Topological methods in quantum gravity

by
Artem Starodubtsev

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Abstract

The main technical problem with background independent approaches to quantum gravity is inapplicability of standard quantum field theory methods. New methods are needed which would be adapted to the basic principles of General Relativity. Topological field theory is a model which provides natural tools for background independent quantum gravity. It is exactly soluble and, at the same time, diffeomorphism invariant. Applications of topological field theory to quantum gravity include description of boundary states of quantum General Relativity, formulation of quantum gravity as a constrained topological field theory, and a new perturbation theory which uses topological field theory as a starting point. The later is the central theme of the thesis. Unlike the traditional perturbation theory it does not require splitting metric into a background and fluctuations, it is exactly diffeomorphism invariant order by order, and the coupling constant of this theory is dimensionless. We describe the basic ideas and techniques of this perturbation theory as well as inclusion of matter particles, boundary states, and other necessary tools for studying scattering problem in background independent quantum gravity.

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1 Introduction

One of the major challenges of today's theoretical physics is to find a theory of quantum gravity, the theory that will unify Einstein's General Relativity and Quantum Mechanics. The standard perturbative quantization which has proved to be successful for all other interactions has failed for gravity. The Feynmann diagrams for graviton modes, which are small excitations around a flat spacetime background, contain ultraviolet divergences which cannot be canceled via renormalization. Different possible interpretations of the origin of this problem lead to different approaches to quantum gravity.

The traditional perturbative approach to quantum gravity starts with fixing a background metric g_{0ab} and defining excitations around it

$$g_{ab} = g_{0ab} + h_{ab}. \tag{1}$$

The constraints of General Relativity generating four dimensional diffeomorphisms are then linearized in h_{ab}, which makes them look very much like those of abelian gauge theory. This constraints can easily be solved and as a result one obtains ordinary local field theory with two degrees of freedom per point. This is used as the zero order approximation for perturbation theory. The only apparent difference between this theory and local field theories for other interactions is the presence of extra derivatives in the interaction term. As a result, the role of the coupling constant is played by Newton's constant, G, which is dimensionful. This can be interpreted as that the theory becomes strongly coupled in the ultraviolet regime and breaks down at the Plank scale $l_{pl} \sim \sqrt{G}$.

To solve this problem, String Theory [1] suggests keeping the above perturbation theory unaltered at the scales much larger than l_{pl}, and modify it by replacing local excitations (gravitons) with excitations of a finite size (strings) at the scales of the order of l_{pl}. This reduces the number of degrees of freedom of the theory at short distances, thus improving ultraviolet behavior. Such approach has an advantage that existing field theory techniques can still be used, and it is easy to make a connection to known physics above the Plank scale which is described by some local field theory. On the other hand, measuring distances at which the theory has to be modified requires a metric. To this end one generally takes the metric g_{0ab}, which defines the expansion (1). So an auxiliary structure, which physics must be independent of, enters the very definition of the theory. To solve this problem it has been suggested that there exists a theory which unifies all different string theories in various backgrounds, called M-theory. To find this theory, one needs a better understanding of non-perturbative physics.

One can however notice, that taking into account non-perturbative effects, even for ordinary General Relativity, leads to a picture that is very different from the traditional approach to quantum gravity which uses a local field theory limit. First, as follows from diffeomorphism invariance, the degrees of freedom of General Relativity must be relational, i.e. can only be associated to points labeled by some actual physical events. Second, from the equivalence principle it follows that the degrees of freedom of GR must be non-local (or quasilocal), i.e. could be restricted to a finite region, but not to a point. This features are not seen in the local field theory limit, which approximates the diffeomorphism transformations with linearized ones. As a result the traditional perturbative approach leads to overcounting the degrees of freedom of the theory. This suggests that the ultraviolet problem could be solved simply by recovering the correct counting of the degrees of freedom of General Relativity, without any modification. For this one has to restore the original non-linear diffeomorphism invariance of the theory. In perturbation theory, however, it would take infinitely many steps, and the theory would break down before it is achieved. This also suggests that a non-perturbative quantization is needed.

Non-perturbative quantization of General Relativity is known as Loop Quantum Gravity [2]. It includes a canonical version [3] as well as a path integral (or spin-foam) version [4]. The advantage of this approach is that it is explicitly background independent: no fixed metric enters the definition of the theory. The existence of a minimum possible length directly follow from both versions of this approach [5], thus supporting the idea that gravity, when treated non-perturbatively, can regulate its ultraviolet behavior itself. The main problem with Loop Quantum Gravity is that doing physics non-perturbatively is always hard, as it requires solving the theory exactly. But General Relativity is too complicated to give a comprehensive classifications of all its exact solutions. In particular, it is not clear what solutions should be taken and in what combination to reproduce an ordinary semiclassical spacetime. This problem is known as the problem of the classical limit [6].

The above problems are technical in essence, and to address them one needs to have some mathematical tools which are adapted to the structure of General Relativity. To do this one can take any model that is exactly soluble and at the same time invariant with respect to all the basic symmetries of General Relativity. An example of such models is provided by a class of theories known as Topological Field Theories.

A Topological Field Theory (TFT) [7] is a theory that has exactly as many local symmetries as fields. This allows one to gauge away all the local degrees of freedom. The number of remaining degrees of freedom is finite and is related to the topology of the manifold. This makes such a theory exactly soluble. On the other hand topological field theory has some basic features of General Relativity that the local field theory limit of GR misses. These are exact diffeomorphism invariance, positive degree form structure of the theory, and non-locality of its fundamental excitations.

Solving TFT exactly requires some specific methods, which are very different from the methods of standard quantum field theory. These methods involve explicit identification of the non-local degrees of freedom of the theory, and applying quantization rules, which could be both canonical or path integral based, to the corresponding variables. This provides well defined results in all the known cases. On the other hand application of standard quantum field theory methods to TFT often leads to inconsistent results.

A well known example of TFT is General Relativity in 2+1 dimensions. Einstein's Equations in 2+1 dimensions imply that the Riemann curvature tensor vanishes locally. The theory therefore doesn't have local degrees of freedom. It can be quantized non-perturbatively and was shown to be exactly soluble [8]. On the other hand the traditional perturbation theory for 2+1 dimensional gravity based on the expansion (1) fails exactly the same way as in 3+1 dimensions, even though 2+1 dimensional graviton doesn't have "propagating modes".

In dimensions higher than three General Relativity is not a topological field theory, as it has infinitely many degrees of freedom. However, TFT can be considered as an approximation to General Relativity in which all the local degrees of freedom are frozen. This limit is in a way opposite to the local field theory limit which has too many degrees of freedom, and some of them have to be removed to recover General Relativity. On the other hand every degree of freedom of TFT is also a degree of freedom of General Relativity and to recover the later one has to reintroduce the missing local degrees of freedom back in the theory. One therefore may argue that TFT could be a useful tool for studying various aspects of quantum gravity, even though it doesn't provide a complete solution for the full theory by itself.

Topological field theory may be relevant to General Relativity in different ways [9]. It may provide a theory describing the dynamics of boundary observables of General Relativity in a closed form [10, 11]. It may give rise to a state sum model which, via imposing some constraints, can be converted in a state sum model for quantum gravity [12]. Finally, it may provide a staring point for a perturbation theory of quantum General Relativity, when the local physical degrees of freedom are made from gauge degrees of freedom via symmetry breaking [13]. All these possibilities will be touched upon in the present paper.

In section 2 we review how to reformulate 2+1 and 3+1 dimensional gravity as gauge theories. This reformulation is essential for establishing the relation between General Relativity and topological field theory. We also consider a generalized action, which contains General Relativity and a certain topological field theory as different sectors. These results are based on paper [**?**].

In section 3 we review how to describe propagating point particles in 2+1 and 3+1 dimensional TFT. This allows one to introduce matter degrees of freedom into the theory without loosing its exact solubility. 3+1 dimensional results are based on paper [49].

In section 4 we describe how the global conserved quantities of General Relativity can be related to local charges of a certain topological field theory induced on a boundary. This provides a specific point of view on how to describe propagating excitations of non-perturbative General Relativity. This results are based on paper [59]

In section 5 we provide the basic ideas of a new perturbation theory for quantum gravity which uses a topological field theory as a starting point. It is exactly diffeomorphism invariant order by order and has a very small dimensionless coupling constant, thus avoiding the main problems of the traditional perturbative approach. This results are based on paper [13]

In section 6 we review the basic techniques necessary for making explicit calculations in this perturbation theory. This includes construction of spinfoam models for TFT coupled to particles, different approaches to removing divergences from the theory, and a setup of the scattering problem of matter particles due to quantum gravity effects, and its diagram techniques. An argument for renormalizibility of this perturbation theory is given. This is based on a forthcoming paper [56] The results are based on an unfinished work and are incomplete. However, they can provide a first concise picture of background independent perturbation theory for quantum gravity.

2 Gravity as a gauge theory

2.1 2+1 gravity: BF and Chern-Simons formulation

In this part we recall briefly the properties of 3 dimensional Euclidean gravity in the first order formalism and fix some notations.

We consider the first order formalism for 3d gravity. The field variables are the triad frame field e^i_μ ($i = 1, 2, 3$) and the spin connection ω^i_μ. The metric is reconstructed as usual from the triad $g_{\mu\nu} = e^i_\mu \eta_{ij} e^j_\nu$ where $\eta = (+, +, +)$ for Euclidean gravity. In the following, we will denote by e^i, ω^i the one-forms $e^i_\mu dx^\mu, \omega^i_\mu dx^\mu$. We also introduce the SU(2) Lie algebra generator J_i, taken to be i times the Pauli matrices, satisfying

$$[J_i, J_j] = -2\epsilon_{ijk}\, \eta^{kl} J_l, \tag{2}$$

where ϵ_{ijk} is the antisymmetric tensor. The trace is such that $\mathrm{tr}(J_i J_j) = -2\delta_{ij}$. One defines the Lie algebra valued one-forms $e = e^i J_i$ and $\omega = \omega^i J_i$. The action is

$$S[e, \omega] = -\frac{1}{8\pi G}\int_M \epsilon_{ijk}(\, e^i \wedge F^{jk}(\omega) + \frac{\Lambda}{3} e^i \wedge e^j \wedge e^k) = \frac{1}{16\pi G}\int_M \mathrm{tr}(e \wedge F(\omega) + \frac{\Lambda}{3} e \wedge e \wedge e), \tag{3}$$

where $\wedge$ is the antisymmetric product of forms and $F(\omega) = d\omega + \omega \wedge \omega$ is the curvature of ω, and Λ is a cosmological constant. The equations of motion of this theory are

$$d_\omega e = 0, \tag{4}$$

$$F(\omega) + \Lambda e \wedge e = 0, \tag{5}$$

where $d_\omega = d + [\omega, \cdot]$ denotes the covariant derivative. If M possess some boundaries ∂M, the variation of the action is not zero on-shell but contains a boundary contribution

$$\delta S = \frac{1}{16\pi G}\int_{\partial M} \mathrm{tr}(e \wedge \delta\omega). \tag{6}$$

This boundary term vanishes if one fixes the value of the connection on ∂M. The gauge symmetries of the continuum action (3) are the Lorentz gauge symmetry

$$\omega \;\rightarrow\; g^{-1} dg + g^{-1} \omega g, \tag{7}$$

$$e \;\rightarrow\; g^{-1} e g, \tag{8}$$

locally parameterized by a group element g, and the translational symmetry locally parameterized by a Lie algebra element ϕ

$$\omega \;\rightarrow\; \omega + \Lambda[e, \phi], \tag{9}$$

$$e \;\rightarrow\; e + d_\omega \phi, \tag{10}$$

which holds due to the Bianchi identity. identity $d_\omega F = 0$. This supposes that $\phi = 0$ on ∂M. The infinitesimal diffeomorphism symmetry is equivalent on-shell to this symmetry when we restrict to non-degenerate configurations $det(e) \neq 0$. The action of an infinitesimal diffeomorphism generated by a vector field ξ^μ can be expressed as the combination of an infinitesimal Lorentz transformation with parameter $\omega_\mu \xi^\mu$ and a translational symmetry with parameter $e_\mu \xi^\mu$.

The conjugate phase space variables are the pull-back of (ω, e) on a two dimensional spacelike surface, their Poisson brackets being

$$\{\omega^i_\mu, e^j_\nu\} = \delta^{ij}\epsilon_{\mu\nu}. \tag{11}$$

The generator of the translational gauge symmetry is given by the pull-back of the curvature on the two dimensional slice, whereas the pull-back of the torsion generates the Lorentz gauge symmetry.

The first order formalism for 3d gravity provides an example of a topological field theory which is called BF-theory. BF-theory can be defined in any dimension d of spacetime and for arbitrary local gauge group G. Its basic variables are 1-form connection field A and $d-2$-form field B, both taking the values in the Lie algebra of G. The action principle for BF-theory is

$$S = \int_M \mathrm{tr}(B \wedge F(A)), \tag{12}$$

where $F(A) = dA + A \wedge A$ is the curvature of the connection. One of the equations of motion of this theory

$$F(A) = 0 \tag{13}$$

means that the connection is locally flat everywhere. This is the reason why the theory doesn't have local degrees of freedom, and could be exactly solved in any dimension.

The action (3) for $\Lambda = 0$ is an example of the action (12), where the dimension of space is three and the role of B-field is played by the vierbein.

There is another possibility write down the action for 3-dimensional gravity in a BF form, which is much more similar to the approach which we will be using in 4 dimensions. This approach, however, requires a non-zero cosmological constant. For example for positive cosmological constant one could extend the gauge group from $SO(3)$ to $SO(4)$. In addition to the tree rotation generators (2) it will include three "translation" generators P_i, satisfying the following commutation relations

$$[J_i, J_j] = -2\epsilon_{ijk}\,\eta^{kl} J_l \tag{14}$$

$$[P_i, P_j] = -2\epsilon_{ijk}\,\eta^{kl} J_l \tag{15}$$

$$[P_i, J_j] = -2\epsilon_{ijk}\,\eta^{kl} P_l \tag{16}$$

We can define the $SO(4)$-connection to be composed of the $SO(3)$-connection and the vierbein,

$$A = \frac{1}{l} e^i P_i + \omega^i J_i, \tag{17}$$

where the parameter of the dimension of length $l = 1/\sqrt{\Lambda}$ is needed for matching dimensions. The curvature of this connection has a form

$$F(A) = \frac{1}{l} d_\omega e^i P_i + (F^{ij}(\omega) + \frac{1}{l^2} e^i \wedge e^j)\epsilon_{ijk} J^k, \tag{18}$$

i.e. it contains information on both curvature and torsion.

It is easy to see that an action principle of the form (12) for connection (17) and curvature (18), where trace is defined either by

$$\mathrm{tr}(J^i J^j) = 2\delta^{ij}, \quad \mathrm{tr}(P^i P^j) = 2\delta^{ij}, \quad \mathrm{tr}(J^i P^j) = 0 \tag{19}$$

or by

$$\mathrm{tr}(J^i J^j) = 0, \quad \mathrm{tr}(P^i P^j) = 0, \quad \mathrm{tr}(J^i P^j) = 2\delta^{ij} \tag{20}$$

results in the same equations of motion (4,5) for connection and vierbein as the action (3). It also contains an equation for auxiliary B-field

$$d_A B = 0. \tag{21}$$

Therefore, one can say that such an action is classically equivalent to the action (3). On the other hand the Poisson bracket structure of this action is different from (11). The $SO(3)$ connection and the vierbein are no longer canonically conjugate, they commute with each other. Therefore the quantization of the action (12) will be different from quantization of the action (3).

Another possible reformulation of 3-dimensional gravity is Chern-Simons theory. This is a 3d topological field theory which has as its only variable a connection for an arbitrary gauge group G. Its action principle is

$$S = \frac{\kappa}{8\pi} \int_M \mathrm{tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A), \tag{22}$$

where κ is a dimensionless coupling constant. Similar to BF-theory, its equation of motion is the local flatness condition for A, $F(A) = 0$.

One can show [14] that if in the action (22) we choose the connection (17) for the $SO(4)$ gauge group and $\kappa = 1/(G\sqrt{\Lambda})$ for the coupling constant, and also define the trace by (20) we recover the action (3) for 3d gravity. It has the same equations of motion (4,5) and the same simplectic form (11). Therefore, this action is equivalent to first order General Relativity even at quantum level, provided the we do not impose the condition of invertibility of the vierbein.

Another possibility is to use the definition of trace in (19) instead of (20). The resulting action is

$$S = \frac{1}{16\pi G\sqrt{\Lambda}} \int_M \mathrm{tr}(\omega \wedge d\omega + \frac{2}{3}\omega \wedge \omega \wedge \omega + \Lambda e d_\omega e), \tag{23}$$

where the trace is now defined the same way as in (3). It results in the same equations of motion (4,5) as ordinary 3d gravity, but has a different simplectic form. Also, unlike ordinary 3d gravity, it does not allow limit $\Lambda \to 0$. So for studying this action it is essential to have a nonzero cosmological constant. In section 4 we will see that the action of the form (23) describes in a natural way the dynamics of boundary observables of four dimensional General Relativity. There we will consider such an action in more detail.

2.2 3+1 gravity: MacDowell-Mansouri formulation

In this section we review the construction of [20], which allows to reformulate four dimensional General Relativity as a gauge theory. Most of the results of the thesis involve this formulation.

Macdowell-Mansouri action is a gauge theory for the $SO(5)$ gauge group. We consider a Euclidian theory with positive cosmological constant; however in the classical theory everything can be directly generalized to a Lorentzian signature. For writing down the action it is convenient to use γ-matrices (See Appendix A):

$$\gamma^A = \gamma^{A\dagger}, \quad \{\gamma^A, \gamma^B\} = 2\delta^{AB}, \tag{24}$$

where $A, B = 1, 2...5$, and $\{.,.\}$ means anticommutator. The ten generators of SO(5) group can then be represented as

$$T^{AB} = T^{AB\dagger}, \quad J^{AB} = \frac{i}{4}[\gamma^A, \gamma^B]. \tag{25}$$

The 15 matrices γ^A and J^{AB} form a basis in the space of 4×4 hermitian traceless matrices.

The only variable in the theory is an $so(5)$-valued gauge field A^{AB}. In a background independent theory, the action cannot involve any fixed metric. So, the only form of the action possible in four dimensions is

$$S_{FF} = \frac{1}{\alpha} \int F^{AB} \wedge F^{CD} \mathrm{tr} \gamma_A \gamma_B \gamma_C \gamma_D. \tag{26}$$

Here

$$F^{AB} = dA^{AB} + A^A_C \wedge A^C_B \tag{27}$$

is the curvature of the SO(5)-connection A^{AB}. The equations of motion following from the action (26)

$$d_A F^{AB} = 0 \tag{28}$$

are trivially satisfied due to the Bianci identity $\nabla \wedge F^{AB} = 0$ and, therefore, the theory (26) is topological.

By a small modification, however, which is a breaking of the SO(5) symmetry down to the SO(4) the action (26) can be turned into that of General Relativity. For example one can insert a γ_5 in the trace in (26)

$$S_{GR} = \frac{1}{\alpha} \int F^{AB} \wedge F^{CD} Tr \gamma_A \gamma_B \gamma_C \gamma_D \gamma_5, \tag{29}$$

where 5 labels some preferred direction. To see that the action (29) is indeed the action of General Relativity let us rewrite it in terms of $4+1$-decomposed indices $A = (i, 5)$, $i = 1, 2..4$:

$$S_{GR} = \frac{1}{\alpha} \int (F^{ij} \wedge F^{kl} \epsilon_{ijkl5}). \tag{30}$$

Also we can decompose the SO(5)-connection

$$A^{ij} = \omega^{ij}, \quad A^{i5} = \frac{1}{l} e^i, \tag{31}$$

where ω^{ij} is an SO(4)-connection, e^i is a tetrad, and l is a constant of dimension of length. Equation (31) leads to the following decomposition of SO(5)-curvature

$$\begin{aligned} F_{ij} &= d\omega_{ij} + \omega_{ik} \wedge \omega^k_j + \frac{1}{l^2} e_i \wedge e_j = F(\omega)_{ij} + \frac{1}{l^2} e_i \wedge e_j \\ F_{i5} &= \frac{1}{l} \nabla \wedge e_i = T_i. \end{aligned} \tag{32}$$

Here $F(\omega)_{ij}$ is an SO(4)-curvature and T_i is a torsion. The action (30) can then be rewritten as

$$S_{GR} = \frac{1}{2G\Lambda} \int (F(\omega)_{ij} + \Lambda e_i \wedge e_j) \wedge (F(\omega)_{kl} + \Lambda e_k \wedge e_l) \epsilon^{ijkl}, \tag{33}$$

where $\Lambda = \frac{1}{l^2}$ is the cosmological constant and $G = \alpha/\Lambda$ is the Newton constant. The action (33) is the action of General Relativity.

2.3 General Relativity with a topological phase

The connection with topological field theory naturally raises the question of whether there might be dynamical transitions between a low energy phase in which gravity is approximately described by general relativity, and a high energy phase, which is topological. Indeed, speculations in this direction have been made for some time. For example, in 1988 Witten proposed that the Hagedorn temperature might represent a transition to a topological phase in which the metric vanishes[16]. This kind of conjecture has recently gained attention again in the context of spin foam models[15].

It has also been noticed that in certain first order action principles for GR and supergravity there are degenerate phases in which the determinate of the metric vanishes[17]. In [18] phase boundaries were studied between regions in which the metric is degenerate and non-degenerate, and were found to be null.

However, to describe a transition between a topological and a gravitational phase dynamically, both must be solutions to the same theory. In [19] it was proposed that this could be done by making the constraints that reduce the gauge symmetry of a $TQFT$ dynamical. In this paper we would like to describe one way in which this can be done.

In the next section we review the basic idea of gravity as a constrained topological field theory and describe a new action principle from which the constraints arise dynamically. In section 3 we describe solutions to those constraints and show how four different theories are recovered. These are two $TQFT$'s: $F \wedge F$ theory and BF theory for $SO(5)$ and two versions of general relativity: the action of Palatini and the action principle for the Ashtekar-Sen variables. In section 4 we study the boundary between a topological and gravitational phase and find that it resembles the conditions imposed on an horizon.

2.3.1 Action principle

It is possible to consider a theory in which the symmetry breaking in MacDowell-Mansouri formalism is not introduced from the beginning but instead induced by the theory itself. This is possible if e.g. the fixed quantity γ_5 in (29) is replaced by a dynamical variable.

Let us consider the following action

$$S'_{GR} = \int B^{AB} \wedge F^{CD} Tr\gamma_A\gamma_B\gamma_C\gamma_D - \frac{\alpha}{2}\int B^{AB} \wedge B^{CD} Tr\gamma_A\gamma_B\gamma_C\gamma_D\Gamma + \int \lambda(\Gamma^2 - 1), \tag{34}$$

where we introduced u(4)-valued (hermitian, but not necessarily traceless) matrices λ and Γ. For general covariance Γ should be a 0-form (scalar) and λ should be a 4-form (scalar density).

Let us first solve the equation for Γ resulting from the variation of the action (34) with respect to λ:

$$\Gamma^2 = 1. \tag{35}$$

As an hermitian 4×4 matrix Γ can be represented as

$$\Gamma = u1 + v_A\gamma^A + w_{AB} i[\gamma^A, \gamma^B], \tag{36}$$

where u, v_A, and w_{AB} are 16 arbitrary real numbers. By substituting (36) into (35) and using the anticommutation relations for γ-matrices (24) and

$$\begin{aligned} \{i[\gamma^A, \gamma^B], \gamma^C\} &= \epsilon^{ABCDE} i[\gamma_D, \gamma_E] \\ \{i[\gamma^A, \gamma^B], i[\gamma^C, \gamma^D]\} &= \frac{1}{2}(\delta^{AC}\delta^{BD} - \delta^{AD}\delta^{BC})1 + \epsilon^{ABCDE}\gamma_E \end{aligned} \tag{37}$$

one finds

$$(u^2+v_Av^A+w_{AB}w^{AB})1+(uv^E+\epsilon^{ABCDE}w_{AB}w_{CD})\gamma_E+(uw^{DE}+\epsilon^{ABCDE}w_{AB}v_C)i[\gamma_D, \gamma_E] = 1 \tag{38}$$

This leads to the following set of equations for u, v_A, and w_{AB}

$$\begin{aligned} u^2 + v_Av^A + w_{AB}w^{AB} &= 1 \\ uv^E + \epsilon^{ABCDE}w_{AB}w_{CD} &= 0 \\ uw^{DE} + \epsilon^{ABCDE}w_{AB}v_C &= 0 \end{aligned} \tag{39}$$

2.3.2 Solutions and phases

In the absence of the general solution to the equations (39) below we will give several examples. As eq. (39) is 16 non-linear equations for 16 parameters it is natural to expect that different solutions to them are disconnected from each other, i.e. cannot be transformed into each other by a continuous change of parameters. The examples are:

1. $u = 1$, $v^A = 0$, $w_{AB} = 0$, which means that

$$\Gamma = 1, \tag{40}$$

i.e. 4×4 unity matrix,

2. $u = 0$, $w_{AB} = 0$, and v_A is an arbitrary 5-dimensional vector such that $v_Av^A = 1$, i.e.

$$\Gamma = \gamma^A v_A, \tag{41}$$

3. $u = 0$, $v^A = 0$, and w_{AB} is an antisymmetric tensor 5×5 such that all the nonzero components of it share one common index and $w_{AB}w^{AB} = 1$, which results in

$$\Gamma = i[\gamma^A, \gamma^B]w_{AB}, \tag{42}$$

4. $u = 1/2$, $v^5 = -1/2$, $w^{12} = w^{34} = 1/2$, all the other components being zero, i.e.

$$\Gamma = \frac{1}{2}(1 - \gamma^5 + i[\gamma^1, \gamma^2] + i[\gamma^3, \gamma^4]) \tag{43}$$

Let us now see what kind of theories the above solutions result in. If we plug the solution (40) into the action (34) then solve the equation for B^{AB} and substitute it back into the action we will obtain the $F \wedge F$ theory for SO(5) group.

$$S_1' = \frac{1}{2\alpha} \int F^{AB} \wedge F^{CD} Tr \gamma_A \gamma_B \gamma_C \gamma_D. \tag{44}$$

Due to the Bianchi identity the bulk equations of motion of this theory are trivial, and therefore the theory is topological.

