $\ell = 2$, $m = 0$, $n = 0$ quasinormal perturbation. This means that the instantaneous frequency will only evolve according to the change in the black hole's mass. Modifying the changing frequency model of Ref. [179] to integrate the phase up to each

time instead of using the instantaneous frequency, we have

$$\phi(t) = \int_{t_0}^t \omega_{200}(t') dt' = \int_{t_0}^t \frac{m_1 \omega_{200}}{m(t')} dt' \quad (2.24)$$

$$= \left(\frac{\omega_{200} m_1}{-2\mathfrak{I}\omega} \right) \log \left(\frac{\exp(-2\mathfrak{I}\omega t) - \Delta M}{\exp(-2\mathfrak{I}\omega t_0) - \Delta M} \right),$$

$$\Psi = \tilde{A} \left[1 + \tilde{Q} \frac{\delta m(t)}{m_2 - m_1} \right] \exp(i\phi(t)). \quad (2.25)$$

We introduce the start time t_0 as a fitting parameter, since we do not know the time of propagation from the horizon to null infinity with high accuracy. Here ω_{200} is the fundamental mode of the initial black hole with mass m_1 , and $m(t) = m_1 + \delta m(t) = m_1 + \Delta m(1 - \exp(2\mathfrak{I}\omega_{200}t))$. The other free parameters are $\tilde{A} \in \mathbb{C}$ and $\tilde{Q} \in \mathbb{C}$ ³.

We consider an example on an evolving background with $\Delta M/M \approx 2\%$ and find the same behavior as in the spinning cases in the overtone amplitude (as was shown in Fig. 2.11) as well as when performing a free frequency fit after filtering out the fundamental and retrograde modes (as shown in Fig. 2.9). This example is thus representative of the behavior described above. We compare how well the changing frequency model does in fitting the ringdown in this example compared to a fixed-frequency model with two modes: (200) and (201). As shown in Fig. 2.12, we find the changing frequency model captures the gravitational wave signal better during the first $10M$, but does worse at later times when we expect the signal to asymptote to a sum of quasinormal modes (while the changing frequency model will asymptote to only the fundamental mode).

Finally, we can illustrate how a changing frequency component would affect our fixed-frequency mode analysis by applying it directly to the waveform from Eq. (2.24). Using the best-fit parameters from the above example and first applying a (200) quasinormal mode filters, and then fitting a free frequency, we obtain the results shown in Fig. 2.13. For the first $\approx 60M$ (after which the results are not well resolved), the analysis identifies a mode with real frequency $\approx \Re\omega_{200}$, but imaginary frequency $\approx \mathfrak{I}\omega_{201}$. However, we note that, in fact, $\mathfrak{I}\omega_{201} \approx 3\mathfrak{I}\omega_{200}$ (to within 2.5%). Some component with the latter decay rate is expected in Eq. (2.24) for a fundamental quasinormal mode with frequency ω_{200} that is interacting with (multiplying in the evolution equation) a changing mass/spin component to the black hole background that goes like $\sim \exp(2\mathfrak{I}\omega_{200}t)$.

³In [179], the value of $\tilde{Q}$ is fit to data, and should be the same for all cases. Since there is some uncertainty in this fit, we make $\tilde{Q} \in \mathbb{C}$ a free parameter to give this model the best chance of matching the data. In fact, we find that setting $\tilde{Q} = 0$ in our integrated phase model does not significantly affect the fit performance (while it does seem to make an improvement in the adiabatic approximation model from Ref. [179]).

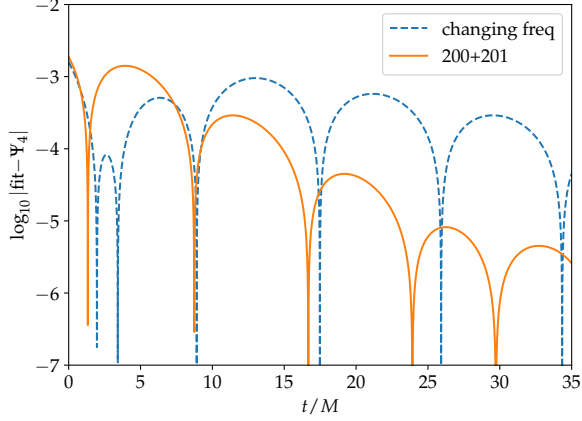


Figure 2.12: A comparison of a changing frequency model [Eq. (2.24)] to a fixed-frequency model with fundamental and overtone (200) + (201). We show the real part of the residual of two fits to Ψ_4 . As only the relative values of the two residuals are important, the overall scale of the y axis is arbitrary.

Based on these results, we can conclude that in the ringdown on the evolving background, for both the spherically symmetric and spinning cases, there are components that one can attribute both to the presence of a changing fundamental frequency, as well to the excitation of an overtone. Because of the similar, and short, decay times of these two features, they are hard to disentangle. On the one hand, one has a decay rate of $3\mathfrak{I}\omega$ for the cubic effect of a mode interacting with the changing black hole properties at twice the mode's decay rate. On the other hand, in the WKB approximation [216], the imaginary component of the quasinormal frequency changes with overtone number as $\propto (n + 1/2)$, which means that, in general, $\mathfrak{I}\omega_{n=1} \approx 3\mathfrak{I}\omega_{n=0}$; this holds to $< 1\%$ for the $a/M = 0.7$ case shown in Fig. 2.9, which shows similar behavior at early times.

2.6 Discussion and Conclusion

2.6.1 Summary of main results

In this Chapter, we have studied the nonlinear effect on a ringdown gravitational wave signal due to the changing black hole mass and spin attendant in the ringdown process. Our analysis is based on promoting the mass and spin in the Teukolsky equation governing

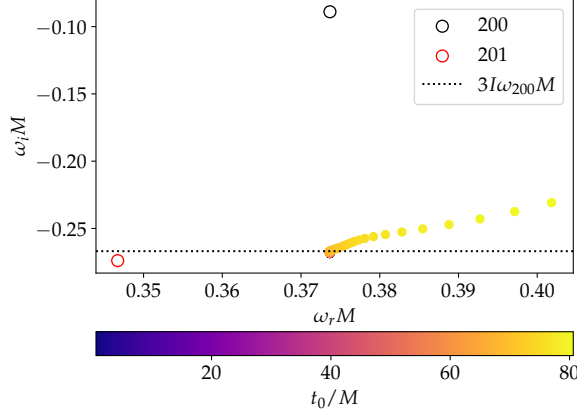


Figure 2.13: The results of a free frequency fit on the changing frequency model given by Eq. (2.24) after filtering out the fundamental mode. The points with $t < 50M$ points are stacked closely together and have $\omega_I \approx \Im\omega_{201} \approx 3\Im\omega_{200}$.

linear metric perturbations to be functions of time, as prescribed by the flux of gravitational wave energy and angular momentum into the black hole due to a quasinormal mode. Though somewhat simplistic, it allows us to isolate and quantify one particular nonlinear effect in black hole ringdown, among many that may be present, and test the consistency of assuming linear evolution in a given regime. We find that, beginning from single least-damped quasinormal mode, this effect causes a phase shift and amplitude diminishment in the fundamental mode, while exciting overtone, retrograde, and higher ℓ modes. In addition, we find evidence that as the black hole background evolves at earlier times (e.g. $t \lesssim 30M$ for $a/M = 0.7$), there is a component to the signal that cannot be described purely as a superposition of remnant quasinormal modes, likely due to a changing frequency contribution.

2.6.2 Comparison to binary black hole ringdown

The results described here can also be taken as a self-consistency check for when a black hole ringdown signal is in the linear regime: a ringdown signal can only be said to be described purely by a sum of quasinormal modes to the degree to which the nonlinear effects described here (as well as others) can be ignored. We can compare the magnitude of the effects found here to the ringdown signals found in comparable mass binary black hole mergers. For a range of binaries, the relative excitation of the first overtone and

retrograde mode have been measured by Ref. [70]. We present those amplitudes, together with our estimate of the contribution due to AIME for a case with a remnant spin of $a/M = 0.7$ (computed as strain amplitudes). For the latter, we include the values both when using initial data corresponding to a fundamental quasinormal of the remnant black hole and corresponding to a quasinormal mode of the initial black hole.

Table 2.1: Estimate of AIME in comparison with total excitation from numerical relativity (NR). The errors shown are due to an uncertainty in reference time.

	h_{221}/h_{220}	h_{220R}/h_{220}
NR	4 ± 1	$10^{-4} \pm 10^{-4}$
AIME (remnant ID)	$(0.34 \pm 50\%) \times \frac{\Delta M/M}{0.01}$	$(2.1 \times 10^{-2} \pm 5\%) \times \frac{\Delta M/M}{0.01}$
AIME (initial BH ID)	$(0.4 \pm 50\%) \times \frac{\Delta M/M}{0.01}$	$(3.5 \times 10^{-3} \pm 5\%) \times \frac{\Delta M/M}{0.01}$

In Table 2.1, $\Delta M/M$ is the mass change in the linear regime due to the absorption of the fundamental (220) mode. As described above, estimating the magnitude of the least-damped quasinormal mode from the postpeak gravitational wave signal implies a $\sim 1\%$ in the black hole mass and $\sim 6\%$ change in the spin, though there is some ambiguity in connecting the time at null infinity with that at the black hole horizon. This is a factor of few smaller than the change of mass of the final apparent horizon found in binary black hole simulations.

In this example, we estimate that the AIME effect excites the first overtone amplitude at $(5-20)\%$ the value found in binary black hole mergers. On the other hand, we estimate the retrograde excitation due to this effect to be $(20-75)\times$ the value fit to numerical relativity simulations. Of course, in a binary black hole merger, various quasinormal modes—in addition to the least-damped one—will be significantly excited by the dynamics of the merger, which will both add (constructively or destructively) with those generated by nonlinear effects in the ringdown, as well as affect how the black hole mass and spin changes at early times. We note, for example, that the ratio between the change in the apparent horizon mass and spin from the simulation shown Fig. 2.1 is noticeably different from that expected from a (220) mode. It would be interesting for future work to calculate the horizon flux due to a superposition of quasinormal modes and see how the resulting evolution of black hole mass and spin affect the results found here.

We can also compare the significance of this third-order effect with the second-order mode doubling effect. The strain amplitudes of mode doubled modes are fit in Refs. [227,

[155, 178, 70] for various numerical relativity simulations. Here we compare to Ref. [70], where they find that these modes are of order $h_{220 \times 220}/h_{220}^2 = (0.125 - 0.175)$, and $h_{220 \times 320}/(h_{220}h_{320}) = (0.4 - 0.6)$. The mode doubled amplitudes are also calculated analytically in [146], where for $a/M = 0.7$, they find $h_{220 \times 220}/h_{220}^2 = (0.1233)$. As an example, in the binary black hole merger described in Sec. 2.2.2 (with mass ratio of 1.2, remnant spin $a/M = 0.69$ and low eccentricity [132]), the measured mode amplitudes are $h_{220} \approx 1$ and $h_{320} \approx 0.03$ [71]. To compare these effects, we can use ratios of the time-integrated strain squared, which would roughly correspond to the square of the signal-to-noise ratio (SNR) assuming white noise. That is, the ratio of the SNR of a particular mode (ℓmn) to that of another mode $(\ell' m' n')$ is given by

$$\frac{\text{SNR}_{\ell mn}}{\text{SNR}_{\ell' m' n'}} = \sqrt{\frac{\int_0^\infty dt |h_{\ell mn}|^2}{\int_0^\infty dt |h_{\ell' m' n'}|^2}} = \frac{|A_{\ell mn}| \sqrt{-\Im \omega_{\ell' m' n'}}}{|A_{\ell' m' n'}| \sqrt{-\Im \omega_{\ell mn}}}, \quad (2.26)$$

where $A_{\ell mn}$ are the amplitudes at $t = 0$. Taking the signal from the peak, with amplitudes from Table 2.1 and the example case referenced above, this gives $\text{SNR}_{221}^{\text{AIME}}/\text{SNR}_{220 \times 220} \approx (2 - 3)$ and $\text{SNR}_{220R}^{\text{AIME}}/\text{SNR}_{220 \times 320} \approx (0.3 - 2)$. Recall that in the overtone case we do not extract a stable amplitude until late times. If we calculate instead the SNR for the overall change in signal with a single quasinormal mode, we find $\text{SNR}^{\text{AIME}}/\text{SNR}_{220 \times 220} \approx 0.2 - 0.5$. According to this naïve approximation, the third-order AIME effect is comparable in magnitude to second-order mode doubling.

Following Ref. [176], we calculate the maximum ringdown SNR such that the pure 220 mode signal and the signal with our AIME result included are indistinguishable, or ρ where the mismatch $\mathcal{M}$ satisfies

$$\mathcal{M}(h_{220}, h_{\text{AIME}}) < \frac{D}{2\rho^2}, \quad (2.27)$$

where D is the number of model parameters. We calculate mismatch in the time domain using

$$\mathcal{M}(f(t), g(t)) = 1 - \frac{\int_{t_0}^{t_f} (f(t) * g(t)) dt}{\sqrt{\int_{t_0}^{t_f} (f(t) * f(t)) dt \int_{t_0}^{t_f} (g(t) * g(t)) dt}}. \quad (2.28)$$

We approximate $\mathcal{M}(h_{220}, h_{\text{AIME}})$ by the mismatch between the signals with and without a changing background, where t_0 is set to the lookback time and $t_f = 60M$. We find the range of SNRs for indistinguishability,

$$\rho < 60 \left(\sqrt{\frac{D}{4}} \right) \left(\frac{0.01}{\Delta M/M} \right). \quad (2.29)$$

For reference, the postpeak SNR of GW150914 is ~ 14 [127], which would correspond to $\Delta M/M \approx 4\%$ in the above. According to our leading-order approximation, this effect is only starting to be relevant for current ringdown analyses from the peak.

2.6.3 Limitations and directions for future work

In this study, we have focused on a particular nonlinear effect in black hole ringdown, due to the changing black hole mass and spin, while ignoring other nonlinear effects. Our numerical evolution code simulated a changing background by changing the evolution operators everywhere on the hyperbolic time slice together, which introduces an unphysical instantaneousness, and ignores the mass and angular momentum of the gravitational perturbation outside the black hole horizon. As well, we have only calculated the leading-order contribution to the black hole mass and spin change, and ignored how backreaction may affect this. While doing so allowed us to isolate, and estimate in a straightforward manner, the effect of the changing black hole mass and spin, it would be desirable for future work to perform a fully relativistic calculation of black hole ringdown that would quantify this effect (as well as other nonlinear ringdown effects). This could be done using numerical relativity simulations of isolated black holes with a perturbation whose amplitude is varied [94, 193, 227], but in the spirit of the calculation performed here, beginning with initial data where the perturbation closely approximates a single quasinormal mode of a Kerr black hole.

In the setup studied here, we found evidence for nonmode content in the ringdown, which we attempted to account for using a simple changing frequency model. A corresponding effect has not so far been identified in the ringdown of binary black hole simulations and may be artificially magnified by our setup, or may be hard to disentangle from the nonlinearities in the merger. However, it might be beneficial, for future work, to further develop such models so that they can better capture mildly nonlinear effects in black hole ringdown signals.

Chapter 3

Self Gravity Effects in Superradiant Cloud Graviational Wave Signals

3.1 Introduction

The black hole superradiant instability occurs for minimally coupled ultralight scalar and vector bosons¹, therefore, it can probe particles with arbitrarily weak couplings to Standard Model particles, complementing terrestrial experiments. Black hole spin measurements and gravitational waves from binary black hole mergers can be used to place constraints on the existence of ultralight bosons [19, 18, 60, 160, 33, 205]. The gravitational waves emitted by a dissipating boson cloud fall in the LIGO-Virgo-KAGRA sensitivity band [2, 11, 23, 14] for boson masses in the range from 10^{-13} to 10^{-11} eV, while boson masses in between 10^{-18} and 10^{-15} eV correspond to LISA's future sensitivity range [15]. Stochastic background searches and blind continuous wave searches can be used to look for signals for some unknown population of black holes that has undergone the superradiant instability [7, 211, 210, 167, 9, 49, 21, 19, 219, 18, 50, 60, 229], though placing constraints in the absence of a detection requires one to make assumptions about the (largely unknown) black hole population. Alternatively, one can consider targeting specific known black holes [206]. In particular, performing follow-up searches of the merger remnants of observed binary black hole mergers has the advantage that the properties of a merger remnant can be estimated from the merger signal, and so specific boson masses can be constrained with minimal assumptions [128, 64, 130]. However, this is feasible only for the fastest evolving

¹The superradiant instability can also occur for massive spin-2 fields [46, 84], we will address the question of spin-two superradiance in Chapter 4.

boson clouds and loudest gravitational wave signals. For these, it is important to have a relativistically accurate gravitational waveform model, a crucial aspect of which we develop in this work.

The gravitational wave signal from a superradiant cloud is mostly monochromatic, but with a small upward drift in the frequency due to the changing mass of the boson cloud. Not accounting for this limits the timescales over which the signal can be coherently tracked by gravitational wave search methods. The frequency shift due to the self-gravity of the cloud was first calculated in Refs. [33] and [128, 32], in the Newtonian limit of the fastest growing vector and scalar modes, respectively. In Ref. [198], this was extended to higher azimuthal modes and compared to a quasirelativistic calculation introduced in Ref. [196]. The latter is still based on evaluating the Newtonian expressions for the frequency shift, but employs relativistic test-field solutions to calculate the energy density. More recently, Ref. [57] used relativistic perturbation theory to estimate this effect, but still treating the perturbation from the cloud’s self-gravity in terms of a Newtonian potential, and found an improvement from the nonrelativistic limit.

As we show below, these previous nonrelativistic estimates for the frequency shift can be off by an order-one factor in the relativistic regime. In this work, we account for the self-gravity effects of the boson cloud around a black hole in a fully relativistic way by leveraging stationary spacetime solutions describing black holes with synchronized complex scalar or vector clouds, first constructed in Refs. [122, 120, 121, 192]. We utilize these to determine the cloud mass dependence of the cloud frequency, and hence, the frequency of the gravitational wave signal in an oscillation-average approximation for $m = 1$ scalar and vector boson clouds. Using this, we update the waveform model **SuperRad**², introduced in Ref. [198]. With these results, the accumulated phase error is reduced by several orders of magnitude, and for vector bosons is on the order of one radian or smaller over the characteristic timescale of the gravitational wave signal in much of the parameter space.

The black hole-boson cloud solutions can also be used to determine how the cloud changes the black hole’s spacetime, and hence, how it affects the redshift, lensing, shadows, etc. associated with the system, as studied in Refs. [120, 161, 195]. Here, we compare the geometry of the scalar cloud case to the vector case in terms of the effect on quantities like the innermost stable circular orbit (ISCO) radius and light ring, and comment on how this may affect measurements of black hole spin.

The layout of this Chapter is as follows. In Sec. 3.2, we provide some background on the nonrelativistic treatment of this problem, and some details on the oscillation-averaged approximation considered here. We describe how we numerically construct solutions in Sec.

²Available at <https://www.bitbucket.org/weast/superrad>.

3.3, with more details included in the Appendices. We present our main results in Sec. 3.4, showing the relativistic correction to the frequency in Sec. 3.4.1, with an updated estimate of the phase error in the superradiance waveform in Sec. 3.4.2, as well as the effect of the presence of a boson cloud on the geometry in Sec. 3.4.3. Finally, we conclude with a discussion of the implications of these results in Sec. 3.5. In the following, we use geometrized units ($G = c = 1$) and the unit-adjusted boson mass $\mu \equiv m_b/\hbar$.

3.2 Preliminaries

3.2.1 Nonrelativistic superradiance

In this section, we review the solutions describing boson clouds arising from superradiance in the nonrelativistic limit and their associated timescales. This is to provide context for the results of this study, as well as to introduce some useful definitions.

First, we note that the nonrelativistic limit for black hole superradiance corresponds to the regime $\alpha \equiv Gm_b M_{\text{BH}}/(c^3 \hbar) \equiv M_{\text{BH}} \mu \ll 1$, where the characteristic length scale of the black hole is much smaller than that of the cloud [the former being given by $\sim \alpha/\mu$ and the latter by $\sim 1/(\alpha\mu)$]. In the nonrelativistic limit, the governing test-field equations can be solved analytically [83, 218, 47, 33] to obtain a spectrum of superradiantly unstable modes. These modes are analogous to the energy states in the hydrogen atom, and therefore, are also labeled by the numbers ℓ , m , and n , corresponding to the orbital angular number, azimuthal number, and number of radial nodes. The frequencies of the bound states for a boson field around a black hole are given by

$$\omega \approx \mu \left(1 - \frac{\alpha^2}{2(n + \ell + 1)^2} + \mathcal{O}(\alpha^3) \right). \quad (3.1)$$

In this study, we will focus on the $(\ell, m, n) = (1, 1, 0)$ mode for the scalar case and the $(\ell, m, n) = (0, 1, 0)$ for the vector case, which correspond to the fastest growing modes at the respective spins. The cloud growth timescale for the fastest growing mode, to leading order in α , scales as [83]

$$\tau_{\text{inst.}}^{\text{sca}} \sim \frac{1}{\alpha^9 (m\Omega_H - \omega)}, \quad (3.2)$$

in the scalar case. In the vector case the fastest growing mode has a growth timescale that scales as [33]

$$\tau_{\text{inst.}}^{\text{vec}} \sim \frac{1}{\alpha^7 (m\Omega_H - \omega)}, \quad (3.3)$$

where in this limit $\omega \approx \mu \ll m\Omega_H$. Once the cloud extracts a sufficient amount of angular momentum from the black hole, the superradiance condition is nearly saturated $\Omega_H \approx \omega/m$, and the cloud dissipates through gravitational radiation. The emission timescale τ_{GW} is the time for the cloud to decrease to half its original mass through gravitational dissipation after saturation. This emission timescale for the fastest growing (and dissipating) modes scale with α to leading order as follows [47, 33]:

$$\begin{aligned}\tau_{\text{GW}}^{\text{sca}} &\sim \alpha^{-14} \left(\frac{M_{\text{BH}}^2}{M_{\text{cloud}}} \right)_{t=t_{\text{sat}}}, \\ \tau_{\text{GW}}^{\text{vec}} &\sim \alpha^{-10} \left(\frac{M_{\text{BH}}^2}{M_{\text{cloud}}} \right)_{t=t_{\text{sat}}},\end{aligned}\tag{3.4}$$

where the mass of the cloud M_{cloud} is evaluated at the time it reaches its maximum. Note that $\tau_{\text{inst}} \ll \tau_{\text{GW}}$. As indicated by the above scalings, vector boson clouds can grow and dissipate on much shorter timescales than scalar clouds.

3.2.2 Nonrelativistic frequency shift

Generally, there will be a contribution to the gravitational binding energy due to the boson cloud's self-gravity, which was neglected in the above expressions. Most notably, the cloud's frequency ω (and hence, also the gravitational wave frequency) is shifted by an amount dependent on the cloud's mass M_{cloud} . We define $\Delta\omega = \omega(M_{\text{cloud}}) - \omega(M_{\text{cloud}} = 0)$ to be this cloud mass dependent frequency shift. As the cloud dissipates through gravitational wave emission and M_{cloud} decreases, this leads to a gradual increase of the gravitational wave frequency over time. Following Refs. [33, 128, 32, 198], this frequency shift was determined in the nonrelativistic regime. There, the cloud's energy density $\rho(\mathbf{r})$ sources the Newtonian potential

$$U(\mathbf{r}) = - \int d^3\mathbf{r}' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}.\tag{3.5}$$

The leading order in α expression for the energy density of the cloud $\rho(\mathbf{r})$ [where, in this limit, the cloud mass is $M_{\text{cloud}} = \int d^3\mathbf{r} \rho(\mathbf{r})$] is given by $\rho(\mathbf{r}) = \mu^2 \psi(\mathbf{r})^2$ and $\rho(\mathbf{r}) = \mu^2 \mathcal{A}_a(\mathbf{r}) \mathcal{A}^a(\mathbf{r})$, using the analytic nonrelativistic solution of the cloud distributions $\psi(\mathbf{r})$ and $\mathcal{A}^a(\mathbf{r})$ from, e.g., Refs. [47, 33]. Here and in the following, we use μ interchangeably for the scalar and vector masses (as done also for ω and m). Note also, we dropped the time-dependencies of $\psi(\mathbf{r})$ and $\mathcal{A}^a(\mathbf{r})$, since these integrate away as discussed below. The

frequency shift is then

$$\begin{aligned}\Delta\omega \frac{M_{\text{cloud}}}{\mu} &\approx \int d^3\mathbf{r} \rho(\mathbf{r}) U(\mathbf{r}) \\ &= -2 \int d^3\mathbf{r} \int_{|\mathbf{r}'| < |\mathbf{r}|} d^3\mathbf{r}' \frac{\rho(\mathbf{r})\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} = 2W,\end{aligned}\tag{3.6}$$

where W is the nonrelativistic expression for the binding energy of the cloud³. We can then perform a multipole expansion for the denominator in Eq. (3.6). When $|\mathbf{r}| \geq |\mathbf{r}'|$, we have that

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \sum_{l=0}^{\infty} \sum_{m=-l}^l \left(\frac{|\mathbf{r}'|^l}{|\mathbf{r}|^{l+1}} \right) \frac{4\pi}{2l+1} Y_{lm}(\Omega) Y_{lm}^*(\Omega'),\tag{3.7}$$

where $Y_{lm}(\Omega)$ are the spherical harmonics, Ω denotes the two angles on the sphere, and $*$ indicates complex conjugation. In the monopole approximation, where only the first term ($l = 0$) in this series is included, Eq. (3.6) gives

$$\Delta\omega \approx -\frac{2\mu}{M_{\text{cloud}}} \int d^3\mathbf{r} \int_{|\mathbf{r}'| < |\mathbf{r}|} d^3\mathbf{r}' \frac{\rho(\mathbf{r})\rho(\mathbf{r}')}{|\mathbf{r}|}.\tag{3.8}$$

In Ref. [198], several higher multipoles were calculated and the error in the monopole approximation (when considering the leading order in α term) is found to be at the percent level in the scalar case, and subleading in α in the vector case. As noted above, after performing this integral, the time dependence of the potential drops out. The final result in the monopole and nonrelativistic approximation is [198]

$$\Delta\omega_{\text{nonrel}}^{\text{sca}} \approx -\frac{93}{512} \frac{M_{\text{cloud}}\alpha^3}{M_{\text{BH}}^2},\tag{3.9}$$

$$\Delta\omega_{\text{nonrel}}^{\text{vec}} \approx -\frac{5}{8} \frac{M_{\text{cloud}}\alpha^3}{M_{\text{BH}}^2}.\tag{3.10}$$

3.2.3 Relativistic frequency shift

The aim of this work is to calculate the self-gravity effects of the cloud, including the frequency shift, in a relativistically accurate way. In principle, this could be calculated using

³N.B. the factor of two difference in Eq. (3.6) compared to some previous references, as discussed in Ref. [198].

black hole perturbation theory by solving for the metric perturbation due to the cloud, and then determining how this shifts the eigenfrequencies of the boson cloud solutions. However, given the intricacies of beyond-leading order black hole perturbation theory, here we opt instead to numerically solve the full governing Einstein equations. For a single (real) bosonic field, the presence of the cloud means that the spacetime is no longer exactly stationary or axisymmetric, so solving for the full, dynamical spacetimes while covering the parameter space (with the accuracy we require) would be infeasible without reducing the dimensionality of the problem. Instead, we calculate the self-gravity effects of the cloud in a related axisymmetric and stationary spacetime, where instead of considering a real scalar or vector field, we consider a complex field with a harmonic time and azimuthal dependence given by $\sim e^{i(m\varphi - \omega t)}$. This is equivalent to considering a superposition of two real bosonic fields (with identical masses) with a relative phase shift of $\pi/2$. In this sense, considering a stationary complex field instead of a real one is an oscillation-averaged approximation. When we further assume that the bosonic field saturates the black hole superradiance condition exactly ($\omega = m\Omega_H$), and thus has no flux through the horizon, then one obtains the solutions found in Refs. [122, 120, 121, 192].

