

On Asymmetric Induced Saturation

by

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Author's Declaration

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of Contributions

The content presented in this thesis is based on an article in preparation, entitled “*On asymmetric induced saturation*,” which is joint work with Xinyue Fan, Sepehr Hajebi, and Sophie Spirkl. Several passages in this thesis are taken verbatim from the above-mentioned manuscript, which are all written by me.

Abstract

Given a graph H , a graph G is H -free if no induced subgraph of G is isomorphic to H . A graph G is H -induced-saturated if G is H -free but deleting or adding any edge in G creates an induced copy of H . The notion of induced saturation originated in a 2012 work by Martin and Smith [23] concerning the extremal properties of H -induced-saturated graphs.

On the structural side, a large body of work since then has been devoted to the study of graphs H for which H -induced-saturated graphs *do* exist in the first place. We say that H is *normal* if there exists an H -induced-saturated graph. It is immediate that complete graphs (except when on two vertices) are not normal because deleting edges from a graph cannot increase its clique number. Similarly, empty graphs (except when on two vertices) are not normal because adding edges to a graph cannot increase its independence number. Beyond these trivial cases, however, characterizing normal graphs is quite difficult: The four-vertex path is the only other graph currently known *not* to be normal, and very few graphs are known to be normal. In particular, it remains open whether all even cycles are normal.

In this thesis, we study the analogous notions with only one of the two operations – edge deletion or addition – required to create an induced copy of H . Given a graph H , we say that a graph G is H -deletion-saturated if G is H -free, has at least one edge, and deleting any edge in G creates an induced copy of H . We say that H is *deletion-normal* if such a graph G exists. (The complementary notions of H -addition-saturated and *addition-normal* are defined similarly.) These “asymmetric” weakenings of induced saturation appear to be more tractable. For example, in contrast to the fact that, as mentioned above, it remains open whether all even cycles are normal, Tennenhouse [28] proved that all even cycles are addition-normal, and with Fan, Sepehr Hajebi, and Spirkl, we proved recently [16] that all even cycles are deletion-normal.

We conjecture that every non-complete graph H is deletion-normal (or equivalently, that every non-empty graph H is addition-normal), and provide evidence for our conjecture by proving it for a variety of graphs H , including all complete multipartite graphs with a unique largest part, line graphs of all trees, all triangle-free graphs with exactly one cycle, and all graphs on at most six vertices.

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I thank my undergraduate supervisor, Ramin Javadi, for introducing me to mathematical research, and Mahmood Manjegani for his support throughout my bachelor's studies.

My heartfelt appreciation goes to my family: to my parents, Sepideh and Majid, for their constant love and support; to my sister, Setareh, whose presence has always been a source of comfort to me in difficult times; and to my sister-in-law, Hoda, for her kindness and support, especially since I moved to Canada.

Finally, special thanks go to my brother, Sepehr, who introduced me to mathematics and has supported me selflessly throughout my life, and in particular throughout the writing of this thesis with his excellent feedback.

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Chapter 1

Introduction

1.1 Background and motivation

This thesis explores the “asymmetric” relaxation of the induced saturation problem on graphs. Given a graph H , a graph G is H -free if G has no induced subgraph isomorphic to H . A graph G is H -induced-saturated if G is H -free and this H -freeness is critical with respect to the full adjacency profile of the graph; that is:

- G is H -free;
- for every $e \in E(G)$, $G - e$ is not H -free; and
- for every $e \in E(\overline{G})$, $G + e$ is not H -free.

Induced saturation originated in a 2012 work by Martin and Smith [23] (see also [27]), where they defined this notion as a property of “trigraphs” and studied its extremal aspects such as Turán-type problems about the maximum number of “undecided” pairs of vertices in an n -vertex trigraph that is H -induced-saturated. (This line of work, in turn, was motivated by the heavily investigated topic of “non-induced” saturation in extremal combinatorics. We omit the details on that front; see [10] for an exhaustive survey.)

Since then, the literature on this topic has evolved around the study of graphs H for which H -induced-saturated graphs exist in the first place. For instance, it is readily seen that if a graph H is complete or empty (except when $|V(H)| = 2$), then there is no H -induced-saturated graph. Bonamy et al. recently proved [6] that, for all other graphs H , an “infinite H -induced-saturated graph” always exists.

Theorem 1.1 (Bonamy, Groenland, Johnston, Morrison, and Scott [6]). *For every (finite) graph H that is neither complete nor empty, there is a countably infinite graph G such that:*

- G does not contain H as a finite induced subgraph;

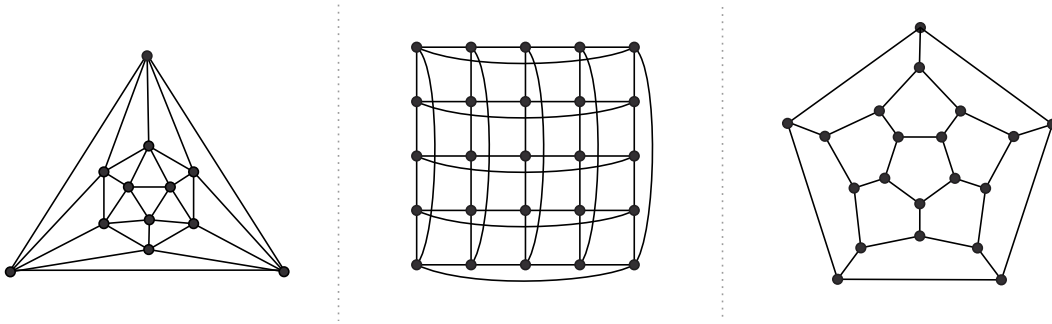


Figure 1.1: The icosahedron (left), the Cartesian product of two 5-cycles (middle), and the dodecahedron (right).

- for every $e \in E(G)$, the infinite graph $G - e$ contains H as an induced subgraph; and
- for every $e \in E(\overline{G})$, the infinite graph $G + e$ contains H as an induced subgraph.

(In fact, they prove something stronger, that there is countably infinite H -free graph G such that any “bounded degree perturbation” of that infinite graph creates an induced copy of H .)

Our focus, in turn, is on graphs H for which there exists a finite H -induced saturated graph. We call such graphs *normal*. It follows that complete graphs and empty graphs (unless on two vertices) are not normal. Martin and Smith [23] also proved that:

Theorem 1.2 (Martin and Smith [23]). *The four-vertex path P_4 is not normal.*

Complete graphs, empty graphs, and the four-vertex path remain the only graphs known not to be normal, and, as far as we know, it could be that all other graphs are normal. However, even for structurally rather simple graphs H , deciding whether H is normal appears to be quite difficult. For instance, it remains wide open whether all even cycles are normal. With Fan, Sepehr Hajebi, and Spirkł, we showed the following; see Figure 1.1. (The case of C_4 was also observed in [3]).

Observation 1.3 (Fan, Hajebi, Hajebi, Spirkł [16]). *Cycles of length 4, 6, 8 and 10 are normal. In particular, the icosahedron is C_4 -induced-saturated, the Cartesian product of two 5-cycles is C_6 -induced-saturated, and the dodecahedron is C_8 -induced-saturated.*

Via computer search, we also found [16] a 28-vertex graph that is C_{10} -induced-saturated. However, the 10-cycle is the largest even cycle currently known to be normal.

Problem 1.4. *Are all even cycles (on 12 or more vertices) normal?*

This is particularly curious because all odd cycles (except for the 3-cycle, which is a complete graph) are known to be normal, and the proof, due to Behrens et al. [3], is surprisingly simple.

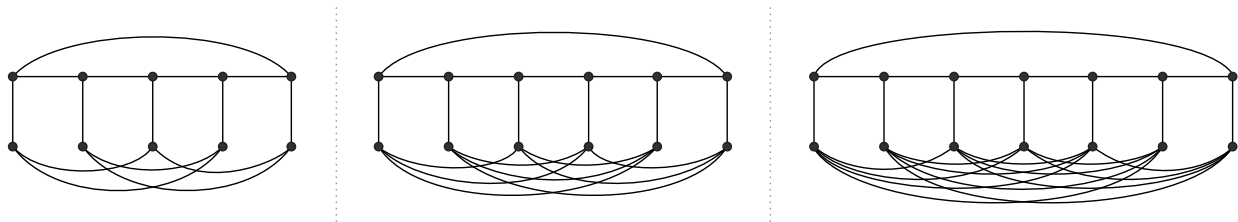


Figure 1.2: From left to right: the graphs D_5 , D_6 , and D_7 .

Theorem 1.5 (Behrens, Erbes, Santana, Yager, and Yeager [3]). *All odd cycles except for the 3-cycle are normal. Specifically, for every integer $t \geq 3$, the graph $L(K_{t,t})$ is C_{2t-1} -induced-saturated.*

Another family of structurally simple graphs H for which the full proof of normality is surprisingly difficult is that of paths. For small paths, the solution is immediate: P_1 is normal (because the null graph is P_1 -induced-saturated), P_2 is normal (because empty graphs on two or more vertices are P_2 -induced-saturated), P_3 is normal (because complete graphs on three or more vertices are P_3 -induced-saturated), and by Theorem 1.2, P_4 is not normal. Moreover, Bonamy et al. [5, 6] showed the following.

Theorem 1.6 (Bonamy, Groenland, Johnston, Morrison, and Scott [5, 6]). *The complement of the icosahedron is P_5 -induced-saturated.*

In contrast, progress on paths with six or more vertices was much slower. First, Raty [26] proved that P_6 is normal by observing that the Clebsch graph is P_6 -induced-saturated. Cho et al. [9] later gave constructions of P_{3s} -induced-saturated graphs for every $s \geq 2$. Only recently did Dvořak [13] find a complete proof that P_t is normal for all $t \geq 6$.

Dvořak’s construction of P_t -induced-saturated graphs is elegant: For each $t \in \mathbb{N}$, define D_t to be the graph obtained from the disjoint union of a t -cycle and its complement by adding a perfect matching between the two. More precisely, D_t has $2t$ vertices $\{u_i, v_i : i \in \{1, \dots, t\}\}$ such that for all distinct $i, j \in \{1, \dots, t\}$, we have:

- u_i, u_j are adjacent if and only if $|i - j| \in \{1, t - 1\}$;
- v_i, v_j are adjacent if and only if $|i - j| \notin \{1, t - 1\}$; and
- u_i, v_j are adjacent if and only if $i = j$.

For instance, D_5 is isomorphic to the Petersen graph (see Figure 1.2).

Dvořak proved in [13] that:

Theorem 1.7 (Dvořak [13]). *For every integer $t \geq 6$, the graph D_{t-1} is P_t -induced-saturated.*

Putting together Theorems 1.2, 1.6, and 1.7 yields the following, which completely resolves the case of paths:

Corollary 1.8. *Let $t \in \mathbb{N}$. Then P_t is normal if and only if $t \neq 4$.*

In addition, some other types of graphs H were shown to be normal in subsequent work by Behrens et al. [3] and by Axenovich and Csikós [2]. Beyond that, a full characterization of all normal graphs remains wide open.

1.2 Our take on induced saturation

Our focus here is on graphs in which H -freeness is only required to be sensitive to *one* of the two operations of deleting or adding edges. Formally, given a graph H , we say that a graph G is *H -deletion-saturated* if

- G is H -free and $E(G) \neq \emptyset$; and
- for every $e \in E(G)$, $G - e$ is not H -free.

Similarly, we say that a graph G is *H -addition-saturated* if

- G is H -free and $E(\overline{G}) \neq \emptyset$; and
- for every $e \in E(\overline{G})$, $G + e$ is not H -free.

We say that H is *deletion-normal* if there exists a graph G that is H -deletion-saturated, and that H is *addition-normal* if there exists a graph G that is H -addition-saturated.

These notions appear to be more tractable than symmetric normality. For example, while it remains open whether all even cycles are normal, it is known that all even cycles are both deletion-normal [16] and addition-normal [28] (although the proofs are rather involved, especially that of deletion-normality). The case of addition was proved in 2016 by Tennenhouse [28]:

Theorem 1.9 (Tennenhouse [28]). *All even cycles are addition-normal.*

With Fan, Sepehr Hajebi, and Spirk, we recently proved the deletion case:

Theorem 1.10 (Fan, Hajebi, Hajebi, Spirk [16]). *All even cycles are deletion-normal.*

Furthermore, in contrast to Theorem 1.2, P_4 is readily seen to be both deletion- and addition-normal: the five-cycle C_5 is P_4 -deletion-saturated, and the two-edge matching $\overline{C}_4$ is P_4 -addition-saturated. In fact, complete graphs on two or more vertices are the only graphs known not to be deletion-normal, and empty graphs on two or more vertices are the only graphs known not to be addition-normal.

We therefore conjecture the following:

Conjecture 1.11. *Every non-complete graph H is deletion-normal. Every non-empty graph H is addition-normal.*

Our contribution is the proof of Conjecture 1.11 for a variety of graphs H , which we believe provides substantial evidence for both statements in Conjecture 1.11. Specifically, our results imply the following. (All definitions will appear in Chapter 2).

Theorem 1.12. *The following graphs, if not complete, are deletion-normal.*

- (a) *The complete bipartite graph $K_{s,t}$ for all $s, t \in \mathbb{N} \cup \{0\}$ with $s < t$.*
- (b) *The complete bipartite graph $K_{t,t}$ where $t = \binom{2r-1}{r-1} + 1$ or $t = \binom{2r}{r} + 1$ for some $r \in \mathbb{N}$.*
- (c) *Every graph obtained by iteratively adding isolated vertices to a deletion-normal graph.*
- (d) *Every graph obtained by iteratively adding vertices of degree at most one to a 2-connected deletion-normal graph.*
- (e) *Line graphs of all trees.*
- (f) *All line^- graphs and biline^- graphs, and the complements of all line^+ graphs and all biline^+ graphs.*
- (g) *All graphs on at most six vertices.*

Each of the subsequent chapters of this thesis is devoted to the proof of one or two of the outcomes of Theorem 1.12: In Chapter 3, we prove 1.12(a) and 1.12(b), in Chapter 4, we prove 1.12(c) and 1.12(d), and in Chapter 5, we prove 1.12(e) and 1.12(f). In fact, instead of each of 1.12(a), 1.12(b), 1.12(c), and 1.12(d), we prove a much stronger result in the corresponding chapter.

Regarding Theorem 1.12 (g), up to isomorphism, there are **202** non-complete graphs on at most six vertices, all depicted in Figure 1.3 (see, for example, [24]). The first six outcomes of Theorem 1.12 (or their strengthenings, to be more precise) establish the deletion-normality of many of them – we include this corollary of each outcome in the corresponding chapter. In particular, we handle

- **23** graphs in Chapter 3;
- **69** graphs in Chapter 4; and
- **85** graphs in Chapter 5.

(Some graphs could be handled in many different ways, in which case we have broken the tie arbitrarily.)

There remains a total of **25** “sporadic” graphs on at most six vertices, each of which requires new ideas for the proof of their deletion-normality. We handle these graphs in Chapter 6.

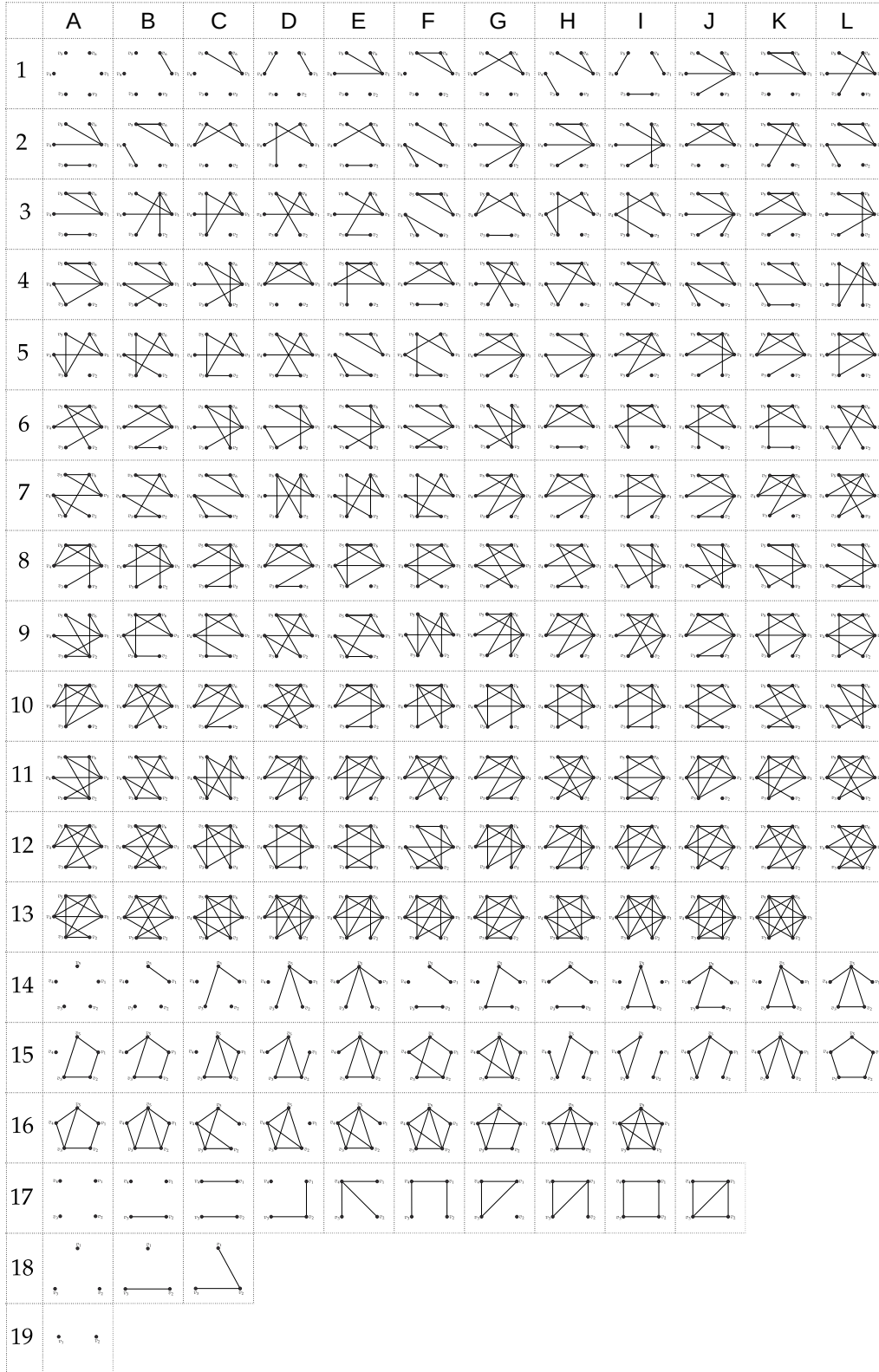


Figure 1.3: All non-complete graphs on at most six vertices (up to isomorphism).

Chapter 2

Preliminaries

We denote the set of all positive integers by $\mathbb{N}$, and for integers k, k' , we write $\{k, \dots, k'\}$ to denote the set of all integers no smaller than k and no larger than k' (thus, we have $\{k, \dots, k'\} \neq \emptyset$ if and only if $k \leq k'$). Graphs in this thesis have finite vertex sets, no loops, and no parallel edges (unless specified otherwise).

Below, we cover some notation and terminology that will be used throughout the thesis. For standard graph-theoretic terminology, the reader is referred to [12].

2.1 Graphs

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For $X \subseteq V(G)$, we use X to denote both the set X of vertices and the induced subgraph $G[X]$ of G with vertex set X , also called *the subgraph of G induced by X* . For $v \in V(G)$, we write $N_G(v)$ for the set of all neighbors of v in G . A *leaf* of G is a degree-one vertex. We write $\overline{G}$ for the complement of G . For $e \in E(G)$, we denote by $G - e$ the graph obtained from G by removing the edge e . For $e \in E(\overline{G})$, we denote by $G + e$ the graph obtained from G by adding the edge e .

We say that two subsets X, Y of $V(G)$ are *complete in G* (or that *X is complete to Y in G*) if $X \cap Y = \emptyset$ and every vertex in X is adjacent to every vertex in Y . We also say that X, Y are *anticomplete in G* (or that *X is anticomplete to Y in G*) if $X \cap Y = \emptyset$ and every vertex in X is nonadjacent to every vertex in Y . For a vertex $x \in V(G)$, we say that a vertex x is *complete (anticomplete) to Y in G* if $\{x\}$ is complete (anticomplete) to Y in G .

A *clique* in G is a set of pairwise adjacent vertices, and a *stable set* is a set of pairwise nonadjacent vertices. The *chromatic number* of G , denoted $\chi(G)$, is the minimum number of stable sets into which $V(G)$ can be partitioned.

For $k \in \mathbb{N}$ and a graph H , we denote by kH the graph obtained from the disjoint union of k isomorphic copies of H . For graphs $H_1, \dots, H_k$, we denote by $H_1 + \dots + H_k$ the graph obtained from the disjoint union of isomorphic copies of $H_1, \dots, H_k$.

2.2 Special graphs

We say that a graph G is *complete* if $E(G)$ is the set of all 2-subsets of $V(G)$, and that G is *empty* if $E(G) = \emptyset$. For every $n \in \mathbb{N}$, we denote the n -vertex complete graph by K_n .

For $m, n \in \mathbb{N} \cup \{0\}$, we denote by $K_{m,n}$ the complete bipartite graph with parts of cardinality m and n . The complete bipartite graph $K_{1,n}$, for $n \in \mathbb{N} \cup \{0\}$, is also called a *star*. In general, for $k \in \mathbb{N}$ with $k \geq 2$ and $n_1, \dots, n_k \in \mathbb{N} \cup \{0\}$, we denote by $K_{n_1, \dots, n_k}$ the complete k -partite graph with parts of cardinality $n_1, \dots, n_k$.

Let P be a path. We write $P = p_1 - \dots - p_k$ to indicate that $V(P) = \{p_1, \dots, p_k\}$, for some $k \in \mathbb{N}$, and that $E(P) = \{p_i p_{i+1} : i \in \{1, \dots, k-1\}\}$. We call p_1 and p_k the *ends* of P . The *length* of P is the number of edges in P . For every $k \in \mathbb{N}$, we denote the unique k -vertex path (that is, the path of length $k-1$) up to isomorphism by P_k . Given a graph G , a *path in G* is an induced subgraph of G that is a path.

Let C be a cycle. We write $C = c_1 - \dots - c_k - c_1$ to indicate that $V(C) = \{c_1, \dots, c_k\}$, for some $k \in \mathbb{N}$, and that $E(C) = \{c_i c_{i+1} : i \in \{1, \dots, k-1\}\} \cup \{c_k c_1\}$. The *length* of C is the number of edges (or equivalently, the number of vertices) in C , and a cycle of length k is also called a *k -cycle*. For every $k \in \mathbb{N}$, we denote the unique k -cycle up to isomorphism by C_k . Given a graph G , the *girth* of G is the shortest length of a cycle that is an (induced) subgraph of G .

2.3 k -graphs

Let $k \in \mathbb{N}$. A *k -graph* (also called a *k -uniform hypergraph*) is a pair $\Gamma = (V(\Gamma), E(\Gamma))$ where $V(\Gamma)$ is a finite set of *vertices* and $E(\Gamma)$ is a set of k -subsets of $V(\Gamma)$, called the *hyperedges*.

For $n \in \mathbb{N}$, we denote by K_n^k the n -vertex *complete k -graph*; that is, the unique k -graph (up to isomorphism) in which every k -subset of the vertex set is a hyperedge. Note that, with this terminology, graphs are exactly the 2-graphs, and $K_n = K_n^2$ for every $n \in \mathbb{N}$.

Let Γ be a k -graph for some $k \geq 2$. The *chromatic number* of Γ is the smallest $c \in \mathbb{N}$ for which there is a partition $(S_1, \dots, S_c)$ of $V(\Gamma)$ such that no hyperedge of Γ is contained entirely in one of the sets $S_1, \dots, S_c$. The *girth* of Γ is the smallest $g \in \mathbb{N}$ for which there are g pairwise distinct hyperedges of Γ whose union contains at most $g(k-1)$ vertices. Given a vertex $v \in V(\Gamma)$, the *degree of v in Γ* is the number of hyperedges of Γ containing v . The *line graph* of a k -graph Γ , denoted $L(\Gamma)$, is the graph (or the 2-graph) with vertex set $E(\Gamma)$ such that distinct hyperedges $f_1, f_2 \in E(\Gamma)$ are adjacent as vertices in $L(\Gamma)$ if and only if $f_1 \cap f_2 \neq \emptyset$. Note that $L(K_n^k)$ for $n \geq 2k$ are the same as the famous *Kneser graphs* [19].