If we use the solution (41) in the action (34) we will get the following result

$$S_2' = \int B^{AB} \wedge F_{AB} - \frac{\alpha}{2} \int B^{AB} \wedge B^{CD} \epsilon_{ABCDE} v^E. \tag{45}$$

This action is very similar to the action (30) except that it includes an additional arbitrary parameter v^A. The appearance of this parameter is an additional gauge freedom in the action. This freedom can be fixed by aligning the vector v^A along some preferred direction. Then the analysis (30-33) can be repeated and the resulting action will be the action of General Relativity. This is the ordinary Palatini action for General Relativity which involves both left-handed and right-handed connections.

We also have the solution (42). After plugging it into (34) the second term in the action will read

$$\frac{i\alpha}{2} \int B^{AC} \wedge B_C^B w_{AB}. \tag{46}$$

As w_{AB} is antisymmetric and the tensor it is contracted with is symmetric the contribution (46) to the action disappears. The resulting action

$$S_3' = \int B^{AB} \wedge F_{AB} \tag{47}$$

is the action of BF-theory for the SO(5)-group. The equations of motion of this theory mean that SO(5) curvature of the connection A_B^A is zero. So although it is also a topological field theory, it is slightly different from the theory (44).

Finally, let us consider the solution (43). The result will be a sum of the results (44) and (45):

$$S_4' = \frac{1}{2\alpha} \int F^{AB} \wedge F_{AB} + \frac{1}{2\alpha} \int F^{AB} \wedge F^{CD} \epsilon_{ABCD5}. \tag{48}$$

This action is the self-dual part of the action of General Relativity, which leads to the Ashtekar canonical formulation with the Immirzi parameter equal to 1 (in the Euclidian theory Ashtekar's variables are real). In the bulk the actions (45) and (48) are equivalent as they differ from each other by canonical transformation. However they may lead to inequivalent field equations on the boundary. It is also interesting to notice that this formalism doesn't seem to lead to any value of the Immirzi parameter other than 1.

These three solutions are disconnected from each other, so a possible phase transition between them must be a first order phase transition.

2.3.3 Conditions on a phase boundary

As it was mentioned the phase transition between different solutions of the above theory is of the first order. Such transition generally occurs via formation of bubbles of a phase B within a medium of a phase A. For two phases to coexist some boundary conditions on a boundary between two phases must be satisfied.

Consider a two-phase mixture of the above model one of which is General Relativity (34) and the other is the Donaldson theory (44). Let the phase boundary be located at $x_1 = 0$. We don't specify whether the direction x_1 is spacelike or null. With the traces of γ-functions calculated its action principle reads

$$S_{2phase} = \frac{1}{2\alpha} \int F^{AB} \wedge F^{CD} \Gamma_{ABCD}, \tag{49}$$

where

$$\Gamma_{ABCD}(x_1) = \delta_{AC}\delta_{BD}\theta(x_1) + \epsilon_{ABCD5}\theta(-x_1), \tag{50}$$

where θx_1 is the step θ-function. The variation of the action (49) will give the equations of GR in the region $x_1 < 0$, the equations of $F \wedge F$ theory (which are trivially satisfied) in $x_1 > 0$ region and the singular contribution to the variation at $x_1 = 0$ resulting from differentiation of θ-functions

$$\delta S_{2phase} = ... + \frac{1}{\alpha} \int F^{AB} \wedge n_1 \wedge \delta A^{CD} \delta(x_1) \Delta_{ABCD} = ... + \frac{1}{\alpha} \int_{x_1=0} F^{AB} \wedge \delta A^{CD} \Delta_{ABCD}. \tag{51}$$

Here ... stays for the bulk regular terms, n_1 is unit vector in x_1-direction and

$$\Delta_{ABCD} = \delta_{AC}\delta_{BD} - \epsilon_{ABCD5}. \tag{52}$$

For the variational principle to be well defined the singular term (51) in the variation must vanish. For this the condition $F^{AB} = 0$ must be satisfied on a phase boundary, and, according to the 4+1-decomposition (32) this means

$$\begin{aligned} f_{\alpha\beta} + \Lambda e_\alpha \wedge e_\beta = 0 \\ T_\alpha = 0. \end{aligned} \tag{53}$$

The second of the equations (53) (zero torsion) is always satisfied in GR, while the first equation is a specific type of isolated horizons boundary conditions. This may suggest that after a suitable generalization the dynamics of the formation of a new phase will be governed by the dynamics of isolated horizons.

As an example of a further generalization of the theory given by (34) one can consider the following action

$$S'_{GR} = \int TrB^{AB}\gamma_A\gamma_B \wedge (F^{CD}T\gamma_C\gamma_D + \beta d\Gamma \wedge d\Gamma) - \frac{\alpha}{2}\int B^{AB}\wedge B^{CD}Tr\gamma_A\gamma_B\gamma_C\gamma_D(\Gamma - \frac{1}{3}\Gamma^3). \tag{54}$$

It is obvious that all the solutions of the theory (34) with constant Γ considered above are also solutions of the theory (54). (54) may have other solutions, but they would be difficult to analyze as that would require to consider situations in which the equation of motion for Γ is not 'decoupled' from those for other variables. In the case of solutions in which Γ is varying in spacetime the term containing derivatives of Γ in (54) would define the shape of the boundaries between domains of different phases.

3 Topological field theory with point sources

Generally coupling a topological field theory with matter, for example with a scalar field, spoils its exact solubility. It also spoils the positive degree form structure of the theory, since in writing down a field theory action one needs to invert the metric. However there is a specific kind of matter, incorporation of which does not require inverting a metric, and when included keeps the topological field theory exactly soluble. These are point particles represented as topological defects that were first introduced in 2+1 dimensional gravity by Deser, Jackew, and 't Hooft [21]. It can be considered as a form of geometrization of matter, as it doesn't require introducing any new degrees of freedom. The particle degrees of freedom are made from gravitational gauge degrees of freedom which are transformed into physical degrees of freedom by symmetry breaking. In this section we will review the construction of [21], show how to extend it to 3+1 dimensional topological field theory, and find a relation of this construction to the formalism of Balachandran [22] for describing point particles coupled to 3+1 dimensional gravity.

3.1 Point particles in 2+1 gravity

Here we review the construction of [21]. The presentation will closely follow that of [23] and [24]. For simplicity we will consider Euclidian gravity

As suggested in [21] the metric associated with a spinning particle of mass m and spin s coupled to 3 dimensional gravity is a spinning cone

$$ds^2 = (dt + 4Gsd\varphi)^2 + dr^2 + (1 - 4Gm)^2 r^2 d\varphi^2, \tag{55}$$

where m is the mass of the particle, s is its spin, t is the Euclidian time coordinate, r is a radial coordinate, and φ is an angular coordinate with identification $\varphi \to \varphi + 2\pi$. The space described by this metric is locally flat. When $4Gm < 1$ it can be identified with a portion of Minkowski space. The spinning cone is obtained by cutting out of the Minkowski space a wedge $0 < \varphi < 2\pi(1 - 4Gm)$, and then identifying the two faces of this wedge by a translation along the t axis of length $8\pi Gs$. Around $r = 0$, which is the location of the particle there is a deficit angle of $8\pi Gm$ and a time offset of length $8\pi Gs$. The mass of the particle then has to be bounded by $1/4G$, as the deficit angle cannot exceed 2π. A frame field for this geometry can be given by

$$e = J_0 dt + (\cos\varphi J_1 + \sin\varphi J_2)dr + ((1 - 4Gm)r(\cos\varphi J_2 - \sin\varphi J_1) + 4GsJ_0)\, d\varphi, \tag{56}$$

and the spin connection by

$$\omega = 2GmJ_0 d\varphi. \tag{57}$$

The torsion and the curvature have a distributional contribution at the location of the particle[1]

$$\begin{aligned} d_\omega e &= 8\pi G s J_0 \delta^2(x) d^2x, && (58)\\ F(\omega) &= 4\pi G m J_0 \delta^2(x) d^2x, && (59)\end{aligned}$$

where the delta function is along the plane $t = const$. Since the torsion is the generator of Lorentz gauge symmetry we see that having a spin means that this symmetry is broken at the location of the particle, also the mass is breaking the translational symmetry at the location of the particle. We can explicitly see that this is the case if we perform a Lorentz transformation labeled by g^{-1} and then a translation labeled by $-\phi$, the equations (58) then become

$$\begin{aligned} d_\omega e &= 4\pi G \jmath \delta^2(x) d^2x, && (60)\\ F(\omega) &= 4\pi G p \delta^2(x) d^2x. && (61)\end{aligned}$$

where $\jmath, p$ are the following Lie algebra elements

$$\begin{aligned} p &= m g J_0 g^{-1}, && (62)\\ \jmath &= 2s g J_0 g^{-1} - m[g J_0 g^{-1}, \phi]. && (63)\end{aligned}$$

p is the momentum of the particle and $\jmath$ the total angular momentum, they satisfy the constraints

$$-\frac{1}{2}\mathrm{tr}p^2 = m^2;\ -\frac{1}{2}\mathrm{tr}(pj) = 2ms. \quad (64)$$

From the canonical point of view these constraints are first class [23]. The mass constraint generates time reparameterization and the spin constraints generate $U(1)$ gauge transformation $g \to gh$. Due to the breaking of symmetry at the location of the particle the gauge degrees of freedom g, ϕ become dynamical (modulo the remnant reparameterization and $U(1)$ gauge symmetry): g describes the Lorentz frame of the particle, ϕ describes the position of the particle. Moreover knowledge of $p, \jmath$ is enough to reconstruct g, ϕ modulo the remnant gauge symmetries. Indeed gH is determined by (62). If we denote by $x_\perp^a$ the position of the particle perpendicular to the momenta, $\phi = \frac{(p\cdot\phi)}{m^2}p + x_\perp$, then $x_\perp = \frac{1}{m^2}[\jmath, p]$, also $\jmath^2 = s^2 + m^2 x_\perp^2$.

We can easily understand the canonical commutation relations of $p, \jmath$ from the equations (60,61). Since the LHS of (60) is the generator of Lorentz transformations and the LHS of (61) is the generator of translational symmetry, these constraints are first class and from their canonical algebra we can easily deduce that the Poisson algebra of $p, \jmath$ is given by

$$\{\jmath^a, \jmath^b\} = -2\epsilon^{abc}\jmath_c,\ \{\jmath^a, p^b\} = -2\epsilon^{abc}p_c,\ \{p^a, p^b\} = 0. \quad (65)$$

[1] we use the distributional identity $dd\varphi = 2\pi\delta^2(\vec{x})dxdy$.

This analysis shows that, instead of treating the gravity degrees of freedom and the particle degrees of freedom as separate entities, we can reverse the logic and consider that the equations (58,59) are defining equations for a spinning particle. This allows us to describe a particle as a singular configuration of the gravitational field giving a realization of matter from geometry. The 'would-be gauge' degrees of freedom [27] are promoted to dynamical degrees of freedom at the location of the particle. This is the point of view we are going to take in this thesis. In order to get equations (58,59) from an action principle we have to add to the gravity action (3) the following terms

$$\overline{S}_{P_{m,s}}(e,\omega) = -\frac{1}{2}\int dt \ \mathrm{tr}[(me_t + 2s\omega_t)J_0], \tag{66}$$

where the integral is along the worldline of the particle. This action describes a 'frozen' particle without degrees of freedom. We have seen in (60,61) that the particle degrees of freedom are encoded in the former gauge degrees of freedom. To incorporate the dynamics of the particle we perform the transformation

$$\omega \to \tilde{\omega} = g^{-1}\omega g + g^{-1}dg, \ \ e \to \tilde{e} = g^{-1}(e + d_\omega\phi)g \tag{67}$$

the action (66) becomes

$$\overline{S}_{P_{m,s}}(\tilde{e},\tilde{\omega}) = -\frac{1}{2}\int dt \ \mathrm{tr}[e_t p + \omega_t j] + S_{P_{m,s}}(g,\phi) \tag{68}$$

where the first term describes the interaction between the particle degree of freedom and gravity. The second term

$$S_{P_{m,s}}(g,\phi) = -\frac{1}{2}m\int dt \mathrm{tr}(g^{-1}\dot{\phi}gJ_0) - s\int dt \mathrm{tr}(g^{-1}\dot{g}J_0), \tag{69}$$

is the action for a relativistic spinning particle in a form first described by Sousa Gerbert [23]. One sees again that the original gauge degrees of freedom are now promoted to dynamical degrees of freedom describing the propagation of a particle.

A similar procedure could be applied for describing particles coupled to the action (12) of three dimensional gravity. It will be analogous to introducing particles in four dimensional BF theory. This will be described in section 3.3.

3.2 Point particles in 3+1 gravity

We now present the formalism developed by Balanchandran et al. [22] which allow to describe the motion of spinning particles coupled to gravity. As we will see, in this formalism the degrees of freedom of the particle – Lorentz frame and position – are identified with the gauge parameters of gravity: local Lorentz transformations and diffeomorphism invariance. The particle is breaking the symmetry of General relativity and the gauge degree of freedom becomes dynamical at the location of the particle. This is similar to what happens in three dimensions.

The gravitational field is encoded in a SO$(4,1)$ connection

$$A_\mu = \left(\frac{1}{2}e_\mu{}^a\,\gamma_a\gamma_5 + \frac{1}{4}\omega_\mu{}^{ab}\,\gamma_{ab}\right) \tag{70}$$

where the index a run from 0 to 3 (see appendix A for notation). The 4-dimensional de Sitter group acts by conjugation on its Lie algebra, the orbits of this action are labeled by two numbers (m,s), which are the mass and spin of the particle. For each orbit we choose a fixed representative element of the 4-dimensional de Sitter Lie algebra $J_0 \equiv m\gamma_1\gamma_5/2 + s\gamma_2\gamma_3/4$.

The Lorentz Lie algebra so$(3,1)$ is identified with the subalgebra of so$(4,1)$ generated by γ_{ab}. The Lagrangian of a single particle propagating in a gravitational field is characterized by an embedding of its worldline $z(\tau)$ and a function $g(\tau)$ valued in the Lorentz group $g = \exp(\alpha^{ab}\frac{\gamma_{ab}}{4})$. Let us denote by $A^g = g^{-1}Ag + g^{-1}dg$ the gauge transformation of A. The lagrangian takes the simple form

$$L(z,h) = \mathrm{tr}\left(J_0 A^h_\tau(\tau)\right) \qquad S = \int d\tau\, L, \tag{71}$$

where τ parameterizes the world line and $A_\tau(\tau) \equiv A_\mu(z(\tau))\,\dot{z}^\mu$. Consider now the equations of motion that follow from the Lagrangian (71). The variation over h gives (ignoring total derivatives)

$$\delta L = \mathrm{tr}\left(h^{-1}\delta h([J_0, A^h])\right), \tag{72}$$

$$= \mathrm{tr}\left(\delta h h^{-1}(D_\tau J)\right), \tag{73}$$

where we introduced the SO$(4,1)$ covariant derivative along the world line of the particle

$$D_\tau \equiv \frac{d}{d\tau} + [A_\tau, \cdot], \tag{74}$$

and we defined

$$J \equiv g\, J_0\, g^{-1}. \tag{75}$$

The components of J can be expressed in terms of the particle's momenta p_a and spin s_{ab}

$$J = \frac{1}{2}p_a\,\gamma^a\gamma_5 + \frac{1}{4}s_{ab}\,\gamma^{ab}. \tag{76}$$

Since h is restricted to be in the Lorentz subgroup of SO$(4,1)$ the equations (72) constrain only the spin part of J and give the spin precession equation

$$D_\tau J^{ab} = \nabla_\tau s^{ab} + e^a_\tau p^b - e^b_\tau p^a = 0, \tag{77}$$

with $\nabla_\tau \equiv \frac{d}{d\tau} + [\omega_\tau, \cdot]$ the Lorentz connection and $e^a_\tau = e^a_\mu \dot{z}^\mu$. Note that by construction the momenta and spin satisfy the orthogonality condition

$$s^{ab} p_b = 0 \tag{78}$$

The variation over z gives

$$\begin{aligned} \frac{\delta L}{\delta z^\mu} &= -\frac{d}{d\tau}\mathrm{tr}(J A_\mu) + \mathrm{tr}(J \partial_\mu A_\nu)\dot{z}^\nu, &(79)\\ &= -\mathrm{tr}\left(D_\tau J A_\mu\right) + \mathrm{tr}\left(J F_{\mu\nu}(A)\right)\dot{z}^\nu, &(80) \end{aligned}$$

where

$$\begin{aligned} F_{\mu\nu}(A) &\equiv \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu], &(81)\\ &= T^a_{\mu\nu}\gamma_a\gamma_5/2 + (R^{ab}_{\mu\nu}(\omega) - e^a_\mu e^b_\nu)\gamma_{ab}/4. &(82) \end{aligned}$$

$T^a = de^a + \omega^a{}_b \wedge e^b$ is the torsion, and $R^{ab} = d\omega + [\omega, \omega]/2$ the Lorentz curvature.

If one uses the equation (72), equation (80) written in components reads

$$\nabla_\tau p_\mu = \frac{1}{2} s_{ab} R^{ab}_{\mu\nu}\dot{z}^\nu + p_a T^a_{\mu\nu}\dot{z}^\nu. \tag{83}$$

This is the Mathiasson-Papapetrou equations describing the dynamics of a spinning particle in the presence of torsion[2] When the torsion is zero we recover the usual Mathiasson-Papapetrou equation. When the spin is also zero we recover the usual geodesic equation.

The difference with the coupling of particles in 3 dimensional gravity described in the previous section is that in three dimensions information about propagation is encoded in the local translation group and the particle action is invariant with respect to diffeomorphisms. In four dimensions, on the other hand information about the position of the particle is encoded in the diffeomorphism group, the action depends on the embedding of the worldline of a particle in a manifold, and there is no analog of the local translation group. As the particle moves in curved spacetime where the curvature changes from point to point, the state of the particle changes, too.

The absence of an analog of the local translation group in four dimensional gravity is the main obstacle in describing quantum particles in curved spacetime [25].

[2] We can write this equation in the usual form if we introduce the affine connection $\Gamma_{\mu\nu}{}^\rho$, which is related to the spin connection ω^{ab}_μ by the identity $\partial_\mu e^a_\nu + \omega_\mu{}^a{}_b e^b_\nu = \Gamma_{\nu\mu}{}^\rho e^a_\rho$. It can be written in terms of the Christofel symbol Γ as $\Gamma_{\mu\nu\rho} = \Gamma_{\mu\nu\rho} + T_{\rho\{\mu\nu\}} - \frac{1}{2}T_{\mu\nu\rho}$ and the MP equation reads

$$\nabla_\tau \psi_\mu = \frac{1}{2} s_{ab} R^{ab}_{\mu\nu}\dot{z}^\nu, \tag{84}$$

where $p_\mu \equiv p_a e^a_\mu$ and $\nabla_\mu p_\nu \equiv \partial_\mu p_\nu - \Gamma^\rho_{\mu\nu} p_\rho$.

The source of the problem is maybe inconsistency between a quantum mechanical description of particles and a classical description of spacetime geometry. One of the basic ideas of Loop Quantum Gravity, and one of the main topics of this thesis (see section 6) is that in quantum theory it is more natural to describe curvature of spacetime not as a continuous field but as a dynamical system of sources of a certain topological field theory. The corresponding description of particles that is compatible with this picture is based on topological field theory. This is what the next section is devoted to.

3.3 Point particles in 3+1 dimensional BF theory

In this section we consider how to introduce matter particles as extrinsic sources in four dimensional topological field theory. It is much more analogous with particles in 3d gravity (section 3.1) than with particle in four dimensional gravity (introduced in the previous section). In particular it has all the advantages of the presence of local translational symmetry in the theory. So the constructions of this section will basically repeat those of section 3.1, with some modifications which we will emphasize along the way.

We start with four dimensional BF-theory for the $SO(5)$ gauge group whose action in very similar to the action (12)

$$S = \int B^{IJ} \wedge F_{IJ}(A) = \int \mathrm{tr}(B \wedge F(A)), \tag{85}$$

with the difference that the B-field is now a two form. Here $B = B^{IJ} T_{IJ}$ and $A = A^{IJ} T_{IJ}$, where

$$T_{IJ} = \frac{1}{4}[\gamma_I, \gamma_J] \tag{86}$$

are $so(5)$-generators in the fundamental representation and γ_I are γ-matrices satisfying $\{\gamma_I, \gamma_J\} = 2\delta_{IJ}$. And the equations of motion are

$$d_A B = 0, \quad F(A) = 0. \tag{87}$$

The gauge symmetry of the action (85) includes local $SO(5)$-symmetry

$$\begin{aligned} A &\rightarrow g^{-1} dg + g^{-1} A g \\ B &\rightarrow g^{-1} B g, \end{aligned} \tag{88}$$

where g is an $SO(5)$-valued scalar field and some 10-dimensional translational symmetry (10 is the number of generators of $so(5)$)

$$\begin{aligned} A &\rightarrow A \\ B &\rightarrow B + d_A \phi, \end{aligned} \tag{89}$$

where ϕ is an $so(5)$-valued 1-form field.

The general solution to equations (87) can be written as

$$A = g^{-1}dg, \quad B = g^{-1}d\phi g, \tag{90}$$

where g and ϕ are arbitrary due to the above gauge freedom.

We will study the above theory with extrinsic sources added. We will be primary interested in distributional sources that transform covariantly with respect to diffeomorphisms. These can be defined by adding lower dimensional integrals to the action (85):

$$S' = \int_M tr(B \wedge F(A)) + \int_S tr(P_0 B) + \int_L tr(J_0 A), \tag{91}$$

where S is some 2-dimensional submanifold of M (worldsheet of a string) and L is some 1-dimensional submanifold of M(worldline of a particle), P_0 and J_0 are some fixed elements of $so(5)$. The equations of motion then become

$$\begin{aligned} d_A B &= J_0 \delta^3(x) d^3x \\ F(A) &= P_0 \delta^2(x) d^2x, \end{aligned} \tag{92}$$

where $\delta^3(x)$ is concentrated on L and $\delta^2(x)$ is concentrated on S. Since P_0 and J_0 are fixed the gauge symmetry on S and L is broken. By performing a gauge transformation equations (92) turn into

$$\begin{aligned} d_A B &= J \delta^3(x) d^3x \\ F(A) &= P \delta^2(x) d^2x, \end{aligned} \tag{93}$$

where

$$\begin{aligned} P &= g P_0 g^{-1} \\ J &= g J_0 g^{-1} - [g P_0 g^{-1}, \phi_{*L\wedge S}], \end{aligned} \tag{94}$$

where $\phi_{*L\wedge S}$ means that the component of ϕ is taken in the direction which together with the basis in S and the basis in L forms a basis in M.

In this section we consider a special case with no stringy extrinsic sources, which means that $P_0 = 0$ in (91) and all the subsequent formulas. We consider only local particle sources which carry charges of $so(5)$-symmetry that is the symmetry of the vacuum solution of (Euclidian) General Relativity. This must give us a description of particles propagating in the vacuum spacetime with gravity effects ignored.

The third term in the action (91) when L is timelike

$$\bar{S}_P(A) = \int_L tr(J_0 A) = \int_L dt \mathrm{tr}(J_0 A_t) \tag{95}$$

describes a point charge of $SO(5)$-group (or particle). Dynamics of the particle comes from the gauge degrees of freedom. To incorporate them explicitly we can perform a gauge transformation,

$$A \rightarrow \tilde{A} = g^{-1}Ag + g^{-1}dg \tag{96}$$

and the action (95) then becomes

$$\bar{S}_P(\tilde{A}) = \int_L dt\mathrm{tr}(JA_t) + S_P(g) \tag{97}$$

where the first term with J given by (94) describes the covariant coupling between the particle and BF theory and the second term

$$S_P = 2\int_L dt\mathrm{tr}(g^{-1}\dot{g}J_0), \tag{98}$$

describes the dynamics of the particle.

This action is analogous to the spin part of the action (69) for a particle in three dimensional gravity. The difference is that the gauge group is now $SO(5)$ which has two Casimirs (mass and spin) and the information about the two Casimirs is encoded in the extrinsic source J_0. It is convenient to take

$$J_0 \equiv m\gamma_1\gamma_5/2 + s\gamma_2\gamma_3/4 \tag{99}$$

as in the previous section.

To put (97) in a more conventional form let us rewrite it explicitly, distinguishing the rotation transformations generated by T_{ij}, $i,j = 1,..4$ and translation transformations generated by $T_{\mu 5}$. By introducing the scalars $J^{IJ} = \mathrm{tr}(JT^{IJ})$, $J = J^{IJ}T_{IJ}$ and recalling that $A^{5i} = \sqrt{\Lambda}e^i$ and also introducing the 'momentum' $P^i = \sqrt{\Lambda}J^{5i}$ we can rewrite (97) as

$$\bar{S}_P(\tilde{A}) = \int_L dt(A_t^{IJ}J_{IJ}) + S_P(g) = \int_L dte_t^iP_i + \int_L dt\omega_t^{ij}J_{ij} + S_P(g) \tag{100}$$

J^{IJ} in the above equations must satisfy the constraints

$$J^{IJ}J_{IJ} = C_2 \tag{101}$$

and

$$J^{IJ}J^{KL}\epsilon_{IJKLM}\epsilon^{MABCD}J_{AB}J_{CD} = C_4, \tag{102}$$

where C_2 and C_4 are the quadratic and fourth power Casimir operators of the $so(5)$ algebra.