Applying results from these stationary solutions to the gravitational wave-emitting cases is justified by the fact that the gravitational radiation timescale is always much longer than the oscillation period, and the energy density of the boson cloud solutions (in the Newtonian limit) is axisymmetric to a large degree. However, this does introduce a small theoretical error. We explicitly estimate this in Appendix B.3.1, and include this in our overall error estimates, as discussed below. We also compare the relativistic frequency calculation to the frequency obtained in a numerical relativity evolution without symmetry assumptions in Appendix B.3.1. We find the two are consistent, though the uncertainties in the latter are significantly larger than the expected error in the former.

3.3 Methods

3.3.1 Construction of solutions

As in Refs. [122, 120, 121, 192], we numerically construct axisymmetric, stationary spacetimes describing a black hole with a (synchronized) boson cloud. To that end, we solve the fully relativistic Einstein-Klein-Gordon and Einstein-Proca equations for scalar and vector boson clouds, respectively. These are, for a complex massive scalar field Φ coupled

to gravity,

$$G_{ab}/(8\pi) = \nabla_b \Phi^* \nabla_a \Phi + \nabla_a \Phi^* \nabla_b \Phi - g_{ab} (\mu^2 \Phi^* \Phi + \nabla_c \Phi \nabla^c \Phi^*), \quad (3.11)$$

$$\nabla_a \nabla^a \Phi = \mu^2 \Phi, \quad (3.12)$$

where G_{ab} is the Einstein tensor associated with the metric g_{ab} . For the case of a complex massive vector field A_b with associated field strength tensor

$$F_{ab} = \nabla_a A_b - \nabla_b A_a, \quad (3.13)$$

the Einstein-Proca equations are

$$G_{ab}/(8\pi) = \frac{1}{2} \mu^2 (A_b A_a^* + A_a A_b^*) - \frac{1}{2} \mu^2 A_c A^{*c} g_{ab} - \frac{1}{4} g_{ab} F_{cd}^* F^{cd} - \frac{1}{2} g^{cd} (F_{ac}^* F_{db} + F_{bc}^* F_{da}), \quad (3.14)$$

$$\nabla^a F_{ab} = \mu^2 A_b. \quad (3.15)$$

We do not consider any additional nongravitational interactions. Imposing axisymmetry and stationarity of the metric, the spacetime admits the Killing vectors φ_a and t_a , respectively. This assumption implies that the Lie derivative of the scalar field along the Killing vector fields satisfies

$$\mathcal{L}_\varphi \Phi = im\Phi, \quad \mathcal{L}_t \Phi = -i\omega\Phi, \quad (3.16)$$

and similarly for the vector A_a . We further assume the synchronicity condition between the cloud frequency and the black hole horizon, $\omega = m\Omega_H$; i.e., the bosonic fields saturate the black hole superradiance condition. In this study, we restrict to solutions corresponding to the endpoint of the fastest growing superradiant unstable modes: we set $m = 1$.

Our general approach to numerically constructing these solutions is similar to Refs. [122, 120, 121, 192], though the particular form for the metric ansatz that we use and some other details are different, and we give the full expressions for the elliptic equations that are solved in Appendix B.2. As in Refs. [122, 120, 121, 192], we use a grid covering the angular domain and a compactified radial coordinate that goes from the black hole horizon to spatial infinity. We solve these discretized partial differential equations using an adapted version of the Newton-Raphson method that was initially developed for boson star initial data in Ref. [197]. The resulting data is publicly available at [149]. For more details on how these solutions are constructed, see Appendix B.2.

3.3.2 Covering the parameter space

We construct black hole-boson cloud solutions across the two dimensional parameter space labeled by the frequency ω and the black hole radius (in units of the boson mass μ) by beginning from some seed solution and making small changes to these input quantities to relax to a nearby solution and then repeating. For each solution, we record the frequency ω , the black hole mass M_{BH} and angular momentum J_{BH} , and the total boson cloud mass M_{cloud} and angular momentum J_{cloud} (see Appendix B.1 for how these quantities are calculated). The two dimensional parameter space of solutions can alternatively be parametrized in terms of $M_{\text{cloud}}/M_{\text{BH}}$ and α . In terms of cloud mass, we cover the range going from a maximum of $M_{\text{cloud}}/M_{\text{BH}} \approx 0.12$ down to small values of the cloud mass (as low as $M_{\text{cloud}}/M_{\text{BH}} \approx 0.001$, which can then be linearly extrapolated to zero). These encompass the cloud solutions that might arise through superradiance⁴. In terms of α , we consider values ranging from the maximum allowed values for $m = 1$ ($\alpha \approx 0.5$ for scalars and $\alpha \approx 0.6$ for vectors), down to small values of α , where the scales of this problem become disparate [recall, the black hole radius is $\sim \alpha/\mu$, while the cloud radius is $\sim 1/(\alpha\mu)$], requiring more computational resources to resolve both objects. Due to this limitation, we include results for numerical solutions with $\alpha > \alpha_{\text{min}}$, where $\alpha_{\text{min}} = 0.2$ in the scalar case and $\alpha_{\text{min}} = 0.09$ in the vector case. We reach a lower value of α in the vector case because the fastest growing state is more compact than in the scalar case. In total, we compute ≈ 6000 solutions in the scalar case and ≈ 13000 solutions in the vector case.

We use these numerical solutions to interpolate and extrapolate over the whole physically relevant parameter space. First, a bivariate spline is used to interpolate over these solutions⁵ and calculate the frequency on a regular grid of α and $M_{\text{cloud}}/M_{\text{BH}}$. For each fixed value of α , we then fit ω to the best fit polynomial in cloud mass of the form

$$\frac{\omega}{\mu} = \sum_{q=0}^2 f_q(\alpha) \left(\frac{M_{\text{cloud}}}{M_{\text{BH}}} \right)^q, \quad (3.17)$$

where the choice to include only up to quadratic terms in cloud mass is motivated by the negligible contribution of higher order terms, as detailed in Appendix B.5.2. Then the cloud mass-independent part of the frequency is described by f_0 , and the frequency shift $\Delta\omega = \omega - \omega(M_{\text{cloud}} = 0)$ is described by f_1 and f_2 for each value of α . We compare $f_0(\alpha)$

⁴We estimate whether a cloud could be formed around an isolated black hole via superradiance by checking whether the total angular momentum in the spacetime divided by the total mass squared is less than one. This gives that $M_{\text{cloud}}/M_{\text{BH}} < 0.11$.

⁵We use a fifth order bivariate spline interpolation algorithm from `scipy`.

with test field results and find good agreement (see Appendix B.5 for details). Then we interpolate⁶ $f_q(\alpha)$ over α to return a smooth function that can be evaluated at any α .

Extrapolation of these relativistic results to lower values of $\alpha < \alpha_{\min}$ is done by leveraging the nonrelativistic results reviewed in Sec. 3.2.2. We fit the difference in the coefficients in Eq. (3.17) from their nonrelativistic limit with a higher order polynomial of the form

$$\begin{aligned} f_1(\alpha) + C_{\text{nr}}\alpha^2 &= \sum_{p=3}^6 c_{p,1}\alpha^p, \\ f_2(\alpha) &= \sum_{p=3}^6 c_{p,2}\alpha^p, \end{aligned} \tag{3.18}$$

where C_{nr} is calculated from the nonrelativistic limit (see Sec. 3.2.2), with $C_{\text{nr}} = 93/512$ in the scalar case and $C_{\text{nr}} = 5/8$ in the vector case. We fit for $c_{p,q}$ using points in the low α regime⁷ and extrapolate to lower values to estimate the relativistic correction there. We find that the fit parameters in Eq. (3.18) are order unity or smaller, guaranteeing that the fit will approach the expected nonrelativistic limit at small α .

3.4 Results

3.4.1 Frequency shift

We begin by comparing the relativistic frequency shift for scalar and vector boson clouds calculated in this work to the nonrelativistic approximation described in Sec. 3.2.2. In Fig. 3.1, we plot the quantity $d\omega/dM_{\text{cloud}}$ at $M_{\text{cloud}} = 0$, which means taking only the part of the frequency shift linear in cloud mass and ignoring any higher order parts. (As shown in Appendix B.5.2, the quadratic in cloud mass term in the relativistic case contributes at the percent level.) From the plot, it is apparent that the deviation of the relativistic calculation from the nonrelativistic limit can be as large as 40% – 50% in the scalar case and 70% – 80% in the vector case.

In Fig. 3.1, we also compare our results to the quasirelativistic calculation of the frequency shift used in Ref. [198], which is computed using the energy density computed from

⁶Here we use a cubic spline interpolating algorithm from `scipy`.

⁷We use points with $\alpha < 0.3$ in the scalar case and $\alpha < 0.15$ in the vector case.

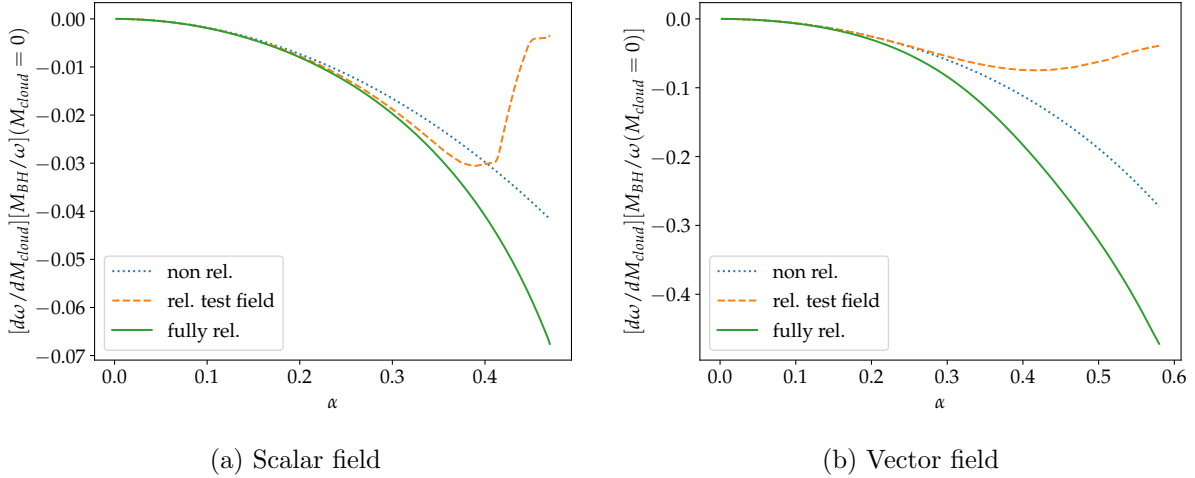


Figure 3.1: The cloud mass dependence of the frequency of oscillation of the cloud around the black hole for scalar bosons (top) and vector bosons (bottom). The leading order nonrelativistic result ($\propto \alpha^3$) is plotted along with the relativistic test field result from Ref. [198] and the fully relativistic result of this Chapter.

the relativistic test field, but still utilizing the Newtonian expression for the frequency shift in terms of this quantity. We find that the test field calculation does well in the scalar case up to moderately high values of α , significantly improving over the nonrelativistic expression up until $\alpha \approx 0.35$. However, for $\alpha > 0.37$ the test field calculation actually becomes less accurate than the leading order nonrelativistic limit⁸. In the vector case, the test field calculation always performs worse than the nonrelativistic limit. It seems that in this case including only some relativistic corrections can actually give a worse result than the purely nonrelativistic one, though there may be a more accurate way to generalize the Newtonian expression for the frequency shift to the relativistic regime.

We next comment on the estimated errors in the fully relativistic calculation of the frequency shift. As described in Appendix B.3, we use a combination of numerical error due to finite resolution, theoretical error due to our oscillation-averaged approximation, and interpolation error due to the finite number of numerical points to estimate a total error in the evaluation of the frequency shift. A summary of these different error estimates can be found at the beginning of Appendix B.3. As can be seen there, in the scalar case,

⁸In Fig. 6 of Ref. [198], there is a plotting error in the test field calculation shown due to the use of an expression for the black hole spin that is inaccurate for high spin or high α .

the relative error in the frequency shift ranges from a few times 0.1% to 1% across the parameter space, and is dominated by the interpolation error. For the vector case, on the other hand, the theoretical error is larger, and dominates over the numerical error and the interpolation error in much of the parameter space. The relative error in the vector frequency shift ranges from 0.001% at low α to 1% at the highest values of α considered.

3.4.2 Implications for gravitational wave searches for ultralight bosons

The more accurate relativistic frequency shift calculations described in the previous section can be used to better model the frequency evolution of the gravitational wave signal from an oscillating boson cloud. We have incorporated these results into **SuperRad**, a waveform model for superradiant clouds described in Ref. [198]. In Fig. 3.2, we plot example waveforms for scalar and vector boson cases for an initial black hole with properties consistent with the remnant from the binary black hole merger GW200308 [10]. Here we can see the enhanced frequency shift in the fully relativistic calculation compared to the nonrelativistic estimate, corresponding to a larger change in frequency and a stronger chirp as the cloud dissipates through gravitational radiation.

We roughly estimate the phase error over one gravitational wave emission timescale with the relativistic calculation of the frequency shift, using the estimated relative error $\delta\omega/\Delta\omega$, as described in Appendix B.3. Recalling that the gravitational wave frequency is given by twice the oscillation frequency of the cloud, the additional gravitational wave phase offset due to the frequency shift is

$$\Delta\phi = 2 \int_0^{\tau_{\text{GW}}} dt [\omega(t) - \omega(0)], \quad (3.19)$$

where the time $t = 0$ corresponds the end of the superradiant instability phase and the peak of the signal, before the cloud starts dissipating, and τ_{GW} is the gravitational wave emission timescale. We translate our estimate of the error in the frequency shift into an estimate of the phase error in the updated waveform using

$$E(\Delta\phi) = 2 \int_0^{\tau_{\text{GW}}} dt \delta\omega \approx \Delta\phi \left(\frac{\delta\omega}{\Delta\omega} \right)_{t=0}. \quad (3.20)$$

The accumulated phase offset $\Delta\phi$ for scalar and vector clouds with different values of α , as well as the estimated error in this quantity, is shown in Fig. 3.3. We include both a case

where the initial black hole is rapidly spinning, to illustrate the behavior at high values of α , as well as a case with a more moderate spin typical of the remnant of a comparable mass binary black hole merger. The figure illustrates that, except for scalar bosons and low values α , where the gravitational wave signal does not evolve significantly over the maximum one year period considered, this phase offset is significant, and the changing frequency needs to be taken into account. For the relativistically correct frequency shift calculated here, we estimate that the theoretical error in much of the parameter space for vector bosons is on the order of a few radians over the characteristic timescale of the gravitational wave signal. This is nearing the regime where theoretical modelling errors would no longer be an impediment to coherent gravitational wave searches, although there are other reasons that such searches may not be practical, including uncertainty in black hole parameters and computational expense.

The estimated phase error in the scalar case is much larger compared to the vector case over much of the parameter space, even when the relativistically correct frequency shift is included. This is because of the longer gravitational wave emission timescale in the scalar case. While the relative error in the frequency shift is comparable to the vector case, when integrated over a much longer period of time it leads to a larger phase error.

In addition to the ground based detector sources discussed above, we consider an example relevant for LISA in Fig. 3.4. Constituents of massive black hole binaries are expected to have higher dimensionless spins than stellar-mass black holes, and may also be promising candidates for binary follow-up searches for gravitational wave from vector clouds around merger remnants [198]. Following Ref. [139], when the phase error is the dominant source of error, a model with a maximum phase error of order $1/\rho_{\text{max.}}$ is sufficiently accurate for a signal with maximum signal-to-noise ratio (SNR) of $\rho_{\text{max.}}$. In Fig. 3.4, we plot the phase error for the vector case using two different models. One model uses the relativistic frequency shift calculated in this work, and the other uses the leading order nonrelativistic frequency shift. The phase error corresponding to a sufficiently accurate model for a signal with $\rho_{\text{max.}} = 10$ is approximately $1/\rho_{\text{max.}} = 0.1$. From the plot, it can be seen that while the nonrelativistic shift model is sufficient for $\alpha < 0.115$ when $\rho_{\text{max.}} = 10$, the relativistic shift model covers the larger range of $\alpha < 0.18$. It is evident that the improved waveform model is preferred for $0.115 < \alpha < 0.18$ or for higher SNR signals at lower α .

3.4.3 Geometry

In addition to changing the oscillation frequency, the boson cloud will change the spacetime geometry around the black hole, which we can characterize using the solutions we construct.

Here, we focus on calculating the change in the light ring and the ISCO of the black hole in the presence of a boson cloud, as these are related to how observations of accreting black holes are used to measure the black hole spin. There has been significant work on this topic in the past. A full null ray-tracing analysis was carried out for black holes with vector boson clouds in Ref. [195], ISCO frequencies were calculated for black holes with scalar hair in Ref. [120], and predictions for $K\alpha$ emission lines have been calculated using null-ray-tracing in Ref. [161] for black holes with scalar hair. Here, we calculate equatorial null and test particle orbits around black holes with both scalar and vector clouds.

For all the black hole-cloud solutions we construct, we calculate the position of the equatorial light ring and ISCO around a black hole in the presence of a superradiant cloud as is described in Appendix B.4, using the circumferential radius to record our measured radii in a coordinate independent way. We compare the light ring and ISCO radii of a black hole with a scalar or vector cloud with those of a Kerr black hole with either the same black hole parameters (measured using the horizon quantities) or the same total mass and angular momentum (in particular we use the ADM mass [17]). The results for the deviation of the light ring from the Kerr values are shown in Fig. 3.5 and the results for the ISCO radius are shown in Fig. 3.6. As one might expect, the biggest effect comes from the comparison where the global quantities are kept fixed, due to the significant amount of mass and angular momentum that can be carried in the cloud. However, for the vector case at high α , the cloud can have non-negligible effect on the near horizon geometry, shifting the light ring and ISCO radius by up to a few percent.

In addition to calculating the location of the ISCO for the black hole-boson cloud systems, we calculate the ISCO frequency, and the redshift of light emitted by an observer traveling along the ISCO as measured by another observer at rest at infinity. In particular, we calculate the redshift for a light ray emitted tangent to the observer's trajectory in the same direction (prograde to the black hole's spin), which we call z_{ISCO} , and show the deviation in this quantity from its Kerr value in Fig. 3.6, either fixing the black hole or the global mass and angular momentum. This shows similar behavior to the ISCO radius. We comment on how the deviations in these quantities from their Kerr values might be related to spin mismeasurement in the presence of a boson cloud in the next section.

The numerical values for the light ring radius, ISCO radius, frequency, and redshift (for light emitted both in the same or opposite direction as the emitter orbiting at the ISCO) of black holes with scalar and vector boson clouds are publicly available⁹.

⁹Data files with the geometric quantities and usage examples can found at <https://www.bitbucket.org/weast/superrad>.

3.4.4 Implications for Kerr parameter measurements

One method for inferring black hole spins is to measure how the emission from accreting black holes is redshifted (or blue shifted) [153, 182, 181]. Roughly speaking, the maximum redshift is taken to come from the ISCO since any material inside the ISCO is not expected to contribute to the observed electromagnetic spectrum under the assumption that the accretion disk is thin and oriented on the equator. Since the presence of a cloud can affect the position and redshift at the ISCO, we wish to estimate how much it would affect these spin measurements.

There are two main methods for measuring black hole spins from spectra, $K\alpha$ line broadening and continuum fitting. The $K\alpha$ line broadening method yields a dimensionless redshift measurement that can be used to directly fit the dimensionless spin [182]. The continuum fitting method gives a dimensionful ISCO radius measurement that must then be scaled by an independent mass measurement, often from distant orbiting dynamics [153]. A mass measurement using distant orbiting dynamics would give a value close to the total mass (e.g. in a black hole-cloud system), rather than the central black hole mass.

A complete analysis of how the presence of a boson cloud would affect an accreting black hole would require solving for accretion disk profiles on the black hole-cloud solutions and tracing out the emission that they source, which is beyond the scope of this work. We further note that here we are restricting to our axisymmetric and stationary spacetimes, and neglecting the effect of an oscillating component to the gravitational potential that would be present in the case of single (real) scalar or vector field (see, e.g., Ref. [66]). However, as a rough estimate of the discrepancy between the dimensionless spin $\chi = J_{\text{BH}}/M_{\text{BH}}^2$ one would infer from such methods when assuming a Kerr black hole when a boson cloud is present, we consider how the value of z_{ISCO} , or the ratio $R_{\text{ISCO}}/M_{\text{ADM}}$, maps to a dimensionless black hole spin assuming Kerr. We define the difference between this estimated black hole spin $\chi_{\text{est.}}$ and the true dimensionless spin χ of a black hole surrounded by a boson cloud as

$$\Delta\chi = \chi_{\text{est.}} - \chi. \quad (3.21)$$

We plot $\Delta\chi$ in Fig. 3.7 for the both the case where one combines the dimensionful ISCO radius with the global mass (left panels), and the case based on the dimensionless redshift z_{ISCO} (right panels). When considering $R_{\text{ISCO}}/M_{\text{ADM}}$, both the scalar and boson case show a similar overestimate in the dimensionless spin up to moderately high values, that is mostly just due to the overestimate of the black hole mass. Interestingly, for the most relativistic vector boson cases, the effect on the geometry near the black hole becomes significant, and causes this quantity to give an underestimate of the black hole spin. Using

the redshift at the ISCO has a smaller effect, leading to a maximum overestimate in the dimensionless spin of $\sim 1\%$ in the scalar case, and 5% in the vector case.

3.5 Discussion and Conclusion

In this work, we have used fully general relativistic solutions to study the self-gravity effects of boson clouds that might form around black holes through the superradiant instability. In particular, we have focused on how the frequency of the cloud’s oscillation changes due to the nonvanishing mass of the cloud. This impacts the frequency shift and evolution of the gravitational waves emitted by the clouds. We found that relativistic effects can enhance this frequency shift by up to a factor of two over nonrelativistic estimates obtained previously. Further, our current fully relativistic computation of the frequency shift is accurate up to uncertainties at the percent level or better. The remaining errors are due both to finite numerical resolution, as well as theoretical error arising from an oscillation-average approximation taken here.

Using this fully relativistic frequency shift, we improved the gravitational waveform model **SuperRad** [198]. This allows for accurate modeling of the loudest, and thus fastest evolving, boson cloud gravitational wave signals (i.e., those where the change in frequency during the observation time is most significant). This is particularly relevant for targeting gravitational waves emitted by ultralight vector boson clouds around binary black hole remnants. Currently, proposed searches for such signals are based on semicoherent search methods, which are fast and more robust to modeling errors, at the expense of have a reduced sensitivity compared to fully coherent searches [130]. The results here can be used to better tune the parameters of such searches, but may also be leveraged to design more sensitive search techniques. This is relevant not only for current ground-based gravitational wave detectors, but also for supermassive black hole mergers and space-based detectors such as LISA.

Most previous continuous wave searches for signals from boson clouds have focused on scalar bosons in the small α regime (for example, in Ref. [7] the largest value of α considered is $\alpha = 0.15$) and often use frequency binning methods that assume the frequency variation within a timestep (or a minimum coherence time) is below a given threshold. In these cases, the relativistic correction to the rate of frequency evolution, and therefore the correction to the maximum α constrained, is at the level of only a few percent. However, for larger values of α , this correction will be more significant. For example, Ref. [229] estimated the detectability of scalar boson clouds with $\alpha \leq 0.5$ for blind searches, and Ref. [130] estimated the detectability of targeted searches for vector boson clouds with $\alpha \lesssim 0.2$.