Observe that in the case $k = 2$, the above notions match the analogous ones for graphs (we will mention the line graph of a graph again in the next section). Note also that if the girth of Γ is larger than 2, then Γ is *linear*; that is, every two hyperedges in Γ share at most one vertex.

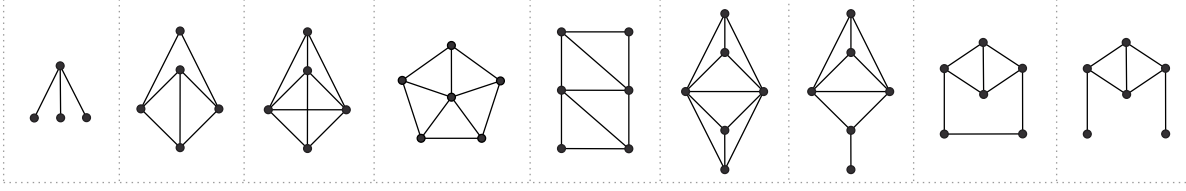


Figure 2.1: Induced subgraph obstructions to line graphs.

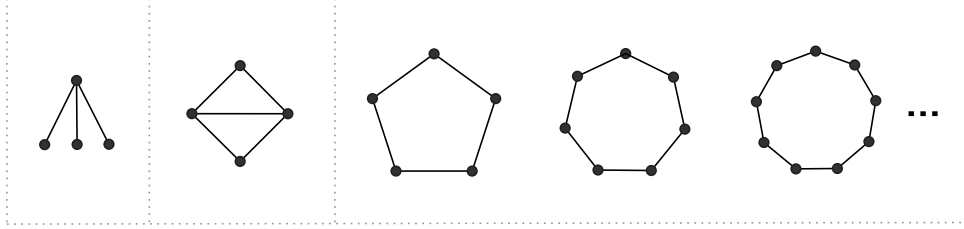


Figure 2.2: Induced subgraph obstructions to biline graphs.

2.4 Line graphs

Given a graph H , the *line graph* of H , denoted $L(H)$, is the graph with vertex set $E(H)$ such that distinct edges $e, e' \in E(H)$ of H are adjacent as vertices in $L(H)$ if and only if e and e' share an end in H . We say that a graph G is a *line graph* if there exists a graph H such that $G = L(H)$, and that G is a *biline graph* if there is a bipartite graph H such that $G = L(H)$. There are well-known characterizations of both line graphs and biline graphs in terms of their forbidden induced subgraphs. (Recall that the *claw* is the star $K_{1,3}$, the *diamond* is the graph obtained from K_4 by removing an edge, and an *odd hole* is an induced cycle of length odd length at least five. See Figure 2.2.)

Theorem 2.1 (Beineke [4]). *A graph G is a line graph if and only if G has no induced subgraph isomorphic to any of the nine graphs in Figure 2.1.*

Theorem 2.2 (See Peterson [25]). *A graph G is a biline graph if and only if G is claw-free, diamond-free, and odd-hole-free.*

We are also interested in graphs that are “an edge away” from being a (bi)line graph. We say that a graph G is a *line⁻graph* if G is not a line graph, but there exists $e \in E(\overline{G})$ such that $G + e$ is a line graph. We also say that G is a *line⁺graph* if G is not a line graph, but there exists $e \in E(G)$ such that $G - e$ is a line graph. Intuitively, line⁻graphs are graphs that are “an edge less than” a line graph, and line⁺graphs are graphs that are “an edge more than” a line graph. Similarly, we say that G is a *biline⁻graph* if G is not a biline graph, but there exists $e \in E(\overline{G})$ such that $G + e$ is a biline graph, and that G is a *biline⁺graph* if G is not a biline graph, but there exists $e \in E(G)$ such that $G - e$ is a biline graph. Again, this roughly means that biline⁻graphs are graphs that are “an edge less than” a biline graph, and biline⁺graphs are graphs that are “an edge more than” a biline graph.

Chapter 3

Complete multipartite graphs

In this chapter, we prove Theorem 1.12(a) and (b). In fact, we prove two stronger results about complete multipartite graphs.

We say that a complete multipartite graph H is *lopsided* if H has a unique part with the maximum number of vertices. It follows that a lopsided complete multipartite graph H is a complete graph if and only if $|V(H)| = 1$.

Instead of Theorem 1.12(a), we prove that:

Theorem 3.1. *Every lopsided complete multipartite graph with at least two vertices is deletion-normal.*

Also, instead of Theorem 1.12(b), we prove that:

Theorem 3.2. *Let $r \in \mathbb{N}$ and let*

$$t \in \left\{ \binom{2r-1}{r-1} + 1, \binom{2r}{r} + 1 \right\}.$$

Then, for every $s \in \mathbb{N}$, the graph $K_{\underbrace{t, \dots, t}_{s+1}}$ is deletion-normal.

The proof of Theorem 3.1 uses a construction from [7], which, for every $t \geq 2$, yields K_{t+1} -free graphs with arbitrarily large chromatic number whose K_t -free induced subgraphs have small chromatic number. We use this construction, for a suitable choice of t based on the graph H from Theorem 3.1, and choose a K_{t+1} -free graph G from the construction with chromatic number large enough (again, depending on H). We make this graph G “edge-maximal” with respect to remaining K_{t+1} -free, and we show that the resulting graph is $\overline{H}$ -addition-saturated.

For Theorem 3.2, we take different approaches based on the choice of r and t . For $t = 2$ (meaning $r = 1$ and $t = \binom{2r-1}{r-1} + 1$), the proof uses the disjoint union of multiple copies of the icosahedron. For the other cases, the proof relies on the line graphs of complete k -graphs, and variants of them, as well as corresponding variants of the Erdős–Ko–Rado Theorem [15] regarding the maximum cardinality of a clique in such graphs.

3.1 The lopsided case

We need the following result from [7]:

Theorem 3.3 (Briański, Davies, Walczak [7]). *For all $c, t \in \mathbb{N}$ with $c \geq t \geq 2$, there is a K_{t+1} -free graph G with $\chi(G) = c$ such that for every K_t -free induced subgraph G' of G , we have $\chi(G') < 27t^3$.*

We can now give a proof of Theorem 3.1:

Proof of Theorem 3.1. We show that for every lopsided complete multipartite graph H with at least two vertices, $\overline{H}$ is addition-normal. Let $s \in \mathbb{N} \cup \{0\}$ be such that H has $s+1$ parts. Then $\overline{H}$ has $s+1$ components, each of which is a clique. In particular, if $\overline{H}$ is connected, then $\overline{H}$ is a complete graph on two or more vertices, which is clearly addition-normal. Therefore, since H is lopsided, we may assume that for some $s, t \in \mathbb{N}$, $\overline{H}$ has exactly $s+1$ components, one of which is isomorphic to K_{t+1} , and each of the remaining s components is a complete graph on t or fewer vertices. In other words,

(1) K_{t+1} is isomorphic to an induced subgraph of $\overline{H}$, and $\overline{H}$ is isomorphic to an induced subgraph of $\overline{K_{t+1, \underbrace{t, \dots, t}_s}}$.

By (1), in order to prove that $\overline{H}$ is addition-normal, it suffices to show that there exists a K_{t+1} -free graph G such that for every $xy \in E(\overline{G})$, the graph $G + xy$ has an induced subgraph isomorphic to $\overline{K_{t+1, \underbrace{t, \dots, t}_s}}$.

For $t = 1$, the edgeless graph G with $s+2$ vertices has this property. So we may assume that $t \geq 2$. Let

$$c = 27(st + 2)t^3.$$

Then $c \geq t \geq 2$ and we can apply Theorem 3.3 to c and t . It follows that there is a K_{t+1} -free graph G^- with $\chi(G^-) = 27t^3(st + 2)$ such that for every K_t -free induced subgraph G' of G^- , we have $\chi(G') < 27t^3$. Let G be a graph containing G^- as a spanning subgraph such that G is K_{t+1} -free, and subject to this property, $E(G)$ is maximal with respect to inclusion. Our goal is to show that G is the desired graph.

Let $\gamma \in \mathbb{N} \cup \{0\}$ and let $X \subseteq V(G)$. By an X -chain of length γ in G we mean a $(\gamma + 1)$ -tuple $(X_0, \dots, X_\gamma)$ of subsets of $V(G)$ such that:

- $X_0 = X$;
- X_i is a t -vertex clique in G for every $i \in \{1, \dots, \gamma\}$; and

- $X_0, \dots, X_\gamma$ are pairwise anticomplete in G .

It follows that there is always an X -chain of length zero in G .

For every $X \subseteq V(G)$, let $\gamma_X \in \mathbb{N} \cup \{0\}$ be the maximum length of an X -chain in G . We claim that:

- (2) *Let $X \subseteq V(G)$ such that $G[X]$ has no isolated vertex. Then we have*

$$\chi(G^-) < 27t^3(|X| + t\gamma_X + 1).$$

Suppose not. Let $\gamma = \gamma_X$. Let $(X_0, \dots, X_\gamma)$ be an X -chain of length γ in G . Let $U = X_0 \cup \dots \cup X_\gamma$, let $N = \bigcup_{v \in U} N_G(v)$ and let $M = V(G) \setminus N$. Note that for each $i \in \{0, 1, \dots, \gamma\}$, the graph $G[X_i]$ has no isolated vertex (for $i = 0$, this follows from the assumption of (2), and for $i \in \{1, \dots, \gamma\}$, this follows from the fact that X_i is a t -vertex clique and $t \geq 2$). We deduce that in the graph G , every vertex in U has a neighbor in U ; in particular, $U \subseteq N$ and U and M are anticomplete in G . On the other hand, since G is K_{t+1} -free, it follows that for every $v \in U$, the graph $G[N_G(v)]$ is K_t -free, and so $G^-[N_G(v)]$ is K_t -free as well (because G^- is a subgraph of G). Thus, by the choice of G^- , we have $\chi(G^-[N_G(v)]) < 27t^3$ for every $v \in U$. Since $|U| = |X| + t\gamma$, it follows that $\chi(G^-[N]) < 27t^3(|X| + t\gamma)$. Therefore, since $\chi(G^-) \geq 27t^3(|X| + t\gamma_X + 1)$ and (M, N) is a partition of $V(G)$, it follows that $\chi(G^-[M]) > 27t^3$, and so by the choice of G^- , there is a t -vertex clique Y in G^- such that $Y \subseteq M$. In particular, Y is a t -vertex clique in G because G^- is a subgraph of G . But now since U and M are anticomplete in G , it follows that $(X_0, \dots, X_\gamma, Y)$ is an X -chain of length $\gamma + 1$ in G , contrary to the maximality of γ . This proves (2).

We now finish the proof. Let $xy \in E(\overline{G})$ and let $G^+ = G + xy$. By the choice of G , there is a $(t+1)$ -vertex clique X in G^+ with $x, y \in X$ such that x and y are the only two vertices in X that are non-adjacent in G . In particular, since $t \geq 2$, it follows that in the graph G , every vertex in X has a neighbor in X . Recall also that

$$\chi(G^-) = 27t^3(st + 2) = 27t^3(t + 1 + (s - 1)t + 1) = 27t^3(|X| + (s - 1)t + 1).$$

Hence, by (2), $\gamma_X \geq s$, and so there is an X -chain $(X_0, \dots, X_s)$ of length s in G . But now $G^+[X_0 \cup \dots \cup X_s]$ is isomorphic to $\overline{K_{t+1, t, \dots, t}}$. This completes the proof of Theorem 3.1. ■

3.2 Some balanced cases

We prove Theorem 3.2 in three steps. First, we handle the case

$$r = 1 \quad \text{and} \quad t = \binom{2r-1}{r-1} + 1 = 2.$$

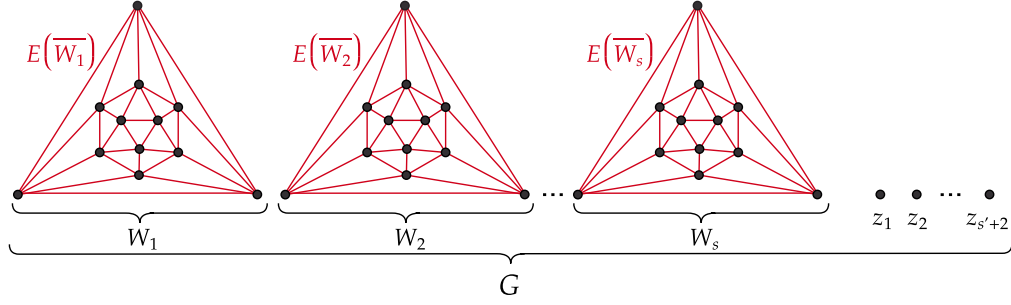


Figure 3.1: Proof of Theorem 3.4.

Indeed, in this case, we prove something stronger:

Theorem 3.4. *For all $s, s' \in \mathbb{N} \cup \{0\}$, the graph $K_{\underbrace{2, \dots, 2}_{s+1}, \underbrace{1, \dots, 1}_{s'}}$ is normal.*

Proof. It suffices to show that

$$H = \overline{K_{\underbrace{2, \dots, 2}_{s+1}, \underbrace{1, \dots, 1}_{s'}}}$$

is normal. If $s = 0$, then the empty graph on $s' + 2$ vertices is H -saturated, and we are done.

So we may assume that $s \geq 1$. Let G be a graph with exactly $s + s' + 2$ components such that G has s components $W_1, \dots, W_s$ each of which is isomorphic to the complement of the icosahedron, and the remaining $s' + 2$ components of G are isolated vertices; say $z_1, \dots, z_{s'+2}$. See Figure 3.1. Our goal is to prove that G is H -saturated.

By Observation 1.3 (and since $K_{2,2}$ is the 4-cycle), each of $W_1, \dots, W_s$ is $\overline{K_{2,2}}$ -saturated, and in particular $\overline{K_{2,2}}$ -free. Now,

- For each $i \in \{1, \dots, s\}$, fix an edge $u_i v_i \in E(W_i)$.
- For every $w \in V(G) \setminus \{z_1, \dots, z_{s'+2}\}$, say $w \in V(W_i)$ for $i \in \{1, \dots, s\}$, fix an edge $u_w v_w$ of W_i such that $u_w w, v_w w \notin E(G)$.

Note that the choice of $u_w v_w$ in the second bullet is possible because the non-neighbors of w in W_i induce a 5-cycle (which in turn is because the neighbors of every vertex in the icosahedron induce a 5-cycle).

First, we show that G is H -free. Suppose not. Then, in particular, there are $s + 1$ pairwise anticomplete two-vertex cliques $\{a_1, b_1\}, \dots, \{a_{s+1}, b_{s+1}\}$ in G . In particular, since $W_1, \dots, W_s$ are the components of G on two or more vertices, it follows that there exist

$i \in \{1, \dots, s\}$ and $j, j' \in \{1, \dots, s+1\}$ such that $\{a_j, b_j\}, \{a_{j'}, b_{j'}\} \subseteq V(W_i)$. But now $W_i[\{a_j, b_j, a_{j'}, b_{j'}\}]$ is isomorphic to $\overline{K_{2,2}}$, contrary to the fact that W_i is $\overline{K_{2,2}}$ -free.

Next, we show that for every $xy \in E(G)$, the graph $G - xy$ has an induced subgraph isomorphic to H . Since $W_1, \dots, W_s$ are pairwise distinct components of G , and the only ones on two or more vertices, it follows that there exists exactly one $i \in \{1, \dots, s\}$ for which $x, y \in V(W_i)$, and thus $xy \in E(W_i)$. Since W_i is $\overline{K_{2,2}}$ -saturated, it follows that $W_i - xy$ has an induced subgraph M isomorphic to $\overline{K_{2,2}}$. Therefore, the subgraph of $G - xy$ induced by

$$\left(\bigcup_{j \in \{1, \dots, s\} \setminus \{i\}} \{u_j, v_j\} \right) \cup V(M) \cup \{z_1, \dots, z_{s'}\}$$

is isomorphic to H , as desired.

It remains to prove that for every $xy \in E(\overline{G})$, the graph $G + xy$ has an induced subgraph isomorphic to H . First, consider the case where there are $i, j \in \{1, \dots, s\}$ for which $x \in V(W_i)$ and $y \in V(W_j)$. Assume that $i = j$. Then $xy \in E(\overline{W_i})$. Since W_i is $\overline{K_{2,2}}$ -saturated, it follows that $W_i + xy$ has an induced subgraph M isomorphic to $\overline{K_{2,2}}$. Thus, the subgraph of $G + xy$ induced by

$$\left(\bigcup_{k \in \{1, \dots, s\} \setminus \{i\}} \{u_k, v_k\} \right) \cup V(M) \cup \{z_1, \dots, z_{s'}\}$$

is isomorphic to H , as required. Consequently, we may assume that $i \neq j$. But then the subgraph of $G + xy$ induced by

$$\left(\bigcup_{k \in \{1, \dots, s\} \setminus \{i, j\}} \{u_k, v_k\} \right) \cup \{u_x, v_x\} \cup \{x, y\} \cup \{u_y, v_y\} \cup \{z_1, \dots, z_{s'}\}$$

is isomorphic to H . Next, consider the case where there are $i \in \{1, \dots, s\}$ and $j \in \{1, \dots, s' + 2\}$ such that $x \in V(W_i)$ and $y = z_j$. Choose an s' -subset S of $\{z_1, \dots, z_{s'+1}\} \setminus \{y\}$. Then the subgraph of $G + xy$ induced by

$$\left(\bigcup_{k \in \{1, \dots, s\} \setminus \{i\}} \{u_k, v_k\} \right) \cup \{u_x, v_x\} \cup \{x, y\} \cup S.$$

is isomorphic to H . Finally, consider the case where there are $i, j \in \{1, \dots, s' + 2\}$ for which $x = z_i$ and $y = z_j$. Then the subgraph of $G + xy$ induced by

$$\left(\bigcup_{k \in \{1, \dots, s\}} \{u_k, v_k\} \right) \cup \{z_1, \dots, z_{s'+2}\}$$

is isomorphic to H . This completes the proof of Theorem 3.4. ■

Second, we handle the case

$$r \geq 2 \quad \text{and} \quad t = \binom{2r-1}{r-1} + 1.$$

The proof will rely on the Erdős-Ko-Rado Theorem [15], which determines the maximum cardinality of a clique in the line graph of complete uniform hypergraphs.

Theorem 3.5 (Erdős, Ko, Rado [15]). *For all $n, r \in \mathbb{N}$ with $n \geq 2r$, the maximum cardinality of a clique in $\mathcal{L}(K_n^r)$ is $\binom{n-1}{r-1}$.*

Theorem 3.6. *Let $r, s \in \mathbb{N}$ with $r \geq 2$. Let*

$$t = \binom{2r-1}{r-1} + 1$$

and let

$$n = (2r+1)(s+1) - 1.$$

Then $\overline{\mathcal{L}(K_n^r)}$ is $K_{\underbrace{t, \dots, t}_{s+1}}$ -deletion-saturated.

Proof. Let

$$L = \mathcal{L}(K_n^r) \quad \text{and} \quad H = \overline{K_{\underbrace{t, \dots, t}_{s+1}}}.$$

Our goal is to prove that L is H -addition-saturated.

First, we show that L is H -free. Suppose not. Then there are $s+1$ pairwise anticomplete cliques $W_1, \dots, W_{s+1}$ in L , each of cardinality t ; in particular, $W_1, \dots, W_{s+1} \subseteq E(K_n^r)$. For each $i \in \{1, \dots, s+1\}$, let $U_i = \bigcup_{f \in W_i} f \subseteq V(K_n^r)$. Since $W_1, \dots, W_{s+1}$ are pairwise anticomplete in L , it follows that $U_1, \dots, U_{s+1}$ are pairwise disjoint. Also, since $|W_1| = \dots = |W_{s+1}| = t > \binom{2r-1}{r-1}$, it follows from Theorem 3.5 that $|U_i| \geq 2r+1$ for every $i \in \{1, \dots, s+1\}$. But then $|V(K_n^r)| \geq |U_1 \cup \dots \cup U_{s+1}| \geq (2r+1)(s+1) = n+1$, a contradiction.

Next, we prove that for every $f_1 f_2 \in E(\overline{L})$, the graph $L + f_1 f_2$ is not H -free. By choice, f_1 and f_2 are disjoint hyperedges of K_n^r . Let $U = f_1 \cup f_2$ and let $U' = V(K_n^r) \setminus U$. Then $|U| = 2r$ and $|U'| = n - 2r = (2r+1)s$. Fix a partition $(U'_1, \dots, U'_s)$ of U' into s subsets with $|U'_1| = \dots = |U'_s| = 2r+1$.

Let $u \in f_1 \subseteq U$ be fixed, let C be the set of all r -subsets of U containing u and let $D = C \cup \{f_2\}$. Then $D \setminus \{f_2\} = C$ is a clique of cardinality $\binom{2r-1}{r-1}$ in L , and $f_1 \in C$. Since U is the disjoint union of f_1 and f_2 , and since $|f_1| = |f_2| = r$, it follows that, other than f_1 , every r -subset of U intersects f_2 . In particular, $D \setminus \{f_1\} = (C \setminus \{f_1\}) \cup \{f_2\}$ is also a clique of cardinality $\binom{2r-1}{r-1}$ in L . Therefore, D is a clique of cardinality $\binom{2r-1}{r-1} + 1 = t$ in $L + f_1 f_2$.

For each $i \in \{1, \dots, s\}$, fix a vertex $u'_i \in U'_i$. Since $|U'_i| = 2r + 1$ and $r \geq 2$, the number of all r -subsets of U'_i containing u'_i is

$$\binom{2r}{r-1} = \frac{2r}{r+1} \binom{2r-1}{r-1} > \binom{2r-1}{r-1}.$$

It follows that there is a set D'_i of r -subsets of U'_i containing u'_i such that $|D'_i| = \binom{2r-1}{r-1} + 1 = t$. In particular, D'_i is a clique of cardinality t in L , and so in $L + f_1 f_2$.

Moreover, for all $(f, f'_1, \dots, f'_s) \in D \times D'_1 \times \dots \times D'_s$, the hyperedges $f, f'_1, \dots, f'_s$ of K_n^r are pairwise disjoint because $f \subseteq U$, $f'_i \subseteq U'_i$ for each $i \in \{1, \dots, s\}$, and $U, U'_1, \dots, U'_s$ are pairwise disjoint. Consequently, $D, D'_1, \dots, D'_s$ are pairwise anticomplete in L ; in particular, $D, D'_1, \dots, D'_s$ are pairwise disjoint. Since $f_1, f_2 \in D$, we deduce that $D, D'_1, \dots, D'_s$ are pairwise anticomplete in $L + f_1 f_2$.

In conclusion, we have shown that $D, D'_1, \dots, D'_s$ are $s + 1$ pairwise anticomplete cliques in $L + f_1 f_2$, each of cardinality t . Hence, the subgraph of $L + f_1 f_2$ induced by $D \cup D'_1 \cup \dots \cup D'_s$ is isomorphic to H . This completes the proof of Theorem 3.6. $\blacksquare$

Finally, we handle the case

$$t = \binom{2r}{r} + 1.$$

The proof here will use a “product” extension of the Erdős-Ko-Rado Theorem due to Frankl [17]. In order to state that result, we need a definition. For $k \in \mathbb{N}$ and graphs $G_1, \dots, G_k$, let $G_1 \boxtimes \dots \boxtimes G_k$ be the graph with vertex set

$$V(G_1) \times \dots \times V(G_k)$$

such that *distinct* vertices $(u_1, \dots, u_k)$ and $(v_1, \dots, v_k)$ are adjacent in $G_1 \boxtimes \dots \boxtimes G_k$ if and only if for some $i \in \{1, \dots, k\}$, either $u_i = v_i$ or $u_i v_i \in E(G_i)$.

The “product” extension of the Erdős-Ko-Rado Theorem is as follows (note that for $k = 1$, the result below is identical with the original version of Erdős-Ko-Rado, namely Theorem 3.5).

Theorem 3.7 (Frankl [17]). *Let $k \in \mathbb{N}$ and let $n_1, r_1, \dots, n_k, r_k \in \mathbb{N}$ such that $n_i \geq r_i$ for all $i \in \{1, \dots, k\}$. Then the maximum cardinality of a clique in $\mathcal{L}(K_{n_1}^{r_1}) \boxtimes \dots \boxtimes \mathcal{L}(K_{n_k}^{r_k})$ is*

$$\max \left\{ \frac{r_1}{n_1}, \dots, \frac{r_k}{n_k} \right\} \times \binom{n_1}{r_1} \times \dots \times \binom{n_k}{r_k}.$$

Theorem 3.8. *Let $r, s \in \mathbb{N}$. Let*

$$t = \binom{2r}{r} + 1$$

and let

$$n = (2r + 1)(s + 1) - 1.$$

Then $\overline{\mathbb{L}(K_n^r) \bowtie \mathbb{L}(K_{s+2}^1)}$ is $\underbrace{K_{t, \dots, t}}_{s+1}$ -deletion-saturated.

