Low Order Controllers for Sampled-Data Systems

by

Mauro Rossi

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Abstract

The application of a digital controller to a continuous-time plant results in a closed loop system that contains both continuous-time and discrete-time signals; such a system is referred to as a *sampled-data system.*

In this thesis, we consider finite-dimensional linear time-invariant plants, and the emphasis is placed on designing *low order* linear time-varying digital controllers that are straightforward to design and easy to implement. We consider two basic controller structures: static generalized sampled-data hold function (GSHF) controllers and linear periodic controllers (LPCs) that consist of a sampler, a low order linear discrete-time compensator, and a zero-order-hold function.

We consider three control problems. The first problem is the combined gain/phase margin problem, which can be viewed as a robust stabilization problem. We show that it is possible to design a static GSHF controller, which can be implemented with a low order LPC, that can provide a gain margin as large as desired and any desired phase margin up to 90 degrees. An analysis of the tolerance of such a controller to unstructured uncertainty in the nominal model is also presented. This controller suffers from poor intersample behaviour, so we also present another low order LPC that has good intersample behaviour while providing a gain margin as large as desired and any desired phase margin up to 90 degrees. The second problem is the model reference control problem (MRCP), where the goal is to track a class of reference signals despite the presence of noise. We show that there exists a static GSHF controller that solves the MRCP when the single-input, single-output plant is minimum phase. Finally, in our third problem we look at an optimal step tracking problem for an arbitrary multi-input, multi-output plant. We show that it is possible to design a low order LPC that not only provides near LQR-optimal step tracking for the nominal plant, but also provides step tracking when there is some uncertainty in the gain of the plant.

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Chapter 1

Introduction

Control systems play an important role in virtually all areas of technology, from manufacturing, computers and communications, right through to the financial and entertainment industries. The main objective of a control system is to provide an input or control signal to a real physical system, so that the output of the system behaves in a desired manner. To attain this goal, the control system designer typically follows four steps: (1) quantify the desired behaviour in mathematical specifications, (2) develop an idealized mathematical representation or model for the real system, (3) design a controller based on the model that meets the specifications, and (4) implement the controller and verify that the specifications are met. There are three issues that must be kept in mind throughout the control system design process.

First, while developing an idealized mathematical model, many simplifying assumptions (e.g. linearization, order reduction) are typically made and accurate estimates of the actual parameters of the system are not always available. Hence, there is almost always some degree of uncertainty in the idealized mathematical model. Designing controllers that account for uncertainty in the model of the system is commonly referred to as *robust control*.

Secondly, since most physical systems are analog in nature, the resulting mathematical

model commonly takes the form of a set of differential equations. Furthermore, it is usually intuitive to express the design specifications – commonly measured by some *performance index* (e.g. a cost function, tracking error) – in the continuous-time or analog domain. However, with the advent of fast and inexpensive digital technology, it is desirable to implement the controller digitally. A digital controller typically consists of three components: a sampling operation, discrete-time signal processing, and a digital to analog conversion (or hold operation). Conventionally, one fixes the sampling and hold operations, synchronizes these operations in time with a fixed sampling period, and then designs the discrete-time signal processing component (controller). Hence, the combination of the analog model and the digital controller contains both continuous-time and discrete-time signals; such systems are commonly referred to as *sampled-data systems.* To simplify the model of the system, approximations are typically made so that the model can be expressed as a finite set of linear time-invariant (LTI) differential equations. To simplify the controller synthesis, one can restrict the discrete-time controller to be LTI and only consider the behaviour of the sampled-data system at the sampling instances, thus yielding an overall closed loop system that is discrete-time LTI. Recently, some results have been presented where intersample behaviour is taken into account while still restricting the discrete-time component of the controller to be LTI. In this thesis, we will consider intersample behaviour and we will not restrict the discrete-time component of our controller to be LTI.

Finally, the third issue that the control system designer must address is the complexity of the controller. In order to say that one controller is simpler than another, we must define a measure of the complexity; a reasonable one is that of the degree of difficulty of implementation. Since we will be considering linear periodic time-varying controllers, we will use the order of the controller state and the periodicity of the controller parameters as our measure of the complexity. In this thesis, we will emphasize the design of *low order* controllers, but it will turn out that the periodicity of the controller parameters is usually at least as high as the order of the plant. Hence, the overall complexity of the

controllers we propose will be of the same order as that of the controllers designed using other LTI techniques.

Therefore, to summarize the above, we will look at designing *low order* LPCs for LTI systems in a *sampled-data* setting, that are in some respects superior to LTI controllers when looking at issues related to *robust control* and *performance.*

1.1 A Brief Literature Survey

Before going on to state the specific objectives of our work, it would be prudent to discuss some of the motivating and related work that can be found in the literature. First, one naturally might ask why it would be desirable to use a linear time-varying (LTV) controller instead of an LTI controller. It was shown by Feintuch and Francis [18] and by Shamma and Dahleh [46] that for LTI discrete-time systems, LTV controllers do not lead to any improved performance for problems of uniform optimal control. Furthermore, Khargonekar et al. [30] showed that under some weak assumptions, if there does not exists an LTI controller that will stabilize every plant in a specified set (i.e. additive uncertainty set) then there does not exist a LTV controller that will do so either.

However, LTV controllers have been shown to be superior in attaining other control objectives. For example, it is well known that given an unstable non-minimum phase LTI plant, the maximum attainable gain and phase margin provided by an LTI controller is bounded [33]. However, this limitation is not present if we allow the controller to be LTV (e.g. continuous-time case [35], discrete-time case [31], sampled-data case [21]). Note however, that in these approaches, the resulting controller may have a high order, and by our earlier discussion, can be considered relatively complex.

Motivated by a desire for low order controllers, Kabamba [26] investigated a second approach to sampled-data control that uses a generalized sampled-data hold function (GSHF) instead of the classical zero-order-hold function. This gives rise to GSHF con-

trollers, and the ideas behind this approach can be traced back to work done by Chammas and Leondes [11]. Kabamba [26] showed that GSHF controllers are useful in the area of pole assignment, optimal noise rejection, model matching, decoupling and robustness. A more detailed discussion of the literature related to the application of GSHF controllers to the gain margin problem and the model matching problem can be found in Chapters 3 and 5, respectively. On a related topic of sampled-data low order controllers, Madievski and Anderson [36] and Anderson et al.[2] investigate methods for approximating a high order continuous-time LTI controller by a low order discrete-time LTI controller (with a sampler and zero-order-hold) using fast multirate sampling and classical order reduction techniques. For the low order discrete-time periodic pole placement problem, see [1, 28, 37]. A multi-rate discrete-time periodic controller that can be synthesized to achieve pole placement was also discussed in [5] and [23].

As noted by Feuer and Goodwin [19], one major limitation of the GSHF controller is the poor intersample performance that results from the fact that as the sampling period of the GSHF controller tends to zero, the gain of the hold function, and in turn the control signal, typically becomes large. Juan and Kabamba [25] and Werner [54] attempt to improve the intersample behaviour by selecting the hold function in an optimal fashion. Furthermore, the study of the robustness properties provided by GSHF controllers has been studied in some detail in the literature [26, 22, 40, 10], but there are no results to our knowledge that describe what happens to the tolerance of the controller to unstructured uncertainty in the model as the sampling period of the GSHF controller tends to zero. We will address this issue in this thesis.

Before leaving this section, we note some other related work that the reader should be made aware of. There has been a vast amount of work done in the area of optimal sampled-data control, and we list but a few papers in each specific area. For the case where the digital controller consists of a sampler, an LTI discrete-time compensator and a zero order hold, there has been a number of results presented in $\mathscr{L}_1$ optimal control

[17, 6], H_∞ optimal control [9, 24, 27, 49, 50], and H_2 optimal control [8, 7, 13, 32]. When the discrete-time component is allowed to be LTV, there have also been results presented in the areas of H_2 and H_∞ optimal control [53, 52, 14, 42]. A general mathematical tool that is used in obtaining some of the previous results, commonly referred to as the lifting technique, has been developed in [50, 56, 8]. Recently, a new discretization-based solution to the sampled-data H_∞ control problem that does not directly using the lifting technique was given in [51]. There the authors relate their technique to the lifting technique and show that both methods lead to identical synthesis equations. Finally, a survey paper by Araki [4] provides a comprehensive outline of many of the developments in digital control theory made previous to 1993.

1.2 Our Objectives

In this thesis, we would like to find low order sampled-data LPCs that can provide robust control and/or provide a desired performance. We will propose two different controller structures to meet our objectives: a static GSHF controller (Chapters 3 and 5), and a low order LPC that consists of a sampler, a linear periodic discrete-time compensator, and a zero-order-hold (Chapters 4 and 6). We restrict ourselves to considering finite-dimensional LTI plants and all of the low order controllers that we propose will be capable of attaining goals that LTI controllers cannot achieve, but as expected it typically comes at a cost.

The first problem that we will consider (Chapters 3 and 4) is the robust stabilization problem commonly referred to as the combined gain/phase margin problem. We know that for unstable non-minimum phase LTI plants, the maximum attainable gain and phase margin provided by an LTI controller is bounded [33]. While it has been shown that it is possible to design a static GSHF controller that can provide a gain margin as large as desired [58], there are no results in the literature that address the issue of

designing a static GSHF controller to provide a desired phase margin. In our results, we use a significantly different approach to that proposed in [58] to incorporate the phase margin specification into our design, and show that for any n^{th} order multi-input, multi-output (MIMO) LTI continuous-time plant, it is possible to design a static GSHF controller that provides a gain margin as large as desired and any desired phase margin up to 90 degrees. We use the fact that the GSHF controller provides a combined gain and phase margin to show that the GSHF controller will be capable of tolerating moderate additive dynamic perturbations to the nominal model, even as the sampling period tends to zero. A drawback of this first GSHF controller is that we typically require the sampling period to be small in order to stabilize even the nominal plant. Hence, we also propose an alternate GSHF controller that will be capable of stabilizing the nominal model for almost all sampling periods, and will recover the gain/phase margin properties of the first GSHF controller as the sampling period tends to zero. We then show that it is possible to implement each of the proposed static GSHF controllers with a sampler, a low order n-periodic discrete-time compensator, and a zero-order-hold function.

A major drawback of the static GSHF controller is poor intersample performance, which can be attributed to the fact that as the sampling period becomes small, the gains of the hold function, and therefore the input to the plant, becomes large. Hence, we go on to show that it is possible to design an LPC that consists of a sampler, a low order p-periodic ($p > n$) discrete-time compensator, and a zero-order-hold function, that provides satisfactory intersample behaviour while providing a gain margin as large as desired and any desired phase margin up to 90 degrees. Unfortunately, while this controller can be made very tolerant to the structured uncertainty defined in the gain/phase margin problem, it becomes less and less tolerant to unstructured dynamic uncertainty in the nominal model as the sampling period becomes small.

We then turn our attention to issues related to performance by considering the tracking and disturbance rejection problem commonly referred to as the Model Reference

Control Problem (MRCP). In the MRCP, the control system designer chooses a reference model that embodies the desired behaviour and the objective is to find a controller that makes the plant behave like the reference model. Since we know that for non-minimum phase LTI plants there are limits to the best achievable closed loop performance even when the controller is nonlinear and time-varying [38], we will assume that the plant is minimum phase. Furthermore, we assume that both the plant and reference model are single-input, single-output (SISO). We then go on to show that it is possible to design a static GSHF controller that ensures that the plant output tracks the reference model output as well as desired, and as mentioned before, it is possible to implement such a controller using a sampler, a first order n-periodic discrete-time compensator, and a zero-order-hold. Since GSHF controllers typically suffer from poor intersample performance, this result is surprising and to our knowledge, is the first result that uses static GSHF controllers to solve the sampled-data MRCP (e.g. see [26, 41] for the application of GSHF controllers to solve a weaker discrete-time MRCP).

Finally, we address the issue of tracking for a larger class of plants for a control problem that is common in industry. Specifically, we will pose an optimal step tracking problem where we do not assume that the state of the plant can be measured, and the objective is to track step reference signals in an optimal fashion. The optimality criterion will be similar to the standard LQR cost function, and we begin by converting the optimal step tracking problem into the standard LQR problem. We then show that it is possible to design a low order LPC that can provide near optimal performance for the nominal plant and can still provide stability, even when there is some uncertainty in the gain of the plant.

1.3 Thesis Outline

In Chapter 2 we will introduce notation, state some well known results that will be used throughout the thesis, and prove some preliminary mathematical results. In Chapter 3 we show that it is possible to design a GSHF controller that can solve the gain/phase margin problem and that under some mild conditions, this GSHF controller can also tolerate unstructured uncertainty in the model of the plant. In Chapter 4, we show that it is possible to design a low order linear periodic controller that can solve the gain/phase margin problem, while providing satisfactory intersample performance. In Chapter 5 we show how to design a GSHF controller that solves the MRCP for a SISO minimum phase plant. In Chapter 6 we pose an optimal step tracking problem and show that there exists a low order LPC that solves this problem. Moreover, this controller can provide stability even when there is some uncertainty in the gain of the plant. A summary follows in Chapter 7 where we outline the contributions made in this work and propose some areas of future study.

Chapter 2

Mathematical Background

In this chapter, we will introduce some notation and present some preliminary mathematical results that will be used throughout the thesis.

2.1 Notation

Let $\mathbf{R}$ denote the set of real numbers, $\mathbf{R}^+$ denote the set of non-negative real numbers, $\mathbf{C}$ denote the set of complex numbers, $\mathbf{C}^-$ denote the set of complex numbers with a real part less than zero, $\mathbf{Z}$ denote the set of integers, and $\mathbf{Z}^+$ denote the set of non-negative integers. Let $\mathbf{C}^n$ denote the set of all $n \times 1$ vectors with elements in $\mathbf{C}$, and $\mathbf{C}^{n\times m}$ denote the set of all $n \times m$ matrices with elements in $\mathbf{C}$. Similarly, let $\mathbf{R}^n$ and $\mathbf{R}^{n\times m}$ denote the set of $n \times 1$ vectors and $n \times m$ matrices with elements in $\mathbf{R}$. We will denote the n-dimensional identity matrix by I_n, or simply I when n is immaterial.

The real and imaginary parts of $z \in \mathbf{C}$ will be written as $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$, respectively. The complex conjugate transpose of $A \in \mathbf{C}^{n\times m}$ will be denoted by A^* and the transpose of $A \in \mathbf{R}^{n\times m}$ will be denoted by A^T. The set of eigenvalues of $A \in \mathbf{C}^{n\times n}$ will be called the spectrum of A and will be denoted by $\mathrm{sp}(A)$. We say that A is *stable* if $\mathrm{sp}(A) \subset \mathbf{C}^-$; otherwise, we say that A is *unstable*.

If $\mathrm{sp}(Q) \subset \mathbf{R}$, then let

$$\begin{aligned} \lambda_{min}(Q) &:= \min\{\lambda : \lambda \in \mathrm{sp}(Q)\}, \\ \lambda_{max}(Q) &:= \max\{\lambda : \lambda \in \mathrm{sp}(Q)\}. \end{aligned}$$

We say that $Q \in \mathbf{C}^{n\times n}$ is *Hermitian* if $Q = Q^*$; it is well known that a Hermitian matrix Q satisfies the following:

i) $\mathrm{sp}(Q) \subset \mathbf{R}$,
ii) for every $x \in \mathbf{C}^n$, we have $\mathrm{Im}(x^*Qx) = 0$,
iii) for every $x \in \mathbf{C}^n$, we have $\lambda_{min}(Q)x^*x \leq x^*Qx \leq \lambda_{max}(Q)x^*x$.

The matrix $Q \in \mathbf{R}^{n\times n}$ is positive definite if $x^TQx > 0$ for all non-zero $x \in \mathbf{R}^n$, in which case we will write $Q > 0$.

2.2 Norms and Spaces

Let $z \in \mathbf{C}^n$ and denote the i^{th} element of z as z_i. We define the p-norm of $z \in \mathbf{C}^n$ as

$$\|z\|_p := \begin{cases} \left(\sum_{i=1}^{n} |z_i|^p\right)^{1/p} & 1 \leq p < \infty \\ \max_i |z_i| & p = \infty. \end{cases}$$

Since all norms on $\mathbf{C}^n$ are equivalent (e.g. see [15, §II.2]) we will use the 2-norm (Euclidean norm) on $\mathbf{C}^n$ for this work. Namely, for $z \in \mathbf{C}^n$ we will denote

$$\|z\| := \|z\|_2 = \sqrt{(z^*z)};$$

it is easy to prove that the induced norm of $A \in \mathbf{C}^{n\times n}$ satisfies

$$\|A\| := \sup\{\|Az\| : z \in C^n, \|z\| \neq 0\} = \max\{\sqrt{\lambda} : \lambda \in \mathrm{sp}(A^*A)\}.$$

We denote $\mathscr{C}(\mathbf{R}^+, \mathbf{R}^n)$ as the set of all continuous functions mapping $\mathbf{R}^+$ to $\mathbf{R}^n$. We say that $f : \mathbf{R}^+ \to \mathbf{R}^n$ is *piecewise continuous* if there exists a sequence $0 \leq t_1 < t_2 < \cdots$ so that

i) $\lim_{t\to\infty} t_i = \infty$,

ii) f is continuous for $t \in \mathbf{R}^+ - \cup_{i=1}^{\infty}\{t_i\}$, and

iii) for each t_i, the left-hand and right-hand limits of $f(t)$ as $t \to t_i$ exist and are finite.

If $f : \mathbf{R}^+ \to \mathbf{C}^n$, or $f : \mathbf{R}^+ \to \mathbf{C}^{n\times m}$, then we can generalize the concept of continuity and piecewise continuity in the natural way by stacking the real and imaginary parts of $f(\cdot)$ into one vector. Let the set of all piecewise continuous functions $f : \mathbf{R}^+ \to \mathbf{C}^{n\times m}$ be denoted as $\mathbf{PC}(\mathbf{R}^+, \mathbf{C}^{n\times m})$, or simply $\mathbf{PC}$ when $\mathbf{R}^+$ and $\mathbf{C}^{n\times m}$ are immaterial. Now define

$$\mathbf{PC}_\infty := \left\{ f \in \mathbf{PC} : \operatorname*{ess\,sup}_{t\in[0,\infty)} \|f(t)\| < \infty \right\},$$

and $\mathbf{PC}_\infty^1$ to be the set of those elements $f \in \mathbf{PC}_\infty$ which are absolutely continuous and whose derivative[a] $\dot{f}$ belongs to $\mathbf{PC}_\infty$. For such an f, we have

$$f(t) = f(t_0) + \int_{t_0}^{t} \dot{f}(\tau)d\tau, \quad t \geq t_0 \geq 0,$$

e.g. see Theorem 15 [15, pg. 231].

