

Insurance Pricing, and Optimal Strategy Design under Distributional Model Uncertainty

by

Ziyue Shi

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Examining Committee Membership

The following served on the Examining Committee for this thesis. The decision of the Examining Committee is by majority vote.

- External Examiner: Qihe Tang
Professor, School of Risk and Actuarial Studies
University of New South Wales
- Supervisor(s): David Landriault
Professor, Department of Statistics and Actuarial Science
University of Waterloo
- Fangda Liu
Associate Professor, Department of Statistics and Actuarial Science
University of Waterloo
- Internal Member: Jun Cai
Professor, Department of Statistics and Actuarial Science
University of Waterloo
- Chengguo Weng
Professor, Department of Statistics and Actuarial Science
University of Waterloo
- Internal-External Member: Stanko Dimitrov
Professor, Department of Management Science and Engineering
University of Waterloo

Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

The objective of this thesis is to develop theoretically sound and practically applicable solutions to two optimal reinsurance models, and to investigate the properties of the premium with background risk. Specifically, Chapter 2 introduces model uncertainty into the optimal reinsurance problem, while Chapter 3 focuses on solving the optimal reinsurance problem within the framework of a performance-based variable premium. Chapter 4 examines the properties of the indifference premium for an insurable risk in the presence of a background risk, considering both scenarios with and without model uncertainty.

In Chapter 2, we explore a non-cooperative optimal reinsurance problem incorporating likelihood ratio uncertainty, aiming to minimize the worst-case risk of the total retained loss for the insurer. We establish a general relation between the optimal reinsurance strategy under the reference probability measure and the solution in the worst-case scenario. This relation can be generalized to insurance design problems quantified by tail risk measures. We provide a sufficient and necessary condition for when the problem using reference measure has at least one common optimal solution with its worst-case counterpart. As an application of this relation, optimal policies for the worst-case scenario quantified by the expectile risk measure are determined. Additionally, we explore the corresponding cooperative problem and compare its value function with that of the non-cooperative model.

In the literature, insurance and reinsurance pricing is typically determined by a premium principle, characterized by a risk measure that reflects the policy seller's risk attitude. Building on the work of Meyers (1980) and Chen et al. (2016), Chapter 3 propose a new performance-based variable premium scheme for reinsurance policies, where the premium depends on both the distribution of the ceded loss and the actual realized loss. Under this scheme, the insurer and the reinsurer face a random premium at the beginning of the policy period. Based on the realized loss, the premium is adjusted into either a “reward” or “penalty” scenario, resulting in a discount or surcharge at the end of the policy period. We characterize the optimal reinsurance policy from the insurer's perspective under this new variable premium scheme. In addition, we formulate a Bowley optimization problem between the insurer and the monopoly reinsurer. Numerical examples demonstrate that, compared to the expected-value premium principle, the reinsurer prefers the variable premium scheme as it reduces the reinsurer's total risk exposure.

Chapter 4 investigates the properties of the indifference premium for an insurable risk in the presence of a background risk. We establish conditions that guarantee the uniqueness of the premium and provide general properties of the indifference premium without imposing specific assumptions on the dependence structure between the insurable risk and the background risk. Furthermore, we demonstrate that the dependence structure between these risks significantly affects the behavior of the indifference premium. In particular, we analyze the indifference premium under various dependence structures, including independence, stochastic monotonicity, and mutual exclusivity. We further explore the impact of model uncertainty on the indifference premium by deriving equivalent conditions for the dependence structures between the insurable and the background risks and their corresponding uniform random variables. Finally, we illustrate our findings using expected utility preference operators as representative examples.

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Chapter 1

Introduction

In this chapter, we present a review of the relevant literature on optimal reinsurance, risk measures, and model uncertainty. We begin by summarizing foundational works and recent advances in optimal reinsurance design. Then, we summarize the theoretical foundations of risk measures, discussing their development and properties. Finally, we examine the the expanding literature addressing model uncertainty in insurance and risk management, with particular emphasis on robust optimization frameworks and their applications in actuarial science.

1.1 Optimal reinsurance model

Reinsurance plays a crucial role in economic strategy for both insurance companies and society. From the perspective of the insurance company itself, reinsurance serves to control indemnity losses, ensure financial stability, and increase business capacity. From the societal perspective, reinsurance plays an irreplaceable role in global loss reduction. Through reinsurance, substantial losses resulting from natural disasters, man-made accidents (such as terrorism, nuclear incidents, aerospace mishaps, etc.), and business risks can be reimbursed by insurance companies. This emphasizes the necessity for more research, both in theory and practice, on reinsurance.

The study on reinsurance design was pioneered by [Borch \(1960\)](#). Since then, optimal

reinsurance problems have garnered significant attention in the fields of insurance and actuarial science. As of today, reinsurance continues to be an important risk management tool for insurance companies, and research on optimal reinsurance design remains vibrant. This thesis is dedicated to studying various optimal reinsurance problems aimed at fulfilling different requirements of the insurer or the reinsurer across diverse targets.

Let us start by introducing some concepts of reinsurance before delving into the optimal reinsurance problem. Reinsurance is a legal contractual agreement between insurance companies, distinct from direct insurance where the contract is signed between an insured individual or entity and an insurance company. The entity selling the reinsurance contract to another insurer is known as the *reinsurer*, and the counterparty is referred to as the *insurer* or *cedent*. In accordance with the reinsurance contract, the insurer can transfer part of her risk to the reinsurer. In return, the insurance company pays reinsurance premium to the reinsurer for this contractual arrangement. Naturally, as the insurer transfers more risk, her exposure to losses decreases, but the premium payments increase. This trade-off leads to the exploration of optimal reinsurance design.

In this thesis, we focus on the reinsurance problems over a single time period, which is known as the *static models* or *single period models*. Here is a common setup of static optimal reinsurance models. Denote X as the (aggregate) underlying risk faced by the insurer over a certain time period. Let $I(X)$ denote part of the underlying risk covered by the reinsurer and $R_I(X) = X - I(X)$ denote the remaining part paid by the insurer. In return for this coverage, the insurer compensates the reinsurer by paying a reinsurance premium, computed using the premium principle π . This premium principle is typically a function of $I(X)$ and is denoted as $\pi(I(X))$. The function $I(\cdot)$ is referred as the *indemnity function* or the *ceded loss function*, while $R_I(\cdot)$ is known as the *retained loss function*. In the presence a reinsurance contract, the total risk exposure $T_I(X)$ faced by the insurer is given by $T_I(X) = R_I(X) + \pi(I(X))$.

The optimal reinsurance problem is usually formulated as an optimization problem, aiming to identify an indemnity function that satisfies specific optimization targets. An optimal reinsurance model typically comprises several essential components: an optimization objective, an admissible strategy set, and a reinsurance premium principle. In the subsequent part of this section, we will provide a literature review concerning these components.

1.1.1 Optimization objective

According to the characteristics of the objective function, the following categories represent the three commonly-used objectives in static optimal reinsurance models.

- (i) Variance minimization of total risk exposure.
- (ii) Expected utility maximization of final wealth.
- (iii) Risk measure minimization of total risk exposure.

It is worth noting that these three models are nested within one another. Specifically, variance minimization models form a subset of expected utility maximization models, which themselves are special cases of risk measure minimization models. In this subsection, we provide a brief literature review and summarize the main results from the perspective of each of these modeling frameworks.

[Borch \(1960\)](#) pioneered the study of the optimal reinsurance problem. In their study, the optimization objective is to minimize the variance of the insurer's risk exposure. Let $\mathcal{X}$ be a convex cone of random variables and $X \in \mathcal{X}$. With some constraints, the typical variance minimization model is formulated as

$$\min_I \text{Var}(X - I(X)). \quad (1.1)$$

In their work, premium is calculated by the expected value principle and admissible ceded losses are assumed to satisfy the principle of indemnity. In this context, stop-loss reinsurance, characterized by full insurance above a constant deductible, is identified as the optimal policy. For a detailed discussion on the variance minimization model, refer to works such as [Gerber \(1979\)](#), [Bowers et al. \(1997\)](#), and [Kaas et al. \(2008\)](#).

As an extension of [Borch \(1960\)](#), [Arrow \(1963\)](#) achieves a parallel result by setting the optimization objective as maximizing the expected utility of the insurer's final wealth. Here is the general form of expected utility maximization model in optimal reinsurance problems. Let's denote w_0 as the initial wealth of the insurer. After reinsurance arrangement, the terminal wealth of the insurer becomes $w_0 - X + I(X) - \pi(I(X))$. From the insurer's

perspective, she aims to find the optimal reinsurance treaty to maximize her expected utility, formulated as:

$$\max_I \mathbb{E}[v(w_0 - X + I(X) - \pi(I(X)))]. \quad (1.2)$$

Here, the indemnity function I is chosen from a certain admissible policy set, v represents the utility function, and π is the reinsurance premium. It is reasonable to assume that the insurer is risk-averse, as she is seeking risk-sharing, which implies an increasing concave utility function. If we take $u(X - I(X)) \triangleq v(w_0 - X + I(X) - \pi(I(X)))$ and let $u(X - I(X)) = \{X - I(X) - \mathbb{E}[X - I(X)]\}^2$, the expected utility maximization problem (1.2) is reduced to the variance minimization problem (1.1). Following [Arrow \(1963\)](#), a significant amount of literature has been dedicated to the problems of maximizing the expected utility of the insurer's terminal wealth. Readers may refer to [Gerber and Pafum \(1998\)](#) for a review of utility functions in the context of insurance applications. For additional relevant literature, refer to works such as [Yang and Zhang \(2005\)](#), [Zhang and Siu \(2012\)](#), [Liang and Bayraktar \(2014\)](#), and the references contained therein.

Risk measure minimization problems under a particular risk measure ρ usually take the following formulation:

$$\min_I \rho(X - I(X) + \pi(I(X))). \quad (1.3)$$

Risk measures, such as the Value-at-Risk (VaR) and the Tail Value-at-Risk (TVaR) have generated tremendous interests in quantitative finance ([Artzner et al. \(1999\)](#), [Philipp \(2000\)](#), [Dhaene et al. \(2006\)](#)). However, they have not been studied as an objective in optimal reinsurance problem until [Cai and Tan \(2007\)](#) investigate the stop-loss reinsurance under the expectation premium principle to minimize VaR and conditional tail expectation (CTE) of the total risks of an insurer. Then, [Cai et al. \(2008\)](#) generalize these results by obtaining the optimal ceded loss function among the set of increasing and convex reinsurance functions. Motivated by the above works, [Tan et al. \(2009\)](#) extend the model by considering more general premium principles and investigate the quota-share reinsurance in addition to the stop-loss reinsurance. For more recent risk measure minimization problems, readers can refer to, for instance, [Balbás et al. \(2009\)](#), [Cheung et al. \(2014\)](#), [Li \(2018\)](#), [Cheung et al. \(2019a\)](#), and references therein.

1.1.2 Admissible policy set

In the literature, various settings for the admissible ceded loss function set are considered, reflecting different practical situations. It is essential for the indemnity function to be non-decreasing; otherwise, the insurer could potentially exploit the situation by manipulating the reported loss to obtain greater indemnity from the reinsurer. Thus, the ceded loss functions are often assumed to satisfy the *principle of indemnity*, i.e., $0 \leq I(x) \leq x$. [Cai et al. \(2008\)](#) adopts an indemnity function set that encompasses all increasing convex loss functions. The technical assumption of convexity is employed to solve the problems using tools from convex analysis. This type of admissible set is characterized as

$$\hat{\mathcal{I}} = \{0 \leq I(x) \leq x : I(x) \text{ is an increasing convex function}\}. \quad (1.4)$$

However, under this admissible loss function set, the reinsurer is required to pay a higher proportion per unit of insurable risk for a larger loss. In practice, reinsurers may be reluctant to take on excessive tail risk, and this convex assumption appears to be quite stringent.

There are other ways to define the admissible policy set. The reinsurance contract serves as a tool to assist the insurer in mitigating risk, and ideally, it should not generate additional profit for the insurer. In other words, the rise in the indemnity function should not surpass the increase in the loss. In order to avoid this kind of moral hazard, [Huberman et al. \(1983\)](#) suggests the set of admissible indemnity functions should satisfy 1-Lipschitz continuity. In specific, the admissible policy set $\mathcal{I}$ is defined by

$$\mathcal{I} = \{0 \leq I(x) \leq x : I(x) \text{ is non-decreasing and } 0 \leq I(x) - I(y) \leq x - y\}. \quad (1.5)$$

It can be easily verified that $I \in \mathcal{I}$ is equivalent to $R_I \in \mathcal{I}$, highlighting that the reinsurance market commonly favors contracts with both non-decreasing and continuous ceded and retained loss functions. In this thesis, we choose $\mathcal{I}$ as the admissible policy set and refer to an indemnity function as “feasible” if it meets the conditions specified in $\mathcal{I}$.

To include non-continuous functions in the admissible set, define

$$\bar{\mathcal{I}} = \{0 \leq I(x) \leq x : R_I(x) \text{ is an increasing left continuous function}\}. \quad (1.6)$$

The assumption of an increasing ceded loss function R_I is employed to encourage the insurer to scrutinize claims more diligently as the loss increases. The left continuity property

assumption is employed for technical reasons to facilitate the solution of the problem. This type of admissible set encompasses some non-continuous policies that have been identified as optimal solutions in certain optimal reinsurance problems. For example, truncated stop-loss policies, which belongs to $\bar{\mathcal{I}}$, have been proven to be optimal under specific settings, as discussed in [Kaluszka and Okolewski \(2008\)](#) and [Bernard and Tian \(2009\)](#).

Mathematically, the three sets are nested, i.e., $\hat{\mathcal{I}} \subset \mathcal{I} \subset \bar{\mathcal{I}}$. The choice of the admissible set has an effect on the optimal reinsurance strategy. For instance, in the optimal reinsurance model where the optimization objective is the Value-at-Risk (VaR) of the insurer's total risk exposure, and the expected value premium principle is used to calculate the reinsurance premium, [Cai et al. \(2008\)](#) demonstrates that the stop-loss reinsurance is the optimal indemnity function among others in $\hat{\mathcal{I}}$, while [Chi and Tan \(2011\)](#) shows that a reinsurance constructed by a limited stop-loss function is superior to other policies in $\mathcal{I}$, and the truncated stop-loss policy is the optimal in $\bar{\mathcal{I}}$.

Many early theoretical researches regarding reinsurance decision-making focus on some specific reinsurance strategies. Until now, these conventional reinsurance treaties are popular in actuarial science. Given the expression of the reinsurance strategies, the literature aims to determine the optimal parameters. For example, [Taksar and Markussen \(2003\)](#), [Luo et al. \(2008\)](#), and [Yuen et al. \(2015\)](#) analyze the optimal reinsurance and investment problem under the assumption of proportional reinsurance, and [Hipp and Vogt \(2003\)](#) and [Zhang et al. \(2020\)](#) study optimal stop-loss reinsurance. The research boom on the traditional reinsurance strategies, such as proportional reinsurance and stop-loss reinsurance, is mainly due to their simplicity and applicability to insurance practice. In addition, as it gives the smallest variance of the insurer retained risk, stop-loss reinsurance is always optimal in a variety of reinsurance problems.

1.1.3 Premium principles

The determination of insurance premium and/or reinsurance premium is another essential part in optimal reinsurance problems. Premium principles are methods or rules used to determine the amount the policy seller should charge to assume a given risk. The choice of premium principle has a direct impact on the policy seller's risk management and solvency, and serves as a mechanism through which the seller can influence the behavior

of policyholders. The admissible pricing rules include, for example, linear pricing criteria as in [Borch \(1962\)](#) and [Bühlmann \(1984\)](#), mean-variance pricing criteria as in [Kaluszke \(2001\)](#) and [Cai et al. \(2008\)](#), Choquet pricing criteria as in [Wang \(1995\)](#) and [Wang \(1996\)](#), utility-based principle as in [Bühlmann et al. \(1977\)](#) and [Gerber \(1982\)](#), and convex premium principle as in [Deprez and Gerber \(1985\)](#) and [Ghossoub et al. \(2023\)](#).

Premium calculation principle is not used to price actual insurance contracts, but to produce an actuarial benchmark in the theoretical research. It is natural to require that this benchmark should, at least to some extent, be consistent with market prices. [Young \(2004\)](#) introduces several desired properties for an insurance premium principle. Let $\pi(X) : \mathcal{X} \rightarrow \mathbb{R}$ denote the premium principle of the insurable risk $X \in \mathcal{X}$. In the following, we recall some desirable properties of the premium principle.

- (P1) *Law-invariance*: $\pi(X)$ depends solely on the distribution of X for all $X \in \mathcal{X}$.
- (P2) *No unjustified risk loading*: $\pi(c) = c$ for all $X \in \mathcal{X}$ and all constant $c \in \mathbb{R}$.
- (P3) *Translation invariance*: $\pi(X + c) = \pi(X) + c$ for all $X \in \mathcal{X}$ and all constant $c \in \mathbb{R}$.
- (P4) *No rip-off*: $\pi(X) \leq \text{ess-sup} X$ for all $X \in \mathcal{X}$.
- (P5) *Risk loading*: $\pi(X) \geq \mathbb{E}[X]$ for all $X \in \mathcal{X}$.
- (P6) *Positive homogeneity*: $\pi(cX) = c\pi(X)$ for all $X \in \mathcal{X}$ and all constant $c > 0$.
- (P7) *First stochastic order consistency*: $\pi(X_1) \leq \pi(X_2)$ for all $X_1, X_2 \in \mathcal{X}$ such that $X_1 \preceq_{st} X_2$.
- (P8) *Convex order consistency*: $\pi(X_1) \leq \pi(X_2)$ for all $X_1, X_2 \in \mathcal{X}$ such that $X_1 \preceq_{cx} X_2$.
- (P9) *Monotonicity*: If $X_1(\omega) \leq X_2(\omega)$ for all $\omega \in \Omega$, then $\pi(X_1) \leq \pi(X_2)$.
- (P10) *Subadditivity*: $\pi(X_1 + X_2) \leq \pi(X_1) + \pi(X_2)$ for all $X_1, X_2 \in \mathcal{X}$.
- (P11) *Additivity*: $\pi(X_1 + X_2) = \pi(X_1) + \pi(X_2)$ for all $X_1, X_2 \in \mathcal{X}$.
- (P12) *Superadditivity*: $\pi(X_1 + X_2) \geq \pi(X_1) + \pi(X_2)$ for all $X_1, X_2 \in \mathcal{X}$.

- (P13) *Additivity for independent risks:* $\pi(X_1 + X_2) = \pi(X_1) + \pi(X_2)$ for all $X_1, X_2 \in \mathcal{X}$ such that X_1 and X_2 are independent.
- (P14) *Additivity for comonotonic risks:* $\pi(X_1 + X_2) = \pi(X_1) + \pi(X_2)$ for all $X_1, X_2 \in \mathcal{X}$ such that X_1 and X_2 are comonotonic.¹
- (P15) *Continuity:* Let $X \in \mathcal{X}$, then $\lim_{a \rightarrow 0^+} \pi(\max(X-a, 0)) = \pi(X)$ and $\lim_{a \rightarrow \infty} \pi(\min(X-a, 0)) = \pi(X)$.

Law-invariance (P1) is a property commonly satisfied by most desirable premium principles. It is appealing due to its simplicity, as it depends solely on the characteristics of the risk being priced. The no unjustified risk loading condition (P2) states that the premium does not charge a risk loading when the risk is certain. The translation invariance condition (P3) states that the premium for $X + a$ should be the premium for X increased by the fixed amount a . The no rip-off condition (P4) ensures that the risk is transferable by requiring the premium not to exceed the worst possible loss, while the risk loading condition (P5) guarantees contract profitability by requiring the premium to exceed the expected loss. The positive homogeneity (P6) suggests that the premium principle is sensitive only to the composition of portfolios of risks and not to their size, which is useful under the no-arbitrage condition. First stochastic order consistency (P7) implies that risks that always yield higher losses should always be insured at a higher price. Convex order consistency (P8) reflects the principle that risks ranked higher in the convex order should be charged higher premiums. Monotonicity (P9) implies that liabilities which consistently result in higher losses should always command higher insurance premiums.

Properties (P10) to (P14) show how premium calculation criteria deals with portfolios of dependent risks, which is an important aspect of the premium principles. Subadditivity (P10) has the consequence that the pooling of risks always produces diversification benefits, which is widely accepted in imperfect markets. However, subadditivity remains controversial, as it is often criticized for potentially underestimating the risk arising from aggregating positive dependent positions. Additivity (P11) is a stronger form of positive homogeneity. Superadditivity (P12) offers an alternative perspective by emphasizing the

¹Two random variables X and Y are comonotonic if there exists $\Omega_0 \in \mathcal{F}$ with $\mathbb{P}(\Omega_0) = 1$ and, for all $\omega, \omega' \in \Omega_0$, it holds true $(X(\omega) - X(\omega'))(Y(\omega) - Y(\omega')) \geq 0$.

increased aggregate risk that can arise from dependent risks. It is also a direct consequence of superadditivity that a premium principle will not be positively homogenous and thus be sensitive to portfolio size. Additivity for independent risks (P13) and additivity for comonotonic risks (P14) imply that no diversification benefit arises when the pooled risks are independent or perfectly positively dependent. Finally, continuity (P15) is a technical property that proves useful in the characterization of certain premium principles.

In nearly all studies on optimal reinsurance, premiums are calculated based on a risk measure, which depends on the distribution of the ceded loss to the reinsurer. There are more general premium principles that are not necessarily law-invariant. In insurance practice, certain pricing methods, such as retrospective rating, consider both the distribution of the ceded loss and the actual realized loss. Retrospective rating adjusts the premium based on the policyholder’s actual losses during the policy period, with higher premiums reflecting greater losses. This approach incentivizes policyholders to manage risks effectively. It is commonly applied in workers’ compensation, general liability, and auto liability insurance. Under a retrospective rating plan, the premium increases linearly with the realized loss, but to prevent the premium from being excessively low for the seller or prohibitively high for the buyer, the plan includes both a minimum threshold and a maximum cap. Determining parameters in this premium can be complex and the premium introduces additional randomness beyond the retained and ceded losses for both the insurer and reinsurer. Despite its practical application in insurance, the retrospective rating plan premium is seldom addressed in theoretical research due to its complexity. [Mahler \(1976\)](#) examines the miscalculation in the standard retrospective rating premiums due to the failure to account for the cross effects of various elements when calculating the basic premium. [Meyers \(1980\)](#) proposes a modified formula for calculating the basic premium that considers the claim severity distribution. [Lee \(1988\)](#) illustrates the mathematical idea behind the retrospective rating by a graphical approach. In [Chen et al. \(2016\)](#), the authors investigate an optimal retrospective rating plan by minimizing the policyholder’s risk exposure in the sense of convex order. They approach the problem from the policyholder’s perspective and conclude that the insurer favors a policy priced using the expected value premium principle, a specific form of the retrospective rating plan premium. We highlight that, under the retrospective rating plan premium, the premium is random at the time of signing the policy. The actual amount can only be observed at the end of the policy period

when the realized loss amount is available. The randomness of the retrospective rating plan premium is the key feature distinguishing it from other classical premium principles which determine the premium as a constant for an insurance policy.

1.2 Risk measures

We work with an atomless probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathbb{P}$ is the reference probability measure. Without specifying otherwise, all risks are quantified under the reference probability measure $\mathbb{P}$. Let L^q be the set of all random variables in $(\Omega, \mathcal{F}, \mathbb{P})$ with finite q -th moment, $q \in (0, \infty)$, L^0 be the set of all random variables, and L^∞ be the set of essentially bounded random variables. Throughout, for any random loss X , a positive value of X represents a financial loss, F_X denotes the distribution function of X , $S_X = 1 - F_X$ is the survival function of X , and the (left-continuous) quantile function is given by $F_X^{-1}(p) = \inf\{x \in \mathbb{R} : F_X(x) \geq p\}$ for $p \in (0, 1)$. By convention, we define $F_X^{-1}(0) = \text{ess-inf}(X)$ and $F_X^{-1}(1) = \text{ess-sup}(X)$ as the essential infimum and essential supremum of X , respectively. Let U_X be a uniform random variable on $[0, 1]$ such that $F_X^{-1}(U_X) = X$ almost surely (a.s.). The existence of such uniform random variable U_X for any X is established, for example, in Lemma A.32 of [Föllmer and Schied \(2016\)](#). Let $X \stackrel{d}{=} Y$ if the random variables X and Y have the same distribution. For $x, y \in \mathbb{R}$, $x \vee y = \max\{x, y\}$, $x \wedge y = \min\{x, y\}$, and $(x - y)_+ = \max\{0, x - y\}$.

Let $\mathcal{X}$ be a convex cone of random variables containing L^∞ , and a risk measure ρ is a functional that maps $\mathcal{X}$ to $\mathbb{R}$. In this thesis, we focus on *law-invariant* risk measures, meaning that $\rho(X)$ depends solely on the distribution of X . Mathematically, ρ has the law-invariance property if $\rho(X) = \rho(Y)$ for all $X, Y \in \mathcal{X}$ such that $X \stackrel{d}{=} Y$. A large class of risk measures commonly used in insurance and finance literature is the monetary risk measure, which is often employed to quantify the capital reserve.

Definition 1.2.1 (Monetary Risk Measure). A law-invariant risk measure ρ is defined as a *monetary risk measure* if it satisfies the following two properties:

- (i) *Monotonicity*: $\rho(X) \leq \rho(Y)$ if $X \leq Y$ a.s., $X, Y \in \mathcal{X}$.
- (ii) *Translation-invariance*: $\rho(X - m) = \rho(X) - m$ for any $m \in \mathbb{R}$ and $X \in \mathcal{X}$.

Comonotonic additivity is of particular importance for insurance pricing:

- (iii) *Comonotonic additivity*: $\rho(X + Y) = \rho(X) + \rho(Y)$ if $X, Y \in \mathcal{X}$ are comonotonic.

For economic interpretations of these properties, we refer to [Artzner et al. \(1999\)](#), [Föllmer and Schied \(2016\)](#) and [Delbaen \(2012\)](#). In this thesis, we would like to impose as little requirement on the risk measures as possible to aim for great generality. In particular, we will not assume any form of convexity or quasi-convexity in the majority of the thesis.

The two most popular classes of risk measures used in banking and insurance practice are the *Value-at-Risk* (VaR) and the *Tail Value-at-Risk* ($TVaR$); for a general treatment of these risk measures in risk management and optimization, we refer to [Pflug and Römisch \(2007\)](#) and [McNeil et al. \(2015\)](#).

Definition 1.2.2 (Value-at-Risk). For a confidence level $\alpha \in [0, 1]$, VaR has two versions: the right-quantile (VaR_α^R) and the left-quantile (VaR_α^L). The left- VaR at level $\alpha \in [0, 1]$ is defined as

$$VaR_\alpha^L(X) = \inf\{x \in \mathbb{R} : F_X(x) \geq 1 - \alpha\}, \quad X \in L^0,$$

and the right- VaR at level $\alpha \in [0, 1]$ is defined as

$$VaR_\alpha^R(X) = \inf\{x \in \mathbb{R} : F_X(x) > 1 - \alpha\}, \quad X \in L^0,$$

where $\inf(\emptyset) = \infty$. In addition, let $VaR_0^L = \text{ess-sup}$ and $VaR_1^R = \text{ess-inf}$.

In risk management practice, one typically does not distinguish between VaR_α^R and VaR_α^L as they are identical for random variables with a continuous inverse distribution function at α . However, their subtle difference leads to interesting observations. Both VaR_α^R and VaR_α^L will be referred to as $(1 - \alpha)$ -quantiles or $VaRs$ in this thesis.

Definition 1.2.3 (Tail Value-at-Risk). The *Tail Value-at-Risk* ($TVaR$) at a given confidence level $\alpha \in (0, 1)$ is defined as

$$TVaR_\alpha(X) = \frac{1}{\alpha} \int_0^\alpha VaR_u(X) du, \quad X \in L^1.$$

Next we define the family of *Range-Value-at-Risk* (RVaR), introduced by [Cont et al. \(2010\)](#) as a family of robust risk measures.

Definition 1.2.4 (Range-Value-at-Risk). The RVaR at confidence level (p, q) , $0 \leq q \leq p \leq 1$, is defined as

$$\text{RVaR}_{(p,q)}(X) = \begin{cases} \frac{1}{p-q} \int_q^p \text{VaR}_u(X) \, du, & \text{if } q < p, \\ \text{VaR}_p^L(X), & \text{if } q = p, \end{cases} \quad X \in L^1. \quad (1.7)$$

Both VaR_α^R and VaR_α^L are connected to TVaR_α via an optimization formula; see [Rockafellar and Uryasev \(2000\)](#) and [Pflug \(2000\)](#). For $q \in [0, 1)$, the Tail Value-at-Risk can be formulated as $\text{TVaR}_q = \text{RVaR}_{(1,q)}$.

Whenever RVaR and TVaR appear in this paper, their domain is set to $\mathcal{X} = L^1$ to guarantee finiteness. In [Embrechts et al. \(2018\)](#), the parameters p and q of RVaR are allowed to be greater than 1 (leading to $\text{RVaR}_{(p,q)} = -\infty$). For the results in our thesis, it is sufficient to consider only the case $p \leq 1$.

Some standard properties of risk measures, in addition to (i) monotonicity, (ii) translation-invariance, (iii) comonotonic additivity, and (iv) law-invariance, are listed below.

- (v) *Convexity*: $\rho(\lambda X + (1 - \lambda)Y) \leq \lambda \rho(X) + (1 - \lambda)\rho(Y)$ for $\lambda \in [0, 1]$ and $X, Y \in \mathcal{X}$.
- (vi) *Positive homogeneity*: $\rho(\lambda X) = \lambda \rho(X)$ for $\lambda > 0$ and $X \in \mathcal{X}$.
- (vii) *Subadditivity*: $\rho(X + Y) \leq \rho(X) + \rho(Y)$ for $X, Y \in \mathcal{X}$.

Using the standard terminology in [Föllmer and Schied \(2016\)](#), a risk measure ρ is a *convex risk measure* if it is monetary and convex, and it is a *coherent risk measure* if it is monetary, convex, and positively homogeneous. It is well known that a law-invariant convex risk measure on L^∞ is convex-order consistent; see e.g., [Föllmer and Schied \(2016, Corollary 4.65\)](#). The risk measures TVaR_α , $\alpha \in [0, 1)$ belong to the RVaR family, and they are the only ones in that family that are coherent, convex, or convex-order consistent (see e.g., [Wang et al. \(2020b\)](#)).

The class of distortion risk measures, which satisfy comonotonic additivity, is commonly used in the insurance literature.

Definition 1.2.5 (Distortion Risk Measure). A *distortion risk measure*, denoted by ρ^g , is defined as

$$\rho^g(X) = \int_0^{+\infty} g(S_X(x)) dx - \int_{-\infty}^0 (1 - g(S_X(x))) dx, \quad X \in \mathcal{X}, \quad (1.8)$$

where $g : [0, 1] \mapsto [0, 1]$ is a non-decreasing function with $g(0) = 0$ and $g(1) = 1$, and g is called the *distortion function* of ρ^g .

We always assume that $\mathcal{X}$ is a set of random variables such that $\rho^g(X)$ is finite. Via a quantile representation (e.g., [Dhaene et al. \(2012\)](#)), all of VaR^L , VaR^R , TVaR and RVaR belong to the class of distortion risk measures; indeed, all comonotonic-additive monetary risk measures are distortion risk measures (e.g., Theorem 1 of [Wang et al. \(2020b\)](#)). For $p \in (0, 1)$, a distortion risk measure ρ^g is a p -tail risk measure if and only if it dominates VaR_p^R , which is equivalent to $g(p) = 1$.

Using the terminology of [Weber \(2018\)](#), a risk measure ρ is VaR-type with parameter $\alpha \in (0, 1)$ if it satisfies

$$\rho(X) = \rho \left(X \mathbb{1}_{\{X \leq \text{VaR}_\alpha^L(X)\}} + \text{VaR}_\alpha^L(X) \mathbb{1}_{\{X > \text{VaR}_\alpha^L(X)\}} \right), \quad X \in \mathcal{X}.$$

In other words, such a risk measure ρ ignores the tail part of the distribution with probability α . It is easy to check that ρ is VaR-type with parameter α if and only if $X \mapsto \rho(-X)$ is an α -tail risk measure.

We follow the mathematical framework of tail risk measures developed by [Liu and Wang \(2021\)](#), where the following terminology for tail risks and tail risk measures is introduced. For a random variable $X \in \mathcal{X}$ and $p \in (0, 1]$, X_p presents the *tail risk of X* beyond its $(1 - p)$ -quantile, defined by

$$X_p = F_X^{-1}(1 - p + pU_X).$$

One can easily check

$$\mathbb{P}(X_p \leq x) = \mathbb{P}(X \leq x | U_X \geq 1 - p) = \frac{(\mathbb{P}(X \leq x) - (1 - p))_+}{p}, \quad x \in \mathbb{R}. \quad (1.9)$$

We assume that $X \in \mathcal{X}$ implies $X_p \in \mathcal{X}$. This assumption is satisfied by common choices of $\mathcal{X}$, such as $\mathcal{X} = L^q$, $q \in [0, \infty]$.

Definition 1.2.6 (Liu and Wang (2021)). For $p \in (0, 1)$, a risk measure ρ is a *p-tail risk measure* if $\rho(X) = \rho(Y)$ for all $X, Y \in \mathcal{X}$ satisfying $X_p \stackrel{d}{=} Y_p$. In this case, we simply say that ρ is a *tail risk measure*, and the value p is a *tail parameter* of ρ .²

Tail risk measures are risk measures that depend solely on the tail distribution. For a p -tail risk measure ρ , there exists a unique law-invariant risk measure ρ^* , called the *generator*, such that $\rho(X) = \rho^*(X_p)$ for all $X \in \mathcal{X}$. Conversely, for each generator ρ^* , the p -tail risk measure induced by ρ^* is also unique. Note that every number q in $[p, 1)$ is a tail parameter of ρ , and thus the concept of p -tail risk measure gets weaker as p increases. Therefore, we sometimes look for the smallest tail parameter of a tail risk measure, if it exists.

It is immediate from Definition 1.2.6 that VaRs, TVaR and RVaR are generally tail risk measures. In particular, for $\alpha, \beta > 0$ and $\alpha + \beta < 1$, VaR_α^R has a smallest tail parameter α , VaR_α^L has a tail parameter γ for all $\gamma > \alpha$ but not $\gamma = \alpha$, TVaR_β has a smallest tail parameter β , and $\text{RVaR}_{\alpha, \beta}$ has a smallest tail parameter $\alpha + \beta$. In addition to VaRs, TVaR and RVaR, many other tail risk measures are studied in the recent risk management literature, such as the Glue-VaR (Belles-Sampera et al. (2014)) and the Gini Shortfall (Furman et al. (2017)); see the examples in Liu and Wang (2021). The notions of tail risk measures and the VaR-type risk measures introduced by Weber (2018) are complementary.

1.3 Model uncertainty

Typically, an optimal reinsurance model is formulated as an optimization problem, with the aim of identifying indemnity functions that satisfy specific optimization targets. In such optimization problems, the ways of quantifying risk exposure level is one of the most crucial concerns, which is typically conducted under the assumption that the distribution of the underlying loss variable is known or given. However, this assumption is

²Here, we use a “small p ” convention, which is different from Liu and Wang (2021), that a p -tail risk measure is a $(1 - p)$ -tail risk measure using the terminology of Liu and Wang (2021). The choice of “small p ” convention is in agreement with Embrechts et al. (2018), and it makes many results in the thesis more concise. This choice is suggested by two anonymous referees.

often unrealistic, as the actual loss distribution is generally unavailable in most practical scenarios. Estimating distributions from available data is particularly challenging for catastrophic risks due to limited data, making such estimates unreliable. Moreover, there is no universal consensus on which loss distribution or model should be used in real-world applications. This uncertainty motivates us to explore distributional model uncertainty in determining optimal reinsurance policies.

The selection of the uncertainty set of distributions is a central question in problems incorporating model uncertainty. Two common approaches to constructing uncertainty sets are moment-based and distance-based methods. Moment-based sets include distributions whose moments, such as mean and variance, meet specific conditions. For instance, [Ghaoui et al. \(2003\)](#) study the worst-case Value-of-Risk (VaR) in a portfolio optimization problem with only bounds on the mean and covariance matrix. [Chen et al. \(2011\)](#) demonstrate that the worst-case Tail Value-of-Risk (TVaR) shares the same closed-form solution as the worst-case VaR within a set of distributions with the same mean and covariance. These results are extended by [Li \(2018\)](#) to law-invariant coherent risk measures. [Liu et al. \(2020\)](#) determine the worst-case law-invariant convex risk functionals for a risk with a given mean and higher moments. A distance-based uncertainty set incorporates measures that are close to the reference measure according to a chosen distance. [Embrechts et al. \(2020\)](#) model the heterogeneous beliefs by different probability measures. [Liu et al. \(2022\)](#) introduce uncertainty into risk-sharing models, characterizing either by the likelihood ratio between probability measures (i.e., the Radon–Nikodym derivative) or by the Wasserstein metric between distributions. The study of uncertainty sets that combine moment and Wasserstein distance constraints can be found in [Bernard et al. \(2024\)](#), which apply isotonic projections to derive expressions for the distortion risk measure value and the distribution of random losses in worst-case and best-case scenarios. The study of isotonic projections was introduced to the field of quantitative finance by [Rüschendorf and Vanduffel \(2020\)](#). Following the research findings of [Bernard et al. \(2024\)](#), [Cai et al. \(2024\)](#) use the same uncertainty set and further impose stop-loss and limited-loss functions on the random variable, characterizing the worst-case distribution.

In the research area of insurance and actuarial science, optimal insurance and reinsurance design problems under distributional uncertainty become more important nowadays. [Hu et al. \(2015\)](#) introduce model uncertainty to the optimal reinsurance framework by

considering the set of all distributions with the same first two moments. [Asimit et al. \(2017\)](#) consider an uncertainty set including finite alternative distributions and determine the optimal reinsurance strategies. [Birghila and Pflug \(2019\)](#) use Wasserstein metric to define the uncertainty set and impose a premium constraints into the uncertainty model. In [Liu and Mao \(2022\)](#), moments constraints are adopted, and the authors aim to minimize the worst-case VaR of the insurer's total risk exposure. Recently, [Boonen and Jiang \(2024\)](#) determine the optimal insurance strategy when the insurer faces uncertainty about the underlying distribution and considers all the distributions within a Wasserstein ball of a reference distribution.

Before exploring an optimal reinsurance problem involving likelihood ratio uncertainty in Chapter 2, we first introduce some results pertaining to likelihood ratio uncertainty. Let $\mathcal{P}$ be the set of all probability measures that are absolutely continuous with respect to $\mathbb{P}$. Given $\lambda \in (0, 1]$, define

$$\mathcal{P}_\lambda = \{\mathbb{Q} \in \mathcal{P} : d\mathbb{Q}/d\mathbb{P} \leq 1/\lambda\}. \quad (1.10)$$

Here, λ is a parameter representing the degree of uncertainty faced by an agent.³ In particular, $\lambda = 1$ corresponds to $\mathcal{P}_1 = \{\mathbb{P}\}$, that is, there is no uncertainty. In optimal reinsurance models, risk measure ρ is usually assumed to be law-invariant, therefore, evaluating the insurer's risk measure under a probability measure $\mathbb{Q} \in \mathcal{P}_\lambda$ is equivalent to determining the distribution of X under $\mathbb{Q}$. For each $\mathbb{Q} \in \mathcal{P}_\lambda$, let $X_\mathbb{Q}$ be a random variable whose distribution under $\mathbb{P}$ is the same as the distribution of X under $\mathbb{Q}$. The existence of such $X_\mathbb{Q}$ is guaranteed by taking $(F_X^\mathbb{Q})^{-1}(U_X)$ where $F_X^\mathbb{Q}$ is the distribution of X under $\mathbb{Q}$ and U_X is a uniform random variable under $\mathcal{P}$. Thus, the insurer's risk exposure under $\mathbb{Q}$ can be evaluated by using the distribution of $X_\mathbb{Q}$ under $\mathbb{P}$. For the insurer adopting ρ and $\mathcal{P}_\lambda$ to model her uncertainty concerns, the *robust risk measure* induced by ρ and $\mathcal{P}_\lambda$ is defined as

$$\bar{\rho}^{\mathcal{P}_\lambda}(Y) = \sup_{\mathbb{Q} \in \mathcal{P}_\lambda} \rho(Y_\mathbb{Q}), \quad Y \in \mathcal{X}. \quad (1.11)$$

³Generally, we may consider the set $\mathcal{P}_\phi = \{\mathbb{Q} \in \mathcal{P} : \mathbb{E}[\phi(d\mathbb{Q}/d\mathbb{P})] \leq 1\}$ where $\phi : [0, \infty] \rightarrow [0, \infty]$ is a convex function; this includes $\mathcal{P}_\lambda$ with $\phi(x) = \infty \times \mathbb{1}_{\{x > 1/\lambda\}}$, and the set generated by the Kullback-Leibler divergence with $\phi(x) = \beta x \log x$ for some $\beta > 0$. However, explicit results on $\bar{\rho}^{\mathcal{P}_\phi}$ are not available for general choices of ϕ .