To see the physical meaning of C_2 and C_4, let us rewrite equations (101) and (102) using notation as in (100). Below we will also assume that the cosmological constant Λ is small and consider the leading order in Λ. Thus

$$C_2 = J^{ij}J_{ij} + J^{5i}J_{5i} = J^{ij}J_{ij} + \frac{1}{\Lambda}P^iP_i \approx \frac{1}{\Lambda}P^iP_i \tag{103}$$

and

$$\begin{aligned} C_4 &= (J^{ij}J^{kl}\epsilon_{ijkl})^2 + J^{ij}J^{5k}\epsilon_{ijkl}\epsilon^{lmnp}J_{5m}J_{np} \\ &= (J^{ij}J^{kl}\epsilon_{ijkl})^2 + \frac{1}{\Lambda}J^{ij}P^k\epsilon_{ijkl}\epsilon^{lmnp}P_mJ_{np} \approx \frac{1}{\Lambda}J^{ij}P^k\epsilon_{ijkl}\epsilon^{lmnp}P_mJ_{np} \end{aligned} \tag{104}$$

From the way the particles are coupled to the connection in (100) it is clear that P_μ in the above equations is space-time momentum and $J_{\mu\nu}$ is angular momentum. From (103) one can see the Casimir C_2 gives rise to the mass of the particle:

$$m^2 = \Lambda C_2. \tag{105}$$

The last equation for the mass relates two well-known problems in particle physics. Explaining why the cosmological constant is small would also help to explain why masses of elementary particles are small. In the non-relativistic limit, when $P^\mu \approx (m, 0, 0, 0)$, the casimir C_4 in (104) can be rewritten as

$$C_4 = \frac{m^2}{\Lambda}J^{ab}\epsilon_{abc}\epsilon^{cde}J_{de} = C_2s^as_a, \tag{106}$$

where $a, b = 1, 2, 3$-$SO(3)$-indices and $s^a = \epsilon^{abc}J_{bc}$ is the spin in the rest frame of the particle. Thus we have the expression for the spin

$$s^2 = \frac{C_4}{C_2}. \tag{107}$$

This Casimirs are directly related to mass and spin in (99)

A general $SO(5)$ transformation $g = \exp(\alpha^{IJ}T_{IJ})$,where α^{IJ} are real numbers, can be represented as a composition of pure $SO(4)$ rotation and pure translation

$$g = g_tg_r. \tag{108}$$

Here $g_t = \exp(\alpha^{i5}T_{i5})$ are translations and $g_t = \exp(\alpha^{ij}T_{ij})$ are pure rotations. We consider the limit in which all translations g_t commute with each other thereby forming an abelian subgroup. This limit corresponds to the limit of small cosmological constant.

We can plug (108) into (98) to get

$$S_P = 2\int_L dt \mathrm{tr}(g_r^{-1} g_t^{-1}(\dot{g}_t g_r + g_t \dot{g}_r) J_0) = 2\int_L dt \mathrm{tr}(g_r^{-1}\dot{\alpha}_t g_r J_0) + 2\int_L dt \mathrm{tr}(g_r^{-1}\dot{g}_r J_0). \tag{109}$$

Or recalling the definition of momentum and angular momentum entering (100) one gets

$$S_P = 2\int_L dt \dot{\alpha}_i P^i + 2\int_L dt \mathrm{tr}(g_r^{-1}\dot{g}_r T^{ij}) J_{ij} \tag{110}$$

with the constraints on P_i and J_{ij} given by (103) and (104). It is clear that the first term in (110) is the action of a spinless relativistic particle propagating in four-dimensional spacetime and the second term in (110) describes the dynamics of spin. We should notice that for (110) to describe the most general particle the algebraic element J_0 in (109) must be a linear combination of both translation and rotation as in (99). If J_0 is a pure translation (110) gives the description of a spinless particle, if J_0 is a pure rotation (110) gives the description of a particle having only spin and no energy-momentum.

Now one has to point out the difference between equations of motion following from the action (109) and equations (83) and (77) derived in the previous section. The equation (77) is the same for BF-theory, provided that using the local flatness condition we can put all the components of the connection, including the vierbein to zero. Instead of equation (83) for gravity, which follows from the variation of the worldline of the particle, we have the following equation

$$\nabla_\tau p_a = s_{ab} e^b_\tau. \tag{111}$$

following from the variation with respect to local translation. The two equations are identical if the following equation holds along the particle worldline

$$s_{ab} e^b_\tau e^a_\mu = \frac{1}{2} s_{ab} R^{ab}_{\mu\nu} \dot{z}^\nu + p_a T^a_{\mu\nu} \dot{z}^\nu. \tag{112}$$

In the case of BF theory where curvature is locally zero this equation is automatically satisfied.

We also need to describe the effect of inclusion particles on the fields of BF-theory. As we can see from equations (93) in the presence of point sources only, $P_0 = 0$, the curvature of spacetime is still zero everywhere, so unlike in three dimensional gravity inclusion of point particles doesn't change geometry of spacetime. It does however change the solution for the B-field, the equation for which is now

$$\star\, d_A B = J \int dt \delta^4(x - z(t))\, \dot{z} \tag{113}$$

The general solution for the connection is the same as in (90), while the solution for B-field can be written as

$$B = g^{-1}\left(d\phi + \tilde{B}\right)g \tag{114}$$

where $\tilde{B}$ is a particular solution of the equation (in components)

$$\epsilon^{\mu\nu\rho\sigma}\,\partial_\nu\,\tilde{B}_{\rho\sigma} = \tilde{J}\,\delta(x - z(t))\,\dot{z}^\mu, \quad \tilde{J} = gJg^{-1} \tag{115}$$

By a coordinate transformation we can always make the vector $\dot{z}^\mu$ equal $(1, 0, 0, 0)$ so that we find.

$$\epsilon^{ijk}\,\partial_i\,\tilde{B}_{jk} = \tilde{J}\,\delta^3(x - z) \tag{116}$$

The most natural form of the particular solution to this equation is the Wilson line solution. If we choose an arbitrary spacelike curve $\gamma^i(s)$, where $s \in (0, \infty)$ is a parameter of the curve, which begins at the location of the particle, $\gamma^i(0) = z^i$, and ends at spatial infinity, then the following expression will be a solution to equation (116)

$$\tilde{B}_{jk}(x) = \tilde{J}\epsilon_{ijk}\int_\gamma ds\frac{d\gamma^i}{ds}\delta^3(x - \gamma). \tag{117}$$

More generally, the solution (117) can be rewritten as

$$g^{-1}\tilde{B}_{jk}(x)g = B_{jk} = W^{-1}_{\gamma,m,s}(A)\epsilon_{ijk}\frac{\delta}{\delta A_i(x)}W_{\gamma,m,s}(A), \tag{118}$$

where

$$W_{\gamma,m,s}(A) = P\exp(\int_\gamma A_i^{(m,s)}(x(\gamma)))dx^i \tag{119}$$

is a Wilson line along γ taken in a representation labeled by mass m and spin s. This solution will be used in further constructions, in particular in taking into account the back reaction of B field on geometry of spacetime, when we incorporate the terms that turn BF-theory into General Relativity.

4 Boundary observables of General Relativity and topological field theory

4.1 Canonical General Relativity, fixed boundary

4.1.1 Boundary terms in the action

In general the variation of an action for General Relativity in a bounded region contains a surface term, which must vanish for the variational principle to be well defined. This can be achieved by imposing some boundary conditions plus adding a surface term to the action.

There are many possible choices of the surface term in the action. Different boundary conditions require different surface terms. But they do not determine them completely. The surface term also depends on the choice of canonical variables. When we perform a canonical transformation in the theory, this is equivalent to the addition of a total derivative term to the Lagrangian. This term doesn't change the equations of motion in the bulk, but it may have a nontrivial effect in the boundary. This is another piece of ambiguity in the choice of the surface term.

Here we will follow the proposal of Smolin [10, 11] relating quantum gravity in the bulk region with topological quantum field theory on the boundary. The self-dual boundary conditions considered there were shown to be satisfied on black hole horizons [36] and therefore were extremely useful for studying black hole mechanics. However they are not very suitable for our purposes. They do not allow us to define energy and momentum and as a consequence the boundary theory does not completely capture the symmetry of the vacuum.

Therefore we choose another set of boundary conditions. To have a sensible definition of energy and momentum we need to fix the metric on the boundary. We must also choose what canonical variables to use and this is motivated by our interest in the Kodama state. First, we must choose the Ashtekar variables in their original form, with the Immirzi parameter equal to i, since only for these variables the constraints take a simple form of which the Kodama state is a solution. Second, it is well known that General relativity can be obtained from breaking $SO(4,1)$-symmetry in topological field theory down to $SO(3,1)$ [20]. The resulting form of the GR action has definite implications as to what the boundary theory should be. Given that $SO(4,1)$ is the symmetry of the Kodama state it is natural to choose this form of the boundary action.

Here it is worth reviewing how a breaking of $SO(4,1)$ symmetry in TQFT leads to General Relativity, and what kind of action it results in. For convenience, we will use spinorial notation. Our starting point will be a topological field theory for $Sp(4)$ group which is locally isomorphic to $SO(4,1)$. The action depends on the

$Sp(4)$-connection A_α^β, where $\alpha, \beta = 0, ..3$.

$$S = \frac{1}{8\pi G\Lambda} \int Tr F_\beta^\alpha P_\gamma^\beta \wedge F_\delta^\gamma P_\alpha^\delta. \tag{120}$$

Here $F_\beta^\alpha = dA_\beta^\alpha + A_\gamma^\alpha \wedge A_\beta^\gamma$ is the $Sp(4)$-curvature 2-form and P_β^α is a fixed symmetry breaking 0-form matrix.

$Sp(4)$ quantities can be decomposed by using $SL(2,C)$ index notation $A = 0, 1$, $A' = 0, 1$. The $Sp(4)$ connection can be represented as

$$A_\alpha^\beta = \begin{pmatrix} A_A^B & \frac{1}{l} e_A^{B'} \\ \frac{1}{l} e_{A'}^B & A_{A'}^{B'} \end{pmatrix}, \tag{121}$$

where A_A^B and $A_{A'}^{B'}$ are left-handed and right-handed $SL(2,C)$-connections respectively, $e_{A'}^B$ is a tetrad and l is a fixed parameter having dimension of length. Similarly one can decompose $Sp(4)$-curvature

$$F_\alpha^\beta = \begin{pmatrix} F_A^B & F_A^{B'} \\ F_{A'}^B & F_{A'}^{B'} \end{pmatrix}. \tag{122}$$

Here

$$F_A^B = f_A^B + \frac{1}{l^2} e_A^{A'} \wedge e_{A'}^B, \tag{123}$$

where f_A^B is $SU(2)_L$-curvature of the connection A_A^B, and

$$F_{A'}^B = \nabla \wedge e_{A'}^B \tag{124}$$

is the torsion.

To get an action (the canonical form of which is Ashtekar's) one has to restrict to a purely self-dual $SL(2,C)$-connection, which means that we should choose P_β^α in (120) to be

$$P_\alpha^\beta = \begin{pmatrix} \delta_A^B & 0 \\ 0 & 0 \end{pmatrix}. \tag{125}$$

The resulting bulk action is then

$$S_{Bulk} = \frac{1}{8\pi G\Lambda} \int (f_A^B + \Lambda e_A^{A'} \wedge e_{A'}^B) \wedge (f_B^A + \Lambda e_B^{A'} \wedge e_{A'}^A), \tag{126}$$

where $\Lambda = \frac{1}{l^2}$ is the cosmological constant. This will be the basic bulk action for the rest of the paper.

As we mentioned before in order to be able to define energy and momentum we choose to fix the metric on the boundary

$$\delta e_{A'}^A \Big|_S = 0. \tag{127}$$

We can now calculate the variation of the action (176) subject to this condition.

$$\delta S_{Bulk} = \frac{1}{4\pi G\Lambda}\int (f_A^B + \Lambda e_A^{A'} \wedge e_{A'}^B) \wedge e_B^{B'} \wedge \delta e_{B'}^A - \frac{1}{4\pi G\Lambda}\int \nabla \wedge (f_A^B + \Lambda e_A^{A'} \wedge e_{A'}^B) \wedge \delta A_B^A$$
$$+ \frac{1}{4\pi G\Lambda}\int_S (f_A^B + \Lambda e_A^{A'} \wedge e_{A'}^B) \wedge \delta A_B^A. \quad (128)$$

For the variational principle to be well defined the boundary term in (128) has to vanish, which can be achieved by adding a boundary term to the action (126):

$$S = S_{Bulk} + S_S, \quad (129)$$

where

$$S_S = \frac{1}{6\pi G\Lambda} S_{CS}[A] + \frac{1}{4\pi G}\int_S e_A^{A'} \wedge A_B^A \wedge e_{A'}^B + S[e], \quad (130)$$

where $S_{CS}[A]$ is the Chern-Simons action of the connection A and $S[e]$ is a surface action, which depends purely on the metric. It can be checked that the the variation of the action (129) subject to the condition (127) has no surface term for arbitrary choices of $S[e]$ in (130). The later can be fixed by the requirement that the total action (129) be covariant, i.e. that the gauge and diffeomorphism invariance at the boundary be broken by the boundary conditions and not by the action itself. However diffeomorphism and gauge symmetries are partially broken due to the presence of the boundary. We can keep only invariance with respect to diffeomorphisms tangent to the boundary and with respect to rotations in the tangent space of the boundary. To be explicit let us introduce an arbitrary unit vector field on the boundary s^μ and its spinorial representation $s_{A'}^A = s^\mu e_{\mu A'}^A$, $s_{A'}^A s_B^{A'} = \delta_B^A$. By using it one can parametrize the self-dual part of the tetrad $e_{A'}^A \wedge e_B^{A'}$ by a traceless triad $\sigma_B^A = e_{A'}^A s_B^{A'} - 1/2 e_{A'}^C s_C^{A'} \delta_B^A$:

$$e_{A'}^A \wedge e_B^{A'} = \sigma_C^A \wedge \sigma_B^C + s \wedge \sigma_B^A. \quad (131)$$

The second term in the r.h.s. of (130) can then be rewritten in the form

$$\int_S e_A^{A'} \wedge A_B^A \wedge e_{A'}^B = \int_S \sigma_A^B \wedge A_C^A \wedge \sigma_B^C + \int_S s \wedge A_B^A \wedge \sigma_A^B. \quad (132)$$

The second term in the r.h.s of (132) does not admit a covariant extension and has to be removed. This can be done by choosing the vector field s on the boundary to be the unit normal to this boundary, which makes the above term disappear automatically. This means that the triad σ_B^A is chosen to be the projection of the tetrad $e_{A'}^A$ on the surface S. The remaining term in the r.h.s of (132) by a specific choice of $S[e]$ can be completed to the term with a covariant derivative of σ. The covariant form of the boundary action is thus

$$S_S = \frac{1}{6\pi G\Lambda} S_{CS}[A] + \frac{1}{4\pi G}\int_S \sigma_A^B \wedge \nabla \wedge \sigma_B^A \tag{133}$$

In [11] two possible sets of boundary conditions dual to each other were studied. One could either fix connection on the boundary $\delta A_A^B\Big|_S = 0$ or choose self-dual boundary conditions $F_A^B - \Lambda e_A^{A'} \wedge e_{A'}^B$ and leave the connection loose. Similarly, instead of fixing the metric on the boundary, the action principle (129) can be made consistent by imposing the following set of conditions

$$\nabla \wedge \sigma_B^A\Big|_S = 0 \tag{134}$$

As in the case of free varying lapse and shift functions we cannot define energy and momentum; the condition (134) must imply that energy and momentum is zero. In the next section we will see that this is indeed the case.

4.1.2 Quasilocal quantities and the algebra of boundary observables

In this section we will study the boundary theory defined by the action (133) in more detail and relate its observables with the quasilocal energy, momenta, and angular momenta of the bulk theory.

A theory of the form (133) was considered by Witten [37] along with the action of 2+1 gravity. It can be rewritten as a Chern -Simons action

$$S_S = \frac{1}{6\pi G\Lambda} S_{CS}(a) \tag{135}$$

for SO(3,1)-connection

$$a = A_i \mathcal{J}^i + \sqrt{\Lambda}\sigma_i \mathcal{P}^i, \tag{136}$$

where

$$[\mathcal{J}^i, \mathcal{J}^j] = \epsilon^{ijk}\mathcal{J}_k,\ [\mathcal{J}^i, \mathcal{P}^j] = \epsilon^{ijk}\mathcal{P}_k,\ [\mathcal{P}^i, \mathcal{P}^j] = \epsilon^{ijk}\mathcal{J}_k \tag{137}$$

are the generators of the SO(3,1) group. This means that the constraints of the theory (133) form an SO(3,1) algebra with respect to the boundary simplectic form. In this the theory is similar to 2+1 dimensional gravity. Also its constraints have the same form

$$\begin{aligned} C_B^A &= \epsilon^{\alpha\beta}(F_{\alpha\beta B}^A + \Lambda\sigma_{\alpha C}^A \sigma_{\beta B}^C) \\ H_B^A &= \epsilon^{\alpha\beta}\nabla_\alpha \sigma_{\beta B}^A, \end{aligned} \tag{138}$$

where the indices $\alpha, \beta = 1, 2$ are two-dimensional spatial manifold indices and $\epsilon^{\alpha\beta}$ is the completely antisymmetric tensor. It differs however form (2+1) gravity by the fact that the gauge and the diffeomorphism constraints have traded places. Also different are the canonical commutation relations between basic variables. Now they are

$$\begin{aligned}
\{A^A_{\alpha B}, A^C_{\beta D}\} &= 3\pi G\Lambda\epsilon_{\alpha\beta}(\delta^A_D\delta^C_B + \epsilon^{AC}\epsilon_{BD}) \\
\{\sigma^A_{\alpha B}, \sigma^C_{\beta D}\} &= 2\pi G\epsilon_{\alpha\beta}(\delta^A_D\delta^C_B + \epsilon^{AC}\epsilon_{BD}) \\
\{A^A_{\alpha B}, \sigma^C_{\beta D}\} &= 0.
\end{aligned} \tag{139}$$

It is known that the consistent quantization of the Chern-Simons theory require the quantum deformation of the gauge group. So the fact that the boundary theory is a Chern-Simons theory for SO(3,1) group with the coupling constant $\kappa = \frac{6\pi}{G\Lambda\hbar}$ means that the symmetry group of the vacuum is now q-deformed with $q = \exp(\frac{1}{\kappa})$. This means that particles inserted in punctures of the boundary theory will propagate in q-deformed spacetime.

In the rest of this section we will relate the constraints (138) with quasilocal observables of the bulk theory. This relation will involve projection of spinors on surfaces which may be spacelike or timelike. For this some useful notations are introduced below.

Let Σ be an arbitrary surface, which may be spacelike or timelike. Let n_a be the unit normal vector to this surface, $n_a n^a = \pm 1$, and $n^A_{A'} = n^a e^A_{aA'}$ its spinorial representation. $n^A_{A'}$ can be considered as an Hermitian metric for spinors on Σ, which allows us to introduce an operation of Hermitian conjugation for such spinors

$$\mu^\dagger_A = n^{A'}_A \bar{\mu}_{A'}, \tag{140}$$

where bar means complex conjugation. This operation is involutive $(\mu^\dagger_A)^\dagger = \pm\mu_A$, where + stands for a timelike surface and - for a spacelike one.

The operation of Hermition conjugation allows one to define a new type of connection on the surface Σ. In four dimensions the only relevant completely covariant connection is the torsion free one. We will denote it simply by ∇:

$$\nabla \wedge e^{A'}_A = de^{A'}_A + A^{-B}_A \wedge e^{A'}_B - e^{B'}_A \wedge A^{+A'}_{B'} = 0. \tag{141}$$

Here A^{-B}_A and $A^{+A'}_{B'}$ are anti-self-dual and self-dual parts of the torsion-free connection which act on unprimed (left-handed) and primed (right-handed) spinors respectively. They are related to each other by complex conjugation:

$$A^{+A'}_{B'} = \bar{A}^{-A'}_{B'}. \tag{142}$$

Along with the torsion-free covariant derivative one can define define purely anti-self-dual and purely self-dual covariant derivatives. Let $\mu^{AA'}$ be an arbitrary spinor with one primed and one unprimed indices. Then we define

$$\begin{aligned}
\nabla^-_a \mu^{AA'} &= \partial_a \mu^{AA'} + A^{-A}_{aB} \mu^{BA'} \\
\nabla^+_a \mu^{AA'} &= \partial_a \mu^{AA'} + A^{+A'}_{aB'} \mu^{AB'}.
\end{aligned} \tag{143}$$

These "covariant" derivatives are not completely covariant. The first of them restricts the gauge covariance to anti-self dual transformations and the second to self-dual. However they may give rise to fully covariant derivatives when projected on the surface.

Below, unless otherwise stated, Σ is a spacelike slice of spacetime, $n_{A'B}$ is the spinorial representation of timelike unit normal vector and $e^A_B = e^{AA'} n_{A'B} - \frac{1}{2} e^{CA'} n_{A'C} \delta^A_B$ is the triad on Σ.

Let us first show that the torsion of the (anti)self-dual connection defined by (143) projected on a timelike surface S is ADM energy-momentum (generally S is supposed to be taken to infinity although it doesn't necessarily have to). Let us consider the second term on the r.h.s of (130), which is the only term dependent on metric needed for consistency of the action principle if we don't care about covariance. If we take a spatial slice Σ and make a 3+1 decomposition this action will read

$$S_S = ... + \int dt \int e^{A'}_{tA} \wedge A^A_B \wedge e^B_{A'}. \tag{144}$$

The ADM energy and momentum are the coefficients in front of lapse and shift functions in the above integral, and given that $e^{A'}_{tA} n^{A'}_A = N$ and $e^{A'}_{tA} e^{A'}_{iA} = N_i$ they are

$$\begin{aligned}
E_{ADM} &= \frac{1}{4\pi G} \int_{S\cap\Sigma} e^A_B \wedge A^B_A \\
(P_{ADM})^A_B &= \frac{1}{4\pi G} \int_{S\cap\Sigma} (e^A_C \wedge A^C_B - \frac{1}{2} e^C_D \wedge A^D_C \delta^A_B),
\end{aligned} \tag{145}$$

where the symmetric pare of spinorial indices A, B labels 3 spacelike directions. The same expressions written in Ashtekar canonical variables can be found e.g. in [38]. Now taking into account that the total connection is torsion-free it is easy to see that the expressions entering the first and the second integral in (145) are components of the torsion of the connection (143), $\nabla^+ \wedge e^{A'}_A$, projected on $n^{A'}_A$ and orthogonal to $n^{A'}_A$ respectively.

The above expressions for energy and momenta are simple and have been proved to be zero in the vacuum, however they are not covariant and do not form any algebraic structure from the point of view of boundary theory. Below we consider covariant expressions for energy and momenta given by the boundary constraints (138).

First, let us notice that the constraints from the first line in (138) define the quasilocal angular momenta of the bulk theory.

$$J^A_B = \frac{1}{4\pi G}\int_{\Sigma\cap S}\left(\frac{1}{\Lambda}F^A_B + \sigma^A_C \wedge \sigma^C_B\right) \tag{146}$$

Indeed C^A_B in (138) are boundary terms resulting from the variation of the Gaussian constraints of the bulk theory, generating local Lorentz transformations. One can fix a tetrad on the boundary so that it includes a gauge condition aligning the intrinsic Lorentz frame with the global basis of boundary spacetime. So the intrinsic Lorentz transformations are identified with global ones and the operator generating them becomes the angular momentum of the theory.