Considering the typical remnant of a comparable mass, low-spin binary black hole merger, the strain is maximized for $\alpha \approx 0.17$, where the relativistic frequency shift correction is $\approx 10\%$ in the vector boson case.

In this work, we have not considered bosonic fields with nongravitational self-interactions or interactions with other fields, which may give rise to additional radiation channels or affect the saturation of the superradiant instability and the expected gravitational wave signal [102, 32, 91, 92, 72, 199, 165, 166, 75]. For future work, it would be interesting to see whether relativistic black hole-boson cloud solutions like those considered here can be used to incorporate the impact of these interactions on the observational signatures (see, e.g., Ref. [123]).

Finally, we have also studied how the boson cloud will change the black hole space-time, and commented on how this may affect spin measurements based on observations of accreting black holes. As is discussed in Ref. [131], since the two main methods of spin measurement from redshifts, continuum fitting and $K\alpha$ line broadening, give different biases in the presence of a boson cloud, the difference between the values of these two measurements can be used to probe the existence of extra matter located close to the black hole. As was assumed in Ref. [131], we find here that the discrepancy in the spin from mapping the dimensionless redshift to its value in an isolated black hole is subdominant to the discrepancy due to the use of the dimensionful ISCO measurement in the scalar case. In the vector case, however, the former can be more significant at moderately high spin due to the stronger effect on the geometry in the vicinity of the black hole horizon, potentially giving rise to a more sensitive probe for the presence of a boson cloud.

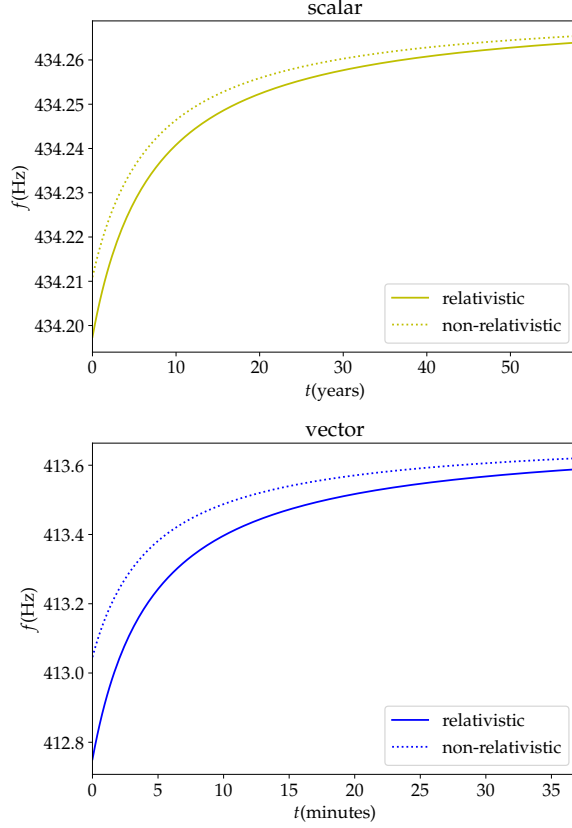
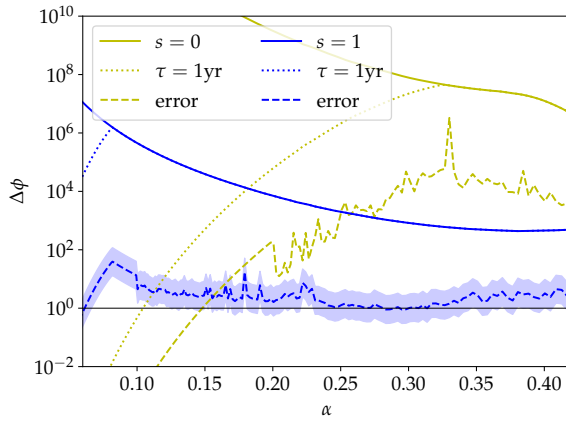
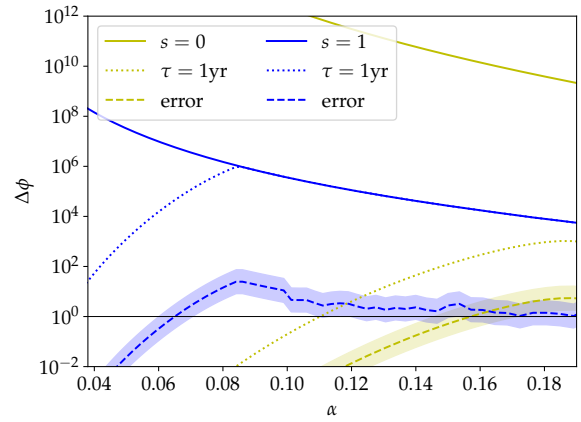


Figure 3.2: The frequency evolution for an example waveform with an initial black hole of dimensionless spin $\chi_{\text{BH}} = 0.91$ and mass $M_{\text{BH}} = 47 M_{\odot}$, and an ultralight boson with mass $m_b = 9.1 \times 10^{-13}$ eV (corresponding to $\alpha = 0.32$). Here f is the gravitational wave frequency, plotted from the maximum cloud mass of $M_{\text{cloud}}/M_{\text{BH}} \approx 3\%$ (5%) to the point where $M_{\text{cloud}}/M_{\text{BH}} \approx 0.3\%$ (0.5%) for the scalar (vector) case.



(a) Initial black hole with dimensionless spin $\chi_{\text{BH}} = 0.99$.



(b) Initial black hole with dimensionless spin $\chi_{\text{BH}} = 0.7$.

Figure 3.3: The phase offset after integrating the gravitational wave signal over its emission timescale, or one year if the gravitational wave timescale is longer than that. In each case the initial black hole mass is $M_{\text{BH}} = 50 M_{\odot}$. The jump in error at low α is where the relativistic shift calculation is extrapolated. The shaded region around the error in phase offset corresponds to an uncertainty in the error of a factor of three, indicating that the error shown here is an order of magnitude estimate.

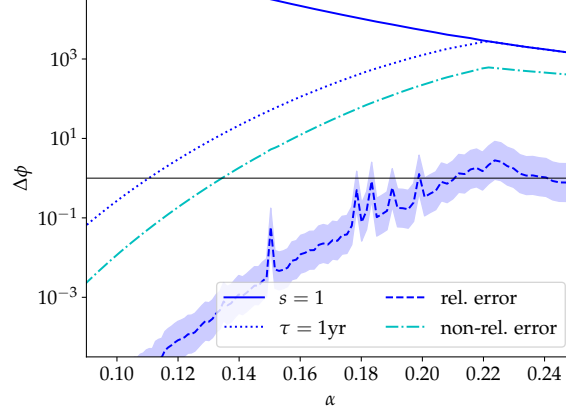


Figure 3.4: The phase offset after integrating the gravitational wave signal over its emission timescale, or one year if the gravitational wave timescale is longer than that. Here we consider a LISA source, with initial black hole mass $M_{\text{BH}} = 5 \times 10^5 M_{\odot}$ and initial dimensionless spin $\chi = 0.85$. We compare the total accumulated phase, the error in the relativistic waveform, and the error in a waveform using the leading order nonrelativistic frequency shift.

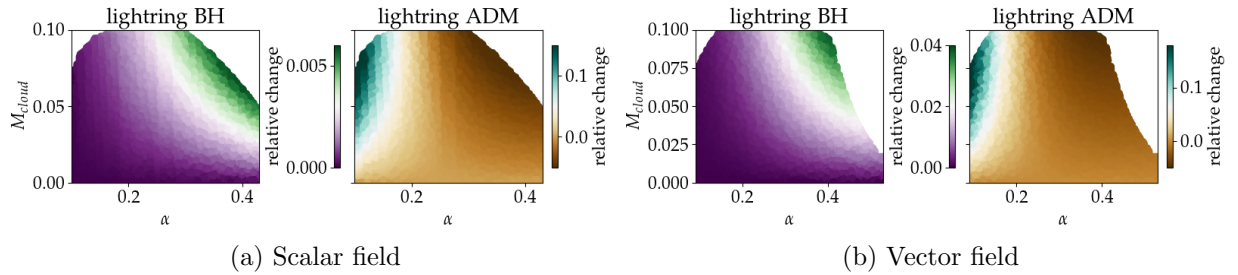


Figure 3.5: The relative difference in the circumferential radius of the light ring for a black hole with a boson cloud compared to a Kerr black hole with the same black hole mass and spin (left panels) or the same global mass and angular momentum (right panels).

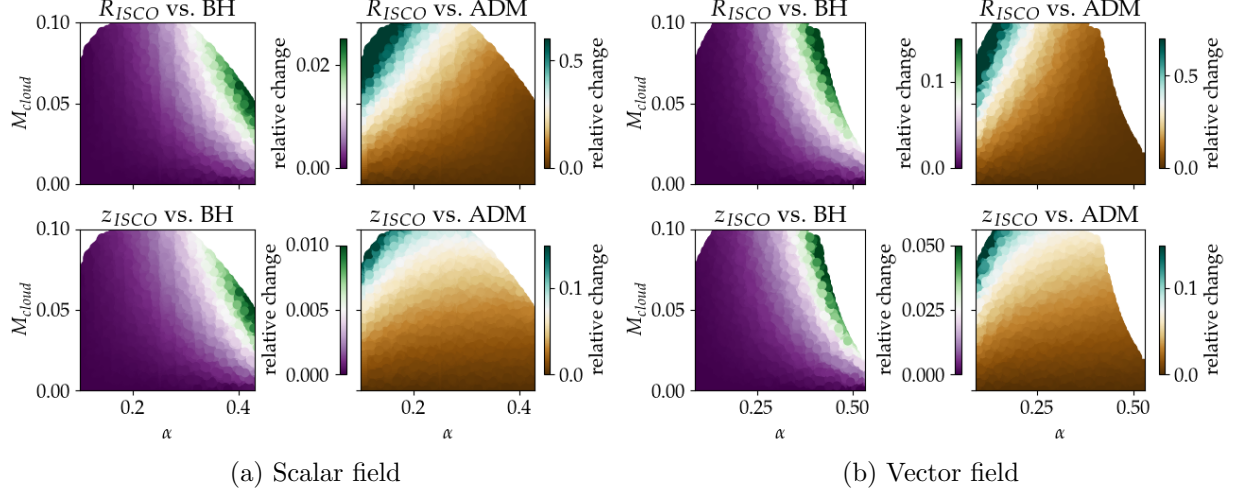


Figure 3.6: Here we show the ISCO radius and the redshift from the ISCO to infinity in the presence of a boson cloud, compared to the ISCO radius and redshift from the ISCO of a Kerr black hole with either the same central black hole's properties or the same global mass and angular momentum. The quantity plotted is the relative difference in ω_1 and ω_2 where ω_1 is the frequency measured by the observer traveling along the ISCO and ω_2 is the frequency measured by a stationary observer at infinite distance. The result plotted here is for light emitted tangent to the observer's motion on the ISCO and in the same (prograde) direction.

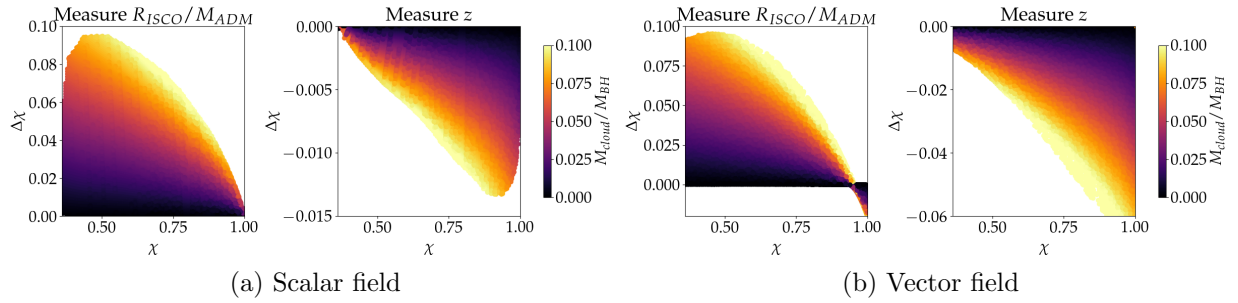


Figure 3.7: The effect of the presence of a boson cloud on the dimensionless spin measurements using a known mass and redshift measurements. The results shown are only for clouds that could be formed through superradiance around an isolated black hole. The spin shown here is the spin of the central black hole after the cloud's formation.

Chapter 4

Nonlinear Evolution of the Spin-2 Black Hole Superradiant Instability

4.1 Introduction

General Relativity (GR) predicts a massless spin-2 graviton as the mediator of gravity. However, massive spin-2 fields have been proposed in a number of contexts including modifications of GR [74], quantum theories of gravity, and dark matter models [16, 26, 148, 107, 129, 147, 136]. Among the theories giving rise to massive spin-2 field are those that add quadratic curvature terms to the Einstein-Hilbert action (quadratic gravity) [203, 204], as well as theories that introduce an additional metric, like nonlinear massive gravity [80, 79], bimetric gravity [114, 113], or other multi-metric theories.

Spinning black holes can trigger superradiant instabilities in the presence of massive bosonic fields [222, 154, 201]. For spin-0 and spin-1 fields, the instability extracts energy and angular momentum from the black hole, spinning down the hole, and gradually reducing the instability rate, until a quasi-equilibrium with a boson cloud is reached. The boson cloud slowly dissipates due to gravitational wave emission, while on even longer timescales higher mode clouds can grow, spinning down the black hole further. This phenomenon has been used to constrain the existence of beyond the Standard Model physics through the observation of black hole spin measurements as well as searches for the gravitational waves sourced by the oscillating clouds [19, 18, 60, 160, 33, 24, 58, 82, 167, 75, 211, 210, 1].

The nonlinear development of black hole superradiant instabilities has been studied in several scenarios. A reflecting boundary, which arises naturally in the context of an

asymptotically Anti-de Sitter spacetime, can lead to instability for a charged scalar field around a charged black hole [43, 191] or for gravitational waves around a spinning black hole [69, 68]. Studies in those settings found that their respective instabilities eventually saturate, resulting in a quasi-stable final state (though in some cases other instabilities arise on longer timescales). Similarly, the minimally-coupled massive vector case was studied nonlinearly in Refs. [93, 90], and the evolution and saturation of the instability was found to be quasi-adiabatic to a good approximation, even for relativistic cases or when the cloud grows to order 10% of the black hole mass. Given this holds for the vector case, we expect the quasi-adiabatic approximation to be even better in the minimally-coupled, massive scalar case, since the associated superradiant timescales are even longer.

This picture can change, however, when considering nonlinear effects besides gravity. Additional bosonic interactions or coupling to other matter can lead to radiation, black hole absorption, or other sources of dissipation that halt the exponential growth of the cloud. This includes nonlinear interactions for a scalar boson [110, 32, 165, 102] or a coupling to charged matter in the vector case [199, 59, 102]. In the interacting scalar case [21, 220], or the Higgs-Abelian model of a massive vector [91, 92, 52], nonlinear effects can become strong enough to lead to an explosive phenomenon referred to as a bosonova that disrupts the growth of the cloud (though in the scalar case, dissipative effects can hinder these values from being reached [32, 165]).

The nonlinear regime of spin-2 superradiance has not been explored. In this case, there is no minimal way to couple such a field to gravity, and additional interactions (depending on the particular theory chosen) will necessarily be present. In addition, the linear spin-2 superradiant instability is parametrically faster than the scalar or vector case. For these reasons, one does not necessarily expect the phenomenology of the nonlinear evolution to be as simple as in the minimally coupled lower boson spin cases. Determining what happens in the spin-2 case is not only theoretically interesting in the context of understanding nonlinear superradiance, but an important step in attempting to use black hole observations to probe the existence of such fields. Here, we investigate the behavior of massive spin-2 black hole superradiance using full nonlinear simulations of quadratic gravity. We choose this particular theory because it admits any vacuum GR solution as a solution, while also containing a massive spin-2 degree of freedom [204] and having a known well-posed initial value formulation [163]. In contrast to the other cases described above, we find that nonlinear effects do not hinder the growth of the superradiant instability. The growing spin-2 cloud increases the black hole spin, accelerating the instability, and eventually leading to a super-extremal apparent horizon.

4.2 Linearized spin-2 Superradiance

The linear evolution of a massive spin-2 field H_{ab} with mass μ on a Ricci-flat spacetime is described by the equations of motion [54, 152, 46],

$$\nabla^c \nabla_c H_{ab} = \mu^2 H_{ab} - 2R_{acbd} H^{cd}, \quad H_a^a = \nabla_a H^{ab} = 0, \quad (4.1)$$

where ∇_a is the usual covariant derivative operator. Unless otherwise stated, we assume units with $G = c = 1$ throughout. As in the spin-0 and spin-1 cases, there is a linear superradiant instability in the spin-2 massive boson around a Kerr black hole with non-zero spin. For a mode solution (in, e.g., Boyer-Lindquist coordinates) of the form

$$H_{ab}(t, r, \theta, \varphi) = h_{ab}(r, \theta) e^{-i(\omega t - m\varphi)}, \quad (4.2)$$

where $\omega = \omega_R + i\omega_I \in \mathbb{C}$ and $m \in \mathbb{Z}$, the superradiantly unstable modes satisfy the condition $0 < \omega < m\Omega_H$ where Ω_H is the horizon angular velocity. The linear superradiant instability has been studied perturbatively in the $M_{\text{BH}}\mu \ll 1$ limit [46], and with $M_{\text{BH}}\mu \leq 0.8$ for the fastest growing superradiant mode [84], where M_{BH} is the black hole mass and μ is the spin-2 boson mass. In Ref. [95], the linear instability rates were calculated in the relativistic part of the parameter space for dimensionless spins $\chi \leq 0.998$ and $M_{\text{BH}}\mu \leq 2$.

In addition to the superradiant instability (and in contrast to the spin-0 and spin-1 cases), massive spin-2 fields exhibit an axisymmetric ($m = 0$) linear instability around even non-spinning black holes when $\alpha := \mu M$ (where M is the spacetime mass) is sufficiently small. This instability was first found in Ref. [25], and has been studied in Refs. [46, 157, 142, 119, 95, 116, 118]. In much of the space of boson masses and black hole parameters, this instability is faster than the superradiant instability. However, for sufficiently large α , the $m = 0$ instability becomes negligible or absent and superradiant instabilities dominate. For example, when $\chi = 0.99$, the $m = 0$ instability dominates for $\alpha \lesssim 1.4$ [95].

4.3 Nonlinear Theory

While the equations of motion for a massive spin-2 field on a black hole background are unique at the linear level, there is no unique, minimal way to nonlinearly couple such a field to gravity. One nonlinear theory which leads to a well-posed initial value problem is quadratic gravity, which has the following action:

$$S = \int d^4x \sqrt{-g} \left(R - \frac{1}{2\mu^2} C^{abcd} C_{abcd} + \frac{1}{6m_0^2} R^2 \right), \quad (4.3)$$

where g is metric determinant, R is the Ricci scalar, and C_{abcd} is the Weyl tensor. This theory, also known as Stelle gravity, admits any vacuum solution to the Einstein equations as a solution, and is renormalizable around flat space [203]. However, as a fourth-order (non-degenerate) theory, Ostrogradsky’s theorem implies the Hamiltonian is unbounded from below [215] with the massive spin-2 degree of freedom acting as a ghost. Of particular relevance for this study is that this theory can still be formulated in a way that leads to well-posed initial value problem [163, 156], which means we can use it to address questions regarding the backreaction of the superradiant instability. An alternative nonlinear theory with a massive spin-2 field is massive bi-gravity [114, 113], though in that case, general well-posed evolution schemes are currently lacking [81, 137].

In quadratic gravity, the traceless part of the Ricci tensor $\tilde{R}_{ab}$ acts as a spin-2 degree of freedom with mass μ , while R obeys the scalar wave equation with mass m_0 . In particular, linearizing perturbations on a Ricci-flat solution gives Eq. (4.1), with $H_{ab} = \tilde{R}_{ab}$ [157]. Since in this study we are beginning with a Ricci-flat background solution, and the scalar superradiant instability operates on much longer timescales than we consider here, we expect that R should remain negligible regardless of its mass. Therefore, we assume $R = 0$ below (or equivalently, consider Einstein-Weyl gravity).

As mentioned above, in addition to the superradiant instability, spin-2 bosons exhibit an axisymmetric ($m = 0$) instability. In Ref. [95], it was shown that, depending on the black hole parameters and the sign of the initial perturbation, the nonlinear evolution of this instability in Einstein-Weyl gravity results in either a superspinning black hole, a zero mass naked singularity formed in finite time, or simply another black hole with a larger mass and spin surrounded by a cloud of the spin-2 field. Here, we show that even in the region of the parameter space where there is no $m = 0$ instability, superradiant instabilities can also result in superspinning apparent horizons.

4.4 Methods

We investigate the nonlinear evolution and backreaction of the spin-2 superradiant instability by numerically evolving quadratic gravity as in Ref. [95]. See Appendix C.1 for more details. We begin from initial conditions describing a Kerr black hole perturbed by a superradiantly unstable mode. We fix the initial dimensionless spin of the black hole to be $\chi = 0.99$, and consider two values for the spin-2 mass: $\mu M = 1.5$ and 1.9 , where M is the initial black hole mass. For these parameters, the fastest growing superradiant modes are the $m = 2$ and $m = 3$, with instability rates $\omega_I = 1.6 \times 10^{-3}/M$ and $1.2 \times 10^{-4}/M$, respectively (the $m = 0$ and 1 modes are stable).

We construct the desired superradiant mode solutions by evolving a massive spin-2 perturbation on the black hole background for several e-foldings. After this time, the test-field profile is strongly dominated by the fastest growing superradiant instability. The test-field data is then rescaled as desired, and is used as initial data for full nonlinear evolution. We check that the errors introduced by starting with such test-field initial data is sufficiently small by considering several different amplitudes. For the evolution simulations, we use a grid spacing of $dx = 0.02M$ around the black hole, though we have also considered a lower resolution to check numerical convergence. Assuming third order convergence, we estimate an error in the instability rate of $\approx 15\%$, and an error in the black hole physical parameters M_{BH} and $J_{\text{BH}}/A_{\text{BH}}$ of $\approx 4\%$. More details regarding these are given in Appendices C.2 and C.3.

Since we are restricting to solutions with $R = 0$, $\tilde{R}_{ab}$ is equal to the Einstein tensor, and is conserved $\nabla_a \tilde{R}^{ab} = 0$ through the Bianchi identity. It is therefore instructive to treat it analogously to the stress energy tensor and define an effective energy and angular momentum, decomposed into azimuthal components, as

$$\begin{aligned} E^{m=m'} &= \frac{1}{8\pi} \int d^3x \sqrt{\gamma} n^b t^a \tilde{R}_{ab} e^{-im'\varphi}, \\ J^{m=m'} &= \frac{1}{8\pi} \int d^3x \sqrt{\gamma} n^b \varphi^a \tilde{R}_{ab} e^{-im'\varphi}. \end{aligned} \quad (4.4)$$

Here $E := E^{m=0}$ and $J := J^{m=0}$ correspond to the total energy and angular momentum, respectively, and higher m components encode the azimuthal dependence of the effective stress energy tensor. In Eq. (4.4), γ is the determinant of the spatial metric, n^a is the normal to the spacelike hypersurface, t^a is the Killing vector associated with the stationary background spacetime, and φ^a is the Killing vector associated with the background axisymmetry. At the linear level, E and J are conserved up to fluxes through the black hole horizon. Furthermore, for superradiant mode solutions (where $\tilde{R}_{ab} \sim e^{im\varphi}$ with $m \neq 0$), these quantities are both zero at the linear level. This means that a study of the nonlinear behavior of the initial perturbation is needed to determine the leading order backreaction on the black hole's mass and angular momentum.

In order to better understand the behavior of the solutions, E^m and J^m with $m \neq 0$ are calculated. Those modes can be used to determine which superradiant modes are growing, as well as the complex frequencies of those modes.

During the evolution, we also track the black hole apparent horizon and calculate its area A_{BH} and associated angular momentum J_{BH} , using this to define a mass M_{BH} through the Christodoulou formula (see Appendix C.1 for details). We note that we calculate the

horizon angular momentum with respect to the coordinate direction of the original black hole spin, and there can be gauge dependence to this quantity when the spacetime differs significantly from axisymmetry.

4.5 Results

We numerically evolve the full quadratic gravity equations for several superradiant spin-2 perturbations to a black hole with initial dimensionless spin $\chi = 0.99$: $m = 2$ with $\alpha = 1.5$ and $m = 3$ with $\alpha = 1.9$. We find that as the massive spin-2 cloud grows, the black hole spin and the cloud growth rate both increase. For $m = 2$ with $\alpha = 1.5$, where (due to the faster instability rate) we follow the growth of the superradiant instability as it strongly backreacts, we find that the black hole apparent horizon eventually becomes super-extremal.

We begin by describing the properties of the superradiant perturbations during the weakly nonlinear growth phase. To linear order in the perturbation amplitude, the effective energy and angular momentum defined in Eq. (4.4) is zero, but from the evolution we can extract the leading order (quadratic in the amplitude) contribution to these quantities. We find that both E and J are negative in all the cases we consider.

This is illustrated in Fig. 4.1, where we show an effective energy density $\rho = n^b t^a \tilde{R}_{ab}/(8\pi)$ for the initial perturbation with $m = 2$ and $\alpha = 1.5$ at two different times, approximately one e-folding time apart. At the earlier time, the nonlinear effects are small and the linear $m = 2$ behavior is most apparent in the effective energy density. At the later time, where nonlinear effects are stronger, there is a noticeable skew towards negative values in the azimuthal angle dependence.