By Proposition 2 of [Liu et al. \(2022\)](#), $\bar{\rho}^{\mathcal{P}_\lambda}$ is a λ -tail risk measure generated by ρ . This result is formally stated below.

Proposition 1.3.1 ([Liu et al. \(2022\)](#)). *For $\alpha, \lambda \in (0, 1)$, let ρ be a monotone α -tail risk measure generated by ρ^* . Then the following statements hold.*

- (i) $\bar{\rho}^{\mathcal{P}_\lambda}$ is a monotone $(\alpha\lambda)$ -tail risk measure generated by ρ^* .
- (ii) $\bar{\rho}^{\mathcal{P}_\lambda}$ is a monotone λ -tail risk measure generated by ρ .
- (iii) If ρ is continuous from above with respect to a.s. convergence, then so is $\bar{\rho}^{\mathcal{P}_\lambda}$.
- (iv) If ρ or ρ^* is convex-order consistent, then so is $\bar{\rho}^{\mathcal{P}_\lambda}$.

Next, let us turn to the special case of VaR and TVaR. For $\mathcal{Q} \subset \mathcal{P}$, write

$$\overline{\text{VaR}}_\alpha^{R, \mathcal{Q}} = \sup_{Q \in \mathcal{Q}} (\text{VaR}_\alpha^R)^Q, \quad \overline{\text{VaR}}_\alpha^{L, \mathcal{Q}} = \sup_{Q \in \mathcal{Q}} (\text{VaR}_\alpha^L)^Q, \quad \text{and} \quad \overline{\text{TVaR}}_\alpha^{\mathcal{Q}} = \sup_{Q \in \mathcal{Q}} \text{TVaR}_\alpha^Q.$$

Corollary 1.3.2 ([Liu et al. \(2022\)](#)). *For $\lambda \in (0, 1]$, we have the following formulas.*

$$\overline{\text{VaR}}_\alpha^{R, \mathcal{P}_\lambda} = \text{VaR}_{\alpha\lambda}^R, \quad \overline{\text{VaR}}_\alpha^{L, \mathcal{P}_\lambda} = \text{VaR}_{\alpha\lambda}^L, \quad \text{and} \quad \overline{\text{TVaR}}_\alpha^{\mathcal{P}_\lambda} = \text{TVaR}_{\alpha\lambda}. \quad (1.12)$$

Chapter 2

Worst-case reinsurance strategy with likelihood ratio uncertainty

2.1 Introduction

In optimal reinsurance frameworks with model uncertainty, the main focus is to determine the optimal reinsurance strategies in the worst-case scenario, which is a common approach to address model uncertainty in robust risk management introduced by [Scarf et al. \(1957\)](#). However, it remains unclear how the optimal strategy in a worst-case scenario differs from that in a regular scenario without the concern of model uncertainty. In practice, the insurer needs to conduct multiple scenario tests including regular-case and worst-case scenarios. Establishing a relation between the optimal solutions in different scenarios can effectively support the insurer's risk management in the following key aspects: (1) directly derive the optimal reinsurance problem in the worst-case scenario from the solution in the regular-case scenario, if such a solution is already known; (2) assess the adequacy of the insurer's reinsurance coverage in the worst-case scenario; (3) quantify the robustness of the insurer's optimal strategies in different scenarios.

To address those concerns, we explore in this work an optimal reinsurance problem in the worst-case scenario incorporating likelihood ratio uncertainty set, as introduced in [Liu et al. \(2022\)](#). The insurer chooses a reference probability measure which can reflect the

regular-case scenario in the best way, and determines her optimal reinsurance strategy. Any alternative probability measure in the uncertainty set should agree with the reference one on events with zero probability. For non-zero probability random events, the likelihood ratio between the reference and alternative measures should be bounded by a pre-determined tolerance level. For a law-invariant risk measure, which assigns the same risk value to random variables with the same distribution, each probability measure in the uncertainty set leads to a specific distribution of the underlying random loss, thereby affecting the risk exposure quantified by the risk measure. The insurer's objective is to minimize the total loss exposure in the worst-case scenario by choosing the optimal reinsurance policy which is modified from the solution in the regular-case scenario. To the best of our knowledge, our model is the first one solving the optimal reinsurance strategy in the worst-case scenario using the likelihood ratio uncertainty set.

Furthermore, contributions of our results are four-folds. First, we establish the relation between solutions in the regular-case and worst-case scenario for any law-invariant monetary risk measure, which is of special interests to the insurer when the robust risk measure in the worst-case scenario, such as robust expected, has no close-form formulation. It is essential to highlight that this relation is not limited to problems involving likelihood ratio uncertainty; it is applicable to any optimal reinsurance problem characterized by tail risk measures. For the general framework of tail risk measures, we refer to [Liu and Wang \(2021\)](#). Second, we demonstrate that for certain distortion risk measures, such as TVaR, the optimal reinsurance strategy remains the same in both the worst-case and regular-case scenarios. Such a characterization of risk measures is desirable for insurers concerned with distributional model uncertainty. Third, we propose two concepts of robustness in term of the solvency gaps and worst-case strategies, respectively, for the insurer, and characterize the robustness for Range Value-at-Risk (RVaR) which covers regulatory risk measures VaR and TVaR as special cases. Finally, we propose an alternative risk sharing model in which the reinsurer is cooperative with the insurer in term of the model uncertainty and re-calculate premium in different scenarios.

This section is organized as follows. Section [2.2](#) presents the preliminaries. In Section [2.3](#), we examine the relation between the optimal solution to a reinsurance design problem and its worst-case counterpart under likelihood ratio uncertainty. We present the necessary and sufficient conditions under which the two problems yield the same solution. The

robustness of the solution is analyzed in this section. Building on this relation, we present the robust solution to the optimal reinsurance problem quantified by the expectile risk measure in Section 2.4. In Section 2.5, we explore the cooperative reinsurance problem with likelihood ratio uncertainty and compare its results with those of its non-cooperative counterpart. Finally, Section 2.6 provides the conclusion of the paper.

2.2 Preliminaries

In a reinsurance contract, the underlying risk X faced by the insurer is shared between the insurer and the reinsurer. Given an indemnity function denoted by I , the risk $I(X)$ is ceded to the reinsurer, while $R_I(X) = X - I(X)$ is retained by the insurer. To avoid the issue of moral hazard, it is commonly take the set of all feasible indemnity functions defined in equation (1.5):

$$\mathcal{I} = \{0 \leq I(x) \leq x : I(x) \text{ is non-decreasing and } 0 \leq I(x) - I(y) \leq x - y\}.$$

In exchange for transferring risk, the reinsurer charges a reinsurance premium, denoted by $\pi(I(X))$. Thus, the insurer's total loss becomes $\ell(X; I) = R_I(X) + \pi(I(X))$ after entering the reinsurance contract I . In a regular case, when the distribution of X is given by $\mathbb{P}$, the optimal reinsurance design problem for the insurer using the risk measure ρ can be expressed as

$$\min_{I \in \mathcal{I}} \rho(R_I(X) + \pi(I(X))). \quad (2.1)$$

Optimization problems developed from (2.1) have been intensively discussed in the literature. Obviously, the choice of the risk measure ρ , the underlying risk X , and the reinsurance premium principle π play crucial roles in (2.1). In this current work, we define the triplet (ρ, π, X) as a *reinsurance setting*, and subsequently consider the optimal reinsurance strategy for the insurer.

Assumption 1. The risk X is a non-negative random variable with a continuous support.

Assumption 2. Assume that ρ is a law-invariant monetary risk measure, and the premium principle is given by $\pi = (1 + \theta)\rho^g$, where $\theta \geq 0$ is a safety loading, and ρ^g is a distortion risk measure induced by a continuously differentiable distortion function g .

The above two assumptions are commonly used in the literature on insurance/reinsurance design. Note that $\pi = (1 + \theta)\rho^g$ encompasses popular premium principles, such as the expected-value premium by setting $\rho^g = \mathbb{E}$, and Wang's premium by setting $\theta = 0$.

Definition 2.2.1 (Regular-case strategy). Given a reinsurance setting (ρ, π, X) , a *regular-case strategy* for the insurer, denoted by $I_{[\rho, \pi, X]}^*$, is the optimal reinsurance solution to the problem (2.1) under the reference probability measure $\mathbb{P}$, i.e.,

$$I_{[\rho, \pi, X]}^* \in \arg \min_{I \in \mathcal{I}} \{ \rho(R_I(X) + \pi(I(X))) \}. \quad (2.2)$$

For a law-invariant risk measure ρ , the “true” distribution of X is required for quantification. The distribution of X under $\mathbb{P}$ represents the regular-case scenario for the insurer. In robust risk management, the insurer needs to quantify the distribution of X in the worst-case scenario as well. Mathematically, we use the likelihood ratio to construct the uncertainty set of probability measures defined in equation (1.10).

2.3 Non-cooperative reinsurance design in the worst-case scenario

In this section, we assume a given reinsurance setting (ρ, π, X) and uncertainty set $\mathcal{P}_\lambda$, and propose a non-cooperative reinsurance model for the insurer in the worst-case scenario. Specifically, we are going to define the worst-case strategy for the insurer and investigate its relation with the regular-case strategy. In the literature, it is a commonly accepted assumption for the problem (2.1), and therefore for the regular-case strategy, that the insurer and the reinsurer use the same probability measure for risk quantification, meaning a basic agreement on the probabilities of random events from both parties. We follow the same assumption to define the optimization problem in the worst-case scenario to be comparable with problem (2.1). Meanwhile, as the counter-party, the reinsurer does not collaborate with the insurer regarding model uncertainty and determines the reinsurance premium under the reference probability measure $\mathbb{P}$, i.e., $\pi(Y) = \pi(Y_{\mathbb{P}})$ for all $Y \in \mathcal{X}$.

On the other hand, the insurer adopts the uncertainty set $\mathcal{P}_\lambda$. For any alternative probability measure $\mathbb{Q} \in \mathcal{P}_\lambda$, the insurer uses $X_{\mathbb{Q}}$ to evaluate her total risk after implementing a reinsurance policy $I \in \mathcal{I}$, and her total risk exposure under $\mathbb{Q}$ becomes

$\ell(X_{\mathbb{Q}}; I) = R_I(X_{\mathbb{Q}}) + \pi(I(X))$ for $I \in \mathcal{I}$. Thus, the insurer's optimization problem in the worst-case scenario can be written as

$$\min_{I \in \mathcal{I}} \sup_{\mathbb{Q} \in \mathcal{P}_{\lambda}} \rho(R_I(X_{\mathbb{Q}}) + \pi(I(X))). \quad (2.3)$$

2.3.1 Optimal strategy in the worst-case scenario

To solve problem (2.3), we first establish a connection between its objective function and the robust risk measure.

Lemma 2.3.1. *If $f(x)$ is a deterministic function such that $\{x : f(x) \leq y\}$ is a Borel set for any $y \in \mathbb{R}$, then the distributions of $f(X_{\mathbb{Q}})$ and $f(X)_{\mathbb{Q}}$ are the same under the reference measure $\mathbb{P}$.*

Proof. Arbitrarily take $y \in \mathbb{R}$. Let $A = \{x : f(x) \leq y\}$, which is a Borel set. The distribution of $f(X_{\mathbb{Q}})$ under $\mathbb{P}$ is given by $\mathbb{P}(f(X_{\mathbb{Q}}) \leq y) = \mathbb{P}(X_{\mathbb{Q}} \in A) = \mathbb{Q}(X \in A)$, where the last equality follows from the construction. In addition, the distribution of $f(X)_{\mathbb{Q}}$ under $\mathbb{P}$ is $\mathbb{P}(f(X)_{\mathbb{Q}} \leq y) = \mathbb{Q}(f(X) \leq y) = \mathbb{Q}(X \in A) = \mathbb{P}(f(X_{\mathbb{Q}}) \leq y)$. Since y is taken arbitrarily, we conclude that $f(X_{\mathbb{Q}})$ follows the same distribution as $f(X)_{\mathbb{Q}}$ under $\mathbb{P}$. $\square$

It is easy to see that all indemnity functions and the associated retained loss functions satisfy the conditions in Lemma 2.3.1. Therefore, the two random variables $R_I(X)_{\mathbb{Q}}$ and $R_I(X_{\mathbb{Q}})$ share the same distribution under $\mathbb{P}$. Since $\pi(I(X))$ is a constant for a given I and X , and ρ is law-invariant, Lemma 2.3.1 implies, for $\mathbb{Q} \in \mathcal{P}_{\lambda}$, $\rho(R_I(X_{\mathbb{Q}}) + \pi(I(X))) = \rho(R_I(X)_{\mathbb{Q}} + \pi(I(X)))$. It follows that

$$\sup_{\mathbb{Q} \in \mathcal{P}_{\lambda}} \rho(R_I(X_{\mathbb{Q}}) + \pi(I(X))) = \sup_{\mathbb{Q} \in \mathcal{P}_{\lambda}} \rho(R_I(X)_{\mathbb{Q}} + \pi(I(X))) = \bar{\rho}^{\mathcal{P}_{\lambda}}(R_I(X) + \pi(I(X))),$$

and the problem (2.3) can be further written as

$$\min_{I \in \mathcal{I}} \bar{\rho}^{\mathcal{P}_{\lambda}}(R_I(X) + \pi(I(X))), \quad (2.4)$$

where the robust risk measure $\bar{\rho}^{\mathcal{P}_{\lambda}}$ is adopted by the insurer. The optimal solution to problem (2.4) represents the insurer's optimal strategy in the worst-case scenario.

Definition 2.3.1 (Worst-case strategy). Given a reinsurance setting (ρ, π, X) , a *worst-case strategy* with respect to the uncertainty set $\mathcal{P}_\lambda$, denoted by $I_{[\bar{\rho}^{\mathcal{P}_\lambda, \pi, X}]}^*$, is an optimal solution to problem (2.4), i.e.,

$$I_{[\bar{\rho}^{\mathcal{P}_\lambda, \pi, X}]}^* \in \arg \min_{I \in \mathcal{I}} \bar{\rho}^{\mathcal{P}_\lambda}(R_I(X) + \pi(I(X))). \quad (2.5)$$

Theorem 2.3.1 below presents the first main result of this paper, where we characterize the worst-case strategy $I_{[\bar{\rho}^{\mathcal{P}_\lambda, \pi, X}]}^*$ by identifying how it differs from the regular-case strategy. To proceed, we first introduce a modified premium principle

$$\bar{\pi}_\lambda \triangleq \begin{cases} (1 + \theta)g(\lambda)\rho^{\bar{g}_\lambda}, & \text{if } g(\lambda) > 0, \\ 0, & \text{if } g(\lambda) = 0, \end{cases}$$

where g satisfies Assumption 2, and $\bar{g}_\lambda(t) = \frac{g(t\lambda)}{g(\lambda)}$, $0 \leq t \leq 1$, is a well-defined distortion function when $g(\lambda) > 0$.

Theorem 2.3.1. *Let Assumptions 1 and 2 hold. Define $X_{\lambda,0} = X_\lambda - v_\lambda$, where $v_\lambda \triangleq \text{VaR}_\lambda(X)$. Then, the insurer's worst-case strategy, which is the optimal solution to problem (2.4), is given by*

$$I_{[\bar{\rho}^{\mathcal{P}_\lambda, \pi, X}]}^*(x) = \begin{cases} (x - v_\lambda + a^*)_+, & x < v_\lambda, \\ I_{[\rho, \bar{\pi}_\lambda, X_{\lambda,0}]}^*(x - v_\lambda) + a^*, & x \geq v_\lambda, \end{cases} \quad (2.6)$$

where $a^* = \inf\{0 \leq a \leq v_\lambda : g(S_X(v_\lambda - a)) \geq \frac{1}{1+\theta}\}$, and $I_{[\rho, \bar{\pi}_\lambda, X_{\lambda,0}]}^*$ is the insurer's regular-case strategy to the following problem:

$$\min_{I \in \mathcal{I}} \rho(R_I(X_{\lambda,0}) + \bar{\pi}_\lambda(I(X_{\lambda,0}))). \quad (2.7)$$

Furthermore, if $\pi = (1 + \theta)\mathbb{E}$, then $a^* = \left(v_\lambda - \text{VaR}_{\frac{1}{1+\theta}}(X)\right)_+$ and $\bar{\pi}_\lambda = (1 + \theta)\lambda\mathbb{E}$.

Proof. The proof of Theorem 2.3.1 is presented in Appendix 2.7. □

Since the robust risk measure $\bar{\rho}^{\mathcal{P}_\lambda}$ is a λ -tail risk measure generated by ρ , from Liu et al. (2022) we know that the insurer's worst-case scenario occurs under the worst-case

probability measure $\mathbb{Q}^*$ such that $X_{\mathbb{Q}^*} \stackrel{d}{=} X_\lambda$ under $\mathbb{P}$. Mathematically, $\mathbb{Q}^*$ can be defined via its Radon-Nikodym derivative as follows: $\frac{d\mathbb{Q}^*}{d\mathbb{P}} = \frac{\mathbb{1}_{\{U_X > 1-\lambda\}}}{\lambda}$. It is easy to see that $\mathbb{Q}^* \in \mathcal{P}_\lambda$, and $X_{\mathbb{Q}^*} \stackrel{d}{=} X_\lambda$ under $\mathbb{P}$ because

$$\mathbb{P}(X_{\mathbb{Q}^*} > z) = \mathbb{Q}^*(X > z) = \frac{\mathbb{P}(X > z, U_X > 1 - \lambda)}{\lambda} = \frac{\min\{\mathbb{P}(X > z), \lambda\}}{\lambda} = \mathbb{P}(X_\lambda > z).$$

Under Assumption 2, $\pi = \rho^g$ is a distortion risk measure, and thus the premium principle $\bar{\pi}_\lambda$ is also a distortion risk measure, induced by the modified distortion function $\bar{g}_\lambda$. For example, the expected-value premium principle $\pi(Y) = (1 + \theta)\mathbb{E}[Y]$ is commonly used in insurance literature. In this case, $\bar{\pi}_\lambda(X) = (1 + \theta)\lambda\mathbb{E}[X]$ is an expected-value premium principle with an adjusted risk loading of $(1 + \theta)\lambda - 1$. Following Theorem 2.3.1, if the insurer has a closed-form solution for the regular-case strategy $I_{[\rho, \pi, X]}^*$ for a given reinsurance setting (ρ, π, X) , then $I_{[\rho, \bar{\pi}_\lambda, X_{\lambda, 0}]}^*$ can be easily determined using the distribution of the tail risk $X_{\lambda, 0}$ and the modified premium principle $\bar{\pi}_\lambda$. The worst-case strategy follows directly from (2.6).

For small loss amounts $x \leq v_\lambda$, the worst-case strategy in (2.6) follows a stop-loss policy with a retention level $v_\lambda - a^*$. Indeed, among all indemnity functions satisfying $I(v_\lambda) = a^*$, the worst-case strategy gives the smallest ceded loss on the loss range $[0, v_\lambda]$, which saves reinsurance premium for the insurer. For large loss amounts $x > v_\lambda$, the type of the worst-case strategy depends on ρ and $\bar{\pi}_\lambda$. As we will see later, under different settings, the worst-case strategy can be full coverage policy, limited-loss policy, or more complex structures.

Example 2.3.1. (RVaR-based worst-case strategy) In this example, we assume that the insurer adopts a Range Value-at-Risk (RVaR). Given a pair of confidence levels (p, q) , $0 \leq q < p \leq 1$,

$$\text{RVaR}_{(p, q)}(Y) = \frac{1}{p - q} \int_q^p \text{VaR}_u(Y) \, du, \quad Y \in \mathcal{X}, \quad (2.8)$$

is a distortion risk measure induced by the distortion function $g(t; p, q) = \left(\mathbb{1}_{\{t \geq q\}} \frac{t - q}{p - q} \right) \wedge 1$, $t \in [0, 1]$. In particular, $\text{TVaR}_p = \text{RVaR}_{(p, 0)}$ and VaR_p can be considered a special case when $p = q$.

When the insurer adopts $\rho = \text{RVaR}_{(p, q)}$ and $\pi = (1 + \theta)\mathbb{E}$, the insurer's RVaR-based

optimization problem in the regular-case scenario is given by

$$\min_{I \in \mathcal{I}} \text{RVar}_{(p,q)}(R_I(X) + (1 + \theta)\mathbb{E}[I(X)]). \quad (2.9)$$

It is easy to verify that the regular-case strategy, i.e., the solution to problem (2.9), is given by

$$I_{[\text{RVar}_{(p,q)}, (1+\theta)\mathbb{E}, X]}^*(x) = \begin{cases} 0, & \text{if } p \geq \frac{1}{1+\theta}, \\ \left(x - \text{VaR}_{\frac{1}{1+\theta}}(X)\right)_+ - \left(x - \text{VaR}_{\frac{q}{1-(p-q)(1+\theta)}}(X)\right)_+, & \text{if } p < \frac{1}{1+\theta}. \end{cases} \quad (2.10)$$

Next, we consider the worst-case scenario, where the insurer's problem is given by

$$\min_{I \in \mathcal{I}} \sup_{\mathbb{Q} \in \mathcal{P}_\lambda} \text{RVar}_{(p,q)}(R_I(X_{\mathbb{Q}}) + (1 + \theta)\mathbb{E}[I(X)]) = \min_{I \in \mathcal{I}} \overline{\text{RVar}}_{(p,q)}^{\mathcal{P}_\lambda}(R_I(X) + (1 + \theta)\mathbb{E}[I(X)]). \quad (2.11)$$

We now verify that the optimal solution to problem (2.11) is

$$I_{[\overline{\text{RVar}}_{(p,q)}^{\mathcal{P}_\lambda}, (1+\theta)\mathbb{E}, X]}^*(x) = \begin{cases} 0, & \text{if } \lambda p \geq \frac{1}{1+\theta}, \\ \left(x - \text{VaR}_{\frac{1}{1+\theta}}(X)\right)_+ - \left(x - \text{VaR}_{\frac{\lambda q}{1-(p-q)(1+\theta)\lambda}}(X)\right)_+, & \text{if } \lambda p < \frac{1}{1+\theta}. \end{cases} \quad (2.12)$$

To this end, we first note that problem (2.7) in Theorem 2.3.1 becomes

$$\min_{I \in \mathcal{I}} \text{RVar}_{(p,q)}(R_I(X_{\lambda,0}) + (1 + \theta)\lambda\mathbb{E}[I(X_{\lambda,0})]). \quad (2.13)$$

Applying result (2.10) to solve problem (2.13), we obtain

$$I^*(x) = \begin{cases} 0, & \text{if } \lambda p \geq \frac{1}{1+\theta}, \\ \left(x - \text{VaR}_{1 \wedge \frac{1}{(1+\theta)\lambda}}(X_{\lambda,0})\right)_+ - \left(x - \text{VaR}_{\frac{q}{1-(p-q)(1+\theta)\lambda}}(X_{\lambda,0})\right)_+, & \text{if } \lambda p < \frac{1}{1+\theta}. \end{cases} \quad (2.14)$$

Then, by Theorem 2.3.1, we have

$$I_{[\overline{\text{RVar}}_{(p,q)}^{\mathcal{P}_\lambda}, (1+\theta)\mathbb{E}, X]}^*(x) = \begin{cases} (x - v_\lambda + a^*)_+, & x < v_\lambda, \\ I^*(x - v_\lambda) + a^*, & x \geq v_\lambda, \end{cases}$$

where $a^* = \left(v_\lambda - \text{VaR}_{\frac{1}{1+\theta}}(X) \right)_+$. If $(1+\theta)\lambda \geq 1/p \geq 1$, then $a^* = 0$. Combining $I^*(x) = 0$, no-insurance is again the optimal solution to the worst-case problem (2.11). If $1/p > (1+\theta)\lambda \geq 1$, then $a^* = 0$, and $I^*(x) = (x - \text{VaR}_{\frac{1}{1+\theta}}(X) + v_\lambda)_+ - (x - \text{VaR}_{\frac{\lambda q}{1-(p-q)(1+\theta)\lambda}}(X) + v_\lambda)_+$. Thus, $I^*_{[\text{RVaR}_{(p,q),\lambda}^\lambda, (1+\theta)\mathbb{E}, X]}(x)$ is given by the second case in (2.12). If $(1+\theta)\lambda < 1$, then $a^* = v_\lambda - \text{VaR}_{\frac{1}{1+\theta}}(X)$, and $I^*(x) = x - (x - \text{VaR}_{\frac{\lambda q}{1-(p-q)(1+\theta)\lambda}}(X) + v_\lambda)_+$. The indemnity function in the second case of (2.12) is the optimal solution.

Remark 1 (Optimal strategy using a tail risk measure). The results of Theorem 2.3.1 can also be applied to characterize the optimal solutions to problem (2.1) with a tail risk measure under $\mathbb{P}$. Quantifying tail risk has been a crucial consideration in modern risk management and insurance regulation. Tail risk measures such as VaR and TVaR are often used in (2.1) to capture the tail behaviour of total risk.

For a general λ -tail risk measure ρ , we denote its generator by ρ^* , i.e., $\rho(Y) = \rho^*(Y_\lambda)$ for all $Y \in \mathcal{X}$. Mathematically, the structure of problem (2.1) using ρ is equivalent to that of (2.4), as both employ a λ -tail risk measure. Therefore, the optimal solution to problem (2.1) under $\mathbb{P}$ can also be expressed by (2.6):

$$I^*_{[\rho, \pi, X]}(x) = \begin{cases} (x - v_\lambda + a^*)_+, & \text{for } x < v_\lambda, \\ I^*_{[\rho^*, \bar{\pi}_\lambda, X_{\lambda,0}]}(x - v_\lambda) + a^*, & \text{for } x \geq v_\lambda, \end{cases} \quad (2.15)$$

where $a^* = \inf\{0 \leq a \leq v_\lambda : g(S_X(v_\lambda - a)) \geq \frac{1}{1+\theta}\}$ and

$$I^*_{[\rho^*, \bar{\pi}_\lambda, X_{\lambda,0}]} \in \arg \min_{I \in \mathcal{I}} \rho^*(R_I(X_{\lambda,0}) + \bar{\pi}_\lambda(I(X_{\lambda,0}))).$$

The expression (2.15) breaks down the optimal solution $I^*_{[\rho, \pi, X]}$ into the “left tail”, which focuses on small losses, and the “right tail”, which addresses large losses. This decomposition helps the insurer capture the best strategy regarding to different loss amounts. To illustrate this, consider a simple case where $\rho = \text{VaR}_\lambda$. Under Assumption 1, $\rho = \text{VaR}_\lambda = \text{VaR}_\lambda^+ \triangleq \inf\{x \in \mathbb{R} : F_X(x) > p\}$. By Liu and Wang (2021), VaR_λ^+ is a λ -tail risk measure generated by the essential infimum, i.e., $\text{VaR}_\lambda^+(Y) = \text{ess-inf}(Y_\lambda)$ for $Y \in \mathcal{X}$. Therefore, $I^*_{[\text{ess-inf}, \bar{\pi}_\lambda, X_{\lambda,0}]}$ is a solution to $\min_{I \in \mathcal{I}} \text{ess-inf}(R_I(X_{\lambda,0}) + \bar{\pi}_\lambda(I(X_{\lambda,0}))) = \min_{I \in \mathcal{I}} \bar{\pi}_\lambda(I(X_{\lambda,0}))$. Intuitively, VaR_λ does not account for losses that exceed the quantile level $1 - \lambda$. From this “risk tail” optimization problem, it becomes evident that the optimal strategy for “right

tail" losses is no insurance, i.e., $I_{[\text{ess-inf}, \pi_\lambda, X_{\lambda,0}]}^* = 0$. Then (2.15) gives that $I_{[\text{VaR}_\lambda, \pi, X]}^*(x) = (x - v_\lambda + a^*)_+ - (x - v_\lambda)_+$.

2.3.2 Comparison of regular-case and worst-case strategies

Following the general relation between the regular-case and worst-case strategies characterized in Theorem 2.3.1, a natural question is whether two strategies being the same. Intuitively, since the distributions of the underlying loss X differ between the regular-case and worst-case scenarios, the insurer may need to adopt different strategies. However, as we will see in the following VaR-based and TVaR-based problems, the two optimal strategies can either be identical or differ. Take an uncertainty set $\mathcal{P}_\lambda$ and a confidence level $p \in (0, 1)$, and assume $\pi = (1 + \theta)\mathbb{E}$ with $p(1 + \theta)\lambda < 1$.

- (VaR-based worst-case problem.) The VaR-based problem

$$\min_{I \in \mathcal{I}} \text{VaR}_p(R_I(X) + (1 + \theta)\mathbb{E}[I(X)])$$

has been well-studied in the literature, see for example Cheung et al. (2014), and it has solution

$$I_{[\text{VaR}_p, (1+\theta)\mathbb{E}, X]}^*(x) = \left(x - \text{VaR}_{\frac{1}{1+\theta}}(X)\right)_+ - (x - \text{VaR}_p(X))_+. \quad (2.16)$$

The VaR-based worst-case problem is expressed as

$$\min_{I \in \mathcal{I}} \overline{\text{VaR}}_p^{\mathcal{P}_\lambda}(R_I(X) + (1 + \theta)\mathbb{E}[I(X)]).$$

Note that VaR can be derived from RVaR by taking $p = q \in (0, 1)$. Substituting $p = q$ into (2.12), we find that the worst-case strategy is

$$I_{[\overline{\text{VaR}}_p^{\mathcal{P}_\lambda}, (1+\theta)\mathbb{E}, X]}^*(x) = \left(x - \text{VaR}_{\frac{1}{1+\theta}}(X)\right)_+ - (x - \text{VaR}_{p\lambda}(X))_+. \quad (2.17)$$

Obviously, the regular-case strategy (2.16) is smaller than the worst-case strategy (2.17). In other words, the regular-case strategy does not provide sufficient protection in the worst-case scenario for the insurer.

- (TVaR-based worst-case problem.) The TVaR-based optimization problem is formulated as

$$\min_{I \in \mathcal{I}} \text{TVaR}_p(R_I(X) + (1 + \theta)\mathbb{E}[I(X)])$$

with solution $I_{[\text{TVaR}_p, (1+\theta)\mathbb{E}, X]}^*(x) = \left(x - \text{VaR}_{\frac{1}{1+\theta}}(X)\right)_+$. The TVaR-based worst-case problem is $\min_{I \in \mathcal{I}} \overline{\text{TVaR}}_p^{\mathcal{P}_\lambda}(R_I(X) + (1 + \theta)\mathbb{E}[I(X)])$. Note that, for $p \in (0, 1)$, $\text{RVaR}_{(p,0)} = \text{TVaR}_p$. We can apply (2.12) with $q = 0$ to verify that

$$I_{[\overline{\text{TVaR}}_p^{\mathcal{P}_\lambda}, (1+\theta)\mathbb{E}, X]}^*(x) = \left(x - \text{VaR}_{\frac{1}{1+\theta}}(X)\right)_+ = I_{[\text{TVaR}_p, (1+\theta)\mathbb{E}, X]}^*(x), \quad (2.18)$$

i.e., if the insurer adopts TVaR, the regular-case strategy remains optimal in the worst-case scenario, and the insurer does not need to change her strategy in difference scenarios.

We can see from the above examples, comparing the regular-case and worst-case strategies under a given risk measure ρ reveals any potential shortcomings of the regular-case strategy in the worst-case scenario. Conversely, a conservative insurer may prefer a risk measure that results in the same strategies in both regular and worst-case scenarios.

For a general risk measure ρ , neither the regular-case strategy nor the worst-case strategy has a mathematically tractable closed-form expression. To facilitate comparison between the two strategies in this section, we further assume that $\rho = \rho^h$ is a distortion risk measure induced by h , and that the expected-value premium principle is adopted. The problem (2.1) can be rewritten as

$$\min_{I \in \mathcal{I}} \left\{ \rho^h(R_I(X)) + (1 + \theta)\mathbb{E}[I(X)] \right\}, \quad \text{for some } \theta \geq 0, \quad (2.19)$$

and its corresponding worst-case problem becomes

$$\min_{I \in \mathcal{I}} \sup_{\mathbb{Q} \in \mathcal{P}_\lambda} \rho^h(R_I(X_\mathbb{Q}) + (1 + \theta)\mathbb{E}[I(X)]) = \min_{I \in \mathcal{I}} \left\{ \overline{\rho^h}^{\mathcal{P}_\lambda}(R_I(X)) + (1 + \theta)\mathbb{E}[I(X)] \right\}. \quad (2.20)$$

The following proposition characterizes the distortion function h such that the regular-case and worst-case strategies are identical.

Proposition 2.3.2. *Let Assumption 1 hold. Assume $\pi = (1 + \theta)\mathbb{E}$ for some $\theta \geq 0$, and $\rho = \rho^h$ is a distortion risk measure induced by a continuously differentiable distortion function h . Define*

$$\tilde{h}(p) = \begin{cases} h(p/\lambda), & 0 \leq p < \lambda, \\ 1, & \lambda \leq p \leq 1. \end{cases} \quad (2.21)$$

The following statements hold.

- (a) *Problems (2.19) and (2.20) have at least one common optimal solution for a given $\lambda \in (0, 1)$ if and only if there exists $p_0 \in [0, \frac{1}{1+\theta}]$ such that $h(p) \geq (1 + \theta)p$ for $p \in [0, p_0]$ and $\tilde{h}(p) \leq (1 + \theta)p$ for $p \in [p_0, 1]$.*
- (b) *Problems (2.19) and (2.20) have at least one common optimal solution for any $\lambda \in (0, 1)$ if and only if $h(p) \geq (1 + \theta)p$ for $p \in [0, \frac{1}{1+\theta}]$.*

Proof. The proof of Proposition 2.3.2 is presented in Appendix 2.7. □

In Proposition 2.3.2, the insurer's risk measure ρ^h is not necessarily coherent. If the insurer selects a concave h , which makes ρ^h a coherent distortion risk measure, the results of Proposition 2.3.2 can be stated more precisely.

Corollary 2.3.3. *Let Assumption 1 hold. Assume $\pi = (1 + \theta)\mathbb{E}$ for some $\theta \geq 0$, and $\rho = \rho^h$ is a coherent distortion risk measure induced by a continuously differentiable concave distortion function h . The following statements hold.*

- (a) *Problems (2.19) and (2.20) have at least one common optimal solution for a given $\lambda \in (0, 1)$ if and only if one of the two conditions holds: (i) ρ^h is a $\frac{1}{1+\theta}$ -tail risk measure; (ii) $h'(0) \leq \lambda(1 + \theta)$.*
- (b) *Problems (2.19) and (2.20) have at least one common optimal solution for all $\lambda \in (0, 1)$ if and only if ρ^h is a $\frac{1}{1+\theta}$ -tail risk measure.*

Remark 2. Following assumptions required in Corollary 2.3.3, we can identify the type of reinsurance policy in cases (a) and (b). Suppose that ρ^h is a $\frac{1}{1+\theta}$ -tail risk measure. By Theorem F.1 in Liu and Wang (2021), $h(t) = 1$ for $t \in [p, 1]$. Since h is concave, the

solution to both problems (2.19) and (2.20) is the stop-loss policy $(x - \text{VaR}_{\frac{1}{1+\theta}}(X))_+$. In this case, the insurer focuses on the tail risk and already chooses full coverage on it in the regular-case strategy. Therefore, the regular-case strategy remains to be the optimal in the worst-case scenario. On the other hand, if $h'(0) \leq \lambda(1 + \theta)$, together with the concavity of h , we can verify that $\tilde{h}(t) \leq (1 + \theta)t$ for $t \in [0, 1]$. Therefore, the optimal solution to (2.19) and (2.20) is no-insurance, meaning that all reinsurance policies are overpriced for the insurer even in the worst-case scenario.

2.3.3 Robustness of the reinsurance setting using RVaR

For an uncertainty set $\mathcal{P}_\lambda$, the value of λ determines its size in the sense that a smaller λ results in a more severe worst-case scenario, as it is selected from a larger $\mathcal{P}_\lambda$. Since the worst-case strategy does not necessarily align with the regular-case strategy, the change in optimal strategies is critical for the insurer from a risk management perspective. If the change in the worst-case strategy is continuous in some sense with respect to λ , the insurer would only need to make small adjustments to the current optimal strategy as the worst-case scenario worsens. In such cases, the reinsurance setting is robust with respect to the worst-case strategy.

It is widely accepted that a monetary risk measure ρ can be used for capital calculation, and $\rho(R_I(X) + \pi(I(X)))$ represents the total capital required by the insurer after entering the reinsurance contract I . In the regular-case scenario, let $R_{[\rho, \pi, X]}^*(x) = x - I_{[\rho, \pi, X]}^*(x)$. Then, the risk measure value $\rho(R_{[\rho, \pi, X]}^*(X) + \pi(I_{[\rho, \pi, X]}^*(X)))$, using the regular-case strategy $I_{[\rho, \pi, X]}^*$ defined in (2.2), is the *perceived risk value* for the insurer. However, if the worst-case scenario occurs, the insurer adopting the regular-case strategy will experience the *actual risk value* $\bar{\rho}^{\mathcal{P}_\lambda}(R_{[\rho, \pi, X]}^*(X) + \pi(I_{[\rho, \pi, X]}^*(X)))$. The difference

$$\begin{aligned} \Delta_{sg}^{\mathcal{P}_\lambda} &\triangleq \underbrace{\bar{\rho}^{\mathcal{P}_\lambda}(R_{[\rho, \pi, X]}^*(X) + \pi(I_{[\rho, \pi, X]}^*(X)))}_{\text{actual risk}} - \underbrace{\rho(R_{[\rho, \pi, X]}^*(X) + \pi(I_{[\rho, \pi, X]}^*(X)))}_{\text{perceived risk}} \\ &= \bar{\rho}^{\mathcal{P}_\lambda}(R_{[\rho, \pi, X]}^*(X)) - \rho(R_{[\rho, \pi, X]}^*(X)) \end{aligned} \quad (2.22)$$

is the *solvency gap* for the insurer when the worst-case scenario occurs. Although the premium cancels out in this expression, it affects the optimal strategies and, consequently, the solvency gap. When λ approaches 1, the insurer may expect the solvency gap to

be small enough, implying that the reinsurance setting is robust concerning the solvency gap. For a general discussions on solvency gap and risk measure robustness, we refer to [Embrechts et al. \(2022\)](#).

Definition 2.3.2. Consider a reinsurance setting (ρ, π, X) that satisfies Assumption 1.

1. (ρ, π, X) is *robust with respect to the solvency gap*, as defined in (2.22), if $\lim_{\lambda \rightarrow 1} \Delta_{sg}^{\mathcal{P}_\lambda} = 0$;
2. (ρ, π, X) is *robust with respect to the worst-case strategy* for $\mathcal{P}_{\lambda_0}$ if, for any $\delta > 0$, there exists a neighborhood Λ_0 of λ_0 such that whenever $\lambda, \gamma \in \Lambda_0 \cap (0, 1]$, there exist $I_{[\bar{\rho}^{\mathcal{P}_\lambda}, \pi, X]}^*$ and $I_{[\bar{\rho}^{\mathcal{P}_\gamma}, \pi, X]}^*$ satisfying

$$\left| I_{[\bar{\rho}^{\mathcal{P}_\lambda}, \pi, X]}^*(x) - I_{[\bar{\rho}^{\mathcal{P}_\gamma}, \pi, X]}^*(x) \right| \leq \delta, \quad \text{for all } x \in [0, \text{ess sup } X].$$

In insurance regulation, VaR and TVaR are two of the most important risk measures. RVaR can be seen as a family of risk measures that bridge VaR and TVaR. Therefore, we focus on the optimization problem using RVaR and present the following proposition to characterize the robustness of the reinsurance setting $(\rho = \text{RVaR}, \pi = (1 + \theta)\mathbb{E}, X)$.

Proposition 2.3.4. Let Assumption 1 hold and $\theta \geq 0$.