The rest of the constraints (138) are components of the torsion of the purely self-dual connection, the triad σ^A_B which is the tetrad $e^{AA'}$ projected on S. The connection entering this torsion can be equally understood as the four-dimensional torsion-free connection:

$$\nabla^- \wedge \sigma^A_B = \nabla \wedge \sigma^A_B. \tag{147}$$

From the fact that the four-dimensional covariant derivative annihilates the spacetime tetrad $e^{AA'}$ it follows that only the derivative of the normal vector $n_{A'B}$ contributes to the torsion (147). This means that the torsion(147) is related to the extrinsic curvature of Σ.

$$\begin{aligned}\nabla \wedge \sigma^A_B &= \nabla(n_{A'B}) \wedge e^{AA'} = (dn_{A'B} + A^{-C}_B n_{A'C} - A^{+C'}_{A'} n_{C'B}) n^{A'}_D \wedge \sigma^{AD} \\ &= (A^{+A}_C - A^{-A}_C) \wedge \sigma^C_B\end{aligned} \tag{148}$$

In the last line we introduced a self-dual connection acting on left-handed spinors

$$A^{+A}_B = n^A_{A'} dn^{A'}_B - n^A_{B'} A^{+B'}_{A'} n^{A'}_C = n^A_{A'} \nabla^+ n^{A'}_B \tag{149}$$

There are some simple relations between self-dual and anti-self-dual connections acting on spinors of the same chirality. First, it follows from (148) that

$$A^{+A}_B - A^{-A}_C = iK^A_B, \tag{150}$$

where K^A_B is the tensor of extrinsic curvature of Σ. On the other hand the usual reality condition for Ashtekar variables means that

$$A^{+A}_B + A^{-A}_C = \Gamma^A_B, \tag{151}$$

where Γ^A_B is the connection which is torsion-free on Σ:

$$\nabla_\Gamma \wedge \sigma^A_B = 0. \tag{152}$$

Similarly one can introduce an anti-self-dual connection acting on right-handed spinors

$$A^{-A'}_{B'} = n^{A'}_{A} \nabla^{-} n^{A}_{B'}. \tag{153}$$

Now we should relate H^A_B in (138) to energy and momenta. Let $\tau^B_{\mu A}$, $\mu = 0, 1, 2$ be 3 generator of SL(2,R) group, which is the restriction of SL(2,C) on timelike slice $\mu = 0$, corresponding to rotation, and $\mu = 1, 2$ corresponding to boosts. By using (148) and (150) it is easy to show that

$$\begin{aligned} i\tau^B_{0A} * (\nabla \wedge \sigma^A_B)\big|_{\Sigma\cap S} &= \tau^B_{0A} * (K^A_C \wedge \sigma^C_B)\big|_{\Sigma\cap S} = det(\sigma\big|_{\Sigma\cap S}) K^\alpha_\alpha \\ i\tau^B_{iA} * (\nabla \wedge \sigma^A_B)\big|_{\Sigma\cap S} &= \tau^B_{iA} * (K^A_C \wedge \sigma^C_B)\big|_{\Sigma\cap S} = det(\sigma\big|_{\Sigma\cap S}) K^i_t, \end{aligned} \tag{154}$$

where * denotes hodge dual with respect to the volume form on $\Sigma \cap S$. In the r.h.s. of the first line of (155) we recognize the density which when integrated over $\Sigma \cap S$ gives rise to the Brown-York quasilocal energy [39] and in the second line we find quasilocal momentum. Thus we can write down the relation between the constraints H^A_B from (138) and the quasilocal energy-momentum as follows

$$\begin{aligned} E &= i\tau^B_{0A} \frac{1}{4\pi G} \int \nabla \wedge \sigma^A_B, \\ P_i &= i\tau^B_{iA} \frac{1}{4\pi G} \int \nabla \wedge \sigma^A_B. \end{aligned} \tag{155}$$

The expressions are not simply related to the basic canonical variables of the bulk theory and they do not vanish in the vacuum. However they can be related to simple ADM expressions (145) by using a reference spacetime [39]

$$\begin{aligned} E_{ADM} &= E - E_{ref} \\ (P_{ADM})^A_B \tau^B_{iA} &= P_i - (P_i)_{ref}, \end{aligned} \tag{156}$$

where the subscript ref means calculated in a reference spacetime. This simple form can be restored at the cost of covariance. The advantage of this form is that quantities with subscript ref are non-dynamical (they are c-numbers), and we can calculate bulk commutators of quasilocal quantities by using simple ADM expressions. A more detailed description of Brown-York energy in Ashtekar variables can be found in the paper of Lau [40].

4.1.3 Einstein's equations as a local conservation law

It is interesting to notice that the torsion of the connection (143) with the spacetime tetrad

$$T^{A'}_{A} = \nabla^{-} \wedge e^{A'}_{A} \tag{157}$$

(the projection of which on a boundary defines the ADM energy) is a locally conserved quantity in a covariant sense. Indeed the covariant divergence of $T^{A'}_A$ vanishes due to Einstein's equations:

$$\nabla \wedge T^{A'}_A = \nabla \wedge \nabla^- \wedge e^{A'}_A = F^{-B}_A \wedge e^{A'}_B = 0. \tag{158}$$

Here F^{-B}_A is the curvature of the self-dual connection. The quantities C^A_B from (138) (which define quasilocal angular momentum when projected on the boundary) are also locally conserved due to the total connection being torsion-free:

$$\nabla \wedge C^A_B = \nabla \wedge \left(\frac{1}{\Lambda} F^A_B + \sigma^A_C \wedge \sigma^C_B\right) = \nabla \wedge (e^A_{A'} \wedge e^{A'}_B) = 0. \tag{159}$$

From (158,159) it follows that the complete set of equations of GR is simply equivalent to the condition of conservation of $T^{A'}_A$ and C^A_B. Therefore all of Einstein's equations can be put in the form of a local conservation law.

One of the conserved quantities, $T^{A'}_A$, is not a tensor (it does not transform covariantly with respect to right-handed gauge transformations). This is like reexpressing the divergence-free condition of the stress-energy tensor as a genuine conservation law for some pseudotensor. In the present situation we can however rewrite all the equations as the conservation of covariant quantities. This can be done on an arbitrary slice of spacetime. To each such slice one can associate a triad σ^A_B which is the projection of the tetrad $e^A_{A'}$ on it. The torsion of σ^A_B, $H^A_B = \nabla \wedge \sigma^A_B$, is a covariant quantity. In particular, if the slice is timelike this is a constraint (138) of the boundary theory (133). The covariant divergence of H^A_B is equivalent to a subset of Einstein's equations:

$$\nabla \wedge H^A_B = \nabla \wedge \nabla \wedge \sigma^A_B = F^A_C \wedge \sigma^C_B = 0 \tag{160}$$

These equations are not necessarily defined on a single slice. One can consider a one parameter family of slices or foliation of the whole spacetime to define it everywhere. One foliation is however not enough to recover the whole set of Einstein's equations (equations with components normal to the foliation are still missing). At least two foliations which are different everywhere are required.

Equations (158,159) pulled back on a spatial slice Σ form the complete set of constraints of GR. Formally all the constraints are now Gaussian – they say that the electric field C^A_B and some other field $T^{A'}_A$ are divergence-free. Quantum-mechanically this form of constraints is very difficult to treat because of complicated dependence of H^A_B and even $T^{A'}_A$ on the basic canonical variables. However it provides us with some intuition about what generic solutions of the bulk theory must look like. If we introduce a perturbation on the boundary that has a certain energy and momentum it has to be continued into the bulk as a certain flux of energy-momentum. Constraint equations simply mean that the lines of such fluxes must be continuous.

4.2 BF theory, movable boundary

In the previous section we propose a specific way of describing propagating modes of non-perturbative General Relativity. They can be defined on a surface on which certain boundary conditions are fixed and represented as local charges of a topological field theory described on this surface.

However this definition has a certain drawback. The gauge group of a resulting topological field theory is $SO(3,1)$, which is only a subgroup of the symmetry needed to describe all possible propagations. This is a consequence of fixing the boundary conditions on a certain surface. This surface cannot be moved, as fixing boundary conditions on a different surface would mean changing the physical system under consideration. As a result, topological field theory defined on this surface cannot incorporate a symmetry with respect to translations and rotations transverse to it.

To be able to describe excitations propagating in all possible directions one needs either to define the excitations in the bulk of spacetime or define them on a boundary that is movable in a transverse direction.

The first possibility is often impossible in a gravitational context. This is because the conserved charges describing propagation of particles, such as mass and spin, cannot be defined in the bulk. An exception is a BF theory of the type (12), where charges can be defined in the bulk as in section 3.3 due to the local flatness condition (87).

In this section we consider BF theory with a cosmological term. As we will see in section 5 this term is necessary to regulate a partition function in a physical way. The action of such BF theory reads

$$S = \int_M (B^{IJ} \wedge F_{IJ} + \frac{\beta}{2} B^{IJ} \wedge B_{IJ}), \tag{161}$$

and its equations of motion are

$$\begin{aligned} F_{IJ} + \beta B_{IJ} &= 0 \\ d_A B^{IJ} &= 0. \end{aligned} \tag{162}$$

Notice that due to the Bianchi identity the first equation (162) implies the second. One can say that the β-term in (161) can mimic the back reaction of the B-field on geometry, which occurs in General Relativity, but in a very simplified way.

If we want to incorporate point charges representing propagating particles into this theory, we will immediately find that this is impossible in the bulk of spacetime. The reason behind this is the relation between the two equations (162). When we introduce a point extrinsic charge to the action (161)

$$S_{particle} = S + \int dt \mathrm{tr}(J_0 A_t) \tag{163}$$

the equations (162) become

$$\begin{aligned} F_{IJ} + \beta B_{IJ} &= 0 \\ d_A B^{IJ} &= J_0^{IJ} \delta(x - z(t)). \end{aligned} \tag{164}$$

From this two equations and the Bianchi identity it follows that $J_0 = 0$, so a non-trivial charge cannot be introduced.

There is however a possibility to put a point charge on a boundary of the system. If we consider a variation of the action (161) in the presence of a boundary there will be an additional contribution on the boundary

$$\delta S = \delta S_{bulk} + \int_{\partial M} B_{IJ} \delta A^{IJ}. \tag{165}$$

Now if the worldline of the particle in (163) completely belongs to the boundary of M then instead of equations (164) we have

$$\begin{aligned} F_{IJ} + \beta B_{IJ} &= 0 \\ d_A B^{IJ} &= 0 \\ B^{IJ}\Big|_{\partial M} &= J_0^{IJ} \delta(x - z(t)). \end{aligned} \tag{166}$$

This is equivalent to equation

$$\frac{1}{\beta} F^{IJ} = J_0^{IJ} \delta(x - z(t)). \tag{167}$$

on the boundary. This equation doesn't force the charge J_0 to be zero and so allows us to introduce a particle.

One can see that equation (167) is the equation of $SO(4,1)$ Chern-Simons theory with extrinsic charges. This is a consequence of the equivalence of the theory (161) with a Chern-Simons thery. This Chern-Simons theory could be used to cancel the boundary contribution to the variation (165) similar to what was done in section 4.1, thus making the bulk theory self-contained. Indeed, if we modify the action (161) by adding a Chern-Simons term

$$S' = S + \frac{1}{\beta} \int_{\partial M} Y_{CS}(A) \tag{168}$$

Then the boundary contribution to the variation of this action vanishes identically due to the first of the bulk equations of motion (166), and the theory becomes a purely bulk theory. Notice that we didn't have to impose any condition on the field on the boundary, it is enough that the bulk equations of motion are satisfied. This means that the boundary could be moved arbitrarily without changing the properties of the physical system considered. The sources on the surface can now be considered as the sources for the Chern-Simons theory for $SO(4,1)$ gauge group.

The main difference from the result of section 4.1 is that the possible motion of the particles is no longer constrained to any subspace in the deSitter spacetime; they can move in all directions. This is seen from the fact that the gauge group now is $SO(4,1)$, which describes all the possible motions of de Sitter spacetime.

This is possible because the surface on which the Chern-Simons theory is defined is no longer fixed. One can picture it the following way. Let us introduce coordinates $\{x_i\}$ parameterizing the deSitter spacetime. We can choose one of these coordinates x_1 and define a one parameter family of surfaces Σ_{x_1} which slices the deSitter spacetime across this coordinate. On these surfaces we can define a family of Chern-Simons theories with extrinsic $SO(4,1)$ charges

$$J = g_{x_1}^{-1} J_0 g_{x_1}, \tag{169}$$

where J_0 is a charge defined on some selected reference surface, for example $x_1 = 0$, and

$$g_{x_1} = e^{x_1 T^{15}}, \tag{170}$$

where T^{15} is the translation generator in x_1 direction. Clearly all such Chern-Simons theories are equivalent and can be identified, because the transformations of the charge (169) are included in the gauge transformations of the theory.

So all the motions in the four dimensional deSitter spacetime can be described by three dimensional Chern-Simons theory. It is well known that a consistent quantization of the Cern-Simons theory requires the quantum deformation of the gauge group. So for $\beta \neq 0$ the propagation of particles in BF-theory will be described not by Special Relativity, but by Doubly Special Relativity [45], with the deformation parameter related to β. What role is played by β in quantum General Relativity will be clear in the next section.

5 Perturbative General Relativity from topological field theory

In a famous paper [8] Witten has shown that in 2+1 dimensional gravity, if we don't do the expansion (1), if we treat the whole geometry quantum mechanically, thus keeping the theory generally covariant, we can avoid the ultraviolet problem. One can even show that the theory exactly soluble.

The natural question that arises, which is also addressed in [8], is whether we can do the same in 3+1 dimensions. The immediate problem then is the following. If we look at the action of 2+1 dimensional General Relativity in the triad-Palatini representation

$$S = \int \mathrm{tr}(e \wedge dA + e \wedge A \wedge A) \tag{171}$$

we see that the lowest order term in it is quadratic. Thus the theory is nearly linear and we can apply standard quantum field theory techniques to it. On the other hand the action of 3+1 dimensional General Relativity in the tetrad-Palatini representation looks like

$$S = \int \mathrm{tr}(e \wedge e \wedge dA + e \wedge e \wedge A \wedge A). \tag{172}$$

The lowest order term in it is cubic. Standard quantum field theory techniques are not applicable anymore. The only way out seems to be to get a quadratic term in the action via the expansion (1), which leads to a non-renormalizible theory. The conclusion of [8] is that 3+1 dimensional General Relativity is non-renormalizible because it is too non-linear.

One of the questions that we address in this paper is: how non-linear is 3+1 dimensional General Relativity?

Below, to avoid complications with using a non-compact group we will consider Euclidian gravity with positive cosmological constant.

5.1 MacDowell-Mansouri type BF-theory: How non-linear is 4 dimensional General Relativity?

There are several formulations of 4 dimensional general relativity known which do contain a quadratic term in the action [46]. They are based on BF-theory plus a term which breaks topological symmetry. A well known example is the so called Plebanski action[3]:

$$S = \int (B^{\mu\nu} \wedge F_{\mu\nu}(\omega) + \phi^{\mu\nu\alpha\beta} B_{\mu\nu} \wedge B_{\alpha\beta}) \tag{173}$$

[3] A similar formulation of gravity works in any dimension [47]

The first term in (173) is a BF-theory, which is an exactly soluble theory. One could think that we can use it as a free field theory and treat the remaining term as a perturbation. There is a problem with such a perturbation theory. This is the fact that the second term actually imposes some constraints on $B^{\mu\nu}$, as $\phi^{\mu\nu\alpha\beta}$ is a Lagrangian multiplier. In a path integral this term becomes a delta function of the constraints, and to treat it as a perturbation we would have to expand a delta function in a power series around zero. But such an expansion doesn't exist. What we would need for a perturbation theory is an action principle in which General Relativity would be represented as an exactly soluble theory plus a regular interaction term. Such a formulation does exist and this is the McDowell-Mansouri formulation of General Relativity [20] rewritten as a BF-theory. This kind of action principle was also considered in [48].

Our starting point will be the BF-theory for the $SO(5)$ group. Let $T^{IJ} = -T^{JI}$ be ten generators of $so(5)$ Lie algebra, where $I, J = 1, ..., 5$ (see appendix A for our conventions and physical interpretation of other gauge groups). The basic dynamical variables are an $so(5)$-connection A^{IJ} and an $so(5)$-valued 2-form field B^{IJ}. The action principle is then

$$S = \int B^{IJ} \wedge F_{IJ}. \tag{174}$$

Here $F_{IJ} = dA_{IJ} + A_I^K \wedge A_{KJ}$ is the $so(5)$ curvature.

The equations of motion following from the action (174)

$$\begin{aligned} F_{IJ} &= 0 \\ d_A B_{IJ} &= 0 \end{aligned} \tag{175}$$

mean that the connection A^{IJ} is flat.

Now the statement is that if we break the $SO(5)$ symmetry in the theory (174) down to $SO(4)$ we get the action of General Relativity.

We add an extra term to the action (174) which depends only on B-field and contains a fixed $SO(5)$ vector v^A pointing in some preferred direction.

$$S_1 = \int (B^{IJ} \wedge F_{IJ} - \frac{1}{2} B^{IJ} \wedge B^{KL} \epsilon_{IJKLM} v^M). \tag{176}$$

The $SO(5)$ symmetry is not a symmetry of the action (176). It is broken down to $SO(4)$, the subgroup of $SO(5)$ rotations that leave v^I immovable. For simplicity we choose $v^I = (0, 0, 0, 0, \alpha/2)$, where α is a fixed dimensionless constant. The action (176) then becomes

$$S_1 = \int (B^{IJ} \wedge F_{IJ} - \frac{\alpha}{4} B_{IJ} \wedge B_{KL} \epsilon^{IJKL5}), \tag{177}$$

To show that (177) is the action of General Relativity we introduce the following notation for $4+1$-decomposition. Let $i, j = 1, 2, ..4$ be four dimensional indices such that $\epsilon^{ijkl} = \epsilon^{ijkl5}$. Then we can introduce an $so(4)$-connection $\omega^{ij} = A^{ij}$ and its curvature $R^{ij}(\omega) = d\omega^{ij} + \omega^i_k \wedge \omega^{kj}$. Also, we can introduce a frame field $e^i = lA^{i5}$, where l is a constant with dimensions of length, giving rise to a four-dimensional metric $g_{\mu\nu} = e^i_\mu e_{\nu i}$.

In the above notation we have the following decomposition of $so(5)$-curvature:

$$F^{ij}(A) = R^{ij}(\omega) - \frac{1}{l^2} e^i \wedge e^j$$
$$F^{i5}(A) = \frac{1}{l} d_\omega e^i. \tag{178}$$

The equations of motion of (177) for B^{5i} impose the torsion to vanish $d_\omega e^i = 0$. This determines uniquely the connection ω to be the spin connection. Since the action is quadratic in the fields B^{ij} we can solve the equations of motion for B^{ij} and substituting them back into action, we find

$$S_1 = \frac{1}{4\alpha} \int F^{ij} \wedge F^{kl} \epsilon_{ijkl}, \tag{179}$$

where we used the notations introduced above. Finally, using (178) one can rewrite (179) as

$$\begin{aligned} S_1 &= \frac{1}{4\alpha} \int (R^{ij} - \frac{1}{l^2} e^i \wedge e^j) \wedge (R^{kl} - \frac{1}{l^2} e^k \wedge e^l) \epsilon_{ijkl} \\ &= S_P + \frac{1}{4\alpha} \int R^{ij}(\omega) \wedge R^{kl}(\omega) \epsilon_{ijkl} \end{aligned} \tag{180}$$

Here

$$S_P = -\frac{1}{2G} \int (R^{ij}(\omega) \wedge e^k \wedge e^l - \frac{\Lambda}{6} e^i \wedge e^j \wedge e^k \wedge e^l) \epsilon_{ijkl} \tag{181}$$

is the Palatini action[4] of General Relativity with nonzero cosmological constant. The role of the Newton constant[5] is played by $G = \alpha l^2$ and the cosmological constant is $\Lambda = 3/l^2$. The constant α in (179) is the square of the ratio of the Planck length over the cosmological radius, $\alpha = G\Lambda/3 \sim 10^{-120}$. It is dimensionless and extremely small which makes it a good parameter for perturbative expansion.

The second term in the r.h.s. of (180) is the integral of the Euler class. It is topological and its variation vanishes identically due to Bianchi identity. Thus the action (176) indeed describes General Relativity.

[4]The normalizations are such that when written in the metric variables the Palatini action is of the usual form

$$S_P = -\frac{1}{G} \int \sqrt{g}(R - 2\Lambda), \tag{182}$$

R being the scalar curvature.

[5]We work in units where c and $16\pi\hbar = 1$, so G means $16\pi G\hbar = l_p$ which is the Planck length.

The main result of this section is that General Relativity in four dimensions admits an action principle (177) that is just slightly non-linear, exactly as non-linear as that of 3 dimensional gravity. Also, (177) has a form of exactly soluble theory plus a small correction. The correction is so small that even if we neglect it we should give a good approximation to the observed reality. And indeed we do, because the equations of motion in this case are (175) which in the case of $SO(4,1)$ gauge group have the only solution which is the deSitter spacetime, which is very close to what we see. This formulation of gravity is strikingly similar to a formulation of QCD in terms of the Lagrangian $\mathrm{tr}(B \wedge F + g^2 B \wedge \star B)$. The difference comes from having a quadratic form contracting the B fields that is strictly positive and background dependent.

Despite the smallness of α, however, there are many situations in which the second term in the action (177) leads to noticeable effects. This happens when some of the components of B-field are large. Then the second term in (177) which is quadratic in B cannot be neglected as compared to the first term which is linear in B, even though multiplied by a tiny constant. In classical theory the B field becomes large, for example, when we couple gravity to massive matter sources [49].

In quantum theory we have to take into account large fluctuations of B-field, thus including the regime in which the theory becomes strongly coupled. This may lead to a breakdown of perturbation theory. Most visible is the contribution from the components of B that form the orbit of the translational gauge group of the free field theory, $B = d_A \phi$, which is broken by the interaction term. To avoid this problem we need to find a way to suppress large fluctuations of the B-field in a path integral. This can be done by a very natural modification of the action principle considered in the next section.

5.1.1 Introducing the Immirzi Parameter

In the previous section we have described gravity in terms of a symmetry breaking perturbation of topological BF theory. In this section we generalize this construction to the case where the topological field theory is BF with a 'cosmological term'. As we will see this is necessary in order to regulate in a physical way our perturbative expansion. We will also see that at the classical level this allows us to introduce naturally another dimensionless parameter that appears in $4D$ gravity, the so called Barbero-Immirzi parameter.

The action principle for $SO(5)$ BF theory with a cosmological term is [50]

$$S = \int B^{IJ} \wedge F_{IJ} - \frac{\beta}{2} B^{IJ} \wedge B_{IJ}. \tag{183}$$

The equations of motion following from this action are

$$\begin{aligned} F_{IJ} &= \beta B_{IJ}, \\ d_A B_{IJ} &= 0. \end{aligned} \tag{184}$$

Note that the first equation implies the second one. This theory is invariant under local $SO(5)$ transformations, it is topological due to the additional 'translational' symmetry labeled by a one form Φ^{IJ} valued in the Lie algebra[6]

$$\delta A^{IJ} = \beta \Phi^{IJ}, \qquad \delta B^{IJ} = d_A \Phi^{IJ}. \tag{187}$$

The gauge invariant observables of this theory are therefore gauge invariant functions of $B^{IJ} - F^{IJ}/\beta$. As before, we add an extra term to the action that breaks the gauge symmetry down to $SO(4)$ and also breaks translational symmetry. Our proposal for a gravity action is

$$S_2 = \int (B^{IJ} \wedge F_{IJ} - \frac{\beta}{2} B^{IJ} \wedge B_{IJ} - \frac{\alpha}{4} B_{IJ} \wedge B_{KL} \epsilon^{IJKL5}). \tag{188}$$

We can solve the equations of motion[7] for B^{IJ}

$$B^{ij} = \frac{1}{\alpha^2 - \beta^2}(\frac{\alpha}{2}\epsilon^{ijkl} F_{kl} - \beta F^{ij}), \tag{189}$$

$$B^{5i} = \frac{1}{\beta} F^{5i}, \tag{190}$$

and substituting them back into the action (188), we get

$$S_2 = \int \left(\frac{\alpha}{4(\alpha^2 - \beta^2)} F^{ij} \wedge F^{kl} \epsilon_{ijkl} - \frac{\beta}{2(\alpha^2 - \beta^2)} F^{ij} \wedge F_{ij} + \frac{1}{\beta} F^{5i} \wedge F_{5i} \right). \tag{191}$$

Using (178) and introducing the Nieh-Yan class $C = d_\omega e^i \wedge d_\omega e_i - R^{ij} \wedge e_i \wedge e_j$ [51], we can rewrite this action in terms of gravity variables

$$S_2 = \widetilde{S}_P + \int \left(\frac{\alpha}{4(\alpha^2 - \beta^2)} R^{ij}(\omega) \wedge R^{kl}(\omega) \epsilon_{ijkl} - \frac{\beta}{2(\alpha^2 - \beta^2)} R^{ij}(\omega) \wedge R_{ij}(\omega) + \frac{1}{\beta} C \right). \tag{192}$$

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[6] The non linear transformations corresponding to this infinitesimal symmetry are given by

$$A \rightarrow A + \beta\phi, \tag{185}$$

$$B \rightarrow B + d_A \phi + \frac{\beta}{2}[\phi, \phi] = B + \frac{F(A + \beta\phi) - F(A)}{\beta}. \tag{186}$$

[7] We restrict to the case $\alpha^2 \neq \beta^2$. Considering this case will lead to a self dual formulation of gravity.

The last term is an integral of a linear combination of the Euler class, the Pontryagin class and the Nieh-Yan class. These are integer valued topological invariants with trivial local variation. The first term of action (192)

$$\widetilde{S}_P = -\frac{1}{2G}\int \left(R^{ij}(\omega)\wedge e^k \wedge e^l \epsilon_{ijkl} - \frac{\Lambda}{6} e^i \wedge e^j \wedge e^k \wedge e^l \epsilon_{ijkl} - \frac{2}{\gamma} R^{ij}(\omega) \wedge e_i \wedge e_j \right) \tag{193}$$

is the Cartan-Weyl action of General Relativity with nonzero cosmological constant and a nonzero Immirzi parameter γ, which is dimensionless [52]. The initial parameters α, β, l are related to the physical parameters as follows

$$\frac{1}{l^2} = \frac{\Lambda}{3}, \qquad \alpha = \frac{G\Lambda}{3(1-\gamma^2)}, \qquad \beta = \frac{\gamma G\Lambda}{3(1-\gamma^2)}. \tag{194}$$

This are not all the terms that could be included in the action: the constants in front of topological terms could be varied independently. However topological terms do not affect the dynamics of the local degrees of freedom of the theory, so we do not need a control over them. The term proportional to γ is not topological (its variation is non zero), it doesn't affect the classical equation of motion when $\gamma^2 \neq 1$. It plays no role in the classical theory of gravity and it is therefore not constrained experimentally. It is important to note however that this term, similarly to the theta term in non abelian gauge theory, breaks CP symmetry. Since this fact seems to have been unnoticed let us explain it in more detail. Suppose that we perform an orientation reversing diffeomorphism of our spacetime, lets call it a C-transformation. All the terms in the action change signs since they are 4-forms, so C is not a symmetry of our gravity action. Lets now consider the discrete Lorentz transformations $g_{ij} = \mathrm{diag}(-+++)$ or $g_{ij} = \mathrm{diag}(+---)$, that we respectively called T or P transformation. The T transformation changes only the sign of e^0 and ω^{0i} leaving all the other fields invariant. The first two terms in the action change sign under P or T since they contain one epsilon tensor contracting the Lorentz indices but the last term does not. The action is not invariant under P or T but if we now consider CP (or CT) we see that the first two terms in the action are left invariant whereas the last one changes sign. In other word CP does not affect G or Λ but changes the sign of the Immirzi parameter. The CP symmetry is therefore realized only if $\gamma = 0$ or ∞. When $\gamma = 0$, which is the case studied previously, we recover the case of usual metric gravity, where the torsion is identically 0. When $\gamma = \infty$ we recover the usual Cartan-Weyl gravity where the torsion is free to fluctuate. Any other value of γ leads to a CP violation mechanism in quantum gravity that is worth exploring.