For what follows, we could consider the more general class of locally (Lebesgue) integrable functions instead of $\mathbf{PC}(\mathbf{R}^+, \mathbf{C}^{n\times m})$, but this would require extra mathematics which would detract from the clarity of the presentation without significantly generalizing our results. Hence, let the subset $\mathscr{L}_p(\mathbf{R}^+, \mathbf{C}^n)$ of $\mathbf{PC}(\mathbf{R}^+, \mathbf{C}^n)$ consist of all $f \in \mathbf{PC}(\mathbf{R}^+, \mathbf{C}^n)$ satisfying

$$\|f\|_p := \begin{cases} \left(\int_0^\infty \|f(t)\|^p dt\right)^{1/p} < \infty & 1 \leq p < \infty \\ \operatorname*{ess\,sup}_{t\in\mathbf{R}^+} \|f(t)\| < \infty & p = \infty; \end{cases}$$

[a] If f is absolutely continuous, then it is well known that $\dot{f}$ exists almost everywhere.

to simplify the notation, we write $\mathscr{L}_p(\mathbf{C}^n)$ when $\mathbf{R}^+$ is immaterial and $\mathscr{L}_p$ when $\mathbf{R}^+$ and $\mathbf{C}^n$ are immaterial. In this thesis, we will primarily be interested in the case when $p = 2$ and $p = \infty$. The $\mathscr{L}_\infty$ induced gain of a system

$$G : \mathbf{PC} \to \mathbf{PC}$$

is defined by

$$\|G\|_\infty := \sup \left\{ \frac{\|Gu\|_\infty}{\|u\|_\infty} : u \in \mathbf{PC}_\infty, \|u\|_\infty \neq 0 \right\}.$$

If $f \in \mathscr{L}_\infty(\mathbf{R}^+, \mathbf{R}^n)$, then the Laplace transform of f, denoted $\mathcal{L}\{f\}$ or $F(s)$, is given by

$$\mathcal{L}\{f\} := \int_0^\infty f(t)\mathrm{e}^{-st}dt,$$

for all $s \in \mathbf{C}$ for which the integral is defined. Let $\mathcal{H}_\infty$ be the set of all complex valued functions $F(s)$ of a complex variable s which are analytic and bounded in the open right half-plane $\mathrm{Re}(s) > 0$. and define

$$\|F\|_\infty := \sup\{|F(s)| : \mathrm{Re}(s) > 0\}.$$

The subset of $\mathcal{H}_\infty$ consisting of real-rational functions will be denoted by $\mathcal{RH}_\infty$. If $F(s)$ is real-rational, then $F \in \mathcal{RH}_\infty$ if and only if $F(\infty)$ is finite (proper), and $F(s)$ is finite for $\mathrm{Re}(s) \geq 0$ (stable). Furthermore, by the Maximum Modulus Theorem, it is well known that

$$\|F\|_\infty = \sup\{|F(jw)| : w \in \mathbf{R}\}.$$

Let $\mathcal{RL}_\infty$ be the set of real-rational complex valued functions $F(s)$ of a complex variable s which are analytic and bounded on the imaginary axis $\mathrm{Re}(s) = 0$.

The set of all sequences on $\mathbf{Z}^+$ taking values in $\mathbf{C}^n$ will be denotes as $\mathscr{S}(\mathbf{Z}^+, \mathbf{C}^n)$, and we write $f \in \mathscr{S}(\mathbf{Z}^+, \mathbf{C}^n)$ as $\{f(k)\}$. The subspace $\ell_p(\mathbf{Z}^+, \mathbf{C}^n)$ of $\mathscr{S}(\mathbf{Z}^+, \mathbf{C}^n)$ consists of all $f \in \mathscr{S}(\mathbf{Z}^+, \mathbf{C}^n)$ satisfying

$$\|f\|_p := \begin{cases} \left(\sum_{k=0}^{\infty} \|f(k)\|^p \right)^{1/p} < \infty & 1 \leq p < \infty \\ \sup_{k \in \mathbf{Z}^+} \|f(k)\| < \infty & p = \infty; \end{cases}$$

to simplify the notation, we write $\ell_p(\mathbf{C}^n)$ when $\mathbf{Z}^+$ is immaterial, and ℓ_p when $\mathbf{Z}^+$ and $\mathbf{C}^n$ are immaterial. If $f \in \mathcal{S}(\mathbf{R}^+, \mathbf{R}^n)$, then the z-transform of f, denoted $\mathcal{Z}\{f\}$ or $F(z)$, is given by

$$\mathcal{Z}\{f\} := \sum_{k=0}^{\infty} f(k) z^{-k},$$

for all $z \in \mathbf{C}$ for which the summation is defined.

2.3 Order of a Function

We say that the function $f(T)$ is of *order* T^j, and write $f(T) = \mathcal{O}(T^j)$, if there exists a constant $c_1 > 0$ and $T_1 > 0$ so that

$$\|f(T)\| \leq c_1 T^j, \quad T \in (0, T_1).$$

Lemma 2.1

If $f_1(T)$ and $f_2(T)$ are of order T^i and T^j, respectively, then

i) $f_1(T) + f_2(T) = \mathcal{O}(T^{\min\{i,j\}})$,

ii) $f_1(T) f_2(T) = \mathcal{O}(T^{i+j})$, and

iii) $(I - f_1(T))^{-1} = I + \mathcal{O}(T^i)$.

Proof:

The first two results are straightforward, so we only prove the third result. For $\|\Delta\| < 1$,

$$(I - \Delta)^{-1} = \sum_{k=0}^{\infty} \Delta^k.$$

Since $f_1(T) = \mathcal{O}(T^i)$, it follows that there exists a $c_1 > 0$ and $T_1 > 0$ so that

$$\|f_1(T)\| \leq c_1 T^i, \quad T \in (0, T_1),$$

which means that there exists a $T_2 \in (0, T_1)$ so that

$$\|f_1(T)\| \leq \tfrac{1}{2}, \quad T \in (0, T_2).$$

Hence, for $T \in (0, T_2)$ we have

$$\begin{aligned} \|(I - f_1(T))^{-1} - I\| &= \left\| \sum_{k=1}^{\infty} f_1(T)^k \right\| \\ &\leq \frac{\|f_1(T)\|}{1 - \|f_1(T)\|} \\ &\leq 2\|f_1(T)\|, \end{aligned}$$

so since $f_1(T) = \mathcal{O}(T^i)$, it follows that $(I - f_1(T))^{-1} - I = \mathcal{O}(T^i)$ as well, and our third result follows. ∎

2.4 A Simple Convergence Result

Lyapunov type arguments are used in many of the proofs presented in this thesis, so it will be useful to state the following convergence result.

Lemma 2.2 If there exists a $\lambda \in \mathbf{R}$ and $V \in \mathbf{PC}^1_\infty(\mathbf{R}^+, \mathbf{R})$ satisfying

$$\dot{V}(t) + \lambda V(t) \leq 0, \ t \geq 0,$$

then

$$V(t) \leq \mathrm{e}^{-\lambda t} V(0), \ t \geq 0.$$

Proof:

The following proof is based on the results found in Section 3.5.5 of [48]. Since $V \in \mathbf{PC}^1_\infty$, it follows that

$$W(t) := \dot{V}(t) + \lambda V(t) \tag{2.1}$$

is an element of $\mathbf{PC}_\infty$. Solving the first order differential equation (2.1) for $V(t)$ we get

$$V(t) = \mathrm{e}^{-\lambda t} V(0) + \int_0^t \mathrm{e}^{-\lambda(t-\tau)} W(\tau) d\tau, \ t \geq 0,$$

but since $W(t) \leq 0$ for $t \geq 0$, it follows that

$$V(t) \leq \mathrm{e}^{-\lambda t} V(0), \ t \geq 0.$$

∎

In our results, we choose $V(t) \geq 0$ and show that there exists a $\lambda > 0$ satisfying the conditions of Lemma 2.2. Hence, we can claim that $V(t)$ approaches zero at an exponential rate.

2.5 A Riccati Equation Result

In Chapters 3 and 4, we will be using a discrete-time LQR approach to design our controllers. However, we formulate the discrete-time LQR problem so that as the sampling period tends to zero, the discrete-time solution approaches the solution of a continuous-time LQR problem. This is basically done by first showing that the discrete-time Riccati equation can be written as a perturbed continuous-time Riccati equation. Thus we will need the following result that relates the solution of a Riccati equation to its perturbed counterpart.

Suppose that (A, B) is controllable and that $Q \in \mathbf{R}^{n\times n}$ and $R \in \mathbf{R}^{m\times m}$ are positive definite symmetric matrices. Let P be the unique positive definite symmetric solution of the continuous-time Riccati equation

$$PA + A^T P - PBR^{-1}B^T P + Q = 0; \tag{2.2}$$

the existence of such a P follows from the facts that $(\sqrt{Q}, A)$ is observable and (A, B) is controllable – e.g. see Theorem 12.2 and Lemma 12.2 in [55]. If we define

$$F := -R^{-1}B^T P,$$

then it is well known that $A+BF$ is stable. Now suppose that $\Delta_Q \in \mathbf{R}^{n\times n}$ is symmetric, but not necessarily positive definite, and satisfies

$$Q + \Delta_Q > 0;$$

since $(\sqrt{Q + \Delta_Q}, A)$ is observable and (A, B) is controllable, we know that there exists a positive definite symmetric $P_\Delta \in \mathbf{R}^{n\times n}$ satisfying the perturbed continuous-time

algebraic Riccati equation

$$P_\Delta A + A^T P_\Delta - P_\Delta B^T R^{-1} B P_\Delta + (Q + \Delta_Q) = 0. \tag{2.3}$$

If we define

$$F_\Delta := -R^{-1} B^T P_\Delta,$$

then it is well known that $A + BF_\Delta$ is stable. The following preliminary result states the intuitive result that if $Q + \Delta_Q \approx Q$, then $P_\Delta \approx P$.

Lemma 2.3 There exists a constant $c_1 > 0$ such that

$$\|P_\Delta - P\| \leq \max\left\{\left(\frac{\|P\| + c_1\|\Delta_Q\|}{\lambda_{min}(Q) - \|\Delta_Q\|}\right)\|\Delta_Q\|, c_1\|\Delta_Q\|\right\}.$$

Proof:

Consider the equation

$$\dot{w}(t) = Aw(t) + Bv(t), \quad w(0) = w_0 \in \mathbf{R}^n, \tag{2.4}$$

and let the nominal and perturbed cost functionals be defined as

$$\begin{aligned} J(v, w_0) &:= \int_0^\infty [w^T(t)Qw(t) + v^T(t)Rv(t)]dt, \\ J_\Delta(v, w_0) &:= \int_0^\infty [w^T(t)(Q + \Delta_Q)w(t) + v^T(t)Rv(t)]dt. \end{aligned}$$

From Section 12.3 of [55], we know that the nominal optimal cost

$$\inf_{v \in \mathscr{L}_2} J(v, w_0) = w_0^T P w_0,$$

and the corresponding nominal optimal control law is

$$v(t) = Fw(t).$$

Similarly, the perturbed optimal cost

$$\inf_{v \in \mathscr{L}_2} J_\Delta(v, w_0) = w_0^T P_\Delta w_0,$$

and the corresponding perturbed optimal control law is

$$v(t) = F_\Delta w(t).$$

Let us first find an upper bound on $\lambda_{max}(P_\Delta - P)$. To do this, we set

$$v(t) = Fw(t) \quad \Rightarrow \quad w(t) = e^{(A+BF)t}w_0.$$

Since this control law minimizes J but not necessarily not J_Δ, it follows that

$$\begin{aligned} w_0^T P_\Delta w_0 &\leq \int_0^\infty [w^T(t)(Q+\Delta_Q)w(t) + v^T(t)Rv(t)]dt \\ &= \left(\int_0^\infty [w^T(t)Qw(t) + v^T(t)Rv(t)]dt\right) + \int_0^\infty w^T(t)\Delta_Q Q w(t)dt \\ &= w_0^T P w_0 + w_0^T \left[\int_0^\infty e^{(A+BF)^T t}\Delta_Q e^{(A+BF)t}dt\right] w_0 \\ &\leq w_0^T P w_0 + \underbrace{\int_0^\infty \|e^{(A+BF)t}\|^2 dt}_{=:\, c_1} \|\Delta_Q\| \times \|w_0\|^2, \end{aligned}$$

which means

$$w_0^T(P_\Delta - P)w_0 \leq c_1\|\Delta_Q\| \times \|w_0\|^2 \tag{2.5}$$

$$\Rightarrow \qquad \lambda_{max}(P_\Delta - P) \leq c_1\|\Delta_Q\|. \tag{2.6}$$

Note that c_1 is independent of Δ_Q and that since $A + BF$ is stable, c_1 is finite.

Since $P_\Delta - P$ may not be positive definite, we must show that $\lambda_{min}(P_\Delta - P)$ is also bounded below by a function of $\|\Delta_Q\|$. To do this, we set

$$v(t) = F_\Delta w(t), \quad \Rightarrow \quad w(t) = e^{(A+BF_\Delta)t}w_0.$$

Since this control law minimizes J_Δ but not necessarily not J, it follows that

$$\begin{aligned} w_0^T P w_0 &\leq \int_0^\infty [w^T(t)Qw(t) + v^T(t)Rv(t)]dt \\ &= \left(\int_0^\infty [w^T(t)(Q+\Delta_Q)w(t) + v^T(t)Rv(t)]dt\right) - \int_0^\infty w^T(t)\Delta_Q w(t)dt \\ &= w_0^T P_\Delta w_0 + w_0^T \left[\int_0^\infty e^{(A+BF_\Delta)^T t}(-\Delta_Q)e^{(A+BF_\Delta)t}dt\right] w_0, \end{aligned}$$

which means

$$\begin{aligned} w_0^T(P - P_\Delta)w_0 &\leq w_0^T \left[\int_0^\infty e^{(A+BF_\Delta)^T t}(-\Delta_Q)e^{(A+BF_\Delta)t}dt\right] w_0 \\ &\leq \|\Delta_Q\| \int_0^\infty \|e^{(A+BF_\Delta)t}w_0\|^2 dt. \end{aligned} \tag{2.7}$$

To get a bound on the the last term in the above equation, we note that since $v(t) = F_\Delta w(t)$ optimizes J_Δ, we have

$$\begin{aligned} w_0^T P_\Delta w_0 &= \int_0^\infty [w^T(t)(Q+\Delta_Q)w(t) + v^T(t)Rv(t)]dt \\ &\geq \int_0^\infty w^T(t)(Q+\Delta_Q)w(t)\ dt \\ &= \int_0^\infty w_0^T e^{(A+BF_\Delta)^T t}(Q+\Delta_Q)e^{(A+BF_\Delta)t}w_0 dt \\ &\geq \cdot\ \lambda_{min}(Q+\Delta_Q)\int_0^\infty \|e^{(A+BF_\Delta)t}w_0\|^2 dt, \end{aligned}$$

so it follows that

$$\begin{aligned} \int_0^\infty \|e^{(A+BF_\Delta)t}w_0\|^2 dt &\leq \frac{w_0^T P_\Delta w_0}{\lambda_{min}(Q+\Delta_Q)} \\ &\leq \frac{w_0^T P_\Delta w_0}{\lambda_{min}(Q) - \|\Delta_Q\|} \\ &\leq \frac{\|P\| + c_1\|\Delta_Q\|}{\lambda_{min}(Q) - \|\Delta_Q\|}\|w_0\|^2. \quad \text{[by equation (2.5)]} \end{aligned} \tag{2.8}$$

Combining (2.7) and (2.8) we get

$$\begin{aligned} & w_0^T(P_\Delta - P)w_0 \geq -\left(\tfrac{\|P\|+c_1\|\Delta_Q\|}{\lambda_{min}(Q)-\|\Delta_Q\|}\|\Delta_Q\|\right)\|w_0\|^2 \\ \Rightarrow \quad & \lambda_{min}(P_\Delta - P) \geq -\left(\tfrac{\|P\|+c_1\|\Delta_Q\|}{\lambda_{min}(Q)-\|\Delta_Q\|}\|\Delta_Q\|\right). \end{aligned} \tag{2.9}$$

Hence, from (2.6) and (2.9) we have

$$\begin{aligned} & \mathrm{sp}(P_\Delta - P) \in \left[-\left(\tfrac{\|P\|+c_1\|\Delta_Q\|}{\lambda_{min}(Q)-\|\Delta_Q\|}\|\Delta_Q\|\right), c_1\|\Delta_Q\|\right] \\ \Rightarrow \quad & \|P_\Delta - P\| \leq \max\left\{\left(\tfrac{\|P\|+c_1\|\Delta_Q\|}{\lambda_{min}(Q)-\|\Delta_Q\|}\right)\|\Delta_Q\|, c_1\|\Delta_Q\|\right\}. \end{aligned}$$

■

Given Lemma 2.3, it is trivial to prove the intuitive result that if $\Delta_Q \to 0$, then the solution P_Δ of the perturbed Riccati equation (2.3) approaches the solution P of the nominal Riccati equation (2.2).

2.6 The General Closed Loop Configuration

In this thesis, we will consider plants that can be represented by the linear time-invariant (LTI) differential equation

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + Ew(t), \quad x(0) = x_0 \in \mathbf{R}^n \\ y(t) &= Cx(t), \\ e(t) &= y_{ref}(t) - y(t), \qquad\qquad t \in R^+; \end{aligned} \tag{2.10}$$

here $x(t) \in \mathbf{R}^n$ is the state, $u(t) \in \mathbf{R}^m$ is the control input, $y(t) \in \mathbf{R}^r$ is the plant output, $w(t) \in \mathbf{R}^v$ is the disturbance, $y_{ref}(t) \in \mathbf{R}^r$ is the reference signal, $e(t) \in \mathbf{R}^r$ is the tracking error, and A, B, C and E are constant matrices of appropriate dimensions with elements in $\mathbf{R}$. We assume that (A, B) is controllable, (C, A) is observable and that our controller can only measure y and y_{ref}, and can only excite u.

We say that $\lambda \in \mathbf{C}$ is a *transmission zero* of (2.10) if

$$\text{rank} \begin{bmatrix} A - \lambda I & B \\ C & 0 \end{bmatrix} < n + \min\{r, m\}.$$

We say that (2.10) is *minimum phase* if all the transmission zeros are in $\mathbf{C}^-$; otherwise, we say that (2.10) is *non-minimum phase*. When $m = r = 1$, the transmission zeros are the zeros of $C(sI - A)^{-1}B$.

Suppose that $C(sI - A)^{-1}B$ is not identically zero, and observe that if $s \in \mathbf{C}$ is not an eigenvalue of A, then

$$C(sI - A)^{-1}B = \frac{C\text{adj}(sI - A)B}{\det(sI - A)}.$$

We say the *relative degree* of the plant is the order of the plant n less the highest degree of the elements of $C\text{adj}(sI - A)B$. It can easily be shown that if the relative degree of the plant is q, then

$$CB = CAB = \cdots = CA^{q-2}B = 0, \text{ and } CA^{q-1}B \neq 0.$$

The linear time-varying (LTV) controllers that we will propose in this thesis can be viewed as linear operators that map e to u. Hence, with G_c denoting the LTV controller

and P denoting the LTI plant, we will consider the general closed loop configuration illustrated in Figure 2.1. Since the plant is strictly proper, under a modest restriction

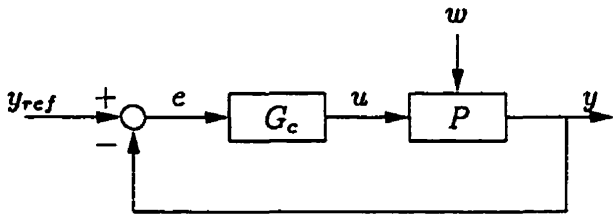


Figure 2.1: General closed loop configuration.

on the controller, the closed loop system is well-posed. In Chapters 3, 4, and 6, we set $w(t) = 0$, $y_{ref}(t) = 0$, and consider $u \in \mathbf{PC}$, which means $e \in \mathscr{C}$. In Chapter 5 we assume $y_{ref}(t)$ is the output of a strictly proper stable finite-dimensional LTI system driven by a bounded piecewise continuous input so that $y_{ref} \in \mathbf{PC}^1_\infty$, $w \in \mathbf{PC}_\infty$, and consider $u \in \mathbf{PC}$. Again, this means that $e \in \mathscr{C}$.