- (a) The reinsurance setting $(\text{RVaR}_{(p,q)}, (1 + \theta)\mathbb{E}, X)$ is robust with respect to the solvency gap for all $0 < p < 1$ and $0 \leq q \leq p$.
- (b) If $q \neq 0$, the reinsurance setting $(\text{RVaR}_{(p,q)}, (1 + \theta)\mathbb{E}, X)$ is robust with respect to worst-case strategy of $\mathcal{P}_{\lambda_0}$ for all $\lambda_0 \in (0, 1]$. If $q = 0$, then $\text{RVaR}_{(p,0)} = \text{TVaR}_p$, and the reinsurance setting $(\text{TVaR}_p, (1 + \theta)\mathbb{E}, X)$ is robust with respect to the worst-case strategy of $\mathcal{P}_{\lambda_0}$ at $\lambda_0 = 1$ and for all $\lambda_0 \in (0, 1)$ with $\lambda_0 \neq \frac{1}{p(1+\theta)}$.

Proof. We first consider robustness with respect to worst-case strategies. Assume $q \neq 0$ and $\lambda_0 \in (0, 1]$. We rely on (2.12) to determine the worst-case strategy $I_\lambda^* = I_{[\text{RVaR}_{(p,q)}^{\mathcal{P}_\lambda}, (1+\theta)\mathbb{E}, X]}^*$ for each $\lambda \in (0, 1]$.

(i) If $(1 + \theta)\lambda_0 p > 1$, there exists a neighborhood Λ_0 of λ_0 such that $(1 + \theta)\lambda p > 1$ for all $\lambda \in \Lambda_0 \cap (0, 1]$. Then $I_\lambda^*(x) = 0$, $x \geq 0$, for all $\lambda \in \Lambda_0 \cap (0, 1]$. Clearly, the reinsurance setting $(\text{RVaR}_{(p,q)}, (1 + \theta)\mathbb{E}, X)$ is robust with respect to the worst-case strategy of $\mathcal{P}_{\lambda_0}$.

(ii) If $(1 + \theta)\lambda_0 p = 1$, then for $x \geq 0$,

$$I_\lambda^*(x) = \begin{cases} 0, & \text{if } \lambda \geq \lambda_0, \\ (x - \text{VaR}_{\frac{1}{1+\theta}}(X))_+ - (x - \text{VaR}_{\frac{\lambda q}{1-\lambda(p-q)(1+\theta)}}(X))_+, & \text{if } \lambda < \lambda_0. \end{cases}$$

Under Assumption 1, $\lim_{\lambda \rightarrow \lambda_0} \text{VaR}_{\frac{\lambda q}{1-\lambda(p-q)(1+\theta)}}(X) = \text{VaR}_{\frac{\lambda_0 q}{1-\lambda_0(p-q)(1+\theta)}}(X) = \text{VaR}_{p\lambda_0}(X)$

and

$\lim_{\lambda \rightarrow \lambda_0} \text{VaR}_{p\lambda}(X) = \text{VaR}_{p\lambda_0}(X)$. For any neighborhood $\Lambda_0 \subseteq (0, 1)$ of λ_0 , and $\lambda, \gamma \in \Lambda_0$, we have

$$\begin{aligned} & \sup_{x \geq 0} |I_\lambda^*(x) - I_\gamma^*(x)| \\ & \leq \max \left\{ \text{VaR}_{\frac{\lambda q}{1-\lambda(p-q)(1+\theta)}}(X) - \text{VaR}_{\frac{1}{1+\theta}}(X), \text{VaR}_{\frac{\gamma q}{1-\gamma(p-q)(1+\theta)}}(X) - \text{VaR}_{\frac{1}{1+\theta}}(X) \right\} \\ & = \max \left\{ \text{VaR}_{\frac{\lambda q}{1-\lambda(p-q)(1+\theta)}}(X), \text{VaR}_{\frac{\gamma q}{1-\gamma(p-q)(1+\theta)}}(X) \right\} - \text{VaR}_{p\lambda_0}(X) \\ & \rightarrow 0, \quad |\Lambda_0| \rightarrow 0. \end{aligned}$$

Therefore, for any $\delta > 0$, we can choose Λ_0 small enough such that $\sup_{x \geq 0} |I_\lambda^*(x) - I_\gamma^*(x)| \leq \delta$.

(iii) If $(1 + \theta)\lambda_0 p < 1$, there exists a neighborhood Λ_0 of λ_0 such that $(1 + \theta)\lambda p < 1$ for all $\lambda \in \Lambda_0 \cap (0, 1]$. In this case, the same argument as in (ii) applies.

Combining these three cases, for any $\lambda_0 \in (0, 1)$, the reinsurance setting $(\text{RVaR}_{(p,q)}, (1 + \theta)\mathbb{E}, X)$ is robust with respect to the worst-case strategy of $\mathcal{P}_{\lambda_0}$.

Next, assume $q = 0$. In this case, $\text{RVaR}_{(p,q)} = \text{TVaR}_p$ for some $0 < p < 1$. Take $\lambda_0 \in (0, 1]$.

(iv) If $\lambda_0 \neq \frac{1}{(1+\theta)p}$, then the same arguments in cases (i) and (iii) apply. Therefore, the reinsurance setting $(\text{TVaR}_p, (1 + \theta)\mathbb{E}, X)$ is robust with respect to the worst-case strategy of $\mathcal{P}_{\lambda_0}$.

- (v) If $\lambda_0 = 1 = \frac{1}{(1+\theta)p}$, then $\mathcal{P}_{\lambda_0} = \{\mathbb{P}\}$, and $I_{\lambda_0}^*$ can be any feasible contract. In particular, we can choose $I_{\lambda_0}^*(x) = (x - \text{VaR}_{\frac{1}{1+\theta}}(X))_+$. For any $\lambda < \lambda_0$, we have $(1+\theta)p\lambda < 1$ and $I_{\lambda}^*(x) = (x - \text{VaR}_{\frac{1}{1+\theta}}(X))_+ = I_{\lambda_0}^*(x)$ for $x \geq 0$.
- (vi) If $\lambda_0 = \frac{1}{(1+\theta)p} \in (0, 1)$, take $\lambda < \lambda_0$. Then $(1+\theta)p\lambda < 1$, and $I_{\lambda}^*(x) = (x - \text{VaR}_{\frac{1}{1+\theta}}(X))_+$ for $x \geq 0$ is the unique worst-case strategy. Meanwhile, take $\gamma > \lambda_0$. Then $(1+\theta)p\gamma > 1$, and $I_{\gamma}^* = 0$ is the unique worst-case strategy. For any neighborhood Λ_0 of λ_0 , there exists $\lambda, \gamma \in \Lambda_0$ such that $\lambda < \lambda_0 < \gamma$. Thus, for any $x > \text{VaR}_{\frac{1}{1+\theta}}(X)$, we have $|I_{\lambda}^*(x) - I_{\gamma}^*(x)| = x - \text{VaR}_{\frac{1}{1+\theta}}(X) > 0$. Therefore, $(\text{TVaR}_p, (1+\theta)\mathbb{E}, X)$ is not robust with respect to the worst-case strategy of $\mathcal{P}_{\lambda_0}$.

The robustness of $(\text{RVaR}_{(p,q)}, (1+\theta)\mathbb{E}, X)$ with respect to the solvency gap can be directly obtained by substituting the regular-case strategy into the expression in (2.22), and is therefore omitted here. $\square$

2.4 Optimal reinsurance design with the worst-case expectile

In this section, we apply the results from Section 2.3 to solve an optimal reinsurance design problem using an expectile risk measure, which has recently gained attention in statistics and actuarial science. Among the two most widely used regulatory risk measures, VaR is often criticized for not satisfying sub-additivity, which makes it non-coherent. In contrast, while TVaR is coherent, it is not elicitable, a key property for model comparison and validation. Expectiles, however, can be both coherent and elicitable. For further discussion on the properties of risk measures, including coherence and elicibility, we refer to Bellini et al. (2014), Ziegel (2016), Liu et al. (2022), Fissler et al. (2024), and references therein.

Definition 2.4.1 (Expectile). The expectile risk measure at a confidence level $\alpha \in (0, 1)$ is defined as

$$\mathcal{E}_{\alpha}(Y) \triangleq \arg \min_{m \in \mathbb{R}} \left\{ \alpha \mathbb{E} \left[(Y - m)_+^2 \right] + (1 - \alpha) \mathbb{E} \left[(m - Y)_+^2 \right] \right\}, \quad Y \in L^2.$$

In Cai and Weng (2016), it has been shown that

$$\mathcal{E}_\alpha(Y) = \mathbb{E}[Y] + \beta \mathbb{E}[(Y - \mathcal{E}_\alpha(Y))_+] \quad \text{with} \quad \beta = \frac{2\alpha - 1}{1 - \alpha}, \quad (2.23)$$

and that $\mathcal{E}_\alpha(Y)$ is the unique constant satisfying this equation. As demonstrated in Ziegel (2016), $\mathcal{E}_\alpha$ is a coherent risk measure for $\alpha \geq 1/2$. Therefore, we assume $\alpha \geq 1/2$, which is equivalent to $\beta \geq 0$.

With the expectile risk measure, the insurer's problem (2.1) becomes

$$\min_{I \in \mathcal{I}} \mathcal{E}_\alpha(R_I(X) + \pi(I(X))),$$

which was studied in Cai and Weng (2016) with a budget constraint and mild assumptions on the premium principle π . In the present work, we focus on the expected-value premium principle, i.e., $\pi = (1 + \theta)\mathbb{E}$, and write the *expectile-based optimization problem* (2.1) as

$$\min_{I \in \mathcal{I}} \mathcal{E}_\alpha(R_I(X) + (1 + \theta)\mathbb{E}[I(X)]), \quad \text{where } \alpha \geq \frac{1}{2}. \quad (2.24)$$

Furthermore, we propose the worst-case version of this problem as:

$$\min_{I \in \mathcal{I}} \sup_{Q \in \mathcal{P}_\lambda} \mathcal{E}_\alpha(R_I(X_Q) + (1 + \theta)\mathbb{E}[I(X)]) = \min_{I \in \mathcal{I}} \bar{\mathcal{E}}_\alpha^{\mathcal{P}_\lambda}(R_I(X) + (1 + \theta)\mathbb{E}[I(X)]), \quad (2.25)$$

where $\bar{\mathcal{E}}_\alpha^{\mathcal{P}_\lambda}$ represents the robust risk measure induced by $\mathcal{E}_\alpha$. To solve (2.25), we first generalize the result of the optimal reinsurance policies of (2.24) in Cai and Weng (2016), in which θ is assumed to be strictly positive, to the more general case of $\theta \geq -1$.

Lemma 2.4.1. *Under the reinsurance setting $(\mathcal{E}_\alpha, (1 + \theta)\mathbb{E}, X)$ with $\alpha \geq 1/2$ and $-1 \leq \theta \leq 0$, the regular-case strategy is the full reinsurance, i.e., $I_{[\mathcal{E}_\alpha, (1 + \theta)\mathbb{E}, X]}^*(x) = x$, $x \geq 0$.*

Proof. The proof of this lemma is presented in Appendix 2.7. □

Using Lemma 2.4.1 and Theorem 3.2 of Cai and Weng (2016), the insurer can apply Theorem 2.3.1 to solve the problem (2.25) and determine the worst-case strategy. Let $v_\alpha = \text{VaR}_\alpha(X)$ for $\alpha \in [0, 1]$, $d_\lambda^0 = v_{\frac{1+\beta}{\beta(1+\theta)} - \frac{\lambda}{\beta}} - v_\lambda$, and $\tilde{d}_\lambda$ be the root of $G(d)$ for a given $\lambda \in (0, 1]$, where

$$G(d) = d - \frac{1}{\lambda} \mathbb{E}[(X - v_\lambda)_+] - \frac{\beta}{\lambda} \mathbb{E}[(X - d - v_\lambda)_+], \quad d \in (0, \infty). \quad (2.26)$$

Furthermore, let $d_\lambda^* = d_\lambda^0 \wedge \tilde{d}_\lambda$, and let m_λ^* be the solution to $K(d_\lambda^*, m) = d_\lambda^*$, where

$$K(d, m) = \frac{1}{\lambda} \mathbb{E}[(X - v_\lambda)_+] - \frac{1}{\lambda} \mathbb{E}[(X - d - v_\lambda)_+] + \frac{1 + \beta}{\lambda} \mathbb{E}[(X - m - v_\lambda)_+], \quad m \geq d. \quad (2.27)$$

Theorem 2.4.1. *Let Assumption 1 hold. Under the reinsurance setting $(\mathcal{E}_\alpha, (1 + \theta)\mathbb{E}, X)$ with $\alpha \geq 1/2$ and $\theta > 0$, the solution to the expectile-based worst-case problem (2.25) can be formulated as*

$$I_{[\mathcal{E}_\alpha^{\mathcal{P}_\lambda, (1+\theta)\mathbb{E}, X}]}^*(x) = \begin{cases} 0, & \text{if } \frac{1+\beta}{1+\theta} \leq \lambda \leq 1, \\ (x - d_\lambda^* - v_\lambda)_+ - (x - m_\lambda^* - v_\lambda)_+, & \text{if } \frac{1}{1+\theta} < \lambda < \frac{1+\beta}{1+\theta}, \\ (x - v_{\frac{1}{1+\theta}})_+, & \text{if } 0 < \lambda \leq \frac{1}{1+\theta}. \end{cases} \quad (2.28)$$

Proof. By Theorem 2.3.1, we begin by solving the following problem:

$$\min_{I \in \mathcal{I}} \mathcal{E}_\alpha(R_I(X_{\lambda,0}) + (1 + \theta')\mathbb{E}[I(X_{\lambda,0})]), \quad (2.29)$$

where $X_{\lambda,0} = X_\lambda - v_\lambda$ and $\theta' = (1 + \theta)\lambda - 1 \in (-1, \theta]$. When $\theta' \leq 0$, by Lemma 2.4.1, the optimal indemnity function is given by $I_{[\mathcal{E}_\alpha, (1+\theta')\mathbb{E}, X_{\lambda,0}]}^*(x) = x$. When $\theta' > 0$, according to Theorem 3.2 of Cai and Weng (2016), the problem (2.29) can be solved by a one-layer indemnity function $I_{d,m}(x) = (x - d)_+ - (x - m)_+$, where $0 \leq d = \mathcal{E}_{I_{d,m}}^R(X_{\lambda,0}) \leq m \leq v_0$, and $\mathcal{E}_{I_{d,m}}^R(X_{\lambda,0})$ is the α -expectile of $R_{I_{d,m}}(X_{\lambda,0}) = X_{\lambda,0} - I_{d,m}(X_{\lambda,0})$.

Using (2.23), we have

$$\begin{aligned} \mathcal{E}_{I_{d,m}}^R(X_{\lambda,0}) &= \mathbb{E}[R_{I_{d,m}}(X_{\lambda,0})] + \beta \mathbb{E}[(R_{I_{d,m}}(X_{\lambda,0}) - \mathcal{E}_{I_{d,m}}^R(X_{\lambda,0}))_+] \\ &= \mathbb{E}[R_{I_{d,m}}(X_{\lambda,0})] + \beta \mathbb{E}[(X_{\lambda,0} - m)_+] \\ &= \frac{1}{\lambda} \mathbb{E}[(X - v_\lambda)_+] - \frac{1}{\lambda} \mathbb{E}[(X - d - v_\lambda)_+] + \frac{1 + \beta}{\lambda} \mathbb{E}[(X - m - v_\lambda)_+] = K(d, m). \end{aligned}$$

The last equality holds because $S_{X_{\lambda,0}}(x) = \frac{S_X(x+v_\lambda)}{\lambda}$. For a given d , the value of m can be determined by the equation $d = \mathcal{E}_{I_{d,m}}^R(X_{\lambda,0})$, or equivalently, by $d = K(d, m)$. Thus, $m = m(d)$ can be considered as a function of d . Define $\mathcal{D} = \{d : 0 \leq d \leq m(d) \leq \infty \text{ such that } K(d, m(d)) = d\}$. Cai and Weng (2016) demonstrates that $\mathcal{D} \neq \emptyset$ and $m = m(d)$ is strictly decreasing with respect to d over the domain $\mathcal{D} = [0, \tilde{d}_\lambda]$, where $\tilde{d}_\lambda$ is the unique root of the function $G(d) \triangleq d - K(d, d)$ over $(0, \infty)$, which can be reformulated

as in (2.26). For a given λ , it is easy to gain from $d = K(d, m)$ that $m(0) = v_0 - v_\lambda$ and $m(\tilde{d}_\lambda) = \tilde{d}_\lambda$.

Define $d_\lambda^0 = v \frac{1+\beta}{\beta(1+\theta)} - \frac{\lambda}{\beta} - v_\lambda$. When $\theta' > 0$, from Example 4.1 of Cai and Weng (2016), we have the following: (i) If $\theta' \geq \beta$, i.e., $\lambda \geq \frac{1+\beta}{1+\theta}$, then $d_\lambda^* = \tilde{d}_\lambda$, $m_\lambda^* = m(\tilde{d}_\lambda) = \tilde{d}_\lambda$ and thus, $I_{[\mathcal{E}_\alpha, (1+\theta')\mathbb{E}, X_{\lambda,0}]}^*(x) = 0$. (ii) If $\theta' < \beta$, i.e., $\frac{1}{1+\theta} < \lambda < \frac{1+\beta}{1+\theta}$, then $d_\lambda^* = d_\lambda^0 \wedge \tilde{d}_\lambda$ and m_λ^* obtained by solving $K(d_\lambda^*, m) = d_\lambda^*$, we have $I_{[\mathcal{E}_\alpha, (1+\theta')\mathbb{E}, X_{\lambda,0}]}^*(x) = (x - d_\lambda^*)_+ - (x - m_\lambda^*)_+$.

By Theorem 2.3.1, we have

$$I_{[\bar{\mathcal{E}}_\alpha^{\mathcal{P}_\lambda}, (1+\theta)\mathbb{E}, X]}^*(x) = \begin{cases} (x - v_\lambda + a^*)_+, & x < v_\lambda, \\ I_{[\mathcal{E}_\alpha, (1+\theta')\mathbb{E}, X_{\lambda,0}]}^*(x - v_\lambda) + a^*, & x \geq v_\lambda, \end{cases} \quad (2.30)$$

where $a^* = \left(v_\lambda - v \frac{1}{1+\theta}\right)_+$. Then, we have

- (i) If $\lambda \geq \frac{1+\beta}{1+\theta}$, then $I_{[\mathcal{E}_\alpha, (1+\theta')\mathbb{E}, X_{\lambda,0}]}^*(x) = 0$, $a^* = 0$, and thus, $I_{[\bar{\mathcal{E}}_\alpha^{\mathcal{P}_\lambda}, (1+\theta)\mathbb{E}, X]}^*(x) = 0$.
- (ii) If $\frac{1}{1+\theta} < \lambda < \frac{1+\beta}{1+\theta}$, it follows that $I_{[\mathcal{E}_\alpha, (1+\theta')\mathbb{E}, X_{\lambda,0}]}^*(x) = (x - d_\lambda^*)_+ - (x - m_\lambda^*)_+$, $a^* = 0$, and $I_{[\bar{\mathcal{E}}_\alpha^{\mathcal{P}_\lambda}, (1+\theta)\mathbb{E}, X]}^*(x) = (x - d_\lambda^* - v_\lambda)_+ - (x - m_\lambda^* - v_\lambda)_+$.
- (iii) If $\lambda \leq \frac{1}{1+\theta}$, then $I_{[\mathcal{E}_\alpha, (1+\theta')\mathbb{E}, X_{\lambda,0}]}^*(x) = x$, $a^* = v_\lambda - v \frac{1}{1+\theta}$, and $I_{[\bar{\mathcal{E}}_\alpha^{\mathcal{P}_\lambda}, (1+\theta)\mathbb{E}, X]}^*(x) = (x - v \frac{1}{1+\theta})_+$.

Then, we obtain the optimal indemnity function $I_{[\bar{\mathcal{E}}_\alpha^{\mathcal{P}_\lambda}, (1+\theta)\mathbb{E}, X]}^*$, as formulated in the theorem. $\square$

In the following proposition, we discuss the robustness of the reinsurance setting $(\mathcal{E}_\alpha, (1+\theta)\mathbb{E}, X)$ with respect to the worst-case strategy for $\mathcal{P}_{\lambda_0}$.

Proposition 2.4.2. *Let Assumption 1 hold. For a given $\lambda_0 \in (0, 1]$ and $\lambda_0 \neq \frac{1}{1+\theta}$, the reinsurance setting $(\mathcal{E}_\alpha, (1+\theta)\mathbb{E}, X)$ with $\alpha \geq \frac{1}{2}$ and $\theta > 0$ remains robust with respect to the worst-case strategy for $\mathcal{P}_{\lambda_0}$.*

Proof. Let $I_\lambda^* \triangleq I_{[\bar{\mathcal{E}}_\alpha^{\mathcal{P}_\lambda}, (1+\theta)\mathbb{E}, X]}^*$ for simplicity. By Implicit Function Theorem, we have that $\tilde{d}_\lambda$ is a continuous function of λ . Assumption 1 ensures that d_λ^0 is continuous with respect to λ . Thus, $d_\lambda^* = \tilde{d}_\lambda \wedge d_\lambda^0$ and $m(d_\lambda^*)$ exhibit continuity concerning $\lambda \in (0, 1]$.

- (i) When $\lambda_0 \in (\frac{1+\beta}{1+\theta}, 1]$, by (2.28), there exists a small neighbour Λ_0 of λ_0 such that $I_\lambda^* = 0$ for $\lambda \in \Lambda_0$.
- (ii) When $\lambda_0 = \frac{1+\beta}{1+\theta}$, we have $d_{\lambda_0}^0 = v_0 - v_{\lambda_0}$ and $G(d_{\lambda_0}^0) = v_0 - \text{TVaR}_{\lambda_0}(X) \geq 0$. Since $G(\tilde{d}_{\lambda_0}) = 0$ and $G(d)$ is an increasing function of d , we have $\tilde{d}_{\lambda_0} \leq d_{\lambda_0}^0$. Thus, $d_{\lambda_0}^* = \tilde{d}_{\lambda_0} \wedge d_{\lambda_0}^0 = \tilde{d}_{\lambda_0}$ and $m_{\lambda_0}^* = m(d_{\lambda_0}^*) = \tilde{d}_{\lambda_0}$. By (2.28), for any neighborhood $\Lambda_0 \subseteq (\frac{1}{1+\theta}, 1)$ of λ_0 , and $\lambda, \gamma \in \Lambda_0$, we have $\sup_{x \geq 0} |I_\lambda^*(x) - I_\gamma^*(x)| \leq \max \{m(d_\lambda^*) - d_\lambda^*, m(d_\gamma^*) - d_\gamma^*\} \rightarrow 0$ as $|\Lambda_0| \rightarrow 0$.
- (iii) When $\lambda_0 \in (\frac{1}{1+\theta}, \frac{1+\beta}{1+\theta})$, for any neighborhood $\Lambda_0 \subseteq (\frac{1}{1+\theta}, \frac{1+\beta}{1+\theta})$ of λ_0 and $\lambda, \gamma \in \Lambda_0$, by (2.28) we have $\sup_{x \geq 0} |I_\lambda^*(x) - I_\gamma^*(x)| \leq |(m_\lambda^* - d_\lambda^*) - (m_\gamma^* - d_\gamma^*)| \rightarrow 0$ as $|\Lambda_0| \rightarrow 0$.
- (iv) When $\lambda_0 = \frac{1}{1+\theta}$, let $\lambda \rightarrow \lambda_0$, we have $d_\lambda^0 \rightarrow 0$, $d_\lambda^* = d_\lambda^0 \wedge \tilde{d}_\lambda \rightarrow 0$, and thus, $m_\lambda^* \rightarrow v_0 - v_{\lambda_0}$. If $v_0 < \infty$, for any neighborhood Λ_0 of λ , take $\lambda, \gamma \in \Lambda_0$. By (2.28), we have $\sup_{x \geq 0} |I_\lambda^*(x) - I_\gamma^*(x)| \leq \max \{v_0 - v_{\lambda_0} - m_\lambda^* + d_\lambda^*, v_0 - v_{\lambda_0} - m_\gamma^* + d_\gamma^*\} \rightarrow 0$ as $|\Lambda_0| \rightarrow 0$. Conversely, if $v_0 = \infty$, then for any $\delta > 0$ and any neighborhood Λ_0 of λ_0 , there exists $\lambda, \gamma \in \Lambda_0$ and $\lambda < \lambda_0 < \gamma$ such that for $x > m_\gamma^* + \delta$, we have $|I_\lambda^*(x) - I_\gamma^*(x)| > x - v_{\lambda_0} - m_\gamma^* + d_\gamma^* > \delta$. This implies that $(\mathcal{E}_\alpha, (1+\theta)\mathbb{E}, X)$ is not robust with respect to the worst-case strategy of $\mathcal{P}_{\lambda_0}$ by Definition 2.3.2.
- (v) When $\lambda_0 \in (0, \frac{1}{1+\theta})$, there exists a neighborhood Λ_0 of λ_0 such that $I_\lambda^* = (x - v_{\frac{1}{1+\theta}})_+$ for $\lambda \in \Lambda_0$.

Therefore, when $\lambda_0 \in (0, 1]$ and $\lambda_0 \neq \frac{1}{1+\theta}$, for any $\delta > 0$, there exists a sufficiently small Λ_0 such that $\sup_{x \geq 0} |I_\lambda^*(x) - I_\gamma^*(x)| \leq \delta$. This implies that $(\mathcal{E}_\alpha, (1+\theta)\mathbb{E}, X)$ is robust with respect to the worst-case strategy of $\mathcal{P}_{\lambda_0}$ by Definition 2.3.2. $\square$

Example 2.4.1 (Expectile-based reinsurance design with Pareto distribution). Assume that X follows a type II Pareto distribution with shape parameter ζ and scale parameter η , i.e. $X \sim \text{Pa}(\zeta, \eta)$, with the survival function $S_X(x) = \left(\frac{\eta}{x+\eta}\right)^\zeta$, where $x \geq 0$. We adopt the parameter settings: $\zeta = 3$, $\eta = 2$, and $\theta = 3$. Then $\text{VaR}_p(X) = \eta p^{-1/\zeta} - \eta$, $\mathbb{E}[X] = \frac{\eta}{\zeta-1} < \infty$, and $\mathbb{E}[X \wedge x] = \frac{\eta}{\zeta-1} - \frac{\eta^\zeta}{(\zeta-1)(x+\eta)^{\zeta-1}}$ for $x \geq 0$.

Figure 2.1 illustrates the optimal retention levels $d_\lambda^* + v_\lambda$ and $m_\lambda^* + v_\lambda$ for the worst-case strategy $I_{[\mathcal{E}_\alpha^{\mathcal{P}_\lambda}, (1+\theta)\mathbb{E}, X]}^*$ as outlined in Theorem 2.4.1. The shaded blue area indicates the

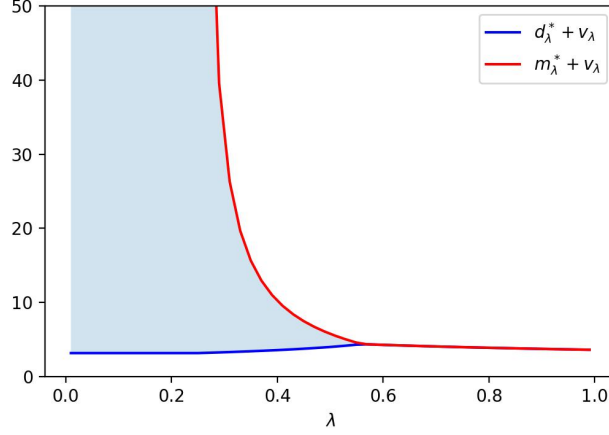


Figure 2.1: Optimal ceded loss with respect to λ when $X \sim \text{Pa}(\zeta, \eta)$. The parameters are $\zeta = 3, \eta = 2, \beta = 2, \theta = 3$.

portion of the loss ceded to the reinsurer. When $\lambda \leq \frac{1}{1+\theta} = 0.25$, the blue line representing $d_\lambda^* + v_\lambda$ aligns with $v_{\frac{1}{1+\theta}}$, and the red line representing $m_\lambda^* + v_\lambda$ extends to infinity. In this scenario, the worst-case strategy simplifies to the stop-loss function described in Theorem 2.4.1. From Figure 2.1, it is evident that as the uncertainty set expands, the insurer cedes more of the medium-sized losses to the reinsurer. Both retention levels change smoothly with respect to λ , supporting the result in Proposition 2.4.2, which asserts that the reinsurance setting is robust concerning the worst-case strategy of $\mathcal{P}_\lambda$.

2.5 Cooperative reinsurance design with likelihood ratio uncertainty

In the non-cooperative problem (2.3), the reinsurance premium is calculated under the reference probability measure and can be considered constant in the insurer's objective function, regardless of the probability measure used by the insurer. In this section, we discuss a cooperative model in which both the insurer and the reinsurer consider the distributional model uncertainty in the reinsurance design procedure while they share a

common belief regarding to the probability measure in different scenarios. In other words, although the insurer and the reinsurer have concerns of uncertainty, they remain to behave cooperatively whenever an alternative measure $\mathbb{Q} \in \mathcal{P}_\lambda$ is given. Such cooperative formulation is adopted for reinsurance design problem in [Liu and Mao \(2022\)](#). We again assume that the uncertainty set of probability measures is given by (1.10) for a given $\lambda \in (0, 1]$. Thus, the insurer's optimization problem in the worst-case scenario can be written as

$$\min_{I \in \mathcal{I}} \sup_{\mathbb{Q} \in \mathcal{P}_\lambda} \rho(R_I(X_{\mathbb{Q}}) + \pi(I(X_{\mathbb{Q}}))). \quad (2.31)$$

It is worth noting that problem (2.31) can be viewed as a comonotonic risk-sharing problem between market participants in the worst-case scenario. To see this, since ρ is translation-invariant, we have $\rho(R_I(X_{\mathbb{Q}}) + \pi(I(X_{\mathbb{Q}}))) = \rho(R_I(X_{\mathbb{Q}})) + \pi(I(X_{\mathbb{Q}}))$, where the insurer and reinsurer share the total loss X under the probability measure $\mathbb{Q}$. They use the risk measures ρ and π , respectively, to quantify their own risk exposure. Thus, problem (2.31) provides the optimal risk-sharing strategy for the insurer and reinsurer in the worst-case scenario. Furthermore, we can easily derive a lower bound for problem (2.31):

$$\min_{I \in \mathcal{I}} \sup_{\mathbb{Q} \in \mathcal{P}_\lambda} \rho(R_I(X_{\mathbb{Q}}) + \pi(I(X_{\mathbb{Q}}))) \geq \sup_{\mathbb{Q} \in \mathcal{P}_\lambda} \min_{I \in \mathcal{I}} \{\rho(R_I(X_{\mathbb{Q}})) + \pi(I(X_{\mathbb{Q}}))\}. \quad (2.32)$$

For an arbitrary monetary risk measure ρ , neither problem on the left nor the right side of (2.32) may be mathematically tractable. To clarify this further, in the remainder of this section, we assume that the insurer uses a distortion risk measure $\rho = \rho^h$ to quantify her risk exposure. We then demonstrate that the inequality in (2.32) holds as an equality, and then characterize the optimal reinsurance strategy in the worst-case scenario.

Theorem 2.5.1. *Let Assumptions 1 and 2 hold, and assume that $\rho = \rho^h$ is induced by a continuously differentiable distortion function h . Then, the minimum of the cooperative problem (2.31) is*

$$\rho^{h \wedge [(1+\theta)g]}(X_\lambda), \quad (2.33)$$

which is achieved by the optimal reinsurance contract $I^*(x) = \int_0^x (I^*)'(y) dy$, $y \geq 0$, with

$$(I^*)'(x) = \begin{cases} 0, & \text{if } h(\mathbb{P}(X_\lambda > x)) < (1+\theta)g(\mathbb{P}(X_\lambda > x)), \\ 1, & \text{if } h(\mathbb{P}(X_\lambda > x)) > (1+\theta)g(\mathbb{P}(X_\lambda > x)), \\ \eta(x), & \text{o/w,} \end{cases} \quad (2.34)$$

with any function $\eta : [0, \infty) \rightarrow [0, 1]$.

Furthermore, the minimum of the cooperative problem (2.31) is always greater than that of the non-cooperative problem (2.3), i.e.,

$$\min_{I \in \mathcal{I}} \sup_{\mathbb{Q} \in \mathcal{P}_\lambda} \{\rho(R_I(X_{\mathbb{Q}})) + \pi(I(X_{\mathbb{Q}}))\} \geq \min_{I \in \mathcal{I}} \sup_{\mathbb{Q} \in \mathcal{P}_\lambda} \{\rho(R_I(X_{\mathbb{Q}})) + \pi(I(X))\}. \quad (2.35)$$

Proof. Note that $h \wedge [(1 + \theta)g] = \min\{h, (1 + \theta)g\}$ is a non-decreasing and continuous function with $(h \wedge [(1 + \theta)g])(0) = 0$ and $(h \wedge [(1 + \theta)g])(1) = 1$. Therefore, $h \wedge [(1 + \theta)g]$ is a valid distortion function. For any $\mathbb{Q} \in \mathcal{P}_\lambda$, if $\rho = \rho^h$ and $\pi = (1 + \theta)\rho^g$, it is well-known that $\min_{I \in \mathcal{I}} \{\rho(R_I(X_{\mathbb{Q}})) + \pi(I(X_{\mathbb{Q}}))\} = \rho^{h \wedge [(1 + \theta)g]}(X_{\mathbb{Q}})$. Therefore, (2.31) can be rewritten as

$$\begin{aligned} \min_{I \in \mathcal{I}} \sup_{\mathbb{Q} \in \mathcal{P}_\lambda} \{\rho(R_I(X_{\mathbb{Q}})) + \pi(I(X_{\mathbb{Q}}))\} &\geq \sup_{\mathbb{Q} \in \mathcal{P}_\lambda} \min_{I \in \mathcal{I}} \{\rho(R_I(X_{\mathbb{Q}})) + \pi(I(X_{\mathbb{Q}}))\} \\ &= \sup_{\mathbb{Q} \in \mathcal{P}_\lambda} \rho^{h \wedge [(1 + \theta)g]}(X_{\mathbb{Q}}) = \rho^{h \wedge [(1 + \theta)g]}(X_\lambda). \end{aligned} \quad (2.36)$$

On the other hand, from the proof of Proposition 2 in Liu et al. (2022), for any $\mathbb{Q} \in \mathcal{P}_\lambda$ and any random variable Y , we have $\mathbb{Q}(Y \leq y) \geq \mathbb{P}(Y_\lambda \leq y)$ for all $y \in \mathbb{R}$. This implies that

$$\mathbb{P}(I(X_{\mathbb{Q}}) \leq x) = \mathbb{P}(I(X)_{\mathbb{Q}} \leq x) = \mathbb{Q}(I(X) \leq x) \geq \mathbb{P}(I(X)_\lambda \leq x) = \mathbb{P}(I(X_\lambda) \leq x), \quad x \geq 0.$$

By the law-invariance and monotonicity of ρ^g , we have $\rho^g(I(X_{\mathbb{Q}})) \leq \rho^g(I(X_\lambda))$. Similarly, we can verify that $\rho^h(R_I(X_{\mathbb{Q}})) \leq \rho^h(R_I(X_\lambda))$. Then it follows that, for any $I \in \mathcal{I}$,

$$\sup_{\mathbb{Q} \in \mathcal{P}_\lambda} \{\rho^h(R_I(X_{\mathbb{Q}})) + (1 + \theta)\rho^g(I(X_{\mathbb{Q}}))\} \leq \rho^h(R_I(X_\lambda)) + (1 + \theta)\rho^g(I(X_\lambda)) = \rho^{h \wedge [(1 + \theta)g]}(X_\lambda).$$

Consequently,

$$\min_{I \in \mathcal{I}} \sup_{\mathbb{Q} \in \mathcal{P}_\lambda} \{\rho(R_I(X_{\mathbb{Q}})) + \pi(I(X_{\mathbb{Q}}))\} \leq \min_{I \in \mathcal{I}} \{\rho^h(R_I(X_\lambda)) + (1 + \theta)\rho^g(I(X_\lambda))\} = \rho^{h \wedge [(1 + \theta)g]}(X_\lambda), \quad (2.37)$$

where the optimal indemnity such that the last equality holds is defined by (2.34). Inequalities (2.36) and (2.37) together imply (2.33), and I^* is the optimal reinsurance contract.

Next, we show (2.35). If h and g are distortion functions, it is straightforward to verify that $h \wedge (1 + \theta)g$ is also a distortion function. For function $f \in \{h, g, h \wedge (1 + \theta)g\}$, define

$$\bar{f}(t) = \begin{cases} f(t/\lambda), & 0 \leq t < \lambda, \\ 1, & \lambda \leq t \leq 1. \end{cases}$$

Under Assumption 2, the problem (2.3) can be transformed into

$$\min_{I \in \mathcal{I}} \left\{ \sup_{\mathbb{Q} \in \mathcal{P}_\lambda} \rho(R_I(X_{\mathbb{Q}}) + \pi(I(X))) \right\} = \min_{I \in \mathcal{I}} \left\{ \rho^{\bar{h}}(R_I(X)) + (1 + \theta)\rho^g(I(X)) \right\} = \rho^{\bar{h} \wedge (1 + \theta)g}(X).$$

Thus,

$$\rho^{h \wedge (1 + \theta)g}(X_\lambda) = \overline{(\rho^{h \wedge (1 + \theta)g})}^{\mathcal{P}_\lambda}(X) = \rho^{\bar{h} \wedge (1 + \theta)\bar{g}}(X) = \rho^{\bar{h} \wedge (1 + \theta)g}(X) \geq \rho^{\bar{h} \wedge (1 + \theta)g}(X). \quad (2.38)$$

The first equality holds because $\bar{\rho}^{\mathcal{P}_\lambda}$ is a λ -tail risk measure generated by ρ , the second equality holds because $(\bar{\rho}^f)^{\mathcal{P}_\lambda}$ is a distortion risk measure with distortion function $\bar{f}$, and the last inequality holds because $\bar{g} \geq g$. The inequality (2.35) follows from (2.38). $\square$

Example 2.5.1 (Comparison of Problem (2.3) and Problem (2.31) with Pareto distribution). Assume that X follows the type II Pareto distribution, i.e., $X \sim \text{Pa}(\zeta, \eta)$ with the survival function $S_X(x) = \left(\frac{\eta}{x + \eta}\right)^\zeta$, $x > 0$. Let $\pi = (1 + \theta)\mathbb{E}$ with $\theta \geq 0$. Table 2.1 provide a theoretical comparison of the objective functional values $\rho^{h \wedge (1 + \theta)g}(X_\lambda)$ and $\bar{\rho}^{\mathcal{P}_\lambda}(\ell(X; I_{[\bar{\rho}^{\mathcal{P}_\lambda}, \pi, X]}^*))$ for $\rho^h = \text{RVaR}_{(p, q)}$. For two special cases when $\rho^h = \text{VaR}_\alpha$, TVaR_α , the comparison of the value functions of Problems (2.3) and (2.31) for different sizes of the uncertainty set are displayed in Figures 2.2 and 2.3.

Table 2.1: $\rho = \text{RVaR}_{(p, q)}$

		$\rho^{h \wedge (1 + \theta)g}(X_\lambda)$	$\bar{\rho}^{\mathcal{P}_\lambda}(\ell(X; I_{[\bar{\rho}^{\mathcal{P}_\lambda}, \pi, X]}^*))$
$(1 + \theta)p > 1$	$\lambda < \frac{1}{(1 + \theta)p}$	$\text{RVaR}_{(\lambda p, \lambda q)}(X)$	$\frac{1}{\lambda(p - q)} \int_{\lambda q}^{c_2} v_u \, du + (1 + \theta) \int_{c_2}^{\frac{1}{1 + \theta}} v_u \, du$
	$\lambda \geq \frac{1}{(1 + \theta)p}$		$\text{RVaR}_{(\lambda p, \lambda q)}(X)$
$(1 + \theta)p \leq 1$		$\frac{1}{\lambda(p - q)} \int_{\lambda q}^{c_1} v_u \, du + \frac{1 + \theta}{\lambda} \int_{c_1}^{\frac{\lambda}{1 + \theta}} v_u \, du$	$\frac{1}{\lambda(p - q)} \int_{\lambda q}^{c_2} v_u \, du + (1 + \theta) \int_{c_2}^{\frac{1}{1 + \theta}} v_u \, du$

The parameter $c_1 = \frac{\lambda q}{1 - (1 + \theta)(p - q)}$ and $c_2 = \frac{\lambda q}{1 - \lambda(1 + \theta)(p - q)}$.