Proof. Recall that $\mathbb{L}(K_{s+2}^1)$ is just an empty graph on $s+2$ vertices (that is, isomorphic to $(s+2)K_1$). Let

$$L = \mathbb{L}(K_n^r) \bowtie \mathbb{L}(K_{s+2}^1) \quad \text{and} \quad H = \overline{K_{t, \dots, t}}_{s+1}.$$

Then our goal is to prove that L is H -addition-saturated.

First, we show that:

(3) L is H -free.

Suppose not. Then there are $s+1$ pairwise anticomplete cliques $W_1, \dots, W_{s+1}$ in L , each of cardinality t ; in particular,

$$W_1, \dots, W_{s+1} \subseteq E(K_n^r) \times E(K_{s+2}^1).$$

For each $i \in \{1, \dots, s+1\}$ and every $j \in \{0, 1\}$, let

$$U_{i,j} = \bigcup_{(f_0, f_1) \in W_i} f_j$$

and let $n_{i,j} = |U_{i,j}| \geq 1$. In particular, we have $U_{i,0} \subseteq V(K_n^r)$ and $U_{i,1} \subseteq V(K_{s+2}^1)$. Since $W_1, \dots, W_{s+1}$ are pairwise anticomplete in L , it follows that for every $j \in \{0, 1\}$, $U_{1,j}, \dots, U_{s+1,j}$ are pairwise disjoint. In particular, we have:

- $\sum_{i=1}^{s+1} n_{i,0} \leq n$; and
- $\sum_{i=1}^{s+1} n_{i,1} \leq s+2$.

Since $n < (2r+1)(s+1)$, by the first bullet, there exists $i \in \{1, \dots, s+1\}$ for which $n_{i,0} \leq 2r$. Also, by the second bullet, we have $n_{i,1} \leq 2$. Since $|W_i| = t$, it follows that $\mathbb{L}(K_{2r}^r) \bowtie \mathbb{L}(K_2^1)$ has a clique of cardinality t . Thus, by Theorem 3.7,

$$t \leq \max \left\{ \frac{r}{2r}, \frac{1}{2} \right\} \times \binom{2r}{r} \times \binom{2}{1} = \binom{2r}{r},$$

which is a contradiction. This proves (3).

Next, we show that:

(4) For every $xy \in E(\overline{L})$, the graph $L + xy$ is not H -free.

We may write $x = (e_0, e_1)$ and $y = (f_0, f_1)$ such that e_0 and f_0 are disjoint hyperedges of K_n^r , and e_1 and f_1 are disjoint hyperedges of K_{s+2}^1 . Let $U = e_0 \cup f_0$. Then $|U| = 2r$. Let $u \in e_0 \subseteq U$ be fixed, let C be the set of all r -subsets of U containing u ; in particular, $e_0 \in C$ and $f_0 \notin C$. Let

$$\mathcal{C} = C \times \{e_1, f_1\}$$

and let

$$\mathcal{D} = \mathcal{C} \cup \{y\}.$$

Note that $\mathcal{D} \setminus \{y\} = \mathcal{C}$ is a clique in L with $x \in \mathcal{C}$. Since U is the disjoint union of e_0 and f_0 , and since $|e_0| = |f_0| = r$, it follows that, other than e_0 , every r -subset of U intersects f_0 . Consequently, for every $z = (g_0, g_1) \in \mathcal{C} \setminus \{x\}$, we have $zy \in E(L)$, because either $g_0 \neq e_0$, in which case we have $g_0 \cap f_0 \neq \emptyset$, or $g_0 = e_0$ and $g_1 = f_1$ (since $z \neq x$). In other words, $\mathcal{D} \setminus \{x\} = (\mathcal{C} \setminus \{x\}) \cup \{y\}$ is also a clique in L . We deduce that $\mathcal{D}$ is a clique in $L + xy$. Moreover,

$$|\mathcal{D}| = |\mathcal{C}| + 1 = 2 \binom{2r-1}{r-1} + 1 = \binom{2r}{r} + 1 = t.$$

Let $U' = V(K_n^r) \setminus U$. Then $|U'| = n - 2r = (2r+1)s$. Fix a partition $(U'_1, \dots, U'_s)$ of U' into s subsets with $|U'_1| = \dots = |U'_s| = 2r+1$. Also, let $E(K_{s+2}^1) \setminus \{e_1, f_1\} = \{h_1, \dots, h_s\}$. For each $i \in \{1, \dots, s\}$, since $|U'_i| = 2r+1$, the number of all r -subsets of U'_i is

$$\binom{2r+1}{r} = \frac{2r+1}{r+1} \binom{2r}{r} > \binom{2r}{r}.$$

So we can choose a set D'_i of r -subsets of U'_i such that $|D'_i| = \binom{2r}{r} + 1 = t$. In particular,

$$\mathcal{D}'_i = D'_i \times \{h_i\}$$

is a clique of cardinality t in L , and so in $L + xy$.

Moreover, for all $((g, h), (g_1, h_1), \dots, (g_s, h_s)) \in \mathcal{D} \times \mathcal{D}'_1 \times \dots \times \mathcal{D}'_s$,

- the hyperedges $g, g_1, \dots, g_s$ of K_n^r are pairwise disjoint because $g \subseteq U$, $g_i \subseteq U'_i$ for each $i \in \{1, \dots, s\}$, and $U, U'_1, \dots, U'_s$ are pairwise disjoint; and
- $h, h_1, \dots, h_s$ are pairwise distinct hyperedges of K_{s+2}^1 because $h \in \{e_1, f_1\} = E(K_{s+2}^1) \setminus \{h_1, \dots, h_s\}$, and thus $h, h_1, \dots, h_s$ are pairwise disjoint.

Consequently, $\mathcal{D}, \mathcal{D}'_1, \dots, \mathcal{D}'_s$ are pairwise anticomplete in L ; in particular, $\mathcal{D}, \mathcal{D}'_1, \dots, \mathcal{D}'_s$ are pairwise disjoint. Since $x, y \in \mathcal{D}$, we deduce that $\mathcal{D}, \mathcal{D}'_1, \dots, \mathcal{D}'_s$ are pairwise anticomplete in $L + xy$.

In conclusion, we have shown that

$$\mathcal{D}, \mathcal{D}'_1, \dots, \mathcal{D}'_s$$

are $s + 1$ pairwise anticomplete cliques in $L + xy$, each of cardinality t . Hence, the subgraph of $L + xy$ induced by $\mathcal{D} \cup \mathcal{D}'_1 \cup \dots \cup \mathcal{D}'_s$ is isomorphic to H . This proves (4).

The result now follows from (3) and (4). This completes the proof of Theorem 3.8. ■

3.3 Corollary for graphs on at most six vertices

Among the 202 non-complete graphs on at most six vertices, there are exactly **23** complete multipartite graphs; see Figure 3.2, where these graphs are depicted with the same labeling as in Figure 1.3, and with a multipartition of their vertices given.

In particular, the only complete multipartite graphs on at most six vertices that are *not* lopsided are:

- Entry 11C (isomorphic to $K_{3,3}$). This graph is handled by Theorem 3.2 for

$$r = 1, \quad t = \binom{2r}{r} + 1 = 3 \quad \text{and} \quad s = 1.$$

- Entry 13H (isomorphic to $K_{2,2,2}$). This is handled by Theorem 3.4 for

$$s = 2 \quad \text{and} \quad s' = 0.$$

- Entry 13J, (isomorphic to $K_{2,2,1,1}$). This is handled by Theorem 3.4 for

$$s = 1 \quad \text{and} \quad s' = 2.$$

- Entry 13K, (isomorphic to $K_{2,1,1,1,1}$). This is handled by Theorem 3.4 for

$$s = 0 \quad \text{and} \quad s' = 4.$$

- Entry 16H, (isomorphic to $K_{2,2,1}$). This is handled by Theorem 3.4 for

$$s = s' = 1.$$

- Entry 17I, (isomorphic to $K_{2,2}$). This is handled by Theorem 3.4 for

$$s = 1 \quad \text{and} \quad s' = 0.$$

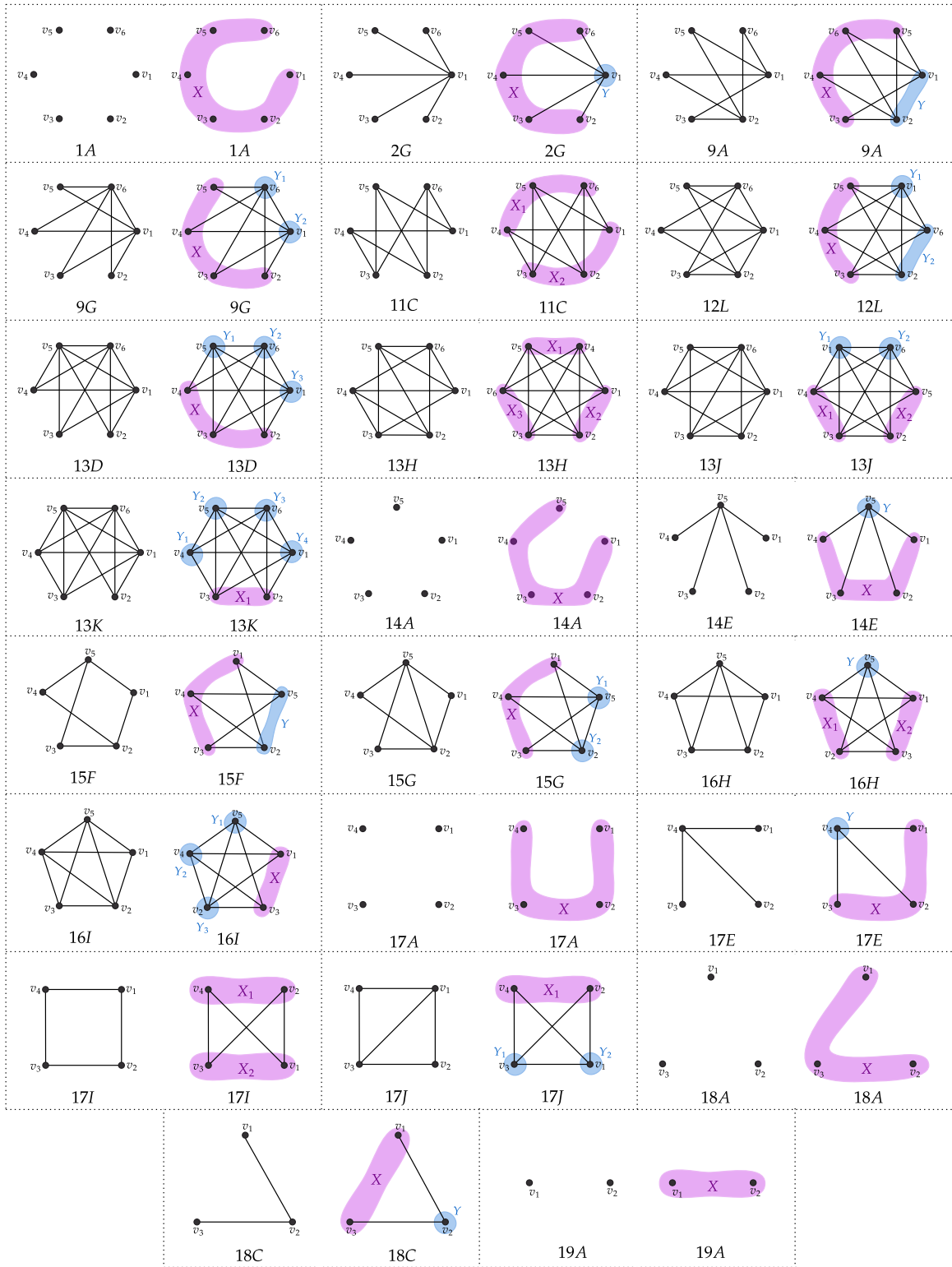


Figure 3.2: All non-complete complete multipartite graphs on at most six vertices.

Chapter 4

Adding vertices of degree at most one

In this chapter, we prove outcomes (c) and (d) of Theorem 1.12. Note that the following is equivalent to Theorem 1.12(c):

Theorem 4.1. *Let G, H be graphs. Let u be an isolated vertex of G and let v be an isolated vertex of H . Assume that $G \setminus \{u\}$ is $(H \setminus \{v\})$ -deletion-saturated. Then G is H -deletion-saturated.*

Proof. First, we show that G is H -free. Suppose for a contradiction that G has an induced subgraph H' isomorphic to H . Let $f : V(H) \rightarrow V(H')$ be an isomorphism between H and H' . Since $G \setminus \{u\}$ is $(H \setminus \{v\})$ -free, it follows that H' is not an induced subgraph of $G \setminus \{u\}$. In other words, we have $u \in V(H')$. Since $G \setminus \{u\}$ is $(H \setminus \{v\})$ -free, it follows that $H' \setminus \{u\}$ is not isomorphic to $H \setminus \{v\}$, and so $f(v) \neq u$. In particular, u and $f(v)$ are distinct isolated vertices of H' . It follows that $H' \setminus \{u\}$ and $H' \setminus \{f(v)\}$ are isomorphic, which in turn implies that $H' \setminus \{u\}$ and $H \setminus \{v\}$ are isomorphic. But then $H' \setminus \{u\}$ is an induced subgraph of $G \setminus \{u\}$ isomorphic to $H \setminus \{v\}$, a contradiction. We deduce that G is H -free.

Now, let $e \in E(G)$. Then $e \in E(G \setminus \{u\})$. Since $G \setminus \{u\}$ is $(H \setminus \{v\})$ -deletion-saturated, it follows that $(G \setminus \{u\}) - e$ has an induced subgraph H'' isomorphic to $H \setminus \{v\}$. Hence, the induced subgraph of $G - e$ with vertex set $V(H'') \cup \{u\}$ is isomorphic to H . This completes the proof of Theorem 4.1. ■

Instead of Theorem 1.12(d), we prove a substantially stronger statement, which requires a few definitions. Let G be a graph. Recall that a *block* in G is a maximal 2-connected induced subgraph of G (we consider complete graphs on one or two vertices to be 2-connected, as well). We denote by $\mathcal{B}(G)$ the set of all blocks of G . We say that a graph H is *omnipresent in G* if for every $u \in V(H)$ and every $v \in V(G)$, there is an injection $\psi : V(H) \rightarrow V(G)$ such that $\psi(u) = v$ and ψ is an isomorphism between H and $G[\psi(V(H))]$. For instance, K_1 is omnipresent in G if and only if G is non-null, and K_2 is omnipresent in G if and only if G has no isolated vertex.

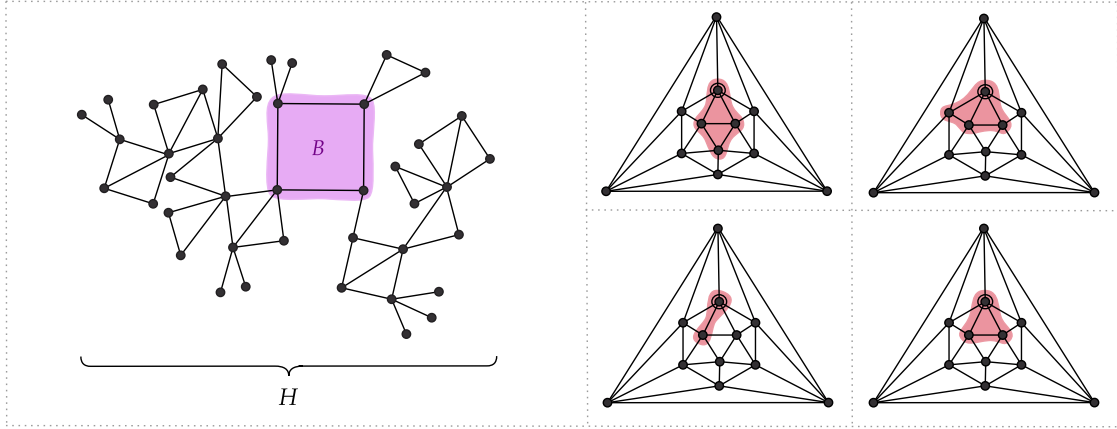


Figure 4.1: Left: Graph H with a governing block B . Right: All blocks of H except for B are omnipresent in the icosahedron.

Given a graph H , we say that a block B of H is *governing* if there is a B -deletion-saturated graph W such that every block $B' \in \mathcal{B}(H) \setminus \{B\}$ is omnipresent in W .

We will prove the following strengthening of Theorem 1.12(d):

Theorem 4.2. *Every graph with a governing block is deletion-normal.*

For example, the graph H in Figure 4.1 has a block B which is a 4-cycle; the other blocks of H are all isomorphic to K_2 , K_3 or the diamond. Therefore, since we saw in Observation 1.3 that the icosahedron is C_4 -saturated (and thus C_4 -deletion-saturated), and since K_2 , K_3 and the diamond are all omnipresent in the icosahedron, it follows that the graph H in Figure 4.1 is deletion-normal.

The idea of the proof of Theorem 4.2 is as follows: Given a graph H with a governing block B , we begin with a B -deletion-saturated graph W . We take a $|W|$ -graph Γ of sufficiently large girth and chromatic number (depending on H), and we build a graph G from Γ by turning each hyperedge of Γ into an isomorphic copy of W . We then use the large girth and chromatic number of the underlying hypergraph Γ to prove that G is H -deletion-normal.

It is straightforward to see that Theorem 4.2 implies Theorem 1.12(d):

Theorem 4.3. *Every graph obtained by iteratively adding vertices of degree at most one to a 2-connected deletion-normal graph is deletion-normal.*

Proof. Let H be a graph obtained by iteratively adding vertices of degree at most one to a 2-connected deletion-normal graph B . Then B is deletion-normal and every block in $\mathcal{B}(H) \setminus \{B\}$ is isomorphic to K_1 or K_2 . In particular, since B is connected, we may assume that there is a B -deletion-saturated graph W which is also connected, and so both K_1 and K_2 are omnipresent in W . It follows that B is a governing block of H , which along with Theorem 4.2 implies that H is deletion-normal, as desired. ■

For example, from Theorem 4.3, it follows that every triangle-free “unicyclic” graph is deletion-normal. A graph is *unicyclic* if it has exactly one cycle. In particular, every triangle-free unicyclic graph is obtained by iteratively adding vertices of degree at most one to a cycle on four or more vertices. Recall also that by Theorems 1.5 and 1.10, all cycles on four or more vertices are deletion-normal. Hence,

Corollary 4.4. *Every triangle-free unicyclic graph is deletion-normal.*

Another consequence of Theorem 4.3 is that for every 2-connected deletion-normal graph B , the disjoint union of B and any forest is deletion-normal. This is somewhat curious because deletion-normality remains open for most forests (and appears to be quite hard to prove).

Corollary 4.5. *The disjoint union of a forest and a 2-connected deletion-normal graph is deletion-normal.*

4.1 Hypergraph tools

The proof of Theorem 4.2 uses the fact that there are uniform hypergraphs of arbitrarily large girth and chromatic number, a result first proved by Erdős and Hajnal [14]. Several other constructions were subsequently found; see, for example, also [1] and [22].

Theorem 4.6 (Erdős and Hajnal, Corollary 13.4 in [14]). *For all $c, g, k \in \mathbb{N}$, there is a k -graph of girth larger than g and chromatic number larger than c .*

From Theorem 4.6, it follows that there are uniform hypergraphs of arbitrarily large girth and minimum degree. (The proof is easy but we include it for the sake of completeness.)

Lemma 4.7. *For all $d, g, k \in \mathbb{N}$, there is a k -graph of girth larger than g in which every vertex has degree larger than d .*

Proof. By Theorem 4.6, there exists a k -graph Γ with girth larger than g and with chromatic number larger than $d + 1$. Let Γ' be a k -graph with $V(\Gamma') \subseteq V(\Gamma)$ and $E(\Gamma') \subseteq E(\Gamma)$ such that Γ' has chromatic number larger than $d + 1$, and subject to this property, $V(\Gamma') \cup E(\Gamma')$ is minimal with respect to inclusion. We claim that Γ' is the k -graph desired in Lemma 4.7. Note that Γ' is k -uniform because Γ is. Also, every cycle in Γ' is also a cycle in Γ , which in turn implies that Γ' has girth larger than g because Γ does. It remains to show that every vertex in Γ' has degree larger than d . Suppose for a contradiction that some vertex $v \in V(\Gamma')$ has degree at most d . Let $\mathcal{E}$ be the set of all hyperedges of Γ' that contain v ; then $|\mathcal{E}| \leq d$. Let Γ'' be the k -graph with $V(\Gamma'') = V(\Gamma') \setminus \{v\}$ and $E(\Gamma'') = E(\Gamma') \setminus \mathcal{E}$. Then Γ'' is a k -graph with $V(\Gamma'') \cup E(\Gamma'')$ strictly contained in $V(\Gamma') \cup E(\Gamma')$. From this and the choice of Γ' , it follows that Γ'' has chromatic number at most $d + 1$. This means there is a

partition $(S_1, \dots, S_{d+1})$ of $V(\Gamma'')$ such that no hyperedge of Γ'' is entirely contained in one of $S_1, \dots, S_{d+1}$. Moreover, since $|\mathcal{E}| \leq d$, it follows that for some $j \in \{1, \dots, d+1\}$, none of the sets $(f \setminus \{v\} : f \in \mathcal{E})$ is entirely contained in S_j . Let $S'_j = S_j \cup \{v\}$ and let $S'_i = S_i$ for each $i \in \{1, \dots, d+1\} \setminus \{j\}$. Then no hyperedge in $E(\Gamma'') \cup \mathcal{E} = E(\Gamma')$ is entirely contained in one $S'_1, \dots, S'_{d+1}$. But now Γ' has chromatic number at most $d+1$, a contradiction. This completes the proof of Lemma 4.7. $\blacksquare$

We also need a somewhat more “tangible” certificate of large girth in uniform hypergraphs. Let $k \geq 2$, let Γ be a k -graph and let $m \geq 2$ be an integer. By a *ring* in Γ we mean an ordered $(m+1)$ -tuple $R = (f_1, \dots, f_{m+1})$ where:

- $f_1, \dots, f_m, f_{m+1}$ are hyperedges of Γ such that $f_1 = f_{m+1}$ but there are at least two distinct hyperedges among $f_1, \dots, f_m$; and
- there are m pairwise distinct vertices $v_1, \dots, v_m$ of Γ such that for every $i \in \{1, \dots, m\}$, we have $v_i \in f_i \cap f_{i+1}$.

The *length* of R is the number m (which is at least two).

Lemma 4.8. *Let $g, k \in \mathbb{N}$ and let Γ be a k -graph. Assume that there is a ring of length at most g in Γ . Then Γ has girth at most g .*

Proof. Let $(f_1, \dots, f_{m+1})$ be a ring of length m in Γ with m as small as possible. Then we have $2 \leq m \leq g$. By definition, there are m pairwise distinct vertices $v_1, \dots, v_m$ of Γ such that for every $i \in \{1, \dots, m\}$, we have $v_i \in f_i \cap f_{i+1}$. Since every hyperedge of Γ has cardinality exactly k , it follows that:

$$(5) \quad \text{We have } \left| \bigcup_{i=1}^m f_i \right| \leq m(k-1).$$

Moreover, we claim that:

$$(6) \quad \text{The hyperedges } f_1, \dots, f_m \text{ are pairwise distinct.}$$

Suppose not. Choose $i, \mu \in \{1, \dots, m-1\}$ with μ as small as possible such that $f_i = f_{i+\mu}$. Since there are at least two distinct hyperedges among $f_1, \dots, f_m$, it follows that $m \geq 3$. If $\mu = 1$, then $(f_1, \dots, f_i, f_{i+2}, \dots, f_{m+1})$ is a ring of length $m-1$ in Γ , a contradiction to the minimality of m . So we may assume that $2 \leq \mu \leq m-1$. Note that by the minimality of μ , there are at least two distinct hyperedges among $f_i, \dots, f_{i+\mu-1}$. But now $(f_i, \dots, f_{i+\mu})$ is a ring in Γ of length $\mu \leq m-1$, again a contradiction to the minimality of m . This proves (6).

From (5) and (6), we deduce that $f_1, \dots, f_m$ are m pairwise distinct hyperedges of Γ whose union contains at most $m(k-1)$ vertices. Hence, Γ has girth at most $m \leq g$. This completes the proof of Lemma 4.8. $\blacksquare$

4.2 The main proof

The hypergraph-theoretic tools from the previous section are used in the proof of the following lemma, which is the key step in the proof of Theorem 4.2.