Even if it doesn't affect the classical theory the Immirzi parameter deeply affects the quantum theory and labels inequivalent quantizations in the context of kinematical loop quantum gravity. Indeed it is known for a long time (see [53] for a review) that this parameter modifies the symplectic structure and this modification is not unitarily implementable at the quantum level. It affects the prediction of the spectra of geometrical operators and plays a key role in the black hole entropy calculation [54]. This calculation suggests that γ and $1-\gamma^2$ are of order unity. One should keep in mind however, that the above conclusions are based on kinematical considerations, i.e. before Hamiltonian constraint is applied. And one open problem in this context is to understand whether the Immirzi parameter really leads to inequivalent quantization once the dynamics is fully taken into account or whether it can just be reabsorbed into a redefinition of the Newton constant. This point has already been raised at the kinematical level in [55] where a seemingly more covariant approach to loop gravity leads to a geometrical spectra independent of the Immirzi parameter.

A more direct way to understand why the Immirzi parameter should affect quantization is to notice that $2/\gamma$ is proportional to the torsion square since $\int R^{ij}\wedge e_i \wedge e_j = \int d_\omega e^i \wedge d_\omega e_i$ up to a boundary term. γ therefore controls the width of fluctuation of the torsion at the quantum level. We have already remarked that if $\gamma = 0$, which is the case of metric gravity studied in the previous section, the torsion is not allowed to fluctuate. The mean value of the torsion is always equal to 0 irrespective of γ; this is why it doesn't affect classical gravity. However a naive semiclassical calculation shows that one should expect the two point function of the torsion to be proportional to γ. Therefore γ controls how strongly we suppress (or not) the torsion fluctuations in the path integral.

In our context the Immirzi parameter appears to act as a physical regulator. The role of the Immirzi parameter in this theory and its relevance to the physical predictions will be explored in more detail in our next paper [56].

There is not yet any preferred experimental value for γ, whether it is 0, ∞ or the value suggested by loop quantum gravity. Anyway, in all these cases[8] α and β are at most of the order $G\Lambda$, which is a tiny number.

5.2 Formal setup for perturbation theory

We first concentrate on the case $\beta = 0$. Let us rewrite the action (177) in an index free form

$$S_{GR} = \int \mathrm{tr}(B\wedge F(A) - \frac{\alpha}{4} B\wedge B\gamma_5). \tag{195}$$

Here $B = B^{IJ}T_{IJ}/2$ and $A = A^{IJ}T_{IJ}/2$, where

$$T_{IJ} = \frac{1}{4}[\gamma_I, \gamma_J] \tag{196}$$

[8] if $\gamma \to \infty$ both α and β are sent to zero while the ratio β^2/α tends to a finite value $G\Lambda/3$

are $so(5)$-generators in the fundamental representation and γ_I are γ-matrices satisfying $\{\gamma_I, \gamma_J\} = 2\delta_{IJ}$. The insertion of γ_5 in the second term of (195) breaks $SO(5)$ symmetry down to $SO(4)$.

We will be calculating the path integral for the action (195)

$$Z_{GR} = \int \mathcal{D}A\mathcal{D}B \exp(iS_{GR}). \tag{197}$$

Following [57] we will treat the BF term in (195) as free field theory and the second term as a perturbation. Define the generating functional which is the path integral for the BF theory with an extrinsic source as

$$Z(J) = \int \mathcal{D}A\mathcal{D}B \exp\Big(i \int \mathrm{tr}(B \wedge F(A) - B \wedge J)\Big), \tag{198}$$

where J is an $so(5)$-valued 2-form field. Then the path integral for General Relativity can be obtained by including the interaction by differentiating with respect to the sources.

$$Z_{GR} = \exp\Big(i \int \mathrm{tr}(\frac{\alpha}{4}\frac{\delta}{\delta J} \wedge \frac{\delta}{\delta J}\gamma_5)\Big) Z(J)\Big|_{J=0}. \tag{199}$$

The perturbation theory can be obtained by expanding the exponent in (199) in a power series

$$Z_{GR} = \sum_n \frac{1}{n!}\Big(i \int \mathrm{tr}(\frac{\alpha}{4}\frac{\delta}{\delta J} \wedge \frac{\delta}{\delta J}\gamma_5)\Big)^n Z(J)\Big|_{J=0}. \tag{200}$$

As $\alpha \ll 1$ we expect the sum to be dominated by the lowest order terms.

5.2.1 Computing the generating functional

We know show that the generating BF functional can be exactly evaluated. We start with

$$Z(J) = \int \mathcal{D}A\mathcal{D}B \exp\Big(i \int \mathrm{tr}(B \wedge F(A) - \frac{\beta}{2} B \wedge B - B \wedge J)\Big). \tag{201}$$

Since the action is quadratic in the B field we can integrate it out by replacing it by its classical value

$$\beta B^{IJ} = F^{IJ}(A) - J^{IJ}, \tag{202}$$

and so the action becomes

$$S_J = \frac{1}{2\beta} \int \mathrm{tr}\Big((F(A) - J) \wedge (F(A) - J)\Big). \tag{203}$$

Its equations of motion are

$$d_A J = 0, \tag{204}$$

and we denote by $\mathcal{M}_J$ the solution space. In order to solve these equations lets introduce a linear operator mapping Lie algebra valued 1-forms to Lie algebra valued three forms

$$L_J : \Omega_1(\mathcal{G}) \rightarrow \Omega_3(\mathcal{G}) \tag{205}$$

$$A \rightarrow [J, A]. \tag{206}$$

The space of three forms $L_{\mu\nu\rho} \in \Omega_3(\mathcal{G})$ is isomorphic to the space of densitized vectors $\tilde{L}^\alpha = 1/2\epsilon^{\alpha\mu\nu\rho} L_{\mu\nu\rho}$. L_J is a square matrix whose matrix elements can be explicitly written as

$$\tilde{L}_J^{(\nu AB)}\binom{\mu}{CD} = \epsilon^{\alpha\beta\mu\nu} J^{[A}_{\alpha\beta[C}\delta^{B]}_{D]}. \tag{207}$$

For a generic J we expect L_J to be invertible. In this case there is a unique connection solution of (204)

$$a_J = L_J^{-1}(dJ) \tag{208}$$

We expand $A = a_J + a$ and the action around this solution

$$2\beta S_J = \int_M \mathrm{tr}\big((F(a_J) - J) \wedge (F(a_J) - J) + 2(d_{a_J} a + \frac{1}{2}[a, a]) \wedge (F(a_J) - J) \tag{209}$$

$$+(d_{a_J} a + \frac{1}{2}[a, a]) \wedge (d_{a_J} a + \frac{1}{2}[a, a]).\big) \tag{210}$$

This expansion can be drastically simplified. First we can integrate by part the second term in the action using the equation of motion and the Bianchi identity $d_{a_J} J = d_{a_J} F(a_J) = 0$. We can also integrate by parts the third term in the action by introducing the Chern-Simons functional

$$CS_J(a) = \mathrm{tr}\big(a \wedge d_{a_J} a + \frac{1}{3} a \wedge [a,a]\big), \tag{211}$$

its derivative is given by

$$dCS_J(a) = \mathrm{tr}\big((d_{a_J} a + \frac{1}{2}[a, a]) \wedge (d_{a_J} a + \frac{1}{2}[a, a]) + [a, a] \wedge F(a_J)\big). \tag{212}$$

The action (209) can then be written as a sum of a boundary term

$$\int_{\partial M} CS_J(a) + 2\mathrm{tr}\big(a \wedge (F(a_J) - J)\big) \tag{213}$$

and a bulk action that remarkably is quadratic

$$2\beta S_J = \int_M \mathrm{tr}\big((F(a_J) - J) \wedge (F(a_J) - J) + a \wedge [J, a]\big). \tag{214}$$

We can then get an exact expression for the generating functional

$$Z(J) = \frac{\exp\left(\frac{i}{2\beta}\int_M \mathrm{tr}(F(a_J) - J)\wedge(F(a_J)-J)\right)}{\sqrt{\det L_J}}. \tag{215}$$

In the denominator we have the determinant of the operator $L_J(x,y) \equiv L_J(x)\delta(x,y)$.

If J is such that L_J is not invertible we can still carry out the computation. In this case $\mathcal{M}_J$, the space of solutions of (204), is an affine space of non zero dimension, its tangent space is the kernel of L_J. The action (203) now possess an extra gauge invariance

$$\delta A = a, \quad \text{with } a \in \ker(L_J). \tag{216}$$

We denote by a_J any solution of (204), and expand as before $A = a_J + a$, we still get the quadratic action (214). The integration over a now gives

$$Z(J) = \int_{\mathcal{M}_J/G_J} da_J \frac{\exp\left(\frac{i}{2\beta}\int_M \mathrm{tr}(F(a_J) - J)\wedge(F(a_J)-J)\right)}{\sqrt{\det' L_J}}. \tag{217}$$

where $\det'$ denotes the determinant of L_J acting on a orthogonal subspace of $Ker(L_J)$, $G_J = \{g/gJg^{-1} = J\}$ is the subgroup preserving J, and $\mathcal{M}_J/G_J$ is the space of solutions modulo gauge transformation.

5.3 Topological effective action

We now want to study further the path integral and the effect of the gauge symmetry-breaking term. We suppose in the following that $\beta = 0$. The gauge parameters are pairs g, ϕ where $g \in SO(5)$ and ϕ is a Lie-algebra-valued one form. The BF action $S_{BF}(B,A)$ is invariant under the transformation

$$A \rightarrow {}^gA = gAg^{-1} + gdg^{-1}, \tag{218}$$

$$B \rightarrow g(B - d_A\phi)g^{-1}. \tag{219}$$

We can split the integration over A, B into an integration over the gauge equivalence class $[A], [B]$ of the BF symmetry and an integration over the gauge parameters g, ϕ.

The integration measure decomposes, by the standard Faddev-Popov argument as $\mathcal{D}A\mathcal{D}B = \mathcal{D}[A]\mathcal{D}[B]\mathcal{D}g\mathcal{D}\phi$ and the path integral becomes

$$Z = \int \mathcal{D}[A]\mathcal{D}[B] e^{S_{BF}(B,A) + s(A,B)}. \tag{220}$$

where $s(A,B)$ is an effective action obtained by integration over the gauge degrees of freedom, explicitly

$$e^{is(A,B)} = \int \mathcal{D}g\mathcal{D}\phi e^{-\frac{\alpha}{4}\int d^4x \mathrm{tr}(|B - d_A\phi|\gamma(g))} \tag{221}$$

where we define $\gamma(g) \equiv g^{-1}\gamma_5 g$ which is a unit vector[9] in $\mathbb{R}^5$ and for every $SO(5)$ valued 2 form B we define the vector density $|B|(x) = |B|_M(x)\gamma^M$ by[10]

$$\mathrm{tr}(B \wedge B\gamma_M) = |B|_M d^4x \tag{223}$$

In order to understand the form of this effective action we will look at the partial effective action obtained by integrating only g and only ϕ. As we will see each partial integration is one loop exact, which means that it localizes on its classical solutions and that its evaluation is given by its stationary phase evaluation (see [58] for a discussion of localization in QFT). This is clear for the integration over ϕ since the action is quadratic in ϕ but it also happens for the integration over g.

5.3.1 Spontaneous symmetry breaking

The action $\int \mathrm{tr}(|B - d_A\phi|\gamma(g))$ is ultralocal for g and it doesn't contain any derivative acting on g. We can therefore understand the localization property of the path integral by looking at the final dimensional analog

$$\int_{SO(4)} dg e^{i\mathrm{tr}(|B|\gamma(g))} = \frac{e^{i||B||}}{2||B||} - \frac{e^{-i||B||}}{2||B||}. \tag{224}$$

The RHS of this expression is the semi classical evaluation of the integral. This is clear since the equation of motion of $\mathrm{tr}(|B|\gamma(g))$ gives $[\gamma(g), |B|] = 0$. The solutions when $||B||^2 \equiv |B|_M|B|^M \neq 0$ are given by

$$\gamma(g) = \pm\frac{|B|}{||B||}. \tag{225}$$

The action evaluated on this solution is $\pm||B||$ which reproduces the terms in the exponential. The denominator comes from the evaluation of the quadratic fluctuations around the solution.

Similarly, the equations of motion of the continuum action $\int \mathrm{tr}(|B|\gamma(g))$ (supposing $\phi = 0$ for simplicity) are $[\gamma(g), |B|] = 0$. The solutions when $||B||^2 \equiv |B|_M|B|^M \neq 0$ are given by

$$\gamma(g)(x) = \pm\frac{|B|(x)}{||B||(x)}. \tag{226}$$

[9] γ_5 is left invariant by an $SO(4)$ subgroup so $\gamma(g)$ determines a point in $S^4 = SO(5)/SO(4)$

[10] If we spell out the indices this reads

$$|B| = \gamma^M \epsilon_{IJKLM}\epsilon^{\mu\nu\rho\sigma} B^{IJ}_{\mu\nu} B^{KL}_{\rho\sigma} \tag{222}$$

The sign is a priori x dependent, but if we restrict to continuous solutions for g we have only two solutions[11].

The localization property of the integral therefore suggests that the gravity effective action obtained by integration over g is (if we keep one branch)

$$S_{GR} = \int \mathrm{tr}(B \wedge F(A)) - \frac{\alpha}{4}||B||(x)d^4x. \tag{227}$$

This action is now $SO(5)$ invariant whereas gravity is only $SO(4)$ invariant. It sounds therefore strange at first that we can recover gravity from this action. In order to understand this let us first check that the equations of motion for this action are equivalent to Einstein's equations.

The action is defined for all B. However it is differentiable only when $||B||(x) \neq 0$ which we now suppose holds true. The equations of motion are given by

$$F^{IJ} = \frac{\alpha}{2}\epsilon^{IJKLM}B_{KL}n_M, \tag{228}$$

$$d_A B^{IJ} = 0. \tag{229}$$

where we denote $n_M \equiv \frac{|B|_M}{||B||}$. Given this unit vector we define

$$e^I \equiv d_A n^I = dn^I + A^{IJ}n_J, \quad \omega^{IJ} \equiv A^{IJ} + n^I e^J - n^J e^I, \tag{230}$$

$$b^I \equiv B^{IJ}n_J, \qquad b^{IJ} \equiv B^{IJ} + n^I b^J - n^J b^I. \tag{231}$$

We denote by F^{IJ} (respectively R^{IJ}) the curvature of the connection A (respectively ω). We have the following identities

$$F^{IJ}n_J = d_A e^I = d_\omega e^I, \tag{232}$$

$$R^{IJ} - 2e^{[I} \wedge e^{J]} = F^{IJ} - 2d_\omega e^{[I}n^{J]}, \tag{233}$$

where the bracket denotes antisymetrisation. From these identities it is clear that $R^{IJ}n_J = 0$, also $d_\omega n_I = 0$, so ω is an $SO(4)$ connection preserving the direction n^I. In terms of the variables (230, 231) the equations of motion read

$$d_\omega e^I = 0, \tag{234}$$

$$\frac{1}{2\alpha}\epsilon_{IJKLM}(R^{IJ} - 2e^I \wedge e^J)n^M = b_{KL}, \tag{235}$$

$$d_A B^{IJ}n_J = d_\omega b^I - b^{IJ} \wedge e_J = 0, \tag{236}$$

$$d_\omega b^{IJ} + 2e^{[I} \wedge b^{J]} = 0. \tag{237}$$

[11] This problem disappears when we consider a negative cosmological constant. In this case $\gamma(g)$ belongs to an hyperboloid and there is only one solution to the equation $[\gamma(g), |B|] = 0$.

The first equation tells us that ω is the spin connection if the frame field e is invertible. If we take the derivative d_ω of (235) we obtain that $d_\omega b_{KL} = 0$ since $d_\omega R^{IJ} = 0$ by the Bianchi identity, $d_\omega e^I = 0$ by the torsion free equation and $d_\omega n_I = 0$ by construction. Equation (237) then implies that $b^I = 0$ when e^I is invertible. This means that $b^{IJ} \wedge e_J = 0$ by equation (236) which is equivalent, due to (235), to the Einstein equation

$$\epsilon_{ijkl}(R^{ij} - 2e^i \wedge e^l) \wedge e^k = 0, \tag{238}$$

where the indices i, j, k label vectors orthogonal to n.

One sees that the equations of motion of the $SO(5)$ invariant theory are equivalent to the Einstein equations when e is invertible. Even if the action is invariant under $SO(5)$ gauge symmetry the solutions of this action spontaneously breaks this symmetry by choosing a preferred direction in the internal space proportional to $|B|$. The same results apply for $\beta \neq 0$.

On shell we have that $||B|| = \frac{1}{\alpha^2}|\epsilon_{ijkl}F^{ij} \wedge F^{kl}|$. Any $SO(4)$ bivector B^{ij} can be decomposed into self dual and anti self dual parts $B = B_+ + B_-$. Using this decomposition for spatial and internal indices of $F^{ij}_{\mu\nu} =$ we can write decompose F as $F = W_+ + W_- + \phi + \phi_0$ where W^{ij}_- is a symmetric traceless tensor labeling the five self dual components of the Weyl tensor, ϕ^{ij} a traceless tensor labeling the trace free part of the Ricci tensor and ϕ_0 is the scalar curvature. In term of these components we have $||B|| = 4!det(e)/\alpha^2((W_+)^2 + (W_-)^2 + (\phi_0)^2 - (\phi)^2)$ The components ϕ, ϕ_0 are zero by the Einstein equation. Thus $||B||$ is zero if and only if the Weyl tensor vanish that is only if $F = 0$ and our spacetime is spherical[12] . We therefore see that the presence of a spontaneous symmetry breaking is equivalent in the Euclidean case to the existence of a non trivial gravitational field.

5.3.2 Gravity as a non local topological theory

We now consider the construction of the effective action coming from the integration of the translational symmetry parameter for g fixed. We discuss the case $\beta = 0$.

$$e^{is(A,B)=} \int \mathcal{D}\phi e^{i\frac{\alpha}{4}\int d^4x \mathrm{tr}(|B - d_A\phi|\gamma_5)}. \tag{239}$$

This integral being quadratic localizes on the classical solution if it exists. The equation of motion are given by

$$d_A\{B, \gamma_5\} = \Delta_A\phi, \tag{240}$$

where Δ_A is the differential operator $\Delta_A\phi = d_A\{d_A\phi, \gamma_5\}$. Using the 4 + 1 decomposition $A^{IJ} = (\omega^{ij}, e^i)$ and $\phi^{IJ} = (\phi^{ij}, \phi^i)$ we can write these equations in

[12]In the Lorentzian case the condition is less restrictive since $W_\pm$ are complex conjugates

components.

$$d_\omega B^{ij} = d_w(d_\omega \phi^{ij} - 2e^{[i} \wedge \phi^{j]}) \tag{241}$$

$$\epsilon_{ijkl} B^{ij} \wedge e^k = \epsilon_{ijkl}(d_\omega \phi^{ij} - 2e^i \wedge \phi^j) \wedge e^k. \tag{242}$$

If Δ_A is invertible we can uniquely solve this equation. Lets denote $\varphi \equiv \Delta_A^{-1}(d_A\{B, \gamma_5\})$ a solution of these equations and define

$$\bar{B} = B - d_A\varphi. \tag{243}$$

By construction $\bar{B}$ is a solution of $d_A\{\bar{B}, \gamma_5\} = 0$. If we insert the previous decomposition in the integral (239) we can factorize $\bar{B}$ out of the integral

$$e^{is(A,B)} = e^{i\frac{\alpha}{4}\int \mathrm{tr}(\bar{B}\wedge\bar{B}\gamma_5)} \int \mathcal{D}\phi e^{-\frac{\alpha}{4}\int d^4x \mathrm{tr}(|d_A\phi|\gamma_5)} = \frac{e^{i\frac{\alpha}{4}\int \mathrm{tr}(\bar{B}\wedge\bar{B}\gamma_5)}}{\sqrt{\det(\Delta_A)}} \tag{244}$$

where Δ_A is the differential operator $\Delta_A\phi = d_A\{d_A\phi, \gamma_5\}$.

One sees that the integration over the gauge modes produces for us an effective action

$$\tilde{s}(A, B) = \frac{\alpha}{4}\int \mathrm{tr}(\bar{B} \wedge \bar{B}\gamma_5) = \frac{\alpha}{4}\int \mathrm{tr}\Big((B \wedge B\gamma_5) + \frac{1}{2}(d_A\{B, \gamma_5\}\Delta_A^{-1} d_A\{B, \gamma_5\})\Big). \tag{245}$$

This action is invariant under the translational gauge symmetry of BF theory which is the symmetry that makes BF theory topological. By construction its partition function is the one of gravity. BF theory does not carry local degrees of freedom whereas gravity does. The catch is that the effective action is a non local observable for BF theory since it involves the propagator of Δ_A. It is important to remark that this action is still quadratic in B since Δ_A is a linear operator. It is not clear however, whether we can explicitly do the g integration of the action (245).

In the derivation of the effective action we have assumed that Δ_A is an invertible operator, or alternatively there is no non trivial solution to the equation

$$d_A\{d_A\phi, \gamma_5\} = 0. \tag{246}$$

We expect it to be true for a generic choice of A (as long as $e = d_A\gamma_5$ is invertible). We are now going to give an argument in favor of this claim, keeping in mind that it will be interesting to have a proper characterization of the connections for which it holds.

Before doing so, let us first study a particular case where on the contrary Δ_A is not invertible. We will now show that if A is a flat $SO(5)$ connection, then the gravitational waves around this connection are in one to one correspondence with the kernel of Δ_A. If we start from a Cartan Weyl formulation of gravity (181), with $\Lambda = 3$, the equation of motions are

$$d_\omega(R^{ij} - e^{[i} \wedge e^{j]}) = 0, \quad \epsilon_{ijkl}(R^{ij} - e^i \wedge e^j) \wedge e^k = 0. \tag{247}$$

In the first equation which comes from variation with respect to ω^{ij} we have added for convenience a term trivial by Bianchi identity. These equations can be written in a compact form

$$d_A\{F(A), \gamma_5\} = 0, \tag{248}$$

using the notation of eq.(178). Given a gravity solution $A = (\omega^{ij}, e^i)$ we can look for 'graviton solutions', i-e infinitesimal perturbation δA such that $A + \delta A$ is a solution of the Einstein equations to first order. The equation for the perturbation is

$$\Delta_A \delta A = [\{F(A), \gamma_5\}, \delta A]. \tag{249}$$

Therefore, if the original space time is a four sphere ($F(A) = 0$), and δA is a graviton solution, then $\phi = \delta A$ is in the kernel of Δ_A. Even in Euclidean space where there is no graviton Δ_A is not invertible around a flat $SO(5)$ connection since infinitesimal diffeomorphisms $\delta A = \mathcal{L}_\xi A$ are in the kernel of Δ_A. Away from a spherical space this is no longer true: the graviton Laplacian is now $\tilde{\Delta}_A = \Delta_A + [\{F(A), \gamma_5\}, \cdot]$, infinitesimal diffeomorphisms $\delta A = \mathcal{L}_\xi A$ are in the kernel of $\tilde{\Delta}_A$ but not of Δ_A. This can be easily understood from the fact that the action $S(A, \phi) = \int \mathrm{tr}(d_A\phi^{ij} \wedge d_A\phi^{kl})\epsilon_{ijkl}$ is not invariant under diffeomorphisms or $SO(4)$ gauge transformations acting on ϕ alone unless A is chosen to be fixed by a combination of diffeomorphism and gauge transformations. This is the case for a flat connection since we have in this case $\mathcal{L}_\xi A - d_A(i_\xi A) = i_\xi(F(A)) = 0$, with ξ a four vector and i_ξ denotes the interior product[13]. In general Δ_A being non invertible means that $S(A, \phi)$ possesses some gauge invariance. We expect all possible gauge invariance of such an action to come from a restriction of the diffeomorphism group times the local rotation group. Only some special connection will have such an invariant subgroup like the flat connection as well as connections of some special holonomy. For a generic A there is no such invariance and therefore we expect Δ_A to be invertible.

[13] In the case of Flat $SO(5)$ connection this action is clearly invariant under the transformation $\delta\phi = d_A\psi$

6 Spinfoam quantization of topological field theories

Now we need to make sense of the path integral in (200). In our perturbative approach the free field theory is a topological field theory. In ordinary QFT the free field theory is Gaussian and the total partition function can be represented as an infinite series of Gaussian integrals with the number of insertions growing with the order. Similarly, in our approach the partition function is an infinite series of topological path integrals with the complexity of moduli space growing with the order. We will see that it is these moduli spaces where the propagating degrees of freedom of General Relativity emerge. The integrals in the series are no longer Gaussian, so we need another method for calculating them. But because they are all topological any method that has proved successful for a topological field theory could be used as a basis for our perturbation theory. Here we will use the spinfoam approach to calculating these path integrals, which is based on discretizing (triangulating) the manifold and then restricting the integration variables to the given triangulation. This definition of the path integral is meaningful because the result is independent of the choice of triangulation due to underlying topological symmetry (provided that the triangulation is complex enough to cover all the moduli spaces of the given topological field theory).