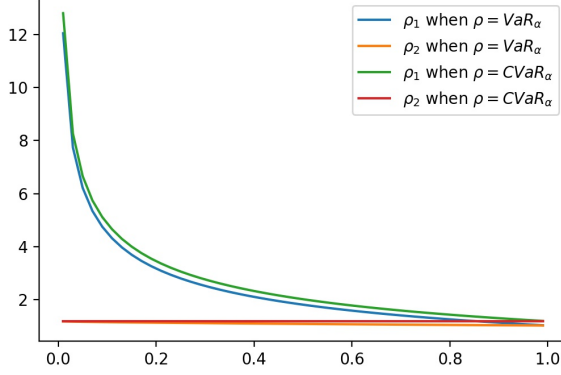


Figure 2.2: For $\lambda \in (0, 1]$, $\rho_1 = \rho^{h \wedge (1+\theta)g}(X_\lambda)$ and $\rho_2 = \bar{\rho}^{\mathcal{P}_\lambda}(\ell(X; I_{[\bar{\rho}^{\mathcal{P}_\lambda, \pi, X}]})^*)$, where $\zeta = 3, \eta = 2, \theta = 0.2$, and $\alpha = 0.05$.

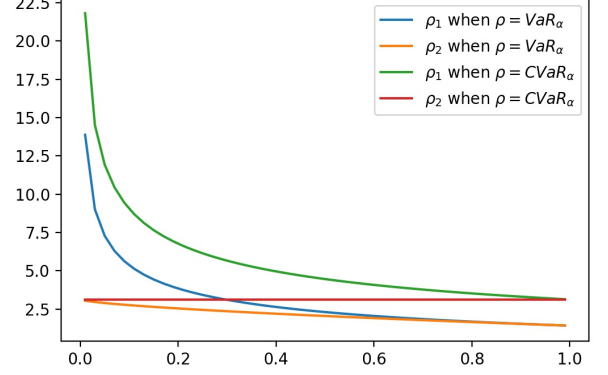


Figure 2.3: For $\lambda \in (0, 1]$, $\rho_1 = \rho^{h \wedge (1+\theta)g}(X_\lambda)$ and $\rho_2 = \bar{\rho}^{\mathcal{P}_\lambda}(\ell(X; I_{[\bar{\rho}^{\mathcal{P}_\lambda, \pi, X}]})^*)$, where $\zeta = 3, \eta = 2, \theta = 4$, and $\alpha = 0.2$.

2.6 Conclusion

In this chapter, we investigate an insurance design problem that seeks to minimize the worst-case risk of the insurer's total loss, accounting for model uncertainty. The worst-case problem involves maximization over probability measures derived from the likelihood ratio uncertainty set. We establish a relation between the optimal strategy in the regular-case strategy, which disregards model uncertainty, and the optimal strategy in the worst-case strategy, which incorporates model uncertainty. Specifically, when the insurer is aware of the regular-case strategy, our results provide a framework for determining the corresponding worst-case strategy in a non-cooperative setting. In a more general context, this relation is applicable to optimal reinsurance models quantified by tail risk measures. Our findings are applicable to a wide range of problems. We focus particularly on the model quantified by the expectile risk measure, where we derive optimal policies for the worst-case problem. We also provide a sufficient and necessary condition for the problem quantified by a distortion risk measure to have common optimal solutions with its robust counterpart. The robustness of the problem is discussed. In addition, we investigate the corresponding cooperative problem and determine the optimal indemnity function. Our

results demonstrate that the value function in the cooperative model exceeds that of the non-cooperative model.

2.7 Appendix

Proof of Theorem 2.3.1. For any $I \in \mathcal{I}$, $\bar{\rho}^{\mathcal{P}_\lambda}(R_I(X)) = \rho(R_I(X)_\lambda)$ because $\bar{\rho}^{\mathcal{P}_\lambda}$ is a λ -tail risk measure generated by ρ . Under Assumption 1, the insurer's objective function in (2.4) can be expressed as

$$\begin{aligned}\bar{\rho}^{\mathcal{P}_\lambda}(\ell(X; I)) &= \bar{\rho}^{\mathcal{P}_\lambda}(R_I(X)) + \pi(I(X)) = \rho(R_I(X)_\lambda) + \pi((I(X) - I(v_\lambda))_+ + I(X) \wedge I(v_\lambda)) \\ &= \rho(R_I(X)_\lambda - R_I(v_\lambda)) + \pi((I(X) - I(v_\lambda))_+) + R_I(v_\lambda) + \pi(I(X) \wedge I(v_\lambda)).\end{aligned}\tag{2.39}$$

It is easy to see that

$$\begin{aligned}\min_{I \in \mathcal{I}} \bar{\rho}^{\mathcal{P}_\lambda}(\ell(X; I)) &= \min_{0 \leq a \leq v_\lambda} \left\{ \min_{I \in \mathcal{I}, I(v_\lambda)=a} \bar{\rho}^{\mathcal{P}_\lambda}(\ell(X; I)) \right\} \\ &\geq \min_{0 \leq a \leq v_\lambda} \left\{ \min_{I \in \mathcal{I}, I(v_\lambda)=a} \{ \rho(R_I(X)_\lambda - R_I(v_\lambda)) + \pi((I(X) - I(v_\lambda))_+) \} \right. \\ &\quad \left. + \min_{I \in \mathcal{I}, I(v_\lambda)=a} \{ R_I(v_\lambda) + \pi(I(X) \wedge I(v_\lambda)) \} \right\}.\end{aligned}\tag{2.40}$$

For any $0 \leq a \leq v_\lambda$, define

$$\min_{I \in \mathcal{I}, I(v_\lambda)=a} \{ R_I(v_\lambda) + \pi(I(X) \wedge I(v_\lambda)) \},\tag{2.41}$$

$$\min_{I \in \mathcal{I}, I(v_\lambda)=a} \{ \rho(R_I(X)_\lambda - R_I(v_\lambda)) + \pi((I(X) - I(v_\lambda))_+) \}.\tag{2.42}$$

If $I_a^* \in \mathcal{I}$ with $I_a^*(v_\lambda) = a$ is a solution to both problems (2.41) and (2.42), then I_a^* is a solution to $\min_{I \in \mathcal{I}, I(v_\lambda)=a} \bar{\rho}^{\mathcal{P}_\lambda}(\ell(X; I))$ in (2.40), and the infinitely dimensional problem (2.4) is reduced to a 1-dimension problem $\min_{0 \leq a \leq v_\lambda} \bar{\rho}^{\mathcal{P}_\lambda}(\ell(X; I_a^*))$. Heuristically, the problem $\min_{I \in \mathcal{I}, I(v_\lambda)=a} \bar{\rho}^{\mathcal{P}_\lambda}(\ell(X; I))$ can be decomposed into the sub-optimization problem (2.41) focusing on “small-loss part” of X (left-tail), and the sub-optimization problem (2.42) focusing on “large-loss part” of X (right-tail).

We first consider the sub-problem (2.42). Note that the function $(I(x) - I(v_\lambda))_+$ is non-decreasing in x . Therefore, $\text{VaR}_t((I(X) - I(v_\lambda))_+) = (I(\text{VaR}_t(X)) - I(v_\lambda))_+$ for any $t \in (0, 1)$. It follows that

$$\begin{aligned}
\frac{1}{1+\theta} \pi((I(X) - I(v_\lambda))_+) &= \int_0^\infty g(\mathbb{P}((I(X) - I(v_\lambda))_+ > t)) \, dt \\
&= \int_0^1 \text{VaR}_t((I(X) - I(v_\lambda))_+) g'(t) \, dt \\
&= \int_0^1 (I(\text{VaR}_t(X)) - I(v_\lambda))_+ g'(t) \, dt \\
&= \int_0^\lambda (I(\text{VaR}_t(X)) - I(v_\lambda)) g'(t) \, dt \\
&= \int_0^1 (I(\text{VaR}_{\lambda u}(X)) - I(v_\lambda)) g'(u\lambda) \lambda \, du \\
&= \int_0^1 (I(\text{VaR}_u(X_\lambda)) - I(v_\lambda)) g'(u\lambda) \lambda \, du,
\end{aligned}$$

where the last equality comes from the fact that, when $p \in (0, 1)$, $\text{VaR}_t(X_p) = \text{VaR}_{pt}(X)$. If $g(\lambda) = 0$, then $g(p) = 0$ and $g'(p) = 0$ for all $p \in (0, \lambda)$. It follows that $\pi((I(X) - I(v_\lambda))_+) = 0$. If $g(\lambda) > 0$, by the continuity of g , $g'(p) > 0$ for p in a neighborhood of λ and then

$$\frac{1}{1+\theta} \pi((I(X) - I(v_\lambda))_+) = g(\lambda) \int_0^1 (I(\text{VaR}_u(X_\lambda)) - I(v_\lambda)) \bar{g}'_\lambda(u) \, du = g(\lambda) \rho^{\bar{g}_\lambda} (I(X_\lambda) - I(v_\lambda)).$$

Meanwhile, since $\text{VaR}_t(X_p) = \text{VaR}_{pt}(X)$, $t, p \in (0, 1)$, and R_I is a non-decreasing function, we have $\text{VaR}_t(R(X)_p) = \text{VaR}_{tp}(R(X)) = R(\text{VaR}_{tp}(X)) = R(\text{VaR}_t(X_p)) = \text{VaR}_t(R(X_p))$ for $t, p \in (0, 1)$. Therefore, $R(X)_\lambda$ and $R(X_\lambda)$ have the same distribution. Using the above expression, we can rewrite the objective function in the problem (2.42) as

$$\begin{aligned}
&\rho(R_I(X)_\lambda - R_I(v_\lambda)) + \pi((I(X) - I(v_\lambda))_+) \\
&= \rho(R_I(X_\lambda) - R_I(v_\lambda)) + (1+\theta)g(\lambda)\rho^{\bar{g}_\lambda}(I(X_\lambda) - I(v_\lambda)) \\
&= \rho(\bar{R}_I(X_{\lambda,0})) + (1+\theta)g(\lambda)\rho^{\bar{g}_\lambda}(\bar{I}(X_{\lambda,0})),
\end{aligned}$$

where $\bar{R}_I(y) \triangleq R_I(y + v_\lambda) - R_I(v_\lambda)$ and $\bar{I}(y) \triangleq I(y + v_\lambda) - I(v_\lambda)$ for all $y \geq 0$. It is easy to see that, $\bar{I} \in \mathcal{I}$ for any $I \in \mathcal{I}$, and conversely, for any $\bar{I} \in \mathcal{I}$, there always exists $I \in \mathcal{I}$ such that $I(x) = \bar{I}(x - v_\lambda) + I(v_\lambda)$ for $x \geq v_\lambda$. Therefore, if I^* is a solution to the

problem (2.42) with $I^*(v_\lambda) = a$, then $\bar{I}^*(x) = I^*(x + v_\lambda) - a$ is a solution to the problem (2.7); if $\bar{I}^* = I^*[\rho, \bar{\pi}_\lambda, X_{\lambda,0}]$ is a solution to the problem (2.7), then any $I \in \mathcal{I}$ satisfying $I(x) = \bar{I}^*(x - v_\lambda) + a$ for $x \geq x_\lambda$ and $I(v_\lambda) = a$ is a solution of the problem (2.42).

Next, we consider the sub-problem (2.41). Take $a \in [0, v_\lambda]$ and $I \in \mathcal{I}$ with $I(v_\lambda) = a$. Consider $f \in \mathcal{I}$ and its retained loss function given below, respectively,

$$f(x) = \begin{cases} (x - R_I(v_\lambda))_+, & x \leq v_\lambda, \\ I(x), & x > v_\lambda, \end{cases} \quad \text{and} \quad R_f(x) = x - f(x) = \begin{cases} x \wedge R_I(v_\lambda), & x \leq v_\lambda, \\ R_I(x), & x > v_\lambda. \end{cases}$$

It is easy to see that $R_f(v_\lambda) = R_I(v_\lambda) = v_\lambda - a$, and $f(x) \leq I(x)$ for all $x \geq 0$. Since π is comonotonic, we have $R_f(v_\lambda) + \pi(f(X) \wedge f(v_\lambda)) \leq R_I(v_\lambda) + \pi(I(X) \wedge I(v_\lambda))$. It says that I is suboptimal to f in the problem (2.41), and thus, f is an optimal solution.

In short, for any $a \in [0, v_\lambda]$, the function

$$I_a(x) \triangleq \begin{cases} (x - v_\lambda + a)_+, & x < v_\lambda, \\ I^*[\rho, \bar{\pi}_\lambda, X_{\lambda,0}](x - v_\lambda) + a, & x \geq v_\lambda, \end{cases} \quad (2.43)$$

satisfies $I_a(v_\lambda) = a$, $I_a(x) = I^*[\rho, \bar{\pi}_\lambda, X_{\lambda,0}](x - v_\lambda) + a$ for $x \geq v_\lambda$, and $I_a(x) \leq I(x)$ for all $x < v_\lambda$ and $I \in \mathcal{I}$ with $I(v_\lambda) = a$. Therefore, I_a is an optimal solution to problems (2.41) and (2.42) simultaneously. Furthermore, I_a is an optimal solution to the inner problem of (2.40).

Finally, we determine the optimal a to the outer problem in (2.40). For each $a \in [0, v_\lambda]$,

$$\begin{aligned} \bar{\rho}^{\mathcal{P}_\lambda}(\ell(X; I_a)) &= \rho(R_{I_a}(X)_\lambda - R_{I_a}(v_\lambda)) + R_{I_a}(v_\lambda) + \pi(I_a(X)) \\ &= \rho(X_{\lambda,0} - I^*_{[\rho, \bar{\pi}_\lambda, X_{\lambda,0}]}(X_{\lambda,0})) + v_\lambda - a + (1 + \theta) \int_{v_\lambda - a}^{v_\lambda} g(S_X(x)) dx + C, \end{aligned}$$

where $\rho(X_{\lambda,0} - I^*_{[\rho, \bar{\pi}_\lambda, X_{\lambda,0}]}(X_{\lambda,0}))$ and constant $C = (1 + \theta) \int_{v_\lambda}^{\infty} g(S_X(x)) I^{*'}[\rho, \bar{\pi}_\lambda, X_{\lambda,0}](x - v_\lambda) dx$ do not dependent on the value of a . Minimizing (2.40) is equivalent to minimizing

$$\min_{0 \leq a \leq v_\lambda} \left\{ -a + (1 + \theta) \int_{v_\lambda - a}^{v_\lambda} g(S_X(x)) dx \right\}, \quad (2.44)$$

which has a minimizer at $a^* = \inf\{0 \leq a \leq v_\lambda : g(S_X(v_\lambda - a)) \geq \frac{1}{1 + \theta}\}$. $\square$

Proof of Proposition 2.3.2. Take $F = F_X$ and $S = S_X$. The objective function in the problem (2.19) is $L_1 = \int_0^\infty [(1 + \theta)S(x) - h(S(x))] I'(x) dx + \rho^h(X)$. By the similar argument in Boonen et al. (2021b), the solutions to the problem (2.19) are

$$I^*(x) \triangleq I^*[\rho, (1 + \theta)\mathbb{E}, X](x) = \int_0^x \xi^*(y) dy, \quad (2.45)$$

where $\xi^*(y) = 1$ if $(1 + \theta)S(y) < h(S(y))$, $\xi^*(y) = 0$ if $(1 + \theta)S(y) > h(S(y))$, and $\xi^*(y) = \xi_1(y)$ can be any function within the range $[0, 1]$ if $(1 + \theta)S(y) = h(S(y))$.

Since $\bar{\rho}^{\mathcal{P}^\lambda}$ is a λ -tail risk measure generated by ρ^h , we know that $\bar{\rho}^{\mathcal{P}^\lambda} = \rho^{\tilde{h}}$ is again a distortion risk measure with distortion function $\tilde{h}$. Similarly, the problem (2.20) has the objective function $L_2 = \int_0^\infty [(1 + \theta)S(x) - \tilde{h}(S(x))] I'(x) dx + \rho^{\tilde{h}}(X)$, and optimal solutions

$$\tilde{I}^*(x) \triangleq I^*[\bar{\rho}^{\mathcal{P}^\lambda}, (1 + \theta)\mathbb{E}, X](x) = \int_0^x \tilde{\xi}^*(y) dy, \quad (2.46)$$

where $\tilde{\xi}^*(y) = 1$ if $(1 + \theta)S(y) < \tilde{h}(S(y))$, $\tilde{\xi}^*(y) = 0$ if $(1 + \theta)S(y) > \tilde{h}(S(y))$, and $\tilde{\xi}^*(y) = \xi_2(y)$ can be any function within the range $[0, 1]$ if $(1 + \theta)S(y) = \tilde{h}(S(y))$.

Let $A \triangleq \{p \in (0, 1) : (1 + \theta)p < h(p)\}$, $E \triangleq \{p \in (0, 1) : (1 + \theta)p = h(p)\}$, $B \triangleq \{p \in (0, \frac{1}{1+\theta}) : (1 + \theta)p > h(p)\}$, $\tilde{A} \triangleq \{p \in (0, 1) : (1 + \theta)p < \tilde{h}(p)\}$, $\tilde{E} \triangleq \{p \in (0, 1) : (1 + \theta)p = \tilde{h}(p)\}$, $\tilde{B} \triangleq \{p \in (0, \frac{1}{1+\theta}) : (1 + \theta)p > \tilde{h}(p)\}$. Because $\tilde{h}(p) \geq h(p)$, we have $A \subset \tilde{A}$ and $\tilde{B} \subset B$.

We first prove that the problems (2.19) and (2.20) have at least one common optimal solution for a given $\lambda \in (0, 1)$ if and only if $\tilde{A} \subset A \cup E$. If $\tilde{A} \not\subset A \cup E$, let $D = \tilde{A} \setminus (A \cup E)$ and denote the indicator function as $\mathbb{1}_{\{\cdot\}}$. It then follows that $\tilde{I}^*(x) = \int_0^x \mathbb{1}_{\{S(y) \in \tilde{A}\}} dy + \int_0^x \xi_2(y) \mathbb{1}_{\{S(y) \in \tilde{E}\}} dy \geq \int_0^x \mathbb{1}_{\{S(y) \in \tilde{A}\}} dy$. Because of the continuity of $\tilde{h}(p)$, there exists $0 < a < b$ such that $(S(b), S(a)) \subset D$ and $\tilde{I}^*(b) \geq \int_0^b \mathbb{1}_{\{S(y) \in D\}} dy + \int_0^b \mathbb{1}_{\{S(y) \in A \cup E\}} dy \geq b - a + I^*(b) > I^*(b)$ meaning that the problems (2.19) and (2.20) have no common optimal solutions when $\tilde{A} \not\subset A \cup E$, which leads to a contradiction. On the other hand, suppose that $\tilde{A} \subset A \cup E$, we can always equate (2.45) and (2.46) by adjusting the values of $\xi_1(y)$ and $\xi_2(y)$. Specifically, let $\xi_1(y) = 1$ when $S(y) \in \tilde{A} \setminus A$, $\xi_2(y) = 0$ when $S(y) \in \tilde{B} \setminus B$, and $\xi_1(y) = \xi_2(y)$ otherwise. Thus, $\tilde{A} \subset A \cup E$ is a necessary and sufficient condition for the problems (2.19) and (2.20) having at least one common optimal solution for a given $\lambda \in (0, 1)$.

- (a) “ $\Leftarrow$ ” In this part, we will prove that condition (i) or (ii) implies $\tilde{A} \subset A \cup E$. (i) If $h(p) \geq (1 + \theta)p$ for $p \in (0, p_0]$ and $\tilde{h}(p) \leq (1 + \theta)p$ for $p \in [p_0, 1]$, where $p_0 \in (0, \frac{1}{1+\theta}]$, we have $\tilde{A} \subset (0, p_0] \subset A \cup E$. (ii) If $\tilde{h}(p) \leq (1 + \theta)p$ for $p \in [0, 1]$, then $\tilde{A} = \emptyset \subset A \cup E$. Thus, the problems (2.19) and (2.20) have at least one common optimal solution for a given $\lambda \in (0, 1)$.

“ $\Rightarrow$ ” The following part proves that $h(p)$ and $\tilde{h}(p)$ are classified into (i) or (ii) under condition $\tilde{A} \subset A \cup E$. If $\tilde{A} \neq \emptyset$, due to the continuity of $\tilde{h}(p)$, there exist $p_u, p_d \in [0, \frac{1}{1+\theta}]$ such that $\tilde{h}(p_u) = (1 + \theta)p_u$, $\tilde{h}(p_d) = (1 + \theta)p_d$, and $\tilde{h}(p) > (1 + \theta)p$ for $p \in (p_d, p_u)$. We first assume $p_d > 0$. If $h(p_d) < \tilde{h}(p_d)$, the continuity of $h(p)$ and $\tilde{h}(p)$ implies that there exists $\delta > 0$ such that $(p_d, p_d + \delta) \subset \tilde{A}$ and $(p_d, p_d + \delta) \not\subset A \cup E$, which contradicts the premise $\tilde{A} \subset A \cup E$. Thus, we have $h(p_d) = \tilde{h}(p_d) = h(\frac{p_d}{\lambda})$. Because $h(p)$ is a continuous and increasing function, $h(p)$ should be a constant for $p \in [p_d, \frac{p_d}{\lambda}]$. Then there exists $\delta' > 0$ such that $(p_d, p_d + \delta') \subset \tilde{A}$ and $(p_d, p_d + \delta') \not\subset A \cup E$, which also leads to a contradiction. Hence, we have $p_d = 0$, which means that $\tilde{A}$ is a connected set with an infimum 0. Let $p_0 \triangleq \sup \tilde{A}$. Then $\tilde{A} = (0, p_0)$, $\tilde{h}(p_0) = (1 + \theta)p_0$, and $p_0 \leq \frac{1}{1+\theta}$. Since $\tilde{A} \subset A \cup E$, we must have $h(p) \geq (1 + \theta)p$ for $p \in (0, p_0]$ and $\tilde{h}(p) \leq (1 + \theta)p$ for $p \in [p_0, 1]$, which is Case (i). If $\tilde{A} = \emptyset$ for $p \in [0, 1]$, it then follows that $\tilde{h}(p) \leq (1 + \theta)p$, which is Case (ii).

- (b) “ $\Leftarrow$ ” If $h(p) \geq (1 + \theta)p$ for $p \in (0, \frac{1}{1+\theta}]$, $\tilde{h}(p) \geq h(p)$ implies $\tilde{A} \subset A \cup E = (0, \frac{1}{1+\theta}]$ for $\lambda \in (0, 1)$.

“ $\Rightarrow$ ” Assume that the problems (2.19) and (2.20) have at least one common optimal solution for any $\lambda \in (0, 1)$. If $B \neq \emptyset$, there exist $p_a \in (0, \frac{1}{1+\theta})$ such that $h(p_a) < (1 + \theta)p_a$. Let $\lambda < p_a$, it then follows that $h(p_a) < (1 + \theta)p_a < \tilde{h}(\lambda) = 1$, which contradicts $\tilde{A} \subset A \cup E$. Thus, $B = \emptyset$, which means $h(p) \geq (1 + \theta)p$ for $p \in (0, \frac{1}{1+\theta}]$. $\square$

Proof of Lemma 2.4.1. For simplicity, write $L_I(X) \triangleq \mathcal{E}_\alpha(R_I(X) + (1 + \theta)\mathbb{E}[I(X)])$ and let $\mathcal{E}_I^R$ denote the α -expectile of $R(X) = X - I(X)$ for $I \in \mathcal{I}$. Define a two-layer function $h_{a,b,m}(x) = x - (x - a)_+ + (x - b)_+ - (x - m)_+$ with parameters $0 \leq a \leq b \leq m \leq \infty$ and $\mathcal{I}_1 = \{h_{a,b,m} \in \mathcal{I} : 0 \leq a \leq b = a + \mathcal{E}_{h_{a,b,m}}^R \leq m \leq \infty\}$. By the similar argument in Theorem 3.1 of Cai and Weng (2016), for $-1 \leq \theta \leq 0$, we have $\min_{I \in \mathcal{I}} L_I(X) = \min_{I \in \mathcal{I}_1} L_I(X)$.

Arbitrarily take and fix $h_{a,b,m} \in \mathcal{I}_1$. Define a one-layer function $I_{d,k}(x) = (x - d)_+ - (x - k)_+$ with $0 \leq d \leq \mathcal{E}_{h_{a,b,m}}^R$, $k = m + d - \mathcal{E}_{h_{a,b,m}}^R$, and $d \leq k$, such that $R_{I_{d,k}}(m) =$

$R_{h_{a,b,m}}(m) = \mathcal{E}_{h_{a,b,m}}^R$ and $R_{I_{d,k}}(x) = R_{h_{a,b,m}}(x)$ when $x > m$. Therefore, for any pair (d, k) , we have

$$\mathbb{E} \left[\left(R_{I_{d,k}}(X) - \mathcal{E}_{h_{a,b,m}}^R(X) \right)_+ \right] = \mathbb{E} \left[\left(R_{h_{a,b,m}}(X) - \mathcal{E}_{h_{a,b,m}}^R(X) \right)_+ \right]. \quad (2.47)$$

Note that $R_{I_{d,k}}(x) \leq R_{h_{a,b,m}}(x)$ when $d = 0$ and $R_{I_{d,k}}(x) \geq R_{h_{a,b,m}}(x)$ when $d = \mathcal{E}_{h_{a,b,m}}^R$. Because $\mathbb{E}[R_{I_{d,k}}(X)]$ is a continuous function of d , there must exist $0 \leq \tilde{d} \leq \mathcal{E}_{h_{a,b,m}}^R$ such that $\mathbb{E}[R_{I_{\tilde{d},k}}(X)] = \mathbb{E}[R_{h_{a,b,m}}(X)]$. Thus, with $\pi = (1 + \theta)\mathbb{E}$, we have $\pi(I_{\tilde{d},k}(X)) = \pi(h_{a,b,m}(X))$. In addition, since

$$\begin{aligned} & \mathbb{E}[R_{I_{\tilde{d},k}}(X)] + \beta \mathbb{E}[(R_{I_{\tilde{d},k}}(X) - \mathcal{E}_{h_{a,b,m}}^R(X))_+] \\ &= \mathbb{E}[R_{h_{a,b,m}}(X)] + \beta \mathbb{E}[(R_{h_{a,b,m}}(X) - \mathcal{E}_{h_{a,b,m}}^R(X))_+] = \mathcal{E}_{h_{a,b,m}}^R(X), \end{aligned}$$

by (2.23), we have $\mathcal{E}_{I_{\tilde{d},k}}^R(X) = \mathcal{E}_{h_{a,b,m}}^R(X)$. Therefore, $L_{I_{\tilde{d},k}}(X) = \mathcal{E}_{I_{\tilde{d},k}}^R(X) + \pi(I_{\tilde{d},k}(X)) = \mathcal{E}_{h_{a,b,m}}^R(X) + \pi(h_{a,b,m}(X)) = L_{I_{h_{a,b,m}}}(X)$. Because $h_{a,b,m}$ is arbitrarily taken, $\min_{I \in \mathcal{I}_1} L_I(X) = \min_{I \in \mathcal{I}_2} L_I(X)$.

For $I_{d,k} \in \mathcal{I}_2$, define m such that $R_{I_{d,k}}(m) = \mathcal{E}_{I_{d,k}}^R(X)$. Then we can get $m \geq k$ and $m = k - d + \mathcal{E}_{I_{d,k}}^R$. The parameter m can be viewed as an implicit function of the other two parameters d and k . Because $\mathcal{E}_{I_{d,k}}^R(X) = \mathbb{E}[R_{I_{d,k}}(X)] + \beta \mathbb{E}[(R_{I_{d,k}}(X) - \mathcal{E}_{I_{d,k}}^R(X))_+] = \mathbb{E}[R_{I_{d,k}}(X)] + \beta \mathbb{E}[(X - m)_+]$, we can get

$$\begin{aligned} \frac{\partial m}{\partial d} &= -1 + \frac{\partial}{\partial d} \mathcal{E}_{I_{d,k}}^R(X) = -1 + \frac{\partial}{\partial d} \{ \mathbb{E}[R_{I_{d,k}}(X)] + \beta \mathbb{E}[(X - m)_+] \} = -1 + S_X(d) - \beta S_X(m) \frac{\partial m}{\partial d} \\ \frac{\partial m}{\partial k} &= 1 + \frac{\partial}{\partial k} \mathcal{E}_{I_{d,k}}^R(X) = 1 + \frac{\partial}{\partial k} \{ \mathbb{E}[R_{I_{d,k}}(X)] + \beta \mathbb{E}[(X - m)_+] \} = 1 - S_X(k) - \beta S_X(m) \frac{\partial m}{\partial k} \end{aligned}$$

which gives $\frac{\partial m}{\partial d} = \frac{-F_X(d)}{1 + \beta S_X(m)} < 0$ and $\frac{\partial m}{\partial k} = \frac{F_X(k)}{1 + \beta S_X(m)} > 0$. Furthermore,

$$L_{I_{d,k}}(X) = \mathcal{E}_{I_{d,k}}^R(X) + (1 + \theta)\mathbb{E}[I_{d,k}(X)] = m - k + d + (1 + \theta) \int_d^k S_X(t) dt.$$

Since $-1 \leq \theta \leq 0$, we have $\frac{\partial}{\partial d} L_{I_{d,k}}(X) = \frac{\partial m}{\partial d} + 1 - (1 + \theta)S_X(d) = \frac{\beta S_X(m)F_X(d)}{1 + \beta S_X(m)} - \theta S_X(d) \geq 0$, and $\frac{\partial}{\partial k} L_{I_{d,k}}(X) = \frac{\partial m}{\partial k} - 1 + (1 + \theta)S_X(k) = \frac{-\beta S_X(m)F_X(k)}{1 + \beta S_X(m)} + \theta S_X(k) \leq 0$, which means $L_{I_{d,k}}(X)$ increases with respect to d while decreases with respect to k . Then $L_{I_{d,m}}(X)$ reaches its minimum at $d = 0$ and $k = m$, indicating $\mathcal{E}_{I_{d,k}}^R = 0$ and $I_{[\mathcal{E}_{\alpha, (1+\theta)\mathbb{E}, X}]^*}(x) = x$ for $x \geq 0$. This completes the proof. $\square$

Chapter 3

Performance-based variable premium scheme and reinsurance design

3.1 Introduction

In the presenting work, we introduce a new performance-based premium scheme for reinsurance policies, referred to as the “reward-and-penalty” variable premium scheme. Inspired by the retrospective rating plan premium, the proposed scheme depends on the realized loss, making it a random variable for both the insurer and reinsurer at the beginning of the reinsurance policy period. As the policy seller, the reinsurer establishes the premium structure at the outset, such that the insurer pays a lower premium if the realized loss is small and conversely, a higher premium if a significant loss occurs. In other words, the actual premium paid by the insurer is contingent on their performance during the policy period. The proposed variable premium scheme may motivate the insurer to strategically choose the reinsurance policy to avoid a large amount of premium. Furthermore, the insurer’s cautions on choosing a reinsurance policy can also benefit the reinsurer through reducing the reinsurer’s risk exposure. Therefore, beyond analyzing the insurer’s response to this premium structure, it is equally valuable to investigate the reinsurer’s behavior under a retrospective rating plan. In summary, this work makes three key contributions. First, we propose a new variable premium scheme based on the performance of the insurable loss. Second, we examine the insurer’s optimal strategy when applying

this scheme. The introduction of additional randomness from the premium complicates the insurer’s optimization problem, making it more mathematically challenging. Lastly, we analyze the impact of this variable premium scheme on the reinsurer’s risk exposure.

It is worth noting that the insurer and the reinsurer often have conflicting interests in a reinsurance policy. One effective approach to addressing these conflicts is by investigating the Bowley optimum between the two parties. The Bowley optimum follows a two-step process. Initially, the policy buyer chooses an optimal indemnity function in accordance with the seller’s pricing rule to maximize their objective. Subsequently, the monopoly seller responds by selecting optimal pricing criteria to maximize their own objective, taking into account the anticipated reactions of the buyer. The Bowley optimum was first introduced into reinsurance by [Bühlmann \(1968\)](#), who derive the solution for one insurer and one reinsurer under quadratic utility assumptions. This work was later extended by [Gerber \(1984\)](#) and [Chan and Gerber \(1985\)](#). For decades, the Bowley solution remained unexplored until [Cheung et al. \(2019b\)](#) extend the foundational work by incorporating a general pricing rule and a comprehensive risk management tool. More recently, [Chi et al. \(2020\)](#) examine a scenario where the insurer is constrained by a two-moment condition set by the reinsurer, while [Li and Young \(2021\)](#) apply the Bowley solution to a mean-variance problem. In addition, [Boonen et al. \(2021a\)](#) and [Boonen and Zhang \(2022\)](#) revisit the Bowley problem with asymmetric information on the insurer’s distortion risk measure. [Ghossoub et al. \(2023\)](#) explore a subclass of convex premium principles, and [Boonen and Ghossoub \(2023\)](#) investigate the relationship between Bowley optimality and Pareto efficiency in optimal reinsurance problems. In this work, we propose a Bowley reinsurance problem incorporating a variable premium scheme. Through numerical analysis, we demonstrate that this variable premium can reduce the reinsurer’s total risk exposure compared to the expected-value premium principle, which is a constant premium scheme.

The following part is organized as follows. In Section [3.2](#), we introduce the reward-and-penalty variable premium scheme and discuss its properties. Section [3.3](#) examines the insurer’s optimization problem under the proposed scheme, providing a characterization of the optimal reinsurance strategy when the insurer uses either a general risk measure or a distortion risk measure to quantify total risk exposure. In Section [3.4](#), we establish a Bowley optimization problem between the insurer and the reinsurer, and numerically analyze the impact of the variable premium scheme from the reinsurer’s perspective. Finally, Section

3.5 presents concluding remarks.

3.2 Reward-and-penalty variable premium scheme

3.2.1 Notation

Let the insurable risk X has a finite mean and should be a pure loss, that is, $X \in L^1$ and $X \geq 0$ a.s. In most reinsurance design problems discussed in the literature, the insurer adopts a risk measure ρ to quantify her total risk exposure $\rho(R_I(X) + \pi(I(X)))$ for a given indemnity function $I \in \mathcal{I}$, and then considers the optimization problem

$$\min_{I \in \mathcal{I}} \rho(X - I(X) + \pi(I(X))). \quad (3.1)$$

The problem (3.1) and its variations have been intensively discussed in the insurance design literature. As this problem is framed from the insurer's perspective, the choice of the insurer's risk measure ρ has attracted significant attention. However, the choice of the reinsurance premium principle π , which represents the reinsurer's risk preference on quantifying her own part of loss, is a crucial component in the formulation of problem (3.1) as well. Properties of the reinsurance premium principle may significantly affect the optimal solution to this problem. Relevant studies can be found in [Young \(1999\)](#), [Kaluszke \(2001\)](#), [Bernard and Tian \(2009\)](#) and references therein.

It is important to note that the reinsurer can leverage the reinsurance premium to influence the insurer's optimal strategy, potentially encouraging more favorable behavior. This influence becomes more pronounced when the reinsurer has greater flexibility to control the premium amount. In other words, transitioning from a constant premium to a variable premium scheme introduces additional randomness, further amplifying the reinsurer's ability to affect the insurer's decisions.

3.2.2 Reward-and-penalty variable premium scheme

Inspired by the rating plan presented in [Meyers \(2004\)](#) and [Chen et al. \(2016\)](#), we propose a “reward-and-penalty” variable premium scheme for the reinsurer. This scheme

defines the premium as a random variable at the outset of the policy period, with its realized value dependent on the insurer's performance. For a given ceded loss $Y \in L^1$, the reinsurer first establishes a benchmark for the premium scheme using the expected-value premium principle. The reinsurer selects a positive risk loading $\theta_0 \geq 0$ and determines the “*benchmark premium*” $\pi_0(Y) = (1 + \theta_0)\mathbb{E}[Y]$. In the second step, the reinsurer implements a performance-based premium plan based on the realized loss amount, $Y = y$. If $y \leq \mathbb{E}[Y]$, this outcome represents a “reward scenario”, and the reinsurer provides a discount on the premium. On the contrary, if $y > \mathbb{E}[Y]$, it triggers a “penalty scenario”, where the insurer needs to pay an extra penalization beyond the benchmark premium. The power of this “reward-and-penalty” premium scheme is controlled by a scheme parameter δ , determined by the reinsurer. Mathematically, the actual premium paid is expressed as $\pi_0(Y) + \delta(y - \mathbb{E}[Y])$ with $\delta \in [0, 1]$. This scheme parameter δ also signifies the increasing rate of the premium. Finally, the reinsurer determines the lower and upper bounds for the premium. To maintain consistent with the benchmark, the reinsurer applies the expected-value premium principle again. Two positive risk loadings, $\theta_1 \in [0, \theta_0]$ and $\theta_2 \in [\theta_0, \infty]$, are selected to calculate the lower and upper bounds of the premium, given by $\pi_1(Y) = (1 + \theta_1)\mathbb{E}[Y]$ and $\pi_2(Y) = (1 + \theta_2)\mathbb{E}[Y]$, respectively. If the actual premium, $\pi_0(Y) + \delta(y - \mathbb{E}[Y])$, falls below the lower bound $\pi_1(Y)$, the reinsurer charges the minimum payable premium $\pi_1(Y)$. Conversely, if the actual premium exceeds the upper bound $\pi_2(Y)$, the premium is capped at the maximum payable premium $\pi_2(Y)$.

Definition 3.2.1 (Performance-based variable premium scheme). Given a non-negative random loss Y with finite mean, the “reward-and-penalty” variable premium scheme on Y is structured as follows

$$\begin{aligned}\pi_Y(y) &\triangleq \min \{ \max \{ \pi_0(Y) + \delta(y - \mathbb{E}[Y]), \pi_1(Y) \}, \pi_2(Y) \} \\ &= \min \{ \max \{ (1 + \theta_0)\mathbb{E}[Y] + \delta(y - \mathbb{E}[Y]), (1 + \theta_1)\mathbb{E}[Y] \}, (1 + \theta_2)\mathbb{E}[Y] \},\end{aligned}\quad (3.2)$$

where $\delta \in [0, 1]$ is the *scheme parameter*, and $\pi_i(Y) = (1 + \theta_i)\mathbb{E}[Y]$ with risk loadings $\theta_i \geq 0$, $i = 0, 1, 2$.

For a given Y , values of $\pi_i(Y)$, $i = 0, 1, 2$, are based on the expected-value premium principle, and serve as key thresholds in the variable premium scheme. Therefore, π_Y in (3.2) can be viewed as a tailor-made premium scheme function for the loss Y , with $\pi_Y(Y)$

being a random variable influenced by the distribution of Y , the scheme parameter δ , and risk loading parameters θ_i , $i = 0, 1, 2$. The proposed variable premium scheme in (3.2) can reduce to the standard expected-value premium principle under specific parameter settings. To demonstrate this, we can take either $\delta = 0$ or $\theta_0 = \theta_1 = \theta_2$, which results in $\pi_Y(y) = (1 + \theta_0)\mathbb{E}[Y]$ for all $y \geq 0$. To avoid redundant explanations, we assume $\delta > 0$ and $\theta_1 \leq \theta_0 < \theta_2$ in the following sections.

Remark 3. [Chen et al. \(2016\)](#) is a pioneering work that studies retrospective rating premium. They assume that the minimum premium is zero and formulate the premium scheme as

$$\pi_{\text{RE}}(Y) = \min \{B + L \cdot Y, (1 + \theta_{\text{max}}\mathbb{E}[Y])\}, \quad (3.3)$$

where $\theta_{\text{max}} \geq 0$, and $B = B(Y)$, called as “basic premium”, is a free variable depending on Y which is determined by solving $\mathbb{E}[\pi_{\text{RE}}(Y)] = (1 + \theta_{\text{RE}})\mathbb{E}[Y]$ for some $\theta_{\text{RE}} \geq 0$. In our work, the variable premium defined in (3.2) can also be written in the form of (3.3) by taking $B = \pi_0(Y) - \delta \cdot \mathbb{E}[Y]$ and $\pi_1(Y) = 0$. However, our work assumes that the benchmark premium $\pi_0(Y)$ is directly given by the expected-value premium principle, and consequently, we have $\pi_Y(\mathbb{E}[Y]) = (1 + \theta_0)\mathbb{E}[Y]$, $\theta_0 \geq 0$. In other words, the value of B in [Chen et al. \(2016\)](#) may vary for different choices of Y with the same expectation. In contrast, within our framework, $\pi_0(Y)$ remains fixed and the form of the premium does not change as long as the expectation of Y is specified.