Lemma 4.9. *For every graph H with a governing block B , there is a graph G such that:*

- (a) G is B -free; and
- (b) *for every $e \in E(G)$ and every connected induced graph H' of H with $V(B) \subseteq V(H')$, $G - e$ has an induced subgraph isomorphic to H' .*

Proof. By definition, since B is governing, there is a B -deletion-saturated graph W such that every block $B' \in \mathcal{B}(H) \setminus \{B\}$ is omnipresent in W . Let $|V(W)| = k \in \mathbb{N}$ and let $h = |V(H)|$. By Lemma 4.7, there exists a k -graph Γ of girth larger than h in which every vertex has degree larger than h . Also, since B is 2-connected and non-complete (because B is deletion-normal), it follows that $h = |V(H)| \geq |V(B)| > 2$, and so Γ has girth larger than 2. In particular,

- (7) *For all distinct $f, f' \in E(\Gamma)$, we have $|f \cap f'| \leq 1$.*

For every hyperedge $f \in E(\Gamma)$, let W_f be a graph with $V(W_f) = f$ that is isomorphic to W and let $\eta_f : V(W_f) \rightarrow V(W)$ be an isomorphism between W_f and W . Let

$$G = \bigcup_{f \in E(\Gamma)} W_f.$$

From (7) and the construction of G , it follows that there is a map $\varphi : E(G) \rightarrow E(\Gamma)$ such that for every $e \in E(G)$, $\varphi(e)$ is the only hyperedge of Γ that contains both ends of e . In particular, $\varphi(e)$ is the only hyperedge of Γ for which we have $e \in E(W_{\varphi(e)})$. Moreover, we claim that:

- (8) *Let $f \in E(\Gamma)$ and let $u, v \in f$ be distinct. Let P be a path of length at most h from u to v which is a subgraph of G . Then for every $e \in E(P)$, we have $\varphi(e) = f$.*

Let $P = v_1 - \dots - v_m$ where $v_1 = u$, $v_m = v$ and $2 \leq m \leq h$. For each $i \in \{1, \dots, m-1\}$, let $f_i = \varphi(v_i v_{i+1})$, let $f_m = f$. We need to show that $f_1 = \dots = f_{m-1} = f_m$. Suppose not. Define $f_{m+1} = \varphi(v_1 v_2)$. Then $f_1, \dots, f_m, f_{m+1}$ are hyperedges of Γ such that $f_1 = f_{m+1}$ but there are at least two distinct hyperedges among $f_1, \dots, f_m$, and $v_1, \dots, v_m$ are m pairwise distinct vertices of Γ such that for every $i \in \{1, \dots, m\}$, we have $v_i \in f_i \cap f_{i+1}$. Therefore, $(f_1, \dots, f_{m+1})$ is a ring of length $m \leq h$ in Γ . But now by Lemma 4.8, Γ has girth at most h , a contradiction. This proves (8).

Our goal is now to show that G satisfies Lemma 4.9. First, we prove that:

(9) G is B -free.

Suppose not. Let B_0 be an induced subgraph of G isomorphic to B . Since B_0 is 2-connected, it follows that $E(B_0) \neq \emptyset$. Choose an edge $e \in E(B_0)$ and let $f = \varphi(e)$. Let M be the component of $B_0[V(B_0) \cap f]$ that contains both ends of e . Since W_f is isomorphic to W , since B_0 is isomorphic to B and since W is B -free, it follows that $V(B_0) \not\subseteq V(W_f) = f$. Consequently, since B_0 is connected, it follows that there is an edge $vw \in E(B_0)$ such that $v \in V(M)$ and $w \in V(G) \setminus f$. Let $f' = \varphi(vw)$. Then $f' \neq f$, and by (7), we have $f \cap f' = \{v\}$. Since M is connected with at least two vertices, it follows that v has a neighbor u in M ; in particular, we have $u, v \in f$ are distinct. On the other hand, since B_0 is 2-connected, it follows that there is a path Q in $B_0 \setminus \{v\}$ from u to w . But now $P = u-Q-w-v$ is a path of length at most $|V(B_0)| = |V(B)| \leq h$ between distinct vertices $u, v \in f$ which is a subgraph of B_0 (and so of G), and vw is an edge of P such that $\varphi(vw) = f' \neq f$, a contradiction to (8). This proves (9).

In view of (9), it remains to show that for every $e \in E(G)$ and every connected induced subgraph H' of H with $V(B) \subseteq V(H')$, $G - e$ has an induced subgraph isomorphic to H' . The proof is by induction on $|V(H')|$ for fixed $e = xy \in E(G)$. Note that B is a block of H' . In fact, since every block of H' is an induced subgraph of a block of H , it follows that B is a governing block of H' .

Let $f_0 = \varphi(e)$ and let $G^- = G - e$. Then we have $x, y \in f_0$, $e \in E(G[f_0]) = E(W_{f_0})$ and

$$G^-[f_0] = G[f_0] - e = W_{f_0} - e.$$

Assume that $|V(H')| = |V(B)|$. Then $H' = B$. Since η_{f_0} is an isomorphism between W_{f_0} and W , it follows that $\eta_{f_0}(x)\eta_{f_0}(y) \in E(W)$ and $W_{f_0} - e$ is isomorphic to $W - \eta_{f_0}(x)\eta_{f_0}(y)$. On the other hand, since W is B -deletion-saturated, it follows that $W - \eta_{f_0}(x)\eta_{f_0}(y)$ has an induced subgraph isomorphic to B , and so isomorphic to H' . Therefore, $G^-[f_0]$ has an induced subgraph isomorphic to H' , as desired.

Now, assume that $|V(H')| > |V(B)|$. Since H' is connected and $B \in \mathcal{B}(H')$, it follows that $\mathcal{B}(H') \setminus \{B\} \neq \emptyset$, and in particular, the block tree of H' has at least two leaves. This means that there is a block $B' \in \mathcal{B}(H') \setminus \{B\}$ such that $V(B')$ contains exactly one cut vertex of H' . Let $u \in V(B')$ be the unique vertex of B' that is a cut vertex of H' . Let $H'' = H' \setminus (V(B') \setminus \{u\})$. Then H'' is connected and $|V(H'')| < |V(H')|$. Moreover, since $\mathcal{B}(H'') = \mathcal{B}(H') \setminus \{B'\}$, it follows that B is a block of H'' , as well. Consequently, by the inductive hypothesis, G^- has an induced subgraph J isomorphic to H'' . Let $\theta : V(H'') \rightarrow V(J)$ be an isomorphism between H'' and J , and let $v = \theta(u) \in V(J) \subseteq V(G)$. Let $\mathcal{E} = \{\varphi(vv') : v' \in N_J(v)\}$. Then every hyperedge in $\mathcal{E}$ contains v , but we have $|\mathcal{E}| < |V(H'')| < h$. Since every vertex of Γ has degree larger than h , it follows that there is a hyperedge $f \in E(\Gamma) \setminus \mathcal{E}$ such that $v \in f$. We further show that:

(10) $f \setminus \{v\}$ and $V(J) \setminus \{v\}$ are anticomplete in G .

Suppose not. Then there exists $u \in f \setminus \{v\}$ such that either $u \in V(J) \setminus \{v\}$ or u has a neighbor in $V(J) \setminus \{v\}$ in G . By the definition of $\mathcal{E}$ and the choice of f , we have $N_J(v) \cap (f \setminus \{v\}) = \emptyset$, and so $u \notin N_J(v)$. Therefore, since J is connected, it follows that there is a path Q in J from v to a vertex $w \in V(J) \setminus \{v\}$ such that $uw \in E(G^-)$. In particular, the unique neighbor v' of v in Q belongs to $N_J(v)$, and so $\varphi(vv') \neq f$. But now $P = u-w-Q-v$ is a path of length at most $|V(Q)| \leq |V(J)| < h$ between distinct vertices $u, v \in f$ which is a subgraph of G^- (and so of G), and vv' is an edge of P such that $\varphi(vv') \neq f$, a contradiction to (8). This proves (10).

We can now finish the proof. By (10), we have $f_0 \neq f$. Since B is a governing block of H' and since $B' \in \mathcal{B}(H') \setminus \{B\}$, it follows that B' is omnipresent in W , and so in W_f . Recall also that $u \in V(B')$ and $v \in f = V(W_f)$. Thus, there is an injection $\psi : V(B') \rightarrow V(W_f)$ such that $\psi(u) = v$ and ψ is an isomorphism between B' and $B'' = W_f[\psi(V(B'))]$. Since $f_0 \neq f$, it follows that B'' is an induced subgraph of G^- . Hence, by (10), $J \cup B''$ is an induced subgraph of G^- isomorphic to H' . This concludes the induction and thus completes the proof of Lemma 4.9. $\blacksquare$

We are now in a position to prove Theorem 4.2:

Proof of Theorem 4.2. Let H be a graph with a governing block B . By definition, there is a B -deletion-saturated graph W such that every block $B' \in \mathcal{B}(H) \setminus \{B\}$ is omnipresent in W . Since B has no isolated vertices, we may assume without loss of generality that W has no isolated vertices either.

We begin by showing that:

(11) *There is a connected graph $\hat{H}$ such that H is an induced subgraph of $\hat{H}$ and B is a governing block of $\hat{H}$.*

If H is connected, then we are done by choosing $\hat{H} = H$. Assume that H is not connected. Let $\hat{H}$ be a graph obtained from H by adding an extra vertex with exactly one neighbor in each component of H . Then $\hat{H}$ is connected and H is an induced subgraph of $\hat{H}$. Moreover, we have $\mathcal{B}(H) \subseteq \mathcal{B}(\hat{H})$, and every block of $\hat{H}$ that is not a block of H is isomorphic to K_2 . Therefore, since B is a governing block of H and since W has no isolated vertices, it follows that B is a governing block of $\hat{H}$, too. This proves (11).

Let $\hat{H}$ be as given by (11). Then, by Lemma 4.9, there is a B -free graph G such that for every $e \in E(G)$, $G - e$ has an induced subgraph isomorphic to $\hat{H}$. But now since B is an induced subgraph of H , and H is an induced subgraph of $\hat{H}$, we deduce that G is H -deletion-saturated. This completes the proof of Theorem 4.2. $\blacksquare$

4.3 Corollary for graphs on at most six vertices

Of the remaining $202 - 23 = 179$ graphs on at most six vertices from Chapter 3,

- The **37** graphs in Figure 4.2 each have at least one isolated vertex, and removing all their isolated vertices leaves a non-complete graph. Thus, by Theorem 4.1, we may assume that these graphs are handled (by removing their isolated vertices and inductively reducing to a smaller graph which has been handled before or will be handled in later chapters).
- The **32** graphs in Figure 4.3 each have a unique non-complete block B such that B is not the entire graph and all other blocks are isomorphic to K_2 . Therefore, by Theorem 4.3, we may assume that these graphs are also handled (by inductively reducing to B , which is a smaller graph that has been handled before or will be handled in later chapters).

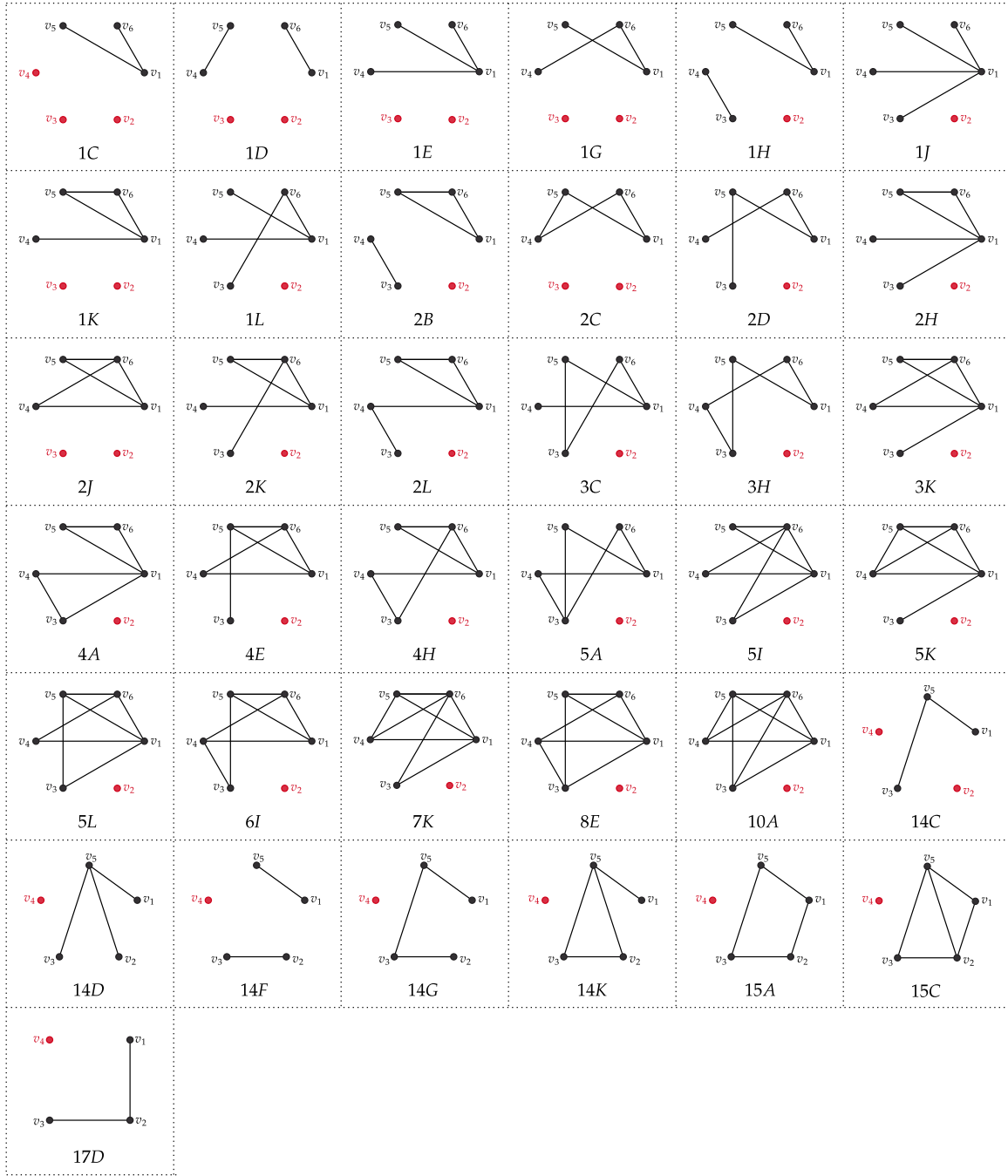


Figure 4.2: These graphs have at least one isolated vertex, and removing all isolated vertices from each graph leaves a non-complete graph.

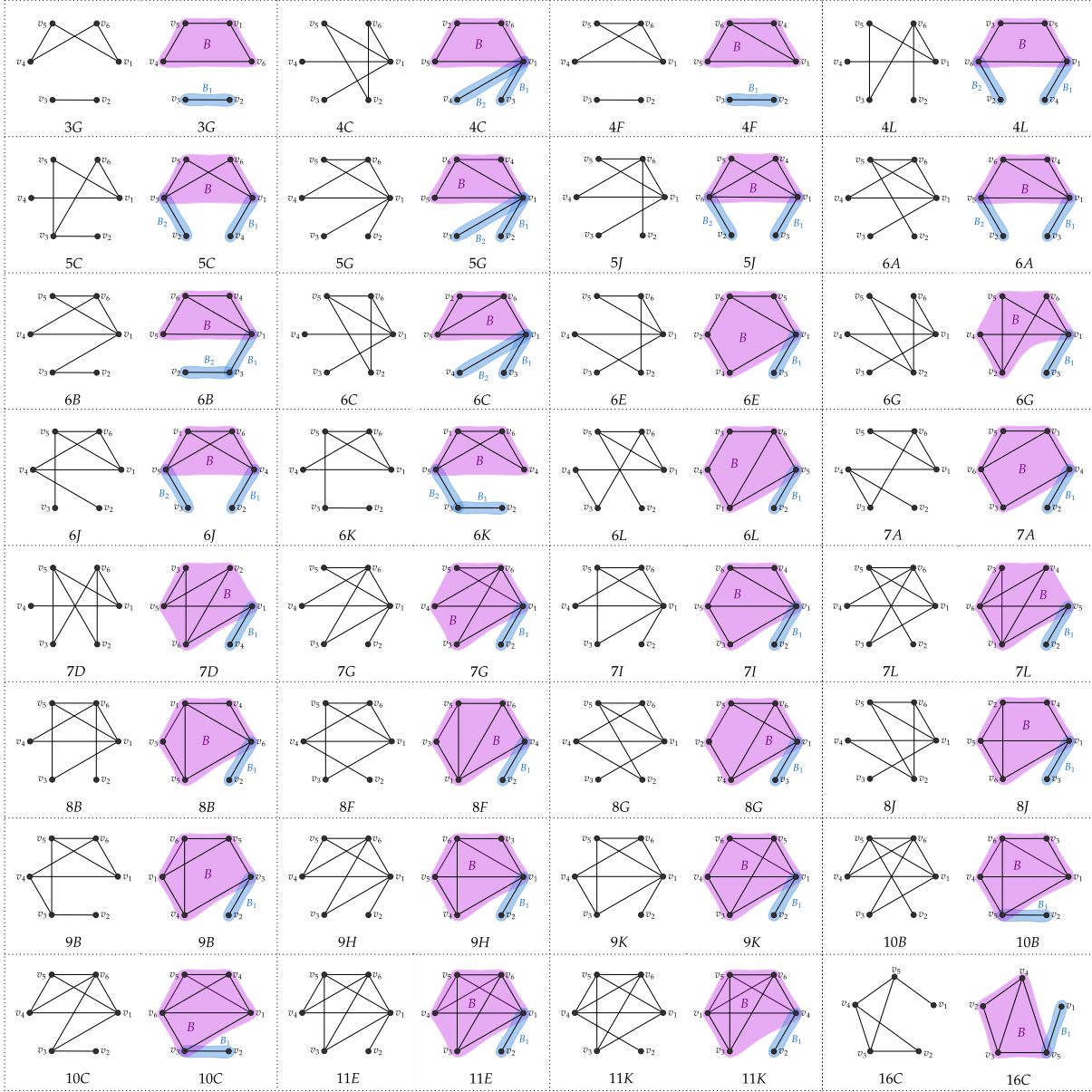


Figure 4.3: These graphs each have a governing block B , and at least one other block, and all blocks other than B are isomorphic to K_2 .

Chapter 5

Using line graphs

Here, we will prove outcomes (e) and (f) of Theorem 1.12.

5.1 Severed pairs

Instead of Theorem 1.12(e), we prove more generally that the line graph of nearly *every* graph with at least two leaves, if not complete, is deletion-normal. Let us give the formal definitions. Given a graph H , a *claw component* of H is a component of H that is isomorphic to $K_{1,3}$, and a *triangle component* of H is a component of H that is isomorphic to K_3 . A pair (x_1, x_2) of vertices in H is said to be *severed* if:

- x_1, x_2 are leaves of H at distance at least three from each other (that is, x_1, x_2 are non-adjacent and the unique neighbors of x_1, x_2 in H are distinct); and
- none of x_1, x_2 belongs to a claw component of H .

We show that:

Theorem 5.1. *Let H be a graph that has a severed pair. Then $L(H)$ is deletion-normal.*

For instance, if H is a tree, then H has a severed pair if and only if H is not a star. Also, conveniently, if H is a tree, then $L(H)$ is a complete graph if and only if H is a star. Since complete graphs are not deletion-saturated, this derives theorem 1.12(e) from Theorem 5.1:

Corollary 5.2. *For a tree H , $L(H)$ is deletion-normal if and only if H is not a star.*

Back to the proof of Theorem 5.1, the idea is to show that $L(K_{h-1})$, for a suitable choice of h depending on H , is H -deletion-saturated. In particular, we choose h so that K_h contains a subgraph whose line graph is isomorphic to $L(H)$, whereas K_{h-1} does not

contain such a subgraph. The choice of h is also sensitive to the fact that line graphs cannot distinguish between claws and triangles (the same fact explains the second bullet condition in the definition of a severed pair). More precisely, we use the following classical result of Whitney [29]:

Theorem 5.3 (Whitney [29]). *Let H_1 and H_2 be connected graphs. Then $\mathsf{L}(H_1)$ and $\mathsf{L}(H_2)$ are isomorphic if and only if either H_1 and H_2 are isomorphic, or one of H_1, H_2 is isomorphic to $K_{1,3}$ and the other is isomorphic to K_3 .*

We also need a definition. The *trimming* of H is the graph obtained from H by first removing all isolated vertices of H and then replacing every claw component of H with a triangle component. From Theorem 5.3, it follows that the trimming of H is exactly the graph with the minimum number of vertices whose line graph is isomorphic to the line graph of H .

Observation 5.4. *Let H be a graph, let $s \in \mathbb{N} \cup \{0\}$ be the number of isolated vertices of H , let $t \in \mathbb{N} \cup \{0\}$ be the number of claw components of H and let $h = |V(H)| - s - t$. Let H' be a graph such that $\mathsf{L}(H)$ and $\mathsf{L}(H')$ are isomorphic. Then $|V(H')| \geq h$ and equality holds if and only if H' is isomorphic to the trimming of H .*

We are now ready to prove Theorem 5.1. The following is a restatement of Theorem 5.1 that explicitly describes the $\mathsf{L}(H)$ -deletion-saturated graph.

Theorem 5.5. *Let H be a graph that has a severed pair. Let $s \in \mathbb{N} \cup \{0\}$ be the number of isolated vertices of H , let $t \in \mathbb{N} \cup \{0\}$ be the number of claw components of H and let $h = |V(H)| - s - t$. Then $\mathsf{L}(K_{h-1})$ is $\mathsf{L}(H)$ -deletion-saturated.*

Proof. By Observation 5.4, there is no subgraph of K_{h-1} whose line graph is isomorphic to the line graph of H . Thus, $\mathsf{L}(K_{h-1})$ is $\mathsf{L}(H)$ -free.

It remains to show that for every edge $ab \in E(\mathsf{L}(K_{h-1}))$ (where $a, b \in E(K_{h-1})$), there is an induced subgraph of $\mathsf{L}(K_{h-1}) - ab$ that is isomorphic to $\mathsf{L}(H)$. Let $a = uv$ and $b = uw$ where $u, v, w \in V(K_{h-1})$ are distinct. Choose a severed pair (x_1, x_2) in H . For each $i \in \{1, 2\}$, let y_i be the unique neighbor of x_i in H . It follows that x_1, y_1, x_2, y_2 are all distinct. Let H_1 be a trimming of H . Then $|V(H_1)| = |V(H)| - s - t = h$ and $x_1, x_2 \in V(H_1)$; in particular, (x_1, x_2) is a severed pair in H_1 , as well. Let H_2 be the graph obtained from H_1 by identifying the vertices x_1 and x_2 into a new vertex x . Then we have $\mathsf{L}(H_1) = \mathsf{L}(H_2) - e_1e_2$ where $e_1 = xy_1 \in E(H_2)$ and $e_2 = xy_2 \in E(H_2)$. Since $|V(H_2)| = |V(H_1)| - 1 = h - 1$, there is a subgraph G of K_{h-1} isomorphic to H_2 with $a = uv \in E(G)$ and $b = uw \in E(G)$, as well as an isomorphism $f : V(G) \rightarrow V(H_2)$ between G and H_2 such that $f(u) = x, f(v) = y_1$, and $f(w) = y_2$. Define the map $\varphi : E(G) \rightarrow E(H_2)$ as follows: for every edge $q_1q_2 \in E(G)$, let $\varphi(q_1q_2) = f(q_1)f(q_2) \in E(H_2)$. Then φ is an isomorphism between $\mathsf{L}(G)$ and $\mathsf{L}(H_2)$ where $\varphi(a) = e_1$ and $\varphi(b) = e_2$. Moreover, note that $\mathsf{L}(G)$ is the induced subgraph of $\mathsf{L}(K_{h-1})$ with vertex set $E(G)$. Hence, $\mathsf{L}(G) - ab$ is an induced subgraph of $\mathsf{L}(K_{h-1}) - ab$ isomorphic to $\mathsf{L}(H_2) - e_1e_2 = \mathsf{L}(H_1) = \mathsf{L}(H)$. This completes the proof of Theorem 5.5. $\blacksquare$

5.2 Near-line-graphs

For Theorem 1.12(f), we also prove the following restatement with explicit descriptions of the corresponding deletion-normal graphs. The key idea in the proof is that the line graphs of complete graphs and balanced complete bipartite graphs are edge-transitive, and so are the complements of those line graphs.