When we try to apply the spinfoam approach to General Relativity directly, the main problem comes from the fact that the amplitude usually proposed depends on the chosen triangulation and some extra work is needed in order to either understand the refinement limit of the model or sum over triangulations by realizing this sum as a perturbative expansion of an auxiliary Field theory [62]. This problem is generally referred to as the continuum limit problem.

The spin foam quantization we are proposing is free of this problem and we want to argue that it leads to an expansion that is triangulation independent. The main idea is that at each order in the perturbation theory the model still carries part of the original BF symmetry. This symmetry allows us to identify triangulations that are identical away from the insertion of the perturbation operator and when properly gauged fixed lead to a triangulation-independent amplitude. The complexity of the minimal triangulation needed for the computation grows with the expansion order. For example in 0-th order our perturbative vacua corresponding to de Sitter space is described by a spin foam with one dual vertex.

We will first consider BF-theory where β (and hence the Immirzi parameter) is 0. It contains infinities, so called bubble divergences, that result from the topological gauge symmetry behind the triangulation independence. We describe the gauge fixing procedure of [61] that removes those divergences. In perturbation theory, however, some of those gauge degrees of freedom are sent to the physical sector, so we cannot fix them any longer and some other treatment of the corresponding divergences is needed. This can be done in the case $\beta \neq 0$ which is much more regular. Indeed it has been conjectured for a long time that the computation of the partition function of the BF model with non-zero 'cosmological term' is realized by a state sum model built on a quantum group with $q = \exp(i\beta)$ roots of unity. This was proved recently by Barrett et al [65] for the case of the group $SU(2)$. Presumably this results holds for any compact group which we will assume in the following. The gauge fixing procedure of [61] can also be extended to this case wherever the gauge symmetry is unbroken, and where it is broken no infinities arise.

6.1 Basics of spinfoam

Fix a triangulation Δ of M and its dual Δ^*. As usual, the connection field A is replaced by group elements g_{e^*} representing the holonomy of A along the dual edges e^* of Δ^*. The B-field is replaced by Lie algebra elements X_f associated to the faces f of Δ and representing the integration of B over these faces. The curvature 2-form is represented as a group element G_f living on the faces f (or dual faces f^*), and is obtained as the ordered product of the group elements g_{e^*} for dual edges $e^* \subset f^*$, upon the choice of a starting dual vertex on the dual face. The discretized action for BF-theory with an extrinsic sources J_f, which are associated to the dual faces, can then be defined to be

$$S_\Delta(J) = \sum_f [\mathrm{tr}(X_f P(G_f e^{J_f}))], \tag{250}$$

where $P(g)$ is a projection of a group element on a Lie algebra defined in Appendix B. The generating functional (198) then becomes

$$Z(J) = \left(\prod_{e^*} \int_{SO(5)} dg_{e^*}\right)\left(\prod_f \int_{so(5)} dX_f\right) \exp\left[iS_\Delta(J)\right]. \tag{251}$$

The integration wrt X_f can then be explicitly performed (as shown in Appendix B), yielding

$$Z(J) = \left(\prod_{e^*} \int_{SO(5)} dg_{e^*}\right) \prod_f \delta(G_f e^{J_f}) \tag{252}$$

In the zeroth order of perturbation theory the partition function is just the generating functional at $J = 0$. By using the Peter-Weyl decomposition for delta functions in (252) and performing integration over the group elements it can be reexpressed in the form of the Ooguri model [63]

$$Z(0) = \sum_{j_f} \sum_{j_t} \prod_f \dim(j_f) \prod_t \dim(j_t) \prod_s (15j)_s, \tag{253}$$

where j_f j_t label representations associated to the faces and tetrahedra respectively and $15j$-symbols are associated to the four simplices s.

The model (253) is generally divergent, the rate of divergence being proportional to the number of closed three-dimensional regions (bubbles) in the dual triangulation. The reason for these divergences is easier to see in the expression (252) at $J = 0$. It contains a product of delta functions of holonomies around all the faces. Not all such holonomies are independent. For each 3-bubble of Δ^* or, equivalently, for each edge e of Δ there is an identity relating the holonomies around the faces forming this bubble.

$$\prod_{f \supset e} (k_{ef})^{-1} G_f^{\epsilon(f,e)} k_{ef} = Id. \tag{254}$$

Here G_f is a holonomy around the face f^* of Δ^* dual to f, k_{ef} is a group element connecting the starting point of the 3-bubble b^* of Δ^* dual to e and the starting point of f^*, and $\epsilon(f,e)$ is a sign chosen so that $\{f^* \subset b^*\} = \partial b^*$. Eq. (254) is a discretized version of the Bianchi identity $d_A F = 0$, which is a condition on a 3-form field. This is why it involves 3-dimensional objects (bubbles).

It has been shown [61] that these divergences come from the gauge symmetry of the model and result from integration over the orbit of the gauge group. There are two possible ways to deal with these divergences: either we remove the integrals over the gauge group by gauge fixing [61] or we modify the model so that the orbit of the gauge group becomes finite, so that the integral over it converges.

6.2 Regulating bubble divergences

6.2.1 Bubble divergences and gauge fixing

In this section we briefly review the gauge fixing procedure of [61] and evaluate the partition function of pure BF theory, which is in turn the partition function of quantum gravity to 0-th order of perturbation theory. This allows us to give a spinfoam representation of the perturbative vacuum of quantum gravity.

The free theory is the pure BF-theory

$$S = \int \mathrm{tr}(B \wedge F(A)). \tag{255}$$

The gauge symmetry of the action (255) includes local $SO(5)$-summetry

$$\begin{aligned} A &\to g^{-1} dg + g^{-1} A g \\ B &\to g^{-1} B g, \end{aligned} \tag{256}$$

where g is an $SO(5)$-valued scalar field, and 10-dimensional translational symmetry

(10 is the number of generators of $so(5)$)

$$\begin{aligned} A &\rightarrow A \\ B &\rightarrow B + d_A\phi, \end{aligned} \tag{257}$$

where ϕ is an $so(5)$-valued 1-form field. This is a very simple gauge group and it has a simple analog in the discretized action.

The 0-th order part of the expansion is just the generating functional (251) with all the sources set to zero, $J = 0$:

$$Z_0 = \left(\prod_{e^*}\int_{SO(5)} dg_{e^*}\right)\left(\prod_f \int_{so(5)} dX_f\right)\exp\left[i\sum_f [\mathrm{tr}(X_f G_f)\right]. \tag{258}$$

Before performing the integration we should remove the variables of which the integrand in independent. The existence of such variables is the consequence of the gauge symmetries (256,257) [61].

The discrete analog the symmetry (256) is parameterized by group elements k_{v^*} living at the dual vertices of the triangulation. It acts as

$$\begin{aligned} g_{e^*} &\rightarrow k^{-1}_{t_{e^*}} g_{e^*} k_{s_{e^*}} \\ G_f &\rightarrow k^{-1}_{st_f} G_f k_{st_f} \\ X_f &\rightarrow k^{-1}_{st_f} X_f k_{st_f} \end{aligned} \tag{259}$$

where s_{e^*} and t_{e^*} denote the dual vertices that are source and target of e^* st_f denotes the dual vertex, which is the starting point for computing curvature on the dual face $f^* \sim f$.

The translational symmetry (257) acts on the B-field as

$$B \rightarrow B + d\phi + [A, \phi]. \tag{260}$$

It is naturally integrated over the faces of the triangulation, $X_f = \int_f B$. The 1-form ϕ is naturally discretized at the edges of the triangulation in terms of a collection of Lie algebra elements Φ_e. Therefore, the discrete transformation should be of the form

$$\delta X_f = \sum_{e\subset f}(\Phi_e + [\Omega_e, \Phi_e]) \tag{261}$$

where the orientation of the edges $e \subset f$ is chosen so that $\{e \subset f\} = \partial f$ and Ω_e is a certain functional of the connection which can be found as in [61].

The reason behind the translational symmetry (261) is the discrete Bianchi identity (254). We now can employ it to isolate the components of X_f of which the partition function is independent.

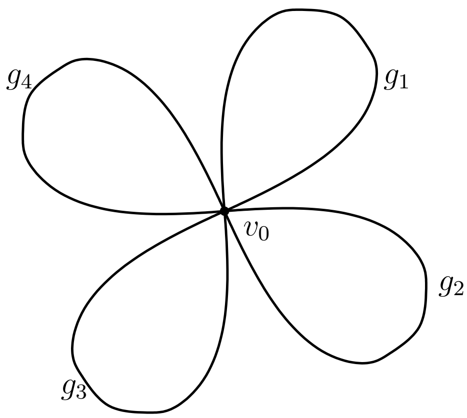

Figure 1: Spinfoam representation of the perturbative vacuum of quantum gravity. It consists of a single dual vertex v_0 with several closed loops g_I attached to it.

Eq. (254) allows us to rewrite a factor in the path integral (258) related to one bubble as

$$\left(\prod_{f\supset e}\int_{so(5)} dX_f\right)\exp\left[i\sum_{f\supset e}[\mathrm{tr}(X_f G_f)\right] = \prod_{f\supset e}\delta(G_f)$$

$$= \left(\prod_{\{f\supset e\}\backslash f_0}\int_{so(5)} dX_f\right)\exp\left[i\sum_{\{f\supset e\}\backslash f_0}\int_{so(5)}[\mathrm{tr}(X_f G_f)]\right]\int_{so(5)} dX_{f_0}\exp[i\mathrm{tr}(X_{f_0} Id)](262)$$

Clearly in the last expression the integrand does not depend on X_{f_0} and the integration w.r.t to it has to be removed. This should be done for every bubble in Δ^*.

The symmetry (259) results in the fact that the integrand in (258) does not depend on the group elements connecting any two invariant vertices. The integration w.r.t. such group elements must also be removed.

Generally we choose a maximum tree in a 1-skeleton of Δ^*, $T^* \subset \Delta^*$, and remove every integral w.r.t. the holonomies along the dual edges $e^* \subset T^*$. Also we fix a maximum tree in the 2-skeleton of Δ, $T \subset \Delta$, and remove every integral w.r.t. X_f such that $f \subset T$. We also should take into account the Faddeev-Popov determinant which was proved to be unity in [61]. For trivial topology the partition function is thus equal to 1.

At the end we get a triangulation containing only one edge and only one dual vertex. This simple spinfoam describes the perturbative vacuum. This is because with no matter and no gravitational excitations there is no physical way to distinguish different points of spacetime. (Fig. 1) There is one copy of $SO(5)$ associated with this vertex. It can be understood as the global $SO(5)$ symmetry of the vacuum.

In the generating functional (251) the gauge symmetry described in this section is broken. Therefore, instead of infinities coming from the integration over the gauge group and which can be represented as $\delta(0)$ we will have delta functions expressing the conservation of current J at every bubble.

$$Z(J) \sim \prod_{b} \delta(\prod_{f \subset b} k_{fb}^{-1} e^{J_f} k_{fb}), \tag{263}$$

where k_{fb} is a group element connecting the starting point of a face and a starting point of a bubble. The corresponding X_f degrees of freedom become physical. Then we have to apply derivatives to (263) and take a limit $J \to 0$. But because of the distributional character of the expression (263) such a limit is ill defined. We have to regularize the expression for the generating functional. This is done in the next section.

6.2.2 Bubble divergences and compactification of the translation group

There is another way to treat the bubble divergences which can be extended beyond free field theory. While for pure BF theory the problem comes from integration over (the translational part of) the gauge group, in perturbation theory it can be interpreted as a strong coupling regime. Indeed, if we look at the action (195) for $\beta = 0$, we see that the free theory (pure BF) is linear in B, while the interaction term is quadratic in B. Thus, even though the coupling constant α is very small, for sufficiently large fluctuations of B-field the interaction term becomes large as compared to free field term and the perturbation theory breaks down. Clearly, the later problem is an artifact of the perturbative expansion, as the interaction term, if treated non-perturbatively, would suppress the large fluctuations of B-field due to its strong oscilatory behavior. But exactly the same could be done by adding the term $\beta \neq 0$ to the free field theory. This modification keeps the free field theory topologically invariant and exactly soluble. Now we can make a perturbative expansion in terms of α. Because the large fluctuations of B-field are now suppressed by the β-term, perturbation theory does not break down, and later in this paper we will argue that it is finite in all orders.

The first step will be the quantization of BF-theory with a 'cosmological' term (183). The basic difference between the translational gauge symmetry of pure BF theory (255) and that of (183) is that (257) just shifts some components of B-field, while as it follows from (6) for $\beta \neq 0$ the translational symmetry acts on B-field non-linearly

$$\delta B = d_A \phi + [\phi, \wedge \phi]. \tag{264}$$

By analogy with gravity in 2+1 dimensions the this can be interpreted as a deformation of the coalgebra to the gauge group. So, it has been conjectured that the computation of the partition function of BF model with non zero 'cosmological term' is realized by a state sum model built on a quantum group with $q = \exp(i\beta)$

roots of unity. The model is known as the Crane-Yetter model [64], and it is obtained from the Ooguri model (253) by replacing the representations and intertwiners with q-deformed ones:

$$Z_{CY} = \left(\frac{\sqrt{\beta}(q^{1/2} - q^{-1/2})}{i\sqrt{2}} \right)^{-2(N_v+N_e)} \sum_{j_f} \sum_{j_t} \prod_f \dim_q(j_f) \prod_t \dim_q(j_t) \prod_s ((15j)_q)_s, \tag{265}$$

where N_v and N_e are numbers of vertices and edges of the triangulation respectively. Recently [65] this conjecture was proven along with triangulation independence of the model.

It is important that the model (265) has no bubble divergences in it, instead it contains a factor of $1/\sqrt{beta}$ per bubble. This can be interpreted as a result of compactification of the translation group after which it acquired a finite volume $1/\sqrt{beta}$.

Below, in the absence of differential calculus on a quantum group, we will be working directly with the action principle (183) and corresponding generating functional. We will use the conjecture of [64] in the discussion on triangulation independence.

6.3 Spinfoam perturbation theory (outline)

In this section we illustrate the basics of spinfoam perturbation theory techniques. We will describe all the possible diagrams relevant to the vacuum-vacuum transiton amplitude to the second order in α. For this we will first need to derive the expression for the generating functional evaluated at the simplest possible triangulation of a 4-sphere.

The simplest possible triangulation of a 4-sphere consists of two 4-simplices glued to each other along five tetrahedra. The dual triangulation consists of two vertices dual to the two 4-simplices and five links connecting them, which are dual to the five tetrahedra. The two dual vertices allow a description of all the possible arrangements of two derivative terms acting on the generating functional. Therefore, the simplest possible triangulation allows to calculate the partition function to the second order of perturbation theory. Higher order contributions would require a more complex triangulation to be calculated.

6.3.1 Generating functional on a lattice

We begin with evaluation of the partition function on the simplest possible triangulation of a 4-manifold, having a boundary that is a 3-sphere. Such triangulation consists of a single dual vertex in the bulk and five dual vertices on the boundary, Fig.3. The bulk vertex v is connected to the boundary vertices by holonomies g_I, $I = 1..5$,

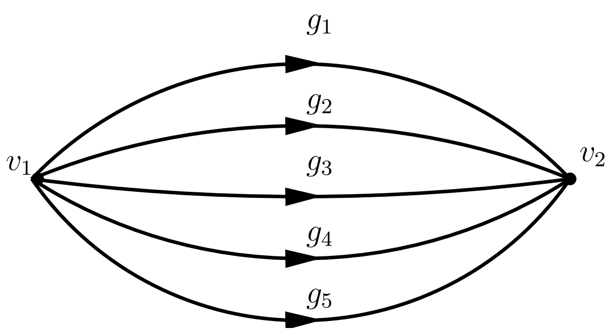

Figure 2: The dual of the simplest possible triangulation of a 4-sphere. Two dual vertices v_1 and v_2 connected with five dual links g_I

and the boundary vertices are interconnected with each other by holonomies h_{IJ}. The holonomies h_{IJ} comprise a triangulation of the boundary, which is different from the simplest possible triangulation by one $1 \to 4$ Pachner move, i.e. a change of the triangulation in which one dual vertex is replaces by four dual vertices..

The picture in Fig.3 can also be considered as a part of larger triangulation restricted to one 4-simplex with the dual faces cut in wedges. The vertex v is the vertex dual to that simplex, the holonomies g_I are parts of dual edges belonging to the simplex, and the holonomies h_{IJ} are those cutting the dual faces into wedges.

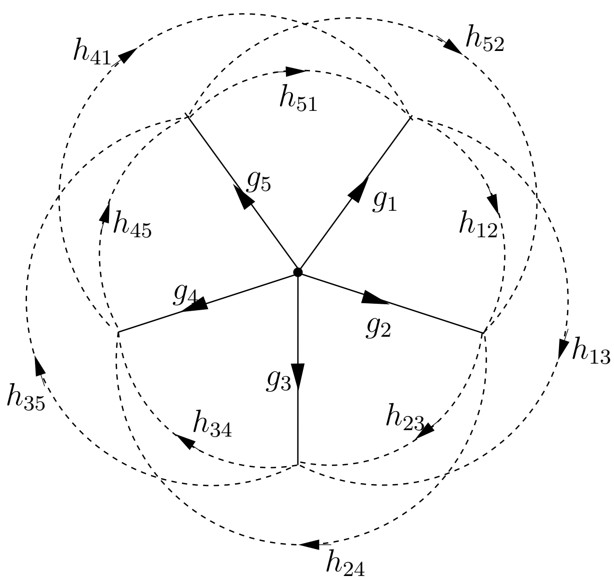

Figure 3: The simplest possible triangulation of a four manifold with a boundary. The holonomies g_I lie inside the bulk while the holonomies h_{IJ} are on the boundary.

We consider the regularized action with a non-zero Immirzi parameter $\beta \neq 0$. To make the formalism more convenient for generalization to an arbitrary triangulation we will use the same generating functional technique as before. First, we calculate the generating functional for $\beta = 0$. Then we convert the term containing β into a derivative term acting on the generating functional. Finally, we calculate the result of applying the derivative term to the generating functional *non-perturbatively*, so that no bubble divergences appear in the intermediate calculation.

We assume the boundary triangulation fixed, so the expression for the generating functional reads

$$Z_0(J,h) = \int \Big(\prod_{I=1\ldots5} dg_I \Big) \Big(\prod_{J<I} dX_{IJ} \Big) e^{iS_0(J,g,h,X)} \tag{266}$$

Here we introduced algebra elements $X_{IJ}^{1,2}$ corresponding to the B-field smeared along the wedges. The holonomy around a wedge can then be written as $G_{IJ} = g_I h_{IJ} (g_J)^{-1}$. All the charges J_{IJ} originate at the vertex v in the center of the given four simplex and are associated with ten wedges w_{IJ}. By Z_0 and S_0 in (266) we mean at $\beta = 0$.

In our calculations we will use the following δ-function identity described in the appendix

$$\int dX e^{i\mathrm{tr}(XP(g))} = \delta(g), \tag{267}$$

where X is an algebra element, g is a group element, and $P(g)$ is a projection of a group element on the Lie algebra described in the appendix.

Then we can write the discretized action as

$$S_0(J,g,h,X) = \sum_{J<I} \mathrm{tr}(X_{IJ} P(g_I h_{IJ} (g_J)^{-1} e^{J_{IJ}})). \tag{268}$$

The path integral (266) is now easy to evaluate by performing integration over X_{IJ} and g_I. This can be done by using the identity (267) and the composition of delta functions. One of the integrations over g_I has to be left out however, and we choose it to be g_5, and we can put $g_5 = g$. The resulting expression contains six δ-functions corresponding to six independent bubbles in the model

$$Z_0(J,h) = \int dg \prod_{i<j} \delta(e^{J_{5i}} e^{J_{ij}} e^{J_{j5}} g h_{5j} h_{ji} h_{i5} g^{-1}). \tag{269}$$

This is the expression to be used for further calculations.

The term containing β in the action can be discretized as

$$S_\beta(X) = \frac{\beta}{2} \sum_{I,J,K,L,M=1\ldots5} \epsilon_{IJKLM} \mathrm{tr}(X^{JK} X^{LM}). \tag{270}$$

This is the discretization of $B \wedge B$ term on one simplex. Notice that ϵ_{IJKLM} factor in this expression has nothing to do with the symmetry breaking ϵ from the previous sections. This is the five dimensional 'ϵ' needed to define a 4-simplex.

So, the generating functional for non-zero β can be rewritten as

$$Z_\beta(J,h) = \exp\left[i\frac{\beta}{2}\sum_{I,J,K,L,M=1...5}\epsilon_{IJKLM}\mathrm{tr}(\frac{\partial}{\partial J_{JK}}\frac{\partial}{\partial J_{LM}})\right] Z_0(J,h) \tag{271}$$

For non-perturbative evaluation of this expression we use the following identity. For arbitrary non-degenerate quadratic form A and a vector variable x

$$e^{i\partial_x A\partial_x} = \frac{1}{det(A)}\int dke^{i(k\partial_x - kA^{-1}k)} \tag{272}$$

and then we can use the fact that the differential operator in the r.h.s. of this expression is a shift operator. In fact, this is the same as a direct evaluation of the path integral containing both terms (268) and (268). However, as we will see, for a complex triangulation the above method is much more convenient, as it allows us to compose all the delta functions first.

First of all, one should notice that the quadratic form in (270) to be inverted is degenerate, so one should extract the non-degenerate part of it. The degeneracy of this quadratic form is easy to see if one introduces the following variables:

$$X^{*JK} = \sum_{I=1...5}\epsilon^{IJKLM}X_{LM}. \tag{273}$$

One can notice that they are not all linearly independent. There are four obvious relations between them

$$\sum_I X^{*IJ} = 0, \tag{274}$$

Only six of the ten X^{*IJ} are independent, and we can take them for the basis X^{*ij}, $i,j = 1...4$. By using these variables the quadratic form in (270) then can be rewritten as

$$\begin{aligned}\sum_{I,J,K,L,M=1...5}\epsilon_{IJKLM}\mathrm{tr}(X^{JK}X^{LM}) &= \sum_{J,K=1...5}\mathrm{tr}(X^{*JK}X_{JK}) \\ &= \sum_{i,j,k,l=1...4}\mathrm{tr}(X^{*ij}X^{*kl})\epsilon_{ijkl}.\end{aligned} \tag{275}$$

To get the last line in this equation we used the relation (274) to express X^{*I5} in terms of X^{*ij}. Thus the quadratic form in (270) can be written as a form depending purely on X^{*ij}, which are independent variables.

For the rest of this section we will consider the example of an abelian group. This will allow us to illustrate the basic structure of the generating functional on a simple example. Many of the results obtained here could be generalized for a non-abelian group. So, let g_I,h_{IJ}, and J_{IJ} all be elements of $U(1)$. We can use the identity (272) to apply the differential operator in a form of the last line of (275) to the generating functional (269). The shift operator

$$T(K^*) = e^{iK^*_{ij}\frac{\partial}{\partial J^*_{ij}}} = e^{iK^*_{ij}(\sum_{I=1...5}\epsilon^{IijKL}\frac{\partial}{\partial J_{KL}})} \tag{276}$$

acting on the generating functional (269) yields

$$T(K^*)Z_0(J,h) =$$

$$\int dg \prod_{i<j} \delta(\prod_{m=1..4} e^{\epsilon^{ipqm}K^*_{pq}} e^{\epsilon_{ijkl}K^*_{kl}} \prod_{m=1..4} e^{\epsilon^{jpqm}K^*_{pq}} e^{J_{5i}} e^{J_{ij}} e^{J_{j5}} g h_{5j} h_{ji} h_{i5} g^{-1}) \tag{277}$$

In an abelian theory the group elements g in the delta functions and integration with respect to g can be neglected. The expression for the regularized generating functional

$$Z_\beta(J,h) = \int d^6K^* e^{i\frac{\beta}{2}\epsilon^{ijkl}K^*_{ij}K^*_{kl}} T(K^*) Z_0(J,h), \tag{278}$$

where the action of the shift operator $T(K^*)$ is given by (277), can be evaluated using the formula (333) from appendix C for regularized delta functions defined on quantum $U(1)$ with $q = e^{i\beta}$:

$$Z_\beta(J,h) = \prod_{i,j,k,l} \epsilon^{ijkl} \delta_\beta(e^{J_{5i}} e^{J_{ij}} e^{J_{j5}} h_{5j} h_{ji} h_{i5}, e^{J_{5k}} e^{J_{kl}} e^{J_{l5}} h_{5k} h_{kl} h_{l5}) \tag{279}$$

These regularized delta functions express conservation of charges J in the bubbles.

The expression (279) describes the generating functional for the simplest possible triangulation of a 4-sphere. The natural question is then: what happens if we use another, more complex triangulation for evaluating the generating functional and its derivative. Because zeroth order of perturbation theory is topologically invariant it is natural to expect that the result is independent of the triangulation, the partition function changes by a factor. The basic idea of the present perturbative approach is that a large part of topological symmetry is preserved in higher orders of perturbation theory. At a finite order of perturbation theory topological symmetry is broken only at a finite number of places (finite number of vertices were the derivative terms act). Because of this we expect that the result must also be independent of triangulation provided that the tringulation is complex enough to describe all the possible symmetry breaking configurations at a given order of perturbation theory.