The first stage of the controller will always be a sampler $S : \mathscr{C} \to \mathscr{S}$ which is defined via

$$\eta = Se \iff \eta[k] = e(kT),\ k \in \mathbf{Z}^+.$$

In some cases, the last stage of the controller will be the zero-order-hold $H : \mathscr{S} \to \mathbf{PC}$ which is defined via

$$y = H\nu \iff y(t) = \nu[k],\ t \in [kT, (k+1)T).$$

As a result of the sampling and hold operations, it will turn our that the following matrices will play an important role in the design of our low order LPC:

$$A_d := \mathrm{e}^{AT},\quad B_d = \int_0^T \mathrm{e}^{A\tau} B d\tau.$$

We now state some important properties of these matrices.

Definition 2.1 The sampling period T is said to be *pathological* if A has two eigenvalues with equal real parts and imaginary parts that differ by an integral multiple of $\frac{2\pi}{T}$; otherwise, the sampling period is *non-pathological*.

Theorem 2.1 If the sampling period T is non-pathological, then

$$(A, B) \text{ controllable} \Rightarrow (A_d, B_d) \text{ controllable}$$
$$(C, A) \text{ observable} \Rightarrow (C, A_d) \text{ observable}$$

Proof:

See Theorem 12 in [29]. ∎

Theorem 2.2 There exists a $T_0 > 0$ so that for every $T \in (0, T_0)$ we have (A_d, B_d) controllable and (C, A_d) observable.

Proof:

See Lemma 8 in [21]. ∎

With minor modifications to the proof of Lemma 8 in [21], the following result that is used in Chapter 4 can be proven:

Theorem 2.3 For every integer $p > n$, there exists a $T_0 > 0$ so that for every $T \in (0, T_0)$, the pair $\left(e^{ApT}, \int_0^{(p-n)T} e^{A\tau} B d\tau\right)$ is controllable.

Chapter 3

Robust Stability: Static GSHF Controllers

3.1 Introduction

The first step in control system design is usually that of obtaining a mathematical model of the plant. However, since there are inaccuracies in all measurements, we end up with a model with uncertainty in the parameters. One of the simplest ways to model this uncertainty is via an uncertain scalar multiplicative gain. To this end, we consider the following robust stabilization problem: with $P_0(s)$ the nominal finite dimensional linear time-invariant (FDLTI) plant model, we would like to find (if possible) a linear controller that will stabilize every system in a set of the form

$$\{\gamma P_0(s) : \ \gamma \text{ is a scalar gain uncertainty}\}.$$

This setup encompasses the classical gain margin problem, phase margin problem, and the combined gain/phase margin problem.

Given a single-input single-output (SISO) unstable non-minimum phase plant $P_0(s)$, it was shown by Khargonekar and Tannenbaum [33] that there is a maximum attainable

gain margin which can be provided by an LTI controller, and that this maximum is a function of the right half plane zeros and poles of $P_0(s)$. However, it turns out that time-varying controllers can do better.

In the continuous-time case, Lee, Meerkov, and Runolfsson [35] showed that for a SISO FDLTI plant, one can design a continuous-time periodic controller to provide a gain margin as large as desired; the controller order equals that of the plant. In the discrete-time case, Khargonekar, Poolla, and Tannenbaum [31] showed, among other things, that for a SISO FDLTI discrete-time bicausal plant with distinct unstable poles, there exists a discrete-time periodic controller that will provide a gain margin as large as desired, as well as a phase margin of up to 90 degrees.

In the sampled-data setting, there have been several approaches. Francis and Georgiou [21] considered the control of a FDLTI continuous-time plant with a sampled-data controller composed of a periodic discrete-time compensator and a zero-order-hold; they showed that for every multi-input multi-output (MIMO) continuous-time plant, there exists such a controller of suitable period that will provide a gain margin as large as desired. A second approach uses generalized hold functions, which gives rise to generalized sampled-data hold function (GSHF) controllers, which have been shown to be useful in the area of pole assignment, optimal noise rejection, model matching, decoupling and robustness [26, 25]. Yan, Anderson, and Bitmead [57] showed that for a MIMO FDLTI continuous-time plant, it is possible to design a dynamic GSHF controller to provide a gain margin as large as desired; indeed, for a given sampling period, they find the GSHF controller that will provide the maximum attainable gain margin. Motivated by a desire for low order controllers, Yang and Kabamba [58] showed that one can design a static GSHF controller to provide a gain margin as large as desired; in fact, they solve a more general multivariable gain margin problem.

In this chapter we consider the general gain/phase margin problem. We will design low order LTV controllers, parameterized by the sampling period T, which can provide

any desired phase margin up to 90 degrees, and have the property that the gain margin goes to infinity as $T \to 0$; in fact, we can prove that for every $\underline{\rho} \in (0,1]$, $\overline{\rho} \in [1,\infty)$, and $\overline{\phi} \in [0, \frac{\pi}{2})$, for sufficiently small T these controllers will stabilize every system in

$$\{\rho e^{j\phi} P_0(s) : \rho \in [\underline{\rho}, \overline{\rho}], \phi \in [-\overline{\phi}, \overline{\phi}]\}.$$

This work can be viewed as an extension of the work of Yang and Kabamba [58] to include a phase margin specification, although our proof is significantly different. In comparison to the work of Yan, Anderson, and Bitmead[57], here the controller is less complex and provides a guaranteed phase margin; on the other hand, it does not provide the maximal attainable gain margin for a given sampling period.

This chapter is organized in the following manner. We begin in Section 3.2 by formulating the problem in terms of designing a MIMO static GSHF controller for a MIMO LTI plant. In Section 3.3, we solve the problem posed in Section 3.2 using a continuous-time approach and illustrate the design method in an example. An improved design algorithm based on a discrete-time approach is presented in Appendix A.1, and through an example the two approaches are compared in Appendix A.2. Finally, in Section 3.4 it is shown that all of the MIMO static GSHF controllers presented can be implemented as a low order sampled-data controller consisting of a sampler, a discrete-time linear periodic compensator, and a zero-order-hold.

3.2 Problem Formulation

Our nominal plant model is

$$\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u(t), \quad x(0) = x_0, \\ y(t) &= C\,x(t), \end{aligned} \tag{3.1}$$

with $x(t) \in \mathbf{R}^n$ the state, $u(t) \in \mathbf{R}^m$ the control input, and $y(t) \in \mathbf{R}^r$ the plant output. Our standing assumption is that (A, B) is controllable and (C, A) is observable.

We capture uncertainty in the model by supposing that the actual system is given by

$$\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u(t), \quad x(0) = x_0, \\ y(t) &= \gamma\,C\,x(t), \end{aligned} \tag{3.2}$$

with $\gamma \in \mathbf{C}$; we represent this system by the triple $(A, B, \gamma C)$. Our parameter γ is assumed to lie in a set of the form

$$\Gamma(\underline{\rho}, \overline{\rho}, \overline{\phi}) := \{\rho e^{j\phi} : \rho \in [\underline{\rho}, \overline{\rho}], \phi \in [-\overline{\phi}, \overline{\phi}]\}.$$

Our goal is to find a controller which will simultaneously stabilize every model in

$$\{(A, B, \gamma C) : \gamma \in \Gamma(\underline{\rho}, \overline{\rho}, \overline{\phi})\}.$$

If $\overline{\phi} = 0$, then we have a gain margin problem; if $\underline{\rho} = \overline{\rho} = 1$, then we have a phase margin problem; the general case is a combined gain and phase margin problem.

We now define the set of controllers that will be considered. With $T > 0$ the sampling period of the generalized hold and $\overline{F} : \mathbf{R} \to \mathbf{R}^{m \times r}$ piecewise continuous and periodic of period T, we consider the static GSHF controller

$$u(t) = \overline{F}(t)\,y(kT), \quad t \in [kT, (k+1)T), \quad k \in \mathbf{Z}; \tag{3.3}$$

we represent the controller by the pair $(\overline{F}, T)$.

Definition 3.1 The GSHF controller (3.3) stabilizes (3.2) if, for every $x_0 \in \mathbf{R}^n$, we have

$$\lim_{t \to \infty} x(t) = 0.$$

Hence our objective is as follows:

> Given the nominal system (A, B, C) and the set $\Gamma(\underline{\rho}, \overline{\rho}, \overline{\phi})$, find a $T > 0$ and a piecewise continuous T-periodic function $\overline{F}(t)$ such that the GSHF controller $(\overline{F}, T)$ stabilizes the plant $(A, B, \gamma C)$ for every $\gamma \in \Gamma(\underline{\rho}, \overline{\rho}, \overline{\phi})$.

We will now reformulate our continuous-time problem as a discrete-time problem and restate our control objective. If we define

$$F := \int_0^T e^{A(T-\tau)}\,B\,\overline{F}(\tau)\,d\tau \in \mathbf{R}^{n \times r}, \tag{3.4}$$

then we can combine (3.2) and (3.3) to get the closed loop discrete-time system

$$x[(k+1)T] = \left(e^{AT} + \gamma FC\right) x[kT],\ k \in \mathbf{Z}^+. \tag{3.5}$$

It is straightforward to show using (3.2)-(3.5) that $x[kT] \to 0$ as $k \to \infty$ if, and only if $x(t) \to 0$ as $t \to \infty$; hence, it follows that the GSHF controller (3.3) stabilizes (3.2) if and only if

$$\mathrm{sp}(e^{AT} + \gamma FC) \subset \{z \in \mathbf{C} : |z| < 1\}.$$

Now our objective can be restated as:

> Given the nominal system (A, B, C) and the set $\Gamma(\underline{\rho}, \overline{\rho}, \overline{\phi})$, find a $T > 0$ and an $F \in \mathbf{R}^{n\times r}$ such that for every $\gamma \in \Gamma(\underline{\rho}, \overline{\rho}, \overline{\phi})$, we have
> $$\mathrm{sp}(e^{AT} + \gamma FC) \subset \{z \in \mathbf{C} : |z| < 1\}.$$

Since (A, B) is controllable, for every $T > 0$ and $F \in \mathbf{R}^{n\times r}$ there exists a piecewise continuous T-periodic function $\overline{F}(t)$ such that (3.4) is satisfied[26]. For example, with

$$W(T) := \int_0^T e^{A\tau} BB^T e^{A^T\tau} d\tau,$$

one possible solution is

$$\overline{F}(t) = B^T e^{A^T(T-t)} (W(T))^{-1} F. \tag{3.6}$$

This solution is not unique and in Section 3.4 we will present another solution for $\overline{F}(t)$ that is easier to implement in practice.

Remark 3.1 Suppose that for a given A, B, C we have found a $T > 0$ and an $F \in \mathbf{R}^{n\times r}$ such that

$$\mathrm{sp}(e^{AT} + FC) \subset \{z \in \mathbf{C} : |z| < 1\}.$$

i) The upper (lower) gain margin provided by the controller (3.3) is the maximum $\overline{\rho} \in [1, \infty)$ (minimum $\underline{\rho} \in (0, 1]$) such that

$$\mathrm{sp}(e^{AT} + \rho FC) \subset \{z \in \mathbf{C} : |z| < 1\},\quad \rho \in [1, \overline{\rho}]\ \ (\rho \in [\underline{\rho}, 1]).$$

The gain margin provided by the controller is $\overline{\rho}/\underline{\rho}$.

ii) The phase margin provided by the controller (3.3) is the maximum $\overline{\phi} \in [0, \frac{\pi}{2})$ such that

$$\mathrm{sp}(e^{AT} + e^{j\phi}FC) \subset \{z \in \mathbf{C} : |z| < 1\}, \quad \phi \in [-\overline{\phi}, \overline{\phi}].$$

iii) If the controller were LTI, then it is straightforward to show that if either the gain or phase margin is small, then the closed loop system is close to instability; a Nyquist argument would work. However, even if both the gain and phase margin are *independently* large, then the closed loop system may still be close to instability, e.g. see Figure 3.1 (a) which is motivated by [16, pp. 52–53]; here simultaneous small changes in the gain and phase of the plant may result in instability.

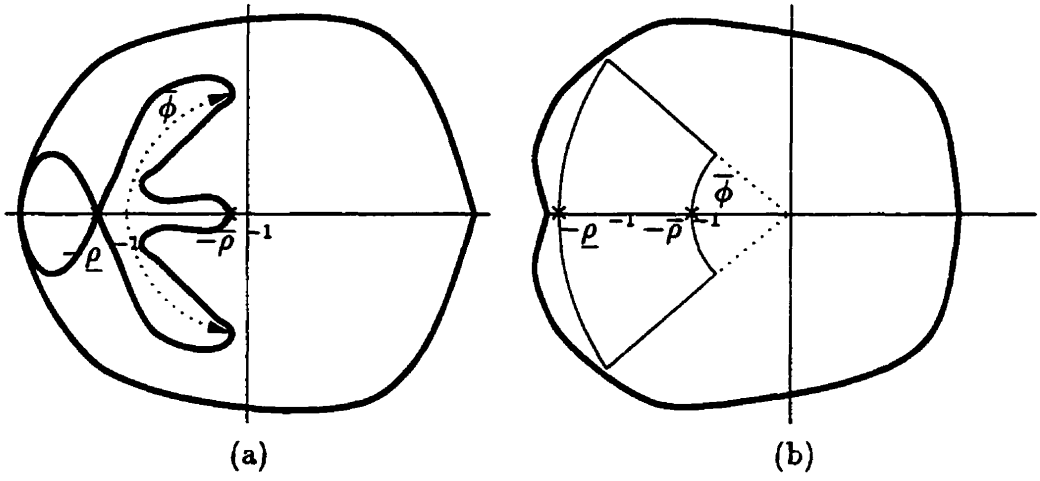


Figure 3.1: The combined gain/phase margin.

However, if the LTI controller provides a large *combined* gain/phase margin, then we are guaranteed to have a better overall stability margin, e.g. see Figure 3.1 (b). Since we are using a linear time-varying controller here, the above arguments are not directly applicable. However, they are applicable to the LTI discrete-time system arising from the sampler, plant, and generalized hold combination. Indeed, observe that if we perturb our continuous-time system model in the frequency domain, the discrete-time counterpart is perturbed accordingly. This will be discussed further in Section 3.5.

□

3.3 Continuous-Time Approach

Define

$$F_0 := T^{-1}F = T^{-1}\int_0^T \mathrm{e}^{A(T-\tau)}\, B\, \overline{F}(\tau)\, d\tau.$$

It follows that the GSHF controller (3.3) stabilizes (3.2) if and only if

$$\mathrm{sp}(\mathrm{e}^{AT} + \gamma T F_0 C) \subset \{z \in \mathbf{C} : |z| < 1\}.$$

This first approach is motivated by the observation that for small T, we have

$$\mathrm{e}^{AT} + \gamma T F_0 C \approx I + T(A + \gamma F_0 C) \approx \mathrm{e}^{(A+\gamma F_0 C)T},$$

so if

$$\mathrm{sp}(A + \gamma F_0 C) \subset \mathbf{C}^-, \tag{3.7}$$

then we might expect that

$$\mathrm{sp}(\mathrm{e}^{AT} + \gamma T F_0 C) \subset \{z \in \mathbf{C} : |z| < 1\}.$$

Conditions similar to (3.7) are commonly encountered in **continuous-time** state feedback problems, and hence the motivation for the name of this approach.

However, the above is an approximate analysis, and here we have plant uncertainty, which further complicates the analysis. To make this precise, we define

$$\Delta(T) := T^{-1}(\mathrm{e}^{AT} - I - AT), \quad T > 0, \tag{3.8}$$

so that

$$\mathrm{e}^{AT} + \gamma T F_0 C = I + T(A + \gamma F_0 C + \Delta(T));$$

notice that $\Delta(T)$ is $\mathcal{O}(T^2)$. If (3.7) holds and $\|\Delta(T)\|$ is sufficiently small, then

$$\mathrm{sp}(A + \gamma F_0 C + \Delta(T)) \subset \mathbf{C}^- = \{s \in \mathbf{C} : \mathrm{Re}(s) < 0\},$$

which means that

$$\mathrm{sp}(\mathrm{e}^{AT} + \gamma T F_0 C) = \mathrm{sp}[I + T(A + \gamma F_0 C + \Delta(T))] \subset \{s \in \mathbf{C} : \mathrm{Re}(s) < 1\}.$$

Unfortunately, this does not imply that

$$\mathrm{sp}(e^{AT} + \gamma T F_0 C) \subset \{z \in \mathbf{C} : |z| < 1\}. \tag{3.9}$$

However, if we can choose F_0 so that $\mathrm{sp}(A + \gamma F_0 C + \Delta(T))$ lies in a sector of the form

$$\mathcal{S}(\theta) := \{\sigma e^{j\omega} : (\tfrac{\pi}{2} + \theta) < \omega < (\tfrac{3\pi}{2} - \theta), \sigma > 0\} \tag{3.10}$$

(see Figure 3.2), then it can be shown that (3.9) would indeed hold, at least for small enough T. This will be investigated in the next subsection.

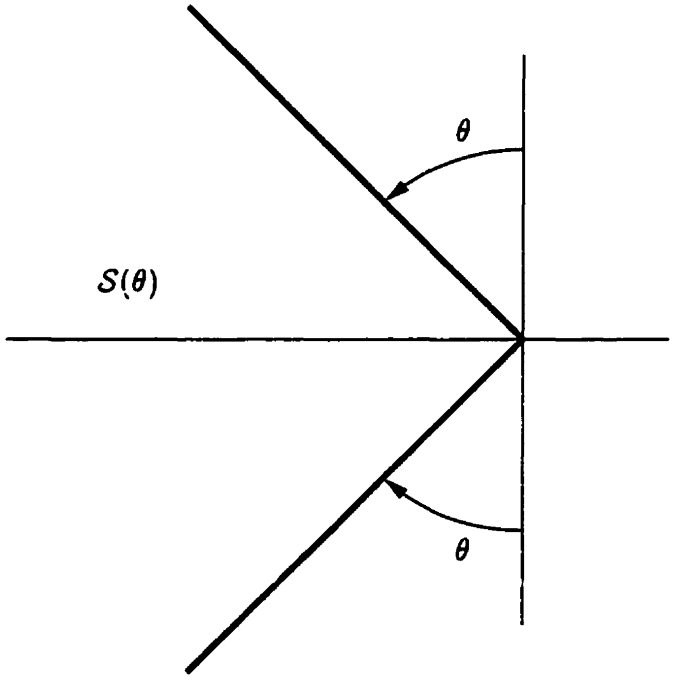


Figure 3.2: The region $\mathcal{S}(\theta)$.