Theorem 5.6. *Let H be a graph on $h \in \mathbb{N}$ vertices. Then the following hold:*

- (a) *If H is a line^- graph, then $\mathsf{L}(K_{2h})$ is H -deletion-saturated.*
- (b) *If $\overline{H}$ is a line^+ graph, then $\overline{\mathsf{L}(K_{2h})}$ is H -deletion-saturated.*
- (c) *If H is a biline^- graph, then $\mathsf{L}(K_{h,h})$ is H -deletion-saturated.*
- (d) *If $\overline{H}$ is a biline^+ graph, then $\overline{\mathsf{L}(K_{h,h})}$ is H -deletion-saturated.*

Proof. First, we prove 5.6(a). Since H is not a line graph, it follows that $\mathsf{L}(K_{2h})$ is H -free. It remains to show that for every edge $ab \in E(\mathsf{L}(K_{2h}))$ (where $a, b \in E(K_{2h})$), there is an induced subgraph of $\mathsf{L}(K_{2h}) - ab$ that is isomorphic to H . Let $a = uv$ and $b = uw$ where $u, v, w \in V(K_{2h})$ are distinct. Since H is a line^- graph, there exists $xy \in E(\overline{H})$ for which $H + xy$ is a line graph. Choose a graph G' with no isolated vertex such that $\mathsf{L}(G') = H + xy$. Since x, y are adjacent in $H + xy$, it follows that there are distinct vertices $u', v', w' \in V(G')$ for which $x = u'v' \in E(G')$ and $y = u'w' \in E(G')$. Since $|E(G')| = |V(H + xy)| = |V(H)| = h$ and G' has no isolated vertex, it follows that G' has at most $2h$ vertices. Therefore, there is a subgraph G of K_{2h} isomorphic to G' with $a = uv \in E(G)$ and $b = uw \in E(G)$, as well as an isomorphism $f : V(G) \rightarrow V(G')$ between G and G' such that $f(u) = u', f(v) = v'$, and $f(w) = w'$. Define the map $\varphi : E(G) \rightarrow E(G')$ as follows: for every edge $q_1q_2 \in E(G)$, let $\varphi(q_1q_2) = f(q_1)f(q_2) \in E(G')$. Then φ is an isomorphism between $\mathsf{L}(G)$ and $\mathsf{L}(G')$ where $\varphi(a) = x$ and $\varphi(b) = y$. Moreover, note that $\mathsf{L}(G)$ is the induced subgraph of $\mathsf{L}(K_{2h})$ with vertex set $E(G)$. Hence, $\mathsf{L}(G) - ab$ is an induced subgraph of $\mathsf{L}(K_{2h}) - ab$ isomorphic to $\mathsf{L}(G') - xy = H$. This completes the proof of 5.6(a).

Second, we prove 5.6(b). Since $\overline{H}$ is not a line graph, it follows that $\mathsf{L}(K_{2h})$ is $\overline{H}$ -free, and so $\overline{\mathsf{L}(K_{2h})}$ is H -free. It remains to show that for every edge $ab \in E(\overline{\mathsf{L}(K_{2h})})$ (where $a, b \in E(K_{2h})$), there is an induced subgraph of $\overline{\mathsf{L}(K_{2h})} - ab$ that is isomorphic to H . Let $a = uv$, and $b = wz$ where $u, v, w, z \in V(K_{2h})$ are distinct. Since $\overline{H}$ is a line^+ graph, there exists $xy \in E(\overline{H})$ for which $\overline{H} - xy$ is a line graph. Choose a graph G' with no isolated vertex such that $\mathsf{L}(G') = \overline{H} - xy$. Since x, y are not adjacent in $\overline{H} - xy$, it follows that there are distinct vertices $u', v', w', z' \in V(G')$ such that $x = u'v' \in E(G')$ and $y = w'z' \in E(G')$. Since $|E(G')| = |V(\overline{H} - xy)| = |V(H)| = h$ and G' has no isolated vertex, it follows that G' has at most $2h$ vertices. Therefore, there is a subgraph G of K_{2h} isomorphic to G' with $a = uv \in E(G)$ and $b = wz \in E(G)$, as well as an isomorphism $f : V(G) \rightarrow V(G')$ such

that $f(u) = u', f(v) = v', f(w) = w'$, and $f(z) = z'$. Define the map $\varphi : E(G) \rightarrow E(G')$ as follows: for every edge $q_1q_2 \in E(G)$, let $\varphi(q_1q_2) = f(q_1)f(q_2) \in E(G')$. Then φ is an isomorphism between $L(G)$ and $L(G')$ where $\varphi(a) = x$ and $\varphi(b) = y$. Moreover, note that $L(G)$ is the induced subgraph of $L(K_{2h})$ with vertex set $E(G)$. Hence, $L(G) + ab$ is an induced subgraph of $L(K_{2h}) + ab$ isomorphic to $L(G') + xy = \overline{H}$, which in turn implies that $\overline{L(G) + ab} = \overline{L(G)} - ab$ is an induced subgraph of $\overline{L(K_{2h}) + ab} = \overline{L(K_{2h})} - ab$ isomorphic to $\overline{L(G')} + xy = H$. This completes the proof of 5.6(b).

Third, we prove 5.6(c). Since H is not the line graph of a bipartite graph, it follows that $L(K_{h,h})$ is H -free. It remains to show that for every edge $ab \in E(L(K_{h,h}))$ (where $a, b \in E(K_{h,h})$), there is an induced subgraph of $L(K_{h,h}) - ab$ that is isomorphic to H . Let $a = uv$ and $b = uw$ where $u, v, w \in V(K_{h,h})$ are distinct. Since H is a biline⁻graph, there exists $xy \in E(\overline{H})$ for which $H + xy$ is the line graph of a bipartite graph. Choose a bipartite graph G' with no isolated vertex such that $L(G') = H + xy$. Since x, y are adjacent in $H + xy$, it follows that there are distinct vertices $u', v', w' \in V(G')$ for which $x = u'v' \in E(G')$ and $y = u'w' \in E(G')$. Since $|E(G')| = |V(H + xy)| = |V(H)| = h$ and G' has no isolated vertex, it follows that G' admits a bipartition with at most h vertices on either side. Therefore, there is a subgraph G of $K_{h,h}$ isomorphic to G' with $a = uv \in E(G)$ and $b = uw \in E(G)$, as well as an isomorphism $f : V(G) \rightarrow V(G')$ between G and G' such that $f(u) = u', f(v) = v'$, and $f(w) = w'$. Define the map $\varphi : E(G) \rightarrow E(G')$ as follows: for every edge $q_1q_2 \in E(G)$, let $\varphi(q_1q_2) = f(q_1)f(q_2) \in E(G')$. Then φ is an isomorphism between $L(G)$ and $L(G')$ where $\varphi(a) = x$ and $\varphi(b) = y$. Moreover, note that $L(G)$ is the induced subgraph of $L(K_{h,h})$ with vertex set $E(G)$. Hence, $L(G) - ab$ is an induced subgraph of $L(K_{h,h}) - ab$ isomorphic to $L(G') - xy = H$. This completes the proof of 5.6(c).

Finally, we prove 5.6(d). Since $\overline{H}$ is not the line graph of a bipartite graph, it follows that $L(K_{h,h})$ is $\overline{H}$ -free, and so $\overline{L(K_{h,h})}$ is H -free. It remains to show that for every edge $ab \in E(L(K_{h,h}))$ (where $a, b \in E(K_{h,h})$), there is an induced subgraph of $\overline{L(K_{h,h})} - ab$ that is isomorphic to H . Let $a = uv$, and $b = wz$ where $u, v, w, z \in V(K_{h,h})$ are distinct. Since $\overline{H}$ is a biline⁺graph, there exists $xy \in E(\overline{H})$ for which $\overline{H} - xy$ is a line graph. Choose a bipartite graph G' with no isolated vertex such that $L(G') = \overline{H} - xy$. Since x, y are not adjacent in $\overline{H} - xy$, it follows that there are distinct vertices $u', v', w', z' \in V(G')$ such that $x = u'v' \in E(G')$ and $y = w'z' \in E(G')$. Since $|E(G')| = |V(\overline{H} - xy)| = |V(H)| = h$ and G' has no isolated vertex, it follows that G' admits a bipartition with at most h vertices on either side. Therefore, there is a subgraph G of $K_{h,h}$ isomorphic to G' with $a = uv \in E(G)$ and $b = wz \in E(G)$, as well as an isomorphism $f : V(G) \rightarrow V(G')$ such that $f(u) = u', f(v) = v', f(w) = w'$, and $f(z) = z'$. Define the map $\varphi : E(G) \rightarrow E(G')$ as follows: for every edge $q_1q_2 \in E(G)$, let $\varphi(q_1q_2) = f(q_1)f(q_2) \in E(G')$. Then φ is an isomorphism between $L(G)$ and $L(G')$ where $\varphi(a) = x$ and $\varphi(b) = y$. Moreover, note that $L(G)$ is the induced subgraph of $L(K_{h,h})$ with vertex set $E(G)$. Hence, $L(G) + ab$ is an induced subgraph of $L(K_{h,h}) + ab$ isomorphic to $L(G') + xy = \overline{H}$, which in turn implies that $\overline{L(G) + ab} = \overline{L(G)} - ab$ is an induced subgraph of $\overline{L(K_{h,h}) + ab} = \overline{L(K_{h,h})} - ab$ isomorphic to $\overline{L(G')} + xy = H$. This completes the proof of 5.6(d). ■

5.3 Corollary for graphs on at most six vertices

Of the $179 - 37 - 32 = 110$ graphs on at most six vertices remaining from Chapters 3 and 4, we have:

- The **34** graphs in Figures 5.1 and 5.2 are all line⁻graphs, and so by Theorem 5.6(a), they are all deletion-normal.
- The **8** graphs in Figure 5.3 are all complements of line⁺ graphs, and so by Theorem 5.6(b), they are all deletion-normal.
- The **2** graphs in Figure 5.4 (specifically, entries 7B and 8I in Figure 1.3) are biline⁻graphs, and so by Theorem 5.6(c), they are deletion-normal.
- The **3** graphs in Figure 5.5 (specifically, entries 8L, 11A and 11D in Figure 1.3) are complements of biline⁺ graphs, and so by Theorem 5.6(d), they are all deletion-normal.
- The **38** graphs in Figure 5.6 are line graphs of graphs with a severed pairs, and so by Theorem 5.1, they are all deletion-normal.

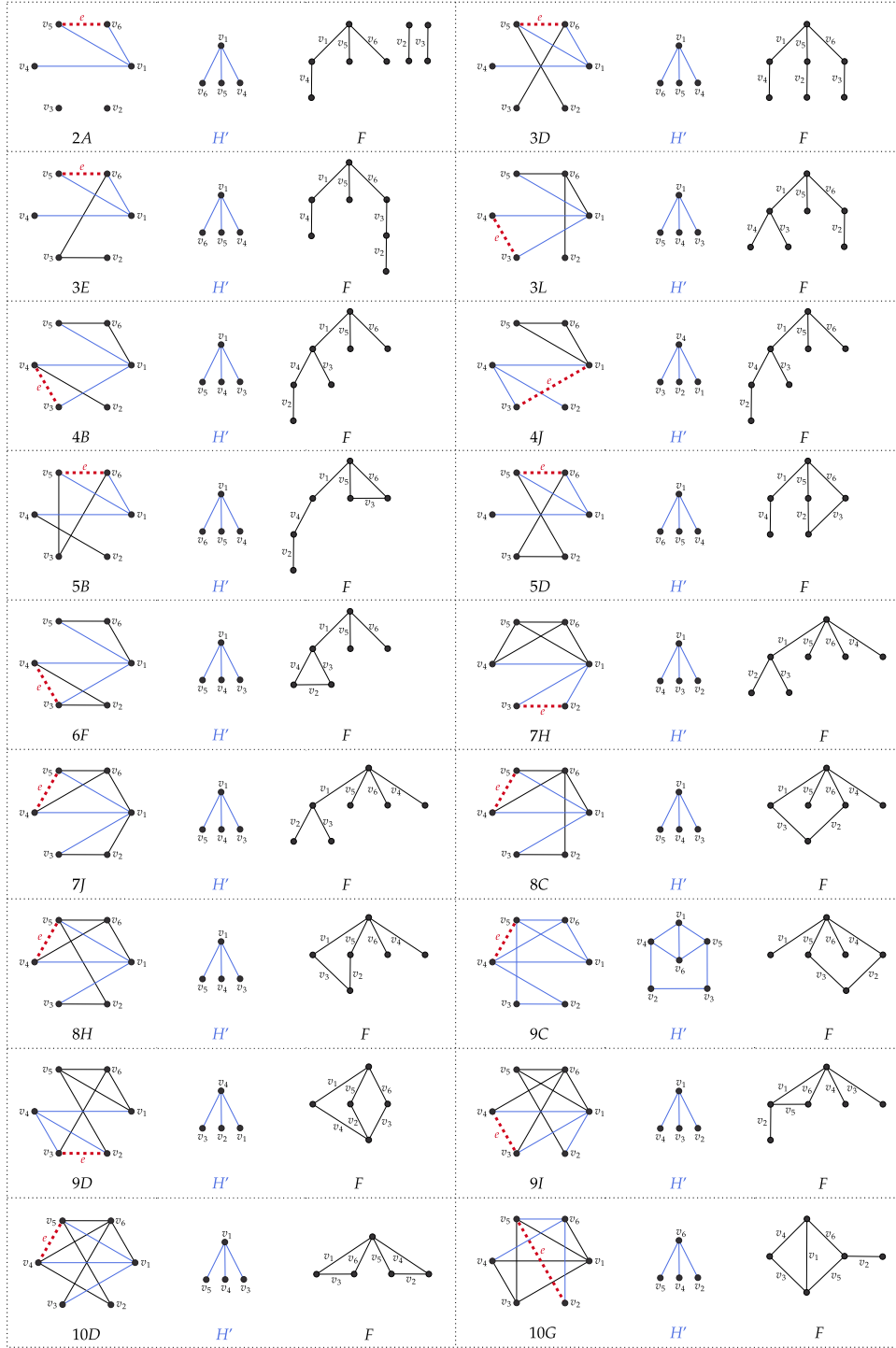


Figure 5.1: These graphs are all line-graphs. In each entry, we have a drawing of the graph H itself with a specified non-edge e , a drawing of an induced subgraph H' of H that by Theorem 2.1 prevents H from being a line graph, and then a drawing of the graph F whose line graph is $H + e$.

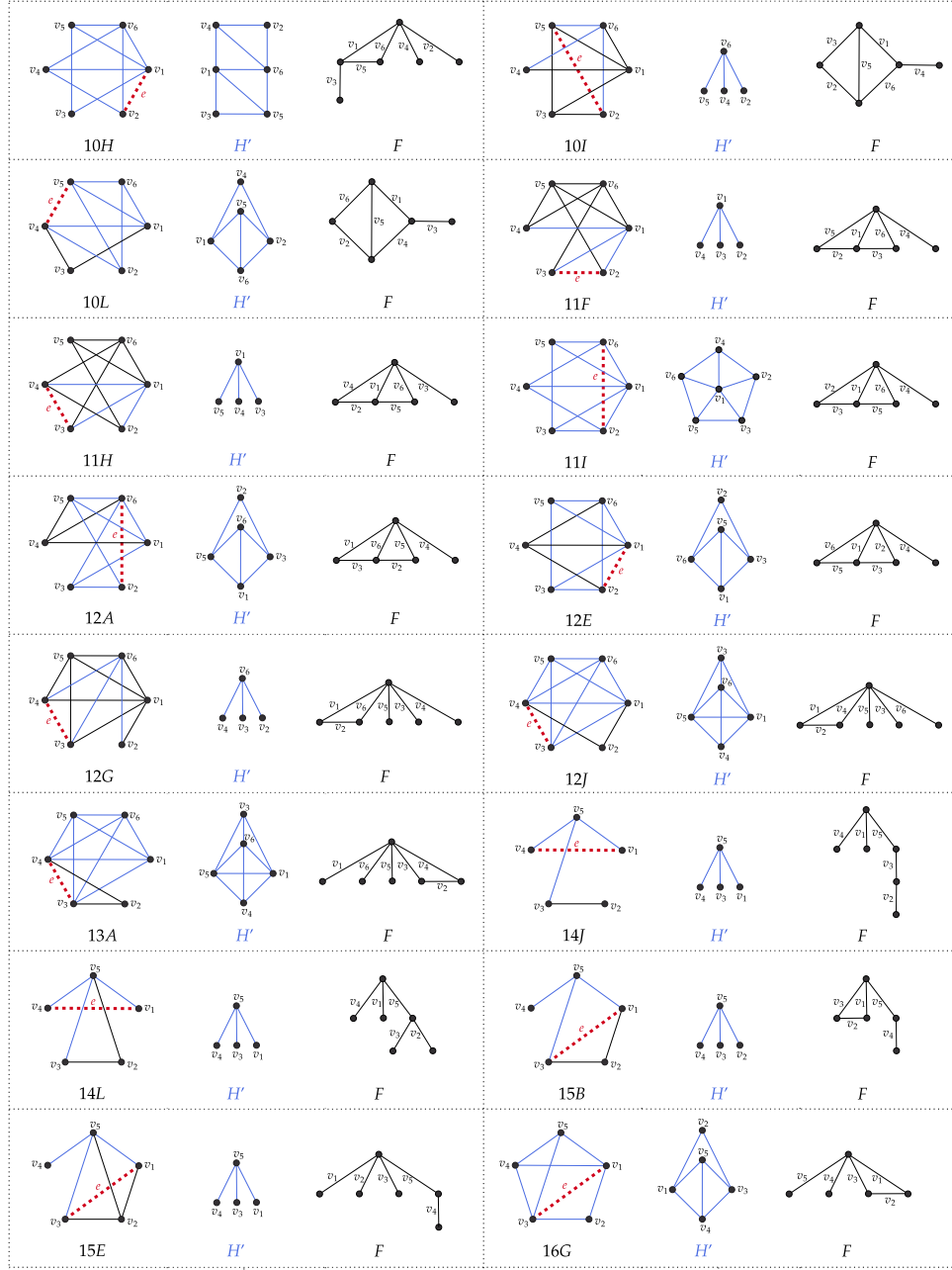


Figure 5.2: These graphs are all line-graphs. In each entry, we have a drawing of the graph H itself with a specified non-edge e , a drawing of an induced subgraph H' of H that by Theorem 2.1 prevents H from being a line graph, and then a drawing of the graph F whose line graph is $H + e$.

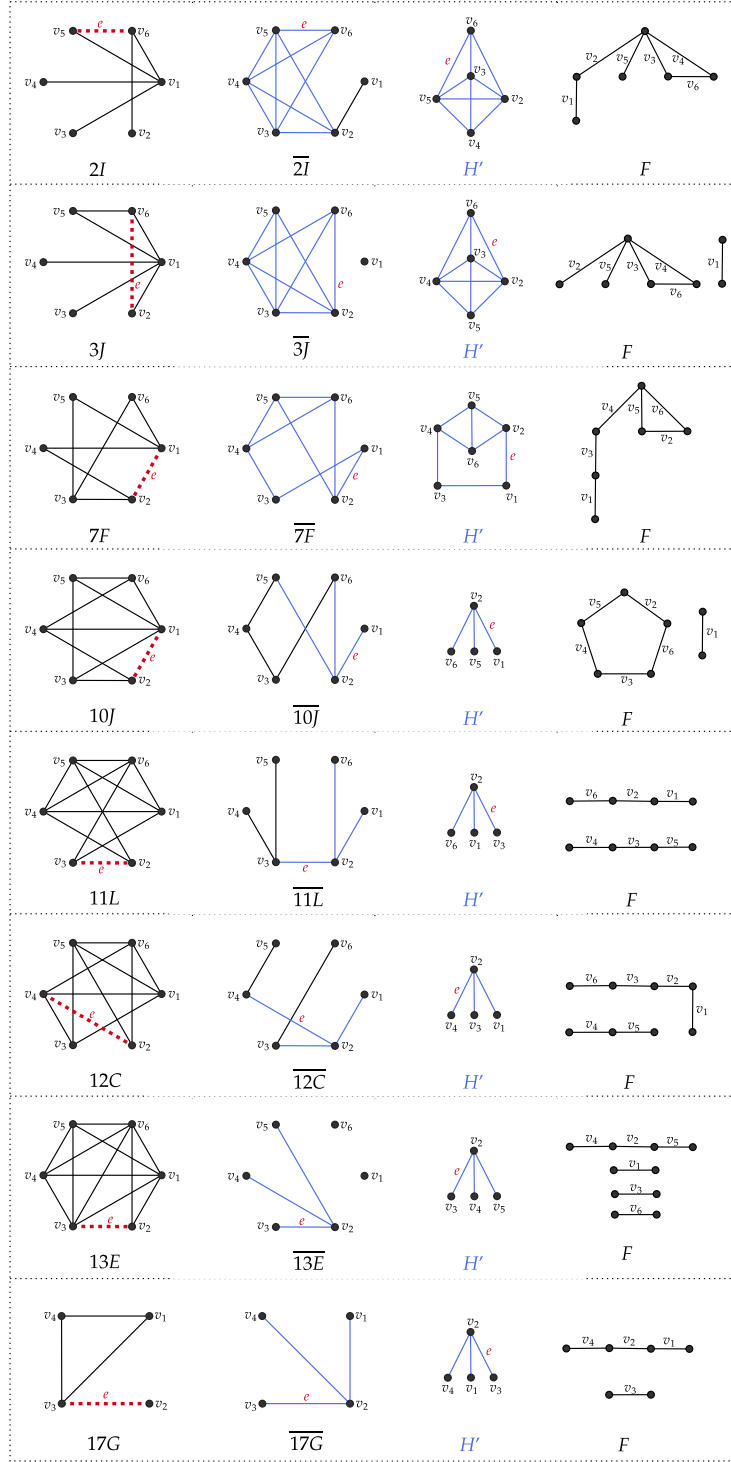


Figure 5.3: These graphs are all complements of line^+ graphs. In each entry, we have a drawing of the graph H itself, a drawing of $\overline{H}$ with a specified edge e of $\overline{H}$, a drawing of an induced subgraph H' of $\overline{H}$ that by Theorem 2.1 prevents $\overline{H}$ from being a line graph, and then a drawing of the graph F whose line graph is $\overline{H} - e$.

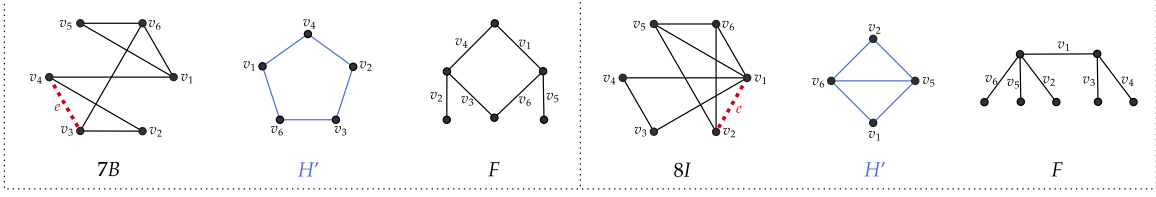


Figure 5.4: These graphs are all biline⁻graphs. In each entry, we have a drawing of the graph H itself with a specified non-edge e , a drawing of an induced subgraph H' of H that by Theorem 2.2 prevents H from being a biline graph, and then a drawing of the bipartite graph F whose line graph is $H + e$.

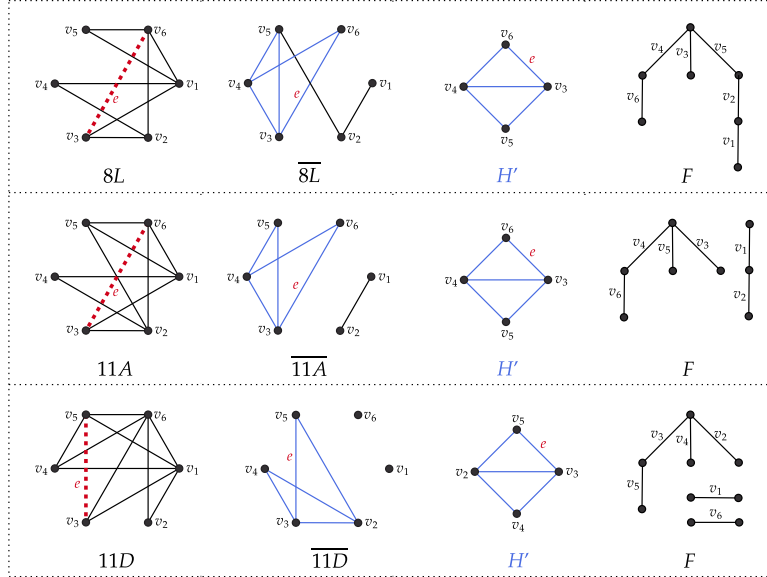


Figure 5.5: These graphs are all complements of biline⁺graphs. In each entry, we have a drawing of the graph H itself, a drawing of $\overline{H}$ with a specified edge e of $\overline{H}$, a drawing of an induced subgraph H' of $\overline{H}$ that by Theorem 2.2 prevents $\overline{H}$ from being a biline graph, and then a drawing of the bipartite graph F whose line graph is $\overline{H} - e$.