To see whether this is actually the case we need an expression for generating functional for an arbitrary triangulation. Any two triangulations of arbitrary manifold can be related to each other by a sequence of Pachner moves. In four dimensions

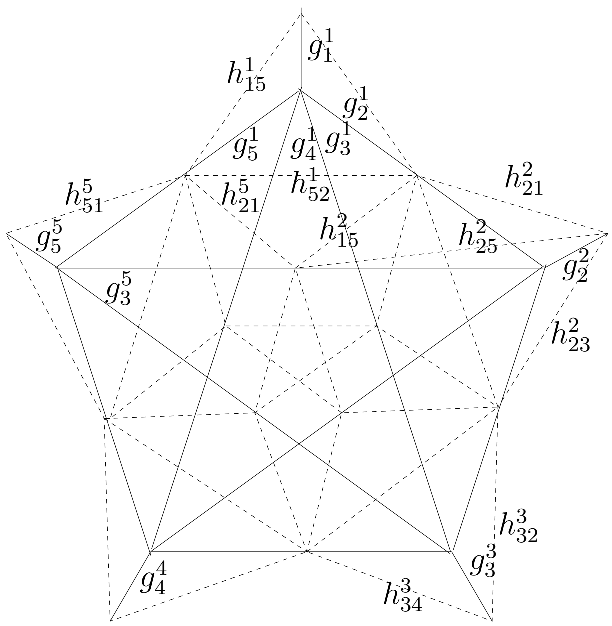

Figure 4: The triangulation of a 4-ball Fig. 3 after a $1 \to 5$ Pachner move, solid lines g_J^I denote the edges of the dual triangulation, dashed lines h_{JK}^I denotes the boundaries of each new simplex (not all of them are depicted), the edges h_{IK}^I can be thought of as belonging to the boundary of the ball.

there are $1 \to 5$, $2 \to 4$ and $3 \to 3$ moves which split one dual vertex into five which replace two dual vertices with four vertices, and which replaces three with three others respectively. We will prove the invariance of the partition function with respect to one type of Pachner move, $1 \to 5$.

Let us consider a triangulation which can be obtained from that on Fig. 3 by $1 \to 5$ Pachner move, see Fig. 4. It consists of five copies of triangulations Fig. 3 glued together. We will label group elements and charges corresponding to each dual vertex on Fig. 4 with a superscript $I = 1...5$. Let $Z_\beta(J^I, h^I)$ is a generating functional corresponding to a part of triangulation around a dual vertex I, and let $Z'_\beta(J, h)$ is the generating functional for the whole triangulation depicted on Fig. 4. The later can be expressed through the former by

$$Z'_\beta(J,h) = \int \left(\prod_I \prod_{J<K\neq I} dh_{JK}^I \right) \left(\prod_I Z_\beta(J^I, h^I) \right) \left(\prod_{I<J<K} \delta(h_{JK}^I h_{IJ}^K h_{KI}^J) \right). \quad (280)$$

Because of the quantum deformation the group elements J and h take on a discrete set of values. Therefore, by integration in (280) we mean a discrete summation and by a delta function – the Kronecker delta symbol, (see appendix C). The later is introduced to describe the gluing of the five simplices into one complex.

The integration in (280) can be explicitly performed to yield

$$Z'_\beta(J,h) = \frac{1}{\beta^2} Z_\beta(\tilde{J},\tilde{h}) \prod_{i=1..4} \delta\left(\prod_M \epsilon^{iMJKL} e^{J^J_{KL}}\right). \tag{281}$$

Here $\tilde{h}_{IJ} = h^I_{IJ} h^J_{IJ}$, $\tilde{J}_{IJ} = J^I_{IJ} + J^J_{IJ}$, $Z_\beta(J,h)$ is the partition function for one simplex (279), and the delta function is the Kronecker delta as in appendix C.

From (281) it directly follows that

$$Z'_\beta(J,h)\Big|_{J=0} = \frac{1}{\beta^2} Z_\beta(\tilde{J},\tilde{h})\Big|_{\tilde{J}=0}. \tag{282}$$

which proves the triangulation independence (up to a finite factor) of the partition function in zeroth order of perturbation theory.

To consider higher orders of perturbation theory, one can apply the interaction term with derivatives to generating functionals evaluated at different triangulations and compare the results. By a direct computation one can check that

$$\sum_{IJKLMN=1..5} \epsilon^{JKLMN} \left(\frac{\partial}{\partial J^I_{KL}} \frac{\partial}{\partial J^I_{MN}}\right) Z'_\beta(J,h)\Big|_{J=0}$$
$$= \frac{1}{\beta^2} \sum_{JKLMN=1..5} \epsilon^{JKLMN} \left(\frac{\partial}{\partial \tilde{J}_{KL}} \frac{\partial}{\partial \tilde{J}_{MN}}\right) Z_\beta(\tilde{J},\tilde{h})\Big|_{\tilde{J}=0}. \tag{283}$$

which proves the triangulation independence to the next order.

6.3.2 The basics of the diagram technique of perturbation theory

In this section we will describe schematically the perturbative expansion considering all the orders which could be based on the simplest possible triangulation of a four sphere. As we will argue this will allow us to see perturbation series up to the second order.

In the previous section we have derived the generating functional for an abelian group. This have illustrated why the perturbative expansion must be triangulation independent. However in an abelian theory we cannot describe one important feature of the present perturbative approach. This is a gauge symmetry breaking by the interaction term and sending some of of the gauge degrees of freedom to the physical sector. For this we need to consider a non-abelian theory.

We do not have an explicit expression for a generating functional for non-abelian theory so far, and it is expected to be much more complicated than abelian. However there is a way of making calculations in this case by employing some results already existing in literature. One can recall a formal expression for the generating functional from section 5.2.1

$$Z_J = \int DA \exp\left(i\frac{1}{2\beta}\int \mathrm{tr}\big((F(A)-J)\wedge(F(A)-J)\big)\right). \tag{284}$$

If we discretize the manifold in (284) and associate the extrinsic charges J to the dual faces, than application of derivatives to the generating functional (284) will yield

$$\frac{\partial}{\partial J_a^{AB}}\cdots\frac{\partial}{\partial J_b^{CD}}Z(J)\Big|_{J=0}$$
$$= \int DA \ln(G_a^{AB})\ldots\ln(G_b^{CD})\exp\left(i\frac{1}{2\beta}\int \mathrm{tr}\big(F(A)\wedge F(A)\big)\right) \tag{285}$$

Here the little indices a, b label the dual faces to which the charges J^{AB} are associated. Capital A, B are $SO(5)$ indices. G_a^{AB} is a holonomy of the connection A^{AB} around a dual face a, i.e. Wilson line. The evaluation of expectation values of Wilson lines with a $F\wedge F$ action already exists in literature (see [65] and refrences therein). It is related to certain knot invariants formed by those Wilson lines and uses the quantum group techniques with $q = e^{i\beta}$, analogously to the previous section. In this paper we will not focus on knot structure of (285), but we will mostly be interested in the tensorial structure of this expression, because this is where the "local" degrees of freedom of General Relativity should appear. Generally an expression (285) is a knot invariant of the holonomies $G_a^{AB}, \ldots$ times some invariant tensor containing the indices $A, B, \ldots$.

The perturbative expantion can be obtained by applying the derivatives from the interaction term to (284). The explicit expression for the interaction term reduced to the triangulation Fig. 2 is (indices 1 and 2 stay for the two dual verices of this triangulation)

$$S_{Int} = \alpha\sum_I \epsilon_{IJKLM}\mathrm{tr}(\gamma_5\frac{\partial}{\partial J^1_{JK}}\frac{\partial}{\partial J^1_{LM}}) - \alpha\sum_I \epsilon_{IJKLM}\mathrm{tr}(\gamma_5\frac{\partial}{\partial J^2_{JK}}\frac{\partial}{\partial J^2_{LM}}). \tag{286}$$

Because we consider perturbative expansion to the second order and the interaction term is the of second order in derivatives, we will be needing the expressions of the derivatives of Z_J up to the fourth order evaluated at$J = 0$.

Thus the derivatives of $Z(J)$ that we need are:

$$\frac{\partial}{\partial J^1_{AB}}Z(J)\Big|_{J=0} = a_1(\beta)\mathrm{tr}(T^{AB}) = 0 \tag{287}$$

$$\frac{\partial}{\partial J^1_{AB}}\frac{\partial}{\partial J^1_{CD}}Z(J)\Big|_{J=0} = a_2(\beta)\mathrm{tr}(T^{AB}T^{CD}) \tag{288}$$

$$\frac{\partial}{\partial J^1_{AB}}\frac{\partial}{\partial J^2_{CD}}Z(J)\Big|_{J=0} = a_2' \int dg \mathrm{tr}(T^{AB} g T^{CD} g^{-1}) \quad (289)$$

$$\frac{\partial}{\partial J^1_{AB}}\frac{\partial}{\partial J^1_{CD}}\frac{\partial}{\partial J^1_{EF}}\frac{\partial}{\partial J^1_{GH}}Z(J)\Big|_{J=0} = a_4(\beta)\mathrm{tr}(T^{AB}T^{CD}T^{EF}T^{GH}) \quad (290)$$

$$\frac{\partial}{\partial J^1_{AB}}\frac{\partial}{\partial J^1_{CD}}\frac{\partial}{\partial J^2_{EF}}\frac{\partial}{\partial J^2_{GH}}Z(J)\Big|_{J=0} = a_4'(\beta)\int dg \mathrm{tr}(T^{AB}T^{CD} g T^{EF}T^{GH} g^{-1}) \quad (291)$$

where $a_1(\beta)$, $a_2(\beta)$, $a_2'(\beta)$, $a_4(\beta)$, $a_4'(\beta)$ are some knot invariants. The intependece of these invariants on the triangulation has also been demonstrated in [65] The integrals over g, the group element connecting the two dual vertices, appears when the two derivatives are acting at two different points. This is an integral of a regular expression over a compact space. So, we can notice that all the above expressions are finite, and the same could be said about the derivatives of $Z(J)$ of any order. On the other hand at $\beta = 0$ these expressions diverge. So, one can say that the parameter β renormalizes the coupling constant of the perturbation theory.

Now we will try to give an expression for perturbative expansion of the partition function of GR up to the second order

$$Z = Z^0 + Z^1 + Z^2 + ... \quad (292)$$

Zero order contribution is simply

$$Z^0 = Z_\beta(J)\Big|_{J=0} = \frac{1}{\beta^3}. \quad (293)$$

The first order contribution appears to be zero. Indeed

$$a_2 \epsilon_{ABCD5}\mathrm{tr}(T^{AB}T^{CD}) = 0 \quad (294)$$

because it contains a contraction of a symmetric tensor with antisymmetric. So

$$Z^1 = 0 \quad (295)$$

The first non-trivial order in this expansion is the second order. We will describe it schematically without numerical evaluation. At this stage it is important to demonstrate where the "local" degrees of freedom of General Relativity come into play.

In the expression

$$Z^2 = S^2_{Int} Z_\beta(J) \quad (296)$$

There are two types of terms. One is when all four derivatives in the squared interaction term act at the same vertex and the other is when two derivatives act on one vertex and two others act on the other vertex. Here we describe the second, the most interesting contribution. We list the terms relevant for this contribution except those which are zero as they contain $\ln B_a$ and those which could be reduced to the first order contributions applied twice and are therefore zero, too. These includes

$$\int dg \epsilon_{ABCD5}\epsilon_{EFGH5}\mathrm{tr}(T^{AB}T^{EF}gT^{CD}T^{GH}g^{-1}) \tag{297}$$

$$\int dg \epsilon_{ABCD5}\epsilon_{EFGH5}\mathrm{tr}(T^{AB}gT^{EF}g^{-1})\mathrm{tr}(T^{CD}gT^{GH}g^{-1}) \tag{298}$$

and

$$\int dg \epsilon_{ABCD5}\epsilon_{EFGH5}\mathrm{tr}(T^{AB}gT^{GH}g^{-1})\mathrm{tr}(T^{CD}gT^{EF}g^{-1}) \tag{299}$$

One can show that due to $SO(4)$ invariance of the last three expressions the integrand does not depend on the $SO(4)$ subgroup of $SO(5)$. The integration is thus actually taken over the coset $SO(5)/SO(4)$. This is where the physical degrees of freedom of General Relativity show up. They are elements of the $SO(5)/SO(4)$ coset which are sent from gauge sector to physical sector by symmetry breaking.

The terms appearing in such a perturbation theory can be expressed in terms of chord diagrams with interchord connections. A typical second order chord diagram is depicted on Fig. 5. The main difference between such chord diagram and a chord diagrams that appear in the theory of Vassiliev invariants [57] is the presence of inrterchord connections described on the diagram as a dotted line. This is precisely the point where the perturbation theory described becomes different from that for topological quantum field theory. Interchord connections encode information of the "local" physical degrees of freedom of General Relativity. To each such connection is associate an element of $SO(5)/SO(4)$ coset and an integration over it.

We will not study the higher orders of perturbation here, but we can notice that the expression (284) for the generating functional is infinitely differentiable at $J = 0$ for $\beta \neq 0$. This follows from explicit calculations of the expectation values of the Wilson lines. So, for $\beta \neq 0$ the expressions for the vacuum loops must be finite to all orders. One can say that a single counterterm related to the Immirzi parameter controls all the divergences appearing in the vacuum loop diagrams.

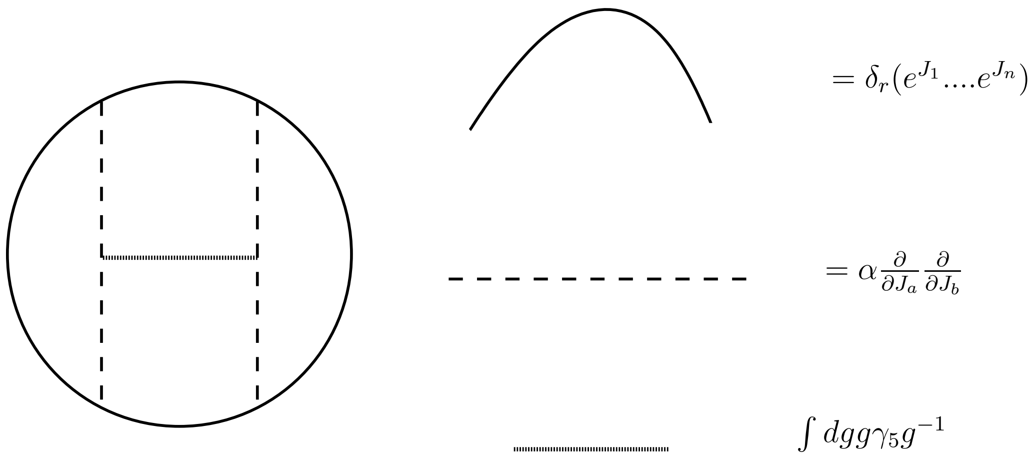

Figure 5: A typical chord diagram. A solid line denotes a flux of extrinsic current J. It should be closed as the current is conserved (a regularized delta function expresses the conservation of the current). A chord, the dashed line, denotes a pare of derivative in the interaction term acting on a functional of extrinsic current at two different places. Each chord has a point of origin. The origins of each two pares of chords are related to each other by a holonomy g. The integral with respect to g is denoted by a dotted line connecting two chords.

7 Discussion

The central part of this thesis is a new background independent perturbation theory for quantum General Relativity. It is very different from the traditional pertuirbation theory, which is based on splitting the metric into a background and fluctuations (1). So there are two natural questions to ask. First, is the new perturbation theory behaves any better in the ultraviolet regime than the traditional one? Second, is there any regime in which the results of two different perturbation theories reproduce each other?

While at the present stage we cannot give a final answer on either of these questions, there are several indications as to what the answer should be.

The main reason behind the non-renormalazibility of the traditional perturbative quantum gravity is the dimensionality of coupling constant. The actual dimensionless coupling is proportional to energy squared and therefore the theory becomes strongly coupled in the ultraviolet regime producing infinitely many type of divergent diagrams. One needs infinitely many counterterms to control all these divergences.

On the other hand the coupling constant of the perturbation theory for quantum gravity proposed here is dimensionless. The only divergences appearing in this theory are very similar to bubble divergences in the Ponzano-Regge model. The difference is that the divergences in the Ponzano-Regge model are infrared, while in the model proposed here they are infrared and ultraviolet at the same time. One may expect that there is some mechanism relating ultraviolet and infrared physics in this theory. We have argued that at least at the level of vacuum loops all the divergences appearing in the theory are controlled by a single counterterm which is related to the Immirzi parameter. This is already very different from the results of the traditional perturbative approach where one needs infinitely many counterterms to remove all the divergences.

This is not yet a proof of renormalizibility, because one has to show that the divergences can be removed from all the possible diagrams, not only from vacuum loops. This has to do with scattering problem which is also related to the second question.

To address the second question, one could think of setting up the problem of graviton-graviton scattering. In attempting this we encounter the main difference between the traditional perturbative approach and the present one. In the traditional approach as a result of linearization of the diffeomorphism transformations graviton can be described as a point particle. So, the ordinary scattering problem for particles can be directly generalized for gravity.

In the present perturbative approach, however, the theory is kept exactly diffeomorphism invariant order by order. As a result the generic non-locality of the basic excitations of the gravitational field appears already at the perturbative level. So, the conventional approach to scattering can no longer be directly applied, some revision of the scattering problem is needed.

However, there is a simpler way to address a scattering problem. Matter fields have local excitations even in a diffeomorphism invariant context. So, we can try to calculate scattering of matter particles, such as Deser Jackew 't Hooft like particles described in this thesis due to quantum gravity effect. In the approach developed we have all the necessary tools to address this problem. This will be a subject of further investigations.

A SO(5) conventions

$T_{IJ} = -T_{JI}$ with $I = 1, \cdots, 5$ are the ten generators of $so(5)$. They satisfy the algebra

$$[T_{IJ}, T_{KL}] = \eta_{JK} T_{IL} - \eta_{IK} T_{JL} + \eta_{IL} T_{JK} - \eta_{JL} T_{IK}, \tag{300}$$

where $\eta_{IJ} = \delta_{IJ}$ in the case of $so(5)$. The corresponding theory of gravity is Euclidean with a positive cosmological constant, i-e 'spherical gravity'. This is the one we focus on in the main text. If we want to describe Lorentzian gravity and/or other sign of cosmological constant one should consider metric of different signatures, the cases of interest for gravity are: $SO(4,1)$, where $\eta = diag(++++-)$ which describes Euclidean gravity with a negative cosmological constant (i-e 'hyperbolic gravity'). $SO(1,4)$ where $\eta = diag(-++++)$ which describes Lorentzian gravity with a positive cosmological constant, i-e 'de Sitter gravity'. $SO(3,2)$ where $\eta = diag(-+++-)$ which describes Lorentzian gravity with a negative cosmological constant, i-e 'AdS gravity'. One can split the generators of in terms of $so(4)$ generators T_{ij}, $i = 1, \cdots, 4$ and 'translation' generators $P_i = T_{5i}/l$, where l is a length scale (cosmological length scale in our context). The algebra reads

$$[T_{ij}, T_{kl}] = \eta_{jk} T_{il} + \cdots, \tag{301}$$

$$[T_{ij}, P_k] = \eta_{ik} P_j - \eta_{jk} P_i, \tag{302}$$

$$[P_i, P_j] = -\frac{\eta_{55}}{l^2} T_{ij}. \tag{303}$$

The $so(5)$ can be represented in terms of γ matrix

$$T_{IJ} = \frac{1}{4}[\gamma_I, \gamma_J] \tag{304}$$

where γ_I are γ-matrices satisfying $\{\gamma_I, \gamma_J\} = 2\eta_{IJ}$.

In this spinorial representation we can easily write the root system and Cartan basis of the Lie algebra of $SO(N)$. Lets consider $SO(2n+1)$ and $SO(2n)$ and lets define

$$\gamma_j^{\pm} = \frac{1}{2}(\gamma_{2j-1} \pm i\gamma_{2j}), \tag{305}$$

which satisfy

$$\{\gamma_i^{\pm}, \gamma_i^{\pm}\} = 0, \quad \{\gamma_i^{\epsilon_i}, \gamma_j^{\epsilon_j}\} = 0, \quad \{\gamma_i^{+}, \gamma_i^{-}\} = 1. \tag{306}$$

We define the Cartan subalgebra to be generated by

$$h_i \equiv \frac{1}{2}[\gamma_i^{+}, \gamma_i^{-}]. \tag{307}$$

which satisfy $h_i^2 = 1/4$. The adjoint action of h_is are diagonalized by the roots generators

$$E_{\epsilon_i e_i + \epsilon_j e_j} \equiv \frac{1}{2}[\gamma_i^{\epsilon_i}, \gamma_j^{\epsilon_j}] = \gamma_i^{\epsilon_i} \gamma_j^{\epsilon_j} \tag{308}$$

where $i < j$ and we denote by e_i the dual basis of the Cartan subalgebra $e_i(h_j) = \mathrm{tr}(h_i h_j) = \delta_{ij}$. One can easily check that if h is any Cartan element we have:

$$[h, E_{\epsilon_i e_i + \epsilon_j e_j}] = (\epsilon_i e_i(h) + \epsilon_j e_j(h)) E_{\epsilon_i e_i + \epsilon_j e_j}. \quad (309)$$

In the case of $SO(2n)$ this generators gives a basis of the Lie algebra and $\pm e_i + \pm e_j, i < j \in [1, \cdots n]$ are all the roots. The positive roots are $e_i + e_j, e_i - e_j, i < j$ and the simple roots are $\{e_1 - e_2, \cdots, e_{n-1} - e_n, e_{n-1} + e_n\}$. The Weyl is of dimension $2^{n-1} n!$ and consists of the group of permutation of and even sign change of e_i.

In the case of $SO(2n+1)$ we consider

$$\gamma = (2H_1) \cdots (2H_n) = (-i)^n \gamma_1 \cdots \gamma_{2n}, \quad (310)$$

which clearly commutes with the Cartan generators, and define

$$E_{\epsilon_i e_i} = \gamma_i^{\epsilon_i} \gamma. \quad (311)$$

For $SO(2n+1)$, $\pm e_i$ and $\pm e_i + \pm e_j, i < j \in [1, \cdots n]$ are all the roots. The positive roots are $e_i, e_i + e_j, e_i - e_j, i < j$ and the simple roots are $\{e_1 - e_2, \cdots, e_{n-1} - e_n, e_n\}$. The Weyl is of dimension $2^n n!$ and consists of the group of permutation of and sign change of e_i.

We now look at representations of $SO(5)$, the roots are $\pm e_1 \pm e_2, e_1, e_2$. The fundamental weights are given by

$$\lambda_1 = e_1, \quad \lambda_2 = \frac{e_1 + e_2}{2} \quad (312)$$

the representations are labeled by one integers and one half integer $(m, s) = m e_1 + s e_2 = (m - s)\lambda_1 + 2s\lambda_2$ which physically correspond to mass and spin. They satisfy the restriction $m \geq s$. The vectorial representation corresponds to $(1, 0)$ the spinorial to $(0, 1/2)$ and the adjoint to $(1, 1)$. The Casimirs are given by the quadratic casimir $C_1 = \sum_{IJ} X_{IJ} X^{IJ}$ and $C_2 = W^I W_I$, with $W^I = \frac{1}{8} \epsilon^{IJKLM} X_{JK} X_{LM}$. The action of this casimir on the representation (m, s) is diagonal and given by

$$C_1 = (m + \frac{3}{2})^2 + (\frac{s+1}{2})^2 - \frac{5}{2}, \quad C_2 = ms(ms + c). \quad (313)$$

where c is a integer which can be evaluated in the adjoint representation.

The character of this representation is given by

$$\chi_{(m,s)}(e^{y_i h_i}) = \frac{\sin[(m + \frac{3}{2})y_1] \sin[(s + \frac{1}{2})y_2] - \sin[(s + \frac{1}{2})y_1] \sin[(m + \frac{3}{2})y_2]}{\sin(\frac{3}{2}y_1) \sin(\frac{1}{2}y_2) \sin\left(\frac{y_1 - y_2}{2}\right) \sin\left(\frac{y_1 + y_2}{2}\right)} \quad (314)$$

and the dimension is given by

$$d_{(m,s)} = \frac{1}{3}(2m + 3)(2s + 1)(m + s + 2)(m - s + 1). \quad (315)$$

B Delta function identity

Lets consider the orthogonal group $G = SO(N)$ and denote $\mathcal{G}$ its Lie algebra and Z_G its center. We define $P : G \to \mathcal{G}$ the following map from the group to the Lie algebra: P is such that $P(gHg^{-1}) = gP(H)g^{-1}$. Using this property it is enough to define it on the Cartan subgroup of $SO(N)$. Lets denote $y = y^i h_i \in \mathcal{H}$ an element in the Cartan subalgebra with h_i the basis (307) of the Cartan Lie algebra and $\exp(iy) \in H$ an element of the Cartan subgroup. In the spinorial representation of $SO(N)$ we have $\exp(iy) = \prod_i (\cos(y_i/2) + i \sin(y_i/2) h_i)$. We define $P(\exp(iy)) = \sum_i \sin(y_i/2) h_i$. This projects all center elements on 0.