Intuitively, when a risk measure π is used as a reinsurance premium principle, it reflects the reinsurer’s risk attitude toward the ceded loss. [Young \(2004\)](#) discusses the desirable properties of premium principles and lists popular ones used in the literature. In this work, we focus on the variable premium scheme, where $\pi_Y(Y)$ is a random variable. Although the properties of risk measures cannot be directly applied to $\pi_Y(Y)$, we can still discuss several desirable properties satisfied by this variable premium scheme.

Proposition 3.2.1. *Assume $\delta \in (0, 1]$ and $0 \leq \theta_1 \leq \theta_0 < \theta_2$ are given, and $Y \in L^1$ is a non-negative random loss. The reward-and-penalty variable premium scheme defined in (3.2) satisfies the following properties.*

- (i) *The distribution of $\pi_Y(Y)$ depends only on the distribution of Y .*
- (ii) *$\pi_Y(Y) \geq \mathbb{E}[Y]$.*

(iii) If $(1 + \theta_2)\mathbb{E}[Y] \leq \text{ess-sup}(Y)$, then $\pi_Y(Y) \leq \text{ess-sup}(Y)$.

(iv) $\pi_{c \cdot Y}(c \cdot Y) = c \cdot \pi_Y(Y)$ for all $c \geq 0$.

(v) Let Y_1 and Y_2 be two non-negative random losses. If Y_1 is smaller than Y_2 in the sense of first-order stochastic dominance (FSD), denoted by $Y_1 \preceq_{FSD} Y_2$,¹ then $\pi_{Y_1}(Y_1) \preceq_{FSD} \pi_{Y_2}(Y_2)$.

(vi) Functions $\pi_Y(y)$ and $y - \pi_Y(y)$ are continuous and non-decreasing in $y \geq 0$, and moreover, random variables Y , $\pi_Y(Y)$ and $Y - \pi_Y(Y)$ are comonotonic.

Proof. (i) Since $\pi_i(Y) = (1 + \theta_i)\mathbb{E}[Y]$, $i = 0, 1, 2$, are law-invariant, it is easy to see from (3.2) that the function $\pi_Y(y)$ is determined by the distribution of Y . Hence, the distribution of $\pi_Y(Y)$ depends only on the distribution of Y .

(ii) Since $\theta_1 \geq 0$, the equation (3.2) implies

$$\pi_Y(y) = \begin{cases} (1 + \theta_1)\mathbb{E}[Y], & \text{if } Y \leq \left(\frac{\theta_1 - \theta_0 + \delta}{\delta}\mathbb{E}[Y]\right)_+, \\ (1 + \theta_0)\mathbb{E}[Y] + \delta(y - \mathbb{E}[Y]), & \text{if } \left(\frac{\theta_1 - \theta_0 + \delta}{\delta}\mathbb{E}[Y]\right)_+ < Y < \frac{\theta_2 - \theta_0 + \delta}{\delta}\mathbb{E}[Y], \\ (1 + \theta_2)\mathbb{E}[Y], & \text{if } y \geq \frac{\theta_2 - \theta_0 + \delta}{\delta}\mathbb{E}[Y]. \end{cases} \quad (3.4)$$

It then follows that $\pi_Y(Y) \geq (1 + \theta_1)\mathbb{E}[Y] \geq \mathbb{E}[Y]$.

(iii) If $(1 + \theta_2)\mathbb{E}[Y] \leq \text{ess-sup}(Y)$, then (3.4) implies $\pi_Y(Y) \leq (1 + \theta_2)\mathbb{E}[Y] \leq \text{ess-sup}(Y)$.

(iv) Take a constant $c \geq 0$. It is easy to check that

$$\begin{aligned} \pi_{cY}(cY) &= \min \{ \max \{ (1 + \theta_0)\mathbb{E}[cY] + \delta(cY - \mathbb{E}[cY]), (1 + \theta_1)\mathbb{E}[cY] \}, (1 + \theta_2)\mathbb{E}[cY] \} \\ &= \min \{ c \cdot \max \{ (1 + \theta_0)\mathbb{E}[Y] + \delta(Y - \mathbb{E}[Y]), (1 + \theta_1)\mathbb{E}[Y] \}, c(1 + \theta_2)\mathbb{E}[Y] \} \\ &= c \min \{ \max \{ (1 + \theta_0)\mathbb{E}[Y] + \delta(Y - \mathbb{E}[Y]), (1 + \theta_1)\mathbb{E}[Y] \}, (1 + \theta_2)\mathbb{E}[Y] \} \\ &= c \cdot \pi_Y(Y). \end{aligned}$$

¹Given two random losses Y_1 and Y_2 , the loss Y_1 is said to be smaller than Y_2 in the sense of first-order stochastic dominance, denoted by $Y_1 \preceq_{FSD} Y_2$, if $F_{Y_1}^{-1}(p) \leq F_{Y_2}^{-1}(p)$ for all $p \in [0, 1]$.

(v) Note that $Y_1 \preceq_{FSD} Y_2$ implies $\Pr(Y_1 \leq z) \geq \Pr(Y_2 \geq z)$ for all $z \geq 0$, and thus $\mathbb{E}[Y_1] \leq \mathbb{E}[Y_2]$. The premium $\pi_{Y_1}(Y_1)$ and $\pi_{Y_2}(Y_2)$ are non-negative random variables structured as in (3.4).

(a) For $0 \leq z < (1 + \theta_1)\mathbb{E}[Y_2]$, if any, we have $\Pr(\pi_{Y_2}(Y_2) \leq z) = 0 \leq \Pr(\pi_{Y_1}(Y_1) \leq z)$.

(b) For $(1 + \theta_1)\mathbb{E}[Y_2] \leq z < (1 + \theta_2)\mathbb{E}[Y_1]$, if any, we have

$$\begin{aligned} \Pr(\pi_{Y_2}(Y_2) \leq z) &= \Pr((1 + \theta_0)\mathbb{E}[Y_2] + \delta(Y_2 - \mathbb{E}[Y_2]) \leq z) \\ &= \Pr\left(Y_2 \leq \frac{1}{\delta}(z - (1 + \theta_0 - \delta)\mathbb{E}[Y_2])\right) \\ &\leq \Pr\left(Y_2 \leq \frac{1}{\delta}(z - (1 + \theta_0 - \delta)\mathbb{E}[Y_1])\right) \\ &\leq \Pr\left(Y_1 \leq \frac{1}{\delta}(z - (1 + \theta_0 - \delta)\mathbb{E}[Y_1])\right) = \Pr(\pi_{Y_1}(Y_1) \leq z) \end{aligned}$$

(c) For $z \geq (1 + \theta_2)\mathbb{E}[Y_1]$, we have $\Pr(\pi_{Y_2}(Y_2) \leq z) \leq 1 = \Pr(\pi_{Y_1}(Y_1) \leq z)$.

Combining three cases, we conclude that $\Pr(\pi_{Y_2}(Y_2) \leq z) \leq \Pr(\pi_{Y_1}(Y_1) \leq z)$ for all $z \geq 0$. It follows that $\pi_{Y_1}(Y_1) \preceq_{FSD} \pi_{Y_2}(Y_2)$.

(vi) From equation (3.4), since $\delta \in (0, 1]$, it follows that both $\pi_Y(y)$ and $y - \pi_Y(y)$ are continuous and non-decreasing functions of y . Thus, Y , $\pi_Y(Y)$ and $Y - \pi_Y(Y)$ are comonotonic. $\square$

We can compare the properties outlined in Proposition 3.2.1 with those properties commonly associated with risk measures for premium principles. For a comprehensive discussion on the economic interpretation of these properties, see Young (2004).

Proposition 3.2.1	Properties for a premium principle π
Property (1)	“Law-invariance”: $\pi(Y_1) = \pi(Y_2)$ if $Y_1 \stackrel{d}{=} Y_2$
Property (2)	“Risk loading”: $\pi(Y) \geq \mathbb{E}[Y]$ for all Y
Property (3)	“No rip-off”: $\pi(Y) \leq \text{ess-sup} Y$ for all Y
Property (4)	“Homogeneity”: $\pi(cY) = c\pi(Y)$ for all Y and constant $c > 0$
Property (5)	“Preserving FSD”: $\pi(Y_1) \leq \pi(Y_2)$ if $Y_1 \preceq_{FSD} Y_2$

The property (6) in Proposition 3.2.1 further implies that functions $\pi_Y(y)$ and $y - \pi_Y(y)$ exhibit 1-Lipschitz continuity. This aligns with the condition for a feasible indemnity function as defined in (1.5). Suppose that the distribution of the underlying loss X faced by the insurer is known in the optimal reinsurance problem, then the distribution of the reinsurer's ceded loss $I(X)$ is determined once the indemnity function $I \in \mathcal{I}$ is specified. The 1-Lipschitz continuity condition on I ensures that X , $I(X)$ and $X - I(X)$ are monotonic random variables. This comonotonic property, also referred to as the “no-sabotage” condition, helps to mitigate moral hazard issues in the reinsurance policy. When the reinsurer applies the reward-and-penalty variable premium scheme in (3.2), the premium variable then becomes

$$\Pi_I(X) \triangleq \pi_{I(X)}(I(X)) = \min \{ \max \{ \pi_0(I(X)) + \delta(I(X) - \mathbb{E}[I(X)]), \pi_1(I(X)) \}, \pi_2(I(X)) \},$$

where $\pi_i(I(X)) = (1 + \theta_i)\mathbb{E}[I(X)]$, $i = 0, 1, 2$. The insurer and reinsurer's total losses become $X - I(X) + \Pi_I(X)$ and $I(X) - \Pi_I(X)$, respectively. Since $I \in \mathcal{I}$ ensures that X , $I(X)$, and $X - I(X)$ are comonotonic, Property (6) in Proposition 3.2.1 further implies that X , $X - I(X) + \Pi_I(X)$ and $I(X) - \Pi_I(X)$ are comonotonic random variables. As a result, the “no-sabotage” condition continues to hold.

In the remainder of the paper, we use the variable premium scheme to formulate optimization problems, as specified in the following assumption.

Assumption 3. Assume $\delta \in (0, 1]$, $(\theta_0 - \delta) \vee 0 \leq \theta_1 \leq \theta_0 < \theta_2 < \infty$, and the non-negative underlying loss $X \in L^1$ is given. For any indemnity function $I \in \mathcal{I}$, the reinsurer employs the following “reward-and-penalty” variable premium scheme for the ceded loss $I(X)$:

$$\begin{aligned} \Pi_I(x) &= \min \{ \max \{ \pi_0(I(X)) + \delta(I(x) - \mathbb{E}[I(X)]), \pi_1(I(X)) \}, \pi_2(I(X)) \} \\ &= \begin{cases} (1 + \theta_1)\mathbb{E}[I(X)], & \text{if } 0 \leq I(x) \leq d_I, \\ (1 + \theta_0)\mathbb{E}[I(X)] + \delta(I(x) - \mathbb{E}[I(X)]), & \text{if } d_I \leq I(x) \leq u_I, \\ (1 + \theta_2)\mathbb{E}[I(X)], & \text{if } I(x) \geq u_I, \end{cases} \end{aligned} \quad (3.5)$$

where $d_I = \frac{\theta_1 - \theta_0 + \delta}{\delta} \mathbb{E}[I(X)]$, $u_I = \frac{\theta_2 - \theta_0 + \delta}{\delta} \mathbb{E}[I(X)]$, and $\pi_i(I(X)) = (1 + \theta_i)\mathbb{E}[I(X)]$ for $i = 0, 1, 2$.

The condition $(\theta_0 - \delta) \vee 0 \leq \theta_1$ in Assumption 3 ensures that $d_I \geq 0$ to prevent trivial cases where the lower threshold for the premium calculation would be negative or meaningless.

3.3 Insurer's optimal reinsurance strategy

In this section, we solve the insurer's optimization problem:

$$\min_{I \in \mathcal{I}} \rho(T_I(X)), \quad \text{where } T_I(X) \triangleq R_I(X) + \Pi_I(X) \text{ for any } I \in \mathcal{I}. \quad (3.6)$$

Under Assumption 3, the premium $\Pi_I(X)$ is again a random variable. Since the amount of the premium paid depends on the realized indemnity payment amount, which cannot be observed at the beginning of the policy period, the insurer needs to consider the influence of a given indemnity function I and the distribution of X on the variable premium. In addition to the variable premium scheme, the choice of the risk measure ρ significantly influences the solution to the problem (3.6). To begin, we impose mild conditions on ρ and characterize the optimal reinsurance policy for the insurer. We then focus on a distortion risk measure ρ , which enables us to further simplify the structure of solutions to (3.6).

3.3.1 Optimal reinsurance design with a general risk measure

This section presents the main result of the paper, where the infinite-dimensional problem (3.6) is reduced to a finite-dimension optimization problem, provided that the risk measure ρ meets the following two conditions:

- (i) Law-invariance: $\rho(X) = \rho(Y)$ if $X \stackrel{d}{=} Y$;
- (ii) Preserving the convex order²: $\rho(X) \leq \rho(Y)$ if $X \preceq_{cx} Y$.

To proceed, define

$$\tilde{\mathcal{I}} = \mathcal{I}_1 \cup \mathcal{I}_2, \quad (3.7)$$

where

$$\mathcal{I}_1 = \left\{ I(x) = (x - d_1)_+ - (x - d_1 - d_I)_+ + (x - d_2)_+ \left| \begin{array}{l} d_I = \frac{\theta_1 - \theta_0 + \delta}{\delta} \mathbb{E}[I(X)] \\ \text{and } 0 \leq d_1 \leq d_1 + d_I \leq d_2 \leq \infty \end{array} \right. \right\},$$

²Given two random losses Y_1 and Y_2 , the loss Y_1 is said to be smaller than Y_2 in the convex order, denoted by $Y_1 \preceq_{cx} Y_2$, if $\mathbb{E}[h(Y_1)] \leq \mathbb{E}[h(Y_2)]$ for all convex functions h such that the expectations exist.

and

$$\mathcal{I}_2 = \left\{ I(x) = (x - d_1)_+ - (x - d_1 - u_I)_+ + (x - d_2)_+ \left| \begin{array}{l} u_I = \frac{\theta_2 - \theta_0 + \delta}{\delta} \mathbb{E}[I(X)] \\ \text{and } 0 \leq d_1 \leq d_1 + u_I \leq d_2 \leq \infty \end{array} \right. \right\}.$$

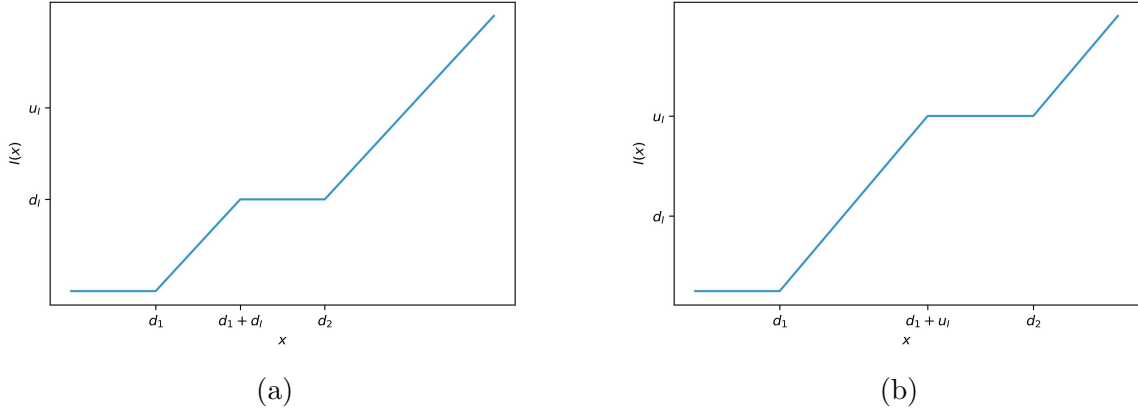


Figure 3.1: Indemnity functions $I(x)$, where (a) $I(x) \in \mathcal{I}_1$, or (b) $I(x) \in \mathcal{I}_2$.

Note that stop-loss functions belong to $\mathcal{I}_1$ and $\mathcal{I}_2$. To see this, we first fix a real value a such that $0 \leq a \leq \mathbb{E}[X]$. There exists $\tilde{d} \geq 0$ such that $\mathbb{E}[(X - \tilde{d})_+] = \int_{\tilde{d}}^{\infty} S(x) dx = a$. Then we can take $d_1 = \tilde{d}$, $d_2 = d_1 + \frac{\theta_1 - \theta_0 + \delta}{\delta} a = d_1 + d_I$, and it is easy to see

$$I(x) = (x - d_1)_+ - (x - d_1 - d_I)_+ + (x - d_2)_+ = (x - d_1)_+ = (x - \tilde{d})_+ \in \mathcal{I}_1.$$

Therefore, $\mathcal{I}_1 \neq \emptyset$. Similarly, we can argue that $(x - \tilde{d})_+ \in \mathcal{I}_2$ and $\mathcal{I}_2 \neq \emptyset$. Consequently, the set $\tilde{\mathcal{I}}$ is well-defined.

The shapes of the indemnity functions in $\mathcal{I}_1$ and $\mathcal{I}_2$ are illustrated in Figure 3.1. The elements in $\tilde{\mathcal{I}}$ represents two-layer insurance policies. We will demonstrate that the optimal indemnity function can be obtained within the set $\tilde{\mathcal{I}}$. To proceed, we first take an intermediate step by verifying that the optimal solution to problem (3.6) belongs to the following three-layer policy set.

$$\begin{aligned} \mathcal{S}_3 = \left\{ f(x) = (x - a)_+ - (x - b)_+ + (x - c)_+ - (x - d)_+ + (x - e)_+, x \geq 0, \right. \\ \left. \text{where } 0 \leq a \leq b = a + d_I \leq c \leq d = c + u_I - d_I \leq e \right\}. \end{aligned} \quad (3.8)$$

Lemma 3.3.1. *Let Assumption 3 hold. Assume that ρ is a law-invariant risk measure that preserves the convex order. For any $I \in \mathcal{I}$, there exists $f_I \in \mathcal{S}_3$ such that $\rho(T_{f_I}(X)) \leq \rho(T_I(X))$. Furthermore, solving problem (3.6) reduces to solving the following problem*

$$\min_{I \in \mathcal{S}_3} \rho(T_I(X)). \quad (3.9)$$

The proof of Lemma 3.3.1 is provided in 3.6. This lemma enables us to focus on the optimal policy within the three-layer policies defined in $\mathcal{S}_3$. Utilizing this result, we can further refine the search for optimal policies. The proof of the following theorem is also presented in 3.6.

Theorem 3.3.1. *Let Assumption 3 hold. Assume that ρ is a law-invariant risk measure that preserves the convex order. For any $I \in \mathcal{I}$, there exists $f_I \in \tilde{\mathcal{I}}$ such that $\rho(T_{f_I}(X)) \leq \rho(T_I(X))$. Furthermore, to solve problem (3.6), it is sufficient to solve the following problem:*

$$\min_{I \in \tilde{\mathcal{I}}} \rho(T_I(X)). \quad (3.10)$$

By Theorem 3.3.1, the problem (3.6) is equivalent to problem (3.10), and the optimal indemnify function belongs to $\tilde{\mathcal{I}}$, which is indexed by a finite number of parameters.

It is worth highlighting that Theorem 3.3.1 requires nothing but law-invariance and preservation of convex order on the risk measure ρ . These assumptions encompass a wide range of risk measures. However, without specifying the exact form of ρ , identifying the optimal solution within $\tilde{\mathcal{I}}$ becomes mathematically challenging. To further investigate the insurer's optimal solution, we impose additional conditions on the risk measure ρ in the following section.

3.3.2 Optimal reinsurance design with a distortion risk measure

In this section, we consider the family of distortion risk measures. Given a confidence level $\alpha \in (0, 1)$, VaR_α is defined as $\text{VaR}_\alpha(X) = F_X^{-1}(1 - \alpha)$, $X \in L^0$, with the distortion function $g(p) = \mathbf{1}_{(\alpha, 1]}(p)$; and TVaR_α is defined as

$$\text{TVaR}_\alpha(X) = \frac{1}{\alpha} \int_0^\alpha \text{VaR}_u(X) \, du = \int_0^\infty g(S_X(x)) \, dx, \quad X \in L^1, \quad (3.11)$$

with the distortion function

$$g(p) = \begin{cases} \frac{p}{\alpha}, & p \leq \alpha, \\ 1, & p > \alpha. \end{cases} \quad (3.12)$$

It is well-known that TVaR_α is a coherent risk measure with a concave distortion function $g(p)$ in (3.12), whereas VaR_α is not coherent with a non-concave distortion function $\mathbf{1}_{(\alpha,1]}$. According to Theorem 3 of Wang et al. (2020a), a distortion risk measure ρ^g is convex order consistent if and only if it is convex,³ which is further equivalent to the concavity of its distortion function g . Additionally, by its definition in (1.8), ρ^g is law-invariant and homogeneous. Therefore, taking a distortion risk measure ρ^g that is convex order consistent is equivalent to assuming that ρ^g is coherent.

The family of distortion risk measures is widely applied in the study of insurance and reinsurance optimization problems. It encompasses popular risk measures, such as VaR and TVaR, which are used to quantify the insurer/reinsurer's risk exposure, as well as premium principles like the expected-value premium principle and Wang's premium principle. Moreover, its expression in (1.8) provides a useful mathematical framework for tracking optimal solutions.

Theorem 3.3.2. *Let $\rho = \rho^g$ be a coherent distortion risk measure and Assumption 3 hold. Then the optimal indemnity function I^* of problem $\min_{I \in \tilde{\mathcal{I}}} \rho(T_I(X))$ is contained in the set $\mathcal{I}_1$. That is,*

$$\min_{I \in \tilde{\mathcal{I}}} \rho(T_I(X)) = \min_{I \in \mathcal{I}_1} \rho(T_I(X)). \quad (3.13)$$

Proof. Let $I(x) = (x - d_1)_+ - (x - d_1 - u_I)_+ + (x - d_2)_+$ with $0 \leq d_1 \leq d_1 + u_I \leq d_2$, which is an element in the policy set $\mathcal{I}_2$. Given $\mathbb{E}[I(X)] = a$, then $d_I = \frac{\theta_1 - \theta_0 + \delta}{\delta}a$ and $u_I = \frac{\theta_2 - \theta_0 + \delta}{\delta}a$. By (3.5), the survival function of $\Pi_I(X)$ is given by

$$S_{\Pi_I(X)}(x) = \begin{cases} 1, & x < (1 + \theta_1)a, \\ S_X\left(\frac{x}{\delta} - \frac{a(1 + \theta_0 - \delta)}{\delta} + d_1\right), & (1 + \theta_1)a \leq x < (1 + \theta_2)a, \\ 0, & x \geq (1 + \theta_2)a, \end{cases} \quad (3.14)$$

³A risk measure ρ satisfies the convexity property if $\rho(\lambda X + (1 - \lambda)Y) \leq \lambda \rho(X) + (1 - \lambda)\rho(Y)$ for any random variables X and Y and constant $\lambda \in [0, 1]$.

from which we can obtain that

$$\rho^g(\Pi_I(X)) = \int_0^\infty g(S_{\Pi_I(X)}(x)) dx = (1 + \theta_1)a + \delta \int_{d_1+d_I}^{d_1+u_I} g(S_X(x)) dx.$$

In addition, $\rho^g(R_I(X)) = \int_0^{d_1} g(S_X(x)) dx + \int_{d_1+u_I}^{d_2} g(S_X(x)) dx$. Note that $R_I(x)$ and $\Pi_I(x)$ are both increasing functions of x , implying that the retained loss $R_I(X)$ and the variable premium scheme $\Pi_I(X)$ are comonotonic random variables. By the comonotonic additivity property of ρ^g , we have

$$\begin{aligned} \rho^g(T_I(X)) &= \rho^g(R_I(X)) + \rho^g(\Pi_I(X)) \\ &= \int_0^{d_1} g(S_X(x)) dx + \int_{d_1+u_I}^{d_2} g(S_X(x)) dx + \delta \int_{d_1+d_I}^{d_1+u_I} g(S_X(x)) dx + (1 + \theta_1)a. \end{aligned}$$

It then follows that

$$\begin{cases} \frac{\partial \rho^g(T_I(X))}{\partial d_1} = g(S_X(d_1)) - \delta g(S_X(d_1 + d_I)) - (1 - \delta)g(S_X(d_1 + u_I)), \\ \frac{\partial \rho^g(T_I(X))}{\partial d_2} = g(S_X(d_2)). \end{cases}$$

Since $\mathbb{E}[I(X)] = \int_{d_1}^{d_1+u_I} S_X(x) dx + \int_{d_2}^\infty S_X(x) dx$, by Implicit Function Theorem, we have

$$\frac{dd_2}{dd_1} = -\frac{\partial \mathbb{E}[I(X)] / \partial d_1}{\partial \mathbb{E}[I(X)] / \partial d_2} = \frac{S_X(d_1 + u_I) - S_X(d_1)}{S_X(d_2)}.$$

Thus, we can get

$$\begin{aligned} \frac{d}{dd_1} \rho^g(T_I(X)) &= \frac{\partial}{\partial d_1} \rho^g(T_I(X)) + \frac{dd_2}{dd_1} \left(\frac{\partial}{\partial d_2} \rho^g(T_I(X)) \right) \\ &= g(S_X(d_1)) - \delta g(S_X(d_1 + d_I)) - (1 - \delta)g(S_X(d_1 + u_I)) \\ &\quad + g(S_X(d_2)) \frac{S_X(d_1 + u_I) - S_X(d_1)}{S_X(d_2)} \\ &\leq [S_X(d_1) - S_X(d_1 + u_I)] \left[\frac{g(S_X(d_1)) - g(S_X(d_1 + u_I))}{S_X(d_1) - S_X(d_1 + u_I)} - \frac{g(S_X(d_2))}{S_X(d_2)} \right] \\ &\leq 0. \end{aligned} \tag{3.15}$$

Here, the first inequality follows from the fact that $g(S_X(d_1 + d_I)) \geq g(S_X(d_1 + u_I))$, while the second inequality is the result of the concavity of g . Thus, $\rho^g(T_I(X))$ is a decreasing

function of d_1 . Thus, the minimum value of $\rho^g(T_I(X))$ is achieved when $d_1 = d_2 - u_I$. This indicates that the stop-loss policy $I^*(x) = (x - d_1^*)_+$ is the optimal solution to the problem $\min_{I \in \mathcal{I}_2, \mathbb{E}[I(X)] = a} \rho(T_I(X))$, with the optimal parameter d_1^* determined by the condition $\mathbb{E}[I^*(X)] = a$. Noting that stop-loss functions are also elements of $\mathcal{I}_1$, it follows that, for a given a , $\min_{I \in \mathcal{I}_1, \mathbb{E}[I(X)] = a} \rho(T_I(X))$ is no larger than $\min_{I \in \mathcal{I}_2, \mathbb{E}[I(X)] = a} \rho(T_I(X))$. Therefore, we conclude that (3.13) holds, which completes the proof. $\square$

Recall from Theorem 3.3.1 that the range of the optimal solution to problem (3.1) is reduced to the set $\tilde{\mathcal{I}}$. By its definition in (3.7), there are two types of reinsurance structure, denoted as $\mathcal{I}_1$ and $\mathcal{I}_2$. Theorem 3.3.2 further refines the range of optimal solutions from $\tilde{\mathcal{I}}$ to $\mathcal{I}_1$ when a coherent distortion risk measure is employed. According to (3.7), any $I \in \mathcal{I}_1$ takes the form $I(x) = (x - d_1)_+ - (x - d_1 - d_I)_+ + (x - d_2)_+$, where $d_I = \frac{\theta_1 - \theta_0 + \delta}{\delta} \mathbb{E}[I(X)]$ and $0 \leq d_1 \leq d_1 + d_I \leq d_2 \leq \infty$. Mathematically, the set $\mathcal{I}_1$ can be partitioned into layers based on the expected value $\mathbb{E}[I(X)]$, i.e., $\mathcal{I}_1 = \bigcup_{0 \leq a \leq \mathbb{E}[X]} \{I \in \mathcal{I}_1 : \mathbb{E}[I(X)] = a\}$. Within each layer, the expected value $\mathbb{E}[I(X)] = a$ is fixed. Consequently, the parameter d_I remains constant for all indemnity functions within this layer. Meanwhile, the values of d_1 and d_2 are restricted by the expected value such that $\mathbb{E}[I(X)] = \left(\int_{d_1}^{d_1 + d_I} + \int_{d_2}^{\infty} \right) S_X(x) dx = a$. Therefore, each layer $\{I \in \mathcal{I}_1 : \mathbb{E}[I(X)] = a\}$ only has one free parameter. Consequently, the problem (3.13) can be reformulated as a two-step minimization problem

$$\min_{0 \leq a \leq \mathbb{E}[X]} \left\{ \min_{I \in \mathcal{I}_1, \mathbb{E}[I(X)] = a} \rho(T_I(X)) \right\},$$

where each inner minimization problem involves a single variable, making it feasible to be solved numerically.

In the following proposition, we take $\rho = \text{TVaR}_\alpha$ in the above problem and derive the closed-form solution to the inner optimization problem. Specifically, we solve the following problem:

$$\min_{I \in \mathcal{I}_1, \mathbb{E}[I(X)] = a} \text{TVaR}_\alpha(T_I(X)). \quad (3.16)$$

Proposition 3.3.3. *Let $\rho = \text{TVaR}_\alpha$ for some $0 < \alpha < 1$, $0 < a \leq \mathbb{E}[X]$, and Assumption 3 hold. Define $d_I = a(\theta_1 - \theta_0 + \delta)/\delta$, $u_I = a(\theta_2 - \theta_0 + \delta)/\delta$, and let $\tilde{d}$ be the solution to $\mathbb{E}[(X - d)_+] = a$. If $\frac{S_X(d + u_I - d_I)}{S_X(d)}$ is an increasing function of d , then the optimal indemnity*

function for problem (3.16) is given by

$$I^*(x) = \begin{cases} (x - \bar{d}_1)_+ - (x - \bar{d}_1 - d_I)_+ + (x - \bar{d}_2)_+, & \text{if } \bar{d}_1 < \tilde{d}, \\ (x - \tilde{d})_+, & \text{if } \bar{d}_1 \geq \tilde{d}, \end{cases} \quad (3.17)$$

where the pair $(\bar{d}_1, \bar{d}_2)$ is the solution to the following equation system

$$\begin{cases} \frac{\alpha - S_X(d_1 + d_I)}{S_X(d_1) - S_X(d_1 + d_I)} - \delta \frac{S_X(d_2 + u_I - d_I)}{S_X(d_2)} + \delta - 1 = 0, \\ \int_{d_1}^{d_1 + d_I} S_X(x) dx + \int_{d_2}^{\infty} S_X(x) dx = a. \end{cases} \quad (3.18)$$

Proof. Let $I(x) = (x - d_1)_+ - (x - d_1 - d_I)_+ + (x - d_2)_+$ with $0 \leq d_1 \leq d_1 + d_I \leq d_2$, which is an element in the policy set $\mathcal{I}_1$. The domain of d_1 is clearly $[0, \tilde{d}]$. Given that $\mathbb{E}[I(X)] = a$, we have $d_I = \frac{\theta_1 - \theta_0 + \delta}{\delta} a$ and $u_I = \frac{\theta_2 - \theta_0 + \delta}{\delta} a$.

Since $\mathbb{E}[I(X)] = \int_{d_1}^{d_1 + d_I} S_X(x) dx + \int_{d_2}^{\infty} S_X(x) dx = a$, by Implicit Function Theorem, d_2 can be treated as an implicit function of d_1 , and

$$\frac{dd_2}{dd_1} = - \frac{\partial \mathbb{E}[I(X)] / \partial d_1}{\partial \mathbb{E}[I(X)] / \partial d_2} = \frac{S_X(d_1 + d_I) - S_X(d_1)}{S_X(d_2)}. \quad (3.19)$$

By (3.5), the survival function of $\Pi_I(X)$ is formulated as

$$S_{\Pi_I(X)}(x) = \begin{cases} 1, & x < (1 + \theta_1)a, \\ S_X\left(\frac{x}{\delta} - \frac{a(1 + \theta_1)}{\delta} + d_2\right), & (1 + \theta_1)a \leq x < (1 + \theta_2)a, \\ 0, & x \geq (1 + \theta_2)a. \end{cases} \quad (3.20)$$

Denote g the distortion function of TVaR_α , as defined in (3.12). Using (3.11) and (3.20), we obtain

$$\text{TVaR}_\alpha(\Pi_I(X)) = \int_0^\infty g(S_{\Pi_I(X)}(x)) dx = (1 + \theta_1)a + \delta \int_{d_2}^{d_2 + u_I - d_I} g(S_X(x)) dx.$$

In addition, we have $\text{TVaR}_\alpha(R_I(X)) = \int_0^{d_1} g(S_X(x)) dx + \int_{d_1 + d_I}^{d_2} g(S_X(x)) dx$. Because of

the comonotonic additivity of TVaR, it then follows that

$$\begin{aligned}
\text{TVaR}_\alpha(T_I(X)) &= \text{TVaR}_\alpha(R_I(X) + \Pi_I(X)) \\
&= \int_0^{d_1} g(S_X(x)) \, dx + \int_{d_1+d_I}^{d_2} g(S_X(x)) \, dx \\
&\quad + \delta \int_{d_2}^{d_2+u_I-d_I} g(S_X(x)) \, dx + (1 + \theta_1)a.
\end{aligned} \tag{3.21}$$

Thus, we have $\frac{\partial}{\partial d_1} \text{TVaR}_\alpha(T_I(X)) = g(S_X(d_1)) - g(S_X(d_1 + d_I))$ and $\frac{\partial}{\partial d_2} \text{TVaR}_\alpha(T_I(X)) = (1 - \delta)g(S_X(d_2)) + \delta g(S_X(d_2 + u_I - d_I))$.

It then follows that

$$\begin{aligned}
\frac{d}{dd_1} \text{TVaR}_\alpha(T_I(X)) &= \frac{\partial}{\partial d_1} \text{TVaR}_\alpha(T_I(X)) + \frac{dd_2}{dd_1} \frac{\partial}{\partial d_2} \text{TVaR}_\alpha(T_I(X)) \\
&= [S_X(d_1) - S_X(d_1 + d_I)] \left\{ \frac{g(S_X(d_1)) - g(S_X(d_1 + d_I))}{S_X(d_1) - S_X(d_1 + d_I)} \right. \\
&\quad \left. - \left[(1 - \delta) \frac{g(S_X(d_2))}{S_X(d_2)} + \delta \frac{g(S_X(d_2 + u_I - d_I))}{S_X(d_2)} \right] \right\}.
\end{aligned} \tag{3.22}$$

Let $v_\alpha \triangleq \text{VaR}_\alpha(X)$. Depending on the value of α , we have the following discussion.

- (i) If $d_1 \leq v_\alpha - d_I$, we have $g(S_X(d_1)) = g(S_X(d_1 + d_I)) = 1$, and thus $\frac{d}{dd_1} \text{TVaR}_\alpha(T_I(X)) \leq 0$, which means $\text{TVaR}_\alpha(T_I(X))$ decreases with d_1 .
- (ii) If $d_1 \geq v_\alpha$, it then follows that

$$\frac{d}{dd_1} \text{TVaR}_\alpha(T_I(X)) = \frac{\delta}{\alpha} [S_X(d_1) - S_X(d_1 + d_I)] \left\{ 1 - \frac{S_X(d_2 + u_I - d_I)}{S_X(d_2)} \right\} \geq 0.$$

- (iii) If $v_\alpha - d_I \leq d_1 \leq v_\alpha$, we have

$$\begin{aligned}
\frac{d}{dd_1} \text{TVaR}_\alpha(T_I(X)) &= \left[\frac{\alpha - S_X(d_1 + d_I)}{S_X(d_1) - S_X(d_1 + d_I)} - \delta \frac{S_X(d_2 + u_I - d_I)}{S_X(d_2)} + \delta - 1 \right] \\
&\quad \cdot \frac{1}{\alpha} [S_X(d_1) - S_X(d_1 + d_I)].
\end{aligned} \tag{3.23}$$

Denote

$$H(d_1) \triangleq \frac{\alpha - S_X(d_1 + d_I)}{S_X(d_1) - S_X(d_1 + d_I)} - \delta \frac{S_X(d_2 + u_I - d_I)}{S_X(d_2)} + \delta - 1.$$

Assuming that $\frac{S_X(d+u_I-d_I)}{S_X(d)}$ is an increasing function of d , it follows from (3.19) that $H(d_1)$ is an increasing function of d_1 . Furthermore, in Case (i), $H(d_1) \leq 0$ when $d_1 = v_\alpha - d_I$ and in Case (ii), $H(d_1) \geq 0$ when $d_1 = v_\alpha$. Hence, there exists $\bar{d}_1 \in \mathbb{R}$ satisfying $H(\bar{d}_1) = 0$ such that $H(d_1) \leq 0$ when $d_1 \leq \bar{d}_1$ and $H(d_1) \geq 0$ when $d_1 \geq \bar{d}_1$. Therefore, from (3.23), we have that $\frac{d}{dd_1} \text{TVaR}_\alpha(T_I(X)) \leq 0$ for $d_1 \leq \bar{d}_1$ and $\frac{d}{dd_1} \text{TVaR}_\alpha(T_I(X)) \geq 0$ for $d_1 \geq \bar{d}_1$.

Combining the above discussion, we have that $\text{TVaR}_\alpha(T_I(X))$ decreases for $d_1 \leq \bar{d}_1$ and increases for $d_1 \geq \bar{d}_1$. If $\bar{d}_1 < \tilde{d}$, the optimal parameters are $d_1^* = \bar{d}_1$ and $d_2^* = \bar{d}_2$, where $(\bar{d}_1, \bar{d}_2)$ are determined by solving (3.18). However, if $\bar{d}_1 \geq \tilde{d}$, the optimal values are $d_1^* = \tilde{d}$ and $d_2^* = d_1^* + d_I$, yielding the optimal indemnity function $I^*(x) = (x - \tilde{d})_+$. This completes the proof. $\square$

Remark 4. For a general distribution, the closed-forms of the optimal parameters are not available. We propose an algorithm, which is designed based on equation (3.22), to numerically calculate the values of those parameters. We use the “bisection” method to obtain the optimal indemnity function for a general risk distribution. Note that the derivative of the objective function admits a unique root of d_1 by equation (3.22). Then d_2 can be uniquely determined by d_1 through the equation $\mathbb{E}[I(X)] = a$ for a given a . The pseudocode of the process is shown in Algorithm 1.