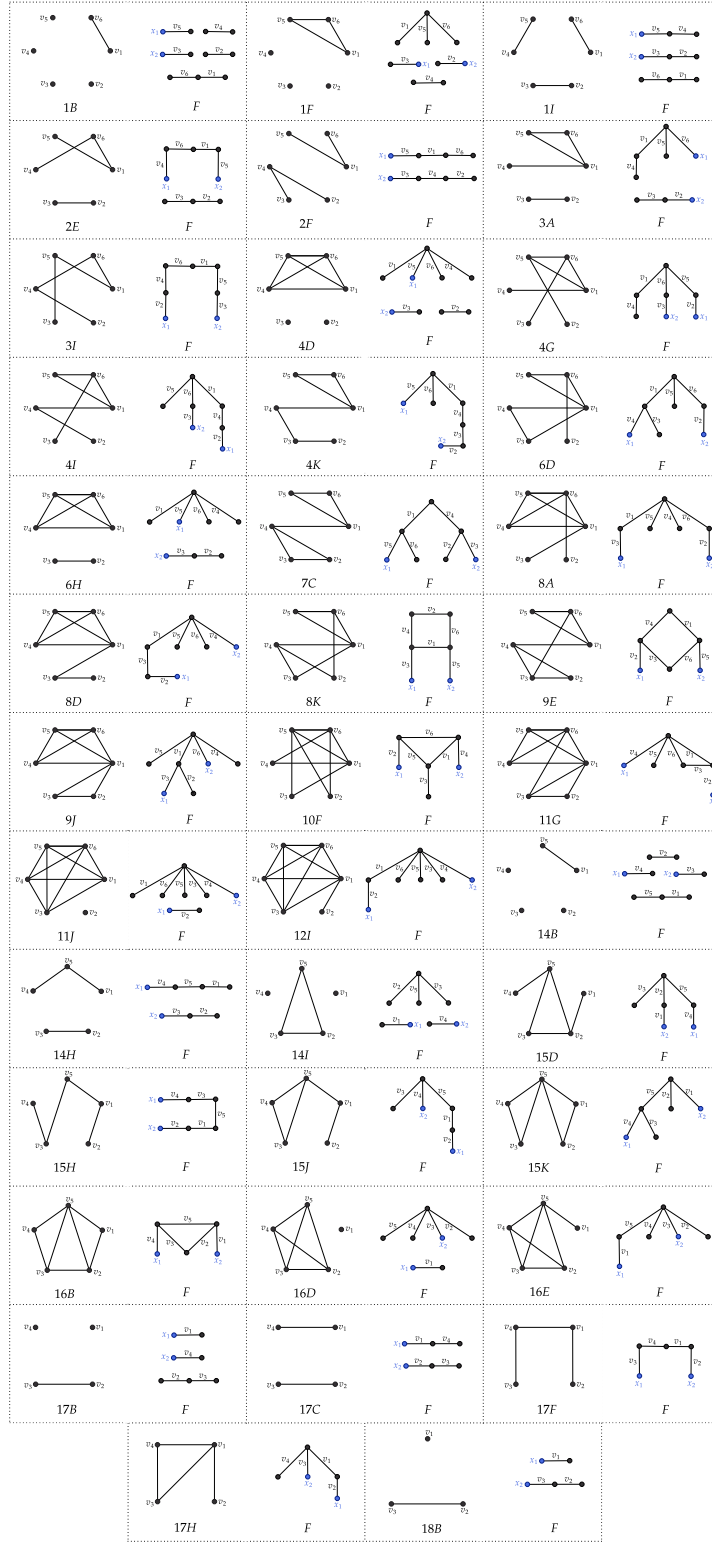


Figure 5.6: Every graph here is the line graph of a graph F with a severed pair. In each entry, we have a drawing of the graph H itself, and then a drawing of the graph F with a severed pair (x_1, x_2) such that $L(F) = H$.

Chapter 6

Sporadic graphs on at most six vertices

There are now $110 - 34 - 8 - 2 - 3 - 38 = \mathbf{25}$ graphs on at most six vertices remaining from Chapters 3, 4 and 5 that need special treatment. These graphs are entries

3B, 3F, 5E, 5F, 5H, 7E, 9F, 9L, 10E, 10K, 11B, 12B, 12D

12F, 12H, 12K, 13B, 13C, 13F, 13G, 13I, 15I, 15L, 16A, 16F

in Figure 1.3. In this section, we handle these 25 graphs in three groups:

- The “easy” cases; namely:

3F, 5F, 9F, 11B, 12B, 12D, 13C, 13F, 13G, 13I, 15L, 16F.

We handle these **12** graphs in Section 6.1.

- Cases that use the icosahedron; namely:

7E, 9L, 12F, 12K, 16A.

We handle these **5** graphs in Section 6.2.

- The “hard” cases; namely:

3B, 5E, 5H, 10E, 10K, 12H, 13B, 15I.

We handle these **8** graphs in Section 6.3.

6.1 Easy cases

For the first 12 graphs, we prove their deletion normality in three groups:

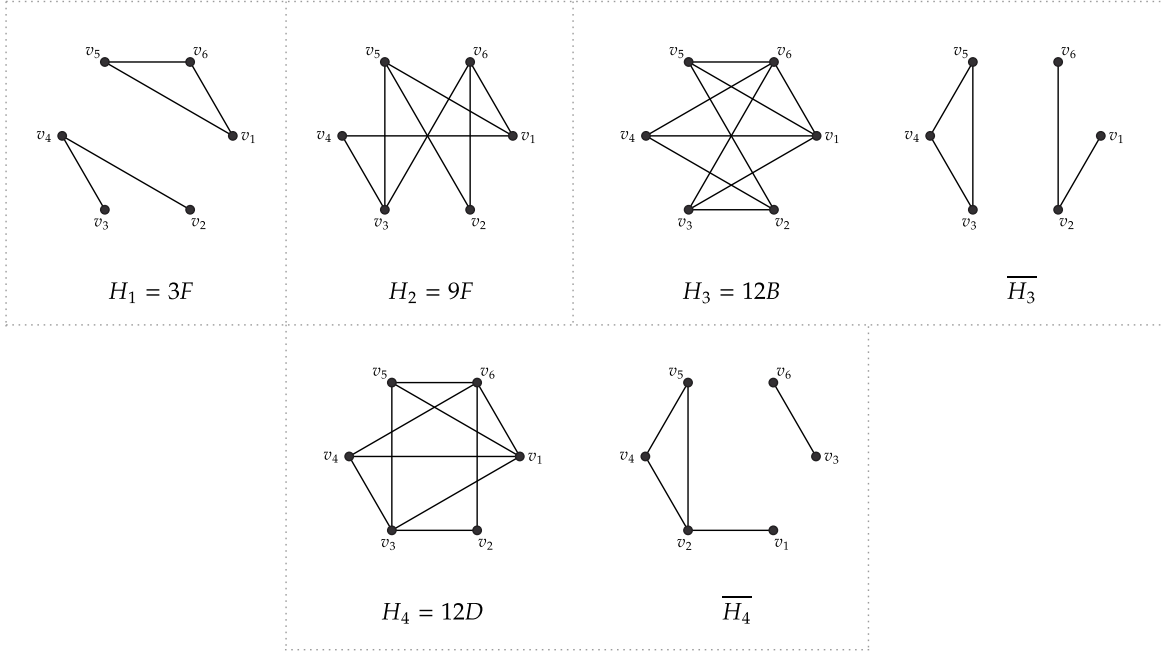


Figure 6.1: Proof of Theorem 6.1. Graphs H_1, H_2, H_3, H_4 , along with the complement of H_3 and H_4 .

Theorem 6.1. *Entries 3F, 9F, 12B and 12D in Figure 1.3 are deletion-normal.*

Proof. Let H_1 be entry 3F in Figure 1.3, let H_2 be entry 9F in Figure 1.3, let H_3 be entry 12B in Figure 1.3, and let H_4 be entry 12D in Figure 1.3.

Note that H_1 is isomorphic to the graph obtained from $2K_3$ by removing an edge. Since $2K_3$ is edge-transitive, it follows that $2K_3$ itself is H_1 -deletion-saturated.

Note that H_2 is isomorphic to the graph obtained from $K_{3,3}$ by removing an edge. Since $K_{3,3}$ is edge-transitive, it follows that $K_{3,3}$ itself is H_2 -deletion-saturated.

Note also that $\overline{H_3}$ is isomorphic to $K_3 + P_3$ and $\overline{H_4}$ is the disjoint union of a two-vertex path and a graph obtained from a triangle by adding a leaf. It is straightforward to observe that $3K_3$ is both $\overline{H_3}$ -addition-saturated and $\overline{H_4}$ -addition-saturated. Hence, $\overline{3K_3} = K_{3,3,3}$ is both H_3 -deletion-saturated and H_4 -deletion-saturated. This completes the proof of Theorem 6.1. ■

Theorem 6.2. *Entries 5F, 11B and 15L in Figure 1.3 are deletion-normal.*

Proof. Note that entry 5F in Figure 1.3 is a 6-cycle, entry 11B is the complement of a 6-cycle, and entry 15L is a 5-cycle. Recall that by Observation 1.3 and Theorem 1.5, all cycles of length at most 10 (except for the 3-cycle) are normal (and thus both deletion-normal and addition-normal). This completes the proof of Theorem 6.2. ■

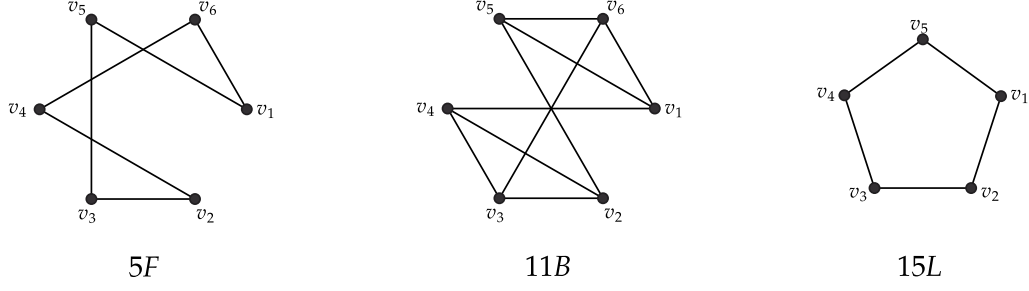


Figure 6.2: Proof of Theorem 6.2. The 6-cycle, the complement of the 6-cycle, and the 5-cycles.

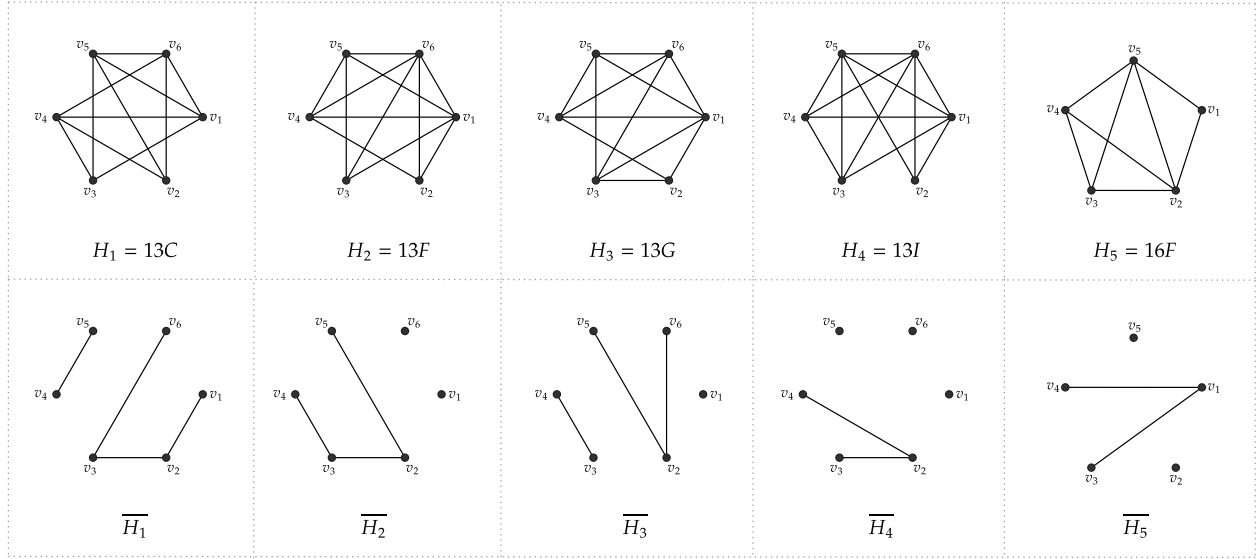


Figure 6.3: Proof of Theorem 6.3. From left to right, the graphs $H_1, \dots, H_5$ (top) and their complements (bottom).

Theorem 6.3. *Entries 13C, 13F, 13G, 13I, and 16F in Figure 1.3 are deletion-normal.*

Proof. Let H_1, H_2, H_3, H_4 , and H_5 , respectively, be the graphs at entries 13C, 13F, 13G, 13I, and 16F in Figure 1.3. Then,

- $\overline{H_1}$ is isomorphic to $P_4 + P_2$;
- $\overline{H_2}$ is isomorphic to $P_4 + 2P_1$;
- $\overline{H_3}$ is isomorphic to $P_3 + P_2 + P_1$;
- $\overline{H_4}$ is isomorphic to $P_3 + 3P_1$;

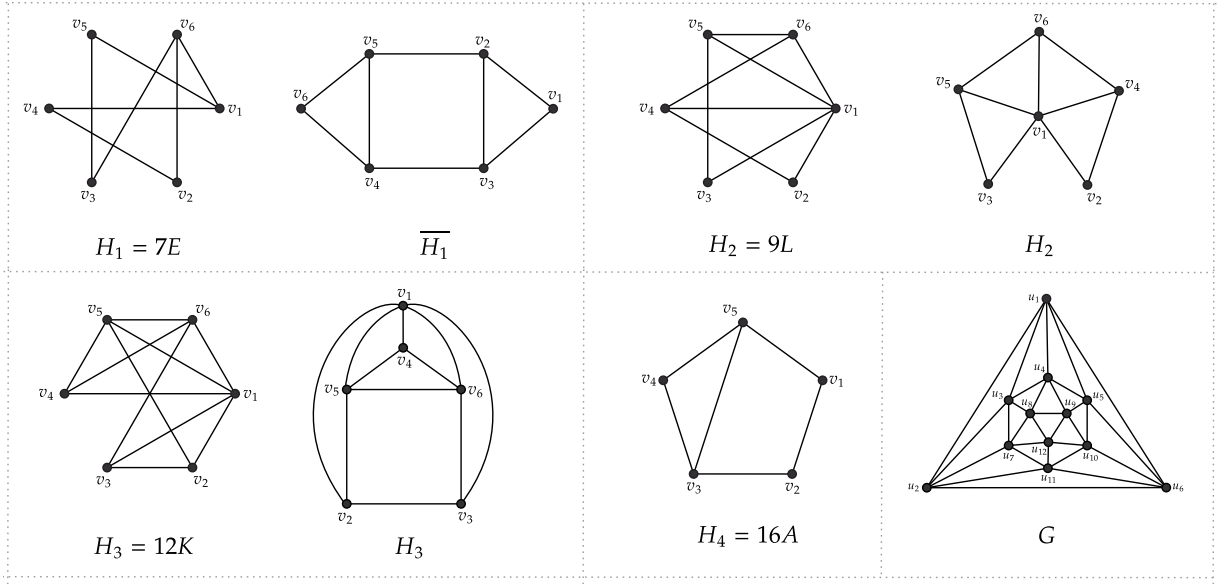


Figure 6.4: Proof of Theorem 6.4: The graph H_1 and its complement, two drawings of H_2 , two drawings of H_3 , the graph H_4 , and the graph G .

- $\overline{H_5}$ is isomorphic to $P_3 + 2P_1$;

Our goal is to show that for each $i \in \{1, \dots, 5\}$, the graph $5P_2$ is $\overline{H_i}$ -addition-normal. Note, in particular, that $5P_2$ is $\overline{H_i}$ -free because $\overline{H_i}$ has at least one vertex of degree two whereas $5P_2$ is 1-regular. Also, for every edge $e \in E(5P_2)$, the graph $5P_2 + e$ is isomorphic to $P_4 + 3P_2$, which is easily seen to contain an induced subgraph isomorphic to $\overline{H_i}$ for every $i \in \{1, \dots, 5\}$. This completes the proof of Theorem 6.3. ■

6.2 Using the icosahedron

The next five graphs will be handled by the following two theorems:

Theorem 6.4. *Entries 7E, 9L, 12K and 16A in Figure 1.3 are all deletion-normal.*

Proof. Let H_1 be entry 7E in Figure 1.3, let H_2 be entry 9L, let H_3 be entry 12K, and let H_4 be entry 16A; see Figure 6.4. Let G be an isomorphic copy of the icosahedron with $V(G) = \{u_1, \dots, u_{12}\}$ and the adjacency as depicted in Figure 6.4, and let G^+ be the graph obtained from G by adding a new vertex u_{13} adjacent to all vertices in $V(G)$.

First, we show that G is $\overline{H_1}$ -addition-saturated (which would in turn imply that $\overline{G}$ is H_1 -deletion-saturated). Note that G is $\overline{H_1}$ -free because the icosahedron, and thus G , is C_4 -free whereas $\overline{H_1}[v_2, v_3, v_4, v_5]$ is a 4-cycle. It remains to show that for every edge $e \in E(\overline{G})$, the graph $G + e$ is not $\overline{H_1}$ -free. There are now two cases up to symmetry:

- $e = u_1u_{11}$. Then, for $S = \{u_1, u_3, u_6, u_7, u_8, u_{11}\}$, it is readily seen that the map $f : V(H_1) \rightarrow S$ with

$$f(v_1) = u_6, \quad f(v_2) = u_1, \quad f(v_3) = u_{11}, \quad f(v_4) = u_7, \quad f(v_5) = u_3, \quad f(v_6) = u_8$$

is an isomorphism between $\overline{H_1}$ and $G[S] + e$.

- $e = u_1u_{12}$. Then, for $S = \{u_1, u_3, u_4, u_7, u_{11}, u_{12}\}$, it is readily seen that the map $f : V(H_1) \rightarrow S$ with

$$f(v_1) = u_4, \quad f(v_2) = u_1, \quad f(v_3) = u_3, \quad f(v_4) = u_7, \quad f(v_5) = u_{12}, \quad f(v_6) = u_{11}$$

is an isomorphism between $\overline{H_1}$ and $G[S] + e$.

Second, we show that G is H_2 -deletion-saturated. Note that G is H_2 -free because the neighborhood of every vertex in the icosahedron, and so in G , induces a 5-cycle, whereas $H_2[N_{H_2}(v_1)]$ is a five-vertex path. Since the icosahedron, and thus G , is edge-transitive, it remains to show that for some edge $e \in E(G)$, the graph $G - e$ is not H_2 -free. We pick $e = u_2u_3$. Then, for $S = \{u_1, \dots, u_6\}$, it is readily seen that the map $f : V(H_2) \rightarrow S$ with

$$f(v_i) = u_i \quad \text{for all } i \in \{1, \dots, 6\}$$

is an isomorphism between H_2 and $G[S] - e$.

Third, we show that G^+ is H_3 -deletion-saturated. Since G is C_4 -free, and since C_4 has no dominating vertex, it follows that G^+ is C_4 -free, too. This implies that G^+ is H_3 -free because $H_3[\{v_2, v_3, v_5, v_6\}]$ is a 4-cycle. It remains to show that for every $e \in E(G^+)$, the graph $G^+ - e$ is not H_3 -free. Since the icosahedron, and thus G , is edge-transitive, there are now two cases up to symmetry:

- $e = u_1u_3$. Then, for $S = \{u_1, u_2, u_3, u_4, u_5, u_{13}\}$, it is readily seen that the map $f : V(H_3) \rightarrow S$ with

$$f(v_1) = u_{13}, \quad f(v_2) = u_3, \quad f(v_3) = u_2, \quad f(v_4) = u_5, \quad f(v_5) = u_4, \quad f(v_6) = u_1$$

is an isomorphism between H_3 and $G[S] - e$.

- $e = u_1u_{13}$. Then, for $S = \{u_1, u_3, u_4, u_5, u_8, u_{13}\}$, it is readily seen that the map $f : V(H_3) \rightarrow S$ with

$$f(v_1) = u_4, \quad f(v_2) = u_1, \quad f(v_3) = u_5, \quad f(v_4) = u_8, \quad f(v_5) = u_3, \quad f(v_6) = u_{13}$$

is an isomorphism between H_3 and $G[S] - e$.

Finally, note that $\overline{H_4}$ is isomorphic to P_5 . By Theorem 1.6, $\overline{G}$ is $\overline{H_4}$ -induced-saturated, and thus G is H_4 -induced-saturated. This completes the proof of Theorem 6.4. $\blacksquare$

Theorem 6.5. *Entry 12F in Figure 1.3 is deletion-normal.*

Proof. Let H be entry 12F in Figure 1.3. Construct a graph G as follows. Let G_1 and G_2 be two disjoint and isomorphic copies of the icosahedron, such that $V(G_1) = \{u_i^1 : i \in \{1, \dots, 12\}\}$, $V(G_2) = \{u_i^2 : i \in \{1, \dots, 12\}\}$, and $f : u_i^1 \mapsto u_i^2$ is an isomorphism between G_1 and G_2 . Let G be the graph with

$$V(G) = \{u_i^1, u_i^2 : i \in \{1, \dots, 12\}\}$$

$$E(G) = E(G_1) \cup E(G_2) \cup \{u_i^1 u_i^2 : i \in \{1, \dots, 12\}\} \cup \{u_i^1 u_j^1 : i, j \in \{1, \dots, 12\}, u_i^1 u_j^1 \in E(G_1)\}.$$

See Figure 6.5.

Our goal is to prove that G is H -deletion-saturated.

(12) G is H -free.

Suppose for a contradiction that G has an induced subgraph H' isomorphic to H . Let $f : V(H) \rightarrow V(H')$ be an isomorphism between H and H' such that $f(v_1) = u_i^k$ and $f(v_2) = u_j^l$ for some $k, l \in \{1, 2\}$ and $i, j \in \{1, \dots, 12\}$. Since $v_1 v_2 \notin E(H)$, the vertices u_i^k and u_j^l are non-adjacent in G . Let $I = N_G(u_i^k) \cap N_G(u_j^l)$. Since $H[N_H(v_1) \cap N_H(v_2)]$ is isomorphic to $2K_2$, it follows that $G[I]$ has an induced subgraph isomorphic to $2K_2$, as well. On the other hand, since u_i^k and u_j^l are non-adjacent in G , by the construction of G , we have $I = (N_{G_1}(u_i^1) \cap N_{G_1}(u_j^1)) \cup (N_{G_2}(u_i^2) \cap N_{G_2}(u_j^2))$. Since G_1 and G_2 are both isomorphic to the icosahedron, it follows that either $N_{G_1}(u_i^1) \cap N_{G_1}(u_j^1) = N_{G_2}(u_i^2) \cap N_{G_2}(u_j^2) = \emptyset$, or both $N_{G_1}(u_i^1) \cap N_{G_1}(u_j^1)$ and $N_{G_2}(u_i^2) \cap N_{G_2}(u_j^2)$ are 2-vertex cliques in G . Thus, by the construction of G , either $I = \emptyset$ or I is a 4-vertex clique in G . But then $G[I]$ is $2K_2$ -free, a contradiction. This proves (12).

(13) For every $e \in E(G)$, the graph $G - e$ has an induced subgraph isomorphic to $\overline{H}$.