For a wave with fixed frequency and azimuthal number, we expect the energy and angular momentum to obey $E/J = \omega_R/m$ (assuming any imaginary frequency component is small enough compared to the real component to be ignored). We find that this also holds for the superradiant spin-2 modes we study here and the quantities defined in Eq. (4.4). In particular, we find that the value of J/E oscillates around a constant value, $J/E \approx 2.3M$ and $2.5M$ for $(m, \alpha) = (2, 1.5)$ and $(m, \alpha) = (3, 1.9)$, respectively. This agrees to within $\approx 5\%$ with the value $m/\omega_R \approx 2.4$ (for both cases) found from either the linear calculation or from the frequency of oscillation of $E^{m=2}$.

We have also looked for gravitational radiation sourced nonlinearly during the growth of the spin-2 field, but find that it is too small as to be reliably extracted from the simulation. Applying the quadrupole formula to the $\alpha = 1.5$, $m = 2$ case to obtain an

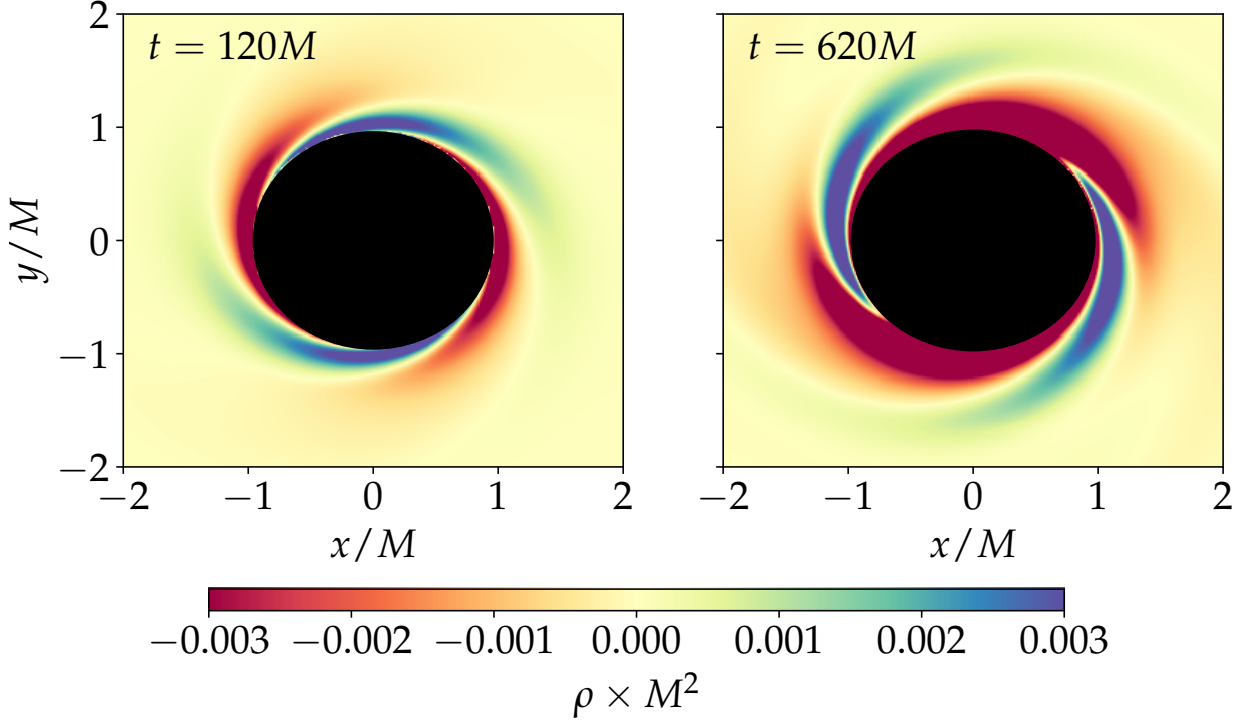


Figure 4.1: The profile of the effective energy density $\rho = n^b t^a \tilde{R}_{ab}/(8\pi)$ of an $m = 2$, $\alpha = 1.5$ perturbation on the equatorial ($z = 0$) plane, where the black hole is at the origin, and the spin points out of the page. It can be seen that there are two negative and two positive energy peaks, consistent with the leading order $m = 2$ azimuthal dependence. The left panel shows the field profile at early times, where the backreaction on the black hole is small. The right panel shows the field approximately one e-folding time later, where the overall negative mono-polar component to the energy density is clearly visible.

order-of-magnitude estimate (and assuming a harmonic time dependence) indicates that any gravitational radiation is likely subdominant to the exponential tail of the spin-2 cloud even at a distance of $\sim 100M$.

Focusing now on the $m = 2$ and $\alpha = 1.5$ case, we begin with a perturbation where $E^{m=2} \approx 6 \times 10^{-3}M$ and follow the superradiant growth as it transitions from the linear instability rate to strongly backreacting on the black hole. We track the mass and angular momentum of the black hole apparent horizon with time and find, consistent with the fact that E and J are negative, that both increase until an apparent horizon can no longer be found. As can be seen in Fig. 4.2, shortly before this occurs, the apparent horizon

becomes super-extremal in the sense that the ratio of angular momentum to area satisfies $J_{\text{BH}}/A_{\text{BH}} > 8\pi$.

Though we cannot rule that it is an artifact of our horizon-finding algorithm, our failure to find an apparent horizon at late times, coupled with the rapidly growing spin of the horizon leading up to this, suggests that the apparent horizon may have disappeared. Without a trapped surface, one can no longer continue the evolution while excising the black hole interior. However, in the lead up to this, we see no sign of diverging curvature or other singularities outside the black hole horizon.

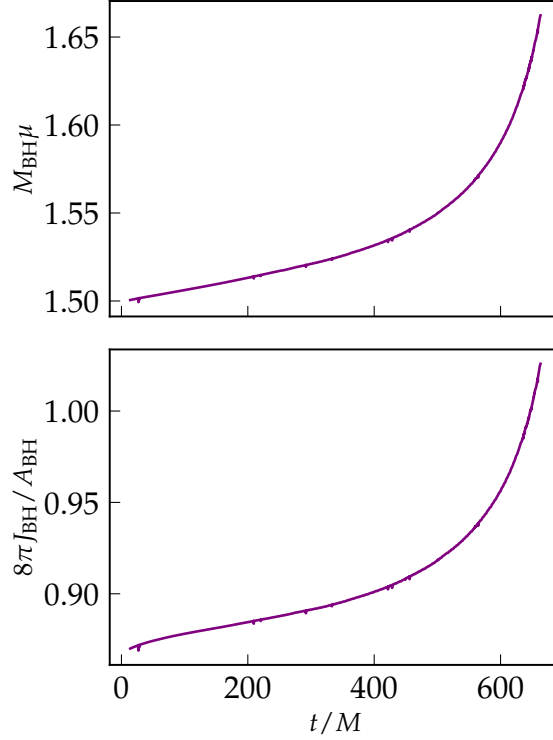


Figure 4.2: The black hole apparent horizon mass (top) and angular momentum relative to its area (bottom) as a function of time. The spin increases with time, and shortly after it reaches extremality ($8\pi J = A_H$) we are no longer able to find an apparent horizon.

The growth of the black hole’s mass and angular momentum closely tracks that of the massive spin-2 cloud. Examining the azimuthal modes of the effective energy density through the calculation of $E^{m=m'}$ according to Eq. 4.4, we find that $m = 2$ initially

dominates over the other azimuthal modes, as expected from the linear instability when nonlinear effects are subdominant. The only other significant m component in the effective energy density is $m = 0$, which determines the total effective energy of the field. We show the growth with time of both of these quantities in Fig. 4.3. The rate at which the field's energy E decreases at early times is twice the superradiant growth rate, consistent with it having a quadratic dependence on the superradiantly unstable mode.

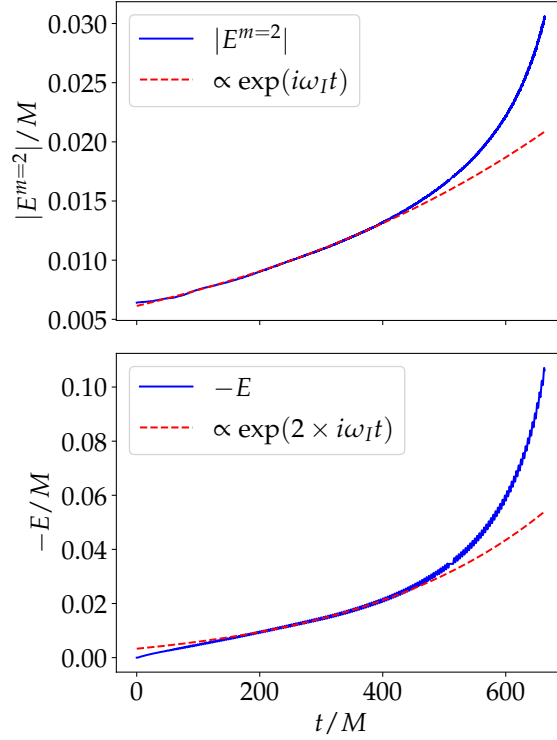


Figure 4.3: The magnitude of the $m = 2$ (top panel) and $m = 0$ (i.e., total; bottom panel) azimuthal component of the effective energy density as a function of time for an initial perturbation with $m = 2$ and $\alpha = 1.5$. The linear instability growth rate for a background with $M_{\text{BH}}\mu = 1.5$ and $\chi = 0.99$ is shown for comparison (dashed lines). After a brief transient period due to the linear initial conditions, the total energy growth at intermediate times is consistent with a quadratic effect associated with the $m = 2$ linear superradiant growth. At $t/M \approx 500$, when the growth rate begins to noticeably differ from the linear prediction, the black hole has changed such that $M_{\text{BH}}\mu = 1.55$ and $\chi = 0.9964$.

From Fig. 4.3, we can see that as the spin-2 cloud grows into the nonlinear regime, and

as M_{BH} and J_{BH} increase, the growth rate of $E^{m=2}$ and E increase significantly compared to the expected linear instability rate. There could be several reasons for this. If we treat the evolution quasi-adiabatically, and imagine that we can still apply a linear instability analysis around a Kerr background to the evolving black hole, then as the black hole spin and mass increase, this indicates that not only will the $m = 2$ instability rate increase, but the $m = 0$ instability will as well. At $t/M \approx 500$, the black hole mass and spin have changed such that $M_{\text{BH}} = 1.55/\mu$ and $\chi_{\text{BH}} = 0.9964$. The linear instability rates¹ for these values are $\omega_I M \approx 0.0051$ and 0.0018 , for $m = 0$ and 2 , respectively. This means that the $m = 0$ linear instability growth rate exceeds the quadratic growth in E associated with the $m = 2$ superradiant instability. Towards the end of the evolution, the growth in E is roughly consistent with the increasing linear $m = 0$ instability rate, though the growth in $E^{m=2}$ is much larger and not consistent with linear behavior. Of course, as the black hole begins to evolve rapidly, the linear, quasi-adiabatic analysis need not apply, and certainly breaks down as the black hole becomes superextremal, so additional nonlinear effects may also be contributing to the enhanced growth rate at late times.

Due to its significantly smaller instability rate, we only evolve the initial perturbation with $m = 3$ and $\alpha = 1.9$ for long enough to measure the values of E and J associated with the massive spin-2 cloud. Nevertheless, the negative values of these quantities suggest that the nonlinear development of the instability will also result in a superspinning black hole in finite time. Furthermore, we note that in this case, based on the linear analysis around a Kerr black hole, we do not expect the $m = 0$ instability to dominate over the $m = 3$ instability, even as the black hole approaches extremal spin.

4.6 Discussion and conclusion

For a minimally coupled massive scalar or vector field, a superradiantly growing cloud decreases the mass and angular momentum of a spinning black hole in such a proportion that the horizon frequency decreases until the superradiant instability shuts off. For a massive spin-2 field, on the other hand, there is no canonical way to couple it to gravity, and no energy conditions to guarantee that this happens. When self-interactions or couplings to other fields are sufficiently strong in the scalar or vector case, they can halt the exponential growth of the cloud before the instability of backreacts significantly on the black hole. In the massive tensor case, such interactions will be required, with their exact form depending on the particular nonlinear theory. In this work, using nonlinear evolutions in quadratic

¹We follow the same procedure as Ref. [95] to calculate linear instability rates.

gravity, we find that the spin-2 superradiant instability does indeed grow until it backreacts on the black hole. However, in this case it spins up the black hole, leading to a run away effect whereby the horizon eventually becomes superextremal. This behavior resembles that found for the axisymmetric ($m = 0$) massive spin-2 instability when evolved nonlinearly in quadratic gravity for some values of black hole spin and initial perturbations [95].

For future work, it would be interesting to determine whether the behavior found here occurs in other nonlinear theories with massive spin-2 fields, for example, if bimetric gravity [113] could be formulated in a way where the nonlinear development of this instability could be addressed. While nominally ghost-free, in contrast to quadratic gravity, this class of theories also generically violates the equivalent of the null energy condition [27]. Furthermore, for bimetric gravity in spherical symmetry, the backreaction of the $m = 0$ massive spin-2 instability can either decrease or increase the area of the black hole horizon, depending on the sign of the perturbation [224], similar to quadratic gravity.

Previous works have considered how one can constrain massive spin-2 fields through the superradiant instability using spin measurements or gravitational wave searches, assuming the instability saturates by spinning down the black hole in a manner similar to the massive spin-0 or spin-1 cases [46, 51, 84] (with perturbative analyses restricted to the regime where the superradiant instability is subdominant to the $m = 0$ instability). In the cases studied here in quadratic gravity, the backreaction of superradiance is quite different; in particular, we do not find that the instability saturates with a Kerr-like black hole. Though this makes it harder to judge the expected phenomenological signatures, it does suggest that observations of Kerr-like black holes can be used to rule out such theories where the Compton wavelength of the massive spin-2 field is commensurate with the black hole radius, as these instability timescale is quite short. Restoring astrophysical units, we recall that the cases studied here correspond to $\mu \sim 2 \times 10^{-11} \text{ eV}(10 M_\odot/M_{\text{BH}})$ with instability e-folding times of 0.03–0.3 sec $M_{\text{BH}}/(10 M_\odot)$.

Chapter 5

Outlook

This thesis employs numerical methods to compute nonlinear corrections to perturbative models of black hole physics. These corrections are important for accurate tests of both beyond-GR and beyond-Standard Model physics. In Chapter 2, we analyze the contribution of a particular nonlinear effect in black hole ringdown. In Chapter 3 we present a nonlinear correction to the gravitational wave signal sourced by a superradiant cloud of scalar or vector bosons. Finally, in Chapter 4, we present the nonlinear dynamics and backreaction of a spin-2 boson field undergoing superradiance.

The contribution of nonlinearities to ringdown has been a subject of significant recent interest. In this study, we make an estimate as to the contribution of a particular third-order effect. We conclude that the magnitude of this contribution can be comparable to that of the second order mode doubling effect. As is demonstrated in Chapter 2, this third order effect can be reasonably well fit using a superposition of linear modes. Considering this, it seems unlikely that third order ringdown effects will be detectable in practice. However, studying these effects offers a guideline for the level to which a ringdown signal can be understood consistently, using only the leading order in perturbation theory. A more precise quantification and comparison of nonlinear effects could further improve the fidelity of ringdown waveform modeling, thereby enhancing the robustness of GR tests in this regime.

Searches for ultralight bosons using black hole superradiance can constrain regions of parameter space beyond the Standard Model that are hard to reach using terrestrial experiments. This study calculates a nonlinear correction to the gravitational waveform model. This correction increases the accuracy of the model, particularly in regimes where the signal amplitude is large. These corrections have been incorporated into the waveform

modeling package **superrad**, which is currently used in gravitational wave searches involving superradiant boson clouds. The most recent of these searches by the LVK collaboration [1] places constraints on boson masses with an order of 50% confidence. Our update to the model may enable the use of more accurate search algorithms in the future, for tighter constraints on the existence of ultralight bosons. Developing and exploring such algorithms is a promising direction for future work.

Ultralight spin-2 bosons are proposed extensions of the Standard Model motivated by the dark matter problem. They arise in several modified gravity theories. We show here that the phenomenology of ultralight spin-2 bosons around black holes may be markedly different from that of scalar or vector bosons. Constraints on the existence of spin-2 bosons have been placed under the assumption that their nonlinear backreaction on the black hole spacetime mimics that of scalar and vector fields. Our findings have implications for current constraints, strengthening them in the case where the Einstein Weyl gravity term is the low energy massive spin-2 perturbation, and casting them into doubt where the spin-2 perturbation is due to another theory. Analyzing the nonlinear behavior of spin-2 superradiance in other theories, particularly nonlinear massive gravity, would be complimentary to this study. Those analyses might offer insight as to whether the phenomenology in the Einstein Weyl gravity case can be understood as generic.

Overall, the findings of this thesis contribute to stronger and more refined tests of GR and beyond-Standard Model physics. Testing these fundamental paradigms is essential if we are to progress toward a more complete theory of gravity, and to uncover potential clues regarding quantum gravitational effects.

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APPENDICES

Appendix A

Nonlinear Ringdown

A.1 Change in a black hole mass and spin due to the absorption of gravitational waves

Here we review how to compute the change in the mass and spin of a black hole from the absorption of a mode solution to the Teukolsky equation. This problem was first examined in [115] and elaborated upon in [207, 175, 209]. A review and extension of those earlier results to the time domain was performed in [172]. Despite these earlier works, for completeness and to set our notation, we provide a detailed review of the relevant calculations here.

First we compute how the black hole area changes due to the absorption of linearized gravitational waves within the NP formalism. We then use the first law of black hole thermodynamics, along with the Fourier decomposition of the linearized gravitational perturbations, to relate the change in the black hole area to the change in the mass and spin of the hole.

A.1.1 Transforming to the Hawking-Hartle tetrad

We begin by deriving the Hawking-Hartle tetrad [115]; see also Sec. 98(d) of Ref. [65]. Following Hawking, Hartle, and Chandrasekhar, we make use of the Newman-Penrose formulation of the Einstein equations [159]. The metric for a Kerr black hole in ingoing

Kerr-Schild coordinates is

$$g_{ab}dx^a dx^b = \left(1 - \frac{2Mr}{\Sigma_{\text{BL}}}\right) dv^2 - 2dvdr + \frac{4aMr \sin^2 \theta}{\Sigma_{\text{BL}}} dv d\varphi + 2a \sin^2 \theta dr d\varphi - \Sigma_{\text{BL}} d\theta^2 - \left(a^2 + r^2 + 2Mr \frac{a^2}{\Sigma_{\text{BL}}} \sin^2 \theta\right) \sin^2 \theta d\varphi^2, \quad (\text{A.1})$$

where $\Sigma_{\text{BL}} \equiv r^2 + a^2 \cos^2 \theta$, $\Delta_{\text{BL}} \equiv (r - r_+)(r - r_-)$, and $r_{\pm} \equiv M \pm \sqrt{M^2 - a^2}$ are the inner and outer horizon radii. The metric (A.1) follows from the Boyer-Lindquist metric via the coordinate transformation

$$dv = dt + \frac{r^2 + a^2}{\Delta_{\text{BL}}} dr, \quad (\text{A.2a})$$

$$d\theta = d\phi + \frac{a}{\Delta_{\text{BL}}} dr. \quad (\text{A.2b})$$

The inverse metric in the transformed coordinates is

$$g^{ab}\partial_a\partial_b = -\frac{a^2 \sin^2 \theta}{\Sigma_{\text{BL}}} \partial_v^2 - \frac{2(r^2 + a^2)}{\Sigma_{\text{BL}}} \partial_v \partial_r - \frac{\Delta_{\text{BL}}}{\Sigma_{\text{BL}}} \partial_r^2 - \frac{2a}{\Sigma_{\text{BL}}} \partial_v \partial_\varphi - \frac{2a}{\Sigma_{\text{BL}}} \partial_v \partial_\theta - \frac{1}{\Sigma_{\text{BL}}} \partial_\theta^2 - \frac{\csc^2 \theta}{\Sigma_{\text{BL}}} \partial_\varphi^2. \quad (\text{A.3})$$

We see that on the black hole horizon, r is a null coordinate. If $a = 0$, v is also a null coordinate, but when $a \neq 0$ it is timelike. The Kinnersley tetrad [134] in ingoing Kerr-Schild coordinates is

$$l_{(K)}^a \partial_a = \frac{2(r^2 + a^2)}{\Delta_{\text{BL}}} \partial_v + \partial_r + \frac{2a}{\Delta_{\text{BL}}} \partial_\varphi, \quad (\text{A.4a})$$

$$n_{(K)}^a \partial_a = -\frac{\Delta_{\text{BL}}}{2\Sigma_{\text{BL}}} \partial_r, \quad (\text{A.4b})$$

$$m_{(K)}^a \partial_a = \frac{1}{2^{1/2}(r + ia \cos \theta)} \left(ia \sin \theta \partial_v + \partial_\theta + \frac{i}{\sin \theta} \partial_\varphi \right). \quad (\text{A.4c})$$

This tetrad is not regular on the black hole horizon. We transform to the Hawking-Hartle tetrad [115], which is regular, via the rotation

$$l_{(HH)}^a \equiv \frac{\Delta_{\text{BL}}}{2(r^2 + a^2)} l_{(K)}^a, \quad (\text{A.5})$$

$$n_{(HH)}^a \equiv \frac{2(r^2 + a^2)}{\Delta_{\text{BL}}} n_{(K)}^a. \quad (\text{A.6})$$

We then have

$$l_{(HH)}^a \partial_a = \partial_v + \frac{\Delta_{\text{BL}}}{2(r^2 + a^2)} \partial_r + \frac{a}{r^2 + a^2} \partial_\varphi, \quad (\text{A.7a})$$

$$n_{(HH)}^a \partial_a = -\frac{r^2 + a^2}{\Sigma_{\text{BL}}} \partial_r, \quad (\text{A.7b})$$

$$m_{(HH)}^a \partial_a = \frac{1}{2^{1/2}(r + ia \cos \theta)} \left(ia \sin \theta \partial_v + \partial_\theta + \frac{i}{\sin \theta} \partial_\varphi \right). \quad (\text{A.7c})$$

A.1.2 Properties of the ingoing Kerr-Schild metric and the Hawking-Hartle tetrad

The Hawking-Hartle tetrad is well defined on the horizon, and we will use it to calculate horizon fluxes. The Ricci-rotation coefficients and Weyl scalars are listed in [186]. As $\kappa = 0$, $l_{(HH)}^a$ is pregeodesic (see [65] Sec. 9); that is, $l^a \nabla_a l^b \propto l^b$. On the black hole horizon, $l_{(HH)}^a$ is tangent to the null generators of the black hole horizon; that is, it can be written as [115, 30]

$$l_{(HH)}^a = \frac{dx^a}{dv}. \quad (\text{A.8})$$

This follows from Eq. (A.2) and the fact that $\Delta_{\text{BL}} = 0$ on the black hole horizon

$$\begin{aligned} \frac{dx^a}{dv} \partial_a &= \frac{dv}{dv} \partial_v + \frac{dr}{dv} \partial_r + \frac{d\theta}{dv} \partial_\theta + \frac{d\varphi}{dv} \partial_\varphi \\ &= \partial_v + \frac{a}{r^2 + a^2} \partial_\varphi. \end{aligned} \quad (\text{A.9})$$

Thus on the black hole horizon the NP derivative operator $D_{(HH)} \equiv l_{(HH)}^a \partial_a$ is

$$D_{(HH)} = \frac{d}{dv}. \quad (\text{A.10})$$

Moreover, it is straightforward to see that

$$\xi_{(v)}^a \partial_a \equiv \partial_v, \quad \xi_{(\varphi)}^a \partial_a \equiv \partial_\varphi \quad (\text{A.11})$$

are Killing vectors for the spacetime. On the black hole horizon we can then write

$$l_{(HH)}^a = \xi_{(v)}^a + \Omega_H \xi_{(\varphi)}^a, \quad (\text{A.12})$$

where the angular velocity of the horizon Ω_H is $\Omega_H \equiv \frac{a}{r_+^2 + a^2} = \frac{a}{2Mr_+}$. We note that $l_{(HH)}^a$ and $m_{(HH)}^a$ commute with each other on the inner and outer horizons,

$$l_{(HH)}^a \nabla_a m_{(HH)}^b = m_{(HH)}^a \nabla_a l_{(HH)}^b. \quad (\text{A.13})$$

A.1.3 Change in the black hole area

The area of the black hole horizon is

$$A(v) = \int d\varphi d\theta \sqrt{s}, \quad (\text{A.14})$$

where s_{AB} is the induced metric on the black hole horizon. We can write

$$s_{AB} \equiv g_{ab} e_A^a e_B^b, \quad (\text{A.15})$$

where e_A^a are the (spacelike, essentially angular) tangent vectors on the horizon, and $d\varphi d\theta$ are the angular coordinates on the horizon. In ingoing Kerr-Schild coordinates, we have

$$e_{(1)}^a \partial_a = \frac{dx^a}{d\theta} \partial_a = \partial_\theta, \quad e_{(2)}^a \partial_a = \frac{dx^a}{d\varphi} \partial_a = \partial_\varphi. \quad (\text{A.16})$$

This implies that

$$s_{AB} d\theta^A d\theta^B = g_{\theta\theta} d\theta^2 + g_{\varphi\varphi} d\varphi^2, \quad \sqrt{s} = (g_{\theta\theta} g_{\varphi\varphi})^{1/2}. \quad (\text{A.17})$$

The e_A^a can be converted to the NP vectors m^a by first defining a complex transformation vector e_A to obtain a complex vector e^a that satisfies

$$e^a = e^A e_A^a, \quad g_{ab} e^a e^b = 0, \quad g_{ab} e^a \bar{e}^b = -1. \quad (\text{A.18})$$

From the form of the Hawking-Hartle tetrad, we see that on the black hole horizon ($\Delta_{\text{BL}} = 0$) we can choose a complex e_A and A such that

$$e^a = m_{(HH)}^a + A l_{(HH)}^a. \quad (\text{A.19})$$

We note that $l_{(HH)}^a$ commutes with the angular vectors e_A^a

$$l_{(HH)}^a \nabla_a e_A^b = e_A^a \nabla_a l_{(HH)}^b. \quad (\text{A.20})$$

Using all these relations, we have [dropping the (HH) suffix to make the equations less cluttered]

$$\begin{aligned} \frac{dA}{dv} &= \int d\varphi d\theta \sqrt{s} \left(\frac{1}{\sqrt{s}} \frac{d\sqrt{s}}{dv} \right) \\ &= \int d\varphi d\theta \sqrt{s} \left(\frac{1}{2} s^{AB} l_{(HH)}^c \nabla_c s_{AB} \right) \\ &= - \int d\varphi d\theta \sqrt{s} g_{ab} \left((\bar{m}^c m^b + m^c \bar{m}^b) (\nabla_c l^a) \right) \\ &= - \int d\varphi d\theta \sqrt{s} (\rho + \bar{\rho}). \end{aligned} \quad (\text{A.21})$$

We have used that $l^c \nabla_c l^a \propto l^a$, $l^a m_a = 0$, $l^a l_a = 0$, and the definition for $\rho \equiv \bar{m}^a (\nabla_a l^b) m_b$ (see Ref. [65] §8). In summary, we have that the change the black hole area is the surface integral over the horizon of twice the real part of the Newman-Penrose scalar ρ in the Hawking-Hartle tetrad in ingoing Kerr-Schild coordinates,

$$\frac{dA}{dv} = - \int d\varphi d\theta \sqrt{s} (\rho_{(HH)} + \bar{\rho}_{(HH)}) . \quad (\text{A.22})$$

We note that the real part of ρ corresponds to the expansion of the null congruence l^a [65], which describes the change rate of the congruence's cross-sectional area [173].