3.3.1 Preliminary Continuous-Time LQR Results

We first investigate how $\mathrm{sp}(A + \rho e^{j\phi} F_0 C + \Delta(T))$ is affected by the perturbation $\Delta(T) \in \mathbf{R}^{n \times n}$ over different ranges of ρ and ϕ. We will choose F_0 using linear quadratic regulator (LQR) theory. To motivate our approach, recall that in the continuous-time

setting an LQR optimal controller provides a lower gain margin of $\frac{1}{2}$, an upper gain margin of infinity, and a 60 degree phase margin [3, pp. 70–76]. It turns out that we can increase the phase margin and decrease the lower gain margin by modifying the standard LQR problem. From the previous section, we saw that the stability of $A+\gamma F_0 C$ plays a crucial role. This matrix is stable if and only if $A^T+\gamma C^T F_0^T$ is stable; finding an F_0 to achieve this is simply a state feedback problem. Hence, with $\overline{\phi} \in [0, \frac{\pi}{2})$ and $\underline{\rho} \in (0,1]$, we define

$$\alpha := 2\cos(\overline{\phi}), \tag{3.11}$$

and consider the auxiliary system

$$\dot{w}(t) = A^T w(t) + \alpha\underline{\rho} C^T v(t),\ w(0) = w_0;$$

with $Q \in \mathbf{R}^{n\times n}$ and $R \in \mathbf{R}^{m\times m}$ satisfying

$$\begin{aligned} Q > 0 \quad &, \quad Q = Q^T, \\ R > 0 \quad &, \quad R = R^T, \end{aligned} \tag{3.12}$$

we wish to find the control law which, for each w_0, minimizes

$$\int_0^\infty w(t)^T Q w(t) + v(t)^T R v(t) dt.$$

Since $(\sqrt{Q}, A^T)$ is observable and $(A^T, \alpha\underline{\rho} C^T)$ is controllable, it follows from Lemma 12.1 and Theorem 12.2 found in [55], that the optimal control law is of the form

$$v(t) = F_0^T w(t),$$

and we can obtain the optimal gain F_0 by first solving the continuous-time algebraic Riccati equation

$$AP_0 + P_0 A^T - P_0(\alpha\underline{\rho} C)^T R^{-1}(\alpha\underline{\rho} C)P_0 + Q = 0 \tag{3.13}$$

for the unique positive definite symmetric solution P_0, and then setting

$$F_0 = -P_0(\alpha\underline{\rho} C)^T R^{-1}. \tag{3.14}$$

Proposition 3.1 If

$$\|\Delta(T)\| < \frac{\lambda_{min}(Q)}{2\|P_0\|},$$

then for every $\rho \in [\underline{\rho}, \infty)$ and $\phi \in [-\overline{\phi}, \overline{\phi}]$ we have

$$\mathrm{sp}(A + \rho e^{j\phi} F_0 C + \Delta(T)) \subset \mathbf{C}^-.$$

Remark 3.2 Proposition 3.1 states that an LQR optimal controller can be designed to simultaneously provide any desired lower and upper gain margin and any desired phase margin up to 90 degrees, even under small perturbations in the A matrix. □

Proof:(of Proposition 3.1)

To prove this, we adopt a Lyapunov argument. Let $\widehat{\rho} \in \left[\frac{1}{\alpha}, \infty\right)$, $\phi \in [-\overline{\phi}, \overline{\phi}]$,

$$A_n := A^T, \quad B_n := (\alpha \underline{\rho} C)^T, \quad F_n := F_0^T, \quad \Delta_n(T) := \Delta(T)^T,$$

and consider

$$\dot{w}(t) = (A_n + \widehat{\rho} e^{j\phi} B_n F_n + \Delta_n(T)) w(t), \quad w(0) = w_0 \in \mathbf{C}^n.$$

Consider the Lyapunov candidate function $V : \mathbf{C}^n \to \mathbf{C}$:

$$V(w) := w^* P_0 w.$$

Since $P_0 = P_0^T$ we know that V is real-valued, so

$$\dot{V}(w(t)) := \frac{\partial}{\partial t} V(w(t))$$

is also real-valued. Expanding $\dot{V}(w(t))$ and using (3.13) and (3.14) to simplify, it follows that

$$\begin{aligned} \dot{V}(w(t)) &= -w(t)^* Q w(t) + (1 - 2\widehat{\rho}\cos(\phi)) w(t)^* (P_0 B_n R^{-1} B_n^T P_0) w(t) \\ &\quad + w(t)^* (\Delta_n(T)^T P_0 + P_0 \Delta_n(T)) w(t). \end{aligned} \tag{3.15}$$

Using the bounds on $\overline{\phi}$, $\widehat{\rho}$, and the fact that $R > 0$, it can easily be shown that the second term on the RHS of equation (3.15) is non positive, so

$$\dot{V}(w(t)) \leq -w(t)^* Q w(t) + w(t)^* (\Delta_n(T)^T P_0 + P_0 \Delta_n(T)) w(t).$$

Now for $w(t) \neq 0$, it follows that

$$\begin{aligned} w(t)^* Q w(t) &\geq \lambda_{min}(Q)\|w(t)\|^2 \\ &> 2\|P_0\| \times \|\Delta_n(T)\| \times \|w(t)\|^2 \qquad \text{(by hypothesis)} \\ &\geq \|\Delta_n(T)^T P_0 + P_0 \Delta_n(T)\| \times \|w(t)\|^2 \\ &\geq |w(t)^*(\Delta_n(T)^T P_0 + P_0 \Delta_n(T))w(t)|, \end{aligned}$$

so for $w(t) \neq 0$, we have

$$\dot{V}(w(t)) < 0.$$

Combining this with (3.15) we see that there exists a positive definite symmetric matrix $U \in \mathbf{R}^{n \times n}$ so that

$$\dot{V}(w(t)) = -w(t)^* U w(t) \quad \Rightarrow \|w(t)\|^2 \leq -\frac{\dot{V}(w(t))}{\lambda_{min}(U)}.$$

Therefore,

$$V(w(t)) \leq \|P_0\| \times \|w(t)\|^2 \leq -\frac{\|P_0\|}{\lambda_{min}(U)} \dot{V}(w(t)),$$

so it follows from Lemma 2.2 that

$$V(w(t)) \leq e^{-\frac{\lambda_{min}(U)}{\|P_0\|}t} V(w_0), \quad t \geq 0.$$

Hence, for every w_0, $V(w(t))$ goes to zero as $t \to \infty$; since P_0 is positive definite, it follows that $w(t)$ goes to zero as $t \to \infty$ as well, which means that

$$\begin{aligned} & \mathrm{sp}(A_n + \widehat{\rho} e^{j\phi} B_n F_n + \Delta_n(T)) \subset \mathbf{C}^- \\ \Rightarrow \quad & \mathrm{sp}(A + \alpha \underline{\rho} \widehat{\rho} e^{j\phi} F_0 C + \Delta(T)) \subset \mathbf{C}^-. \end{aligned}$$

But this holds for $\alpha \underline{\rho} \widehat{\rho} \in [\underline{\rho}, \infty)$, so for $\rho \in [\underline{\rho}, \infty)$ and $\phi \in [-\overline{\phi}, \overline{\phi}]$ we have

$$\mathrm{sp}(A + \rho e^{j\phi} F_0 C + \Delta(T)) \subset \mathbf{C}^-.$$

■

Now we will show that by restricting the perturbation further, we can force the eigenvalues of $A + \rho e^{j\phi} F_0 C + \Delta(T)$ to lie in the sector $\mathcal{S}(\theta)$ given by (3.10).

Proposition 3.2 Let

$$\theta \in \left[0, \min\left\{\tfrac{\pi}{2} - \overline{\phi}, \tan^{-1}\left(\tfrac{\lambda_{min}(Q)}{3\|P_0A^T - AP_0\|}\right), \tan^{-1}\left(\tfrac{\lambda_{min}(Q)}{3\underline{\rho}^2\alpha\sqrt{4-\alpha^2}\|P_0C^TR^{-1}CP_0\|}\right)\right\}\right).$$

If

$$\|\Delta(T)\| < \frac{\lambda_{min}(Q)}{6\|P_0\|}\cos(\theta),$$

then for every $\rho \in [\underline{\rho}, \infty)$ and $\phi \in [-\overline{\phi}, \overline{\phi}]$ we have

$$\mathrm{sp}(A + \rho e^{j\phi}F_0C + \Delta(T)) \subset \mathcal{S}(\theta).$$

Proof:

Let $\widehat{\rho} \in [\frac{1}{\alpha}, \infty)$, $\phi \in [-\overline{\phi}, \overline{\phi}]$, and

$$A_n := A^T, \quad B_n := (\alpha\underline{\rho}C)^T, \quad F_n := F_0^T, \quad \Delta_n(T) := \Delta(T)^T,$$

and consider

$$\dot{w}(t) = e^{j\theta}(A_n + \widehat{\rho}e^{j\phi}B_nF_n + \Delta_n(T))w(t), \quad w(0) = w_0 \in \mathbf{C}^n. \tag{3.16}$$

Consider the Lyapunov candidate function $V : \mathbf{C}^n \to \mathbf{C}$:

$$V(w) := w^*P_0w.$$

Since $P_0 = P_0^T$, it follows that V and $\dot{V}$ are real valued. Expanding $\dot{V}(w(t))$ and simplifying, we have

$$\begin{aligned}\dot{V}(w(t)) &= \cos(\theta)w(t)^*(A_n^TP_0 + P_0A_n)w(t) + j\sin(\theta)w(t)^*(P_0A_n - A_n^TP_0)w(t)\\ &\quad + \widehat{\rho}w(t)^*(e^{-j(\theta+\phi)}F_n^TB_n^TP_0 + e^{j(\theta+\phi)}P_0B_nF_n)w(t)\\ &\quad + w(t)^*(e^{-j\theta}\Delta_n(T)^TP_0 + e^{j\theta}P_0\Delta_n(T))w(t).\end{aligned}$$

Using (3.13) and (3.14), this can be rewritten as

$$\begin{aligned}\dot{V}(w(t)) &= -\tfrac{1}{3}\cos(\theta)w(t)^*Qw(t) + j\sin(\theta)w(t)^*(P_0A_n - A_n^TP_0)w(t)\\ &\quad -\tfrac{1}{3}\cos(\theta)w(t)^*Qw(t) + (\cos(\theta) - 2\widehat{\rho}\cos(\theta+\phi))w(t)^*(P_0B_nR^{-1}B_n^TP_0)w(t)\\ &\quad -\tfrac{1}{3}\cos(\theta)w(t)^*Qw(t) + w(t)^*(e^{-j\theta}\Delta_n(T)^TP_0 + e^{j\theta}P_0\Delta_n(T))w(t).\end{aligned}$$

By hypothesis we have

$$\tan(\theta) \le \frac{\lambda_{min}(Q)}{3\|P_0A^T - AP_0\|} \quad \Rightarrow \quad \frac{1}{3}\cos(\theta)\lambda_{min}(Q) \ge \sin(\theta)\|P_0A_n - A_n^TP_0\|,$$

so

$$\begin{aligned}\tfrac{1}{3}\cos(\theta)w(t)^*Qw(t) &\geq \tfrac{1}{3}\cos(\theta)\lambda_{min}(Q)\|w(t)\|^2 \\ &\geq \sin(\theta)\|P_0A_n - A_n^TP_0\| \times \|w(t)\|^2 \\ &\geq |\sin(\theta)w(t)^*(P_0A - A_n^TP_0)w(t)|. \end{aligned} \tag{3.17}$$

Furthermore,

$$\|\Delta_n(T)\| < \frac{\lambda_{min}(Q)}{.6\|P_0\|}\cos(\theta) \quad \Rightarrow \quad \frac{1}{3}\cos(\theta)\lambda_{min}(Q) > 2\|P_0\| \times \|\Delta_n(T)\|,$$

so for $w(t) \neq 0$, we have

$$\begin{aligned}\tfrac{1}{3}\cos(\theta)w(t)^*Qw(t) &\geq \tfrac{1}{3}\cos(\theta)\lambda_{min}(Q)\|w(t)\|^2 \\ &> 2\|P_0\| \times \|\Delta_n(T)\| \times \|w(t)\|^2 \\ &\geq \|e^{-j\theta}\Delta_n(T)^TP_0 + e^{j\theta}P_0\Delta_n(T)\| \times \|w(t)\|^2 \\ &\geq |w(t)^*(e^{-j\theta}\Delta_n(T)^TP_0 + e^{j\theta}P_0\Delta_n(T))w(t)|. \end{aligned} \tag{3.18}$$

Finally, since $\theta + \phi \in (-\frac{\pi}{2}, \frac{\pi}{2})$, it follows that

$$\begin{aligned}\cos(\theta) - 2\widehat{\rho}\cos(\theta+\phi) &\leq \cos(\theta) - \tfrac{2}{\alpha}(\cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi)) \\ &= \cos(\theta)(1 - \tfrac{2}{\alpha}\underbrace{\cos(\phi)}_{\geq\frac{\alpha}{2}}) + \tfrac{2}{\alpha}\sin(\theta)\underbrace{\sin(\phi)}_{\leq\frac{\sqrt{4-\alpha^2}}{2}} \\ &\leq \tfrac{\sqrt{4-\alpha^2}}{\alpha}\sin\theta, \end{aligned}$$

and from the hypothesis it follows that

$$\begin{aligned}\tan(\theta) &\leq \frac{\lambda_{min}(Q)}{3\frac{\sqrt{4-\alpha^2}}{\alpha}\|P_0B_nR^{-1}B_n^TP_0\|} \\ \Rightarrow \quad \frac{1}{3}\cos(\theta)\lambda_{min}(Q) &\geq \tfrac{\sqrt{4-\alpha^2}}{\alpha}\sin(\theta)\|P_0B_nR^{-1}B_n^TP_0\|, \end{aligned}$$

so

$$\begin{aligned}\tfrac{1}{3}\cos(\theta)w(t)^*Qw(t) &\geq \tfrac{1}{3}\cos(\theta)\lambda_{min}(Q)\|w(t)\|^2 \\ &\geq \tfrac{\sqrt{4-\alpha^2}}{\alpha}\sin(\theta)\|P_0B_nR^{-1}B_n^TP_0\| \times \|w(t)\|^2 \\ &\geq (\cos(\theta) - 2\widehat{\rho}\cos(\theta+\phi))\|P_0B_nR^{-1}B_n^TP_0\| \times \|w(t)\|^2 \\ &\geq (\cos(\theta) - 2\widehat{\rho}\cos(\theta+\phi))|w(t)^*(P_0B_nR^{-1}B_n^TP_0)w(t)|.\end{aligned} \tag{3.19}$$

Therefore, from (3.17)–(3.19) it follows that for $w(t) \neq 0$, we have

$$\dot{V}(w(t)) < 0.$$

Arguing as in the proof of Proposition 3.1, we conclude that

$$\begin{aligned}&\mathrm{sp}(e^{j\theta}(A_n + \widehat{\rho}e^{j\phi}B_nF_n + \Delta_n(T))) \subset \mathbf{C}^- \\ &\Rightarrow \mathrm{sp}(A_n + \widehat{\rho}e^{j\phi}B_nF_n + \Delta_n(T)) \subset \{\sigma e^{j\omega} : (\tfrac{\pi}{2} - \theta) < \omega < (\tfrac{3\pi}{2} - \theta), \sigma > 0\}.\end{aligned}$$

By replacing $e^{j\theta}$ by $e^{-j\theta}$ in (3.16) and modifying the above argument slightly, it can be shown that

$$\mathrm{sp}(A_n + \widehat{\rho}e^{j\phi}B_nF_n + \Delta_n(T)) \subset \{\sigma e^{j\omega} : (\tfrac{\pi}{2} + \theta) < \omega < (\tfrac{3\pi}{2} + \theta), \sigma > 0\}.$$

Hence,

$$\begin{aligned}&\mathrm{sp}(A_n + \widehat{\rho}e^{j\phi}B_nF_n + \Delta_n(T)) \subset \{\sigma e^{j\omega} : (\tfrac{\pi}{2} + \theta) < \omega < (\tfrac{3\pi}{2} - \theta), \sigma > 0\} = \mathcal{S}(\theta) \\ &\Rightarrow \mathrm{sp}(A + \underline{\rho}\alpha\widehat{\rho}e^{j\phi}F_0C + \Delta(T)) \subset \mathcal{S}(\theta).\end{aligned}$$

But this holds for $\alpha\underline{\rho}\widehat{\rho} \in [\underline{\rho}, \infty)$, so for $\rho \in [\underline{\rho}, \infty)$ and $\phi \in [-\overline{\phi}, \overline{\phi}]$, we have

$$\mathrm{sp}(A + \rho e^{j\phi}F_0C + \Delta(T)) \subset \mathcal{S}(\theta). \qquad \blacksquare$$

3.3.2 Controller Design

In this subsection we will use the results of Subsection 3.3.1 to show that the control objective of Section 3.2 can be attained. The proof is constructive, and we provide a design algorithm.