Let $e = u_i^k u_j^l \in E(G)$, where $k, l \in \{1, 2\}$ and $i, j \in \{1, \dots, 12\}$. Again, let $I = N_G(u_i^k) \cap N_G(u_j^l)$. By the choice of H , it suffices to show that there exists $S \subseteq I$ such that $G[S]$ is isomorphic to $2K_2$ (because then the subgraph of $G - e$ induced by $S \cup \{u_i^k, u_j^l\}$ is isomorphic to H). By the construction of G , there are now two cases:

- $i = j$ and $k \neq l$. So we have $e = u_i^1 u_i^2$. Since G_1 is isomorphic to the icosahedron, $G_1[N_{G_1}(u_i^1)]$ is 5-cycle, and so we may choose two non-adjacent vertices $u_s^1, u_t^1 \in N_{G_1}(u_i^1)$ where $s, t \in \{1, \dots, 12\} \setminus \{i\}$. But now $S = \{u_s^1, u_s^2, u_t^1, u_t^2\} \subseteq I$, and $E(G[S]) = \{u_s^1 u_s^2, u_t^1 u_t^2\}$; that is, $G[S]$ is isomorphic to $2K_2$.
- $i \neq j$ and $u_i^1 u_j^1 \in E(G_1)$. Since G_1 is isomorphic to the icosahedron, $N_{G_1}(u_i^1) \cap N_{G_1}(u_j^1)$ is a stable set of cardinality two; say $N_{G_1}(u_i^1) \cap N_{G_1}(u_j^1) = \{u_s^1, u_t^1\}$. But now $S = \{u_s^1, u_s^2, u_t^1, u_t^2\} \subseteq I$, and $E(G[S]) = \{u_s^1 u_s^2, u_t^1 u_t^2\}$; that is, $G[S]$ is isomorphic to $2K_2$.

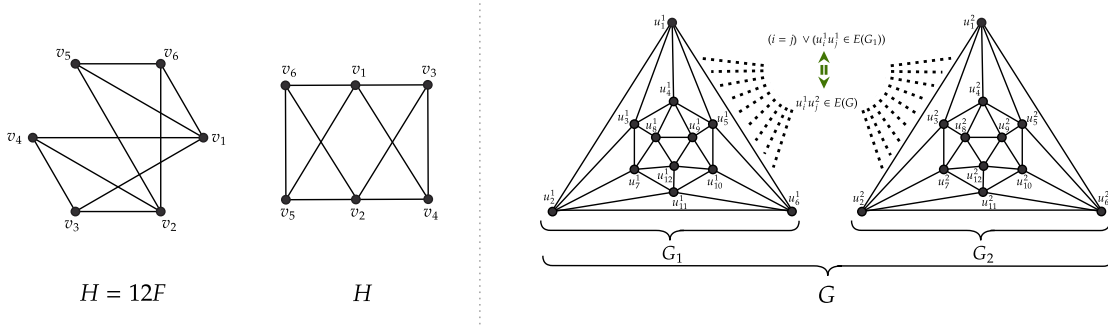


Figure 6.5: Proof of Theorem 6.5: Two drawings of the graph H (left) and the graph G (right).

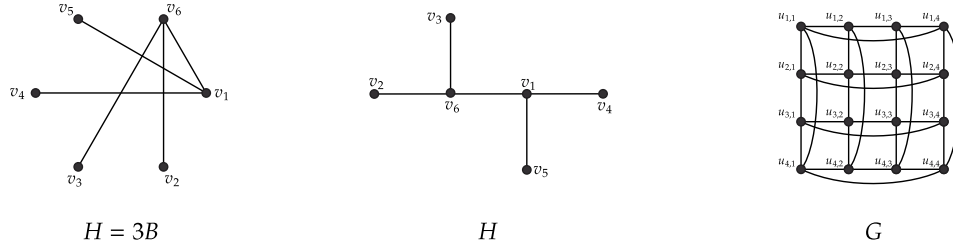


Figure 6.6: Proof of Theorem 6.6. From left to right: Two drawings of the graph H , and the graph G .

This proves (13).

The result is now immediate from (12) and (13). This completes the proof of Theorem 6.5. ■

6.3 Hard cases

We are now left with eight graphs for which the proof of deletion-normality requires completely different methods.

Theorem 6.6. *Entry 3B in Figure 1.3 is deletion-normal.*

Proof. Let H be entry 3B in Figure 1.3. Let G be the graph with vertex set $V(G) = \{u_{i,j} : i, j \in \{1, 2, 3, 4\}\}$ and edge set

$$E(G) = \left(\bigcup_{i=1}^4 \{u_{i,1}u_{i,2}, u_{i,2}u_{i,3}, u_{i,3}u_{i,4}, u_{i,4}u_{i,1}\} \right) \cup \left(\bigcup_{j=1}^4 \{u_{1,j}u_{2,j}, u_{2,j}u_{3,j}, u_{3,j}u_{4,j}, u_{4,j}u_{1,j}\} \right).$$

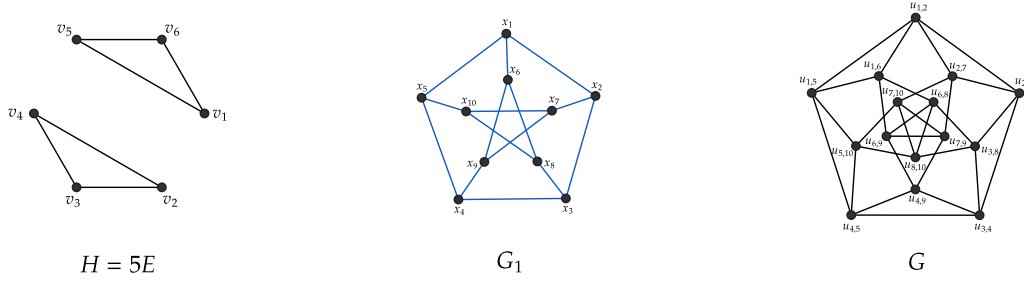


Figure 6.7: Proof of Theorem 6.7. From left to right: Graphs H, G_1 and G .

See Figure 6.6. In particular, G is the Cartesian product of two 4-cycles.

We prove that G is H -deletion-saturated. First, we show that G is H -free. Suppose not. According to the labeling given in Figure 6.6, we have $(N_H(v_1) \cup N_H(v_6)) \setminus \{v_1, v_6\} = \{v_2, v_3, v_4, v_5\}$, which is a four-vertex stable set in H . Therefore, since G has an induced subgraph isomorphic to H , it follows that there is an edge $uu' \in E(G)$ for which $G[(N_G(u) \cup N_G(u')) \setminus \{u, u'\}]$ contains a four-vertex stable set. However, since G is edge-transitive, it is easy to check that for every edge $uu' \in E(G)$, the graph $G[(N_G(u) \cup N_G(u')) \setminus \{u, u'\}]$ is indeed isomorphic to $3K_2$, which does not contain a four-vertex stable set, a contradiction. We deduce that G is H -free.

It remains to show that for every $e \in E(G)$, the graph $G - e$ has an induced subgraph isomorphic to H . Since G is edge-transitive, it suffices to prove the latter statement for only one edge $e \in E(G)$. Let $e = u_{1,1}u_{1,2}$. Let $S = \{u_{1,1}, u_{1,2}, u_{1,3}, u_{1,4}, u_{2,3}, u_{4,4}\}$. Then one can verify that the map $f : V(H) \rightarrow S$ with

$$f(v_1) = u_{1,3}, \quad f(v_2) = u_{1,1}, \quad f(v_3) = u_{4,4}, \quad f(v_4) = u_{1,2}, \quad f(v_5) = u_{2,3}, \quad f(v_6) = u_{1,4}$$

is an isomorphism between H and $G[S] - e$. This completes the proof of Theorem 6.6. ■

Theorem 6.7. *Entry 5E in Figure 1.3 is deletion-normal.*

Proof. Let H be entry 5E in Figure 1.3. Then H is isomorphic to $2K_3$. Let G_1 be the Petersen graph with $V(G_1) = \{x_1, \dots, x_{10}\}$ and the adjacency as depicted in Figure 6.7, and let $G = \mathbb{L}(G_1)$. For each edge $e = x_i x_j \in E(G_1)$, $i, j \in \{1, \dots, 10\}$, we label the vertex of G corresponding to e with $u_{i,j}$; see Figure 6.7.

Our goal is to show that G is $2K_3$ -deletion-saturated. Note that the only graphs of which the line graph is isomorphic to $2K_3$ are $2K_3, K_3 + K_{1,3}$ and $2K_{1,3}$. Since G_1 is triangle-free and 3-regular, and every two non-adjacent vertices in G_1 have exactly one common neighbor, G_1 has no subgraph isomorphic to H' for any $H' \in \{2K_3, K_3 + K_{1,3}, 2K_{1,3}\}$, and thus G is $2K_3$ -free.

It remains to show that for every $e \in E(G)$, the graph $G - e$ has an induced subgraph isomorphic to $2K_3$. By symmetry, we may assume that $e = u_{1,2}u_{2,3} \in E(G)$. Let $T_1 =$

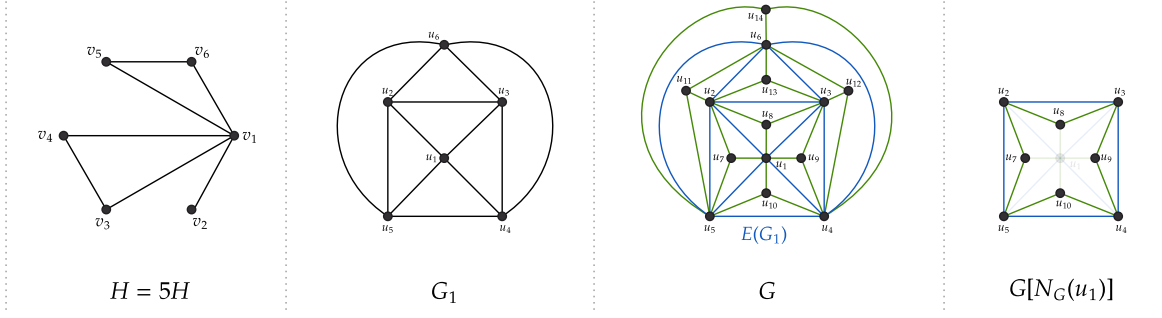


Figure 6.8: Proof of Theorem 6.8. From left to right: Graphs H , G_1 and G , and the subgraph of G induced by the neighborhood of u_1 .

$\{u_{1,2}, u_{1,5}, u_{1,6}\}$ and let $T_2 = \{u_{2,3}, u_{3,4}, u_{3,8}\}$. Then T_1 and T_2 are disjoint triangles in G , and e is the only edge of G with an end in T_1 and an end in T_2 . Hence $G[T_1 \cup T_2] - e$ is isomorphic to $2K_3$. This completes the proof of Theorem 6.7. $\blacksquare$

Theorem 6.8. *Entry 5H in Figure 1.3 is deletion-normal.*

Proof. Let H be entry 5H in Figure 1.3. Construct a graph G as follows. Let G_1 be an isomorphic copy of $L(K_4)$ with $V(G_1) = \{u_1, \dots, u_6\}$ and the adjacency as depicted in Figure 6.8. Note that G_1 has exactly eight triangles. Let G be the graph obtained from G_1 by adding eight pairwise adjacent new vertices $\{u_7, \dots, u_{14}\}$ – one for each triangle of G_1 – and making each vertex in $\{u_7, \dots, u_{14}\}$ adjacent to the three vertices in its corresponding triangle of G_1 , and to no other vertex in G , with the adjacency as depicted in Figure 6.8. (Note, in particular, that $\{u_7, \dots, u_{14}\}$ remains a stable set in G .) It is straightforward to observe that the natural action of the automorphism group of G on $V(G)$ has exactly two orbits, namely $V(G_1) = \{u_1, \dots, u_6\}$ and $V(G) \setminus V(G_1) = \{u_7, \dots, u_{14}\}$. Similarly, the natural action of the automorphism group of G on $E(G)$ has exactly two orbits, namely $E(G_1)$ and $E(G) \setminus E(G_1)$.

Our goal is to show that G is H -deletion-saturated. First, we show that G is H -free. Suppose for a contradiction that G has an induced subgraph H' isomorphic to H with an isomorphism $f : V(H) \rightarrow V(H') \subseteq V(G)$. Let $u = f(v_1)$. Since $N_H(v_1)$ is isomorphic to $2K_2 + K_1$, it follows that $G[N_G(u)]$ has an induced subgraph isomorphic to $2K_2 + K_1$. In particular, since the neighborhood of every vertex u_i for $i \in \{7, \dots, 14\}$ is a triangle in G , it follows that $u = u_i$ for some $i \in \{1, \dots, 6\}$. By symmetry, we may assume that $u = u_1$. But then

$$G[N_G(u)] = G[N_G(u_1)] = G[\{u_2, u_3, u_4, u_5, u_7, u_8, u_9, u_{10}\}]$$

is the graph on the right in Figure 6.8, which is easily observed to be $(2K_2 + K_1)$ -free, a contradiction. We deduce that G is H -free.

It remains to show that for every $e \in E(G)$, the graph $G - e$ has an induced subgraph isomorphic to H . There are now two cases:

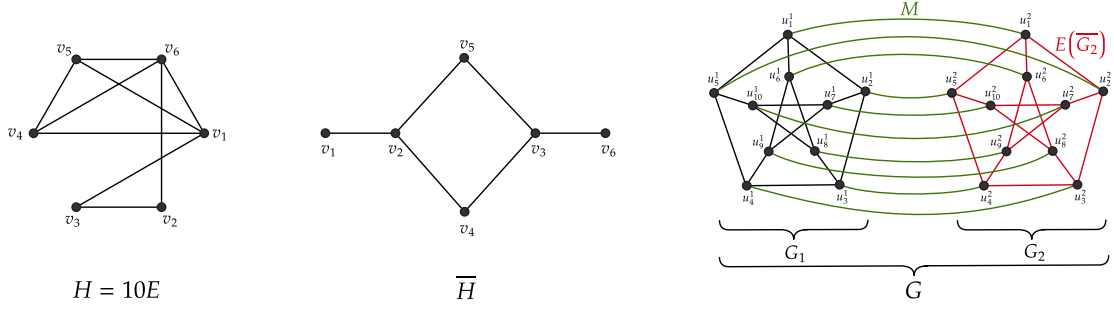


Figure 6.9: Proof of Theorem 6.9. From left to right: The graph H and its complement, and the graph G with a depiction of $E(G_1)$, $E(G_2)$, and M .

- $e \in E(G_1)$. By symmetry, we may assume that $e = u_4u_5$. Let $S = \{u_1, u_4, u_5, u_7, u_8, u_9\}$. Then one can check that the map $f : V(H) \rightarrow S$ with

$$f(v_1) = u_1, \quad f(v_2) = u_8, \quad f(v_3) = u_9, \quad f(v_4) = u_4, \quad f(v_5) = u_5, \quad f(v_6) = u_7$$

is an isomorphism between H and $G[S] - e$.

- $e \in E(G) \setminus E(G_1)$. Due to symmetry, we may assume that $e = u_2u_7$. Let $S = \{u_1, u_2, u_4, u_7, u_8, u_9\}$. Then one can check that the map $f : V(H) \rightarrow S$ with

$$f(v_1) = u_1, \quad f(v_2) = u_7, \quad f(v_3) = u_2, \quad f(v_4) = u_8, \quad f(v_5) = u_9, \quad f(v_6) = u_4$$

is an isomorphism between H and $G[S] - e$.

This completes the proof of Theorem 6.8. ■

Theorem 6.9. *Entry 10E in Figure 1.3 is deletion-normal.*

Proof. Let H be entry 10E in Figure 6.9. Construct a graph G as follows. Let G_1 and G_2 be disjoint graphs such that G_1 is isomorphic to the Petersen graph with $V(G_1) = \{u_1^1, \dots, u_{10}^1\}$ and $E(G_1)$ as shown in Figure 6.9, and $\overline{G_2}$ is isomorphic to the Petersen graph with $V(G_2) = \{u_1^2, \dots, u_{10}^2\}$ and $E(\overline{G_2})$ as shown in Figure 6.9. Let G be the graph obtained from the (disjoint) union of G_1 and G_2 by adding the matching $M = \{u_i^1u_i^2 : i \in \{1, \dots, 10\}\}$. See Figure 6.9.

To prove Theorem 6.9, it suffices to show that G is $\overline{H}$ -addition-saturated.

(14) G is $\overline{H}$ -free.

Suppose for a contradiction that G has an induced subgraph H' isomorphic to $\overline{H}$ where $V(H') = \{v'_1, \dots, v'_6\} \subseteq V(G)$, and $f : v_i \mapsto v'_i$ is an isomorphism between $\overline{H}$ and H' . Let

$I' = \{v'_2, v'_3, v'_4, v'_5\}$. Then $G[I']$ is isomorphic to C_4 . Assume that $I' \cap V(G_1) \neq \emptyset$. Since G_1 is C_4 -free, it follows that $I' \cap V(G_2) \neq \emptyset$. Since, in the graph G , every vertex in $V(G_1)$ has exactly one neighbor in $V(G_2)$ (and vice versa), it follows that $|I' \cap V(G_1)| = |I' \cap V(G_2)| = 2$ and $|E(G[I']) \cap M| = 2$. In particular, we have $I' \cap V(G_1) = \{u_i^1, u_j^1\}$ and $I' \cap V(G_2) = \{u_i^2, u_j^2\}$ for some $i, j \in \{1, \dots, 10\}$. But then exactly one of $u_i^1 u_j^1$ and $u_i^2 u_j^2$ is an edge in G , and so $|E(G[I'])| = 3$, a contradiction.

We deduce that $I' \cap V(G_1) = \emptyset$. Thus, we may write

$$v'_2 = u_{i_1}^2, \quad v'_4 = u_{i_2}^2, \quad v'_3 = u_{i_3}^2, \quad v'_5 = u_{i_4}^2$$

for some distinct $i_1, i_2, i_3, i_4 \in \{1, \dots, 10\}$. In particular, we have

$$E(G[I']) = E(G_2[I']) = \{u_{i_1}^2 u_{i_2}^2, u_{i_2}^2 u_{i_3}^2, u_{i_3}^2 u_{i_4}^2, u_{i_4}^2 u_{i_1}^2\}.$$

There are now two cases:

- $\{v'_1, v'_6\} \subseteq V(G_1)$. Then $v'_1 = u_{i_1}^1$ and $v'_6 = u_{i_3}^1$. But now since $u_{i_1}^2 u_{i_3}^2 \notin E(G_2)$, it follows that $v'_1 v'_6 = u_{i_1}^1 u_{i_3}^1 \in E(G_1) \subseteq E(G)$, which in turn implies that $v'_1 v'_6 \in E(H')$, a contradiction.
- $\{v'_1, v'_6\} \cap V(G_2) \neq \emptyset$. By symmetry, we may assume that $v'_1 \in V(G_2)$. It follows that $\{v'_1, u_{i_2}^2, u_{i_4}^2\}$ is a stable set of cardinality three in G_2 . But then $\overline{G_2}$, and thus the Petersen graph, contains a triangle, a contradiction.

This proves (14).

(15) For every $e \in E(\overline{G})$, the graph $G + e$ has an induced subgraph isomorphic to $\overline{H}$.

It suffices to show that there exists $S \subseteq V(G)$ containing both endpoints of e such that $G[S] + e$ is isomorphic to $\overline{H}$. There are now three cases:

- $e = u_{i_1}^1 u_{i_2}^1$ for some $i_1, i_2 \in \{1, \dots, 10\}$. Then, it is readily observed that, since G_1 is isomorphic to the Petersen graph, there is a path $u_{i_1}^1 - u_{i_2}^1 - u_{i_3}^1 - u_{i_4}^1$ of length three in G_1 from $u_{i_1}^1$ to $u_{i_2}^1$. Thus, $C = u_{i_1}^1 - u_{i_2}^1 - u_{i_3}^1 - u_{i_4}^1 - u_{i_1}^1$ is an induced 4-cycle in $G + e$. Let $N_{G_1}(u_{i_3}^1) \setminus \{u_{i_2}^1, u_{i_4}^1\} = \{u_{i_5}^1\}$, where $i_5 \in \{1, \dots, 10\} \setminus \{i_1, i_2, i_3, i_4\}$. Since the Petersen graph, and so G_1 , has girth five, it follows that $u_{i_5}^1$ is anticomplete to $\{u_{i_1}^1, u_{i_2}^1, u_{i_4}^1\}$ in G_1 . Let $S = \{u_{i_1}^1, u_{i_2}^1, u_{i_3}^1, u_{i_4}^1, u_{i_5}^1, u_{i_1}^2\}$. Then, it is easy to check that the map $f : V(\overline{H}) \rightarrow S$ with

$$f(v_1) = u_{i_1}^2, \quad f(v_2) = u_{i_1}^1, \quad f(v_3) = u_{i_3}^1, \quad f(v_4) = u_{i_2}^1, \quad f(v_5) = u_{i_4}^1, \quad f(v_6) = u_{i_5}^1$$

is an isomorphism between $\overline{H}$ and $G[S] + e$.

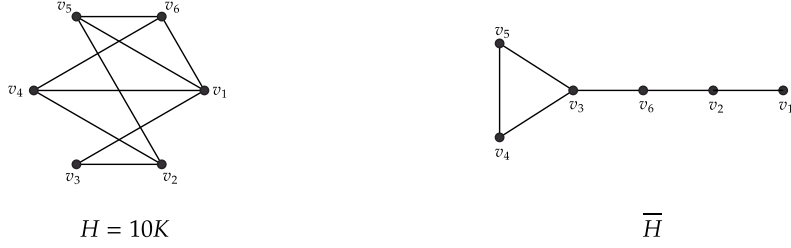


Figure 6.10: Proof of Theorem 6.10. The graph H (left) and its complement (right).

- $e = u_{i_1}^1 u_{i_2}^2$ for some $i_1, i_2 \in \{1, \dots, 10\}$. Since G_1 has girth five, every two vertices in G_1 have at most one common neighbor. Thus, since $u_{i_2}^1$ has degree three in G_1 , we may choose a vertex $u_{i_3}^1 \in N_{G_1}(u_{i_2}^1) \setminus N_{G_1}[u_{i_1}^1]$ where $i_3 \in \{1, \dots, 10\} \setminus \{i_1, i_2\}$. Also, it is easy to check that in G_1 (as in the Petersen graph), for every edge $e' \in E(G_1)$, there is a (unique) induced subgraph $M_{e'}$ of G_1 isomorphic to $3K_2$ where $e' \in E(M_{e'})$. In particular, there are $i_4, i_5 \in \{1, \dots, 10\} \setminus \{i_1, i_2, i_3\}$ for which $u_{i_4}^1 u_{i_5}^1 \in E(M_{u_{i_2}^1 u_{i_3}^1})$. It follows that $C = u_{i_2}^2 - u_{i_4}^2 - u_{i_3}^2 - u_{i_5}^2 - u_{i_2}^2$ is an induced 4-cycle in G_2 (and so in G). Let $S = \{u_{i_1}^1, u_{i_2}^2, u_{i_3}^2, u_{i_4}^2, u_{i_5}^2, u_{i_3}^1\}$. Then, it is easy to check that the map $f : V(\overline{H}) \rightarrow S$ with

$$f(v_1) = u_{i_1}^1, \quad f(v_2) = u_{i_2}^2, \quad f(v_3) = u_{i_3}^2, \quad f(v_4) = u_{i_4}^2, \quad f(v_5) = u_{i_5}^2, \quad f(v_6) = u_{i_3}^1$$

is an isomorphism between $\overline{H}$ and $G[S] + e$.

- $e = u_{i_1}^2 u_{i_2}^2$ for some $i_1, i_2 \in \{1, \dots, 10\}$. Then we have $u_{i_1}^1 u_{i_2}^1 \in E(G_1)$, and so $C = u_{i_1}^1 - u_{i_2}^2 - u_{i_3}^2 - u_{i_4}^2 - u_{i_1}^1$ is an induced 4-cycle in $G + e$. Let $N_{G_1}(u_{i_1}^1) = \{u_{i_2}^1, u_{i_3}^1, u_{i_4}^1\}$ where $i_3, i_4 \in \{1, \dots, 10\} \setminus \{i_1, i_2\}$. Since $N_{G_1}(u_{i_1}^1)$ is a stable set in G_1 , it follows that $u_{i_2}^2 u_{i_4}^2 \in E(G_2)$. Let $S = \{u_{i_1}^1, u_{i_2}^2, u_{i_3}^2, u_{i_4}^2, u_{i_2}^1, u_{i_4}^1\}$. Then, it is easy to check that the map $f : V(\overline{H}) \rightarrow S$ with

$$f(v_1) = u_{i_3}^2, \quad f(v_2) = u_{i_1}^1, \quad f(v_3) = u_{i_2}^2, \quad f(v_4) = u_{i_4}^2, \quad f(v_5) = u_{i_2}^1, \quad f(v_6) = u_{i_4}^1$$

is an isomorphism between $\overline{H}$ and $G[S] + e$.