A.1.4 The Newman-Penrose equations for the change in the black hole area

Note that Eq. (A.22) is technically nonperturbative—we do not rely on expanding the solution to a fixed order in perturbation theory. To relate dA/dv to solutions to the Teukolsky equation ($\Psi_0^{(1)}$), we perturbatively solve the NP equations; see Refs. [115] or [65] Sec. 98(d).

We start with Ref. [65] (310a) and (310b),

$$D\rho = (\epsilon + \bar{\epsilon})\rho + \rho^2 + \sigma\bar{\sigma} + (\bar{\delta} - 3\alpha - \bar{\beta} + \varpi)\kappa - \tau\bar{\kappa} + \Phi_{00}, \quad (\text{A.23})$$

$$D\sigma = (\rho + \bar{\rho} + 3\epsilon - \bar{\epsilon})\sigma + (\delta - \bar{\alpha} - 3\beta - \tau + \bar{\varpi})\kappa + \Psi_0. \quad (\text{A.24})$$

Here $\Phi_{00} \equiv -(1/2)R_{ab}l^a l^b = -(1/2)T_{ab}l^a l^b$ represents the matter field contribution; $\Phi_{00}^{(0)} = 0$ since the background is vacuum Kerr. We will work with a Hawking-Hartle tetrad to all orders in perturbation theory. That is, we assume l^a is pregeodesic ($\kappa = 0$), and we assume $D = d/dv$ to all orders in perturbation theory. The NP equations then are

$$\frac{d\rho_{(HH)}}{dv} = (\epsilon_{(HH)} + \bar{\epsilon}_{(HH)})\rho_{(HH)} + \rho_{(HH)}^2 + \sigma_{(HH)}\bar{\sigma}_{(HH)} + \Phi_{00}, \quad (\text{A.25})$$

$$\frac{d\sigma_{(HH)}}{dv} = (\rho_{(HH)} + \bar{\rho}_{(HH)} + 3\epsilon_{(HH)} - \bar{\epsilon}_{(HH)})\sigma_{(HH)} + \Psi_{0(HH)}. \quad (\text{A.26})$$

As $\sigma_{(HH)}^{(0)} = 0$ and $\rho_{(HH)}^{(0)} = 0$ on the black hole horizon, and $\epsilon_{(HH)}^{(0)}$ is real, to linear order in perturbation theory we see that

$$\frac{d\rho_{(HH)}^{(1)}}{dv} = 2\epsilon_{(HH)}^{(0)}\rho_{(HH)}^{(1)}, \quad (\text{A.27})$$

$$\frac{d\sigma_{(HH)}^{(1)}}{dv} = 2\epsilon_{(HH)}^{(0)}\sigma_{(HH)}^{(1)} + \Psi_{0(HH)}^{(1)}. \quad (\text{A.28})$$

Note that we have assumed that $\Phi_{00}^{(1)} = 0$. This is a reasonable assumption (at least to linear order in perturbation theory) as the matter stress-energy tensor is typically quadratic in the matter fields so the linear perturbation would be zero. The only solution to the first equation is $\rho_{(HH)}^{(1)} = 0$ (with the boundary condition $\rho_{(HH)}^{(1)} = 0$ at $v = \infty$), so we expand the ρ equation to second order to obtain

$$\frac{d\rho_{(HH)}^{(2)}}{dv} = 2\epsilon_{(HH)}^{(0)}\rho_{(HH)}^{(2)} + \left|\sigma_{(HH)}^{(1)}\right|^2 + \Phi_{00(HH)}^{(2)}. \quad (\text{A.29})$$

Setting $\rho_{(HH)}^{(2)} = 0$ and $\sigma_{(HH)}^{(1)} = 0$ at $v = \infty$, we have

$$\rho_{(HH)}^{(2)}(v, \theta, \varphi) = - \int_v^\infty dv' \exp \left[2\epsilon_{(HH)}^{(0)}(v - v') \right] \left(\left| \sigma_{(HH)}^{(1)}(v', \theta, \varphi) \right|^2 + \Phi_{00(HH)}^{(2)}(v', \theta, \varphi) \right). \quad (\text{A.30a})$$

$$\sigma_{(HH)}^{(1)}(v, \theta, \varphi) = - \int_v^\infty dv' \exp \left[2\epsilon_{(HH)}^{(0)}(v - v') \right] \Psi_{0(HH)}^{(1)}(v', \theta, \varphi). \quad (\text{A.30b})$$

We see that from $\Psi_{0(HH)}^{(1)}$ we can obtain $\sigma_{(HH)}^{(1)}$ and then $\rho_{(HH)}^{(2)}$. Finally, we see that $\rho_{(HH)}^{(0)} = \rho_{(HH)}^{(1)} = 0$ on the black hole horizon, and $\rho_{(HH)}^{(2)}$ is real. Then from Eq. (A.22), we have to leading order that [115]

$$\frac{dA^{(2)}}{dv} = -2 \int d\varphi d\theta \sqrt{s} \rho^{(2)}(v, \theta, \varphi). \quad (\text{A.31})$$

A.1.5 Relating change in the black hole area to the change in the black hole mass and spin

The black hole area $A = 8\pi M r_+$ is given by Eq. (A.14).

Note that $\sqrt{s}$ on the black hole horizon is the same in ingoing Kerr-Schild (as we are using here) and Eddington-Finkelstein coordinates (which are used in the evolution code referenced in the main text). Defining $J \equiv aM$, we then have

$$dA = \frac{8\pi (-JdJ + 2M^2 r_+ dM)}{\sqrt{M^4 - J^2}}. \quad (\text{A.32})$$

From this, we conclude that

$$\frac{d^2 A}{dv d\Omega} = \frac{8\pi}{\sqrt{M^4 - J^2}} \left(2M^2 r_+ \frac{d^2 M}{dv d\Omega} - J \frac{d^2 J}{dv d\Omega} \right). \quad (\text{A.33})$$

We consider perturbations in frequency space, that is, perturbations of the form

$$\Psi_0^{(1)}(v, r, \theta, \varphi) = \int d\omega \sum_m e^{im\varphi - i\omega v} \tilde{\Psi}_0(\omega, r, \theta, m). \quad (\text{A.34})$$

As $\xi_{(v)}^a$ and $\xi_{(\varphi)}^a$ are Killing vectors, we have [193, 207]

$$dJ = m \frac{\Re \omega}{|\omega|^2} dE \quad (\text{A.35})$$

for perturbations dJ and dE of the spacetime angular momentum and spacetime energy. With Eq. (A.35) and $E = M$, $J = aM$, we have

$$\frac{d^2 A}{dv d\Omega} = \frac{8\pi (2Mr_+ |\omega|^2 - am \Re \omega)}{\sqrt{M^2 - a^2 |\omega|^2}} \frac{d^2 M}{dv d\Omega}. \quad (\text{A.36})$$

Note that the sign changes for certain values of ω . The sign of the mass change depends on the sign of $\left[\frac{|\omega|^2}{\Re \omega} - m\Omega_H \right]$. This is the superradiant condition in the complex case. Inverting this equation, we obtain Eq. (2.3) in the main text. Using Eq. (A.35) and Eq. (2.3), we have

$$\frac{d^2 J}{dv d\Omega} = \frac{\sqrt{M^2 - a^2 |\omega|^2} m \Re \omega}{8\pi (2Mr_+ |\omega|^2 - am \Re \omega)} \frac{d^2 A}{dv d\Omega}. \quad (\text{A.37})$$

Using $dJ = Mda + adM$ and Eqs. (2.3) and (A.37), we obtain Eq. (2.4) in the main text.

A.1.6 Change in the black hole mass and spin for a gravitational quasinormal mode

We consider a quasinormal mode solution to Ψ_0 ,

$$\Psi_0(v, r, \theta, \varphi) = \mathcal{A} \times e^{im\varphi - i\omega v} R(r) S(\theta), \quad (\text{A.38})$$

where $\mathcal{A}$ is the amplitude of the quasinormal mode, and S is a spin-weighted spheroidal harmonic [208]. We normalize the solution so that $R(r_+) = 1$. On the black hole horizon, we have

$$\epsilon_{(HH)}^{(0)} = \frac{(M^2 - a^2)^{1/2}}{4Mr_+}. \quad (\text{A.39})$$

We can now compute $d^2 A/(dv d\Omega)$ for this solution.

First we compute $\sigma_{(HH)}^{(1)}$ using Eq. (A.30b),

$$\begin{aligned}\sigma_{(HH)}^{(1)}(v, \theta, \varphi) &= -\mathcal{A} S(\theta) e^{im\varphi} \int_v^\infty dv' \exp \left[2\epsilon_{(HH)}^{(0)}(v - v') - i\omega v' \right] \\ &= \mathcal{A} \frac{S(\theta) e^{im\varphi - i\omega v}}{2\epsilon_{(HH)}^{(0)} + i\omega}.\end{aligned}\tag{A.40}$$

From Eq. (A.30a) we have

$$\rho_{(HH)}^{(2)}(v, \theta, \varphi) = -|\mathcal{A}|^2 |S(\theta)|^2 \int_v^\infty dv' \frac{\exp \left[2\epsilon_{(HH)}^{(0)}(v - v') + 2(\Im\omega) v' \right]}{\left| 2\epsilon_{(HH)}^{(0)} - \Im\omega \right|^2 + |\Re\omega|^2}.\tag{A.41}$$

From Eq. (2.1), we conclude that

$$\frac{dA^{(2)}}{dv} = 2|\mathcal{A}|^2 \int d\varphi d\theta \sqrt{s} |S(\theta)|^2 \int_v^\infty dv' \frac{\exp \left[2\epsilon_{(HH)}^{(0)}(v - v') + 2(\Im\omega) v' \right]}{\left| 2\epsilon_{(HH)}^{(0)} - \Im\omega \right|^2 + |\Re\omega|^2}.\tag{A.42}$$

In our case (and in the evolution code coordinates), we have an induced horizon metric s_{AB} such that:

$$\sqrt{s} = 2Mr_+ \sin(\theta).\tag{A.43}$$

Using the fact that the spin-weighted spheroidal harmonics are normalized to have unit norm over the θ integral [208] [Eq. (2.8)], we can integrate over the sphere to obtain

$$\frac{dA^{(2)}}{dv} = 4\pi |\mathcal{A}|^2 \int_v^\infty dv' 2Mr_+ \frac{\exp \left[2\epsilon_{(HH)}^{(0)}(v - v') + 2(\Im\omega) v' \right]}{\left| 2\epsilon_{(HH)}^{(0)} - \Im\omega \right|^2 + |\Re\omega|^2}.\tag{A.44}$$

Now we take $M \neq M(v)$ and $a \neq a(v)$, making this a leading level calculation only. Carrying out this integral assuming a fixed background gives Eq. (2.6) in the main text. Setting $\Im\omega = 0$, we find this matches the pure mode result in [172] [Eqs. (5.14) and (5.22)] up to a coordinate choice.

Say we know the black hole mass at time $v = v_1$. We can then compute the black hole area at time $v = v_2$ using Eq. (2.6)

$$\Delta A \equiv A_2 - A_1 = 2\pi |\mathcal{A}|^2 M r_+ \frac{\exp[2\mathfrak{I}\omega v_2] - \exp[2\mathfrak{I}\omega v_1]}{\mathfrak{I}\omega \left(\epsilon_{(HH)}^{(0)} - \mathfrak{I}\omega \right) \left((2\epsilon_{(HH)}^{(0)} - \mathfrak{I}\omega)^2 + \Re\omega^2 \right)}. \quad (\text{A.45})$$

The approximate change in the black hole mass and spin over that interval are, from Eqs. (2.3) and (2.4):

$$\Delta M \approx \frac{\sqrt{M_0^2 - a_0^2} |\omega|^2}{8\pi (2M_0 r_+ |\omega|^2 - a_0 m \Re\omega)} \Delta A, \quad (\text{A.46})$$

$$\Delta a \approx \frac{\sqrt{M_0^2 - a_0^2} (a_0 |\omega|^2 + m \Re\omega)}{8\pi M_0 (2M_0 r_+ |\omega|^2 - a_0 m \Re\omega)} \Delta A. \quad (\text{A.47})$$

From these formulas it may at first appear that in the extremal limit $dM/dv = da/dv = 0$ for this approximation. However, looking more closely at the expression for dA/dv , we see that, in the real frequency case, this factor cancels out (since $\epsilon_{(HH)}^{(0)} \propto \sqrt{M^2 - a^2}$). In the extremal case, quasinormal modes approach real frequencies. Note that, apart from the extremal case or some fine-tuning of the perturbation, we have $\Delta a = 0$ only for $a_0 = 0$ and a perturbation with $m = 0$.

A.2 Physical perturbing amplitude

In order to apply our flux formulas, we need to relate a strain amplitude at null infinity to the amplitude of Ψ_0 on the black hole horizon. Here we derive a relation between these. First, we relate the strain amplitude to that of the Weyl scalar Ψ_4 at infinity. Then we use numerical solutions to the Teukolsky equation to relate the amplitude at infinity to one on the horizon in the Kinnersley tetrad. We then use a result from [35] to relate Ψ_4 to Ψ_0 on the horizon. We then rotate to the well-defined Hawking-Hartle tetrad. Finally, we account for the effect of choosing a particular time slicing.

A.2.1 Strain amplitudes and Weyl scalars

Since here we are concerned with the amplitude of the Weyl scalars, we estimate the amplitude of Ψ_4 . From the relation $\Psi_4^K(r \rightarrow \infty) = 1/2(d^2 h/dt^2)$, for a single (220)

quasinormal mode we have

$$|\Psi_4^K(r \rightarrow \infty)| = \frac{|h_{220}\omega_{220}^2|}{2} \frac{1}{r}, \quad (\text{A.48})$$

where we recall that the definition of $h_{\ell mn}$ is given by Eq. (2.8).

A.2.2 Moving from null infinity to the horizon

In order to solve the Teukolsky equation, the “long-range potential” in the Teukolsky equation can be removed (see [185] Eq. 6), and a scaling with $\rho^{(0)} = -1/(r - ia \cos \theta)$ introduced ([208] Table 1—this is necessary to have a separable Teukolsky equation). This gives new variables ψ . These ψ are well defined on both the horizon and null infinity. Explicitly,

$$\Psi_4^K \equiv \frac{1}{r} \Delta_{\text{BL}}^{-s} \rho^{(0)4} \psi_4 \quad (\text{A.49})$$

$$r \gg M \approx r^{-1} \psi_4 \quad (\text{A.50})$$

$$\Psi_0^K \equiv \frac{1}{r} \Delta_{\text{BL}}^{-s} \psi \quad (\text{A.51})$$

$$r \gg M \approx r^{-5} \psi_0. \quad (\text{A.52})$$

We relate the value of ψ_4 at null infinity to its value at the horizon. We do this by numerically calculating the radial eigenfunctions of the Teukolsky equation in our evolution hyperbolic coordinates using [185] [this code uses the Kinnersley tetrad, with the variables described in Eq. (A.49)]. We define this ratio

$$C_{\text{to-horizon}} = \left| \frac{\psi_4(r_+)}{\psi_4(\infty)} \right|. \quad (\text{A.53})$$

A.2.3 Relating Weyl scalars Ψ_0 and Ψ_4

We use a result from Ref. [35] using metric reconstruction to relate Ψ_0 and Ψ_4 at infinity in Boyer-Lindquist coordinates and the Kinnersley tetrad. According to this calculation, a solution [Eq. (2.43) in Ref. [35]]

$$\Psi_4^K = \rho^{(0)4} e^{-i\omega t + im\phi} R^{(-2)} {}_2S_{\ell m} \quad (\text{A.54})$$

is equivalent to a solution [Eq. (2.44) in Ref. [35]]

$$\Psi_0^K = \frac{4D'}{\mathfrak{C}'} e^{-i\omega t + im\phi} R^{(2)} {}_2S_{\ell m} + \frac{48i\omega^* M}{\mathfrak{C}'^*} e^{+i\omega^* t - im\phi} R^{(2)} {}_2S_{\ell -m}^*, \quad (\text{A.55})$$

where D' and $\mathfrak{C}'$ are versions of the Starobinsky constant, see Ref. [35] for the complete definitions. Here $*$ indicates the complex conjugate. The prograde m mode is the first term and the retrograde $-m$ mode is the second term. So we have two modes corotating with the black hole with the same frequency, described by two different quasinormal modes. From Ref. [35], these radial functions are normalized such that (for ingoing physical solutions)

$$|R^s(r \rightarrow r_+)| \rightarrow \Delta_{\text{BL}}^{-s}. \quad (\text{A.56})$$

Taking the norm of the corresponding solutions, we have

$$|\Psi_4^K(\infty)| = \mathcal{A} |\rho^{(0)4} \Delta_{\text{BL}}^2| \quad (\text{A.57})$$

$$|\Psi_0^K(\infty)| = \sqrt{\int d\Omega \Psi_0^K \Psi_0^{K*}} \quad (\text{A.58})$$

$$= \mathcal{A} \Delta_{\text{BL}}^{-2} \sqrt{\left| \frac{4D'}{\mathfrak{C}'} \right|^2 + \left| \frac{48\omega M}{\mathfrak{C}'} \right|^2}. \quad (\text{A.59})$$

Note that the cross term (corresponding to the second-order mode) disappears in the norm and so does not change the black hole area. Thus we get the relation between the norms [in terms of the regularised variables from Eq. (A.49)]

$$|\psi_0(r_+)| = |\psi_4(r_+)| \sqrt{\left| \frac{4D'}{\mathfrak{C}'} \right|^2 + \left| \frac{48\omega M}{\mathfrak{C}'} \right|^2}. \quad (\text{A.60})$$

Then we have

$$|\Psi_0^K(r_+)| = \left| \left(\frac{1}{r} \Delta_{\text{BL}}^{-2} \right)_{r_+} \sqrt{\left| \frac{4D'}{\mathfrak{C}'} \right|^2 + \left| \frac{48\omega M}{\mathfrak{C}'} \right|^2} C_{\text{to-horizon}} (r \Delta_{\text{BL}}^{-2} \rho^{(0)-4})_{\infty} \Psi_4^K(\infty) \right|. \quad (\text{A.61})$$

In our case, $C_{\text{to-horizon}}$ is the same for both the m prograde mode and the $-m$ retrograde mode (both corotating with frequency ω). Recall that this relation applies only to the Kinnersley tetrad.

A.2.4 Relating to Hawking-Hartle tetrad

The two tetrads are related through a transformation known as a tetrad rotation; specifically, in the language of Chandrasekhar [65], they are related by a type III rotation. From Ref. [65] Sec. 347, a type III rotation sends the tetrad components $l \rightarrow A^{-1}l$, $n \rightarrow An$, $m \rightarrow e^{i\theta}m$, $\bar{m} \rightarrow e^{-i\theta}\bar{m}$. The Weyl scalars change according to

$$\Psi_0 \rightarrow A^{-2}e^{2i\theta}\Psi_0, \quad \Psi_4 \rightarrow A^2e^{-2i\theta}\Psi_4, \quad (\text{A.62})$$

where A and θ are defined by the specific rotation. The translation from the Hawking-Hartle to the Kinnersley tetrad corresponds to $\theta_{HH \rightarrow K} = 0$ and $A_{HH \rightarrow K} = \frac{\Delta_{\text{BL}}}{2(r^2 + a^2)}$ from (A.5). Using the argument from Ref. [207] [Eq. (4.43)], if Weyl scalars in different tetrads are equal on the horizon, then it is correct to use the same flux through the horizon, and so we can use a tetrad rotation to calculate the input for our flux formulae. From Eq. A.62, Ψ_0 in the Hawking-Hartle tetrad is related to the Kinnersley tetrad by

$$\Psi_0^K(r_+) = (A_{HH \rightarrow K}^{-2}e^{2i\theta_{HH \rightarrow K}}\Psi_0^{HH}(r_+)) \quad (\text{A.63})$$

$$= \left(\frac{4(r^2 + a^2)^2}{\Delta_{\text{BL}}^2} \right)_{r_+} \Psi_0^{HH}(r_+). \quad (\text{A.64})$$

And finally, using Eqs. (A.61) and (A.48),

$$\left| \Psi_0^{(HH)}(r_+) \right| = \left| \frac{C_{\text{to-horizon}}}{16M^2r_+^3} \sqrt{\left| \frac{4D'}{\mathfrak{C}'} \right|^2 + \left| \frac{48\omega M}{\mathfrak{C}'} \right|^2} \left(\frac{|h_{220}\omega^2|}{2} \right) \right|. \quad (\text{A.65})$$

A.2.5 Lookback time

Equation (A.65) relates the amplitude at the horizon in the Hawking-Hartle tetrad to the amplitude of the strain at infinity at a given time, for our particular choice of time slicing, and assuming a quasinormal mode that has existed for all time. However, physically we are interested in the scenario where we have some transient perturbation in the vicinity of the black hole (roughly, say, in the vicinity of the light ring) and we wish to synchronize a gravitational wave signal at infinity with a flux at the horizon arising from the same perturbation. This introduces a “lookback time” ambiguity.

The quantity given in Eq. (A.65) is the amplitude at the horizon in the Hawking-Hartle tetrad when the signal at infinity is assumed to be composed of a quasinormal mode. However, assuming the whole spacetime is described by a quasinormal mode, we

expect the signal at the horizon to be composed of quasinormal modes at an earlier time. At some coordinate time the behavior in the bulk is assumed to be linear. This behavior (we approximate the peak at the light ring) sends signatures to null infinity and to the black hole horizon. The linear behavior should be apparent first at the horizon (in our coordinate choice). Since we take this behavior to be linear, we can account for this effect by considering an increase in initial amplitude corresponding to the change in start time for the linear behavior.

As a rough estimate of the lookback time and its uncertainty, we can take a null geodesic starting at the light ring and calculate the coordinate time of propagation to null infinity T_∞ and to the horizon T_{r_+} . These propagation times are calculated according to Appendix A.3 in the same coordinates as those for the calculation of $C_{\text{to-horizon}}$. The difference in those times corresponds to an estimate of the lookback time, the coordinate time difference to synchronize a signal at infinity with the flux at the horizon. The change in the amplitude of a quasinormal mode associated with this is a factor of

$$C_{\text{lookback}} = e^{\mathfrak{I}\omega(T_\infty - T_{r_+})}. \quad (\text{A.66})$$

Taking an uncertainty in the radial peak of the quasinormal mode of $1M$ (in the Boyer-Lindquist radius) corresponds to an uncertainty in the amplitude of a factor of 2.

A.3 Propagation times

In order to estimate the lookback time connecting a gravitational perturbation near the black hole to a time at null infinity in our particular evolution coordinates, we calculate the propagation times from the light ring to the horizon and null infinity. The evolution coordinates $\{t, R, \theta, \varphi\}$ are related to the usual ingoing Eddington-Finkelstein coordinates by $R = 1/r$ and $t = v - r - 4M \log(r)$ (for more discussion, see, for example, Appendix C of [186]).

For the case of $a/M = 0.7$, if we consider a radially outward, equatorial, null geodesic originating near the light ring at $r = r_{\text{LR}} \pm M/5$, we find the coordinate time to reach null infinity is $T_\infty = (9 \pm 5)M$. We get similar results if we numerically evolve for some narrowly peaked pulse in Ψ_4 at the same position and find the time it takes to reach null infinity.

We can compare this estimate of the lookback time to the calculation using a quasinormal mode. The total phase change of the signal (relative to the original quasinormal mode) at infinity due to a changing background is calculated for several values of ΔM in Fig. A.1. The phase difference begins to grow after $t \approx 7M$, which also provides an

estimate of how long it takes for the part of the quasinormal mode in the near zone that is sensitive to the black hole dynamics to propagate to null infinity. This is comparable, though slightly smaller than the estimate based on null geodesics.