Theorem 3.1 Let $\overline{\phi} \in \left[0, \frac{\pi}{2}\right)$, $\underline{\rho} \in (0,1]$, and $\overline{\rho} \in [1, \infty)$. Then there exists a $T_{max} > 0$ so that for every $T \in (0, T_{max})$, there exists a static GSHF controller $(\overline{F}, T)$ which stabilizes every system in

$$\{(A, B, \gamma C) : \gamma \in \Gamma(\underline{\rho}, \overline{\rho}, \overline{\phi})\}.$$

Proof:

Recall that $\Delta(T)$ has been defined in (3.8) so that

$$e^{AT} + \rho e^{j\phi} T F_0 C = I + T(A + \rho e^{j\phi} F_0 C + \Delta(T)),$$

which means that

$$\begin{aligned} & \mathrm{sp}(e^{AT} + \rho e^{j\phi} T F_0 C) \subset \{z \in \mathbf{C} : |z| < 1\} \\ \Leftrightarrow \quad & \mathrm{sp}[I + T(A + \rho e^{j\phi} F_0 C + \Delta(T))] \subset \{z \in \mathbf{C} : |z| < 1\} \\ \Leftrightarrow \quad & \mathrm{sp}(A + \rho e^{j\phi} F_0 C + \Delta(T)) \subset \left\{\tfrac{z-1}{T} : z \in \mathbf{C}, |z| < 1\right\} =: \mathcal{D}(T). \end{aligned}$$

Let $\overline{\phi} \in \left[0, \frac{\pi}{2}\right)$, $\underline{\rho} \in (0,1]$, and $\overline{\rho} \in [1, \infty)$. Now choose Q and R to satisfy (3.12), and let α, $P_0 > 0$ symmetric and F_0 satisfy (3.11), (3.13), and (3.14). Let

$$\theta \in \left(0, \min\left\{\tfrac{\pi}{2} - \overline{\phi}, \tan^{-1}\left(\tfrac{\lambda_{min}(Q)}{3\|P_0 A^T - A P_0\|}\right), \tan^{-1}\left(\tfrac{\lambda_{min}(Q)}{3\underline{\rho}^2 \alpha \sqrt{4-\alpha^2}\|P_0 C^T R^{-1} C P_0\|}\right)\right\}\right).$$

Since $\|\Delta(T)\| \to 0$ as $T \to 0$, we can choose $T_\Delta > 0$ so that

$$\|\Delta(T)\| < \frac{\lambda_{min}(Q)}{6\|P_0\|} \cos(\theta), \quad T \in (0, T_\Delta].$$

From Proposition 3.2 it follows that for every $\rho \in [\underline{\rho}, \infty)$, $\phi \in [-\overline{\phi}, \overline{\phi}]$ and $T \in (0, T_\Delta]$, we have

$$\mathrm{sp}(A + \rho e^{j\phi} F_0 C + \Delta(T)) \subset \mathcal{S}(\theta).$$

Consider the diagram illustrating $\mathcal{S}(\theta) \cap \mathcal{D}(T)$ in Figure 3.3, and define r as illustrated; it is easily shown that

$$r = 2T^{-1} \sin(\theta).$$

It suffices to show that there exists a $T_{max} \in (0, T_\Delta]$ such that for every $\rho \in [\underline{\rho}, \overline{\rho}]$, $\phi \in [-\overline{\phi}, \overline{\phi}]$, and $T \in (0, T_{max})$, we have

$$\max\{|\lambda| : \lambda \in \mathrm{sp}(A + \rho e^{j\phi} F_0 C + \Delta(T))\} < r.$$

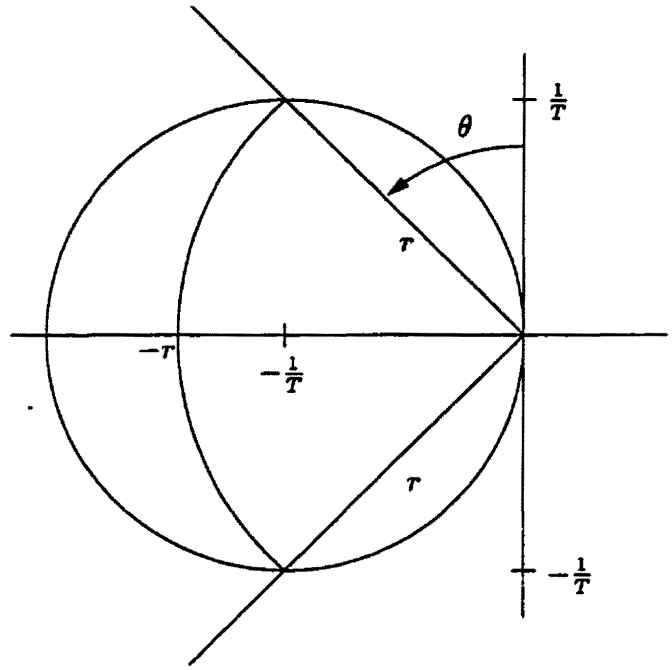


Figure 3.3: The region $\mathcal{S}(\theta) \cap \mathcal{D}(T)$.

But

$$\begin{aligned} \max\{|\lambda| : \lambda \in \mathrm{sp}(A + \rho e^{j\phi} F_0 C + \Delta(T))\} &\leq \|A + \rho e^{j\phi} F_0 C + \Delta(T)\| \\ &\leq \|A\| + \bar{\rho}\|F_0 C\| + \frac{\lambda_{min}(Q)}{6\|P_0\|}\cos(\theta). \end{aligned}$$

Hence, it is sufficient that

$$r = 2T^{-1}\sin(\theta) > \|A\| + \bar{\rho}\|F_0 C\| + \frac{\lambda_{min}(Q)}{6\|P_0\|}\cos(\theta),$$

so define

$$T_{max} := \min\left\{T_\Delta, \frac{2\sin(\theta)}{\|A\| + \bar{\rho}\|F_0 C\| + \frac{\lambda_{min}(Q)}{6\|P_0\|}\cos(\theta)}\right\}.$$

Finally, with $T \in (0, T_{max})$ and $F = TF_0$, we find an $\overline{F}(t)$ which satisfies (3.4). ∎

Remark 3.3 With $\bar{\phi} \in [0, \frac{\pi}{2})$, $\underline{\rho} \in (0, 1]$, and $\bar{\rho} \in [1, \infty)$, we now summarize an algorithm for constructing a static GSHF controller $(\overline{F}, T)$ that stabilizes every system in

$$\{(A, B, \gamma C) : \gamma \in \Gamma(\underline{\rho}, \bar{\rho}, \bar{\phi})\}.$$

i) Choose symmetric positive definite matrices Q and R. With

$$\alpha = 2\cos(\overline{\phi}),$$

find the unique positive definite symmetric matrix P_0 satisfying (3.13), and determine F_0 from (3.14).

ii) Choose

$$\theta \in \left(0, \min\left\{\frac{\pi}{2} - \overline{\phi}, \tan^{-1}\left(\frac{\lambda_{min}(Q)}{3\|P_0A^T - AP_0\|}\right), \tan^{-1}\left(\frac{\lambda_{min}(Q)}{3\underline{\rho}^2\alpha\sqrt{4-\alpha^2}\|P_0C^TR^{-1}CP_0\|}\right)\right\}\right).$$

iii) Find T_Δ such that for every $T \in (0, T_\Delta]$, we have

$$\|\tfrac{1}{T}(e^{AT} - I - AT)\| < \frac{\lambda_{min}(Q)}{6\|P_0\|}\cos(\theta);$$

this is computationally easy to do, especially if A is diagonal.

iv) Define

$$T_{max} = \min\left\{T_\Delta, \frac{2\sin(\theta)}{\|A\| + \overline{\rho}\|F_0C\| + \frac{\lambda_{min}(Q)}{6\|P_0\|}\cos(\theta)}\right\}.$$

v) With $T < T_{max}$, let $F = TF_0$ and use any desired method to find an $\overline{F}(t)$ to satisfy (3.4), e.g. use (3.6).

□

Remark 3.4 The T_{max} obtained in the above algorithm is based on a Lyapunov approach and hence is typically quite conservative; perhaps a better choice for the Lyapunov candidate function in the proof of Proposition 3.2 would result in a less conservative value of T_{max}. However, from a practical point of view, it might be preferable to compute the combined gain margin and phase margin for various values of $T > T_{max}$ using a 2 dimensional search algorithm, and choose the largest sampling period which achieves the desired robustness.

□

3.3.3 An Example

Suppose our nominal plant is

$$(A,B,C) = \left(\begin{bmatrix} -1 & 0 \\ 0 & 0.5 \end{bmatrix}, \begin{bmatrix} \frac{4}{3} \\ -\frac{1}{3} \end{bmatrix}, \begin{bmatrix} 1 & 1 \end{bmatrix} \right),$$

with associated transfer function

$$P_0(s) := C(sI - A)^{-1}B = \frac{s-1}{(s-0.5)(s+1)}.$$

This example is taken from [16, pp. 200–203], where it is shown that an LTI stabilizing compensator can provide at most a phase margin of 38.9°, and the upper and lower gain margins must satisfy $\overline{\rho}/\underline{\rho} \leq 4$.

Using the algorithm outlined in Remark 3.3, we now construct a static GSHF controller to stabilize every system in

$$\{(A,B,\gamma C) : \gamma \in \Gamma(0.75, 6, 70^o)\}.$$

i) Here $\alpha = 2\cos(70^o) = 0.6840403$. Let $Q = I$ and $R = 1$. Then P_0 and F_0 are

$$P_0 = \begin{bmatrix} 0.4974287 & -0.3576478 \\ -0.3576478 & 5.218269 \end{bmatrix}, \quad F_0 = \begin{bmatrix} -0.07171182 \\ -2.493646 \end{bmatrix}.$$

ii) Now

$$\min\left\{\tfrac{\pi}{2} - \overline{\phi}, \tan^{-1}\left(\tfrac{\lambda_{min}(Q)}{3\|P_0A^T - AP_0\|}\right), \tan^{-1}\left(\tfrac{\lambda_{min}(Q)}{3\underline{\rho}^2\alpha\sqrt{4-\alpha^2}\|P_0C^TR^{-1}CP_0\|}\right)\right\} = 0.01949220$$

so choose $\theta = 0.019$.

iii) It is easily verified that for $T \in (0, 0.06492000]$, we have

$$\|\Delta(T)\| = \|\tfrac{1}{T}(e^{AT} - I - AT)\| < \frac{\lambda_{min}(Q)}{6\|P_0\|}\cos(\theta) = 0.03176428.$$

iv) So $T_{max} = \min\left\{0.06491624, \frac{2\sin(\theta)}{\|A\| + \overline{\rho}\|F_0C\| + \frac{\lambda_{min}(Q)}{6\|P_0\|}\cos(\theta)}\right\} = 0.001712.$

v) Hence, choose $T = 0.0017$. As noted in Remark 3.4, this sampling period is quite conservative, so we now propose a method to approximately determine a less conservative sampling period. Using a 2-dimensional bisection search algorithm, we determine

$$\underline{\rho}_{c.t.}(T) := \min\{\,\widehat{\rho} \in (0,1] : \mathrm{sp}(e^{AT} + \rho e^{j\phi} T F_0 C) \subset \{z \in \mathbf{C} : |z| < 1\},$$
$$\rho \in [\widehat{\rho}, 1], \phi \in [-\overline{\phi}, \overline{\phi}]\},$$
$$\overline{\rho}_{c.t.}(T) := \max\{\widehat{\rho} \in [1, \infty) : \mathrm{sp}(e^{AT} + \rho e^{j\phi} T F_0 C) \subset \{z \in \mathbf{C} : |z| < 1\},$$
$$\rho \in [1, \widehat{\rho}], \phi \in [-\overline{\phi}, \overline{\phi}]\};$$

for lack of a better name, we will refer to $\underline{\rho}_{c.t.}(T)$ ($\overline{\rho}_{c.t.}(T)$) as the "combined lower (upper) gain/phase margin" provided by the GSHF controller $(\overline{F}, T)$ using the "continuous-time" approach. A plot of $\underline{\rho}_{c.t.}(T)$ and $\overline{\rho}_{c.t.}(T)$ is provided in Figure 3.4). Observe that for $T = 0.041$ we achieve the desired gain and phase margin.

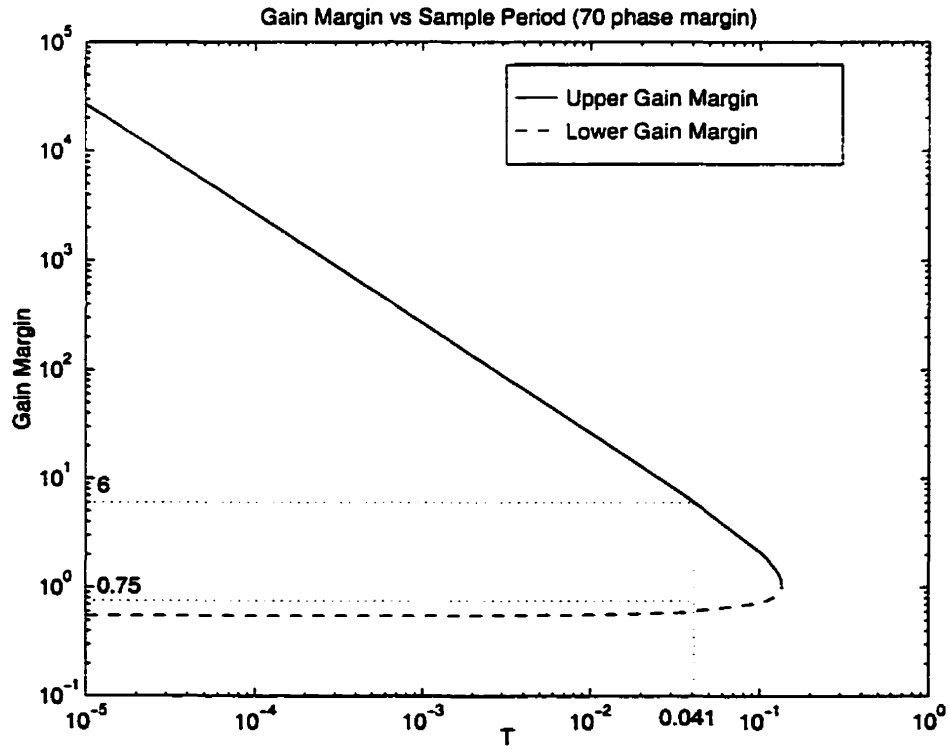


Figure 3.4: Combined upper and lower gain and phase margin as a function of T (c.t. approach).

Using (3.6), we can choose

$$\overline{F}(t) = -2.415424481(10)^4 e^{-(0.041-t)} + 2.342890355(10)^4 e^{0.5*(0.041-t)}.$$

Figure 3.5 illustrates the response of the closed loop system at the GSHF sample points when the initial condition $x_0 = [\ 1 \quad 0\ ]^T$ and the scalar gain uncertainty $\gamma = 4$.

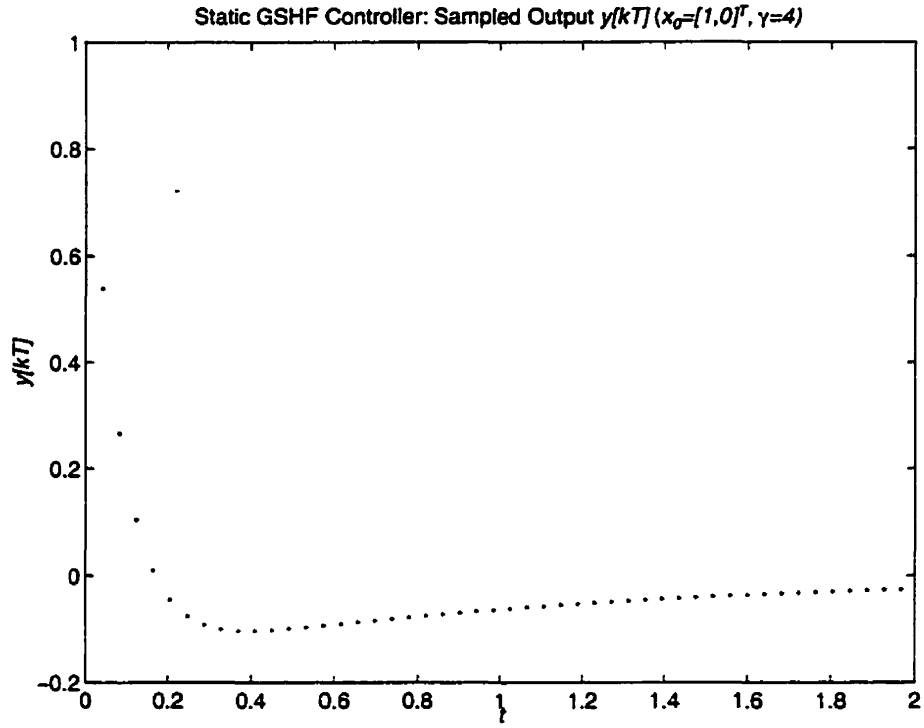


Figure 3.5: Sampled output $y[kT]$ when $x_0 = [1\ 0]^T$ and $\gamma = 4$ (c.t. approach).

While the GSHF controller provides adequate performance at the sample points, it can be seen in Figure 3.6 that the intersample performance is quite poor. This is primarily due to the fact that the generalized hold has large gains.

Remark 3.5 One major drawback of the controller synthesis algorithm presented in Remark 3.3, is that we typically need T to be small even to stabilize the nominal plant. In Appendix A, we show how to design a static GSHF controller based on a discrete-time approach so that for almost all $T > 0$, our controller will at least be capable of stabilizing the nominal plant. Moreover, as $T \to 0$, the new GSHF controller will approach the GSHF controller described in the previous section and inherit its good gain/phase margin properties.

□

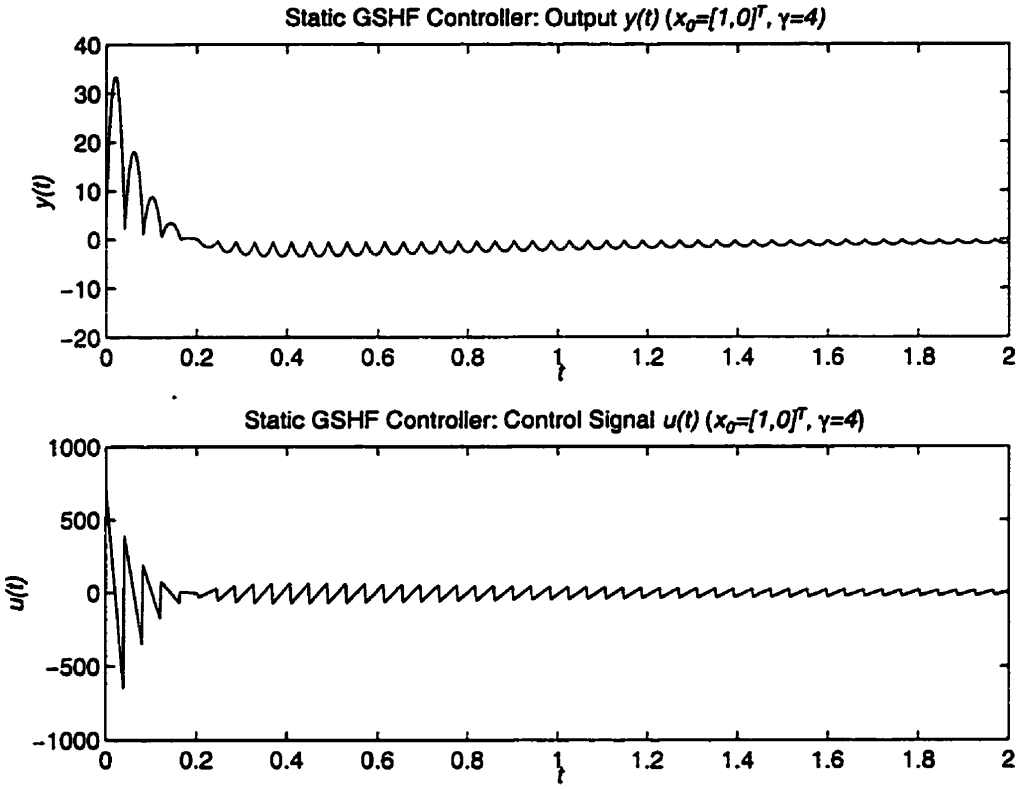


Figure 3.6: Output $y(t)$ and control signal $u(t)$:$x_0 = [1\ 0]^T$ and $\gamma = 4$ (c.t. approach).

3.4 Implementation as a Low Order LPC

Up to this point, we have implemented our controller as a pure GSHF. In practice, this may prove to be difficult to implement, so one might question the benefit of such a control strategy. Recall however, that given $F \in \mathbf{R}^{m\times n}$, the solution of (3.4) is not unique. In this section, we will show how we can pick $\bar{F}(t)$ to be of a simple form so that it can be implemented with a sampler, a zero-order-hold, and a low order discrete-time linear periodic controller. We begin by showing how to implement a SISO GSHF controller as a first order LPC, and then we will extend the result to the MIMO case.