Example 3.3.1. Assume that the loss follows an Exponential distribution, i.e., $X \sim \text{Exp}(1/\mu)$ with survival function $S_X(x) = e^{-x/\mu}$ for $x \geq 0$, and $\mu = \mathbb{E}[X]$. It can be easily verified that $\frac{S_X(d+u_I-d_I)}{S_X(d)}$ is constant, which satisfies the condition in Proposition 3.3.3. For a given $a \in [0, \mu]$, we would like to solve the problem (3.16). Note that $\tilde{d} = \mu \ln \frac{\mu}{a}$ is obtained from $\mathbb{E}[(X - d)_+] = a$. Solving the system of equations in (3.18), we find the optimal parameters

Algorithm 1: Iterative Optimization of the Optimal Parameters of Indemnity Functions for a General Loss Distribution Given δ

Data: Parameters: $\mu, \Delta_a, \Delta_\delta, \theta_0, \theta_1, \theta_2$
 Functions: g, S

Result: a^*, d_1^*, d_2^*

for each δ in $[0, 1]$ with step Δ_δ **do**

for each a in $[0, \mu]$ with step Δ_a **do**

Solve for $\tilde{d}$ such that $\mathbb{E}[(X - \tilde{d})_+] = a$

$d_I \leftarrow \frac{\theta_1 - \theta_0 + \delta}{\delta} \cdot a$

$u_I \leftarrow \frac{\theta_2 - \theta_0 + \delta}{\delta} \cdot a$

if $d_I > 0$ **then**

$d_1^* \leftarrow$ Perform bisection on d_1 between 0 and $\tilde{d}$ until $\frac{d\rho_g(T_I(X))}{dd_1} = 0$ in equation (3.22). For each d_1 , solve d_2 from

$\int_{d_1}^{d_1 + d_I} S(x)dx + \int_{d_2}^{\infty} S(x)dx = a$

else

$d_I \leftarrow 0$

$d_1^* \leftarrow \tilde{d}$

end

Compute $\rho_g(T_I(X))$ in equation (3.21)

end

$a^* \leftarrow$ Select a such that $\rho_g(T_I(X))$ reaches its minimum

$d_1^* \leftarrow$ Select d_1 such that $\rho_g(T_I(X))$ reaches its minimum

$d_2^* \leftarrow$ Solve d_2 such that $\int_{d_1^*}^{d_1^* + d_I} S(x)dx + \int_{d_2}^{\infty} S(x)dx = a^*$

end

$$\begin{aligned}\bar{d}_1 &= \mu \ln \left(1 - \delta + \delta e^{-\frac{d_I}{\mu}} + \delta e^{-\frac{u_I - d_I}{\mu}} - \delta e^{-\frac{u_I}{\mu}} \right) - \mu \ln \alpha, \\ \bar{d}_2 &= -\mu \ln \left(\frac{a}{\mu} - \frac{\alpha(1 - e^{-\frac{d_I}{\mu}})}{1 - \delta + \delta e^{-\frac{d_I}{\mu}} + \delta e^{-\frac{u_I - d_I}{\mu}} - \delta e^{-\frac{u_I}{\mu}}} \right).\end{aligned}$$

It is easy to see that $\bar{d}_1$ and $\bar{d}_2$ are linearly increasing with respect to μ if the ratio $\frac{a}{\mu} = \frac{\mathbb{E}[I(X)]}{\mathbb{E}[X]}$ is fixed. This observation suggests that, given a fixed ratio $\frac{a}{\mu}$, the insurer would like to transfer large losses to the reinsurer, while retain relatively small losses. Figure 3.2 illustrates the optimal indemnity function as defined in (3.17) for $\delta = 1$. When $\bar{d}_1 < \tilde{d}$, the optimal indemnity follows a two-layer structure, while for $\bar{d}_1 \geq \tilde{d}$, it takes a form of a stop-loss function.

Furthermore, we use Algorithm 1 to numerically calculate the optimal solution to the problem (3.13). Under the setting in Figure 3.3, the optimal contract $I^*(x) = (x - d_1^*)_+ - (x - d_1^* - d_I^*)_+ + (x - d_2^*)_+$ is reduced to a stop-loss contract because $d_2^* = d_1^* + d_I^*$. From Figure 3.3-(a), we can see that the value of d_1^* increases linearly with respect to the mean of the exponential loss, and thus, the insurer transfers less loss to the reinsurer. Meanwhile, we calculate $a^* \triangleq \mathbb{E}[I^*(X)]$ and present results in Figure 3.3-(b). It is interesting to see that the ratio $\frac{a^*}{\mu}$ is a constant. This result suggests that it is always optimal for the insurer to transfer a certain percentage of the mean of this exponential loss.

Example 3.3.2. In Chen et al. (2016) the insurer is assumed to adopt the risk-adjusted liability as the optimization objective, and consider the problem

$$\min_{I \in \mathcal{I}} \mathcal{L}(T_I(X)) = \min_{I \in \mathcal{I}} \{ \mathbb{E}[T_I(X)] + \omega \cdot \rho(T_I(X) - \mathbb{E}[T_I(X)]) \}, \quad (3.24)$$

where ρ is a risk measure, $\pi_{\text{RE}}(I(X))$ is the retrospective rating premium defined in (3.3), $T_I(X) = X - I(X) + \pi_{\text{RE}}(I(X))$ is the total risk exposure and $\omega \in (0, 1)$ is known as the cost-of-capital rate. When $\rho = \text{TVaR}_\alpha$ and X follows a Pareto distribution, the results of optimal solutions to (3.24) are presented in Example 5.1 in Chen et al. (2016). To facilitate a comparison between the frameworks used in Chen et al. (2016) and in our work, we follow the same distribution assumption of Example 5.1 in Chen et al. (2016), and match parameters in our performance-based variable premium scheme with those used

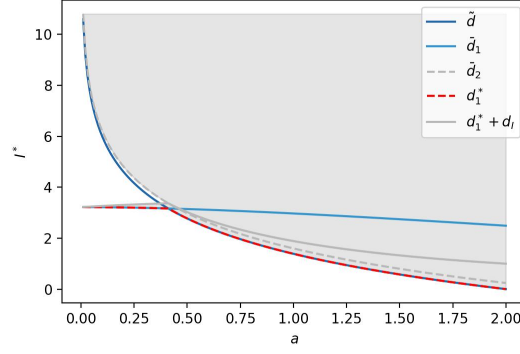


Figure 3.2: The optimal indemnity function among all indemnities with the same expected value a , for the case when $\delta = 1$. The gray area represents the optimal ceded amount. The other parameters are set as follows: $\mu = 2$, $\theta_0 = 1$, $\theta_1 = 0.5$, $\theta_2 = 2$, and $\alpha = 0.2$.

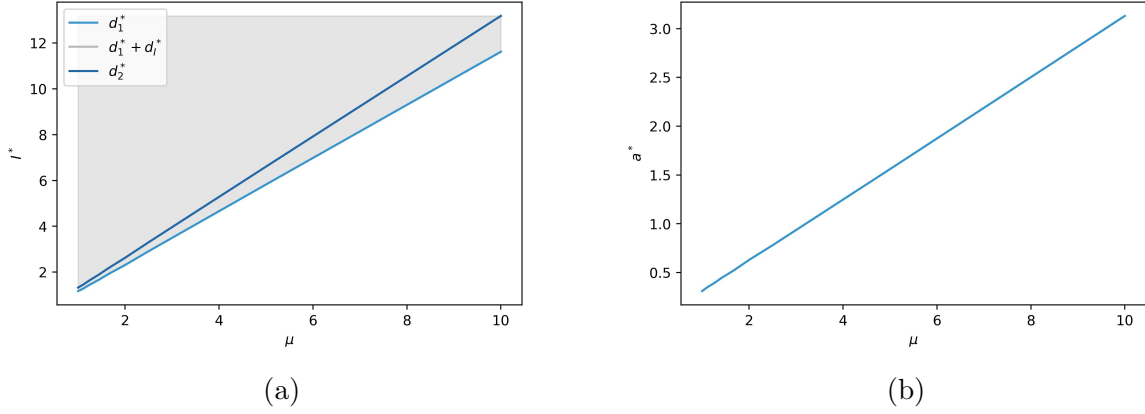


Figure 3.3: (a) The optimal indemnity function I^* with respect to μ . The gray area represents the optimal ceded amount. (b) The optimal expected ceded loss with respect to μ . Here, $X \sim \text{Exp}(1/\mu)$. The parameters are set as follows: $\theta_0 = 1$, $\theta_1 = 0.5$, $\theta_2 = 2$, $\alpha_1 = 0.2$, $\delta = 1$.

$L = \delta/1.025$	d^*	d_c^*
0.1	61.33	66.54
0.3	40.33	67.83
0.5	28.35	67.71
0.7	21.77	66.79
0.9	17.64	66.03

Table 3.1: Comparison of the optimal deductibles for different scheme parameter δ under Pareto loss distribution.

in retrospective rating premium (3.3). Then we numerically solve the problem

$$\min_{I \in \mathcal{I}} \text{TVaR}_\alpha(X - I(X) + \Pi_I(X)). \quad (3.25)$$

The optimal solutions to problems (3.24) and (3.25) are both stop-loss contracts, denoted by $I_c^*(x) = (x - d_c^*)_+$ and $I^*(x) = (x - d^*)_+$, respectively. The comparison of d_c^* and d^* is presented in Table 3.1, in which values of d_c^* come from Example 5.1 in Chen et al. (2016). Clearly, $d^* < d_c^*$ in all five cases, which implies that the insurer in our model shall purchase more reinsurance coverage compared to the case in Chen et al. (2016). This result may be a consequence of that the problem (3.25) aims to minimize TVaR, which is a tail risk measure. Therefore, transferring large losses to the reinsurer can significantly reduce the insurer's right tail risk exposure. Such pattern is more obvious for large δ , i.e., when the variable premium $\Pi_I(X)$ introduces more variation into the insurer's objective function.

3.4 Bowley-Optimal problem

As the seller, the reinsurer determines the parameters δ and θ_i , for $i = 0, 1, 2$, in the variable premium scheme (3.5). The parameter δ governs both the discount and penalty rates within the scheme. When the realized ceded loss $I(x)$ deviates from its expected value $\mathbb{E}[I(X)]$, a larger δ amplifies the impact—resulting in either a greater penalty or a larger discount, depending on whether $\delta(I(x) - \mathbb{E}[I(X)])$ is positive or negative. In other words, the reinsurer can adjust δ to control the premium charged in both the reward and

the penalty scenarios. In the limiting case when $\delta = 0$, the variable premium scheme simplifies to the expected-value premium principle with risk loading θ_0 , resulting in a constant premium $\Pi_I(x) = \pi_0(I(X)) = (1 + \theta_0)\mathbb{E}[I(X)]$.

In Section 3.3, we examine the insurer's optimization problem when the reinsurer implements this variable premium scheme. Obviously, the value of δ selected by the reinsurer influences the insurer's optimal solution, and, in turn, the indemnity function chosen by the insurer impacts the reinsurer's total risk exposure. A natural question arises: what benefit does the reinsurer gain from adopting this variable premium scheme, and what would be the optimal choice of δ from the reinsurer's perspective? To explore this, we first modify Assumption 3 to allow flexibility in the choice of δ .

Assumption 4. Assume that $0 \leq \bar{\theta}_1 \leq \theta_0 < \theta_2$ are given. For any $\delta \in [0, 1]$, take $\theta_1 = (\theta_0 - \delta) \vee \bar{\theta}_1$ and define the corresponding "reward-and-penalty" variable premium scheme as follows. For any $I \in \mathcal{I}$ and $x \geq 0$, the premium scheme is given by

$$\Pi_I^\delta(x) = \min \{ \max \{ \pi_0(I(X)) + \delta(I(x) - \mathbb{E}[I(X)]), \pi_1(I(X)) \}, \pi_2(I(X)) \}, \quad (3.26)$$

where $\pi_i(I(X)) = (1 + \theta_i)\mathbb{E}[I(X)]$ for $i = 0, 1, 2$.

We propose a Bowley optimization problem between the insurer and the reinsurer, structured as a two-step sequential process:

- (i) In the first phrase, the reinsurer chooses the value of $\delta \in [0, 1]$ in the variable premium scheme under Assumption 4. In response, the insurer selects the optimal indemnity function, denoted by I^δ , which minimizes the following insurer's optimization problem

$$\min_{I \in \mathcal{I}} \rho_1(R_I(X) + \Pi_I^\delta(X)), \quad (3.27)$$

where ρ_1 is the insurer's risk measure.

- (ii) In the second phrase, after observing the insurer's choice I^δ as a function of δ , the reinsurer then selects the optimal $\delta \in [0, 1]$, which minimizes the following reinsurer's problem

$$\min_{0 \leq \delta \leq 1} \rho_2(I^\delta(X) - \Pi_{I^\delta}^\delta(X)), \quad (3.28)$$

where ρ_2 is the reinsurer's risk measure.

In the Bowley optimization problem, the reinsurer acts as the leader by setting the prices for reinsurance policies, while the insurer is the price taker. If there exists a value of δ and a reinsurance policy that allow both the insurer and reinsurer to optimize their respective objectives, as outlined in steps (i) and (ii), then we say that this Bowley optimization problem has an optimal solution. Mathematically, we define Bowley-Optimality below.

Definition 3.4.1. A pair $(\delta^*, I^*) \in [0, 1] \times \mathcal{I}$ is said to be Bowley-Optimal (BO) if

- (i) $I^* = I^{\delta^*}$ is the optimal solution to the problem (3.27) with respect to δ^* , that is $I^* \in \arg \min_{I \in \mathcal{I}} \rho_1(R_I(X) + \Pi_I^{\delta^*}(X))$.
- (ii) δ^* is the solution to the problem (3.28), that is $\rho_2(I^*(X) - \Pi_{I^*}^{\delta^*}(X)) \leq \rho_2(\tilde{I}(X) - \Pi_{\tilde{I}}^{\tilde{\delta}}(X))$ for all $\tilde{\delta} \in [0, 1]$ and $\tilde{I} = I^{\tilde{\delta}} \in \arg \min_{I \in \mathcal{I}} \rho_1(R_I(X) + \Pi_I^{\tilde{\delta}}(X))$.

The results from Section 3.3 can be applied to further study the reinsurer's optimization problem in Definition 3.4.1-(ii). Since the expected-value premium principle is a specific case of the variable premium scheme when $\delta = 0$, identifying the optimal solution δ^* in the Bowley-Optimal (BO) problem allows us to determine whether the variable premium scheme provides a greater benefit to the reinsurer compared to the expected-value premium principle.

Mathematically, the process to find the Bowley-Optimal (BO) solution begins by solving the problem (3.27) to determine I^δ for each value of δ . Next, this I^δ is substituted into the problem (3.28) to find the optimal δ^* . Finally, solving (3.27) again using δ^* provides the optimal indemnity function I^{δ^*} . Thus, the pair $(I^* = I^{\delta^*}, \delta^*)$ forms the BO solution. However, due to the complexity in the structure of I^δ , determining δ^* analytically is challenging. Therefore, we rely on numerical examples to explore δ^* and the corresponding BO solution for both the insurer and reinsurer. In Examples 3.4.1, 3.4.2, and 3.4.3, we adopt the following assumption for both parties:

$$\rho_1 = \text{TVaR}_\alpha, \quad \text{and} \quad \rho_2 = \text{TVaR}_\beta, \quad \text{where } 0 < \alpha, \beta < 1. \quad (3.29)$$

We then provide examples with both heavy-tailed and light-tailed distributions of X .

Example 3.4.1 (Bowley-optimal problem under Pareto loss distribution). We assume that the loss X follows the type II Pareto distribution with shape parameter ζ and scale parameter η , i.e. $X \sim \text{Pa}(\zeta, \eta)$. The survival function of X is given by $S_X(x) = \left(\frac{\eta}{x+\eta}\right)^\zeta$ for $x \geq 0$. Assume $\zeta > 1$ to ensure that the expected value $\mathbb{E}[X]$ exists.

Figure 3.4 illustrates the retention levels in I^δ , where $I^\delta \in \arg \min_{I \in \mathcal{I}} \text{TVaR}_\alpha(R_I(X) + \Pi_I^\delta(X))$ is the optimal solution for the insurer based on different values of $\delta \in [0, 1]$. A higher δ leads to a stronger power of discount and penalty. It is evident that as δ increases, the insurer prefers to retain more risks. In other words, $I^{\delta_1} \geq I^{\delta_2}$ for $\delta_1 \leq \delta_2$. In the scenario depicted in Figure 3.4(a), the optimal solutions are stop-loss functions for all values of δ . In contrast, in Figure 3.4(b), I^δ begins as a stop-loss function for small values of δ and transitions into a two-layer function as δ increases.

As illustrated in Figure 3.5, the value function of the insurer $\text{TVaR}_\alpha(R_{I^\delta}(X) + \Pi_{I^\delta}^\delta(X))$ increases with δ . Since the optimal amount ceded decreases as δ rises, the insurer bears more risks from retained loss while facing reduced risk from premium payments. Figure 3.5 indicates that the increased risk from the retained loss outweighs the change in the risk of the premium. Therefore, the insurer favors a constant premium scheme over a variable premium scheme.

Conversely, the objective of the reinsurer $\text{TVaR}_\beta(I^\delta(X) - \Pi_{I^\delta}^\delta(X))$ significantly benefits from adopting the variable premium scheme, as illustrated in Figure 3.6. In Figure 3.6(a), the reinsurer's objective reaches its minimum at $\delta^* = 0.259$, the unique solution for this parameter set. In Figure 3.6(b), however, δ^* is not unique. Similar to the impact of δ on the insurer's value function, the reduction in $\text{TVaR}_\beta(I^\delta(X))$ outweighs the increase in $\text{TVaR}_\beta(\Pi_{I^\delta}^\delta(X))$ as δ increases, resulting in an overall decrease in the reinsurer's objective. In the context of the Bowley optimum, there may exist multiple optimal δ^* that minimize the reinsurer's objective. However, since the insurer's value function rises with increasing δ , the smallest value within the optimal set of δ^* is preferred, as it is more likely to align the interests of both the insurer and reinsurer, facilitating agreement on the reinsurance contract.

When the insurer's risk preference, represented by α , remains constant, the variation in the optimal δ^* as the reinsurer's risk preference, represented by β , changes is shown in Figure 3.7. As β increases, the minimum optimal δ^* initially rises and then falls. When

$\beta = 1$, in which case $\text{TVaR}_1 = \mathbb{E}$, the reinsurer becomes risk neutral and prefers a constant premium scheme. This result suggests that the variable premium scheme helps the reinsurer mitigate risk whenever she is not risk neutral.

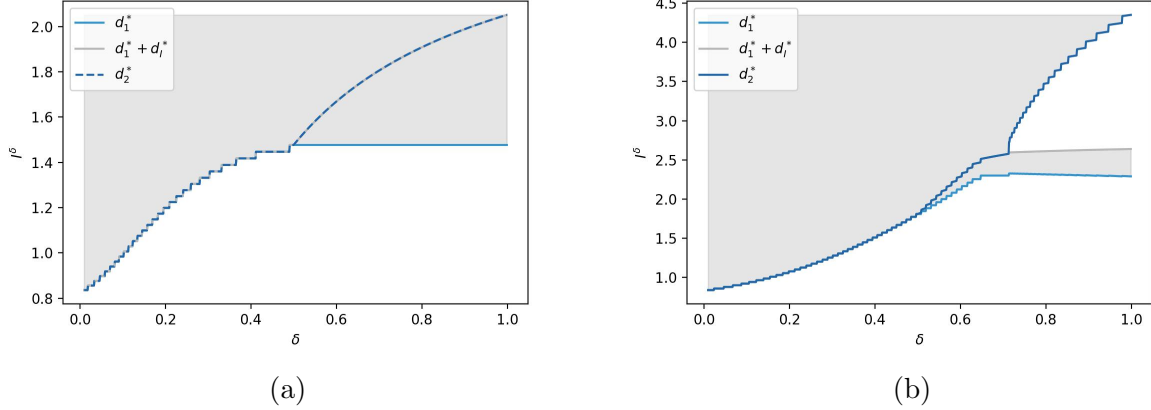


Figure 3.4: The optimal ceded loss function $I^\delta(x) = (x - d_1^*)_+ - (x - d_1^* - d_I^*)_+ + (x - d_2^*)_+$ with respect to δ , where $d_I^* = \frac{\theta_1 - \theta_0 + \delta}{\delta} \mathbb{E}[I^\delta(X)]$. The shaded area represents the ceded portion. Here, $\rho = \text{TVaR}_\alpha$ is adopted to measure the insurer's risk and $X \sim \text{Pa}(\eta, \zeta)$. The parameters are as follows. (a) $\eta = 2$, $\zeta = 2$, $\theta_0 = 1$, $\theta_1 = 0.5$, $\theta_2 = 2$, $\alpha = 0.1$; (b) $\eta = 2$, $\zeta = 2$, $\theta_0 = 1$, $\theta_1 = 0.5$, $\theta_2 = 5$, $\alpha = 0.2$.

Example 3.4.2 (Bowley-optimal problem under Exponential loss distribution). Assume that (3.29) holds and $X \sim \text{Exp}(\lambda)$ with $S_X(x) = e^{-\lambda x}$ for $x \geq 0$.

As illustrated in Figure 3.8-(a), the optimal indemnity function is a stop-loss function for all δ . In Figure 3.8-(b), with a larger value of θ_2 , the function I^δ becomes a two-layer function when δ is large. These findings are consistent with those from the Pareto loss distribution in Example 3.4.1. However, unlike the Pareto distribution with the same expected value, the insurer tends to retain more loss under the Exponential loss distribution. This is intuitive, as the Exponential distribution has a lighter tail compared to the Pareto distribution, implying a lower risk of extreme losses.

With the same expectation and consistent conditions, the optimal δ^* for the Exponential loss distribution is notably smaller than that for the Pareto distribution, as seen in the comparison between Figures 3.7 and 3.9. This is likely because the insurer prefers to retain

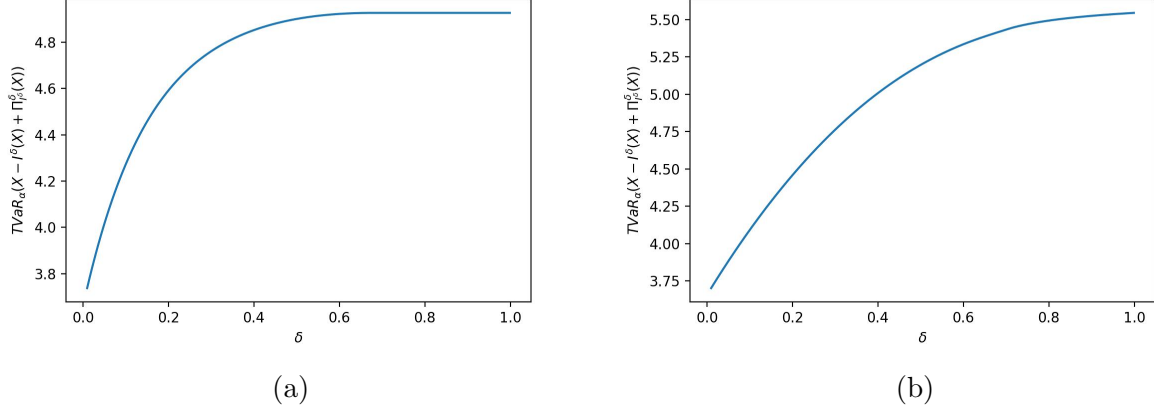


Figure 3.5: The value function of the insurer with respect to δ . Here, $\rho = \text{TVaR}_\alpha$ is adopted by the insurer and $X \sim \text{Pa}(\eta, \zeta)$. The parameters are as follows. (a) $\eta = 2, \zeta = 2, \theta_0 = 1, \theta_1 = 0.5, \theta_2 = 2, \alpha = 0.1$; (b) $\eta = 2, \zeta = 2, \theta_0 = 1, \theta_1 = 0.5, \theta_2 = 5, \alpha = 0.2$.

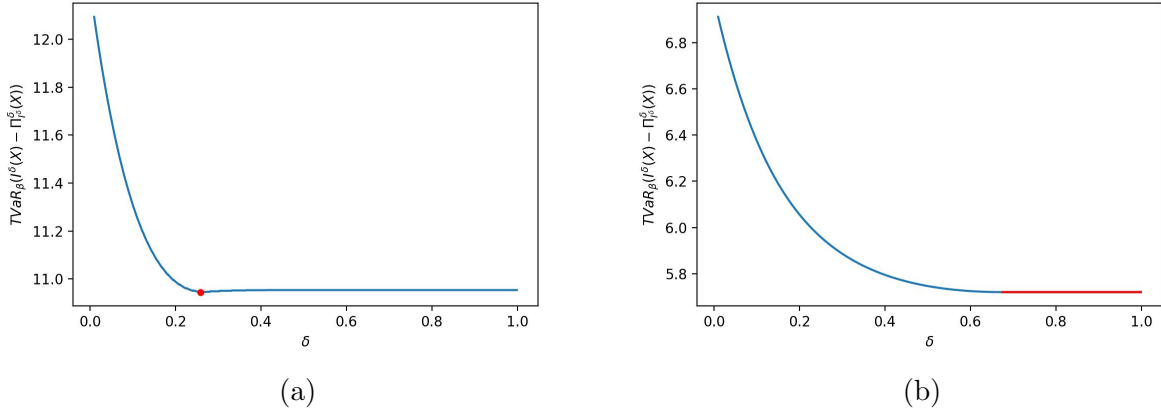


Figure 3.6: Assume $X \sim \text{Pa}(\eta, \zeta)$. The insurer adopts $\rho = \text{TVaR}_\alpha$ and the reinsurer uses $\rho = \text{TVaR}_\beta$ to measure their risk. The indemnity function I^δ minimizes the insurer's objective for a given δ . The figure illustrates reinsurer's objective function with respect to δ when adopting I^δ . The red represents the minimum value. The parameters are as follows. (a) $\eta = 2, \zeta = 2, \theta_0 = 1, \theta_1 = 0.5, \theta_2 = 2, \alpha = 0.1, \beta = 0.05$; (b) $\eta = 2, \zeta = 2, \theta_0 = 1, \theta_1 = 0.5, \theta_2 = 2, \alpha = 0.1, \beta = 0.1$.

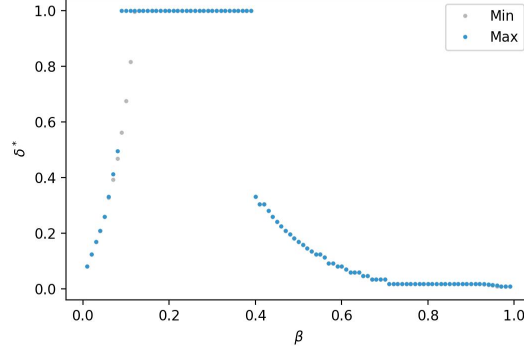


Figure 3.7: The optimal δ^* with respect to β under given α . Here, assume $X \sim \text{Pa}(\eta, \zeta)$ and the insurer and the reinsurer adopt $\rho = \text{TVaR}$ with levels α and β , respectively. The grey dots represent the minimum optimal δ^* , while the blue dots represent the maximum. The dots in between indicate the other optimal values. The parameters are as follows. $\eta = 2$, $\zeta = 2$, $\theta_0 = 1$, $\theta_1 = 0.5$, $\theta_2 = 2$, $\alpha = 0.1$.

more risk under the Exponential distribution. A smaller δ^* may encourage the insurer to cede more loss, enabling the reinsurer to increase income by charging a higher premium.

Example 3.4.3 (Other discussions on the Bowley-optimum). Following Examples 3.4.1 and 3.4.2, we now discuss the impact of $\mathbb{E}[X]$ on the BO solution. We take the Pareto and Exponential distributions to illustrate our findings.

The Pareto distribution is characterized by two parameters, and we vary one while keeping the other constant. Figure 3.10-(a) displays the variation of the optimal δ^* , which achieves the Bowley optimum, in relation to changes in $\mathbb{E}[X]$. Under these parameter settings, the optimal indemnity functions at the Bowley optimum are all stop-loss functions. The deductible of these stop-loss functions, as influenced by $\mathbb{E}[X]$, is shown in Figure 3.10-(b).

As shown in Figure 3.10, when the shape parameter ζ of the Pareto distribution is varied while keeping the scale parameter η fixed, the optimal δ^* and the optimal deductible $d_{\delta^*}^*$ change significantly with shifts in $\mathbb{E}[X]$. Specifically, as the loss distribution becomes heavier-tailed (i.e., when ζ decreases as $\mathbb{E}[X]$ increases), the insurer decides to cede a larger proportion of the loss to the reinsurer, while the reinsurer demands a significantly higher δ^* .

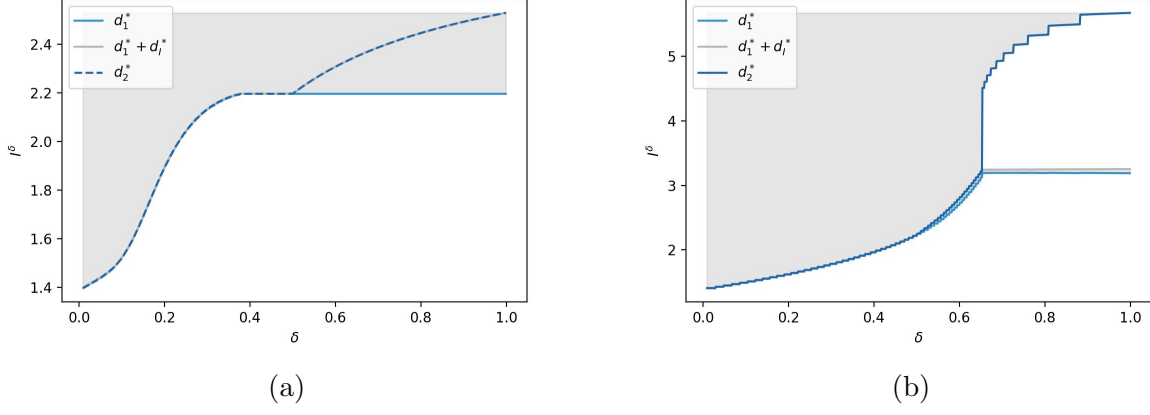


Figure 3.8: The optimal ceded loss function $I^\delta(x) = (x - d_1^*)_+ - (x - d_1^* - d_I^*)_+ + (x - d_2^*)_+$ with respect to δ , where $d_I^* = \frac{\theta_1 - \theta_0 + \delta}{\delta} \mathbb{E}[I^\delta(X)]$. The shaded area represents the ceded portion. Here, $X \sim \text{Exp}(\lambda)$ and the insurer adopts $\rho = \text{TVaR}_\alpha$. The parameters are as follows. (a) $\lambda = 0.5$, $\theta_0 = 1$, $\theta_1 = 0.5$, $\theta_2 = 2$, $\alpha = 0.1$; (b) $\lambda = 0.5$, $\theta_0 = 1$, $\theta_1 = 0.5$, $\theta_2 = 10$, $\alpha = 0.2$.

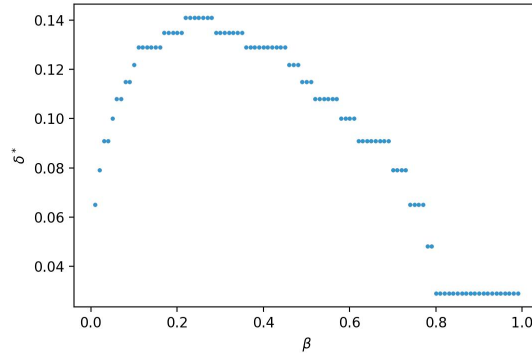


Figure 3.9: The optimal δ^* that minimizes the reinsurer's objective function with respect to β . Here, assume $X \sim \text{Exp}(\lambda)$. The insurer and the reinsurer use TVaR to measure their risk with levels α and β , respectively. The parameters are as follows. $\lambda = 0.5$, $\theta_0 = 1$, $\theta_1 = 0.5$, $\theta_2 = 2$, $\alpha = 0.1$.

In contrast, varying the scale parameter of either the Exponential or Pareto distribution only slightly affects the optimal δ^* . The optimal deductible $d_{\delta^*}^*$ appears proportional to $\mathbb{E}[X]$ when only the scale parameter is altered, but this proportionality breaks down when the shape parameter is varied.

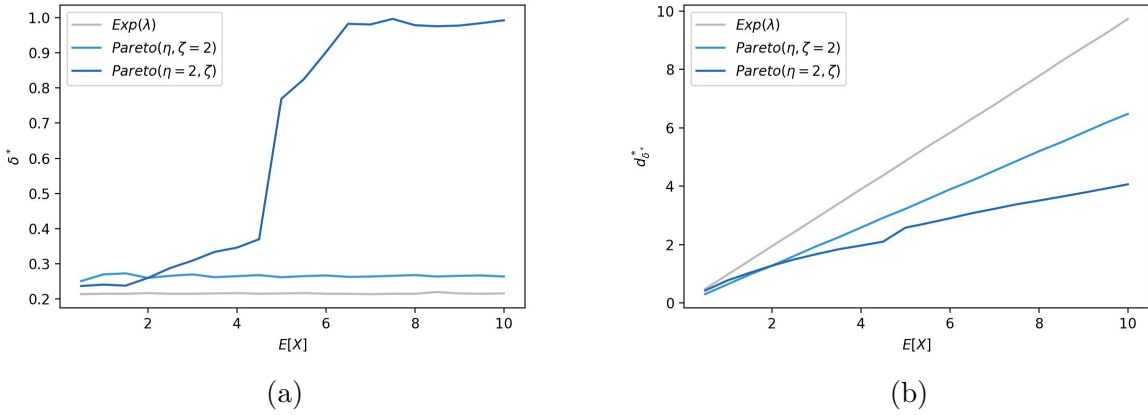


Figure 3.10: (a) The optimal δ^* with respect to $\mathbb{E}[X]$; (b) The optimal deductible $d_{\delta^*}^*$ with respect to $\mathbb{E}[X]$. If $X \sim \text{Exp}(\lambda)$, the results are illustrated by the grey line. If $X \sim \text{Pa}(\eta, \zeta)$, given $\mathbb{E}[X]$, the results of fixing ζ or η while varying the other parameter are represented by the light blue and dark blue lines, respectively. The parameters are as follows. $\theta_0 = 1$, $\theta_1 = 0.5$, $\theta_2 = 2$, $\alpha = 0.1$, $\beta = 0.05$.

Example 3.4.4 (Bowley-optimum with power distortion risk measure). In this example, instead of using TVaR, we assume that both the insurer and reinsurer adopt power distortion risk measures. Specifically, the insurer's risk measure is defined as $\rho_1(X) = \int_0^\infty [S_X(x)]^\alpha dx$ and the reinsurer's risk measure is $\rho_2(X) = \int_0^\infty [S_X(x)]^\beta dx$, where $0 < \alpha, \beta < 1$ represent the risk aversion parameters of the insurer and reinsurer, respectively. The underlying loss is assumed to follow an exponential distribution with scale parameter λ , that is, $X \sim \text{Exp}(\lambda)$.

All results are consistent with those obtained using TVaR, with one key difference: the optimal δ^* for the Bowley-optimal problem decreases as β increases when the power distortion risk measure is used. This is illustrated in Figure 3.11. As the reinsurer approaches

risk neutrality, a variable premium becomes unnecessary to optimize her objective. However, if the reinsurer is highly risk-averse (i.e., when β is small), the variable premium scheme becomes significantly valuable in managing risk exposure.

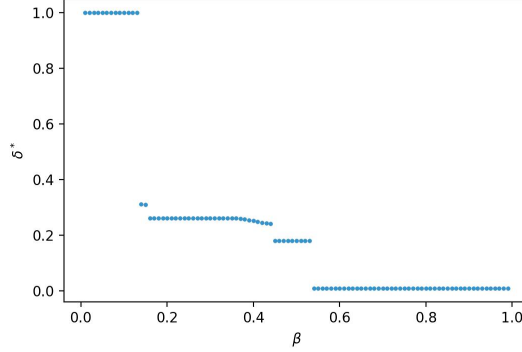


Figure 3.11: The optimal δ^* with respect to β . Here, assume $X \sim \text{Exp}(\lambda)$. The insurer and the reinsurer adopt the power distortion risk measure with levels α and β , respectively. Parameters are as follows. $\lambda = 0.5$, $\theta_0 = 1$, $\theta_1 = 0.5$, $\theta_2 = 2$, $\alpha = 0.2$.

3.5 Conclusion

In this work, we introduce a reward-and-penalty variable premium scheme for reinsurance policies. Unlike classical premium principles, which assign a fixed value based on the distribution of a random loss, the proposed scheme not only accounts for the loss distribution, but also incorporates the realized loss amount. This feature introduces additional randomness for both the insurer and the reinsurer. We begin by discussing the properties of this variable premium scheme and comparing them with the desirable characteristics of the classical premium principles. Next, we establish an optimization framework for the insurer and characterize the optimal reinsurance under mild assumptions regarding the insurer's risk measure. In particular, when the insurer uses distortion risk measure or even TVaR to quantify her risk exposure, we can further formulate the optimal solutions. Moreover, we formulate a Bowley optimization problem to capture the interaction between the insurer and the reinsurer. Through various numerical examples, we demonstrate that

the reinsurer can benefit from adopting the variable premium scheme, as it significantly reduces her total risk exposure compared to the expected-value premium principle, which serves as a limiting case of the proposed variable premium scheme.

3.6 Appendix

Proof of Lemma 3.3.1. For $I \in \mathcal{I}$, if $\mathbb{E}[I(X)] = 0$, then $I(X) = 0$ almost surely. This indemnify function is an element in $\mathcal{S}_3$. When $\delta = 0$, we have $\theta_1 = \theta_0$. Then $\pi(I(x)) = (1 + \theta)\mathbb{E}[I(X)]$ is the expected value principle. Consequently, the problem (3.6) is reduced to the problem (3.1). By Theorem 6.1 in Van Heerwaarden et al. (1989), the optimal solution to the problem (3.1) belongs to the class of stop-loss functions $I_d(x) = (x - d)_+$, where $d \geq 0$. Clearly, we have $I_d(x) \in \mathcal{S}_3$.

In the following part, we assume $\mathbb{E}[I(X)] > 0$, $\delta > 0$, $\theta_1 \leq \theta_0 < \theta_2$. Denote $x_d^I = \sup\{x \geq 0 : I(x) \leq d_I\}$ and $x_u^I = \sup\{x \geq 0 : I(x) \leq u_I\}$. Due to the continuity of $I(x)$, it is clear that $0 \leq x_d^I < x_u^I$. Then the premium (3.5) can be further written as

$$\Pi_I(x) = \begin{cases} (1 + \theta_1)\mathbb{E}[I(X)], & x \leq x_d^I, \\ \delta I(x) + (1 + \theta_0 - \delta)\mathbb{E}[I(X)], & x_d^I < x \leq x_u^I, \\ (1 + \theta_2)\mathbb{E}[I(X)], & x > x_u^I. \end{cases} \quad (3.30)$$

Arbitrarily select and fix $I \in \mathcal{I}$, we are going to show that there exists $f_I \in \mathcal{S}_3$ such that $\mathbb{E}[f_I(X)] = \mathbb{E}[I(X)]$ and T_I is dominated by T_{f_I} in the sense of convex order.

(i) If $x_d^I < x_u^I < \infty$, define

$$f_1(x) = \begin{cases} I(x), & x \leq x_u^I, \\ I(x_u^I) + (x - e)_+, & x > x_u^I, \end{cases} \quad (3.31)$$

where $e \in [x_u^I, \infty)$ is determined by $\mathbb{E}[f_1(X)] = \mathbb{E}[I(X)]$. Such e exists because we note that if $e = x_u^I$, then $f_1(x) \geq I(x)$ for all $x \geq 0$; if $e = \infty$, then $f_1(x) \leq I(x)$ for all $x \geq 0$. Together with the continuity of $\mathbb{E}[f_1(X)]$ as a function of e , we can verify

that there exists e such that $\mathbb{E}[f_1(X)] = \mathbb{E}[I(X)]$. It then follows that

$$\begin{aligned}\mathbb{E}[f_1(X)] &= \mathbb{E}[f_1(X)|X \leq x_u^I]\mathbb{P}(X \leq x_u^I) + \mathbb{E}[f_1(X)|X > x_u^I]\mathbb{P}(X > x_u^I) \\ &= \mathbb{E}[I(X)|X \leq x_u^I]\mathbb{P}(X \leq x_u^I) + \mathbb{E}[f_1(X)|X > x_u^I]\mathbb{P}(X > x_u^I) \\ &= \mathbb{E}[I(X)|X \leq x_u^I]\mathbb{P}(X \leq x_u^I) + \mathbb{E}[I(X)|X > x_u^I]\mathbb{P}(X > x_u^I) = \mathbb{E}[I(X)].\end{aligned}\tag{3.32}$$

Note that $x_u^I < \infty$ implies $\mathbb{P}(X > x_u^I) > 0$, we obtain from (3.32) that

$$\mathbb{E}[f_1(X)|X > x_u^I] = \mathbb{E}[I(X)|X > x_u^I].\tag{3.33}$$

In addition, $d_{f_1} = \frac{\theta_1 - \theta_0 + \delta}{\delta} \mathbb{E}[f_1(X)] = \frac{\theta_1 - \theta_0 + \delta}{\delta} \mathbb{E}[I(X)] = d_I$ and $u_{f_1} = \frac{\theta_2 - \theta_0 + \delta}{\delta} \mathbb{E}[f_1(X)] = \frac{\theta_2 - \theta_0 + \delta}{\delta} \mathbb{E}[I(X)] = u_I$, implying $x_d^{f_1} = x_d^I < \infty$ and $x_u^{f_1} = e \in [x_u^I, \infty)$. By (3.30), (3.31), and (3.33), we obtain

$$\begin{aligned}\mathbb{E}[\Pi_I(X)] &= \mathbb{E}[\Pi_I(X)|X \leq x_u^I]\mathbb{P}(X \leq x_u^I) + \mathbb{E}[\Pi_I(X)|X > x_u^I]\mathbb{P}(X > x_u^I) \\ &= \mathbb{E}[\Pi_{f_1}(X)|X \leq x_u^I]\mathbb{P}(X \leq x_u^I) + (1 + \theta_2)\mathbb{E}[f_1(X)]\mathbb{P}(X > x_u^I) \\ &= \mathbb{E}[\Pi_{f_1}(X)|X \leq x_u^{f_1}]\mathbb{P}(X \leq x_u^{f_1}) + (1 + \theta_2)\mathbb{E}[f_1(X)]\mathbb{P}(X > x_u^{f_1}) = \mathbb{E}[\Pi_{f_1}(X)].\end{aligned}$$

The third equation holds because $\Pi_{f_1}(x) = (1 + \theta_2)\mathbb{E}[f_1(X)]$ for $x \in [x_u^I, x_u^{f_1}]$. It then follows that $\mathbb{E}[T_I(X)] = \mathbb{E}[R_I(X) + \Pi_I(X)] = \mathbb{E}[R_{f_1}(X) + \Pi_{f_1}(X)] = \mathbb{E}[T_{f_1}(X)]$. Since $1 - \delta \geq 0$, we can find that $T_I(x)$ up-crosses $T_{f_1}(x)$. By Lemma 3 in Ohlin (1969), we have

$$T_{f_1}(X) \leq_{cx} T_I(X).\tag{3.34}$$

Then, we define

$$f_2(x) = \begin{cases} f_1(x_d^{f_1}) + (x - c)_+ - (x - (c + u_{f_1} - d_{f_1}))_+, & x_d^{f_1} \leq x < x_u^{f_1}, \\ f_1(x), & \text{otherwise,} \end{cases}$$

where $c \in [x_d^{f_1}, x_u^{f_1})$ is determined by $\mathbb{E}[f_2(X)] = \mathbb{E}[f_1(X)]$. By similar argument, we obtain

$$T_{f_2}(X) \leq_{cx} T_{f_1}(X).\tag{3.35}$$

Finally, define

$$f_3(x) = \begin{cases} (x - a)_+ \wedge d_{f_2}, & x \leq x_d^{f_2}, \\ f_2(x), & x > x_d^{f_2}, \end{cases}$$

where $a \in [0, x_d^{f_2})$ is defined by $\mathbb{E}[f_3(X)] = \mathbb{E}[f_2(X)]$. By similar argument, we have

$$T_{f_3}(X) \leq_{cx} T_{f_2}(X). \quad (3.36)$$

Combining (3.34), (3.35), and (3.36), it then follows that $T_{f_3}(X) \leq_{cx} T_I(X)$.