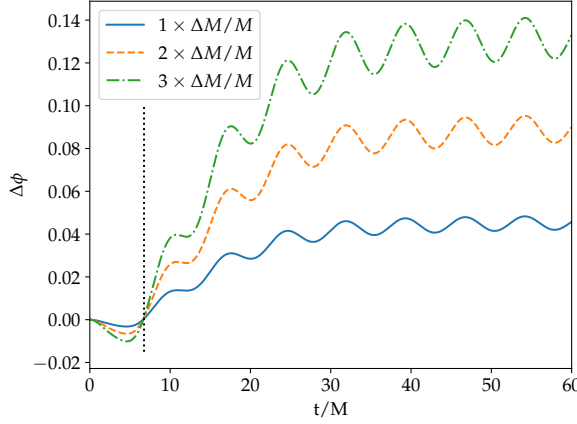


Figure A.1: The change in signal at infinity is shown for $a/M = 0.7$ and various degrees of changing black hole mass, with the lowest mass change $\Delta M/M = 0.7\%$. $\Delta\phi(t)$ is the phase difference in Ψ_4 between the changing background case and the fixed remnant background case as a function of time. As can be seen from the figure, this asymptotes to a constant value, as described in Sec. 2.4. The black dotted line is located at $t = 6.75M$ and shows where the different cases start to diverge.

A.4 Comparison of different quasinormal initial data

In Secs. 2.4 and 2.5.1, we consider a setup where the initial data corresponded exactly to a quasinormal mode of the final black hole, with the initial black hole changing mass and spin due to the absorption of this perturbation to reach these final parameters. In some sense this can be considered a consistency check—if a quasinormal mode of the remnant is present at some time, then we should see this third-order effect. For comparison, here we consider initial data corresponding to a quasinormal mode of the initial black hole. Considering these types of initial data also allows us to compare the excitation of different modes when the black hole evolves according to Eqs. (2.22) and (2.23) versus an instantaneously changing black hole background (i.e. fixing the mass and spin to the remnant values).

In Fig. A.2, we compare the amplitudes of the modes extracted in these two cases (cf.

Fig. 2.11). We find that using an evolving background enhances overtone excitation, while suppressing retrograde excitations. We also find that using these initial data allows us to more confidently extract an overtone compared to using a quasinormal mode of the final black hole. Here it seems that this new mode (with frequency determined by a free frequency fit at $t = 10M$, as described in Sec. 2.5.1) has a more constant amplitude, and so fits the data better, at early times. However, in this case, the physical (221) mode has approximately constant amplitude in the $t_0 \sim (30 - 45)M$ region. This region corresponds to the region where a free frequency fit returns something close to the first overtone frequency. Recall that Fig. 2.10 shows free frequency fits after filtering out fundamental modes for both initial data cases and an evolving background. When using a quasinormal mode of the initial black hole we more confidently identify an overtone amplitude for longer and find a frequency closer to the remnant overtone at later times, compared to using a quasinormal mode of the final black hole. We speculate that this is simply due to the fact that the overtone excitation is larger and hence can be resolved longer. As expected, in the instantaneously changing background case there is no evidence for a new mode or nonquasinormal mode content, and the overtone is extracted cleanly.

In Table A.1, we summarize the change in the amplitude of the (220) quasinormal measured at later times compared to the amplitude of the initial perturbation for both initial data cases and for the fixed and evolving backgrounds. As expected, we find a larger change when using initial data not corresponding to a quasinormal mode of the final black hole. When using such initial data, there is a bigger change for the evolving background compared to the instantaneously changing background. We conclude that the most conservative case to consider, in terms of minimizing nonlinear effects, is that of a perturbation in the form of a remnant quasinormal mode on an evolving background.

Table A.1: Comparing initial data and background cases

Initial data type	Background	$\Delta 220/220$
Remnant quasinormal mode	Evolving	0.011
Initial quasinormal mode	Evolving	0.026
Initial quasinormal mode	Instantaneous	0.018

Finally, we consider an instantaneously changing spherically symmetric background to compare to Ref. [193]. There the amplitude of the inner product between the fundamental 200 mode of a nonspinning black hole with mass M and the overtone 201 mode of a

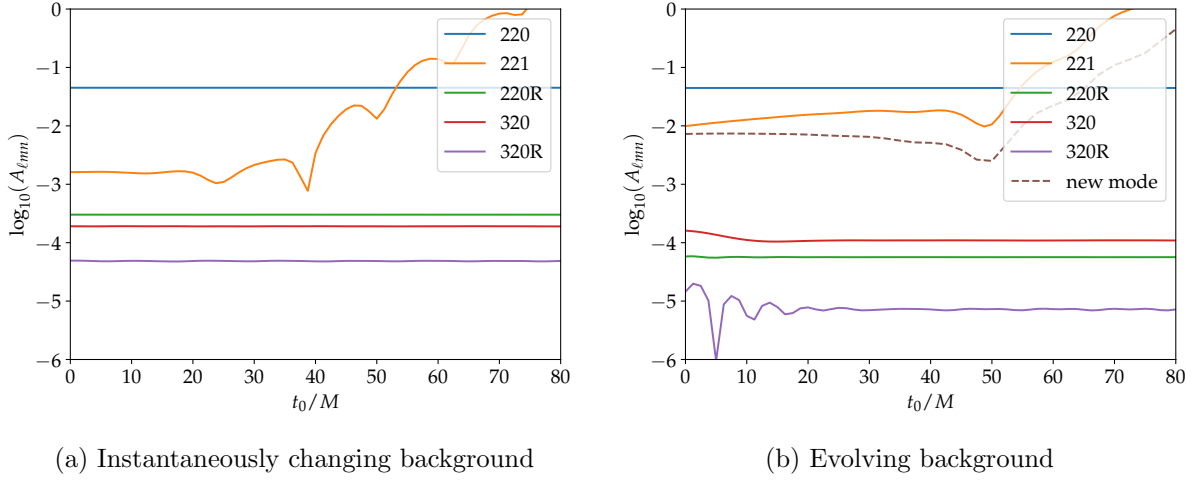


Figure A.2: Comparing linear evolution of the same initial perturbation on two different backgrounds. In both cases the initial perturbation is a quasinormal mode of a black hole with $M_{\text{BH}} = (1 - 0.01)M$ and spin $a_{\text{BH}} = 0.7(1 - 0.063)M$. (b) The case when the black hole evolves from these parameters to have mass M and spin $a/M = 0.7$ [according to Eqs.(2.22) and (2.23)], while (a) shows the case where the black hole has the latter parameters for all time. We see that the changing background enhances overtones, while the retrograde modes are relatively suppressed. We include a fit for a mode frequency $\omega = (\Re\omega_{220}) + i(\Im\omega_{221} + 0.0236)$ for the evolving background case. The new mode has constant amplitude at $t \sim (0 - 20)M$, but the physical (221) mode has a more constant amplitude at $t_0 \sim (30 - 45)M$.

nonspinning black hole with mass $M + \Delta M$ was computed, giving $A_{\text{exc}}^{(201)}/A_{\text{pert}}^{(200)} = 6 \times 10^{-3}$ for a mass change of $\Delta M/M = 10^{-3}$. We expect this calculation to give results equivalent to our mode analysis with background with mass $M + \Delta M$ and $a = 0$ and an initial perturbation corresponding to an $\ell = 2$, $m = 0$, $n = 0$ quasinormal mode of a black hole with mass M . We find, for a mass change of $\Delta M/M = 10^{-3}$, $A_{\text{exc}}^{(201)}/A_{\text{pert}}^{(200)} = 1.5 \times 10^{-3}$. If we take the amplitudes from the start of the simulation in this case (without considering a lookback time), we find $A_{\text{exc}}^{(201)}/A_{\text{pert}}^{(200)} = 3 \times 10^{-3}$. This factor of a few difference could be due to some combination of coordinate differences and lookback time ambiguity.

A.5 Numerical Convergence

The evolution code uses a grid in real space in the radial direction, with n_R grid points, and a spectral grid in the angular directions, with n_ℓ spherical harmonics. The radial finite differencing is fourth order, as is the time stepping. The standard choice for results included in the main text is $n_R = 121$, $n_\ell = 25$. We perform a pointwise convergence test on the numerical evolution code with changing background in Fig. A.3. In this case, we consider three resolutions: $n_R^{\text{LR}} = 121$, $n_R^{\text{MR}} = 241$, and $n_R^{\text{HR}} = 481$. We scale the number of time steps with n_R , while keeping $n_\ell^{\text{LR}} = n_\ell^{\text{MR}} = n_\ell^{\text{HR}} = 25$ fixed. We find the expected fourth-order convergence, showing that the radial/time resolution is dominating the total numerical error. We use this scaling to calculate the numerical accuracy in the signal as a function of time. We find that the relative error is $< 10^{-5}$ for $t < 100M$, though the relative error grows exponentially in time as the quasinormal mode solutions decays exponentially.

We perform a convergence test for the extracted amplitudes $A_{\ell mn}/\mathcal{A}$. These amplitudes are extracted by applying quasinormal mode filters (2.14) to the data and then fitting in the time domain for $t > 10M$ with a fixed-frequency model Eq. (2.17). We find that the fit amplitudes converge, with convergence order in the range 2-5 depending on the time of extraction. We check the convergence of extracted amplitudes for an evolving background in the cases of $a/M = 0.4$ and $a/M = 0.8$, as well as in the instantaneously changing background and a remnant spin of $a/M = 0.7$. To calculate the numerical errors for fit amplitudes in Sec. 2.4, we take two resolutions and assume only first-order convergence.

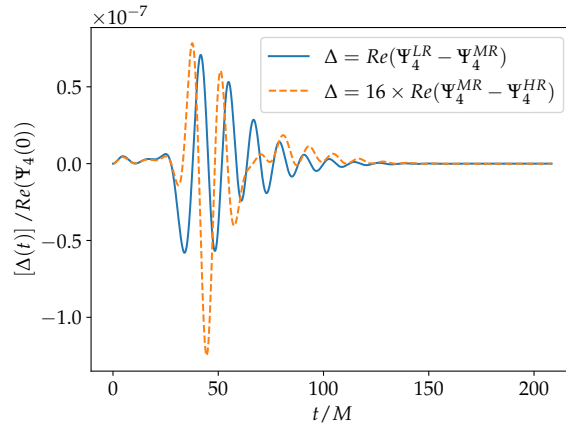


Figure A.3: Convergence plot showing the truncation error in the real part of Ψ_4 extracted at null infinity and projected onto the $\ell = m = 2$ spin-weighted spherical harmonic. The factor of 16 shown is consistent with fourth-order convergence.

Appendix B

Self Gravity Effects in Superradiant Clouds

B.1 Physical quantities

For the black hole-boson cloud solutions constructed in Chapter 3, we calculate several quantities that characterize the mass and angular momentum associated with the black hole and the cloud. To calculate the boson cloud mass, we use the Komar mass integral, as is done in Refs. [120, 121, 192], with

$$M_{\text{cloud}} = - \int_{\Sigma} dS_a (2T_b^a t^b - T t^a), \quad (\text{B.1})$$

where T_{ab} is the stress energy tensor with trace T , the integral is over a spacelike hypersurface Σ , with $dS_a = n_a dV$ proportional to the unit future pointing normal n_a to that hypersurface, and t^b is the Killing vector associated with the stationary spacetime. The cloud angular momentum J_{cloud} is calculated using the Komar integral as in the cloud mass integral in Eq. (B.1), but with t^a replaced by the azimuthal Killing vector φ^a . For both these quantities, we use a special treatment of the inner boundary, to avoid error due to setting the inner radial boundary slightly off the black hole horizon. This involves interpolating the integrand over the last few points before the horizon point then extrapolating to the point on the horizon. We use a linear interpolation because the extrapolated point converges well in that case.

We calculate the black hole mass using two different approaches. We can either calculate the ADM mass and subtract the cloud mass or directly calculate the horizon mass. In

different regimes, these two calculations have better associated error and so we calculate both in order to minimize error as well as to check our mass and angular momentum calculations are consistent. The ADM mass and total angular momentum J can be calculated using the asymptotic behavior of the metric as is described in Ref. [120] (Eq. 3.15). We also calculate the horizon mass as is described in Ref. [120] using the expression

$$M_{\text{BH}} = 2T_H S + 2\Omega_H (J - J_{\text{cloud}}), \quad (\text{B.2})$$

where the Hawking temperature T_H , and the entropy S are calculated according to Ref. [120].

In the scalar case, we use the horizon mass and the Komar cloud mass in all of the parameter space and in the vector case, we use the ADM mass and the Komar cloud mass for high α and the horizon mass and the Komar cloud mass for lower α . In each case, we check that these two definitions give results that are consistent within the numerical error.

B.2 Constructing solutions

B.2.1 Elliptic differential equations

We formulate differential equations using the Einstein-Klein Gordon and Einstein-Proca equations (along with a Lorenz condition in the Proca case). The solution is then constructed from an ansatz. We use metric ansatz with an explicit horizon in the Lewis–Papapetrou coordinates

$$ds^2 = -fNdt^2 + \frac{l}{f} \left[g \left(\frac{dr^2}{N} + r^2 d\theta^2 \right) + r^2 \sin^2 \theta \left(d\varphi - \frac{\omega_m}{r} dt \right)^2 \right], \quad (\text{B.3})$$

where $N = (r - r_H)/r$ vanishes at the horizon radius. This ansatz is stationary and axisymmetric, and so the metric functions f , l , g , and ω_m are real valued functions of only two coordinates (r and θ) and can be solved on a two dimensional grid. Due to the equatorial symmetry, we can restrict the angular domain to the region above the equatorial plane, $\theta \in (0, \pi/2)$. We use a compactified radial coordinate $x = (r - r_H)/r$, and include slight offsets from the boundaries of $0 < x < 1$ and $0 < \theta < \pi/2$ such that our numerical grid boundaries are at $10^{-7} < x < 1 - 10^{-7}$ and $10^{-7} < \theta < \pi/2 - 10^{-7}$. The matter field values are solved simultaneously on the same grid. The scalar field ansatz is

$$\Phi(t, r, \theta, \varphi) = \phi(r, \theta) e^{-i\omega t + im\varphi}. \quad (\text{B.4})$$

Here $\phi(r, \theta)$ is a real valued function, $m \in \mathbb{Z}$, and $\omega \in \mathbb{R}$. The function ϕ is solved along with the metric functions in r - θ space to give a full solution. In the vector case we use the ansatz

$$A_0(t, r, \theta, \varphi) = iV(r, \theta)e^{-i\omega t + im\varphi} \quad (\text{B.5})$$

$$A_r(t, r, \theta, \varphi) = H_1(r, \theta)e^{-i\omega t + im\varphi} \quad (\text{B.6})$$

$$A_\theta(t, r, \theta, \varphi) = H_2(r, \theta)e^{-i\omega t + im\varphi} \quad (\text{B.7})$$

$$A_\varphi(t, r, \theta, \varphi) = i\sin(\theta)H_3(r, \theta)e^{-i\omega t + im\varphi} \quad (\text{B.8})$$

and solve for real valued functions $V(r, \theta)$, $H_1(r, \theta)$, $H_2(r, \theta)$, and $H_3(r, \theta)$. The choice of ω and r_H (in units of μ) determines the solution.

We formulate the scalar (vector) equations of motion by combining Einstein's equation with the Klein Gordon (Proca and Lorentz) equation(s). We get a set of five (eight) second order differential equations, such that we have

$$\partial^2 F_i = \text{Polynomial}(\partial F_j, F_j) \quad (\text{B.9})$$

for each real valued field F_i . We use a metric ansatz that is different from Refs. [122, 120, 121, 192], and we find that, while this ansatz is well behaved in the boson star case, there is a numerical divergence at the horizon in the black hole with boson cloud case. We remove that divergence by including an overall factor of x^3 in all the equations of motion given below. We list the equations of motion for a complex vector field in the presence of a black hole here, these expressions are also available in a notebook accompanying this the. They are

$$\begin{aligned}
& \partial_r^2 V + \frac{\partial_\theta^2 V}{r(r-r_H)} \\
&= -\frac{1}{2r^2(r_H-r)^2 f^2 l} \left[(r_H-r) \sin \theta \left(-\csc \theta \partial_r l \partial_r V r^3 - 2\omega_m \partial_r H_3 \partial_r l r^2 \right. \right. \\
&\quad - 2l \partial_r H_3 \partial_r \omega_m r^2 - 4 \csc \theta l \partial_r V r^2 + r_H \csc \theta \partial_r l \partial_r V r^2 + 2m^2 \csc^3 \theta g l V r \\
&\quad - 2 \cot \theta \csc \theta l \partial_\theta V r - \csc \theta \partial_\theta l \partial_\theta V r - 2l \omega_m \partial_r H_3 r + 2r_H \omega_m \partial_r H_3 \partial_r l r \\
&\quad + 2r_H l \partial_r H_3 \partial_r \omega_m r + 4r_H \csc \theta l \partial_r V r - 4 \cot^2 \theta H_3 l \omega_m - 4 \cot \theta l \omega_m \partial_\theta H_3 \\
&\quad - 2 \cot \theta H_3 \omega_m \partial_\theta l - 2\omega_m \partial_\theta H_3 \partial_\theta l - 2 \cot \theta H_3 l \partial_\theta \omega_m - 2l \partial_\theta H_3 \partial_\theta \omega_m \\
&\quad + 2m \csc \theta H_2 (\omega_m \partial_\theta l + l(2 \cot \theta \omega_m + \partial_\theta \omega_m)) + 4r_H l \omega_m \partial_r H_3 - 2 \csc \theta \\
&\quad \left. H_1 (mr(r_H-r) \omega_m \partial_r l + l(-rr_H \omega + m(2r_H-r) \omega_m + mr(r_H-r) \partial_r \omega_m)) \right) f^2 \\
&\quad + 2(r_H-r) l \left(\mu^2 g l V r^3 + \omega H_1 \partial_r f r^3 + \partial_r f \partial_r V r^3 - r_H \omega H_1 \partial_r f r^2 \right. \\
&\quad - 2m H_1 \omega_m \partial_r f r^2 + 2\omega_m \sin \theta \partial_r f \partial_r H_3 r^2 - r_H \partial_r f \partial_r V r^2 + \partial_\theta f \partial_\theta V r \\
&\quad + 2mr_H H_1 \omega_m \partial_r f r - 2r_H \omega_m \sin \theta \partial_r f \partial_r H_3 r + 2 \cos \theta H_3 \omega_m \partial_\theta f \\
&\quad \left. - H_2 (-r\omega + 2m\omega_m) \partial_\theta f + 2\omega_m \sin \theta \partial_\theta f \partial_\theta H_3 \right) f \\
&\quad + 2rl^2 \left(rgV (-r\omega + m\omega_m)^2 - (r_H-r) H_1 \omega_m \sin^2 \theta (\omega_m - r \partial_r \omega_m) (-r\omega + m\omega_m) \right. \\
&\quad + \omega_m \sin^2 \theta \left(\partial_r \omega_m \partial_r V r^3 + \omega_m \sin \theta \partial_r H_3 \partial_r \omega_m r^2 - \omega_m \partial_r V r^2 \right. \\
&\quad - r_H \partial_r \omega_m \partial_r V r^2 + \partial_\theta \omega_m \partial_\theta V r - \omega_m^2 \sin \theta \partial_r H_3 r \\
&\quad - r_H \omega_m \sin \theta \partial_r H_3 \partial_r \omega_m r + r_H \omega_m \partial_r V r + \cos \theta H_3 \omega_m \partial_\theta \omega_m \\
&\quad \left. \left. - H_2 (-r\omega + m\omega_m) \partial_\theta \omega_m + \omega_m \sin \theta \partial_\theta H_3 \partial_\theta \omega_m + r_H \omega_m^2 \sin \theta \partial_r H_3 \right) \right) \right], \\
&\hspace{15cm} \text{(B.10)}
\end{aligned}$$

$$\begin{aligned}
& \partial_r^2 H_1 + \frac{\partial_\theta^2 H_1}{r(r-r_H)} \\
&= \frac{1}{2(r_H-r)^2} \left[\frac{2 \cot \theta (r_H-r) \partial_\theta H_1}{r} - \frac{4(r_H-r) \partial_\theta H_2}{r^2} \right. \\
&+ \frac{2gl(m\omega_m - \omega r)}{f^3} \left(H_3 \omega_m \sin \theta + rV(r, \theta) \right) \partial_r f + 4(r_H-r) \partial_r H_1 \\
&- \frac{2(r_H-r) \partial_\theta g}{gr} \left(\partial_\theta H_1 - \partial_r H_2 \right) - \frac{(r_H-r) \partial_\theta l}{lr} \left(\partial_\theta H_1 - \partial_r H_2 \right) + \frac{(r_H-r) \partial_\theta l \partial_r H_2}{lr} \\
&- \frac{2(r_H-r)}{fr} \left(\mu^2 g H_1 l r^2 + \partial_\theta f (\partial_r H_2 - \partial_\theta H_1) \right) \\
&- \frac{2m \csc \theta g (r_H-r)}{r} \left(m \csc \theta H_1 - \partial_r H_3 \right) + \frac{(r_H-r)((r_H-r)r \partial_r H_1 - 2\partial_\theta H_2) \partial_r l}{lr} \\
&+ \frac{2m \csc \theta g (r_H-r)}{lr^2} \left(H_3(2l + r \partial_r l) - lr \partial_r H_3 \right) \\
&- \frac{\partial_r f}{flr} \left((r_H-r)(-H_2(2 \cot \theta l + \partial_\theta l) + 2l(m \csc \theta g H_3 - \partial_\theta H_2 + (r_H-r)r \partial_r H_1) \right. \\
&\quad \left. + H_1(2l(r_H-2r) + (r_H-r)r \partial_r l)) \right) \\
&+ \frac{(r_H-r) \partial_r g}{glr} \left(-H_2(2 \cot \theta l + \partial_\theta l) - 2l(\partial_\theta H_2 + r(r-r_H) \partial_r H_1) \right. \\
&\quad \left. + H_1(2l(r_H-2r) + (r_H-r)r \partial_r l) \right) \\
&- \frac{2gl(m\omega_m - \omega r)}{f^2} \left(H_1(m\omega_m - \omega r) - \omega_m \sin \theta \partial_r H_3 - r \partial_r V \right) \\
&- \frac{1}{f^2 r(r-r_H)} \left[2gl \left(H_3 \sin \theta (m(r_H-2r)\omega_m^2 - r(-\omega r + 2m(r_H-r) \partial_r \omega_m) \omega_m \right. \right. \\
&\quad \left. \left. - \omega r^2(r-r_H) \partial_r \omega_m) + r \left(-rV(r, \theta)(-r_H \omega + m\omega_m + m(r_H-r) \partial_r \omega_m) \right. \right. \right. \\
&\quad \left. \left. \left. - (r_H-r)(m\omega_m - \omega r)(\omega_m \sin \theta \partial_r H_3 + r \partial_r V) \right) \right) \right] \\
&- \frac{H_2(r_H-r)}{l^2 r^2} \left(4 \cot \theta l^2 + (2\partial_\theta l + 2 \cot \theta r \partial_r l - r \partial_r \partial_\theta l) l + 2r(\partial_\theta l)(\partial_r l) \right) \\
&+ \frac{H_1(r_H-r)}{l^2 r^2} \left(4(r_H-r) l^2 + r((3r_H-4r) \partial_r l + r(r-r_H) \partial_r^2 l) l + 2(r_H-r) r^2 (\partial_r l)^2 \right) \Big], \\
&\hspace{25em} (B.11)
\end{aligned}$$

$$\begin{aligned}
\partial_r^2 H_2 = & \frac{1}{2(r_H - r)^2} \left[\frac{2(r_H - r)}{f} \left((r_H - r) \partial_r f (\partial_\theta H_1 - \partial_r H_2) - \mu^2 r g H_2 l \right) \right. \\
& - \frac{2gl(m\omega_m - r\omega)}{f^2} \left(H_2(m\omega_m - r\omega) - \sin \theta \omega_m \partial_\theta H_3 - \cos \theta \omega_m H_3 - r \partial_\theta V \right) \\
& + (r_H - r) \left(-2r(r_H - r) \partial_r g l (\partial_\theta H_1 - \partial_r H_2) \right. \\
& + g \left(-r(r_H - r) \partial_r l (\partial_\theta H_1 - \partial_r H_2) - 2l(r_H \partial_\theta H_1 + r(r - r_H) \partial_r \partial_\theta H_1 - r_H \partial_r H_2) \right) \\
& \left. \left. + 2m \csc^2 \theta g^2 l (-m H_2 + \sin \theta \partial_\theta H_3 + \cos \theta H_3) \right) (r g l) \right],
\end{aligned}
\tag{B.12}$$

$$\begin{aligned}
& \partial_r^2 H_3 + \frac{\partial_\theta^2 H_3}{r(r-r_H)} \\
&= \frac{1}{4r(r_H-r)^2 f^2 l} \left[4(r_H-r) f l \left(-m r^2 \csc \theta \partial_r f H_1 + m r r_H \csc \theta \partial_r f H_1 \right. \right. \\
&\quad \left. \left. - m \csc \theta \partial_\theta f H_2 + r^2 \partial_r f \partial_r H_3 + \partial_\theta f \partial_\theta H_3 - r r_H \partial_r f \partial_r H_3 \right. \right. \\
&\quad \left. \left. + \cot \theta \partial_\theta f H_3 - \mu^2 r^2 g H_3 l \right) \right. \\
&\quad \left. - \csc^2 \theta (r_H-r) f^2 \left(H_3 (4l(m^2 g + 1) + \sin(2\theta) \partial_\theta l) \right. \right. \\
&\quad \left. \left. - 2 \sin \theta \left(-2m(r_H-r) H_1 (r \partial_r l + 2l) - \sin \theta ((\partial_\theta H_3) \partial_\theta l + r(r-r_H) (\partial_r H_3) \partial_r l) \right. \right. \right. \\
&\quad \left. \left. + 2l (\cos \theta \partial_\theta H_3 + \sin \theta (r_H \partial_r H_3)) \right) \right. \\
&\quad \left. - 4m H_2 (\sin \theta \partial_\theta l + 2 \cos \theta l) \right) \\
&\quad + 2r l^2 \left(2 \sin \theta \left((r_H-r) H_1 (r \partial_r \omega_m - \omega_m) (m \omega_m - r \omega) \right. \right. \\
&\quad \left. \left. - H_2 \partial_\theta \omega_m (m \omega_m - r \omega) + r^2 \sin \theta (\partial_r H_3) \omega_m \partial_r \omega_m + \sin \theta (\partial_\theta H_3) \omega_m \partial_\theta \omega_m \right. \right. \\
&\quad \left. \left. - r r_H \sin \theta (\partial_r H_3) \omega_m \partial_r \omega_m - r \sin \theta (\partial_r H_3) \omega_m^2 + r_H \sin \theta (\partial_r H_3) \omega_m^2 \right. \right. \\
&\quad \left. \left. + \cos \theta H_3 \omega_m \partial_\theta \omega_m + r^3 \partial_r \omega_m \partial_r V - r^2 r_H \partial_r \omega_m \partial_r V \right. \right. \\
&\quad \left. \left. + r \partial_\theta \omega_m \partial_\theta V - r^2 \omega_m \partial_r V + r r_H \omega_m \partial_r V \right) \right. \\
&\quad \left. - 2g H_3 (m \omega_m - r \omega)^2 \right) \Bigg],
\end{aligned}
\tag{B.13}$$