First suppose that the plant is SISO. The solution (3.6) of equation (3.4) is not unique. In fact, it is easy to show that for every $F \in \mathbf{R}^{n\times 1}$, for almost all $T > 0$ there exists a piecewise constant function $\overline{F}(t)$ which satisfies (3.4) and takes on at most n different values in the interval $[0, T)$. Indeed, let $\overline{T} := \frac{T}{n}$, $f_k \in \mathbf{R}$, and set

$$\overline{F}(t) = f_k, \quad t \in [k\overline{T}, (k+1)\overline{T}), \ k = 0, 1, \cdots, (n-1).$$

With

$$\overline{A} := e^{A\overline{T}}, \quad \overline{B} := \int_0^{\overline{T}} e^{A\tau} B d\tau,$$

for (3.4) to hold we need $f_0, \ldots, f_{n-1}$ to satisfy

$$F = \underbrace{\begin{bmatrix} \overline{A}^{n-1}\overline{B} & \overline{A}^{n-2}\overline{B} & \cdots & \overline{B} \end{bmatrix}}_{=: \mathcal{B}} \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_{n-1} \end{bmatrix}.$$

But for most $\overline{T} > 0$, and for all $\overline{T}$ sufficiently small, $(\overline{A}, \overline{B})$ is controllable (Theorem 2.2), in which case $\mathcal{B}$ is invertible, which means that $f_0, \ldots f_{n-1}$ can be chosen to satisfy (3.4).

Now consider the following implementation of the GSHF controller

$$u(kT + t) = \overline{F}(t)\ y(kT) = \begin{cases} f_0 y(kT), & t \in [0, \overline{T}) \\ f_1 y(kT), & t \in [\overline{T}, 2\overline{T}) \\ \vdots & \\ f_{n-1} y(kT). & t \in [(n-1)\overline{T}, n\overline{T}). \end{cases}$$

Define $(G(k), H(k), J(k), K(k))$ by

$$(G(k), H(k), J(k), K(k)) := \begin{cases} (0, 1, 0, f_0) & , \quad k = 0 \\ (1, 0, f_k, 0) & , \quad k = 1, \cdots, (n-1), \end{cases}$$

and set

$$(G(k+n), H(k+n), J(k+n), K(k+n)) = (G(k), H(k), J(k), K(k)), \quad k \in \mathbf{Z}^+.$$

Then the GSHF controller can be implemented as

$$\begin{aligned} z[k+1] &= G(k)z[k] + H(k)y(k\overline{T}), \quad z[0] = z_0 \in \mathbf{R} \\ u(k\overline{T} + \tau) &= J(k)z[k] + K(k)y(k\overline{T}), \quad \tau \in [0, \overline{T}); \end{aligned} \tag{3.20}$$

we associate this controller with the 5-tuple $(G, H, J, K, \overline{T})$. Hence, the T periodic GSHF controller can be viewed as a first order, n-periodic, discrete-time compensator together with a sampler and a zero-order-hold of period $\overline{T} = \frac{T}{n}$.

Now suppose that the plant is MIMO. Here we can follow an argument very similar to that presented above, but we let $f_i \in \mathbf{R}^{m \times r}$, and observe that the matrix $\mathcal{B}$ has full rank. Hence, we end up with a controller of the form (3.20) but with $z \in \mathbf{R}^m$.

Remark 3.6 If the eigenvalues of A are distinct, then we can precondition our MIMO plant using an approach similar to that used in [37] to convert our MIMO problem to a SISO one, and then design a first order controller to provide our desired robustness.

□

3.5 Unstructured Uncertainty

In the previous sections, we showed that it is possible to design a static GSHF controller that can provide a gain margin as large as desired and any desired phase margin up to 90 degrees. The gain/phase margin problem can be looked upon as a *structured* uncertainty robustness problem, since the uncertainty in the plant is parameterized by two scalar parameters ρ and ϕ. It is not clear if we have tolerance to unmodelled dynamics. To proceed, consider a common unstructured uncertainty model commonly known as the *additive uncertainty model*, in which

- $P_0(s)$ is the transfer function for the nominal model,
- $W(s)$ is a fixed stable weighting transfer function,
- $\Delta(s)$ is a variable stable transfer function,

and we assume that our actual plant lies in a set of the form

$$\mathcal{P}_\beta := \{P(s) : P(s) = P_0(s) + W(s)\Delta(s), \|\Delta\|_\infty \leq \beta\}.$$

If our controller were LTI, then the existence of a combined gain/phase margin could be used to show such a class of unmodelled dynamics can be tolerated. More specifically, suppose the LTI controller $C(s)$ stabilizes every system in

$$\{\rho e^{j\phi} P_0(s) : \rho \in [\underline{\rho}, \overline{\rho}], \phi \in [-\overline{\phi}, \overline{\phi}]\}.$$

If we apply C to the dynamically perturbed model $P_0 + W\Delta$, we can represent the closed loop system by the block diagram illustrated in Figure 3.7.

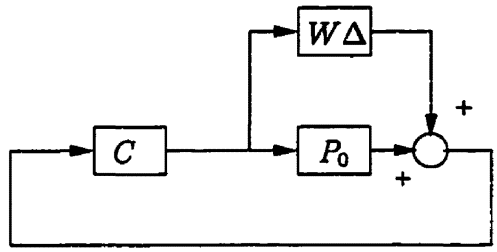


Figure 3.7: Perturbed closed loop system using an LTI controller.

After a loop transformation we obtain the block diagram illustrated in Figure 3.8.

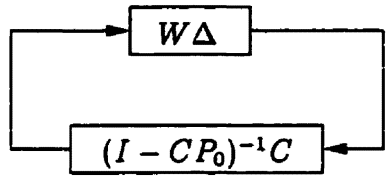


Figure 3.8: Transformed perturbed closed loop system (LTI controller).

Since C stabilizes P_0, the transfer function $(I - CP_0)^{-1}C$ is stable, so by the Small Gain Theorem we maintain stability for all $\Delta \in \mathcal{RH}_\infty$ satisfying

$$\|W\Delta\|_\infty \|(I - CP_0)^{-1}C\|_\infty < 1.$$

If $C \in \mathcal{RL}_\infty$, then this will be the case if

$$\|\Delta\|_\infty < \frac{1}{\|W\|_\infty \cdot \|C\|_\infty} \cdot \frac{1}{\|(I-CP_0)^{-1}\|_\infty}.$$

We can get a bound on the last term by looking at the Nyquist plot of $-CP_0$:

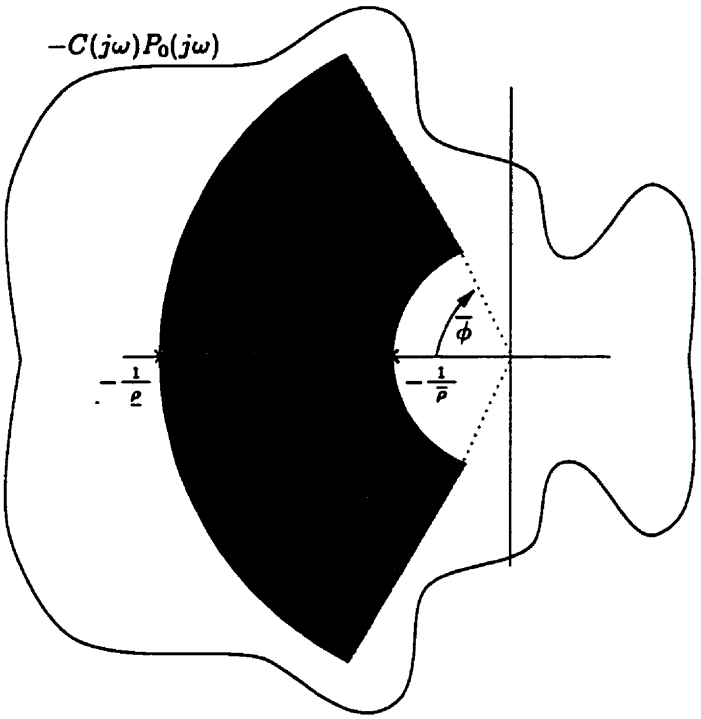


Figure 3.9: LTI Nyquist plot of $-CP_0$.

Indeed, it can be proven that

$$\|(I - CP_0)^{-1}\|_\infty^{-1} > \min\{1/\underline{\rho} - 1, 1 - 1/\overline{\rho}, \sin(\overline{\phi})\};$$

we omit the details since we will be performing a similar analysis later. Since the controller we will be considering is linear time-varying, the above analysis is not directly applicable.

In this section, we would like to see how well the static GSHF controller designed in Section 3.3 tolerates additive perturbations to the nominal plant. We saw in the previous sections that as the sampling period tends to zero, the gain of the hold function becomes very large, so intuitively one might expect that as the sampling period becomes small, the GSHF controller tolerance to unstructured plant uncertainty will tend to zero. Surprisingly, we will be able show that under some conditions on $W(s)$, this intuition is wrong. We will show that if the weighting function W is chosen properly, then our static GSHF controller will tolerate a moderate class of unstructured uncertainty for all sufficiently small sampling periods.

It has been shown by Khargonekar et al.[30] that under some conditions on W, if there does not exist an LTI controller that will stabilize every system in $\mathcal{P}_\beta$, then there does not exists a (possibly) nonlinear time-varying controller that will do so either; hence, our GSHF controller can do no better than an LTI controller. The application of GSHF controllers to the unstructured uncertainty robustness problem has been studied in some detail in the literature (e.g. [26, 22, 40, 10]). While it has been shown that a fixed GSHF controller can tolerate some unstructured uncertainty in the plant, there are few results that describe what happens to the tolerance of the controller to unstructured uncertainty as the sampling period tends to zero. Feuer and Goodwin [19] used amplitude modulation theory to investigate the robustness properties provided by static GSHF controllers when the sampling period is small, and showed that GSHF controllers are typically sensitive to uncertainty found in the high frequency range of the continuous-time frequency response of $P(s)$ (also see [20]).

The question before us now is, if our static GSHF controller is designed to provide good gain/phase margins, then how will the stability of the closed loop sampled-data system be affected by additive perturbations to the nominal plant, as the sampling period tends to zero. Based on the results of Feuer and Goodwin [19], we expect that our static GSHF controller will be sensitive to uncertainty in the high frequency range of the continuous-time frequency response of $P(s)$. Hence, we will impose a condition on $W(s)$ and $\Delta(s)$ (i.e. $W(s)$ has a relative degree of at least $n+2$ and $\|\Delta\|_\infty$ is small enough) so that for high frequencies, $P(j\omega) \approx P_0(j\omega)$.

Since our plant is strictly proper, it is reasonable to assume that the plant uncertainty goes to zero as the frequency increases. From a practical point of view, however, it is hard to justify our requirement that the weighting function W have a relative degree greater than the plant order; unfortunately, it is not clear how to remove this requirement. Note, however, that even if the condition on W is not met, the tools developed in this section will allow us to compute a bound on the perturbation Δ for a fixed sampling period T.

Before presenting any results, let us first outline the approach that will be taken. Let S and H denote the sampling and generalized hold operators respectively. If we apply a static GSHF controller of the form (3.3) to $P(s) \in \mathcal{P}_\beta$, then we can represent our perturbed closed loop system by the block diagram in Figure 3.10.

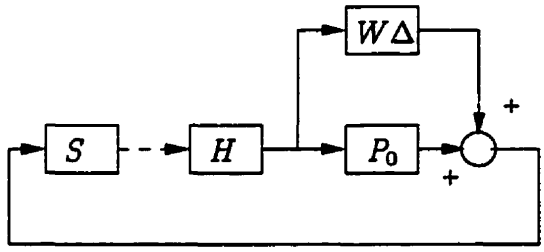


Figure 3.10: Perturbed closed loop system.

Since S and H are linear, we can transform the block diagram shown in Figure 3.10 to the block diagram shown in Figure 3.11. Note that operators $SW\Delta H$ and $(I - SP_0H)^{-1}$ are LTI and can be represented by discrete-time transfer functions.

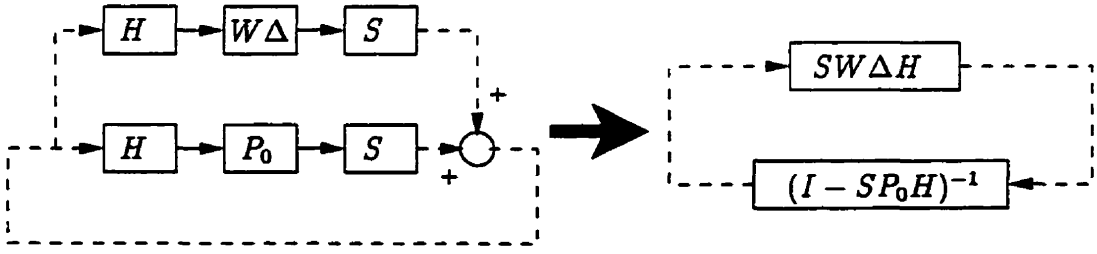


Figure 3.11: Transformed perturbed closed loop system.

We can then use the results of Section 3.3 to show that for all sufficiently small sampling periods, there exists a hold function so that the resulting GSHF controller stabilizes every system in

$$\{\gamma P_0(s) : \gamma \in \Gamma(\underline{\rho}, \overline{\rho}, \overline{\phi})\}.$$

We will then use a discrete-time Nyquist argument to show that for such a hold function, we have

$$\|(I - SP_0H)^{-1}\|_\infty < \frac{1}{\min\{\underline{\rho}^{-1} - 1, 1 - \overline{\rho}^{-1}, \sin(\overline{\phi})\}} =: r.$$

Then all that remains to be shown is that there exists a β, independent of T, so that for every $\Delta \in \mathcal{RH}_\infty$ satisfying $\|\Delta\|_\infty \leq \beta$, we have

$$\|SW\Delta H\|_\infty \leq \tfrac{1}{r}, \tag{3.21}$$

for then

$$\|(SW\Delta H)(I - SP_0H)^{-1}\|_\infty \leq \|SW\Delta H\|_\infty \times \|(I - SP_0H)^{-1}\|_\infty < 1,$$

so using the Small Gain Theorem, we can conclude from Figure 3.11 that we have closed loop stability. Showing that (3.21) holds for sufficiently small T is not trivial. To proceed, we choose the hold function to be a piecewise constant function, which is not restrictive since the resulting GSHF controller will be easy to implement (See Section 3.4). Unfortunately, the H_∞ norm of the frequency response of the resulting hold function blows up as the sampling period tends to zero. However, we will be able to show that $|H(j\omega)|$ is bounded above by a n^{th} order polynomial in ω, independent of the sampling period. Hence, if $W(s)$ has a high enough relative degree, i.e. $|W(j\omega)|$ rolls off at a fast enough rate, then $|H(j\omega)W(j\omega)|$ is bounded, e.g. see Figure 3.12. Therefore, if $\|\Delta\|_\infty$ is sufficiently small, then we would expect that $\|W\Delta H\|_\infty$ is small enough to ensure that $\|SW\Delta H\|_\infty$ will be less than $1/r$.

3.5.1 Choosing the Hold Function

We begin by using the results of Sections 3.3 and 3.4 to choose a specific generalized hold function $\overline{F}(t)$ for our GSHF controller. To this end, with $\underline{\rho} \in (0, 1]$ and $\overline{\phi} \in [0, \frac{\pi}{2})$, pick positive definite symmetric weighting matrices Q and R, let α, P_0 and F_0 satisfy (3.11)–(3.14), and require that F satisfy

$$F = TF_0 = \int_0^T e^{A(T-\tau)} B\overline{F}(\tau)d\tau. \tag{3.22}$$

Then we know from Theorem 3.1 that for sufficiently small sampling periods, the GSHF controller $(\overline{F}, T)$ provides the desired gain/phase margin. Recall that the solution $\overline{F}(\tau)$ of the above equation is not unique. Here we will choose $\overline{F}(\tau)$ to be a piecewise constant

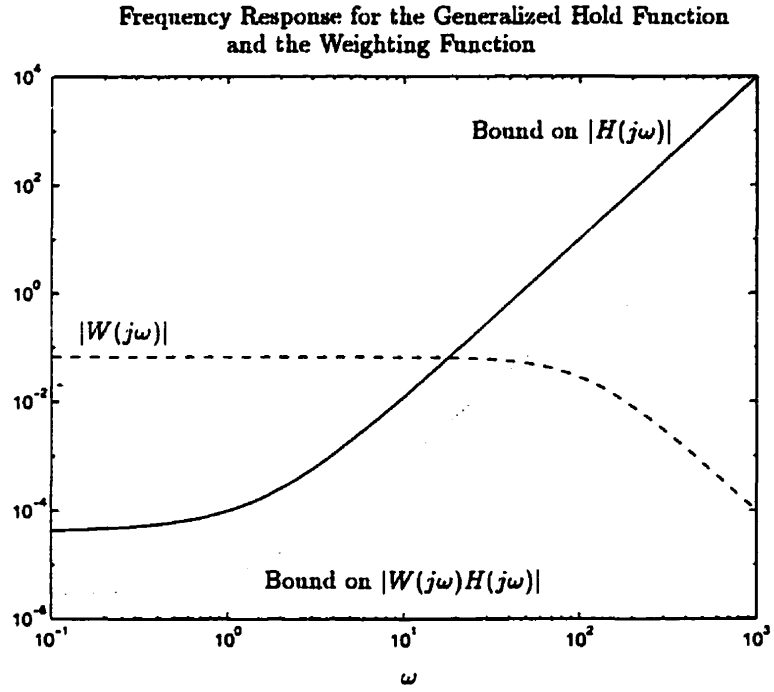


Figure 3.12: Frequency response of $H(s)$, $W(s)$ and $H(s)W(s)$.

function taking on n values over each sample period. Namely, with $\overline{T} := T/n$, we consider hold functions of the form

$$\overline{F}(t) = f_k, \quad t \in [k\overline{T}, (k+1)\overline{T}), \quad k = 0, 1, \cdots, (n-1). \tag{3.23}$$

Using the results from Section 3.4, we know that for every $F \in \mathbf{R}^n$ and for sufficiently small $\overline{T}$, there exists constants f_i so that (3.22) is satisfied. In fact, recall from Section 3.4 that with

$$\overline{A} = \mathrm{e}^{A\overline{T}}, \quad \overline{B} = \int_0^{\overline{T}} \mathrm{e}^{A\tau} B d\tau,$$

we have

$$\begin{bmatrix} f_0 \\ \vdots \\ f_{n-1} \end{bmatrix} = n\overline{T} \begin{bmatrix} \overline{A}^{n-1}\overline{B} & \cdots & \overline{B} \end{bmatrix}^{-1} F_0. \tag{3.24}$$

We will represent the static GSHF controller of this form by the pair $(f_i, \overline{T})$.

3.5.2 Preliminary Results on the Hold Function

We will now examine some properties of the resulting hold function. We know that the sampling period must be small if our GSHF controller is to provide good gain/phase margins, so we begin by determining what happens to the time-domain infinity norm of $\overline{F}$ as the sampling period tends to zero. From earlier observations, we saw that the magnitude of the hold function becomes large as the sampling period becomes small, so the following result that states that $f_i = \mathcal{O}(\overline{T}^{-(n-1)})$, should not come as a surprise. We begin by defining

$$\psi(i,j) := \sum_{k=1}^{i} \frac{j^{k-1}}{(k-1)!(i-k+1)!},$$

and

$$\Psi := \begin{bmatrix} \psi(n,n-1) & \cdots & \psi(n,1) & \frac{1}{n!} \\ \vdots & & \vdots & \vdots \\ \psi(1,n-1) & \cdots & \psi(1,1) & 1 \end{bmatrix}. \tag{3.25}$$

This matrix will be needed to relate the discrete-time controllability matrix associated with the pair $(\overline{A}, \overline{B})$ to the continuous-time controllability matrix associated with the pair (A, B).