This proves (15).

The result is now immediate from (14) and (15). This completes the proof of Theorem 6.9. ■

Theorem 6.10. *Entry 10K in Figure 1.3 is deletion-normal.*

Proof. Let H be entry 10K in Figure 1.3. We will show that $\overline{H}$ is addition-normal (see Figure 6.10).

By Theorem 4.6 applied to $c = 5$, $g = 3$ and $k = 2$, there exists a triangle-free graph of chromatic number larger than 5. Let G be a triangle-free graph with $\chi(G) > 5$ and with $E(G)$ maximal with respect to inclusion. It follows in particular that G is connected.

Our goal is to show that G is $\overline{H}$ -addition-saturated¹. In fact, G is $\overline{H}$ -free because G is triangle-free whereas $\overline{H}$ is not. It remains to show that for every $e \in E(\overline{G})$, the graph $G^+ = G + e$ has an induced subgraph isomorphic to $\overline{H}$.

By the maximality of $E(G)$, there is a triangle T in G^+ containing both ends of e , and in fact every triangle in G^+ contains both ends of e . Let N be the set of all vertices of G^+ with at least one neighbor in T in G^+ . Then $T \subseteq N$, and since G is triangle-free, we have $\chi(G^+[N]) \leq 3$. Thus, $\chi(G^+ \setminus N) \geq \chi(G) - 3 > 2$.

Let C_1 be the component of $G^+ \setminus N$ with maximum chromatic number. Then $\chi(C_1) > 2$, and since G is connected, some vertex in $V(C_1)$ has a neighbor $x_0 \in N \setminus T$ in G^+ . Let $N_0 = N_{G^+}(x_0) \cap V(C_1)$. Then N_0 is a stable set in C_1 , and so $\chi(C_1 \setminus N_0) \geq \chi(C_1) - 1 > 1$.

Let C_2 be the component of $C_1 \setminus N_0$ with maximum chromatic number. Then $\chi(C_2) > 1$, and since C_1 is connected, some vertex in $V(C_2)$ has a neighbor $x_1 \in N_0$ in C_1 . Let $N_1 = N_{G^+}(x_1) \cap V(C_2)$. Then N_1 is a stable set in C_2 , and so $\chi(C_2 \setminus N_1) \geq \chi(C_2) - 1 > 0$. In particular, $V(C_2) \setminus N_1 \neq \emptyset$, and since C_2 is connected, it follows that some vertex $x_3 \in V(C_2) \setminus N_1$ has a neighbor $x_2 \in N_1$ in C_2 .

We deduce that $x_0-x_1-x_2-x_3$ is a four-vertex path in G^+ and x_0 is the only vertex of this path with a neighbor in T in G^+ . If x_0 has exactly one neighbor in T , then $G^+[T \cup \{x_0, x_1, x_2\}]$ is isomorphic to $\overline{H}$. If x_0 has two distinct neighbors $y, y' \in T$, then $G^+[\{y, y', x_0, x_1, x_2, x_3\}]$ is isomorphic to $\overline{H}$. This completes the proof of Theorem 6.10. ■

Theorem 6.11. *Entry 12H in Figure 1.3 is deletion-normal.*

Proof. Let H be entry 12H in Figure 1.3 (see also Figure 6.11). Let $A = \{1, 2, 3\} \times \{1, 2, 3\}$. Let G be the graph with $V(G) = \{u_{i,j}^1, u_{i,j}^2, u_{i,j}^3 : (i, j) \in A\}$ and such that:

- For each $k \in \{1, 2, 3\}$ and all distinct $(i, j), (i', j') \in A$, we have $u_{i,j}^k u_{i',j'}^k \in E(G)$ if and only if $i = i'$ or $j = j'$.
- For every $(k, l) \in \{(1, 2), (2, 3), (3, 1)\}$ and all distinct $(i, j), (i', j') \in A$, we have $u_{i,j}^k u_{i',j'}^l \in E(G)$ if and only if $j \neq i'$.

See Figure 6.11.

¹We remark that the idea in the proof of Theorem 6.10 can be adapted to prove the addition-normality of any graph that is obtained from a triangle and path on two or more vertices by identifying an end of the path with a vertex of the triangle.

This 16-regular 27-vertex graph, known as the *Schläfli graph*, is both vertex-transitive and edge-transitive, and much more. For an integer $k \in \mathbb{N}$, we say that a graph F is *k-ultrahomogeneous* if for every two subsets X_1, X_2 of $V(F)$ with $|X_1| = |X_2| \leq k$ such that $F[X_1]$ and $F[X_2]$ are isomorphic, and every isomorphism $\phi : X_1 \rightarrow X_2$ between $F[X_1]$ and $F[X_2]$, there is an automorphism f of F such that $f|_{X_1} = \phi$. For instance, F is 1-ultrahomogeneous if and only if F is vertex-transitive, and 2-ultrahomogeneous if and only if F is arc-transitive. It is known [8, 11] that the Schläfli graph is 4-ultrahomogeneous.²

Our goal is now to prove that the Schläfli graph G is H -deletion-saturated. To see that G is H -free, suppose for a contradiction that G has an induced subgraph isomorphic to H . It follows in particular that there is a 4-vertex clique $\{x_1, x_2, y_1, y_2\}$ in G and a 2-vertex set Z in G disjoint from $\{x_1, x_2, y_1, y_2\}$ such that Z is anticomplete to $\{x_1, x_2\}$ and complete to $\{y_1, y_2\}$ in G . Since G is 4-ultrahomogeneous, we may assume due to symmetry that

$$x_1 = u_{1,1}^1, \quad x_2 = u_{2,1}^1, \quad y_1 = u_{2,1}^2, \quad y_2 = u_{2,2}^2.$$

But then by the definition of G , the set of all vertices that are anticomplete to $\{x_1, x_2\} = \{u_{1,1}^1, u_{2,1}^1\}$ in G is $\{u_{3,2}^1, u_{3,3}^1, u_{1,1}^2, u_{1,2}^2, u_{1,3}^2\}$, and exactly one of these vertices, namely $u_{3,3}^1$, is complete to $\{y_1, y_2\} = \{u_{2,1}^2, u_{2,2}^2\}$ in G , as well. This is a contradiction to $|Z| = 2$.

It remains to show that for every edge $e \in E(G)$, the graph $G - e$ has an induced subgraph isomorphic to H . Again, since G is also 4-ultrahomogeneous (and in particular arc-transitive), we may assume by symmetry that $e = u_{1,1}^1 u_{1,3}^1$. Let

$$S = \{u_{1,1}^1, u_{2,1}^1, u_{1,3}^1, u_{3,3}^1, u_{2,1}^2, u_{2,2}^2\}.$$

Using the definition of G , it is straightforward to check that the map $f : V(H) \rightarrow S$ with

$$f(v_1) = u_{2,1}^2, \quad f(v_2) = u_{1,1}^1, \quad f(v_3) = u_{2,1}^1, \quad f(v_4) = u_{1,3}^1, \quad f(v_5) = u_{3,3}^1, \quad f(v_6) = u_{2,2}^2$$

is an isomorphism between H and $G[S] - e$. This completes the proof of Theorem 6.11. ■

Theorem 6.12. *Entry 13B in Figure 1.3 is deletion-normal.*

Proof. Let H be entry 13B in Figure 1.3. Construct a graph G as follows. Let G_1 and G_2 be disjoint graphs such that G_1 is a 6-cycle with

$$V(G_1) = \{u_1^1, \dots, u_6^1\} \quad \text{and} \quad E(G_1) = \{u_1^1 u_2^1, u_2^1 u_3^1, u_3^1 u_4^1, u_4^1 u_5^1, u_5^1 u_6^1, u_6^1 u_1^1\}$$

and $\overline{G_2}$ is a 6-cycle with

$$V(\overline{G_2}) = \{u_1^2, \dots, u_6^2\} \quad \text{and} \quad E(\overline{G_2}) = \{u_1^2 u_2^2, u_2^2 u_3^2, u_3^2 u_4^2, u_4^2 u_5^2, u_5^2 u_6^2, u_6^2 u_1^2\}.$$

²In fact, something quite curious is true here: every graph that is 5-ultrahomogeneous is also k -ultrahomogeneous for all $k \in \mathbb{N}$, and the Schläfli graph and its complement are the *only* graphs that are 4-ultrahomogeneous, but not 5-ultrahomogeneous! See [8, 11].

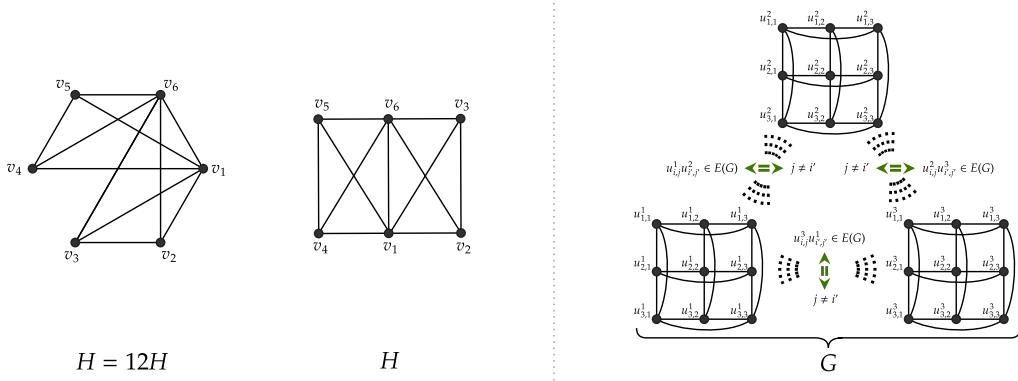


Figure 6.11: Proof of Theorem 6.11. The graph H (left) and the Schläfli graph G (right).

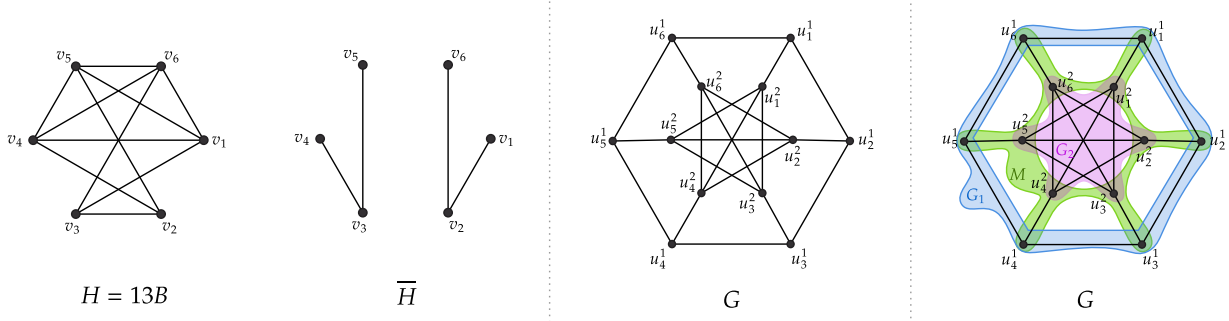


Figure 6.12: Proof of Theorem 6.12. From left to right: The graph H and its complement, the graph G and its construction.

Let G be the graph obtained from the (disjoint) union of G_1 and G_2 by adding the matching $M = \{u_1^1 u_1^2, \dots, u_6^1 u_6^2\}$. See Figure 6.12.

Since $\bar{H}$ is isomorphic to $2P_3$, in order to show that H is deletion-normal, it suffices to show that $2P_3$ is addition-normal, and to that end, we prove that G is $2P_3$ -addition-saturated.

(16) G is $2P_3$ -free.

Suppose for a contradiction that there are two anticomplete induced copies P^1 and P^2 of P_3 in G . Let $I_1 = V(G) \setminus N[V(P^1)]$ and let $I_2 = V(G) \setminus N[V(P^2)]$. Then P^1 is an induced copy of P_3 in $G[I_2]$ and P^2 is an induced copy of P_3 in $G[I_1]$. If $V(P^i) \subseteq V(G_1)$ for some $i \in \{1, 2\}$, then $G[I_i]$ is isomorphic to $2P_2$, and so $G[I_i]$ is P_3 -free, a contradiction. It follows that $V(P^i) \cap V(G_2) \neq \emptyset$ for every $i \in \{1, 2\}$. Moreover, since G_2 , which is the complement of a 6-cycle, is clearly $2P_3$ -free, it follows that for some $i \in \{1, 2\}$, we have $V(P^i) \cap V(G_1) \neq \emptyset$. Thus, without loss of generality, we may assume that P^1 has vertices from both G_1 and G_2 . In particular, we have $E(P^1) \cap M \neq \emptyset$; say $u_1^1 u_1^2 \in E(P^1)$. There are now three cases up to symmetry:

- $P^1 = u_2^1 - u_1^1 - u_1^2$. Then $I_1 = \{u_4^1, u_5^1, u_6^2\}$, and $G[I_1]$ is isomorphic to $\overline{P_3}$.
- $P^1 = u_1^1 - u_1^2 - u_3^2$. Then $I_1 = \{u_4^1, u_5^1, u_2^2\}$, and $G[I_1]$ is isomorphic to $\overline{P_3}$.
- $P^1 = u_1^1 - u_1^2 - u_4^2$. Then $I_1 = \{u_3^1, u_5^1\}$ is a stable set.

But in all three cases, $G[I_1]$ is P_3 -free, a contradiction. This proves (16).

(17) For every $e \in E(\overline{G})$, the graph $G + e$ has an induced subgraph isomorphic to $2P_3$.

We need to prove that there are two anticomplete induced copies P^1 and P^2 of P_3 in $G + e$. First, assume that $e \in E(\overline{G_1})$. Then there are two cases up to symmetry:

- $e = u_1^1 u_3^1$. In this case, $P^1 = u_6^1 - u_1^1 - u_3^1$ and $P^2 = u_4^2 - u_2^2 - u_5^2$ work.
- $e = u_1^1 u_4^1$. In this case, $P^1 = u_1^1 - u_4^1 - u_5^1$ and $P^2 = u_2^2 - u_6^2 - u_3^2$ work.

Next, assume that $e \in E(\overline{G_2})$. Then, by symmetry, we may assume that $e = u_1^2 u_6^2$, in which case $P^1 = u_2^1 - u_3^1 - u_4^1$ and $P^2 = u_5^2 - u_1^2 - u_6^2$ work. Finally, assume that $e = u_i^1 u_j^2$ for some $i, j \in \{1, \dots, 6\}$ with $i \neq j$. Then there are three cases up to symmetry:

- $e = u_1^1 u_2^2$. In this case, $P^1 = u_6^1 - u_1^1 - u_2^2$ and $P^2 = u_4^1 - u_3^1 - u_3^2$ work.
- $e = u_1^1 u_3^2$. In this case, $P^1 = u_6^1 - u_1^1 - u_3^2$ and $P^2 = u_4^1 - u_4^2 - u_2^2$ work.
- $e = u_1^1 u_4^2$. In this case, $P^1 = u_6^1 - u_1^1 - u_4^2$ and $P^2 = u_3^1 - u_3^2 - u_5^2$ work.

This proves (17).

The result is now immediate from (16) and (17). This completes the proof of Theorem 6.12. ■

Theorem 6.13. Entry 15I in Figure 1.3 is deletion-normal.

Proof. Let H be entry 15I in Figure 1.3. Let G be a graph with

$$V(G) = \{u_{a,b}^t : a, b \in \{1, 2\}, t \in \{1, 2, 3, 4\}\}$$

and $E(G) = E_1 \cup E_2$, where

$$E_1 = \{u_{a,b}^t u_{c,d}^t : a, b, c, d \in \{1, 2\}, t \in \{1, 2, 3, 4\}, a \neq c\}$$

$$E_2 = \{u_{a,b}^t u_{a,d}^r : a, b, d \in \{1, 2\}, r, t \in \{1, 2, 3, 4\}, t \neq r, b \neq d\}.$$

See Figure 6.13.

Note that $\overline{H}$ is isomorphic to $K_{2,3}$, so it suffices to show that G is $K_{2,3}$ -addition-saturated. First, we show that G is $K_{2,3}$ -free. Suppose not. Then there are two non-adjacent vertices $x, y \in V(G)$ such that there is a stable set $Z \subseteq N_G(x) \cap N_G(y)$ with $|Z| = 3$. There are now three cases up to symmetry:

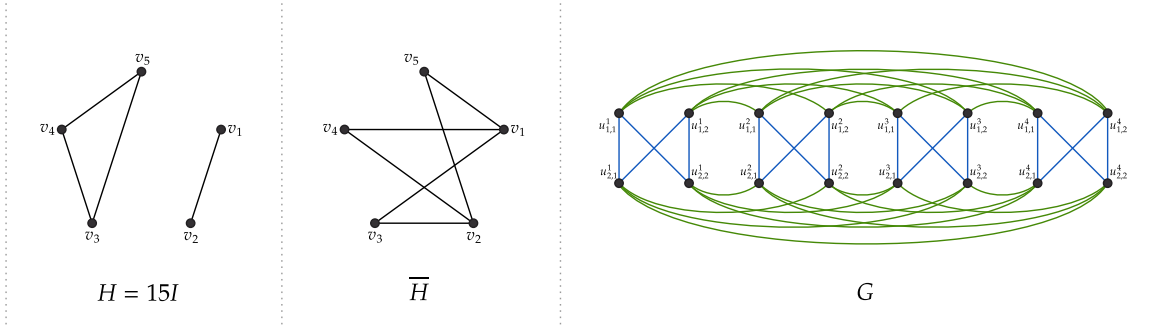


Figure 6.13: Proof of Theorem 6.13. From left to right: The graph H and its complement, and the graph G .

- $x = u_{1,1}^1$ and $y = u_{1,2}^1$. Then $N_G(x) \cap N_G(y) = \{u_{2,1}^1, u_{2,2}^1\}$.
- $x = u_{1,1}^1$ and $y = u_{1,1}^2$. Then $N_G(x) \cap N_G(y) = \{u_{1,2}^3, u_{1,2}^4\}$.
- $x = u_{1,1}^1$ and $y = u_{2,1}^2$. Then $N_G(x) \cap N_G(y) = \{u_{2,2}^1, u_{1,2}^2\}$.

In particular, $|N_G(x) \cap N_G(y)| = 2$, a contradiction. We deduce that G is $K_{2,3}$ -free.

It remains to show that for every edge $e = xy \in E(\overline{G})$, the graph $G + e$ has an induced subgraph isomorphic to $K_{2,3}$. Again, there are now three cases up to symmetry:

- $x = u_{1,1}^1$ and $y = u_{1,2}^1$. Then $Z_1 = \{u_{1,2}^1, u_{1,1}^3\}$ and $Z_2 = \{u_{1,1}^1, u_{1,1}^2, u_{1,1}^4\}$ are two stable sets in $G + e$ such that every vertex in Z_1 is adjacent in $G + e$ to every vertex in Z_2 .
- $x = u_{1,1}^1$ and $y = u_{1,1}^2$. Then $Z_1 = \{u_{2,2}^1, u_{1,1}^2\}$ and $Z_2 = \{u_{1,1}^1, u_{1,2}^1, u_{2,1}^1\}$ are two stable sets in $G + e$ such that every vertex in Z_1 is adjacent in $G + e$ to every vertex in Z_2 .
- $x = u_{1,1}^1$ and $y = u_{2,1}^2$. Then $Z_1 = \{u_{1,1}^1, u_{1,1}^2\}$ and $Z_2 = \{u_{2,1}^2, u_{1,2}^3, u_{1,2}^4\}$ are two stable sets in $G + e$ such that every vertex in Z_1 is adjacent in $G + e$ to every vertex in Z_2 .

This completes the proof of Theorem 6.13. ■

References

- [1] Noga Alon, Alexandr Kostochka, Benjamin Reiniger, Douglas B. West, and Xuding Zhu. Coloring, sparseness and girth. *Israel J. Math.*, 214(1):315–331, 2016.
- [2] Maria Axenovich and Mónika Csikós. Induced saturation of graphs. *Discrete Math.*, 342(4):1195–1212, 2019.
- [3] Sarah Behrens, Catherine Erbes, Michael Santana, Derrek Yager, and Elyse Yeager. Graphs with induced-saturation number zero. *Electron. J. Combin.*, 23(1):Paper 1.54, 23, 2016.
- [4] Lowell W. Beineke. Characterizations of derived graphs. *Journal of Combinatorial Theory*, 9(2):129–135, 1970.
- [5] Marthe Bonamy, Carla Groenland, Tom Johnston, Natasha Morrison, and Alex Scott. Induced saturation for P_5 . <https://tomjohnston.co.uk/blog/2020-05-22-induced-saturation-for-paths.html>.
- [6] Marthe Bonamy, Carla Groenland, Tom Johnston, Natasha Morrison, and Alex Scott. Infinite induced-saturated graphs. Manuscript available at <https://arxiv.org/pdf/2506.08810>, 2025.
- [7] Marcin Briański, James Davies, and Bartosz Walczak. Separating polynomial χ -boundedness from χ -boundedness. *Combinatorica*, 44(1):1–8, 2024.
- [8] Peter J Cameron. 6-transitive graphs. *Journal of Combinatorial Theory, Series B*, 28(2):168–179, 1980.
- [9] Eun-Kyung Cho, Ilkyoo Choi, and Boram Park. On induced saturation for paths. *European J. Combin.*, 91:Paper No. 103204, 12, 2021.
- [10] Bryan L Currie, Jill R Faudree, Ralph J Faudree, and John R Schmitt. A survey of minimum saturated graphs. *The Electronic Journal of Combinatorics*, pages DS19–Oct, 2012.
- [11] A Devillers. Classification of some homogeneous and ultrahomogeneous structures. Ph.D. thesis, Université Libre de Bruxelles, 2002.

- [12] Reinhard Diestel. *Graph theory*, volume 173 of *Graduate Texts in Mathematics*. Springer, Berlin, fifth edition, 2018.
- [13] Vojtěch Dvořák. P_n -induced-saturated graphs exist for all $n \geq 6$. *Electron. J. Combin.*, 27(4):Paper No. 4.43, 6, 2020.
- [14] P. Erdős and A. Hajnal. On chromatic number of graphs and set-systems. *Acta Math. Acad. Sci. Hungar.*, 17:61–99, 1966.
- [15] Paul Erdős, Chao Ko, and Richard Rado. Intersection theorems for systems of finite sets. *Quarterly Journal of Mathematics*, 12:313–320, 1961.
- [16] Xinyue Fan, Sahab Hajebi, Sepehr Hajebi, and Sophie Spirkl. Halfway to induced saturation for even cycles. Manuscript available at <https://arxiv.org/abs/2505.24100>, 2025.
- [17] P. Frankl. An Erdős–Ko–Rado theorem for direct products. *European Journal of Combinatorics*, 17(8):727–730, 1996.
- [18] Fred Galvin. The list chromatic index of a bipartite multigraph. *J. Combin. Theory Ser. B*, 63(1):153–158, 1995.
- [19] Martin Kneser. Aufgabe 360. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 2(27):3–16, 1955.
- [20] Felix Lazebnik and Vasiliy A. Ustimenko. Explicit construction of graphs with an arbitrary large girth and of large size. volume 60, pages 275–284. 1995. ARIDAM VI and VII (New Brunswick, NJ, 1991/1992).
- [21] Nati Linial and Michael Simkin. A randomized construction of high girth regular graphs. *Random Structures & Algorithms*, 58(2):345–369, 2021.
- [22] L. Lovász. On chromatic number of finite set-systems. *Acta Math. Acad. Sci. Hungar.*, 19:59–67, 1968.
- [23] Ryan R. Martin and Jason J. Smith. Induced saturation number. *Discrete Math.*, 312(21):3096–3106, 2012.
- [24] Brendan McKay. Database of graphs. Available at <https://users.cecs.anu.edu.au/~bdm/data/graphs.html>.
- [25] Dale Peterson. Gridline graphs: a review in two dimensions and an extension to higher dimensions. *Discrete Applied Mathematics*, 126(2):223–239, 2003.
- [26] Eero Rätty. Induced saturation of P_6 . *Discrete Math.*, 343(1):111641, 3, 2020.

- [27] Jason J. Smith. Induced saturation number. *ProQuest Dissertations and Theses*, page 75, 2012. Copyright - Database copyright ProQuest LLC; ProQuest does not claim copyright in the individual underlying works; Last updated - 2025-10-31.
- [28] Craig M. Tennenhouse. Induced subgraph saturated graphs. *Theory Appl. Graphs*, 3(2):Art. 1, 14, 2016.
- [29] Hassler Whitney. Congruent Graphs and the Connectivity of Graphs. *Amer. J. Math.*, 54(1):150–168, 1932.