$$\begin{aligned}
& \partial_\theta^2 f + r(r - r_H) \partial_r^2 f \\
&= \frac{1}{8r^2 gl} \left[4\kappa \csc^2 \theta \left(2r(r - r_H) \sin^2 \theta (\partial_\theta H_1 - \partial_r H_2)^2 \right. \right. \\
&\quad + g \left(2m^2 r(r - r_H) H_1^2 - 4mr(r - r_H) \sin \theta (\partial_r H_3) H_1 + 2m^2 H_2^2 \right. \\
&\quad + \cos(2\theta) H_3^2 + H_3^2 - \cos(2\theta) \partial_\theta H_3^2 + \partial_\theta H_3^2 \\
&\quad + r^2 \partial_r H_3^2 - rr_H \partial_r H_3^2 - r^2 \cos(2\theta) \partial_r H_3^2 + rr_H \cos(2\theta) \partial_r H_3^2 \\
&\quad \left. \left. + 2H_3 \sin(2\theta) \partial_\theta H_3 - 4mH_2 (\cos \theta H_3 + \sin \theta \partial_\theta H_3) \right) \right) f^2 \Big] \\
&- \frac{4r_H \partial_r l f}{l} \\
&+ \frac{2 \csc \theta}{r(r - r_H)} \left[4\kappa \sin \theta (\partial_r V)^2 r^4 - \frac{2 \sin \theta \partial_r f \partial_r l r^4}{l} - 4r_H \kappa \sin \theta (\partial_r V)^2 r^3 \right. \\
&\quad - 8 \sin \theta \partial_r f r^3 + \frac{4r_H \sin \theta \partial_r f \partial_r l r^3}{l} + 4\kappa \omega_m \partial_r H_3 \partial_r V r^3 \\
&\quad - 4\kappa \cos(2\theta) \omega_m \partial_r H_3 \partial_r V r^3 + 4\kappa \csc \theta g (-\omega H_3 \sin \theta - mV)^2 r^2 \\
&\quad + 4\kappa \sin \theta (\partial_\theta V)^2 r^2 + 3\kappa \omega_m^2 \sin \theta \partial_r H_3^2 r^2 - \kappa \omega_m^2 \sin(3\theta) \partial_r H_3^2 r^2 \\
&\quad - 4 \cos \theta \partial_\theta f r^2 - \frac{2 \sin \theta \partial_\theta f \partial_\theta l r^2}{l} + 12r_H \sin \theta \partial_r f r^2 \\
&\quad - \frac{2r_H^2 \sin \theta \partial_r f \partial_r l r^2}{l} - 4r_H \kappa \omega_m \partial_r H_3 \partial_r V r^2 \\
&\quad + 4r_H \kappa \cos(2\theta) \omega_m \partial_r H_3 \partial_r V r^2 - 3r_H \kappa \omega_m^2 \sin \theta \partial_r H_3^2 r \\
&\quad + r_H \kappa \omega_m^2 \sin(3\theta) \partial_r H_3^2 r + 4(r - r_H) \kappa H_1^2 (-r\omega + m\omega_m)^2 \sin \theta r \\
&\quad + 4r_H \cos \theta \partial_\theta f r + \frac{2r_H \sin \theta \partial_\theta f \partial_\theta l r}{l} \\
&\quad + 4\kappa H_3 \omega_m \sin(2\theta) \partial_\theta V r + 4\kappa \omega_m \partial_\theta H_3 \partial_\theta V r \\
&\quad - 4\kappa \cos(2\theta) \omega_m \partial_\theta H_3 \partial_\theta V r - 4r_H^2 \sin \theta \partial_r f r \\
&\quad - 8(r - r_H) \kappa H_1 (-r\omega + m\omega_m) \sin \theta (\omega_m \sin \theta \partial_r H_3 + r \partial_r V) r \\
&\quad + 3\kappa \omega_m^2 \sin \theta \partial_\theta H_3^2 - \kappa \omega_m^2 \sin(3\theta) \partial_\theta H_3^2 \\
&\quad + \kappa H_3^2 \omega_m^2 \sin \theta + 4\kappa H_2^2 (-r\omega + m\omega_m)^2 \sin \theta \\
&\quad \left. + \kappa H_3^2 \omega_m^2 \sin(3\theta) + 2\kappa \cos \theta H_3 \omega_m^2 \partial_\theta H_3 \right] \tag{B.14}
\end{aligned}$$

$$\begin{aligned}
& - 2\kappa \cos(3\theta) H_3 \omega_m^2 \partial_\theta H_3 \\
& - 8\kappa H_2 (-r\omega + m\omega_m) \sin \theta \left(\cos \theta H_3 \omega_m + \sin \theta \partial_\theta H_3 \omega_m + r \partial_\theta V \right) \Bigg] \\
& + \frac{8}{(r - r_H) f} \left[(r - r_H) ((\partial_\theta f)^2 + r(r - r_H)(\partial_r f)^2) \right. \\
& + l \left(2\mu^2 \kappa g V^2 r^3 - 2\omega_m \sin \theta ((r - r_H) \sin \theta \partial_r \omega_m - 2\mu^2 r \kappa g H_3 V(r, \theta)) r \right. \\
& \quad \left. + \sin^2 \theta ((\partial_\theta \omega_m)^2 + r(r - r_H)(\partial_r \omega_m)^2) r \right. \\
& \quad \left. + (2\mu^2 r \kappa g H_3^2 + r - r_H) \omega_m^2 \sin^2 \theta \right) \Bigg],
\end{aligned}$$

$$\begin{aligned}
& \partial_\theta^2 g + r(r - r_H) \partial_r^2 g \\
&= \frac{1}{2} \left[-\frac{\kappa f \csc^2 \theta}{r^2 l} \left(6r(r - r_H) \sin^2 \theta (\partial_\theta H_1 - \partial_r H_2)^2 \right. \right. \\
&\quad - g \left(2m^2 r(r - r_H) H_1^2 - 4mr(r - r_H) \sin \theta (\partial_r H_3) H_1 + 2m^2 H_2^2 \right. \\
&\quad + \cos(2\theta) H_3^2 + H_3^2 - \cos(2\theta) (\partial_\theta H_3)^2 + (\partial_\theta H_3)^2 + r^2 (\partial_r H_3)^2 - rr_H (\partial_r H_3)^2 \\
&\quad - r^2 \cos(2\theta) (\partial_r H_3)^2 + rr_H \cos(2\theta) (\partial_r H_3)^2 + 2H_3 \sin(2\theta) \partial_\theta H_3 \\
&\quad \left. \left. - 4mH_2 (\cos \theta H_3 + \sin \theta \partial_\theta H_3) \right) \right) \\
&\quad + 6\mu^2 \kappa g^2 H_3^2 + (r_H - 2r) \partial_r g + \frac{2((\partial_\theta g)^2 + r(r - r_H)(\partial_r g)^2)}{g} \\
&\quad + \frac{g}{l^2} \left(2\mu^2 r(r_H - r) \kappa H_1^2 l^2 - 2\mu^2 \kappa H_2^2 l^2 + 4 \cot \theta (\partial_\theta l) l + 4r(\partial_r l) l \right. \\
&\quad \left. - 2r_H (\partial_r l) l + (\partial_\theta l)^2 + r^2 (\partial_r l)^2 - rr_H (\partial_r l)^2 \right) \\
&\quad - \frac{g}{(r - r_H) f^2} \left((r - r_H) ((\partial_\theta f)^2 + r(r - r_H) (\partial_r f)^2) \right. \\
&\quad + l \left(2\mu^2 \kappa g V^2 r^3 + 2\omega_m \sin \theta (2r \kappa g H_3 V \mu^2 + 3(r - r_H) \sin \theta \partial_r \omega_m) r \right. \\
&\quad - 3 \sin^2 \theta ((\partial_\theta \omega_m)^2 + r(r - r_H) (\partial_r \omega_m)^2) r \\
&\quad \left. \left. + (2\mu^2 r \kappa g H_3^2 - 3r + 3r_H) \omega_m^2 \sin^2 \theta \right) \right) \\
&\quad + \frac{2g}{r(r - r_H) f} \left(\kappa (\partial_r V)^2 r^4 - r_H \kappa (\partial_r V)^2 r^3 + 2\kappa \omega_m \sin \theta (\partial_r H_3) (\partial_r V) r^3 \right. \\
&\quad - 3\kappa \omega_m^2 g H_3^2 r^2 - 3m^2 \kappa \csc^2 \theta g V^2 r^2 + \kappa (\partial_\theta V)^2 r^2 \\
&\quad + \kappa \omega_m^2 \sin^2 \theta (\partial_r H_3)^2 r^2 - 6m\kappa \omega \csc \theta g H_3 V r^2 - r_H (\partial_r f) r^2 \\
&\quad - 2r_H \kappa \omega_m \sin \theta (\partial_r H_3) (\partial_r V) r^2 + (r - r_H) \kappa H_1^2 (-r\omega + m\omega_m)^2 r \\
&\quad - r_H \kappa \omega_m^2 \sin^2 \theta (\partial_r H_3)^2 r + 2\kappa \cos \theta H_3 \omega_m \partial_\theta V r \\
&\quad + 2\kappa \omega_m \sin \theta (\partial_\theta H_3) (\partial_\theta V) r + r_H^2 (\partial_r f) r \\
&\quad - 2(r - r_H) \kappa H_1 (-r\omega + m\omega_m) (\omega_m \sin \theta \partial_r H_3 + r \partial_r V) r \\
&\quad + \kappa \cos^2 \theta H_3^2 \omega_m^2 + \kappa H_2^2 (-r\omega + m\omega_m)^2 \\
&\quad + \kappa \omega_m^2 \sin^2 \theta (\partial_\theta H_3)^2 + \kappa H_3 \omega_m^2 \sin(2\theta) \partial_\theta H_3 \\
&\quad \left. \left. - 2\kappa H_2 (-r\omega + m\omega_m) (\cos \theta H_3 \omega_m + \sin \theta \partial_\theta H_3 \omega_m + r \partial_\theta V) \right) \right],
\end{aligned}$$

(B.15)

$$\begin{aligned}
& \partial_\theta^2 l + r(r - r_H) \partial_r^2 l \\
&= \frac{2\kappa(r - r_H)f}{rg} (\partial_\theta H_1 - \partial_r H_2)^2 + \frac{2\kappa\mu^2 rg l^2 (\sin \theta H_3 \omega_m + rV)^2}{(r - r_H)f^2} \\
&+ \frac{2\kappa rg l (-\omega H_3 - m \csc \theta V)^2}{(r - r_H)f} \\
&+ \frac{1}{2l} \left(-4\kappa\mu^2 g H_3^2 l^2 + r^2 (\partial_r l)^2 + (\partial_\theta l)^2 - rr_H (\partial_r l)^2 \right. \\
&\quad \left. + l(3(r_H - 2r)\partial_r l - 4 \cot \theta \partial_\theta l) \right), \tag{B.16}
\end{aligned}$$

$$\begin{aligned}
& \partial_\theta^2 \omega_m + r(r - r_H) \partial_r^2 \omega_m \\
&= \frac{2}{f} \left(\partial_\theta f \partial_\theta \omega_m + (r - r_H) \partial_r f (r \partial_r \omega_m - \omega_m) \right) \\
&+ \frac{\kappa f \csc^2 \theta}{r^2 l} \left(-2r(r - r_H) H_1 (\sin \theta \partial_r H_3 (2m\omega_m - r\omega) + mr \partial_r V) \right. \\
&\quad + 2mr(r - r_H) H_1^2 (m\omega_m - r\omega) - 2H_2 (\sin \theta \partial_\theta H_3 (2m\omega_m - r\omega) \\
&\quad + \cos \theta H_3 (2m\omega_m - r\omega) + mr \partial_\theta V) + 2mH_2^2 (m\omega_m - r\omega) \\
&\quad + r^2 (\partial_r H_3)^2 \omega_m - r^2 \cos(2\theta) (\partial_r H_3)^2 \omega_m + (\partial_\theta H_3)^2 \omega_m \\
&\quad + 2 \sin(2\theta) H_3 \partial_\theta H_3 \omega_m - \cos(2\theta) (\partial_\theta H_3)^2 \omega_m \\
&\quad - rr_H (\partial_r H_3)^2 \omega_m + rr_H \cos(2\theta) (\partial_r H_3)^2 \omega_m \\
&\quad + 2r^3 \sin \theta (\partial_r H_3) (\partial_r V) - 2r^2 r_H \sin \theta (\partial_r H_3) (\partial_r V) \\
&\quad + 2r \sin \theta (\partial_\theta H_3) (\partial_\theta V) + H_3^2 \omega_m + \cos(2\theta) H_3^2 \omega_m \\
&\quad \left. + 2r \cos \theta H_3 \partial_\theta V \right) \\
&+ \frac{2\omega_m}{r} (\kappa\mu^2 rg H_3^2 + r - r_H) + 2\kappa\mu^2 r \csc \theta g H_3 V \\
&- \frac{3}{2l} \left(\partial_\theta l \partial_\theta \omega_m + (r - r_H) \partial_r l (r \partial_r \omega_m - \omega_m) \right) \\
&- 3 \cot \theta \partial_\theta \omega_m + 2(r_H - r) \partial_r \omega_m. \tag{B.17}
\end{aligned}$$

B.2.2 Boundary conditions

In order to solve the set of elliptic equations given above, we must impose boundary conditions, which we list here for completeness. The θ boundary conditions follow from

regularity at the poles and from the assumed symmetry across the $\theta = \pi/2$ plane (equatorial symmetry), which is used to reduce the domain to $0 < \theta < \pi/2$. At $\theta = 0$ the boundary conditions are

$$\begin{aligned}\partial_\theta f = \partial_\theta l = \partial_\theta \omega_m = V = H_1 = \partial_\theta H_2 = \partial_\theta H_3 = \phi = 0, \\ g = 1,\end{aligned}$$

and at the $\theta = \pi/2$ boundary we have

$$\begin{aligned}\partial_\theta f = \partial_\theta g = \partial_\theta l = \partial_\theta \omega_m = \partial_\theta V = \partial_\theta H_1 \\ = H_2 = \partial_\theta H_3 = \partial_\theta \phi = 0.\end{aligned}$$

In the radial direction, the boundary at the black hole horizon can be derived by using Frobenius's method applied to the equations listed in Sec. B.2.1 (along with the synchronicity condition), while those at spatial infinity ($x = 1$) follow from the assumption of asymptotic flatness. For the $r = r_H$ (the black hole horizon) boundary, we find

$$\partial_r f = \partial_r g = \partial_r l = \partial_r H_i = \partial_r \phi = 0, \quad (\text{B.18})$$

$$\omega_m = m\Omega_H, \quad (\text{B.19})$$

$$\left(V + \frac{\omega}{m} H_3 \sin \theta\right) = 0, \quad (\text{B.20})$$

while at the $r = \infty$ boundary we find

$$f = g = l = 1, \quad \omega_m = V = H_i = \phi = 0.$$

In the scalar case, the above boundary conditions are equivalent to those in Ref. [120]. In the vector case, the boundary conditions are equivalent to those in Ref. [121], with the exception of the condition on the H_1 matter field at the black hole's horizon: $\partial_r H_1(r_H) = 0$ [cf. Ref. [121] which lists $H_1(r_H) = 0$].

B.2.3 Seeding solutions

We discretize the elliptic equations given above with fifth-order accurate finite difference stencils, and solve them using Newton-Raphson relaxation. The relaxation method requires a sufficiently good initial guess for the field values in order to converge. In the scalar case, we use a numerical solution from Ref. [120] as an initial guess when solving for one point in the parameter space (and find a consistent solution), and then find subsequent solutions

by slowly varying the parameters of the system while using the previous solution as an initial guess.

In the vector case, we instead ¹ seed vector solutions using the analytic solution in the nonrelativistic limit [33] superimposed with a Kerr solution for the metric (see Ref. [120] Appendix A for a description of the Kerr metric in these coordinates). In the nonrelativistic limit, the vector matter fields have the solution:

$$\begin{aligned} V &= -\frac{\exp(-r\alpha)\alpha^{5/2}\sin(\theta)}{2\sqrt{\pi}\omega} & H_1 &= \frac{\exp(-r\alpha)\alpha^{3/2}\sin(\theta)}{2\sqrt{\pi}} \\ H_2 &= \frac{\exp(-r\alpha)r\alpha^{3/2}\cos(\theta)}{2\sqrt{\pi}} & H_3 &= \frac{\exp(-r\alpha)r\alpha^{3/2}}{2\sqrt{\pi}}. \end{aligned} \quad (\text{B.21})$$

We note that there are multiple solutions that satisfy the symmetry conditions that we impose here. These solutions can be labeled by mode number n , where the different behavior of the $n = 0$ and $n = 1$ solutions is discussed in Ref. [192]. Here we are interested in the fundamental $n = 0$ mode (which grows the fastest through superradiance when $m = 1$), which can be identified by the absence of nodes in the V component of the matter solution.

B.3 Quantifying errors in frequency shift

In this section, we provide details on how the errors in the cloud mass-dependent shift in the oscillation frequency are estimated. There are multiple sources of error in the frequency shift calculation. The main sources are the following:

- The complex field approximation
- Numerical error in individual solutions
- Interpolation error

¹Relativistic solutions for black hole-vector boson clouds are given in Ref. [192], though we were unable to use them to relax to a solution to our equations, which could be related to the difference in boundary conditions discussed above. However, our solutions do seem equivalent in that the extracted black hole and cloud masses are equivalent for the same choice of frequency and black hole radius.

We take the overall relative error to be

$$\frac{\delta\omega}{\Delta\omega} = |E_{\text{theoretical}}| + |E_{\text{numerical}}| + |E_{\text{fit}}|, \quad (\text{B.22})$$

where $E_{\text{theoretical}}$ is the relative error due to the complex field or oscillation-average approximation, $E_{\text{numerical}}$ is the error from the extraction of physical quantities from each numerical solution, and E_{fit} is the interpolation error. The results of these estimates are shown in Fig. B.1. We outline the calculation of each of these contributions to the overall error in the following.

B.3.1 Complex field approximation

Comparing Newtonian limit term

As discussed in Sec. 3.2.3, we make an axisymmetric or oscillation-averaged approximation to reduce the dimensionality of the problem. The theoretical error arising from making such an approximation can be obtained by comparing the stress-energy tensors of the full complex-field Φ solution to that of a real-field solution ψ . To that end, we obtain a real-field solution by taking the real part of the complex field configuration (adjusted by a factor of $\sqrt{2}$ to obtain a field with the same total energy), such that $\psi = \sqrt{2}\text{Re}(\Phi)$. In the vector case, we proceed in the same way, with $\mathcal{A}_a = \sqrt{2}\text{Re}(A_a)$.

Since the self-gravity correction scales with the binding energy of the cloud [128, 33] (in the Newtonian limit), we will use that quantity to estimate the theoretical error. We introduce the quantity $W^{\ell \leq l'}$, defined as the l' multipole approximation of the nonrelativistic binding energy,

$$W^{\ell \leq l'} = - \int d^3\mathbf{r} \int_{|\mathbf{r}'| < |\mathbf{r}|} d^3\mathbf{r}' \rho(\mathbf{r}) \rho(\mathbf{r}') \times \quad (\text{B.23})$$

$$\sum_{l=0}^{l'} \sum_{m=-l}^l \left(\frac{|\mathbf{r}'|^l}{|\mathbf{r}|^{l+1}} \right) \frac{4\pi}{2l+1} Y_{lm}(\Omega) Y_{lm}^*(\Omega'), \quad (\text{B.24})$$

which is obtained by applying a truncated version of the sum given in Eq. (3.7) to the expression for W given in Eq. (3.6). In computing the above expressions, we use the energy density measured with respect to an observer whose four velocity n_a is normal to slices of constant time

$$\rho = n_a n_b T^{ab}. \quad (\text{B.25})$$

For the volume element, we take $d^3\mathbf{r} = \sqrt{\gamma}d^3x$, where γ is the determinant from the induced spatial metric in our coordinates [see Eq. (B.3)]. We then estimate the theoretical error to be

$$E_{\text{theoretical}} \approx \frac{|W_{\text{Re}}^{\ell \leq 2} - W_{\text{C}}^{\ell \leq 2}|}{W_{\text{C}}^{\ell \leq 0}}, \quad (\text{B.26})$$

where “Re” and “C” indicate the quantity computed with the stress-energy tensors corresponding to the real and complex solutions (ψ and Φ in the scalar case, or $\mathcal{A}_a$ and A_a in the vector case), respectively. In either case, the metric is taken to be identical and equal to that of the full stationary and axisymmetric spacetime. This calculation estimates the error in a nonrelativistic approximation of the self-gravity of the cloud, the binding energy, calculated up to the $l = 2$ multipole. We calculate the next nontrivial multipole in the scalar case ($l = 3$) and find it to be subdominant to the $l = 2$, $m = 2$ contribution at the ratio of 10^{-5} at $\alpha = 0.5$, indicating even higher multipoles are negligible.

Using the estimate (B.26), we obtain subpercent level errors across the parameter space in the vector case, and errors of at most $O(10^{-4})$ in the scalar case, as can be seen in Fig. B.1. This error estimate is a nonrelativistic approximation in both spin cases, and should only be fully accurate in that limit. Because it is a leading order calculation, it will not capture the more quickly growing contributions at higher orders in α , and thus may underestimate the error due to this approximation at high α . However, here we are mainly using it to estimate the relative contribution of nonaxisymmetric components of the cloud’s energy to the self-gravity. We further note that Ref. [57] estimates the frequency shift for the scalar case using relativistic black hole perturbation theory, but with a semi-Newtonian approximation for the gravitational potential and finds results noticeably closer to the relativistic ones presented here than to the leading order nonrelativistic result, suggesting that the frequency shift is well described by taking some leading order nonrelativistic terms. This further motivates us to use our nonrelativistic estimate for the theoretical error in taking an oscillation-averaged approximation, even at higher α . As can be seen in Fig. B.1, the theoretical error is comparable or subdominant to other sources of error in most of the parameter space, and so the exact value of the theoretical error here (up to a factor of a few) does not dramatically impact the total error estimate.

Comparing to numerical relativity

In addition to calculating a nonrelativistic estimate of the theoretical error as is described in the last section, we can also compare to the numerical relativity simulation considered

in Ref. [90]. That study evolved the full Einstein-Proca equations, without symmetry assumptions, through the saturation of the superradiant instability and into the phase where the cloud dissipates through gravitational radiation for one case where initially $\chi_{\text{BH}} = 0.99$ and $\alpha = 0.4$, leading to a vector boson cloud that peaks at $M_{\text{cloud}} \approx 0.06 M_{\text{BH}}$. We compare the gravitational wave frequency found in that study to the test-field frequency with the nonrelativistic frequency shift and fully relativistic frequency shift estimate (using the estimates of the black hole and cloud properties found in the simulation) in Fig. B.2. We find an improved agreement with numerical relativity, compared to the test nonrelativistic shift estimate, when the relativistic frequency shift calculation is included in the waveform. Considering the expected numerical error in the simulations, as well as the fact that there is a small spurious excitation of the first overtone mode as discussed in Ref. [90], which will bias the frequency upward, the difference, at the level of 0.1%, is within the expected bounds.

B.3.2 Numerical error in solutions

We quantify the numerical error $E_{\text{numerical}}$ using a combination of errors in the extracted physical quantities. For the frequency fit, the frequency is a fixed input parameter, and the only extracted quantities are the black hole mass and the cloud mass (described in Appendix B.1). We approximate the relative numerical error in a particular solution by the sum of the relative errors in both the cloud and black hole mass.

We establish the numerical errors in each mass calculation by performing convergence studies. The elliptic solver uses sixth order finite differencing and so the solution in the bulk converges at sixth order. To calculate the mass of the black hole we sometimes use an extrapolation to the horizon point (as described in Appendix B.1), and so have slower convergence.

In the scalar case, we perform a convergence test using three resolutions differing by a factor of 2/3. We find second order convergence in the horizon black hole mass and sixth order convergence in the cloud mass, and we perform Richardson extrapolation to find an overall error. In this case, the black hole mass dominates the numerical error at small α . The reason the black hole mass converges so slowly is because its calculation relies on an extrapolation to the boundary, where the extrapolation error is dominating over the error in each point in the bulk. For the final fit, we use a resolution of $n_x = 391$ radial points and $n_\theta = 46$ angular points. The result of this analysis is shown in comparison with the other sources of error in the top panel in Fig. B.1.