Remark 3.7 Note that Ψ can be written as

$$\Psi = \begin{bmatrix} \frac{1}{0!n!} & \frac{1}{1!(n-1)!} & \cdots & \frac{1}{(n-1)!1!} \\ \frac{1}{0!(n-1)!} & \frac{1}{1!(n-2)!} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ \frac{1}{0!1!} & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} (n-1)^0 & (n-2)^0 & \cdots & 0^0 \\ (n-1)^1 & (n-2)^1 & \cdots & 0^1 \\ \vdots & \vdots & \cdots & \vdots \\ (n-1)^{n-1} & (n-2)^{n-1} & \cdots & 0^{n-1} \end{bmatrix}$$

(here $0^0 := 1$) so since the first matrix is upper triangular with nonzero diagonal elements and the second matrix is a Vandermonde matrix, we can easily show that Ψ is invertible.

□

Proposition 3.3

$$\begin{bmatrix} f_0 \\ \vdots \\ f_{n-1} \end{bmatrix} = n\Psi^{-1} \begin{bmatrix} \frac{1}{\overline{T}^{n-1}} & & & \\ & \ddots & & \\ & & \frac{1}{\overline{T}} & \\ & & & 1 \end{bmatrix} \begin{bmatrix} A^{n-1}B & \cdots & B \end{bmatrix}^{-1} F_0 + \mathcal{O}(\overline{T}).$$

Proof:

If we write

$$\overline{A} = e^{A\overline{T}} = \sum_{i=0}^{\infty} \frac{A^i \overline{T}^i}{i!},$$

and

$$\overline{B} = \int_0^{\overline{T}} e^{A\tau} B d\tau = \sum_{i=1}^{\infty} \frac{A^{i-1} B \overline{T}^i}{i!},$$

then it can be shown that

$$\begin{bmatrix} \overline{A}^{n-1}\overline{B} & \cdots & \overline{B} \end{bmatrix} = \begin{bmatrix} A^{n-1}B & \cdots & B \end{bmatrix} \begin{bmatrix} \overline{T}^n & & \\ & \ddots & \\ & & \overline{T} \end{bmatrix} \Psi + \mathcal{O}(\overline{T}^{n+1}).$$

Substituting this into (3.24) and using the fact that Ψ is invertible (see Remark 3.7), it follows that

$$\begin{aligned} \begin{bmatrix} f_0 \\ \vdots \\ f_{n-1} \end{bmatrix} &= n\overline{T} \left(\begin{bmatrix} A^{n-1}B & \cdots & B \end{bmatrix} \begin{bmatrix} \overline{T}^n & & \\ & \ddots & \\ & & \overline{T} \end{bmatrix} \Psi + \mathcal{O}(\overline{T}^{n+1}) \right)^{-1} F_0 \\ &= n\Psi^{-1} \begin{bmatrix} \frac{1}{\overline{T}^{n-1}} & & \\ & \ddots & \\ & & 1 \end{bmatrix} \begin{bmatrix} A^{n-1}B & \cdots & B \end{bmatrix}^{-1} F_0 + \mathcal{O}(T). \end{aligned}$$

■ .

We now present some results on the weighted average of the hold function

$$\mu_i(T) := T^{-1} \int_0^T \tau^i \overline{F}(\tau) d\tau, \quad i \in \mathbf{Z}^+.$$

We will show that the weighted average of the hold function approaches a constant as the sampling period tends to zero, and that the constant is easily computable. This result will be important when we go on to show that the frequency response of the hold function is bounded above by a n^{th} order polynomial in ω independent of T.

Proposition 3.4 If

$$\begin{bmatrix} c_0 \\ \vdots \\ c_{n-1} \end{bmatrix} := \begin{bmatrix} 1 & & & \\ & (-1)1! & & \\ & & \ddots & \\ & & & (-1)^{n-1}(n-1)! \end{bmatrix} \begin{bmatrix} B & \cdots & A^{n-1}B \end{bmatrix}^{-1} F_0,$$

then

$$\mu_i(T) = \begin{cases} c_i + \mathcal{O}(T) & i = 0, 1, \cdots, n-1 \\ \mathcal{O}(T^{i+1-n}) & i = n, n+1, \cdots. \end{cases}$$

Proof:

Let us first look at $\mu_i(T)$ for $i = n, n+1, \cdots$. By definition

$$\|\mu_i(T)\| = \left\| T^{-1} \int_0^T \tau^i \overline{F}(\tau) d\tau \right\| \leq T^{-1} \int_0^T \tau^i d\tau \|\overline{F}\|_\infty = \frac{T^i}{i+1} \|\overline{F}\|_\infty.$$

But by Proposition 3.3, we know that $\|\overline{F}\|_\infty = \mathcal{O}(T^{-(n-1)})$, so it follows that $\mu_i(T) = \mathcal{O}(T^{i+1-n})$.

Now let us look at $\mu_i(T)$ for $i = 0, 1, \cdots, n-1$. Write (3.22) as

$$T^{-1} \int_0^T \mathrm{e}^{-A\tau} B \overline{F}(\tau) d\tau = \mathrm{e}^{-AT} F_0. \tag{3.26}$$

If we define

$$\Delta_1(\tau) := \mathrm{e}^{-A\tau} - \sum_{i=0}^{n-1} \frac{(-A\tau)^i}{i!}, \quad \text{and}$$

$$\Delta_2(T) := \mathrm{e}^{-AT} - I,$$

then we can write (3.26) as

$$T^{-1} \int_0^T \left[\sum_{i=0}^{n-1} \frac{(-A\tau)^i}{i!} + \Delta_1(\tau) \right] B \overline{F}(\tau) d\tau = (I + \Delta_2(T)) F_0$$

$$\Rightarrow \sum_{i=0}^{n-1} A^i B \tfrac{(-1)^i}{i!} \left(T^{-1} \int_0^T \tau^i \overline{F}(\tau) d\tau \right) + T^{-1} \int_0^T \Delta_1(\tau) B \overline{F}(\tau) d\tau = F_0 + \Delta_2(T) F_0,$$

so along with the definition of $\mu_i(T)$, we have

$$\begin{bmatrix} B & \cdots & A^{n-1}B \end{bmatrix} \begin{bmatrix} 1 & & \\ & \ddots & \\ & & \frac{(-1)^{n-1}}{(n-1)!} \end{bmatrix} \begin{bmatrix} \mu_0(T) \\ \vdots \\ \mu_{n-1}(T) \end{bmatrix} + T^{-1} \int_0^T \Delta_1(\tau) B \overline{F}(\tau) d\tau = F_0 + \Delta_2(T) F_0.$$

Since $\Delta_1(\tau) = \mathcal{O}(\tau^n)$ and $\|\overline{F}\|_\infty = \mathcal{O}(T^{-(n-1)})$, it is possible to show that the second term on the LHS is of order T. By using the fact that $\Delta_2(T) = \mathcal{O}(T)$, it follows that

$$\begin{bmatrix} B & \cdots & A^{n-1}B \end{bmatrix} \begin{bmatrix} 1 & & \\ & \ddots & \\ & & \frac{(-1)^{n-1}}{(n-1)!} \end{bmatrix} \begin{bmatrix} \mu_0(T) \\ \vdots \\ \mu_{n-1}(T) \end{bmatrix} = F_0 + \mathcal{O}(T)$$

$$\Rightarrow \begin{bmatrix} \mu_0(T) \\ \vdots \\ \mu_{n-1}(T) \end{bmatrix} = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & (-1)^{n-1}(n-1)! \end{bmatrix} \begin{bmatrix} B & \cdots & A^{n-1}B \end{bmatrix}^{-1} F_0 + \mathcal{O}(T).$$

■

We are almost ready to prove the key result that the frequency response of the hold function is bounded above by an n^{th} order polynomial, independent of T, but we will first need the following notation and preliminary result relating the gains f_i to the frequency response of the hold function. We define the sampling operator $S : \mathscr{C} \to \mathscr{S}$ via

$$y = Su \iff y(k) = u(kT),$$

and the generalized hold operator $H : \mathscr{S} \to \mathbf{PC}$ via

$$y = Hu \iff y(t) = \overline{F}(t)u(k),\ t \in [kT, (k+1)T).$$

Let $h(t)$ be the "impulse response" of the generalized hold function and $H(s)$ be the associated transfer function:

$$H(s) = \frac{1 - e^{s\overline{T}}}{s} \sum_{i=0}^{n-1} f_i e^{-si\overline{T}};$$

e.g. see Lemma 2 in [22].

We can now state our first main Lemma:

Lemma 3.1 There exists constants $\eta_i > 0$ and $T_1 > 0$ so that

$$|H(j\omega)| \leq T\left(\eta_0 + \eta_1\omega + \cdots + \eta_n\omega^n\right), \quad T \in (0, T_1).$$

Proof:

We will prove this result by first partitioning $H(s)$ into two parts:

$$H_1(s) := \frac{1 - e^{s\overline{T}}}{s}, \quad H_2(s) := \sum_{i=0}^{n-1} f_i e^{-si\overline{T}}. \tag{3.27}$$

For the first part, observe that we can write

$$H_1(j\omega) = \frac{1 - e^{j\omega\overline{T}}}{j\omega} = \left(\overline{T}e^{-\frac{j\omega\overline{T}}{2}}\right)\frac{\sin\left(\frac{\omega\overline{T}}{2}\right)}{\frac{\omega\overline{T}}{2}}.$$

The second term is the well known sinc function $sinc(\frac{\omega\overline{T}}{2})$, and it has the property that $|sinc(\frac{\omega\overline{T}}{2})| \leq 1$. Hence,

$$|H_1(j\omega)| \leq \overline{T}|e^{-\frac{j\omega\overline{T}}{2}}| \times \left|sinc(\tfrac{\omega\overline{T}}{2})\right| \leq \overline{T}, \quad \omega \in \mathbf{R}. \tag{3.28}$$

Let us now look at the second part. We begin by writing

$$\begin{aligned} H_2(j\omega) &= H_2(0) + \int_0^{\omega} \left.\frac{dH_2(s)}{ds}\right|_{s=j\omega_1} d\omega_1 \\ &= H_2(0) + w\left.\frac{dH_2(s)}{ds}\right|_{s=0} + \int_0^{\omega}\int_0^{\omega_1} \left.\frac{d^2H_2(s)}{ds^2}\right|_{s=j\omega_2} d\omega_2 d\omega_1 \\ &= H_2(0) + w\left.\frac{dH_2(s)}{ds}\right|_{s=0} + \cdots + \frac{\omega^{n-1}}{(n-1)!}\left.\frac{d^{n-1}H_2(s)}{ds^{n-1}}\right|_{s=0} \\ &\quad + \int_0^{\omega}\cdots\int_0^{\omega_{n-1}} \left.\frac{d^nH_2(s)}{ds^n}\right|_{s=j\omega_n} d\omega_n \cdots d\omega_1, \end{aligned} \tag{3.29}$$

and noting from the definition of $H_2(s)$ that

$$\left.\frac{d^kH_2(s)}{ds^k}\right|_{s=0} = (-1)^k\overline{T}^k\sum_{i=0}^{n-1} i^k f_i. \tag{3.30}$$

We will now use Proposition 3.4 to come up with an expression for $\overline{T}^k\sum_{i=0}^{n-1} i^k f_i$. Now

$$\mu_0(T) = T^{-1}\int_0^T \overline{F}(\tau)d\tau = \tfrac{1}{n}\sum_{i=1}^{n-1} f_i,$$

so by Proposition 3.4, it follows that

$$\tfrac{1}{n}\sum_{i=1}^{n-1} f_i = c_0 + \mathcal{O}(T) \quad \Rightarrow \quad \sum_{i=1}^{n-1} f_i = nc_0 + \mathcal{O}(T). \tag{3.31}$$

Now

$$\mu_1(T) = T^{-1}\int_0^T \tau \overline{F}(\tau)d\tau = T^{-1}\sum_{i=0}^{n-1} f_i \int_{i\overline{T}}^{(i+1)\overline{T}} \tau d\tau = \tfrac{\overline{T}}{2n}\sum_{i=1}^{n-1} f_i(2i+1)$$

$$\Rightarrow \tfrac{\overline{T}}{n}\sum_{i=1}^{n-1} i f_i = \mu_1(T) - \tfrac{\overline{T}}{2n}\sum_{i=1}^{n-1} f_i.$$

Hence, by using Proposition 3.4 and (3.31), it follows that

$$\tfrac{\overline{T}}{n}\sum_{i=1}^{n-1} i f_i = (c_1 + \mathcal{O}(T)) + \tfrac{\overline{T}}{2}(-c_0 + \mathcal{O}(T)) = c_1 + \mathcal{O}(T)$$

$$\Rightarrow \overline{T}\sum_{i=1}^{n-1} i f_i = nc_1 + \mathcal{O}(T).$$

In a similar fashion, it can be shown that for $k = 2, \cdots, n-1$ we have

$$\overline{T}^k \sum_{i=1}^{n-1} i^k f_i = nc_k + \mathcal{O}(T). \tag{3.32}$$

Let us now look at the last term in (3.29). Since

$$\left|\tfrac{d^n H_2(s)}{ds^n}\right| = \left|\sum_{i=0}^{n-1} f_i(-i\overline{T})^n e^{-si\overline{T}}\right| \le \sum_{i=0}^{n-1} |f_i|(i\overline{T})^n,$$

and $f_i = \mathcal{O}(T^{-(n-1)})$ (see Proposition 3.3), it follows that

$$\tfrac{d^n H_2(s)}{ds^n} = \mathcal{O}(T),$$

so

$$\int_0^{\omega}\cdots\int_0^{\omega_{n-1}} \left.\tfrac{d^n H_2(s)}{ds^n}\right|_{s=j\omega_n} d\omega_n \cdots d\omega_1 = \mathcal{O}(T)\omega^n. \tag{3.33}$$

If we combine (3.29)–(3.33), it follows that

$$|H_2(j\omega)| \le n\left[(|c_0| + \mathcal{O}(T)) + (|c_1| + \mathcal{O}(T))\omega + \cdots + \left(\tfrac{|c_{n-1}| + \mathcal{O}(T)}{(n-1)!}\right)\omega^{n-1} + \mathcal{O}(T)\omega^n\right].$$

Finally, combining this with (3.28), it can be shown that

$$|H(j\omega)| \le T\left[(|c_0| + \mathcal{O}(T)) + (|c_1| + \mathcal{O}(T))\omega + \cdots + \left(\tfrac{|c_{n-1}| + \mathcal{O}(T)}{(n-1)!}\right)\omega^{n-1} + \mathcal{O}(T)\omega^n\right], \tag{3.34}$$

and our result follows. ∎

Remark 3.8 There is an interesting relationship between the weighted average of the hold function $\mu_i(T)$ and the bound on the frequency response of the hold function. As can be seen in (3.34), for sufficiently small sampling periods, the constants η_i can be chosen arbitrarily close to the constants $\frac{c_i}{i!}$. Recall from Proposition 3.4 that c_i is the limiting value of the weighted average of the hold function $\mu_i(T)$, and that it is easy to compute. Hence, we know that as the sampling period becomes small, the bound on the frequency response of the hold function approaches

$$T\left(|c_0| + |c_1|\omega + \cdots + \left(\frac{|c_{n-1}|}{(n-1)!}\right)\omega^{n-1} + \mathcal{O}(T)\omega^n\right).$$

□

3.5.3 Main Results

To prove our main result, it will be easier to break down the problem into two parts. In the first part (Lemma 3.2), we will show that the GSHF controller designed to simultaneously provide an upper gain margin of $\overline{\rho}$, a lower gain margin of $\underline{\rho}$, and a phase margin of $\overline{\phi}$, will also guarantee that

$$\|(I - SP_0H)^{-1}\|_\infty < \frac{1}{\min\{\underline{\rho}^{-1} - 1, 1 - \overline{\rho}^{-1}, \sin(\overline{\phi})\}} =: \tau.$$

Our second result (Lemma 3.3) will show that there exists a constant β so that for every $\Delta \in \mathcal{RH}_\infty$ satisfying $\|\Delta\|_\infty \leq \beta$, we have

$$\|SW\Delta H\|_\infty \leq \tfrac{1}{\tau}$$

for sufficiently small T. Finally, we will combine these two results to get our main result in Theorem 3.3.

Lemma 3.2 If the GSHF controller stabilizes every system in

$$\{(A, B, \gamma C) : \gamma \in \Gamma(\underline{\rho}, \overline{\rho}, \overline{\phi})\},$$

then

$$\|(I - SP_0H)^{-1}\|_\infty < \frac{1}{\min\{\underline{\rho}^{-1} - 1, 1 - \overline{\rho}^{-1}, \sin(\overline{\phi})\}}.$$

Proof:

Define $G_d := SP_0H$ and let $G_d(e^{j\theta})$ be the associated discrete-time frequency response. Let n_1 be the number of eigenvalues of A with real parts greater than zero. Since the GSHF controller stabilizes the nominal plant, it follows that the discrete-time Nyquist plot of $-G_d$ encircles the -1 point in the complex plane n_1 times. Now consider the region defined by

$$\Omega := \{\rho e^{j\phi} : \rho \in [-1/\underline{\rho}, -1/\overline{\rho}], \phi \in [-\overline{\phi}, \overline{\phi}]\},$$

which is illustrated in Figure 3.13.

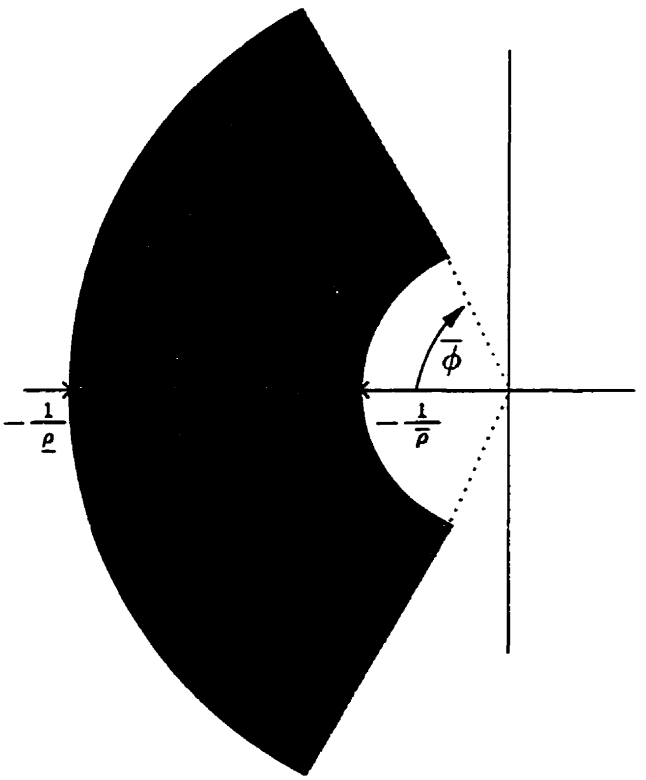


Figure 3.13: The region Ω and $\hat{r}$ defined.