In this case, based on given I , we can always construct $f_I \in \mathcal{S}_3$ in the form of

$$f_I = (x - a)_+ - (x - b)_+ + (x - c)_+ - (x - d)_+ + (x - e)_+$$

with parameters $0 \leq a \leq b = a + d_I \leq c \leq d = c + u_I - d_I \leq e$ such that $\mathbb{E}[f_I(X)] = \mathbb{E}[I(X)]$ and $T_{f_I}(X) \leq_{cx} T_I(X)$.

(ii) If $x_d^I < x_u^I = \infty$, following the same argument, we can find $f_I \in \mathcal{S}_3$ in the form of

$$f_I = (x - a)_+ - (x - b)_+ + (x - c)_+ - (x - d)_+$$

with parameters $0 \leq a \leq b = a + d_I \leq c \leq d = c + u_I - d_I$, such that $\mathbb{E}[f_I(X)] = \mathbb{E}[I(X)]$ and $T_{f_I}(X) \leq_{cx} T_I(X)$.

(iii) If $x_d^I = \infty$, we have $I(x) \leq d_I = \frac{\theta_1 - \theta_0 + \delta}{\delta} \mathbb{E}[I(X)] \leq \mathbb{E}[I(X)]$. This means $I(x) = 0$ almost surely, which is an element in $\mathcal{S}_3$.

Combining the three cases, we conclude that for any $I \in \mathcal{I}$, there always exists $f_I \in \mathcal{S}_3$ such that $\mathbb{E}[f_I(X)] = \mathbb{E}[I(X)]$ and $T_{f_I}(X) \leq_{cx} T_I(X)$. The convex order consistency of ρ gives $\rho(T_{f_I}(X)) \leq \rho(T_I(X))$, which completes the proof. $\square$

Proof of Theorem 3.3.1. Using the similar argument in the proof of Lemma 3.3.1, we only need to consider non-trivial cases when $\mathbb{E}[I(X)] > 0$, $\delta > 0$, and $\theta_1 \leq \theta_0 < \theta_2$.

For any $I \in \mathcal{S}_3$, we are going to show that there exists a function $h \in \tilde{\mathcal{I}}$ satisfying $\mathbb{E}[h(X)] = \mathbb{E}[I(X)]$ and $T_h(X) \leq_{cx} T_I(X)$. Define

$$h_1(x) = \begin{cases} I(x), & x \leq x_d^I, \\ I(x_d^I) + (x - c_1)_+, & x > x_d^I, \end{cases} \quad (3.37)$$

where c_1 is defined by $\mathbb{E}[h_1(X)] = \mathbb{E}[I(X)]$. It is easy to show $c_1 \in [x_d^I, x_u^I - u_I + d_I]$. In specific, if $c_1 = x_d^I$, we have $h_1(x) \geq I(x)$ and thus $\mathbb{E}[h_1(X)] \geq \mathbb{E}[I(X)]$. If $c_1 = x_u^I - u_I + d_I$,

we have $h_1(x) \leq I(x)$ and thus $\mathbb{E}[h_1(X)] \leq \mathbb{E}[I(X)]$. Since $\mathbb{E}[h_1(X)]$ is a continuous function of c_1 , there must exist $c_1 \in [x_d^I, x_u^I - u_I + d_I]$ such that $\mathbb{E}[h_1(X)] = \mathbb{E}[I(X)]$. Therefore, we have $d_{h_1} = \frac{\theta_1 - \theta_0 + \delta}{\delta} \mathbb{E}[h_1(X)] = d_I$ and $u_{h_1} = \frac{\theta_2 - \theta_0 + \delta}{\delta} \mathbb{E}[h_1(X)] = u_I$. In addition, we have $x_d^I \leq x_d^{h_1} = c_1 \leq x_u^{h_1} = c_1 + u_I - d_I \leq x_u^I$. For $f \in \{I, h_1\}$, by (3.30), we have

$$T_f(x) = R_f(x) + \Pi_f(x) = \begin{cases} x - f(x) + (1 + \theta_1)\mathbb{E}[f(X)], & x \leq x_d^f, \\ x - (1 - \delta)f(x) + (1 + \theta_0 - \delta)\mathbb{E}[f(X)], & x_d^f < x \leq x_u^f, \\ x - f(x) + (1 + \theta_2)\mathbb{E}[f(X)], & x > x_u^f. \end{cases}$$

It then follows that

$$\begin{aligned} \mathbb{E}[T_I(X)] - \mathbb{E}[T_{h_1}(X)] &= \mathbb{E}[h_1(X) - I(X) | X \leq x_d^I] \mathbb{P}(X \leq x_d^I) \\ &\quad + (1 - \delta) \mathbb{E}[h_1(X) - I(X) | x_d^I < X < x_u^{h_1}] \mathbb{P}(x_d^I < X < x_u^{h_1}) \\ &\quad + \mathbb{E}[h_1(X) - I(X) | X \geq x_u^{h_1}] \mathbb{P}(X \geq x_u^{h_1}) \\ &\geq \mathbb{E}[h_1(X) - I(X)] = 0. \end{aligned} \tag{3.38}$$

The inequality holds because $h_1(x) \leq I(x)$ when $x \in [x_d^I, x_u^{h_1}]$. Thus, we have

$$\mathbb{E}[T_I(X)] \geq \mathbb{E}[T_{h_1}(X)], \tag{3.39}$$

where the equality holds if and only if $h_1(x) = I(x)$ for $x \geq 0$. In addition, it is easy to find $T_I(X)$ up-crosses $T_{h_1}(X)$.

We then define

$$h_2(x) = \begin{cases} (x - c_2)_+ - (x - c_2 - u_I)_+, & x \leq x_u^I \\ I(x), & x > x_u^I. \end{cases} \tag{3.40}$$

Here, c_2 is determined by $\mathbb{E}[h_2(X)] = \mathbb{E}[I(X)]$. If $c_2 = 0$, we have $h_2(x) \geq I(x)$ and thus $\mathbb{E}[h_2(X)] \geq \mathbb{E}[I(X)]$. If $c_2 = x_d^I - d_I$, we have $h_2(x) \leq I(x)$ and thus $\mathbb{E}[h_2(X)] \leq \mathbb{E}[I(X)]$. Because $\mathbb{E}[h_2(X)]$ is continuous in c_2 , there must exist c_2 such that $\mathbb{E}[h_2(X)] = \mathbb{E}[I(X)]$. Therefore, $c_2 \in [0, x_d^I - d_I]$.

Then, we have $d_{h_2} = \frac{\theta_1 - \theta_0 + \delta}{\delta} \mathbb{E}[h_2(X)] = d_I$ and $u_{h_2} = \frac{\theta_2 - \theta_0 + \delta}{\delta} \mathbb{E}[h_2(X)] = u_I$. It then

follows that $x_d^{h_2} = c_2 + d_{h_2} \leq x_d^I \leq x_u^I$. Thus, we have

$$\begin{aligned} \mathbb{E}[T_I(X)] - \mathbb{E}[T_{h_2}(X)] &= \mathbb{E}[h_2(X) - I(X)|X \leq x_d^{h_2}] \mathbb{P}(X \leq x_d^{h_2}) \\ &\quad + (1 - \delta) \mathbb{E}[h_2(X) - I(X)|x_d^{h_2} < X < x_u^I] \mathbb{P}(x_d^{h_2} < X < x_u^I) \\ &\quad + \mathbb{E}[h_2(X) - I(X)|X \geq x_u^I] \mathbb{P}(X \geq x_u^I) \\ &\leq \mathbb{E}[h_2(X) - I(X)] = 0. \end{aligned} \tag{3.41}$$

The inequality holds because $h_2(x) \geq I(x)$ in $[x_d^{h_2}, x_u^I]$. It then follows that

$$\mathbb{E}[T_I(X)] \leq \mathbb{E}[T_{h_2}(X)]. \tag{3.42}$$

Here, the equality holds if and only if $h_2(x) = I(x)$ for $x \geq 0$. In addition, $T_I(X)$ up-crosses $T_{h_1}(X)$.

Combining (3.39) and (3.42), we can conclude that, for an arbitrarily selected $I \in \mathcal{S}_3$, if $I \in \tilde{\mathcal{I}}$, then (3.10) holds and completes the proof. If $I \in \mathcal{S}_3 \cap \tilde{\mathcal{I}}^c$, then there exists $h_1 \in \mathcal{I}_1$ and $h_2 \in \mathcal{I}_2$ such that $\mathbb{E}[h_1(X)] = \mathbb{E}[h_2(X)] = \mathbb{E}[I(X)]$ and $\mathbb{E}[T_{h_1}(X)] < \mathbb{E}[T_I(X)] < \mathbb{E}[T_{h_2}(X)]$. In the next part, we will show that, based on h_1 and h_2 , there exists $h \in \tilde{\mathcal{I}}$ such that $\mathbb{E}[h(X)] = \mathbb{E}[I(X)]$ and $\mathbb{E}[T_h(X)] = \mathbb{E}[T_I(X)]$.

According to (3.37), we write $h_1(x) = (x - x_1)_+ - (x - x_2)_+ + (x - x_3)_+$ with $0 \leq x_1 < x_2 = x_1 + d_{h_1} \leq x_3$. Define $\tilde{h}_1(x) = (x - \tilde{x}_1)_+ - (x - \tilde{x}_2)_+ + (x - \tilde{x}_3)_+$ with $x_1 < \tilde{x}_1 < \tilde{x}_2 = \tilde{x}_1 + d_{\tilde{h}_1} \leq \tilde{x}_3$ such that $\mathbb{E}[\tilde{h}_1(X)] = \mathbb{E}[h_1(X)]$. Here, $\tilde{h}_2 \in \mathcal{I}_1$. Then, we have $d_{\tilde{h}_1} = \frac{\theta_1 - \theta_0 + \delta}{\delta} \mathbb{E}[\tilde{h}_1(X)] = \frac{\theta_1 - \theta_0 + \delta}{\delta} \mathbb{E}[h_1(X)] = d_{h_1}$. For given $x_1 < \tilde{x}_1$, it is easy to find that $\tilde{x}_3 < x_3$. By the similar argument, we can prove that $\mathbb{E}[T_{\tilde{h}_1}(X)] > \mathbb{E}[T_{h_1}(X)]$ and T_{h_1} up-crosses $T_{\tilde{h}_1}$. With given $\mathbb{E}[\tilde{h}_1(X)]$, $\tilde{x}_2$ and $\tilde{x}_3$ can be determined by $\tilde{x}_1$, and thus $\mathbb{E}[T_{\tilde{h}_1}(X)]$ can be viewed as an increasing function of $\tilde{x}_1$.

By (3.40), write $h_2(x) = (x - y_1)_+ - (x - y_2)_+ + (x - y_3)_+$ with $0 \leq y_1 < y_2 = y_1 + u_{h_2} \leq y_3$. Define $\tilde{h}_2(x) = (x - \tilde{y}_1)_+ - (x - \tilde{y}_2)_+ + (x - \tilde{y}_3)_+$ with $y_1 < \tilde{y}_1 < \tilde{y}_2 = \tilde{y}_1 + u_{\tilde{h}_2} \leq \tilde{y}_3$ such that $\mathbb{E}[\tilde{h}_2(X)] = \mathbb{E}[h_2(X)]$. Here, $\tilde{h}_2 \in \mathcal{I}_2$. Then, we have $u_{\tilde{h}_2} = \frac{\theta_2 - \theta_0 + \delta}{\delta} \mathbb{E}[\tilde{h}_2(X)] = \frac{\theta_2 - \theta_0 + \delta}{\delta} \mathbb{E}[h_2(X)] = u_{h_2}$. For given $y_1 < \tilde{y}_1$, we can find that $\tilde{y}_3 < y_3$. Similarly, we have $\mathbb{E}[T_{\tilde{h}_2}(X)] < \mathbb{E}[T_{h_2}(X)]$ and T_{h_2} up-crosses $T_{\tilde{h}_2}$. Since $\tilde{y}_2$ and $\tilde{y}_3$ can be determined by $\tilde{y}_1$ under given $\mathbb{E}[\tilde{h}_2(X)]$, we then conclude that $\mathbb{E}[T_{\tilde{h}_2}(X)]$ is a decreasing function of $\tilde{y}_1$.

Define a stop-loss function $I_d(x) = (x - d)_+$ such that $\mathbb{E}[I_d(X)] = \mathbb{E}[I(X)]$. Note that

I_d belongs to both $\mathcal{I}_1$ and $\mathcal{I}_2$, and

$$\mathbb{E}[T_{I_d}(X)] = \max_{\tilde{h}_1 \in \mathcal{I}_1, \mathbb{E}[\tilde{h}_1(X)] = \mathbb{E}[I(X)]} \mathbb{E}[T_{\tilde{h}_1}(X)] = \min_{\tilde{h}_2 \in \mathcal{I}_2, \mathbb{E}[\tilde{h}_2(X)] = \mathbb{E}[I(X)]} \mathbb{E}[T_{\tilde{h}_2}(X)]. \quad (3.43)$$

This means for $\mathbb{E}[T_I(X)] \in (\mathbb{E}[T_{h_1}(X)], \mathbb{E}[T_{h_2}(X)])$, there exists $f_I(x) \in \tilde{\mathcal{I}} = \mathcal{I}_1 \cup \mathcal{I}_2$ such that $\mathbb{E}[f_I(X)] = \mathbb{E}[I(X)]$ and $\mathbb{E}[T_{f_I}(X)] = \mathbb{E}[T_I(X)]$. Furthermore, since $T_I(X)$ up-crosses $T_{\tilde{h}_1}(X)$ and $T_{\tilde{h}_2}(X)$ for any $\tilde{h}_1 \in \mathcal{I}_1$ and $\tilde{h}_2 \in \mathcal{I}_2$, by Lemma 3 in [Ohlin \(1969\)](#), we have $T_I(X) \geq_{cx} T_{f_I}(X)$, which completes the proof. $\square$

Chapter 4

Indifference premium with background risk and model uncertainty

4.1 Introduction

Premium principles are mathematical formulations of insurance pricing, ensuring fair compensation that reflects the underlying risk exposure. In the framework of optimal reinsurance design, [Cai et al. \(2008\)](#) derive optimal ceded loss functions using the expectation premium principle. In the context of incentive-compatible contract design, [Landriault et al. \(2024\)](#) propose a performance-based variable premium scheme for reinsurance policies, where the premium is adjusted based on the actual loss during the policy period. The literature has also examined properties of premium principles from an axiomatic perspective. For example, [Reich \(1986\)](#) analyzed several premium principles in light of practical properties such as homogeneity and subadditivity. [Young \(2004\)](#) lists and discusses desirable properties of premium principles. The principle of equivalent utility, originating from the insurer's utility maximization framework (see [Von Neumann and Morgenstern \(1947\)](#)), defines the premium as the amount that equates the utility of taking on risk to the utility of paying compensation. This principle has wide application in both finance and actuarial science. [Moore and Young \(2003\)](#) price equity-linked pure endowments via the principle of

equivalent utility. Building on the equivalent utility pricing principle, [Denuit et al. \(2006b\)](#) generate broad classes of risk measures, of which most classical risk measures appear to be particular cases.

In practical decision-making, economic agents are rarely exposed to isolated risks. Instead, they often face multiple sources of risks simultaneously. One such unavoidable source is background risk, the random factor that cannot be traded or insured against directly. In the context of insurance, background risk may arise from a variety of origins. For individuals, it can include labor income fluctuations, health shocks, or consumption volatility. For insurers, such background risk may represent retained liabilities, existing business lines, or operational exposure that contribute to the firm’s overall exposure. The influence of background risk on decision-making has been extensively studied in the economics and actuarial literature. [Pratt and Zeckhauser \(1987\)](#) and [Gollier and Pratt \(1996\)](#) demonstrate that background risk can increase risk aversion and reduce willingness to take on additional risky assets. [Heaton and Lucas \(2000\)](#) focus on how the presence of background risks, from sources such as labor and entrepreneurial income, influences portfolio allocations. Within actuarial science, [Eeckhoudt and Kimball \(1992\)](#) show how the concept of absolute prudence can be useful to derive comparative statics results of optimal insurance in presence of uninsurable background risk. [Dana and Scarsini \(2007\)](#) discuss the qualitative properties of efficient insurance contracts in the presence of background risk. [Tsanakas \(2008\)](#) shows that the presence of background risk makes risk measurement sensitive to the scale and aggregation of risk.

It is not merely the presence of background risk, but its dependence structure with the new insurable risk that plays a decisive role in determining the properties of optimal risk sharing and premium principles. In the optimal reinsurance framework, [Lu et al. \(2018\)](#) investigate the optimal insurance problem under the stop-loss consistent criterion when the insured faces background risk. [Chi and Wei \(2020\)](#) establish a necessary and sufficient condition for the optimality of an insurance contract under a general dependence structure between insurable and background risks. [Chi and Tan \(2021\)](#) discuss the influence of the stochastic dependence between the background risk and the insurable risk on the insured’s risk transfer decision. Moreover, in the presence of dependence, the premium of a new risk cannot be assessed in isolation from the background exposure. Despite the importance of background risk in insurance decision-making, few studies have examined how it affects

the properties of premium principles.

Our work builds on this growing body of literature by explicitly incorporating both background risk and its dependence structure into the premium evaluation process under the equivalent utility principle. Rather than assuming independence, we allow for other forms of dependence including stochastic increasing and mutually exclusive. This enables us to capture a variety of practical contexts, ranging from positively dependent catastrophic exposures to negatively associated business lines, and to analyze how these dependencies alter the insured's equivalent premium.

This section is organized as follows. Section 4.2 introduces the necessary preliminaries. Section 4.3 discusses the properties of the indifference premium in the presence of a background risk, focusing on various dependence structures including independence, stochastic monotonicity, and mutual exclusivity. In addition, Section 4.4 explores the properties of the indifference premium under model uncertainty. The numerical results are illustrated in section 4.5. Finally, section 4.6 concludes this work.

4.2 Preliminaries

4.2.1 Indifference premium with background risk

Suppose that the insured's preferences on her random wealth position can be characterized by a preference operator $V : \mathcal{X} \rightarrow \mathbb{R}$, such that a higher value of V represents a more favorable position for the insured. For a rational insured, it is natural to assume that V satisfies the following three properties.

- (A1) *Law-invariance*: $V(X)$ depends only on the distribution of X .
- (A2) *Strict consistency with first stochastic order*: $V(X_1) \leq V(X_2)$ for all $X_1, X_2 \in \mathcal{X}$ such that $X_1 \preceq_{st} X_2$, and the equality holds only if $X_1 \stackrel{d}{=} X_2$.
- (A3) *Anti-consistency with convex order*: $V(X_1) \geq V(X_2)$ for all $X_1, X_2 \in \mathcal{X}$ such that $X_1 \preceq_{cx} X_2$, and the equality holds only if $X_1 \stackrel{d}{=} X_2$.

Properties (A2) and (A3) provide an ordering of risks. Preserving first stochastic order is a basic rationality condition, as it ensures the decision-maker prefers higher outcomes to lower ones. Anti-preserving convex order reflects risk aversion, that is, preferring less risky distributions with the same mean. Many premium principles, utility functions, and risk measures are designed specifically to respect these orders to ensure basic economic rationality. For a comprehensive discussion of stochastic orders, see [Shaked and Shanthikumar \(2007\)](#), [Föllmer and Schied \(2016\)](#), and [Levy \(2015\)](#). Common preference operators include expected utility functions and risk measures. An expected utility functional is law-invariant, preserves first-order stochastic dominance if the utility function is non-decreasing, and anti-preserves convex order if and only if the utility function is concave. Similarly, a law-invariant risk measure preserves first-order stochastic dominance when it is monotonic and preserves convex order when it is convex.

Premium principles can be derived from the concept of the indifference premium, the premium amount at which an economic agent is indifferent between accepting and rejecting a risk, originates from expected utility theory, and has deep roots in the study of utility-based pricing in insurance and finance. For more discussion about the indifference premium, see [Pratt \(1964\)](#), [Gerber and Pafum \(1998\)](#), and [Young \(2004\)](#).

Let w_0 denote the policy buyer's initial wealth. Through entering an insurance contract, the insurable risk X can be transferred to the policy seller by paying the premium $\pi(X)$. The buyer's terminal wealth becomes $W = w_0 - \pi(X)$ if she enters the insurance contract, otherwise her terminal wealth is $W = w_0 - X$. The maximum acceptable premium $\pi(X)$ for the buyer is the amount such that she is indifferent between entering the contract or retaining the risk by herself. This indifference can be formally expressed by requiring that the insured's preference operator, measured by a chosen functional V , remain unchanged before and after the contract. Specifically, the premium π is acceptable for the policy buyer if $V(w_0 - X) \leq V(w_0 - \pi(X))$. Since V preserves the first stochastic order, the premium such that the equality holds is the highest premium the buyer can accept. With a preference function V , the *indifference premium* for the insurable risk X , denoted by $\pi(X)$, is the quantity satisfying the following equation

$$V(w_0 - X) = V(w_0 - \pi(X)). \quad (4.1)$$

We can also view equation (4.1) from the policy seller's perspective, as they are nat-

urally more concerned about the risk taken on by selling the contract. If we take V as a translation-invariant and law-invariant risk measure, equation (4.1) can be reformulated as $V(\pi(X) - X) = 0$. This means that the indifference premium $\pi(X)$ for the policy seller is the exact amount charged such that the seller's risk level remains unchanged after the contract is sold.

The best known preference function is the expected utility, defined as $V(X) = \mathbb{E}[u(X)]$, where the utility function u is increasing and concave for a risk averse agent. Then equation (4.1) can further be written as

$$\mathbb{E}[u(w_0 - \pi(X))] = \mathbb{E}[u(w_0 - X)],$$

and the indifference price $\pi(X)$ solved by this equation is known as the *premium principle of equivalent utility*. Further, when the exponential utility is adopted, i.e., $u(x) = (1 - e^{-ax})/a$ with $a > 0$, the indifference premium $\pi(X) = \frac{1}{a} \ln \mathbb{E}[e^{aX}]$ is known as the *exponential premium principle*.

In this paper, we investigate the properties of the indifference premium of X with the existence of the background risk $Y \in \mathcal{X}$. We assume that both Y and X are non-negative random variables representing risks. The background risk represents the non-transferrable risks faced by the policy buyer, such as labor income fluctuations for the individual, and existing business lines for the company. We extend the indifference premium defined by equation (4.1) to a more general formulation to incorporate the background risk.

The definition of the indifference premium with the background risk from the perspective of the policy buyer is described as follows.

Definition 4.2.1 (Indifference Premium). The *indifference premium* $\pi(X)$ of the insurable risk X with the existence of the background risk Y is defined as the solution to

$$V(w_0 - Y - X) = V(w_0 - Y - \pi(X)), \quad (4.2)$$

where V is the preference operator of the policy buyer and w_0 is her initial wealth.

Similar to equation (4.1), the indifference premium in equation (4.2) can be applied from either the policyholder's or the policy seller's perspective. This means the economic agent is indifferent between transacting (buying or selling) the insurance policy and not

transacting it, given the background risk Y . The existence of the background risk and its dependence with the insurable risk affect the insured's preference and thus her choice of taking insurance or not for a certain premium. It is interesting to investigate the effect of the dependence structure between the background and the insurable risks on the indifference premium.

4.2.2 Dependence structures

The dependence structure between the background and insurable risks may cast influence on the indifference premium. In this subsection, we discuss some dependence structures we are interested. In particular, the risks X and Y are assumed to be independent, stochastic increasing, or mutually exclusive. To proceed, we first introduce these three dependence structures.

Definition 4.2.2 (Stochastically increasing). A random variable X_2 is said to be *stochastically increasing* in another random variable X_1 , denoted by $X_2 \uparrow_{st} X_1$, if $\mathbb{E}[f(X_2)|X_1 = x]$ is non-decreasing in x for all x in the support of X_1 and for every non-decreasing function f for which the expectations exist. Equivalently, for any $y \in \mathbb{R}$, the probability $\mathbb{P}(X_2 > y|X_1 = x)$ is non-decreasing in x for x in the support of X_1 .

The notion of stochastic increasing dependence captures a form of positive association between random variables. When the insurable risk X is stochastically increasing in Y , an increase in Y tends to be associated with an increase in X . This structure reflects many real-world scenarios in insurance, such as when an economic downturn simultaneously raises both business-related losses and labor income risk, or when environmental hazards increase the likelihood of both property and operational risks. The concept has been widely used in economics and actuarial science to model positively associated risks and has been shown to influence optimal insurance coverage and pricing (e.g., see [Eeckhoudt et al. \(2005\)](#) and [Chi and Tan \(2021\)](#)).

Definition 4.2.3 (Mutually exclusive). Non-negative random variables X_1 and X_2 are *mutually exclusive* if $\mathbb{P}(X_1 > 0, X_2 > 0) = 0$.

Mutually exclusive risks represent the opposite extreme of dependence. When two risks cannot occur simultaneously, the diversification benefit emerges. This structure is

particularly relevant in portfolios where risks are non-overlapping by design or nature. Modeling such exclusive scenarios allows us to explore the minimal possible aggregate risk and evaluate how this setting influences the insured's valuation of insurance. Prior studies, such as [Dhaene et al. \(2002\)](#) and [Denuit et al. \(2006a\)](#), highlight the importance of mutual exclusivity in optimizing risk sharing and capital allocation.

In the context of indifference pricing, stochastic increasing dependence affects the riskiness of the total position and, consequently, the acceptable premium. When the background and insurable risks are positively dependent, the combined variability faced by the economic agent increases, potentially making insurance more attractive at a given price. In contract, the indifference premium should be lower when the insurable risk is mutually exclusive to the background risk. By comparing the dependence structures of independence, stochastic increasing, and mutual exclusivity, we aim to provide a deeper understanding of how the joint behavior of risks impacts the indifference premium. This analysis can offer valuable insights for insurers and policymakers in assessing the sensitivity of pricing principles to underlying dependence, especially in the presence of non-transferable background risks.

4.3 Properties of indifference premium with background risk

In this section, we explore the properties of the indifference premium $\pi(X)$ under different dependence structures between the insurable risk X and the background risk Y . We aim to examine how the joint behavior of X and Y influences the satisfaction of desirable properties commonly required of premium principles. We begin with a general lemma that provides basic conditions under which the indifference premium is well-defined and satisfies fundamental properties.

Lemma 4.3.1. *If V is law-invariant and strict consistent with first stochastic order, then for given Y , $\pi(X)$ is unique for each $X \in \mathcal{X}$, and $\pi(\cdot)$ satisfies (P2), (P3), and (P4).*

Proof. Suppose V satisfies (A1) and (A2). Assume $\pi(X)$ is not unique for some $X \in \mathcal{X}$, i.e. there exists $\pi_1 < \pi_2$ satisfying the equation $V(w_0 - Y - X) = V(w_0 - Y - \pi)$.

Since $w_0 - Y(\omega) - \pi_1 > w_0 - Y(\omega) - \pi_2$ for all $\omega \in \Omega$, we know $w_0 - Y - \pi_1$ strictly dominates $w_0 - Y - \pi_2$ in the sense of the first stochastic order. Thus, we got from (A2) $V(w_0 - Y(\omega) - \pi_2) < V(w_0 - Y(\omega) - \pi_1)$, which contradict with the assumption that π_1 and π_2 are indifference prices.

(*No unjustified risk loading*). Obviously, $\pi(c) = c$ is a solution to the equation $V(w_0 - Y - c) = V(w_0 - Y - \pi(c))$ and it is the unique solution.

(*Translation invariance*). For any $c \in \mathbb{R}$, we have $V(w_0 - Y - \pi(X) - c) = V(w_0 - Y - (X + c)) = V(w_0 - Y - \pi(X + c))$. This implies that $\pi(X + c) = \pi(X) + c$.

(*No rip-off*). If $\bar{X} = \text{ess-sup} X = \infty$, then (P4) is always satisfied. Suppose $\bar{X} < \infty$ and $\pi(X) > \bar{X}$. There exist $\varepsilon > 0$ such that $\pi(X) - \varepsilon > \bar{X}$. Then, we have $X + Y \leq \pi(X) - \varepsilon + Y$, and thus $X + Y \preceq_{st} \pi(X) - \varepsilon + Y$. It follows that $V(w_0 - X - Y) \geq V(w_0 - \pi(X) + \varepsilon - Y) > V(w_0 - \pi(X) - Y)$, which contradicts the definition of indifference price $\pi(X)$. $\square$

This lemma implies that the indifference premium $\pi(X)$ for the insurable risk X is uniquely determined by the preference operator V and the background risk Y . More precisely, for a given V , the value of $\pi(X)$ is uniquely determined by the joint distribution of Y and X . Equivalently, $\pi(X)$ is fully characterized by the marginal distributions of Y and X together with their dependence structure. Before proceeding, we state the following assumption.

Assumption 5. The insured's objective V satisfies properties (V1), (V2), and (V3).

Let F_Y and F_X denote the marginal distributions of the background risk Y and the insurable risk X , respectively. Let π_{id} represent the indifference premium for X when X and Y are independent, and let π_{st} denote the indifference premium when X is stochastically increasing in Y (i.e., $X \uparrow_{st} Y$). We first present a result comparing the indifference premiums in settings where the insurable risk X is either independent of or stochastically increasing with respect to the background risk Y .

Lemma 4.3.2. *When the marginal distributions F_X and F_Y are given, we have $\pi_{id} \leq \pi_{st}$.*

Proof. According to Chi and Wei (2020), if $X \uparrow_{st} Y$, then X and Y are positively quadrant dependent; that is, for all $x, y \in \mathbb{R}$, $\mathbb{P}(X > x, Y > y) \geq \mathbb{P}(X > x)\mathbb{P}(Y > y)$. Consequently,

there exists a random variable $X' \sim F_X$ that is independent of Y . By Proposition 5.3.9 in [Denuit et al. \(2006a\)](#), it follows that $X' + Y \preceq_{cx} X + Y$. Since V anti-preserves the convex order, we have $V(w_0 - Y - \pi(X')) = V(w_0 - Y - X') \geq V(w_0 - Y - X) = V(w_0 - Y - \pi(X))$. Thus, $\pi(X') \leq \pi(X)$, which implies $\pi_{id} \leq \pi_{st}$. $\square$

The lemma indicates that, for given marginal distributions, assuming dependence between X and Y will always lead to an underestimation of the indifference premium if X is actually stochastic increasing in Y . This result serves as the foundation for the analysis presented in the following subsection. In the remainder of this section, we investigate the properties of the indifference premium under various dependent structures between the background risk and the insurable risks. In particular, we focus on three cases of interest: independence, stochastic increasing, and mutually exclusive.

4.3.1 Independence

Proposition 4.3.1. *If the insurable risk X and the background risk Y are independent, then for given Y , the indifference premium for X satisfies (P1)-(P5), (P7)-(P8), and does not satisfy (P6).*

Proof. (Law-invariance). Assume that $\pi(X)$ depends not solely on the distribution of X , i.e., there exists $X_1, X_2 \in \mathcal{X}$ independent of Y such that $X_1 \stackrel{d}{=} X_2$ and $\pi(X_1) < \pi(X_2)$. Then, we have $X_1 + Y \stackrel{d}{=} X_2 + Y$. Since V is law-invariant, we have $V(w_0 - Y - X_1) = V(w_0 - Y - X_2)$. Because V strictly preserves the first stochastic order, we then have $V(w_0 - Y - X_1) = V(w_0 - Y - \pi(X_1)) > V(w_0 - Y - \pi(X_1)) = V(w_0 - Y - X_2)$, which leads to a contradiction.

(Risk loading). For any convex function φ , since X, Y are independent, we know

$$\begin{aligned} \mathbb{E}[\varphi(X + Y)] - \mathbb{E}[\varphi(\mathbb{E}[X] + Y)] &= \int_0^\infty \mathbb{E}[\varphi(X + y)|Y = y] f_Y(y) dy \\ &\quad - \int_0^\infty \varphi(\mathbb{E}[X] + y) f_Y(y) dy \\ &\geq 0, \end{aligned}$$

where the inequality is an application of Jensen's inequality on $\mathbb{E}[\varphi(X + y)] \geq \varphi(\mathbb{E}[X] + y)$ for all $y \geq 0$. Thus, $\mathbb{E}[X] + Y \preceq_{cx} X + Y$. It follows that $V(w_0 - Y - \pi(X)) = V(w_0 - Y - X) \leq V(w_0 - Y - \mathbb{E}[X])$, and thus $\pi(X) \geq \mathbb{E}[X]$.

(*First stochastic order consistency*). Suppose $X_1 \preceq_{st} X_2$. Note that for each $z \geq 0$, we have

$$\mathbb{P}(X_1 + Y > z) - \mathbb{P}(X_2 + Y > z) = \int_0^z (\mathbb{P}(X_1 > z - y) - \mathbb{P}(X_2 > z - y)) f_Y(y) dy \leq 0.$$

Thus, $X_1 + Y \preceq_{st} X_2 + Y$. It follows that

$$V(w_0 - Y - \pi(X_1)) = V(w_0 - X_1 - Y) \geq V(w_0 - X_2 - Y) = V(w_0 - Y - \pi(X_2))$$

and then $\pi(X_1) \leq \pi(X_2)$.

(*Convex order consistency*). Suppose $X_1 \preceq_{cx} X_2$. For any convex function φ and $y \geq 0$, $\mathbb{E}[\varphi(X_1 + y)] \leq \mathbb{E}[\varphi(X_2 + y)]$. Thus,

$$\mathbb{E}[\varphi(X_1 + Y)] = \mathbb{E}[\mathbb{E}[\varphi(X_1 + Y)|Y]] \leq \mathbb{E}[\mathbb{E}[\varphi(X_2 + Y)|Y]] = \mathbb{E}[\varphi(X_2 + Y)].$$

That is $X_1 + Y \preceq_{cx} X_2 + Y$. Therefore, $V(w_0 - Y - X_1) \geq V(w_0 - Y - X_2)$ and then $\pi(X_1) \leq \pi(X_2)$.

(*Positive homogeneity*). The indifference premium for X does not satisfy positive homogeneity. Assume that $Y_1 \preceq_{st} Y_2$ are two background risks, and both of them are independent of X . Then, $Y_1 + X \preceq_{st} Y_2 + X$. Denote the indifference premiums for X under background risks Y_1 and Y_2 are π_1 and π_2 , respectively. Since V preserves first order stochastic dominance, we have

$$V(w_0 - Y_1 - \pi_1) = V(w_0 - Y_1 - X) \geq V(w_0 - Y_2 - X) = V(w_0 - Y_1 - \pi_2).$$

It then follows that $\pi_1 \leq \pi_2$. In other words, when the insurable and the background risks are independent, the indifference premium for X increases as the background risk becomes larger in the sense of first-order stochastic dominance. Then, we will prove that $\pi(2X) \neq 2\pi(X)$ if X and Y are independent. By Lemma 4.3.2, the indifference premium for X in the presence of the background risk $X + Y$, denoted by π_3 , is greater than the premium for X when the background risk, although having the same distribution as $X + Y$,

is independent of X , denoted as π_4 . Since $Y \preceq_{st} X + Y$, then $\pi(X) \leq \pi_3$. It then follows that

$$\begin{aligned} V(w_0 - Y - \pi(2X)) &= V(w_0 - Y - 2X) = V(w_0 - Y - X - \pi_3) \\ &< V(w_0 - Y - X - \pi_4) \\ &= V(w_0 - Y - \pi(X) - \pi_4). \end{aligned}$$

The last inequality holds because of the translation invariance property of π . Thus, $\pi(2X) > \pi(X) + \pi_4 \geq 2\pi(X)$. $\square$

This proposition demonstrates that, under independence, the indifference premium retains a set of desirable properties. These characteristics are in line with economic intuition and are consistent with those of classical premium principles.

4.3.2 Stochastically increasing

We now consider a more general and practically relevant setting in which the background risk is not independent of the insurable risk, but instead exhibits positive dependence.

Proposition 4.3.2. *If the insurable risk X is stochastically increasing in the background risk Y , i.e. $X \uparrow_{st} Y$, for given Y , the indifference premium for X satisfies (P2) to (P5), and does not satisfy (P1), (P7), and (P8).*

Proof. (Risk loading). For any convex function φ , we have

$$\mathbb{E}[\varphi(Y + X)] = \mathbb{E}[\mathbb{E}[\varphi(Y + X)|Y]] \geq \mathbb{E}[\varphi(Y + \mathbb{E}[X|Y])],$$

where the inequality follows from Jensen's inequality. This implies that $Y + X \succ_{cx} Y + \mathbb{E}[X|Y]$. By the definition of $X \uparrow_{st} Y$, $\mathbb{E}[X|Y]$ is comonotonic with Y . According to Proposition 3.4.29 in [Denuit et al. \(2006a\)](#), it follows that $Y + \mathbb{E}[X|Y] \succ_{sl} Y + \mathbb{E}[X]$, and consequently, $Y + \mathbb{E}[X|Y] \succ_{cx} Y + \mathbb{E}[X]$. Combining results, we obtain $Y + X \succ_{cx} Y + \mathbb{E}[X]$. Since V anti-preserves the convex order, we have

$$V(w_0 - Y - \pi(X)) = V(w_0 - Y - X) \leq V(w_0 - Y - \mathbb{E}[X]),$$

which implies that $\pi(X) \geq \mathbb{E}[X]$.

(*Law-invariance*). Note that when $X \uparrow_{st} Y$, the indifference premium of X is not law-invariant. Here is an example. Let $Y \sim \text{Poisson}(\lambda_1)$, $X_1 = Y$, and $X_2 \sim \text{Poisson}(\lambda_2)$ be independent of Y . Recall that the moment generation function for the Poisson random variable Y is $M_Y(t) = e^{\lambda_1(e^t - 1)}$. Note that $X_1 \uparrow_{st} Y$, $X_2 \uparrow_{st} Y$, and $X_1 \stackrel{d}{=} X_2$ if $\lambda_1 = \lambda_2$. If the expected exponential utility preference $V(X) = \mathbb{E}[(1 - e^{-X})]$ is applied, by equation (4.6), we have that the premium for X_1 is

$$\pi(X_1) = \ln \frac{\mathbb{E}[e^Y e^{X_1}]}{\mathbb{E}[e^Y]} = \ln \frac{M_{X_1}(2)}{M_{X_1}(1)} = \lambda_1 e(e - 1),$$

and the premium for X_2 is

$$\pi(X_2) = \ln \mathbb{E}[e^{X_2}] = \ln M_{X_2}(1) = \lambda_2(e - 1).$$

Then, we have $\pi(X_1) > \pi(X_2)$ for $\lambda_1 = \lambda_2$. This example suggests that the indifference premium π depends on the degree to which X is stochastically increasing in Y .