We want to establish the following identity

$$\int_{\mathcal{G}} dX e^{\mathrm{tr}(XP(G))} = \Delta^2_{\mathcal{G}}(2\rho) \sum_{z \in Z_G} \epsilon_z \delta(Gz). \tag{316}$$

$\rho = 1/2 \sum_{\alpha > 0} \alpha$ and we denote $\Delta_{\mathcal{G}}(X) \equiv \prod_{\alpha > 0} (\alpha | X)$ the product being over all positive roots.[14] The trace is taken in the spinorial representation of $SO(N)$, it is normalized by $\mathrm{tr}(h_i h_j) = \delta_{ij}$, $\delta(G)$ is the delta function on the group associated with the normalized Haar measure, the sum is over all center elements and $\epsilon_z = (-1)^n$ for $SO(2n)$, $\epsilon_z = \prod_i^n (-1)^{(n_i+1)}$ for $SO(2n+1)$ and $z = e^{iy}, y_i = 2\pi n_i$. The measure on the Lie algebra is normalized to be

$$\int_{\mathcal{G}} dX e^{-\mathrm{tr}(X^2)} = \int_{\mathcal{H}} \prod_i^n dX_i \Delta^2_{\mathcal{G}}(X) e^{-\sum_i X_i^2}$$

where the integral on the RHS is over the Cartan subalgebra and we denote $\Delta_{\mathcal{G}}(X) \equiv \prod_{\alpha>0} (\alpha|X)$ the product being over all positive roots.[15] This identity is a consequence of Harish-Chandra-Itzykson-Zuber formula [66]. Lets consider $X = X^i h_i, Y = Y^i h_i \in \mathcal{H}$ two elements in the Cartan subalgebra. We have the identity

$$I(X,Y) \equiv \int_{G/H} dU e^{\mathrm{tr}(XUYU^{-1})} = \Delta_{\mathcal{G}}(\rho) \frac{\sum_w \epsilon_w e^{X_i Y_{w(i)}}}{\Delta_{\mathcal{G}}(X) \Delta_{\mathcal{G}}(Y)}. \tag{317}$$

dU is the normalised invariant measure, the sum is over the Weyl group, $\epsilon_w = \pm 1$ and $\rho = 1/2 \sum_{\alpha > 0} \alpha$.

One of the simplest proof of this formula uses the fact that the integral satisfies the Duistermaat and Heckman hypothesis and is therefore equal to its sationary phase evaluation. The stationary points of $S = \mathrm{tr}(XUYU^{-1})$ are given by $dS = \mathrm{tr}(U^{-1} dU [Y, UXU^{-1}]) = 0$. This implies that U is an element of the Weyl group $U = w$. If one expands the action around such a solution $U = \exp(\phi) w$ one finds

$$S = \mathrm{tr}(XwGw^{-1}) - \frac{1}{2} \mathrm{tr}([\phi, X][\phi, wYw^{-1}]) + \cdots . \tag{318}$$

[14] this is equal to $\prod_{i=1}^n X_i \prod_{i<j} (X_i^2 - X_j^2)$ for $SO(2n+1)$, and to $\prod_{i<j} (X_i^2 - X_j^2)$ for $SO(2n)$.
[15] this is equal to $\prod_{i=1}^n X_i \prod_{i<j} (X_i^2 - X_j^2)$ for $SO(2n+1)$, and to $\prod_{i<j} (X_i^2 - X_j^2)$ for $SO(2n)$.

Expanding $\phi = \sum_\alpha \phi_\alpha E_\alpha$ in the Cartan basis and computing the trace one finds

$$S = \mathrm{tr}(XwGw^{-1}) + \frac{1}{2}\sum_\alpha \phi_\alpha\phi_{-\alpha}(\alpha|X)(\alpha|Y). \tag{319}$$

The stationary phase approximation of the integral is, up to a normalization factor, then given by the RHS of (**??**). The normalisation factor is obtained with the help of the Weyl identity

$$\sum_w \epsilon_w e^{\sum_i X_i\rho_{\epsilon(i)}} = \Delta_G(e^X) \tag{320}$$

by considering $Y = \rho$, $X = \epsilon x$ and letting ϵ go to 0.

In order to prove the delta function identity lets take f an invariant function on the group and consider

$$I = \int_G dG f(G) \int_{\mathcal{G}} dX e^{i\mathrm{tr}(XP(G))}, \tag{321}$$

where dG is the normalized Haar measure. Lets define $\Delta_G(e^y) \equiv \prod_{\alpha>0}\left(e^{\frac{(\alpha|y)}{2}} - e^{-\frac{(\alpha|y)}{2}}\right)$, the product being over all positive roots. We can split both integrals over G and X as integral over the Cartan subgroup and integrals over the corresponding orbits, one gets

$$I = \frac{1}{|W|}\int_H dy \Delta_G^2(e^{iy}) f(e^{iy}) \int_{\mathcal{H}} dX \Delta_{\mathcal{G}}^2(X) \int_{\mathcal{G}} dU e^{i\mathrm{tr}(XUP(G)U^{-1})}. \tag{322}$$

The integral over H is over $y_i \in [0, 4\pi]$, $dy \equiv \prod_i \frac{dy_i}{4\pi}$ denotes the normalized measure and $|W|$ is the volume of the Weyl group. We can use the integral formula (317) with $Y = iP(e^{iy})$ to obtain

$$I = \frac{\Delta_{\mathcal{G}}(\rho)}{|W|}\sum_w \epsilon_w \int_H dy \frac{\Delta_G^2(e^{iy})f(e^{iy})}{\Delta_{\mathcal{G}}(iP(e^{iy}))}\int_{\mathcal{H}}\prod_i dX_i \Delta_{\mathcal{G}}(X) e^{i\sum_i X_i \sin(y_{w(i)}/2)}. \tag{323}$$

The factor $\Delta_{\mathcal{G}}(X)$ can be obtained by acting recursively with derivative of $y_i/2$. If we denote $Y_i = \sin(y_{w(i)}/2)$, then

$$I = \frac{\Delta_{\mathcal{G}}(4\rho)}{|W|}\sum_w \epsilon_w \int_H dy \frac{\Delta_G^2(e^{iy})f(e^{iy})}{\Delta_{\mathcal{G}}(iP(e^{iy}))}\Delta_{\mathcal{G}}\left(\frac{\partial}{i\partial Y_{w(i)}}\right)\int_{\mathcal{H}} dX e^{i\sum_i X_i \sin(y_{w(i)}/2)}. \tag{324}$$

We can reabsorb the sign ϵ_w in the differential operator and perform the sum over the Weyl group. We are left with

$$I = \Delta_{\mathcal{G}}(\rho)\int_H dy \frac{\Delta_G(e^{iy})f(e^{iy})}{\Delta_{\mathcal{G}}(iP(e^{iy}))}\Delta_{\mathcal{G}}\left(\frac{\partial}{i\partial Y_i}\right)\int_{\mathcal{H}} dX e^{i\sum_i X_i \sin(y_i/2)}. \tag{325}$$

The integral over X_i can be performed easily, giving rise to a factor $\prod_i 2\pi\delta(\sin(y_{w(i)}/2))$. This product is equal to $(4\pi)^n \prod_i(\delta(y_i) + \delta(y_i - 2\pi))$. We can then integrate out this delta function, thus

$$I = \Delta_{\mathcal{G}}(\rho) \sum_{\{n_i=0\}}^{1} \Delta_{\mathcal{G}}(i\frac{\partial}{\partial Y_i}) \left[\frac{\Delta_G^2(e^{iy})f(e^{iy})}{\Delta_{\mathcal{G}}(iP(e^{iy}))}\right] |_{y_i=2\pi n_i} \tag{326}$$

$$= \Delta_{\mathcal{G}}(\rho) \sum_{\{n_i=0\}}^{1} \Delta_{\mathcal{G}}(i\frac{\partial}{\partial Y_i}) \left[\Delta_G(e^{iy})\right] \frac{f(e^{iy})\Delta_G(e^{iy})}{\Delta_{\mathcal{G}}(iP(e^{iy}))} |_{y_i=2\pi n_i} . \tag{327}$$

where we have used that all the derivative should act on $\Delta_G(e^{iy})$ in order to get a non zero result. The derivative action on $\Delta_G(e^{iy})$ can be calculated with the Weyl identity (320), when evaluated at $y_i = 2\pi n_i$ it gives an additional factor $\Delta_{\mathcal{G}}(-2\rho) = (-1)^n\Delta_{\mathcal{G}}(2\rho)$ for $SO(2n)$ and $\Delta_{\mathcal{G}}(2\rho)\prod_i(-1)^{n_i+1}$ for $SO(2n)+1$. Finally the ratio $\Delta_G(e^{iy})/\Delta_{\mathcal{G}}(iP(e^{iy}))$ is equal to 2^r where r is the number of positive roots. Overall this gives

$$\Delta_{\mathcal{G}}^2(2\rho) \sum_{z\in Z_G} \epsilon_z f(z). \tag{328}$$

where $\epsilon_z = (-1)^n$ for $SO(2n)$, $\epsilon_z = \prod_i^n(-1)^{(n_i-1)}$ for $SO(2n+1)$ and $z = e^{iy}, y_i = 2\pi n_i$.

C Regularization of delta functions and q-deformation (abelian group)

In this section we consider how to regularize a delta function defined in a previous section using q-deformation. We restrict our consideration to the simplest example which is $U(1)$ group.

For a group element $g = e^{ix}$, the delta function is defined as in the previous section

$$\delta(g) = \int dX e^{XP(g)} = \delta(\sin x) = \sum_n (-1)^n \delta(x + \pi n) \tag{329}$$

We define

$$\delta_\beta(g_1, g_2) = e^{i\beta\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}}\delta(g_1)\delta(g_2) \tag{330}$$

Such expressions naturally show up in the partition function for BF theory with 'cosmological' term β.

(330) can be explicitly calculated by using a formula for the gaussian integral and finite translation operator and the result is a series of gaussian functions of

x_1, x_2:

$$\delta_\beta(g_1, g_2) = \frac{1}{\beta} \sum_{m,n=-\infty}^{\infty} (-1)^{m+n} e^{\frac{i}{\beta}(x_1+\pi m)(x_2+\pi n)} \tag{331}$$

This expression however is ill defined and one needs to find a way to regularize the infinite sum in it.

One can suggest the following regularization scheme. Choose an integer $N \gg 1$. As generally $\beta \ll 1$ we can choose $\beta = \frac{\pi}{2N}$. One can evaluate the sum in (331) at

$$x_1 = 2\pi\beta k_1, \quad x_2 = 2\pi\beta k_2, \tag{332}$$

where k_1 and k_2 are integers. As $\beta \ll 1$, any value of x_1 and x_2 can be well approximated by a value form a subset (332). By restricting the summation in (331) to $\{-M, M\}$ one obtains

$$\delta_\beta(g_1, g_2) = \frac{1}{\beta} \sum_{m,n=-M}^{M} (-1)^{m+n} e^{\frac{i2N}{\pi}(\pi k_1/N+\pi m)(\pi k_2/N+\pi n)} = \frac{1}{\beta} e^{\frac{i2\pi}{N} k_1 k_2} \tag{333}$$

Because the right hand side of (333) does not depend on M the limit $M \to \infty$ is well defined.

This expression for the regularized delta function is regular (with infinitely many regular derivatives) and periodic

$$\delta_\beta(g_1 e^{i2\pi n}, g_2 e^{i2\pi m}) = \delta_\beta(g_1, g_2) \tag{334}$$

and

$$\lim_{\beta \to 0} \delta_\beta(g_1, g_2) = \delta(g_1)\delta(g_2). \tag{335}$$

This construction can be thought of as a quantum deformation of the $U(1)$ group with $q = e^{i\beta}$. The non-commutativity of the coalgebra, the algebra of functions on the group, follows from the definition of non-commutative product in (330).

Differential and integral calculus on this group can be defined as

$$\frac{\partial}{\partial x} \to \frac{q^{\frac{\partial}{\partial x}} - q^{-\frac{\partial}{\partial x}}}{q - q^{-1}} \tag{336}$$

and

$$\int dg f(g) \to \beta \sum_{k=-N}^{N} f(e^{i2\pi\beta k}). \tag{337}$$

It is easy to derive the following relations for regularized delta functions

$$\int dg_1 \delta_\beta(g_{-1} g_2, g_3) \delta_\beta(g_1, g_4) = \delta_\beta(g_2, g_3) \delta_{k_2,k_4} \tag{338}$$

$$\int dg_1 dg_4 \delta_\beta(g_{-1}g_2, g_3)\delta_\beta(g_1, g_4)\delta_\beta(g_5 g_4^{-1}, g_6) = \delta_\beta(g_2, g_3)\delta_\beta(g_5 g_2^{-1}, g_6) \tag{339}$$

These relations are essential for the prove of topological invariance of the quantum deformed state sum model. At the same time all the relations are about regular functions and no divergences appear in the calculations.

References

[1] J. Polchinski ”String theory” Cambridge University Press 1998

[2] C. Rovelli ”Quantum gravity” Cambridge University Press 2004

[3] T. Thiemann, ”Introduction to modern canonical quantum general relativity” gr-qc/0110034

[4] M. P. Reisenberger and C. Rovelli. “Sum-over-surface form of loop quantum gravity”, gr-qc/9612035, Phys. Rev. D 56 (1997) 3490;

J. W. Barrett and L. Crane, “Relativistic spin networks and quantum gravity,” J. Math. Phys. **39**, 3296 (1998) [arXiv:gr-qc/9709028].

J. W. Barrett and L. Crane, “A Lorentzian signature model for quantum general relativity,” Class. Quant. Grav. **17**, 3101 (2000) [arXiv:gr-qc/9904025].

M. P. Reisenberger and C. Rovelli. “Spacetime as a Feynman diagram: the connection formulation”, Class.Quant.Grav., 18:121140, 2001;

A. Perez, “Spin foam models for quantum gravity,” Class. Quant. Grav. **20**, R43 (2003) [arXiv:gr-qc/0301113].

[5] C.Rovelli, L.Smolin ”Discretness of area and volume in quantum gravity” Nucl.Phys.B442 593-622 (1995) gr-qc/9411005

[6] L. Smolin, ”The classical limit and the form of the hamiltonian constraint in nonperturbative quantum gravity,” gr-qc/9609034

Lewandowski, J., and Marolf, D., “Loop constraints: A habitat and their algebra”, (1997),

Gambini, R., Lewandowski, J., Marolf, D., and Pullin, J., “On the consistency os the constraint algebra in spin network quantum gravity”, (September, 1997),

[7] E. Witten, ”Topological quantum field theory”, Commun.Math.Phys.117:353,1988

[8] E. Witten “(2+1)-Dimensional Gravity As An Exactly Soluble System,” Nucl. Phys. B **311**, 46 (1988).

[9] L. Crane ”Topological Field Theory As The Key To Quantum Gravity ” hep-th/9308126

[10] L. Smolin, Linking Topological Quantum Field Theory and Nonperturbative Quantum Gravity, J.Math.Phys. 36 (1995) 6417-6455, gr-qc/9505028

[11] L. Smolin, "A holographic formulation of quantum general relativity", hep-th/9808191, Phys.Rev. D61 (2000) 084007; Yi Ling, Lee Smolin, " Holographic Formulation of Quantum Supergravity", hep-th/0009018, Phys.Rev. D63 (2001) 064010

[12] J. Barrett, L. Crane, "Relativistic spin networks and quantum gravity" J.Math.Phys 39 (1998) 3296-3302, gr-qc/9709028

[13] L. Freidel, A. Starodubtsev, "Quantum gravity in terms of topological observables" hep-th/0501191

[14] A. Achucarro, P.K. Townsend, "A Cern-Simons action for three dimensional anti deSitter supergravity theories", Phys.Lett.B 180, 89 (1986).

[15] S. Alexander, L. Crane, M. D. Sheppeard, " The geometrization of matter proposal in the Barrett-Crane model and resolution of cosmological problems" gr-qc/0306079

[16] J.J. Atick, E. Witten, " The Hagedorn Transition And The Number Of Degrees Of Freedom Of String Theory ", Nucl.Phys. B310 (1988) 291

[17] T. Jacobson, L. Smolin " Nonperturbative Quantum Geometries" Nucl.Phys. B299 (1988) 295; T. Jacobson "1+1 Sector of 3+1 Gravity", gr-qc/9604003, Class.Quant.Grav. 13 (1996) L111

[18] Ingemar Bengtsson, Ted Jacobson " Degenerate Metric Phase Boundaries", gr-qc/9706027, Class.Quant.Grav. 14 (1997) 3109-3121; Erratum-ibid. 15 (1998) 3941-3942

[19] R. Capovilla, J. Dell, T. Jacobson "General Relativity without the metric" Phys.Rev.Lett.63:2325,1989; R. Capovilla, M. Montesinos, V. A. Prieto, E. Rojas, " BF gravity and the Immirzi parameter", gr-qc/0102073, Class.Quant.Grav. 18 (2001) L49-; Erratum-ibid. 18 (2001) 1157

[20] S.W. MacDowell, F. Mansouri, "Unified Geometric theory of gravity and supergravity". Phys.Rev.Lett.38:739,1977., Erratum-ibid.38:1376,1977.

[21] S. Deser, R. Jackiw and G. 't Hooft, "Three-Dimensional Einstein Gravity: Dynamics Of Flat Space," Annals Phys. **152**, 220 (1984).

[22] A.P.Balachandran et al, "Gauge symmetries and fiber bundles: applications to particle dynamics", Lect.Note.Physics 188 1-140 (1983).

[23] P. de Sousa Gerbert, "On Spin And (Quantum) Gravity In (2+1)-Dimensions," Nucl. Phys. B **346**, 440 (1990).

[24] L.Freidel, D.Louapre, "Ponzano-Regge model revisited I: Gauge fixing, observables and interacting spinning particles", Class.Quant.Gravity 21:5685-5726, (2004) hep-th/0401086

[25] R.M.Wald, "Quantum field theory in curvad space time and black hole termodynamics", Chicago Univ. Press 1995.

[26] S. Carlip, "Quantum Gravity In 2+1 Dimensions," Cambridge monographs on Mathematical Physics.

[27] S. Carlip, "The statistical mechanics of the three-dimensional Euclidean black hole," Phys. Rev. D **55**, 878 (1997) [arXiv:gr-qc/9606043].

[28] C. Rovelli, "Loop Quantum Gravity." Living Reviews in Relativity, 1998-1, http://www.livingreviews.org/ and e-print: gr-qc/9710008.

T. Thiemann, Lectures on Loop Quantum Gravity, gr-qc/0210094

[29] T. Thiemann, "Anomaly-free formulation of non-perturbative, four-dimensional Lorentzian quantum gravity." Phys. Lett. **B380** (1996) 257, e-print: gr-qc/9606088.

T. Thiemann, "Quantum Spin Dynamics (QSD)" Class. Quant. Grav. **15** (1998) 839, e-print: gr-qc/9606089.

[30] H. Kodama, "The wave function of the universe" Prog. Theor. Phys 80, 1024 (1988); Phys. Rev. D42(1990)2548.

[31] A. Ashtekar, "New Variables for Classical and Quantum Gravity," Phys. Rev. Lett. 51(18), 2244-2247, (1986)

[32] L. Smolin, "Quantum gravity with a positive cosmological constant", hep-th/0209079

[33] T. Jacobson and L. Smolin, "Nonperturbative quantum geometries." Nucl. Phys. **B299** (1988) 295.

[34] C. Rovelli and L. Smolin, "Knot theory and quantum gravity." Phys. Rev. Lett. 61 (1988) 1155.

C. Rovelli and L. Smolin, "Loop representation of quantum general relativity" Nucl. Phys. **B331** (1990) 80.

[35] C. Rovelli, "The projector on physical states in loop quantum gravity", gr-qc/9806121

[36] K. Krasnov, "On Quantum Statistical Mechanics of a Schwarzschild Black Hole", Gen.Rel.Grav. 30 (1998) 53-68, gr-qc/9605047

[37] E. Witten, Nucl. Phys. B331 (1998) 46.

[38] V. Husain, S. Major, "Gravity and BF theory defined in bounded regions", Nucl.Phys. B500 (1997) 381-401, gr-qc/9703043

[39] J. D. Brown, J. W. York, "Quasilocal Energy and Conserved Charges Derived from the Gravitational Action", gr-qc/9209012

[40] S. R. Lau, "New variables, the gravitational action, and boosted quasilocal stress-energy-momentum", Class.Quant.Grav. 13 (1996) 1509-1540, gr-qc/9504026

[41] B. DeWitt, "Quantum theory of gravity. 1. The canonical theory.", Phys.Rev.160:1113-1148,(1967)

[42] S. Alexandrov, D. Vassilevich, "Area spectrum in Lorentz covariant loop gravity," Phys.Rev. D64 (2001) 044023, gr-qc/0103105,

S. Alexandrov, E. R. Livine, "SU(2) Loop Quantum Gravity seen from Covariant Theory," Phys.Rev. D67 (2003) 044009, gr-qc/0209105

[43] T. Thiemann, QSD VI : "Quantum Poincaré Algebra and a Quantum Positivity of Energy Theorem for Canonical Quantum Gravity", Class.Quant.Grav. 15 (1998) 1463-1485, gr-qc/9705020

[44] S.Major, "Quasilocal Energy for Spin-net Gravity", Class.Quant.Grav. 17 (2000) 1467-1487, gr-qc/9906052

[45] Giovanni Amelino-Camelia, Lee Smolin, Artem Starodubtsev, "Quantum symmetry, the cosmological constant and Planck scale phenomenology" Class.Quant.Grav. 21 (2004) 3095-3110, hep-th/0306134

Laurent Freidel, Jerzy Kowalski-Glikman, Lee Smolin, "2+1 gravity and Doubly Special Relativity" Phys.Rev. D69 (2004) 044001, hep-th/0307085

[46] J. F. Plebanski, "On the separation between Einsteinien substuctures", J. Math. Phys. (12) 1977, 2511. R. Capovilla, T. Jacobson, J. Dell and L. Mason, "Self-dual two forms and gravity," Class. Quant. Grav. **8** (1991) 41. R. De Pietri and L. Freidel, "so(4) Plebanski Action and Relativistic Spin Foam Model," Class. Quant. Grav. **16**, 2187 (1999) [arXiv:gr-qc/9804071]. M. P. Reisenberger, "Classical Euclidean general relativity from *left-handed area = right-handed area*," arXiv:gr-qc/9804061.

[47] L. Freidel, K. Krasnov and R. Puzio, "BF description of higher-dimensional gravity theories," Adv. Theor. Math. Phys. **3**, 1289 (1999) [arXiv:hep-th/9901069].

[48] L. Smolin and A. Starodubtsev, "General relativity with a topological phase: An action principle," arXiv:hep-th/0311163.

[49] L. Freidel, J. Kowalski-Glikman, A. Starodubtsev, "Background independent perturbation theory for gravity coupled to particles: classical analysis". To appear.

[50] J. C. Baez, "Four-Dimensional BF theory with cosmological term as a topological quantum field theory," Lett. Math. Phys. **38**, 129 (1996) [arXiv:q-alg/9507006].

[51] H. T. Nieh and M. L. Yan, "An Identity In Riemann-Cartan Geometry," J. Math. Phys. **23**, 373 (1982). O. Chandia and J. Zanelli, "Torsional topological invariants (and their relevance for real life)," arXiv:hep-th/9708138.

[52] S. Holst, "Barbero's Hamiltonian derived from a generalized Hilbert-Palatini action," Phys. Rev. D **53**, 5966 (1996) [arXiv:gr-qc/9511026].

[53] A. Ashtekar and J. Lewandowski, "Background independent quantum gravity: A status report," Class. Quant. Grav. **21**, R53 (2004) [arXiv:gr-qc/0404018].

[54] A. Ashtekar, J. C. Baez and K. Krasnov, "Quantum geometry of isolated horizons and black hole entropy," Adv. Theor. Math. Phys. **4**, 1 (2000) [arXiv:gr-qc/0005126]. M. Domagala and J. Lewandowski, "Black hole entropy from quantum geometry," Class. Quant. Grav. **21**, 5233 (2004) [arXiv:gr-qc/0407051].

[55] S. Alexandrov and E. R. Livine, "SU(2) loop quantum gravity seen from covariant theory," Phys. Rev. D **67**, 044009 (2003) [arXiv:gr-qc/0209105].

[56] L. Freidel and A. Starodubtsev, "Perturbative gravity via spin foam" To appear.

[57] L. Freidel and K. Krasnov, "Spin foam models and the classical action principle," Adv. Theor. Math. Phys. **2**, 1183 (1999) [arXiv:hep-th/9807092].

[58] E. Witten, "Two-dimensional gauge theories revisited," J. Geom. Phys. **9** (1992) 303 [arXiv:hep-th/9204083].

[59] A. Starodubtsev "Topological excitations around the vacuum of quantum gravity I: the symmetries of the vacuum", hep-th/0306135.

[60] H. Sahlmann, “Some comments on the representation theory of the algebra underlying loop quantum gravity,” arXiv:gr-qc/0207111. H. Sahlmann and T. Thiemann, “Irreducibility of the Ashtekar-Isham-Lewandowski representation,” arXiv:gr-qc/0303074. A. Okolow and J. Lewandowski, “Diffeomorphism covariant representations of the holonomy-flux *-algebra,” Class. Quant. Grav. **20**, 3543 (2003) [arXiv:gr-qc/0302059].

[61] N. Kawamoto, N. Sato and Y. Uchida, “4-dimensional BF gravity on the lattice,” Nucl. Phys. B **574**, 809 (2000) [arXiv:hep-th/9911228]. L. Freidel and D. Louapre, “Diffeomorphisms and spin foam models,” Nucl. Phys. B **662**, 279 (2003) [arXiv:gr-qc/0212001].

[62] R. De Pietri, L. Freidel, K. Krasnov and C. Rovelli, “Barrett-Crane model from a Boulatov-Ooguri field theory over a homogeneous space,” Nucl. Phys. B **574**, 785 (2000) [arXiv:hep-th/9907154].

[63] H.Ooguri, "Topological Lattice Models in Four Dimensions", Mod.Phys.Lett. A7 (1992) 2799-2810 hep-th/9205090

[64] L.Crane, D.Yetter, "A cathegorical construction of 4D TQFT" hep-th/9301062

[65] J. W. Barrett, J. M. Garcia-Islas and J. F. Martins, “Observables in the Turaev-Viro and Crane-Yetter models,” arXiv:math.qa/0411281.

[66] P. Zinn-Justin and J. B. Zuber, “On some integrals over the U(N) unitary group and their large N limit,” J. Phys. A **36**, 3173 (2003) [arXiv:math-ph/0209019].