In the vector case we perform a convergence test using three resolutions differing by

a factor of $3/4$ (where our numerical resolution of $n_x = 261$ and $n_\theta = 35$ is the highest resolution). We find sixth order convergence in the ADM mass and cloud mass, with second order convergence in the horizon black hole mass. We perform Richardson extrapolation to find an overall error. In this case the horizon black hole mass has good convergence at low α , but stops being well resolved at high α ($\alpha > 0.4$), whereas the ADM mass converges well at high α , but stops converging at these resolutions when the cloud becomes bigger, and closer to the compactified boundary, at $\alpha < 0.2$ ². For $\alpha > 0.22$, we use a resolution of $n_x = 261$ and $n_\theta = 35$, and the ADM and cloud mass calculations. We use the horizon black hole mass instead of the ADM mass as well as higher resolution points for $\alpha < 0.22$, with varying resolution so as to make the estimated numerical errors comparable to the theoretical error in that regime. The final error estimates are shown in Fig. B.1. We perform a fit for the frequency shift for three different resolutions of points, and find convergence in the overall fit parameters for the frequency shift as well as in each solution.

There is also some contribution to the overall error from the interpolation between solutions. We evaluate this error E_{fit} by removing each point in turn from the interpolation and calculating the difference between the interpolation and that point. We show the results of this test for the scalar and vector cases in Fig. B.1.

B.3.3 Extrapolation to nonrelativistic solutions

As described in Sec. 3.3, to extrapolate to small α we fit the self gravity correction to the cloud frequency using a polynomial in $M_{\text{cloud}}/M_{\text{BH}}$ and α . We use points with $0.2 < \alpha < 0.3$ in the scalar case and $0.09 < \alpha < 0.14$ in the vector case. By construction, this polynomial reduces to the nonrelativistic result to leading order in the $\alpha \ll 1$ limit. It only remains to quantify the error in the polynomial fit. We do this by first calculating the maximum deviation of the polynomial fit from the full spline interpolation of the data points in the fit region, Δ_{fit} . Then the maximum error in the coefficient for the leading order relativistic correction to the shift is $\delta c_{3,1} = \Delta_{\text{fit}} \alpha^{-3}$. So we can extrapolate the relative error in the frequency shift to lower values of α using

$$\frac{\delta\omega}{\Delta\omega} = \frac{\delta c_{3,1} \alpha^3}{C_{\text{nr}} \alpha^2} = \frac{\Delta_{\text{fit}} \alpha}{C_{\text{nr}}} \quad (\text{B.27})$$

for α below the regime where we have numerical results. In the each spin case we find that the polynomial fit to the data is close to the numerical error for the nearby small α points.

²We check the convergence using a similar study but where the minimum resolution is $n_x = 347$ and $n_\theta = 47$, and find that the ADM mass converges as expected for $\alpha > 0.13$.

B.4 Calculating light ring and ISCO

We find the equatorial light ring by solving the radial geodesic equation together with the normalization of the four velocity. We interpolate our numerical solution between grid points before solving for the radius that gives circular orbits³. We use the proper circumferential radius to convert our measured light ring to standard coordinates

$$R = \frac{1}{2\pi} \int d\varphi \sqrt{g_{\varphi\varphi}(r, \theta = \pi/2)}. \quad (\text{B.28})$$

We calculate the equatorial ISCO by solving the radial geodesic equation, varying the conserved quantities $E = -t_a u^a$ and $L = \phi_a u^a$ (where u^a is a timelike geodesic four velocity, t^a is the stationary Killing vector and ϕ^a is the axial Killing vector) and the estimated ISCO radius to minimize simultaneously $\dot{r}$, $\partial\dot{r}/\partial r$, and $\partial^2\dot{r}/\partial r^2$. The resulting solution gives the ISCO radius, as well as E and L which can be mapped to the four velocity u^a of a particle traveling along the ISCO. This four velocity can then be used to calculate the ISCO orbital frequency as well as the frequency of light measured by that observer. We use the proper time measured by an observer orbiting on the ISCO to define a coordinate independent orbit frequency,

$$\Omega = \frac{d\varphi}{d\tau}. \quad (\text{B.29})$$

The frequency of light measured by that observer is $\omega = -u^a k_a$, where k^a is the four velocity of the light being measured. We choose two initial directions for light k^a being emitted from the ISCO, such that in both cases the light is emitted equatorially and the initial radial component is zero.

Errors in the light ring and ISCO calculations are evaluated using a convergence test and Richardson extrapolation, with results shown in Fig. B.3. As can be seen there, the relative errors are always $< 10^{-3}$ and are $\lesssim 10^{-4}$ except in some corner of the vector boson parameter space.

B.5 Consistency checks

B.5.1 Test field limit

We confirm that in the test field limit our results match those in the literature. We do this by comparing our fit to the constant-in-cloud-mass part of the frequency with the test

³We use a cubic spline interpolation and an optimization algorithm from `scipy`.

field calculation in Ref. [198]. The results of this comparison are shown in Fig. B.4. We find agreement within the expected error bars of our calculation.

B.5.2 Nonlinear dependence of frequency on M_{cloud}

As described in Sec. 3.3.2, when fitting for the cloud mass dependent part of the frequency, we include a term linear in cloud mass as well as a quadratic term in the fit to the frequency shift. As shown in Fig. B.5, the quadratic term corrects the linear term only at the level of a few percent, even when the cloud is significant. We find that including an additional cubic in cloud mass term does not significantly decrease the root mean squared error in the fit, and, as is shown in Fig. B.5, that term is oscillating around zero and does not appear to be well resolved.

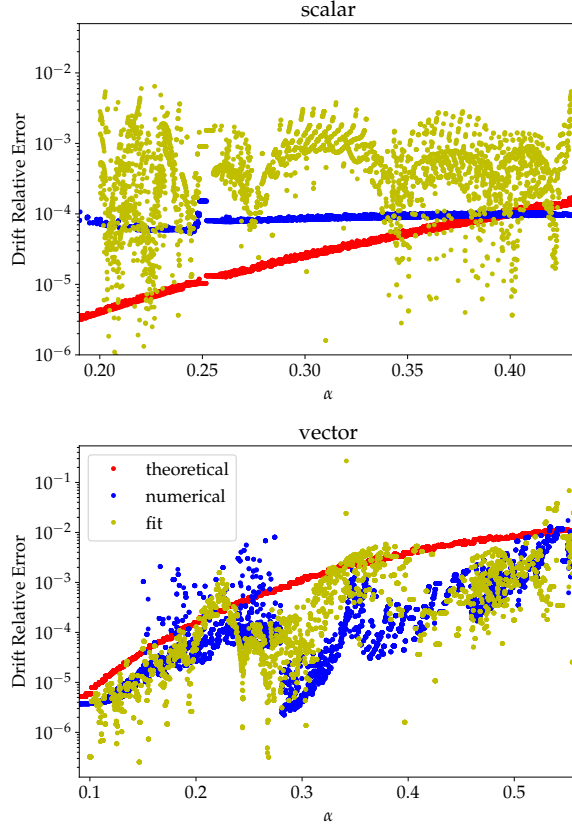


Figure B.1: Estimated errors in the extracted frequency shift relative to the nonrelativistic limit of the frequency shift. We compare the final numerical error in each solution to the estimated theoretical error due to our oscillation averaged approximation, as well as the interpolation error. Here, the numerical and interpolation errors are plotted for solutions with $M_{\text{cloud}}/M_{\text{BH}} > 1\%$ to avoid issues with relative error becoming larger as the frequency shift goes to zero.

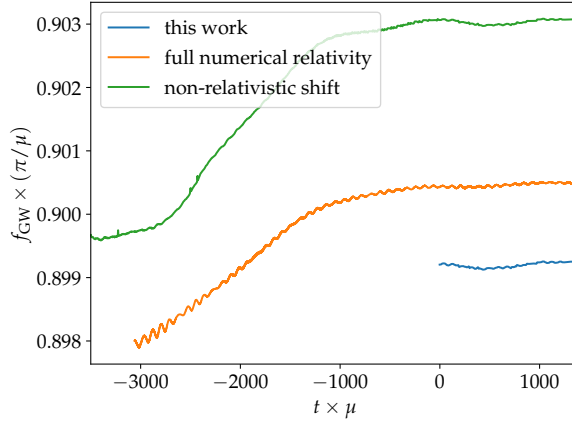


Figure B.2: The frequency of gravitational waves from an oscillating vector boson cloud around a black hole is compared in three cases. The numerical relativity result (averaged over a number of oscillations) is from a simulation including the growth period of the cloud from Ref. [93], and the other two results are using the test field frequency calculation from Ref. [198], and the leading order nonrelativistic frequency shift or the relativistic frequency shift as is calculated here. Here, the superradiant instability phase ends at $t = 0$, after which this frequency shift calculation is valid.

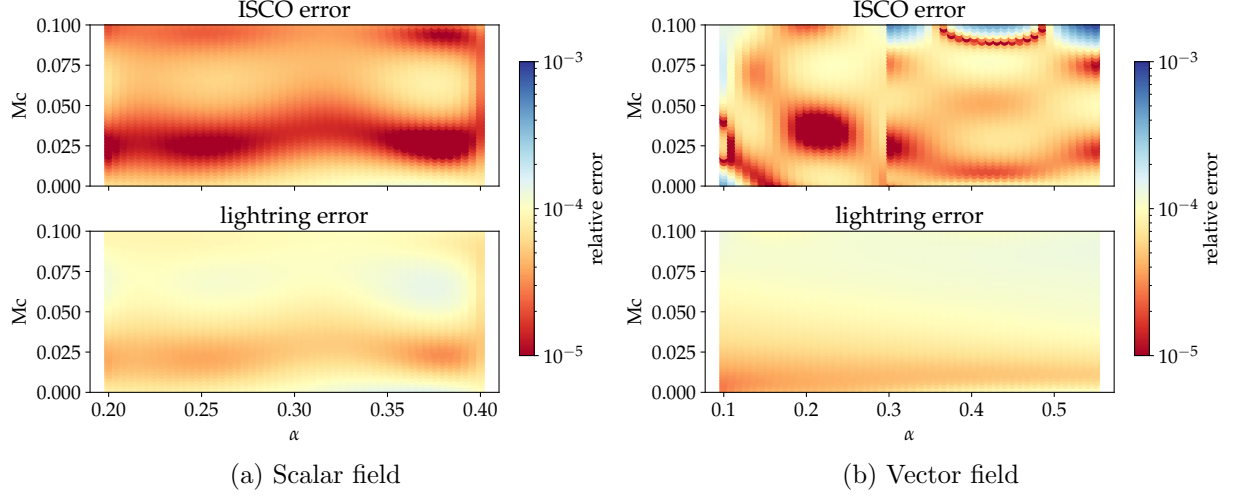


Figure B.3: We perform a convergence test to determine the numerical error in the position of the light ring and ISCO. The relative error in the circumferential radius is shown here for both scalar and vector clouds. The feature at $\alpha = 0.3$ in the ISCO radius for the vector case is due to a change in the convergence order from 6th order at high α to second order for low α .

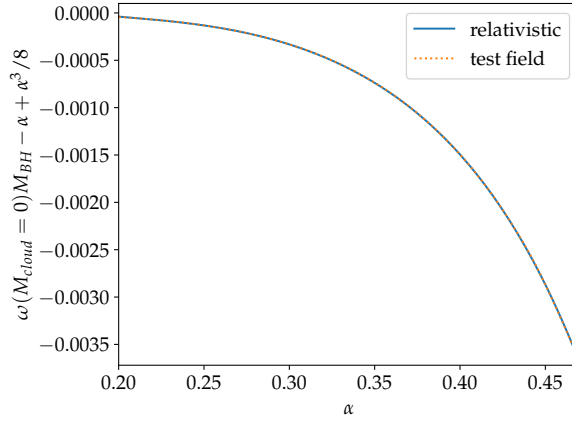
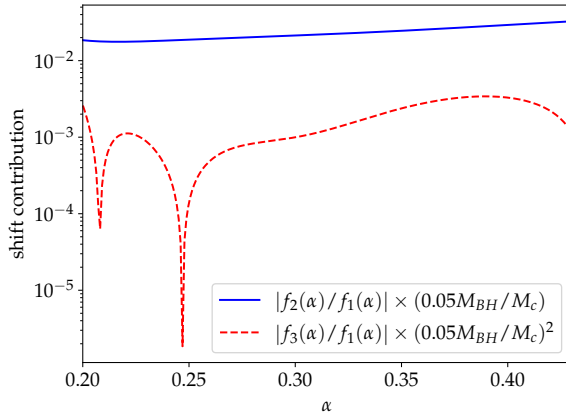
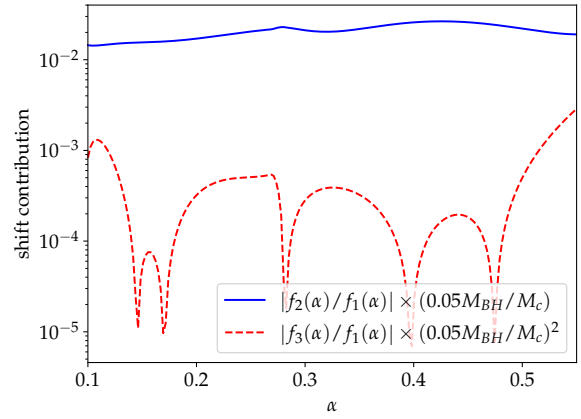


Figure B.4: We show the relativistic correction to the frequency in the test field limit and confirm that our result matches that from the test field calculations in Ref. [198]. This example is from the scalar case, but we see a similar agreement in the vector case.



(a) Scalar field



(b) Vector field

Figure B.5: The contribution to the frequency shift of the second order in cloud mass term, relative to linear term, when the cloud mass is $M_{\text{cloud}}/M_{\text{BH}} = 0.05$. We also include a comparison for the value of a third order term when it is included in the fit described in Eq. (3.17). When this third order term is included, it is found to contribute at below the estimated error. The feature at $\alpha = 0.275$ in panel (b) is due to an interpolation between fits to data at different resolutions.

Appendix C

Spin-2 Superradiance

C.1 Numerical Scheme

We numerically evolve the Einstein-Weyl equations, or equivalently, the quadratic gravity equations with $R \equiv 0$, using $\{g_{ab}, \partial_t g_{ab}, R_{ab}, \partial_t R_{ab}\}$ as the evolution variables. We use the general harmonic formulation [163], where the coordinates x^a satisfy $\square x^a = H^a$ for specified functions $H^a(g_{ab})$. (See Ref. [117] for an alternative formulation.) The metric evolution equation is

$$g^{cd} \partial_c \partial_d g_{ab} = -2\nabla_{(a} H_{b)} + 2H_c \Gamma_{ab}^c - 2\Gamma_{da}^c \Gamma_{cb}^d \quad (\text{C.1})$$

$$- \kappa (n_a C_b + n_b C_a - n_c C^c g_{ab}) \quad (\text{C.2})$$

$$- 2\partial_c g_{d(a} \partial_{b)} g^{cd} - 2R_{ab}. \quad (\text{C.3})$$

Here Γ_{ab}^c are the usual Christoffel symbols, and $C^a = H^a - \square x^a$ is the harmonic constraint [111], with $\kappa \neq 0$ corresponding to the inclusion of constraint damping terms.

The Ricci tensor evolution equation is

$$\square \tilde{R}_{ab} = \mu^2 \tilde{R}_{ab} - 2R_{acbd} \tilde{R}^{cd} + \frac{1}{2} g_{ab} \tilde{R}^{cd} \tilde{R}_{cd} \quad (\text{C.4})$$

$$- \kappa (n_a C_b + n_b C_a - n_c C^c g_{ab}), \quad (\text{C.5})$$

where R_{abcd} is the Riemann tensor, calculated from g_{ab} , $\partial_c g_{ab}$, and $\partial_c \partial_d g_{ab}$, where $\partial_t^2 g_{ab}(g_{ab}, \partial_t g_{ab})$ is calculated according to Eq. (C.1).

We begin the nonlinear evolution using a black hole in Cartesian Kerr-Schild coordinates, and use the gauge $H^a = -(g^{bc} \Gamma_{bc}^a)_{\text{Kerr}}$, where the gauge functions H^a are stationary,

and are fixed by the choice of the Kerr background. Spatial derivatives are calculated using a fourth-order finite difference scheme, and the time evolution uses the fourth-order Runge Kutta algorithm. We use mesh refinement, meaning that there is higher numerical resolution in the vicinity of the black hole. However, the interpolation of the refinement boundaries only converges at third-order with temporal resolution.

The evolution code calculates the position of the black hole’s apparent horizon using a flow method [174], finding the outermost marginally trapped surface. Once the black hole apparent horizon is found, the associated angular momentum is calculated from the contraction of the extrinsic curvature K_{ij} with the approximate rotational Killing vector φ_i [22],

$$J_{\text{BH}} = \frac{1}{8\pi} \int \varphi_i K^{ij} dA_j. \quad (\text{C.6})$$

The Christodoulou mass associated with the horizon is then calculated using the area of the apparent horizon A_{BH} , and this angular momentum,

$$M_{\text{BH}} = \sqrt{\frac{A_{\text{BH}}}{16\pi} + \frac{4\pi J_{\text{BH}}^2}{A_{\text{BH}}}}. \quad (\text{C.7})$$

As noted in the main text, since in this case φ_i is only an approximate rotational Killing vector, J_{BH} as defined here may be mildly coordinate dependent. Following Ref. [174], we excise a region inside the apparent horizon from the numerical domain. This region is dynamically adjusted as the horizon evolves.

We use Cartesian compactified coordinates, with seven levels of mesh refinement, each having approximately 251^3 points for the results shown in the main text. The finest level covers the region $x, y, z \in [-2.5M, 2.5M]$, where the black hole is centered at the origin and initially has polar radius $\approx 1.14M$ and equatorial radius $\approx 1.51M$.

C.2 Dependence on Initial Perturbation Amplitude

We construct initial data for the nonlinear evolution of the full Einstein-Weyl equations corresponding to the fastest growing superradiant mode. We do this by evolving the linearized equations (4.1) with a seed perturbation until the fastest growing superradiant mode dominates and all other modes are negligible. This solution to the linearized equations can then be scaled and used as initial data for the full nonlinear evolution. The initial

perturbation for the nonlinear evolution should ideally be large enough such that the entire evolution can be simulated in a reasonable amount of computer time, while also being small enough that its initial backreaction has a sufficiently small effect on the dynamics and the late time evolution is insensitive to the initial amplitude.

We compare evolutions with different initial amplitudes to verify that the initial backreaction does not meaningfully change the phenomenology. In Fig. C.1, we show the evolution of two different initial amplitudes, differing by a factor of two (2.02 to be exact), where the larger value corresponds to the amplitude used in the main text. From the figure, we see that this increase in initial amplitude, or delay in time before beginning the nonlinear evolution, does not strongly impact the late time behavior. The black hole still spins up until the apparent horizon is no longer found.

C.3 Resolution Study

We use a fourth order evolution code described in Ref. [95] (but without restricting to axisymmetry) to solve the dynamics. We compare evolution with different resolutions to check convergence and evaluate the numerical error in the simulation.

In Fig. C.2, we show the evolution of the same initial perturbation, with $m = 2$ and $\alpha = 1.5$, using two different grid resolutions. The higher resolution curve is the same one shown in Fig. 4.2 in the main text. The grid resolution has a noticeable impact on the initial growth of the field, changing the initial backreaction and therefore the time until the horizon reaches extremality. However, the actual growth rate ω_I is comparable for both cases in the superradiant dominated phase, corresponding to the result from the linear analysis in Ref. [95], as shown for the lower resolution case in Fig. 4.3. In the Figure, a relative time shift is introduced to show the comparable evolution at later times.

In Fig. C.3, we show the total constraint violation for the same two resolutions, as well as a lower resolution evolution with a lower initial amplitude. Here the constraint violation is the norm of the harmonic constraint C^a . From the figure, it is clear that the initial spike in the constraint violation is determined by the initial amplitude and independent of the resolution. So the initial constraint violations are dominated by the effect of using initial data which only solves the linear problem. At later times, we see that the constraint violations for the two different amplitude cases with the same resolution have the same constraint violation when aligned for the same background evolution. We also see that the constraint violation is lower for the higher resolution case at later times. This implies that the constraint violations after the first $\approx 100M$ are dominated by the truncation error.

From Fig. C.2, when a time shift is included in the higher resolution run of $100M$, the mass and spin evolution line up at later time with the lower resolution simulation. Comparing the constraint violations with this time shift we find they are consistent with third order convergence.

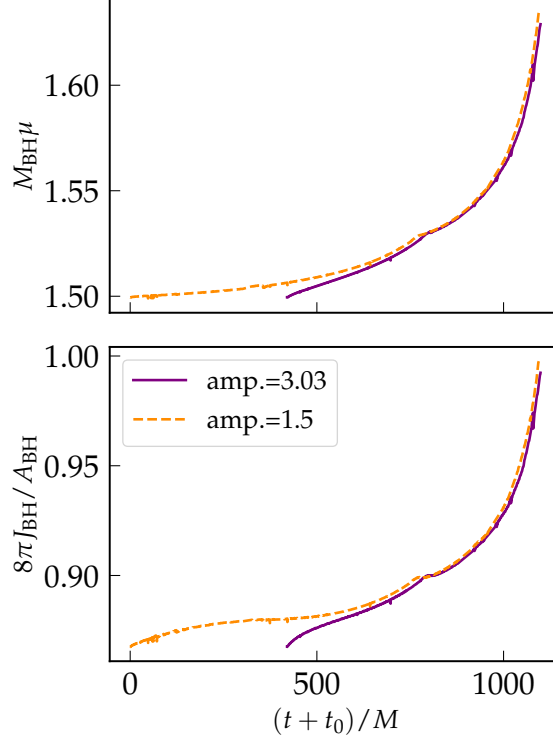


Figure C.1: The apparent horizon parameters are plotted for two cases, where the initial perturbation (with $m = 2$ and $\alpha = 1.5$) is scaled according to the factor ‘amp.’ before beginning the full nonlinear evolution. A time shift $t_0 = 420M$ is introduced in the larger amplitude case to better demonstrate the similarity in evolution. While there are some effects at early times due to the change from linear to nonlinear evolution, the phenomenology is the same and the effect on the late time behavior is not significant. There is a visible kink in the evolution at $t/M \approx 800$. For the simulations in this plot the excision surface was not changed until this time, where excision surface is too close to the apparent horizon for a short window $750 \lesssim t/M \lesssim 800$, affecting the apparent horizon solver. This problem is avoided in the simulation shown in the main text by changing the excision surface at earlier times so that the distance between the excision boundary and the apparent horizon is always sufficient.

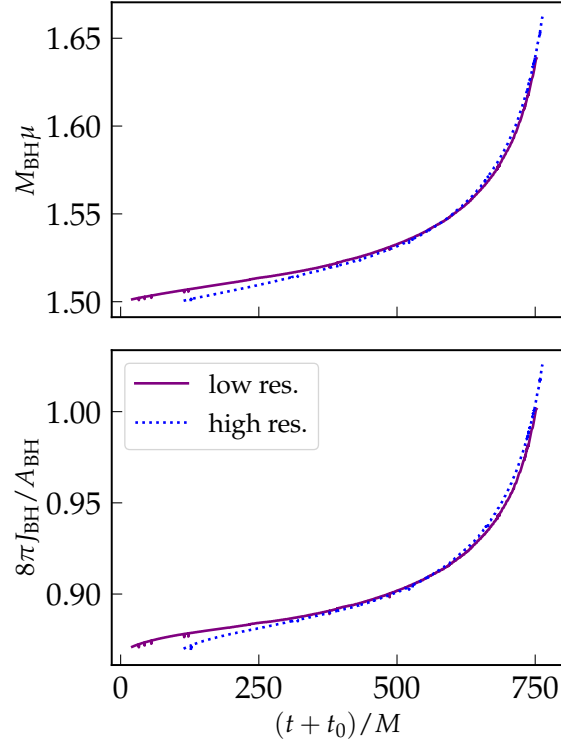


Figure C.2: We compare the evolution of an initial perturbation with $m = 2$ and $\alpha = 1.5$ on a higher resolution grid to a lower resolution grid with grid spacing $\approx 1.3\times$ larger. The backreaction of the initial perturbation is dependent on the resolution of the simulation. A time shift of $t_0 = 100M$ is introduced in the higher resolution curve to better compare the evolution at later times.

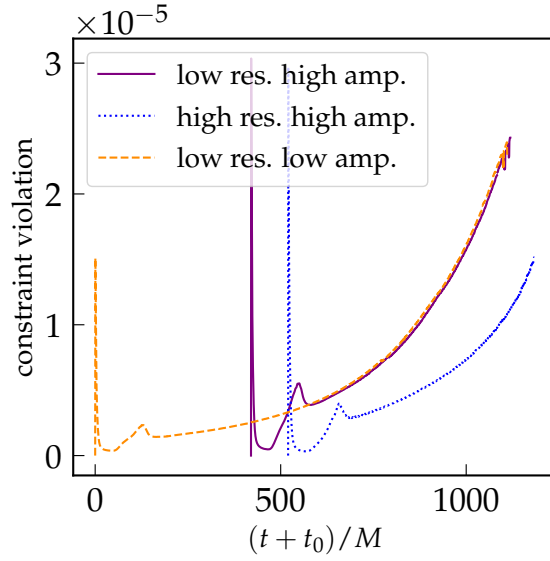


Figure C.3: We compare the constraint violations for all the cases plotted in Figs. C.1 and C.2. (The lower resolution case in Fig. C.2 is the same as the higher amplitude case in Fig. C.1.) The start time for the higher amplitude cases is shifted, as in Figs. C.1 and C.2, to align the background black hole evolution at later times. We have $t_0 = 0, 420M$, and $520M$ for the ‘low res. low amp.’, ‘low res. high amp.’, and ‘high res. high amp.’ cases, respectively.