(*First stochastic order consistency*). In the previous example, $X_1 \preceq_{st} X_2$ if $\lambda_1 < \lambda_2$. As long as $1 < \frac{\lambda_2}{\lambda_1} < e$, we have $\pi(X_1) > \pi(X_2)$. This example implies that π does not preserve first-order stochastic dominance.

(*Convex order consistency*). Consider the following example. Let Y be a two-point random variable, taking the value $-a$ with probability 0.5 and a with probability 0.5. Define $X_1 = Y$, and let X_2 be independent of Y , taking the value $-b$ or b each with probability 0.5. It is evident that both $X_1 \uparrow_{st} Y$ and $X_2 \uparrow_{st} Y$. Furthermore, if $a < b$, it can be verified that $X_1 \preceq_{cx} X_2$. Consider the expected exponential utility preference, $V(X) = \mathbb{E}[1 - e^{-X}]$. According to equation (4.6), the indifference premium for X_1 is given by

$$\pi(X_1) = \ln \frac{\mathbb{E}[e^Y e^{X_1}]}{\mathbb{E}[e^Y]} = \ln \frac{e^{-2a} + e^{2a}}{e^{-a} + e^a},$$

while the indifference premium for X_2 is

$$\pi(X_2) = \ln \mathbb{E}[e^{X_2}] = \ln\left(\frac{1}{2}e^{-b} + \frac{1}{2}e^b\right).$$

Since $\pi(X_1) > \pi(X_2)$ for all $a = b > 0$, there always exists $\varepsilon > 0$ such that $b = a + \varepsilon$ and $\pi(X_1) > \pi(X_2)$. This example demonstrates that the indifference premium may fail to preserve the convex order. $\square$

4.3.3 Mutually exclusive

We now consider the case where the insurable risk X and the background risk Y are mutually exclusive. This dependence structure captures situations where the occurrence of one risk precludes the occurrence of the other, an assumption that, while strong, is applicable in certain insurance portfolios (e.g., claims that arise from disjoint causes). In this case, we analyze the indifference premium under the expected utility framework. Assume $V(w_0 - Y - X) = \mathbb{E}[u(w_0 - Y - X)]$ for some increasing and concave function $u : \mathbb{R} \rightarrow \mathbb{R}$. Let $\tilde{u}(t) = -u(w_0 - t)$. Then

$$V(w_0 - Y - X) = -\mathbb{E}[\tilde{u}(X + Y)], \quad (4.3)$$

and $\tilde{u}$ is an increasing and convex function.

Proposition 4.3.3. *If the insurable risk X and the background risk Y are mutually exclusive and the insured's objective is given by (4.3), the indifference premium for X satisfies (P1) to (P4), (P7) and (P8).*

Proof. (Law-invariance). It is easy to find that the joint distribution of (Y, X) is uniquely determined by its marginal distributions if Y and X are mutually exclusive. Since the preference operator V is law-invariant, then the indifference premium $\pi(X)$ is law-invariant.

(First stochastic order consistency). Suppose Y and X_1 are exclusive, Y and X_2 are exclusive, and $X_1 \preceq_{st} X_2$. Note that, for $i = 1, 2$,

$$\begin{aligned} \mathbb{E}[\tilde{u}(X_i + Y)] &= \mathbb{E}[\tilde{u}(X_i)|Y = 0] \mathbb{P}(Y = 0) + \mathbb{E}[\tilde{u}(Y)|Y > 0] \mathbb{P}(Y > 0) \\ &= \mathbb{E}[\tilde{u}(X_i)|Y = 0] \mathbb{P}(Y = 0) + \tilde{u}(0) \mathbb{P}(Y > 0) \\ &\quad + \mathbb{E}[\tilde{u}(Y)|Y > 0] \mathbb{P}(Y > 0) + \tilde{u}(0) \mathbb{P}(Y = 0) - \tilde{u}(0) \\ &= \mathbb{E}[\tilde{u}(X_i)] + \mathbb{E}[\tilde{u}(Y)] - \tilde{u}(0). \end{aligned}$$

Since $\tilde{u}$ is an increasing function and $X_1 \preceq_{st} X_2$, we have

$$\begin{aligned} V(w_0 - Y - \pi(X_1)) &= -\mathbb{E}[\tilde{u}(X_1 + Y)] = -\mathbb{E}[\tilde{u}(X_1)] - \mathbb{E}[\tilde{u}(Y)] + \tilde{u}(0) \\ &\geq -\mathbb{E}[\tilde{u}(X_2)] - \mathbb{E}[\tilde{u}(Y)] + \tilde{u}(0) \\ &= -\mathbb{E}[\tilde{u}(X_2 + Y)] \\ &= V(w_0 - Y - \pi(X_2)). \end{aligned}$$

Thus, $\pi(X_1) \leq \pi(X_2)$.

(*Convex order consistency*). Suppose Y and X_1 are exclusive, Y and X_2 are exclusive, and $X_1 \preceq_{cx} X_2$. Since $\mathbb{E}[\tilde{u}(X_1)] \leq \mathbb{E}[\tilde{u}(X_2)]$ for convex h , by the same argument as above, we have $\pi(X_1) \leq \pi(X_2)$. $\square$

The proposition shows that, even in the presence of strong negative dependence, such as mutual exclusivity, the indifference premium maintains consistency with monotonicity (P7) and the convex order (P8). These properties are particularly meaningful in risk management, as they ensure the premium appropriately reflects both the size and the tail behavior of the risk being priced.

4.4 Properties of indifference premium with model uncertainty

In this section, we introduce model uncertainty into the framework of indifference premiums. This consideration is particularly relevant in situations where historical data on the new insured risk is limited, leading the agent to adopt a conservative attitude towards the new insurable risk. For the insurable risk X and the background risk Y , we can always find a uniform random variables $U_X \sim U[0, 1]$ and $U_Y \sim U[0, 1]$ such that

$$X = F_X^{-1}(U_X) \text{ almost surely, and } Y = F_Y^{-1}(U_Y) \text{ almost surely.} \quad (4.4)$$

Particularly, we can define $U_X = F_X(X)$ (resp. $U_Y = F_Y(Y)$) if F_X (resp. F_Y) is continuous. Let $\mathcal{P}$ denote the set of admissible distribution functions. In the presence of model uncertainty, it is interesting to characterize the properties of the worst-case indifference premiums in the presence of background risk Y . That is to say, the policyholder seeks to identify the upper bound of indifference premiums corresponding to the least favorable distributions within $\mathcal{P}$, taking into account the existence of background risk Y . Specifically, the maximum possible indifference premium $\bar{\pi}(X)$, representing the worst-case scenario, is defined as the solution to the following equation.

$$\inf_{F \in \mathcal{P}} V(w_0 - F_Y^{-1}(U_Y) - F^{-1}(U_X)) = V(w_0 - F_Y^{-1}(U_Y) - \bar{\pi}(X)). \quad (4.5)$$

This formulation focuses on the uncertainty of the marginal distribution of the insurable risk. The equation (4.5) marks the starting point for introducing model uncertainty into the indifference premium framework with background risk. Many other avenues remain open for incorporating model uncertainty, for instance, accounting for distributional uncertainty in the background risk itself, or even uncertainty in the dependence structure between the background and insurable risks.

In the following propositions, we elucidate the relationship between the dependence structure of X and Y and that of U_X and U_Y , under the cases where X and Y are independent, stochastic increasing, or mutually exclusive.

Proposition 4.4.1. *For any independent $X, Y \in \mathcal{X}$, there exist independent uniform random variables U_X and U_Y such that (4.4) holds. Conversely, for any distribution functions F_X and F_Y , if U_X and U_Y are independent uniform random variables, then X and Y defined via (4.4) are independent.*

Proof. Find $U'_X, U'_Y \in U[0, 1]$ such that U'_X, U'_Y, X, Y are independent. For $x \in \mathbb{R}$, let $F_X(x^-) = \lim_{t \uparrow x} F_X(t)$. Define $U_X = F_X(X^-) + U'_X[F_X(X) - F_X(X^-)]$. When X falls into its atom, then $F_X(X) - F_X(X^-) > 0$ and the factor U'_X spreads the probability uniformly across the interval. If X is taken from the continuous part, then $F_X(X) = F_X(X^-)$ and $U_X = F_X(X)$. We can easily verify that $X = F_X^{-1}(U_X)$ almost surely. Similarly, define $U_Y = F_Y(Y^-) + U'_Y[F_Y(Y) - F_Y(Y^-)]$, and $Y = F_Y^{-1}(U_Y)$ almost surely. Since $\sigma(U_X) \subset \sigma(\sigma(X) \cup \sigma(U'_X))$ and $\sigma(U_Y) \subset \sigma(\sigma(Y) \cup \sigma(U'_Y))$, it follows from Theorem 2.1.5 in Durrett (2019) that U_X and U_Y are independent.

On the other hand, Theorem 2.1.6 in Durrett (2019) guarantees that the independence of U_X and U_Y always implies the independence of $X = F_X^{-1}(U_X)$ and $Y = F_Y^{-1}(U_Y)$. $\square$

Remark 5. The independence of X and Y does not always lead to the independence of U_X and U_Y if either X or Y has an atom point. For example, assume $U \sim U[0, 1]$, define $X = -1_{\{U \leq \frac{1}{2}\}} + 1_{\{U > \frac{1}{2}\}}$, and $Y = -1_{\{U \leq \frac{1}{4} \text{ or } U > \frac{3}{4}\}} + 1_{\{\frac{1}{4} < U \leq \frac{3}{4}\}}$. We can easily verify that $X = F_X^{-1}(U)$ a.s., $Y = F_Y^{-1}(2|U - \frac{1}{2}|)$ a.s., and X and Y are independent. However, $U_X = U$ and $U_Y = 2|U - \frac{1}{2}|$ are dependent.

The following lemma provides the foundation for the subsequent proposition, which characterizes the dependence structure between (X, Y) and (U_X, U_Y) when either pair exhibits stochastic monotonicity.

Proposition 4.4.2. *Assume that $X, Y \in \mathcal{X}$, and U_X and U_Y are uniform random variables defined via (4.4). If F_Y is continuous, then $X \uparrow_{st} Y$ is equivalent to $U_X \uparrow_{st} U_Y$.*

Proof. The continuity of F_Y implies the strict comonotonicity of its quantile function F_Y^{-1} . Then, for given y , we have the events $\{\omega : U_Y(\omega) = y\} = \{\omega : F_Y^{-1}(U_Y(\omega)) = F_Y^{-1}(y)\} = \{\omega : Y(\omega) = F_Y^{-1}(y)\}$ almost surely. It then follows that $\mathbb{P}(U_X > x | U_Y = y) = \mathbb{P}(U_X > x | Y = F_Y^{-1}(y))$. Since $X \uparrow_{st} Y$, we have $\mathbb{P}(U_X > x | Y = F_Y^{-1}(y))$ is an increasing function of y , which implies $U_X \uparrow_{st} U_Y$. Similarly, the converse also holds. $\square$

Proposition 4.4.3. *Assume that $X, Y \in \mathcal{X}$, and U_X and U_Y are uniform random variables defined via (4.4). Then $\mathbb{P}(X > 0, Y > 0) = 0$ if and only if $\mathbb{P}(U_X > F_X(0), U_Y > F_Y(0)) = 0$.*

Proof. Since $\mathbb{P}(X > 0, Y > 0) = 0$, we have that 0 is the mass point of both X and Y . Then event $X > 0$ is equivalent to $U_X > F_X(0)$, and event $Y > 0$ is equivalent to $U_Y > F_Y(0)$ in the sense of probability. $\square$

Propositions 4.4.1, 4.4.2, and 4.4.3 establish that the dependence structure of the background risk Y and the insurable risk X can be replicated by the corresponding uniform random variables U_Y and U_X under mild conditions. Specifically, for any distribution function $F \in \mathcal{P}$, when Y and X are independent, Proposition 4.4.1 implies that there exists U_Y and U_X are independent, and $F_Y^{-1}(U_Y)$ and $F^{-1}(U_X)$ remain independent. Proposition 4.4.2 demonstrates that, provided F_Y is continuous, the condition that X is stochastically increasing in Y is equivalent to $F^{-1}(U_X)$ being stochastically increasing in Y . Finally, for any $F \in \mathcal{P}$ such that $F(0) = F_X(0)$, Proposition 4.4.3 establishes that X and Y are mutually exclusive implies and is implied by $F^{-1}(U_X)$ and $F_Y^{-1}(U_Y)$ being mutually exclusive.

Theorem 4.4.4. *Assume that $\mathcal{P}$ contains n distribution functions. Then there exists a worst-case distribution $\bar{F} \in \mathcal{P}$ at which the infimum in equation (4.5) is attained. Specifically, we have the following results.*

- (i) If U_Y and U_X are independent, then the worst-case indifference premium $\bar{\pi}(X)$ satisfies (P1)-(P5), (P7)-(P8).
- (ii) If F_Y is continuous and $U_X \uparrow_{st} U_Y$, then $\bar{\pi}(X)$ satisfies (P2) to (P5).
- (iii) If $F(0) < F_X(0)$ and $\mathbb{P}(U_X > F(0), U_Y > F_Y(0)) = 0$, then $\bar{\pi}(X)$ satisfies (P1) to (P4), (P7) and (P8).

Proof. If U_Y and U_X are independent, Proposition 4.4.1 demonstrates that $F_Y^{-1}(U_Y)$ is independent of $F^{-1}(U_X)$ for $F \in \mathcal{P}$. Then, it follows from Proposition 4.3.1 that statement (i) holds. By similar arguments, we obtain statements (ii) and (iii). $\square$

Remark 6. If the preference operator is given by the expected utility $V(X) = \mathbb{E}[u(X)]$, where the utility $u(x)$ is a lower semi-continuous and bounded function, and the uncertainty set $\mathcal{P}$ is closed, then the infimum in equation (4.5) is attained and the worst-case premium continues to satisfies the statements in Theorem 4.4.4.

4.5 Numerical examples

In this section, we assume that the preference operator is given by expected utility and compare indifference premiums across various dependent structures and levels of risk aversions. We begin by considering the case where the policyholder adopts the expected exponential utility preference, defined as $V(X) = \mathbb{E}[u(X)]$ with the exponential utility function $u(x) = (1 - e^{-ax})/a$ for risk aversion parameter $a > 0$. The expected exponential utility, also known as the constant absolute risk aversion (CARA) utility, is widely used in actuarial and economic analysis due to its mathematical tractability and analytical convenience, see, for example, Pratt (1978) and Arrow et al. (1965). Under this utility, the indifference premium for X can be derived from Definition 4.2.1 and is formulated as follows:

$$\pi(X) = \frac{1}{a} \ln \frac{\mathbb{E}[e^{aY} e^{aX}]}{\mathbb{E}[e^{aY}]} = \frac{1}{a} \ln \left[\frac{\text{cov}(e^{aY}, e^{aX})}{\mathbb{E}[e^{aY}]} + \mathbb{E}[e^{aX}] \right]. \quad (4.6)$$

Therefore, under the expected exponential utility preference, the premium is independent of the initial wealth w_0 . Moreover, when X and Y are independent, the premium is not affected by the presence of the background risk Y . Given the marginal distributions, let π_{id} , π_{st} , and π_{me} denote the indifference premiums for X in the cases where the insurable risk X and the background risk Y are independent, stochastically increasing ($X \uparrow_{st} Y$), and mutually exclusive, respectively. The following proposition establishes an ordering among these three premiums.

Proposition 4.5.1. *Under the expected exponential utility preference, the indifference premiums satisfy $\pi_{me} \leq \pi_{id} \leq \pi_{st}$.*

Proof. When X and Y are independent, we have $\pi_{id} = \frac{1}{a} \ln \mathbb{E}[e^{aX}]$. When $X \uparrow_{st} Y$, by Definition 4.2.2, for any $x \in \mathbb{R}$, $\mathbb{P}(X > x | Y = y)$ is increasing in y . This is equivalent to $\mathbb{P}(e^{aX} > e^{ax} | e^{aY} = y)$ increasing with y , which further implies that $\pi(e^{aX} > x | e^{aY} = y)$ is an increasing function of y for given $x \in \mathbb{R}$. Then, we have $e^{aX} \uparrow_{st} e^{aY}$. By Chi and Wei (2020), it then follows that $e^{aX} \sim_{PQD} e^{aY}$, i.e., $\mathbb{P}(e^{aX} > x, e^{aY} > y) \geq \mathbb{P}(e^{aX} > x)\mathbb{P}(e^{aY} > y)$ for all x and y . Then, by Lemma 3 in Lehmann (1966), we have $\text{cov}(e^{aY}, e^{aX}) \geq 0$, which implies that $\pi_{id} \leq \pi_{st}$.

Define events $A = \{X = 0, Y > 0\}$, $B = \{X = 0, Y = 0\}$, $C = \{X > 0, Y = 0\}$. When X and Y are mutually exclusive, from Definition 4.2.3, we have that $\mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) = 1$. It then follows that

$$\mathbb{E}[e^{aX} e^{aY}] = \mathbb{E}[e^{aY} 1_A] + \mathbb{E}[e^{aY} 1_B] + 1,$$

and we have

$$\begin{aligned} \mathbb{E}[e^{aX}] \mathbb{E}[e^{aY}] &= \mathbb{E}[e^{aX} 1_B] \mathbb{E}[e^{aY} 1_A] + \mathbb{E}[e^{aX} 1_B] (\mathbb{P}(B) + \mathbb{P}(C)) \\ &\quad + \mathbb{E}[e^{aY} 1_A] (\mathbb{P}(A) + \mathbb{P}(C)) + (\mathbb{P}(A) + \mathbb{P}(C)) (\mathbb{P}(B) + \mathbb{P}(C)). \end{aligned}$$

It then follows that

$$\text{cov}(e^{aX}, e^{aY}) = \mathbb{E}[e^{aX} e^{aY}] - \mathbb{E}[e^{aX}] \mathbb{E}[e^{aY}] = -(\mathbb{E}[e^{aX} 1_B] - \mathbb{P}(B))(\mathbb{E}[e^{aY} 1_A] - \mathbb{P}(A)) \leq 0.$$

Therefore, we obtain $\pi_{me} \leq \pi_{id} \leq \pi_{st}$. □

This result highlights that stronger positive dependence between X and Y (as in the stochastically increasing case) leads to a higher indifference premium, reflecting greater aggregate risk and thus requiring higher compensation for risk transfer. Conversely, negative dependence (as in mutual exclusivity) reduces the overall exposure, resulting in a lower premium. These findings offer a clear quantitative insight into how the structure of dependence affects insurance pricing under utility-based preferences.

In order to characterize the dependence between X and Y , we adopt the Marshall–Olkin Bivariate Exponential distribution, which is chosen for its analytical tractability and flexibility in modeling a broad spectrum of positive dependence structures, making it particularly suitable for actuarial applications (see [Marshall and Olkin \(1967\)](#)). Specifically, let Z_1 , Z_2 , and Z_3 be independent exponential random variables with rates λ_1 , λ_2 , and λ_3 , respectively. We define $X = \min\{Z_1, Z_3\}$ and $Y = \min\{Z_2, Z_3\}$. With this construction, the marginal distributions of X and Y are $\text{Exp}(\lambda_1 + \lambda_3)$ and $\text{Exp}(\lambda_2 + \lambda_3)$, respectively. The dependence between X and Y increases with λ_3 . Specifically, when $\lambda_3 = 0$, X and Y are independent, whereas when $\lambda_1 = \lambda_2 = 0$, X and Y are comonotonic. It is important to note, however, that the Marshall–Olkin model cannot accommodate negative dependence. The joint survival distribution function of (X, Y) is given by

$$\mathbb{P}(X > x, Y > y) = e^{-\lambda_1 x - \lambda_2 y - \lambda_3 \max\{x, y\}}.$$

Let $\lambda \triangleq \lambda_1 + \lambda_2 + \lambda_3$. By Lemma 3.2 in [Marshall and Olkin \(1967\)](#), the correlation between (X, Y) is given by $\frac{\lambda_3}{\lambda}$, and the moment generating function of (X, Y) is formulated as

$$\mathbb{E}[e^{sX} e^{tY}] = \frac{(\lambda - s - t)(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3) + st\lambda_3}{(\lambda - s - t)(\lambda_1 + \lambda_3 - s)(\lambda_2 + \lambda_3 - t)}.$$

Example 4.5.1. We assume that the policyholder adopts the expected exponential utility, and that the dependence structure between X and Y is specified by the Marshall–Olkin Bivariate Exponential model. We impose the conditions $\lambda_1 + \lambda_3 = 2$ and $\lambda_2 + \lambda_3 = 2$, which ensure that the marginal distributions of (X, Y) remain unchanged as the dependence structure transitions from independence to comonotonicity by varying λ_3 from 0 to 2. Under these assumptions, the indifference premium in (4.6) can be reformulated as

$$\pi(X) = \frac{1}{a} \ln \frac{\mathbb{E}[e^{aY} e^{aX}]}{\mathbb{E}[e^{aY}]} = \frac{1}{a} \ln \frac{16 - 4\lambda_3 - 8a + a^2\lambda_3}{2(2 - a)(4 - \lambda_3 - 2a)},$$

where the risk aversion parameter $a < \frac{4-\lambda_3}{2}$.

Figure 4.1 illustrates how the indifference premium and its relative change respond to variations in the correlation between the insurable risk X and the background risk Y . As shown in Figure 4.1-(a), the indifference premium $\pi(X)$ exhibits a clear upward trend as the correlation $\text{corr}(X, Y)$ increases. This reflects the fact that stronger positive dependence between X and Y amplifies the risk exposure faced by the insurer, thereby increasing the required premium. To provide a clearer perspective on the sensitivity of the premium to changes in dependence, Figure 4.1-(b) presents the relative premium, calculated as $\frac{\pi(X) - \pi_{id}}{\pi_{id}}$ with π_{id} being the premium under independence, i.e., $\text{corr}(X, Y) = 0$. The results indicate that even the modest increases in correlation can lead to significant increases in the indifference premium. For instance, at the extreme of perfect positive dependence, i.e., comonotonicity with $\text{corr}(X, Y) = 1$, the premium is approximately 41% higher than under independence. Conversely, if the risks are perfectly negatively dependent, obtained by applying $\text{corr}(X, Y) = -1$ to equation (4.6), the premium can drop to as low as 55% of the independent case. These findings underscore the substantial impact that the dependence structure between X and Y has on the indifference premium. Even when the marginal distributions of the risks remain fixed, the premium can vary dramatically based on their joint behavior. This highlights the importance of accurately modeling dependence in actuarial pricing and risk management.

Figure 4.2 depicts the indifference premium $\pi(X)$ as a function of the risk aversion parameter a , for various levels of correlation between the background risk Y and the insurable risk X . Each curve represents a different level of correlation, ranging from independence, i.e., $\text{corr}(X, Y) = 0$, to perfect positive dependence, i.e., $\text{corr}(X, Y) = 1$. As shown in the figure, for all correlation levels, the indifference premium increases monotonically as the risk aversion parameter a increases. Additionally, for a fixed value of a , the premium rises with the correlation between X and Y . This effect becomes especially pronounced for higher values of a , reflecting the fact that as the policyholder becomes more risk averse, the sensitivity of the premium to the dependence structure becomes substantially greater. These results demonstrate that as the degree of risk aversion grows, accurate modeling of dependence between the insurable risk and background risk becomes increasingly important for premium calculation.

We now consider an alternative preference operator based on the power utility function,

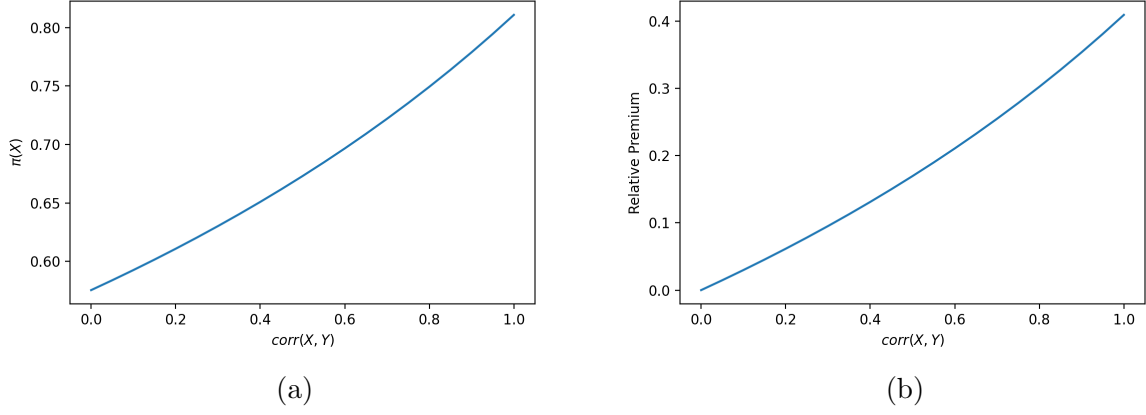


Figure 4.1: (a) Indifference premium for different values of $\text{corr}(X, Y)$. (b) Relative premium $\frac{\pi(X) - \pi_{id}}{\pi_{id}}$ versus $\text{corr}(X, Y)$. Here, the preference operator is $V(X) = \mathbb{E}[(1 - e^{-aX})/a]$ with $a = 1/2$, and (X, Y) is specified by the Marshall–Olkin Bivariate Exponential model.

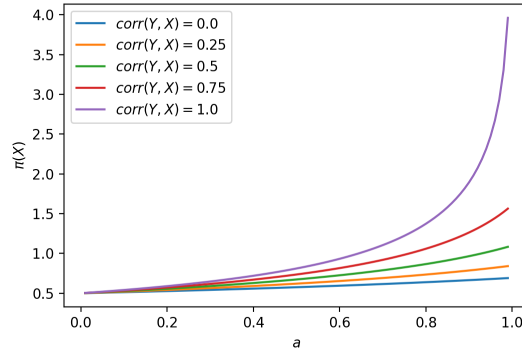


Figure 4.2: Indifference premium versus a . Here, the preference operator is $V(X) = \mathbb{E}[(1 - e^{-aX})/a]$, and (X, Y) follows Marshall–Olkin bivariate exponential distribution.

also known as the constant relative risk aversion (CRRA) utility. This utility reflects the intuitive idea that individuals tend to become less risk-averse as their wealth increases. For further discussions on the properties and applications of power utility see, for example, [Arrow \(1971\)](#), [Cox and Huang \(1989\)](#), and [Bowers et al. \(1997\)](#). The power utility is defined as

$$u(x) = \begin{cases} \frac{x^{1-b}-1}{1-b}, & b \geq 0, b \neq 1, \\ \ln x, & b = 1. \end{cases} \quad (4.7)$$

In contrast to the exponential utility, which often admits closed-form solutions in insurance and finance optimization problems, the power utility function generally leads to more complex optimization problems. As a result, the indifference premium defined in (4.2) typically does not have a closed-form expression. To address this, we employ numerical methods to compute the indifference premium under the power utility framework.

Example 4.5.2. In this example, the policyholder's preferences are modeled using the power utility, with the preference operator given by $V(X) = \mathbb{E}[u(X)]$, where $u(x)$ is defined in equation (4.7). The dependence between the insurable risk X and the background risk Y is captured by the Marshall–Olkin bivariate exponential distribution. To ensure that the marginal distributions of X and Y remain unchanged while varying their dependence, we set $\lambda_1 + \lambda_3 = 2$ and $\lambda_2 + \lambda_3 = 2$, allowing the correlation between X and Y to be adjusted through changes in λ_3 .

Under the power utility framework, Figures 4.3 and 4.4 present the behavior of the indifference premium in relation to dependence structure, risk aversion, and initial wealth. In Figure 4.3, both the absolute and relative premiums for X increase monotonically with the correlation between the insurable risk X and the background risk Y , consistent with intuition that stronger dependence amplifies risk exposure. However, compared to the exponential utility case, the magnitude and sensitivity of the premium to correlation are significantly attenuated, as evidenced by smoother growth and a smaller range of relative premium values.

Figure 4.4 further examines the joint effects of risk aversion and initial wealth. Figure 4.4(a) demonstrates that, as the risk aversion parameter b increases, the indifference premium rises in an approximately linear manner for all levels of correlation, with higher

dependence still yielding higher premiums. Notably, this increase is much less pronounced than under exponential utility. Figure 4.4-(b) reveals that the indifference premium decreases with increasing initial wealth, highlighting that greater wealth reduces the policyholder's sensitivity to risk and lowers the premium they are willing to pay.

The results from both the power utility and exponential utility frameworks consistently demonstrate that the dependence structure between the insurable risk X and the background risk Y exerts a substantial influence on the policyholder's risk preferences and the resulting indifference premium. This impact becomes increasingly pronounced as the degree of risk aversion rises, highlighting the importance of accurately modeling dependence structures in actuarial decision-making, particularly for highly risk-averse individuals.

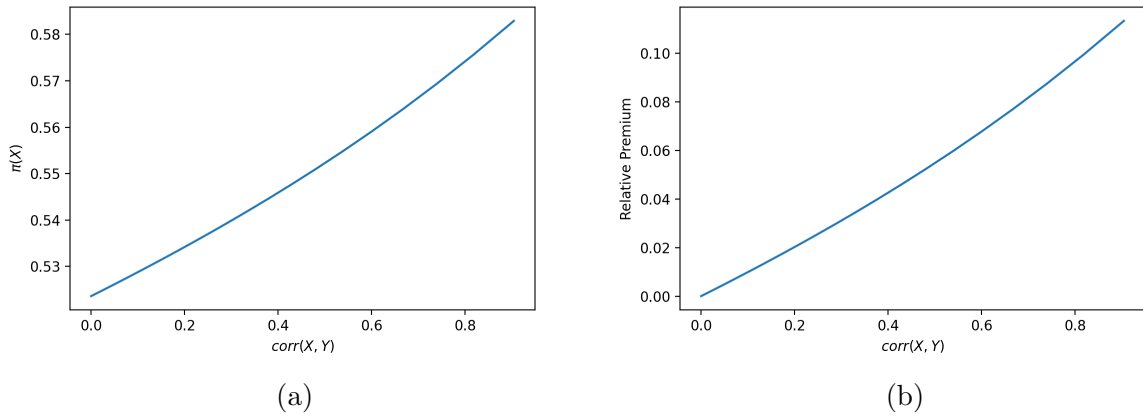


Figure 4.3: (a) Indifference premium versus the correlation between Y and X . (b) Relative premium $\frac{\pi(X) - \pi_{id}}{\pi_{id}}$ as a function of the correlation. Here, the preference operator is the expected power utility with $V(X) = \mathbb{E}[\ln(w_0 - Y - X)]$, and (X, Y) follows the Marshall–Olkin Bivariate Exponential model.

4.6 Conclusion

In this chapter, we examine the properties of the indifference premium for an insurable risk in the presence of background risk. Without imposing restrictive assumptions on the

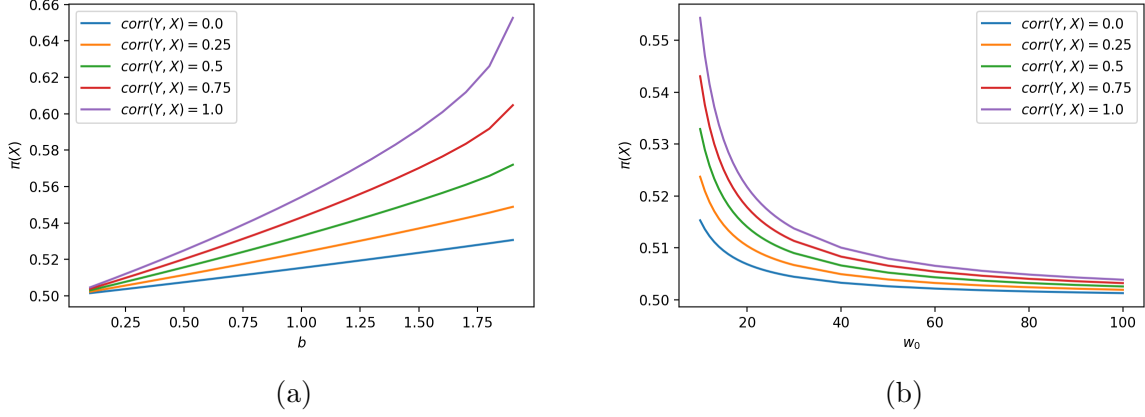


Figure 4.4: (a) Indifference premium versus the risk aversion parameter b . (b) Indifference premium versus the initial wealth w_0 . Here, the preference operator V is the expected power utility, and (X, Y) follows Marshall–Olkin bivariate exponential distribution. The parameters are as follows. (a) $w_0 = 10$. (b) $b = 1$.

dependence structure between the insurable risk and the background risk, we identify the conditions under which the indifference premium is uniquely determined and outlined its general properties. We establish, in particular, that for given marginal distributions, the indifference premium is greater when the insurable risk is stochastically increasing in the background risk than when the risks are independent. Furthermore, we systematically detail the properties of the indifference premium under independence, stochastic monotonicity, and mutual exclusivity. To analyze the impact of model uncertainty, we preserve the dependence structure between Y and X by deriving equivalent conditions for the dependence structures between their corresponding uniform random variables U_Y and U_X . Finally, we illustrate our theoretical findings using expected utility preference operators. Our numerical examples confirm that the indifference premium increases with the correlation between Y and X and with the level of risk aversion; thus, greater risk aversion or higher correlation between the risks results in a higher premium.

Chapter 5

Concluding remarks and future research

This thesis presents a unified investigation into optimal reinsurance and premium design under uncertainty, introducing new perspectives on worst-case strategy, performance-based premiums, and the role of background risk. In Chapter 2, we develop a robust optimization approach to reinsurance design that accounts for model uncertainty through likelihood ratio constraints, establishing a link between classical and worst-case strategies and providing conditions under which the classical strategies are remain robust in the worst-case scenario. Chapter 3 introduces a novel reward-and-penalty variable premium scheme, demonstrating its advantages over traditional constant premium principles and analyzing Bowley optimal reinsurance strategies for both insurer and reinsurer under this framework. Chapter 4 explores the properties of the indifference premium in the presence of background risk, emphasizing the critical influence of dependence structure on premium determination, and highlighting the implications of model uncertainty. Based on the results of this thesis, several directions for future research are proposed below.

5.1 Worst-case reinsurance strategy with heterogeneous beliefs

An intriguing direction for future research involves the reinsurer's risk quantification within optimal reinsurance design problems. In practice, reinsurers may have access to distinct information or may employ different models for the underlying risk, leading to heterogeneous beliefs and potential divergence from the insurer's perspective. These informational and modeling asymmetries can result in different choices of risk distributions or probability measures, thereby influencing optimal contract design. For the studies of classical optimal reinsurance design problems along this direction, we refer to [Boonen and Ghossoub \(2019\)](#), [Chi \(2019\)](#), [Ghossoub \(2019\)](#) and references therein.

Recall that the worst-case problem (2.3) in Chapter 2 under a general uncertainty set $\mathcal{G}$ can be written as

$$\min_{I \in \mathcal{I}} \sup_{F \in \mathcal{G}} \rho(R_I(X^F) + \pi(I(X^{F_0}))), \quad (5.1)$$

where ρ denotes the insurer's risk preference, the premium π is calculated under the reference distribution F_0 , and the premium is not directly affected by the insurer's uncertainty set.

In practical settings, the reinsurer can strategically adjust the premium π to reflect her own risk perspective, thereby directly influencing the policyholder's reinsurance choices. This idea naturally generalizes the problem (5.1). In particular, one can consider a setting in which the reinsurer determines the reinsurance premium to protect her against the worst-case scenarios. Mathematically, the worst-case premium can be formalized as

$$\pi^\uparrow(I(X)) = \sup_{F \in \mathcal{S}} \pi(I(X^F)),$$

where $\mathcal{S}$ denotes the reinsurer's uncertainty set of probability distributions. This approach allows the premium to consider the reinsurer's concerns regarding model uncertainty, ensuring adequate risk compensation across all plausible scenarios.

Under the worst-case premium structure, the insurer's optimization problem (5.1) is fundamentally altered, and the optimization problem for the insurer becomes

$$\min_{I \in \mathcal{I}} \sup_{F \in \mathcal{G}} \rho(R_I(X) + \pi^\uparrow(I(X))). \quad (5.2)$$

To reflect the reinsurer's heterogeneous beliefs, the reinsurer's uncertainty set $\mathcal{S}$ does not necessarily align with the insurer's uncertainty set $\mathcal{G}$, and either $\mathcal{S}$ or $\mathcal{G}$ can be induced by, for examples, ϕ -divergence, Wasserstein metrics, or moments constraints.

It would be of considerable interest to examine whether analogous results to those established in Chapter 2 can be obtained in this extended setting. Moreover, one may formulate equilibrium problems in which the insurer and the reinsurer each operate under their respective uncertainty sets. Addressing these challenging questions is a compelling direction for future research.

5.2 Properties of indifference premium with background risk

Theorem 4.4.4 in Chapter 4 gives the properties of the indifference premium if the uncertainty set $\mathcal{P}$ contains n distribution functions. This result may be generalized to the following theorem. To proceed, we first give the definition of the semi-continuity.

Definition 5.2.1. A functional V is said to be *lower semi-continuous in distribution* if for any sequence of random variables $\{X_n\}$ such that $X_n \xrightarrow{d} X$, it holds that

$$V(X) \leq \liminf_{n \rightarrow \infty} V(X_n).$$

Theorem 5.2.1. *When the preference operator V is lower semi-continuous in distribution and the uncertainty set $\mathcal{P}$ is closed by law, then there exists a worst-case distribution $\bar{F} \in \mathcal{P}$ at which the infimum in equation (4.5) is attained, and the statements in Theorem 4.4.4 hold.*

Proof. Let a sequence of distributions $F_n \in \mathcal{P}$ converge to F_0 almost surely. Denote $X_n \triangleq F_n^{-1}(U_X)$ and $X_0 \triangleq F_0^{-1}(U_X)$, and let $S_n \triangleq Y + X_n$ and $S \triangleq Y + X_0$. For given $x, y \in \mathbb{R}$, the joint distribution of X_n and Y is formulated as $F_{X_n, Y}(x, y) = \mathbb{P}(X_n \leq x, Y \leq y) = \mathbb{P}(F_n^{-1}(U_X) \leq x, F_Y^{-1}(U_Y) \leq y) = \mathbb{P}(U_X \leq F_n(x), U_Y \leq F_Y(y)) = F_{U_X, U_Y}(F_n(x), F_Y(y))$, where the third equality holds due to $F^{-1}(u) \leq x$ is equivalent to $u \leq F(x)$. As long as we can prove that $F_{S_n}(s) = \int \int_{x+y \leq s} dF_{X_n, Y}(x, y) = \int \int_{x+y \leq s} dF_{U_X, U_Y}(F_n(x), F_Y(y)) \rightarrow$

$F_S(s) = \int \int_{x+y \leq s} dF_{U_X, U_Y}(F_0(x), F_Y(y))$, then $S_n \xrightarrow{d} S$. The lower semi-continuity of V indicates that $V(S) \leq \liminf_{n \rightarrow \infty} V(S_n)$. The closeness of $\mathcal{P}$ implies that $S \in \mathcal{P}$. It then follows that the infimum in equation (4.5) is attainable. $\square$

5.3 Optimal reinsurance design with background risk

In addition, I aim to tackle a critical yet understudied optimization problem—how background risk influences decision-making under model uncertainty. In real-world financial and insurance markets, existing business operations inevitably shape the risks of new ventures, especially when their uncertainties are considered. A prime example is reinsurance: a firm underwriting contracts for multiple insurers faces exposure to correlated risks across different policies. The presence of existing contracts can alter the indemnity structure of new agreements, complicating risk assessment and capital allocation. This challenge intensifies when model uncertainty is introduced, as the true dependence structure between risks is often unknown or misestimated. From a risk management perspective, financial firms prioritize resilience against worst-case scenarios, making it essential to investigate the worst-case distributions and optimal decisions under varying dependence structures between legacy and new risks. Unraveling these complex interdependencies will not only advance theoretical insights into robust optimization and risk management but also offer practical solutions for insurers, reinsurers, and financial institutions seeking to navigate an increasingly uncertain and interconnected risk landscape.

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