Since the GSHF controller stabilizes every system in

$$\{(A, B, \gamma C) : \gamma \in \Gamma(\underline{\rho}, \overline{\rho}, \overline{\phi})\},$$

we know that the Nyquist plot of $-G_d$ lies strictly outside the region Ω, for if it did not (e.g. see Figure 3.14), then there would exist a $\rho_1 \in [\underline{\rho}, \overline{\rho}]$ and $\phi_1 \in [-\overline{\phi}, \overline{\phi}]$ so that the

Nyquist plot of $-\rho_1 e^{j\phi_1} G_d$ would intersect the -1 point, which implies that the closed loop system is not stable for this choice of ρ and ϕ.

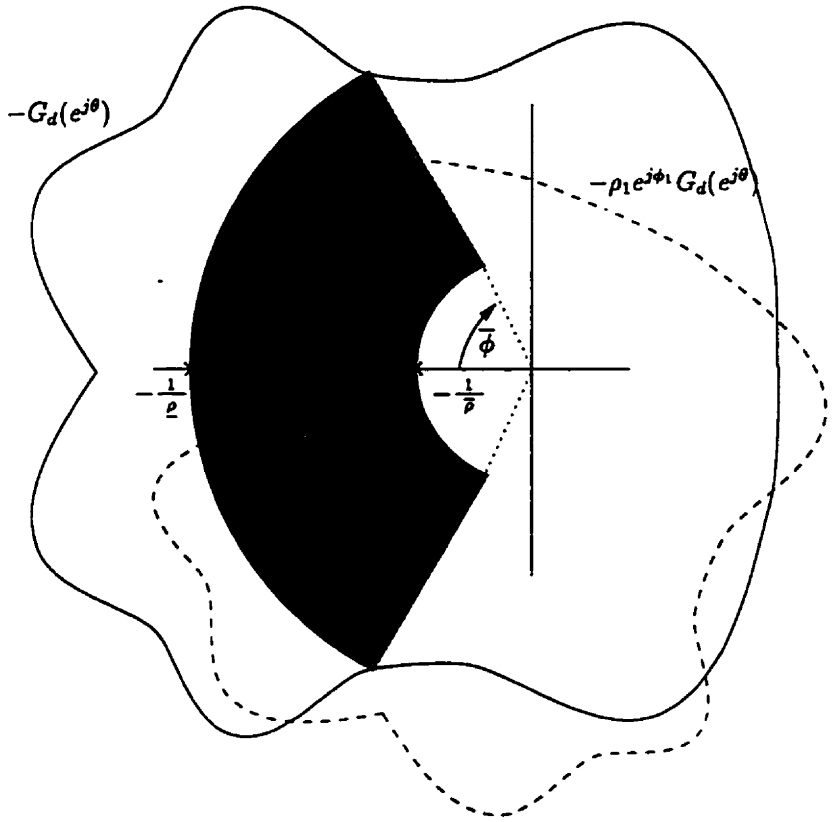


Figure 3.14: Nyquist plot of G_d lies outside Ω.

Define $\hat{r}$ as the largest radius of a circle centered at -1 that is entirely contained in Ω, i.e. see Figure 3.13; it is straightforward to show that

$$\hat{r} := \min\{\underline{\rho}^{-1} - 1, 1 - \overline{\rho}^{-1}, \sin(\overline{\phi})\}.$$

Hence, the distance from the -1 point to $-G_d(e^{j\theta})$ is always greater than $\hat{r}$. In particular,

$$\inf_{\theta} |-1 + G_d(e^{j\theta})| > \hat{r},$$

but

$$\begin{aligned} \inf_{\theta} |-1 + G_d(e^{j\theta})| &= \inf_{\theta} |1 - G_d(e^{j\theta})| \\ &= \left(\sup_{\theta} |[1 - G_d(e^{j\theta})]^{-1}| \right)^{-1} \\ &= \|(1 - G_d)^{-1}\|_{\infty}^{-1}, \end{aligned}$$

so it follows that

$$\|(1-G_d)^{-1}\|_\infty = \|(1-SP_0H)^{-1}\|_\infty < \hat{r}^{-1} = \frac{1}{\min\{\underline{\rho}^{-1}-1, 1-\overline{\rho}^{-1}, \sin(\bar{\phi})\}}.$$

■

Before presenting our second result, we will first need to relate the frequency response of a continuous-time LTI system $G(j\omega)$ to the frequency response of the associated discrete-time LTI system $(SGH)(e^{j\theta})$. Define the sampling frequency as

$$\omega_s := 2\pi/T.$$

Theorem 3.2 The frequency responses $G(j\omega)$ and $(SGH)(e^{j\theta})$ are related by

$$(SGH)(e^{-j\omega T}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} G(j\omega + jk\omega_s)H(j\omega + jk\omega_s).$$

Proof:

See Theorem 3.3.1 in [12] for the proof, and [12, Section 3.3] for related results. ■

We are now ready to state our second result.

Lemma 3.3 For every $W(s)$ of relative degree $n+2$, and every $\hat{r}>0$, there exists a $\beta>0$ and $\widehat{T}>0$ so that for every $T\in(0,\widehat{T})$ and $\Delta\in\mathcal{RL}_\infty$ satisfying $\|\Delta\|_\infty\leq\beta$, we have

$$\|S\Delta WH\|_\infty \leq \hat{r}.$$

Proof:

Fix $\hat{r} > 0$, let $\omega_s = 2\pi/T$, and from Theorem 3.2 it follows that

$$\begin{aligned}|(S\Delta WH)(e^{-j\omega T})| &\leq \frac{1}{T}\sum_{k=-\infty}^{\infty} |\Delta(j\omega + jk\omega_s)W(j\omega + jk\omega_s)H(j\omega + jk\omega_s)| \\ &\leq \frac{\|\Delta\|_\infty}{T}\sum_{k=-\infty}^{\infty} |W[j(\omega + k\omega_s)]H[j(\omega + k\omega_s)]|.\end{aligned}$$

Since $W(s)$ has relative degree $n+2$, it is easy to show that there exists positive constants β_0 and β_1 so that

$$|W(j\omega)| \leq \frac{\beta_0}{1+\beta_1|\omega|^{n+2}},$$

and from Lemma 3.1, it follows that there exists positive constants η_i and T_1 so that

$$|H(j\omega)| \leq T(\eta_0 + \eta_1\omega + \cdots + \eta_n\omega^n), \quad T \in (0, T_1).$$

Hence, there exists positive constants β_2 and β_3 so that

$$|W(j\omega)H(j\omega)| \leq \frac{T\beta_2}{1+\beta_3\omega^2}, \quad T \in (0, T_1),$$

which means that for $T \in (0, T_1)$, we have

$$|(S\Delta WH)(e^{-j\omega T})| \leq \|\Delta\|_\infty \sum_{k=-\infty}^{\infty} \frac{\beta_2}{1+\beta_3(w+k\omega_s)^2}.$$

But for every $\omega \in \mathbf{R}$, there exists an $m \in \mathbf{Z}$ and $\omega_1 \in [0, \omega_s)$ so that

$$\omega = m\omega_s + \omega_1,$$

so

$$\begin{aligned}
&|(S\Delta WH)(e^{-j\omega T})| \\
&\leq \|\Delta\|_\infty \sum_{k=-\infty}^{\infty} \frac{\beta_2}{1+\beta_3[\omega_1+(m+k)\omega_s]^2} \\
&= \|\Delta\|_\infty \sum_{k_1=-\infty}^{\infty} \frac{\beta_2}{1+\beta_3(\omega_1+k_1\omega_s)^2} && (k_1 = m+k) \\
&= \|\Delta\|_\infty \left(\sum_{k_1=0}^{\infty} \frac{\beta_2}{1+\beta_3(\omega_1+k_1\omega_s)^2} + \sum_{k_1=-\infty}^{-1} \frac{\beta_2}{1+\beta_3(\omega_1+k_1\omega_s)^2} \right) \\
&\leq \|\Delta\|_\infty \left(\sum_{k_1=0}^{\infty} \frac{\beta_2}{1+\beta_3(k_1\omega_s)^2} + \sum_{k_1=-\infty}^{-1} \frac{\beta_2}{1+\beta_3[(k_1+1)\omega_s]^2} \right) \\
&= \|\Delta\|_\infty \left(\sum_{k_1=0}^{\infty} \frac{\beta_2}{1+\beta_3(k_1\omega_s)^2} + \sum_{k_2=-\infty}^{0} \frac{\beta_2}{1+\beta_3(k_2\omega_s)^2} \right) && (k_2 = k_1+1) \\
&= 2\|\Delta\|_\infty \sum_{k_1=0}^{\infty} \frac{\beta_2}{1+\beta_3(k_1\omega_s)^2} \\
&\leq 2\|\Delta\|_\infty \left(\beta_2 + \sum_{k_1=1}^{\infty} \frac{\beta_2}{1+\beta_3(k_1\omega_s)^2} \right) \\
&\leq 2\|\Delta\|_\infty \left(\beta_2 + \sum_{k_1=1}^{\infty} \frac{\beta_2}{\beta_3(k_1\omega_s)^2} \right) \\
&= 2\|\Delta\|_\infty \left(\beta_2 + T^2\frac{\beta_2}{(2\pi)^2\beta_3} \sum_{k_1=1}^{\infty} \frac{1}{k_1^2} \right). && (\omega_s = 2\pi/T)
\end{aligned}$$

But it is easily shown that

$$\sum_{k_1=1}^{\infty} \frac{1}{k_1^2} \leq 2,$$

which means

$$\|S\Delta WH\|_\infty \leq 2\|\Delta\|_\infty \left(\beta_2 + T^2 \frac{\beta_2}{2\pi^2\beta_3}\right), \quad T \in (0, T_1).$$

Hence, the result holds with $\widehat{T} = T_1$ and

$$\beta := \left[2\left(\beta_2 + T_1^2 \frac{\beta_2}{2\pi^2\beta_3}\right)\right]^{-1} \hat{r}.$$

■

Finally, we use Lemmas 3.2 and 3.3 to prove our main result.

Theorem 3.3 For every $W(s)$ of relative degree $n+2$, there exists a $\beta > 0$ and $\widehat{T} > 0$ so that for every $T \in (0, \widehat{T})$, the GSHF controller $(f_i, \overline{T})$ stabilizes every system in $\mathcal{P}_\beta$.

Proof:

By Theorem 3.1 and the comments in Section 3.5.1, we know that with $\underline{\rho} \in (0, 1]$, $\overline{\rho} \in [1, \infty)$, and $\overline{\phi} \in [0, \frac{\pi}{2})$, there exists a $T_1 > 0$ so that for every $T \in (0, T_1)$ the associated GSHF controller $(f_i, \overline{T})$ stabilizes every system in

$$\{(A, B, \gamma C) : \gamma \in \Gamma(\underline{\rho}, \overline{\rho}, \overline{\phi})\}.$$

Hence, from Lemma 3.2, we know that for every $T \in (0, T_1)$, the associated GSHF controller $(f_i, \overline{T})$ will also guarantee that

$$\|(I - SP_0H)^{-1}\|_\infty < \frac{1}{\min\{\underline{\rho}^{-1} - 1, 1 - \overline{\rho}^{-1}, \sin(\overline{\phi})\}} =: r.$$

Let $\hat{r} := 1/r$. Then by Lemma 3.3, we know that there exists a $\widehat{T} \in (0, T_1)$ and $\beta > 0$ so that for every $T \in (0, \widehat{T})$ and Δ satisfying $\|\Delta\|_\infty \leq \beta$, we have

$$\|S\Delta WH\|_\infty \leq \hat{r} = 1/r.$$

Hence, for every $T \in (0, \widehat{T})$ and Δ satisfying $\|\Delta\|_\infty \leq \beta$, we have

$$\|(SW\Delta H)(I - SP_0H)^{-1}\|_\infty \leq \|SW\Delta H\|_\infty \times \|(I - SP_0H)^{-1}\|_\infty < 1. \qquad (3.35)$$

If we consider the block diagram of the closed loop system in Figure 3.10, and the equivalent block diagram in Figure 3.11, then by the Small Gain Theorem, we conclude from (3.35) that for every $T \in (0, \widehat{T})$, the associated GSHF controller $(f_i, \overline{T})$ stablizes every system in $\mathcal{P}_\beta$. ∎

Remark 3.9 Before leaving this section, it should be noted that the assumption that Δ be stable was made so that we could easily characterize the set $\mathcal{P}_\beta$. As done in [34], we could have relaxed this assumption and assumed that $P(s)$ was in a set of the form

$$\{P_0 + \Delta W : [S(P_0 + \Delta W)H] \text{ has the same number of unstable poles as } SP_0H,$$
$$\Delta \in \mathcal{RL}_\infty, \|\Delta\|_\infty \leq \beta\}.$$

However, care must be taken to ensure that no unstable poles are cancelled out by pathological sampling. □

3.6 Summary and Concluding Remarks

It has been known for some time that for unstable non-minimum phase LTI plants, there is a maximum gain margin which can be provided by an LTI controller [33]. Here we consider the use of a class of time-varying controllers in solving an extended version of the gain margin problem. We have shown how to design two static GSHF controllers, both of which can provide a gain margin as large as desired and any desired phase margin up to 90 degrees; in fact, the both controllers tolerate a combined gain and phase perturbation in the nominal plant model. The first controller is based on a continuous-time approach and requires that the sampling period be small to stabilize even the nominal plant[a]. The second controller is based on a discrete-time approach and does not suffer from this drawback. These controllers are easy to design, and can be implemented directly as a static GSHF, or indirectly using a first order linear periodic discrete-time controller together with a sampler and a zero-order-hold. We also showed that for a SISO plant,

[a]These results have been published and can be found in [44].

the GSHF controller based on the continuous-time approach can tolerate unmodelled dynamics when an additive unstructured uncertainty model is assumed.

These results complement previous work on the use of static and dynamic GSHF controllers for the gain margin problem [57], [58]. In our approach, however, we can solve the more demanding gain/phase margin problem. We took advantage of the fact that our controller provides a desired phase margin to show that the GSHF controller can tolerate a moderate amount of unmodelled dynamics as well. Unlike the dynamic GSHF controller of [57], our controller does not provide the optimal gain margin for a given sampling period, although it is much simpler to design and to implement. Furthermore, since the controller gain is typically large, this control strategy suffers from poor intersample behaviour.

More work is needed to explore the tradeoff in the design parameters, and to see how sensitive our controller is to other unstructured perturbations in the plant model (e.g. multiplicative uncertainty, feedback uncertainty).

Chapter 4

Robust Stability: Low Order LPC

4.1 Introduction

One of the main limitations of GSHF controllers is that the control signal tends to be large and the intersample behaviour poor. In this chapter, we will show how to design a low order linear periodic controller (LPC) that has satisfactory intersample behaviour, while providing a gain margin as large as desired and any phase margin up to 90 degrees. In fact, we will show that if the sampling period is small, then our LPC control law is very close to a continuous-time state feedback control law.

This chapter is organized in the following manner. In Section 4.2, we formulate the gain/phase margin problem and we motivate the particular approach that will be taken. Section 4.3 describes how a low order LPC can be constructed to implement our control law. The main results follow in Section 4.4, where we show that it is possible to design a low order LPC that has satisfactory intersample performance, while providing a gain margin as large as desired and a phase margin of up to 90 degrees. An example illustrating the low order LPC design method is given in Section 4.5. Additional properties for this controller are also highlighted in this example. Finally, conclusions are made in Section 4.6.

4.2 Problem Formulation

As in Chapter 3, suppose our nominal model is

$$\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u(t), \quad x(0) = x_0, \\ y(t) &= C\,x(t), \end{aligned}$$

with $x(t) \in \mathbf{R}^n$ the state, $u(t) \in \mathbf{R}^m$ the control input, and $y(t) \in \mathbf{R}^r$ the plant output. We capture uncertainty in the model by supposing that the actual system is given by

$$\begin{aligned} \dot{x}(t) &= A\,x(t) + \gamma B\,u(t), \quad x(0) = x_0, \\ y(t) &= C\,x(t); \end{aligned} \tag{4.1}$$

we associate this system with the triple $(A, \gamma B, C)$. With

$$\Gamma(\underline{\rho}, \overline{\rho}, \overline{\phi}) := \{\rho e^{j\phi} : \rho \in [\underline{\rho}, \overline{\rho}], \phi \in [-\overline{\phi}, \overline{\phi}]\},$$

our goal is to find a controller which will simultaneously stabilize every model in

$$\{(A, \gamma B, C) : \gamma \in \Gamma(\underline{\rho}, \overline{\rho}, \overline{\phi})\}.$$

If $\overline{\phi} = 0$, then we have a gain margin problem; if $\underline{\rho} = \overline{\rho} = 1$, then we have a phase margin problem; the general case is a combined gain and phase margin problem. With $T > 0$ the sampling rate, we will consider LPCs of the form

$$\begin{aligned} z[k+1] &= G(k)z[k] + H(k)y(kT), \quad z[0] = z_0 \in \mathbf{R}^l, \\ u(kT+\tau) &= J(k)z[k], \qquad\qquad\qquad \tau \in [0, T); \end{aligned} \tag{4.2}$$

we associate this system with the 4-tuple (G, H, J, T). Here we let p denote the period of the controller parameters G, H, and J, so that pT is the period of the controller (G, H, J, T). Note that (4.2) can be implemented with a sampler, a zero-order-hold, and an l^{th} order periodically time-varying discrete-time system of period p (See Figure 4.1).

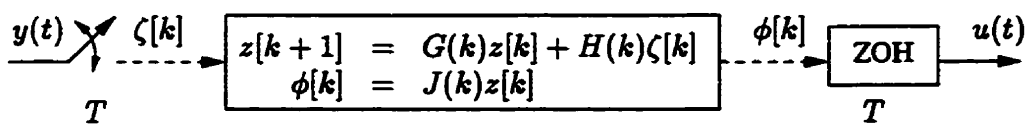


Figure 4.1: Implementation with a sampler and a zero-order-hold.

Here our notion of closed loop stability is the usual one:

Definition 4.1 The LPC (4.2) stabilizes (4.1) if, for every $x_0 \in \mathbf{R}^n$ and $z_0 \in \mathbf{R}^l$, we have

$$\lim_{t\to\infty} x(t) = 0,$$

$$\lim_{k\to\infty} z[k] = 0.$$

Before presenting any results, let us first provide some motivation for the approach adopted in this chapter. With $F \in \mathbf{R}^{m\times n}$, $p > n$, and $\overline{T} := pT$, we consider the control law

$$u(t) = \begin{cases} 0 & t \in [k\overline{T}, k\overline{T}+nT) \\ Fx[k\overline{T}+nT] & t \in [k\overline{T}+nT, (k+1)\overline{T}); \end{cases} \tag{4.3}$$

we will denote the intervals $[k\overline{T}, k\overline{T} + nT)$ as the Estimation Phase and intervals $[k\overline{T} + nT, (k+1)\overline{T})$ as the Control Phase (see Figure 4.2).

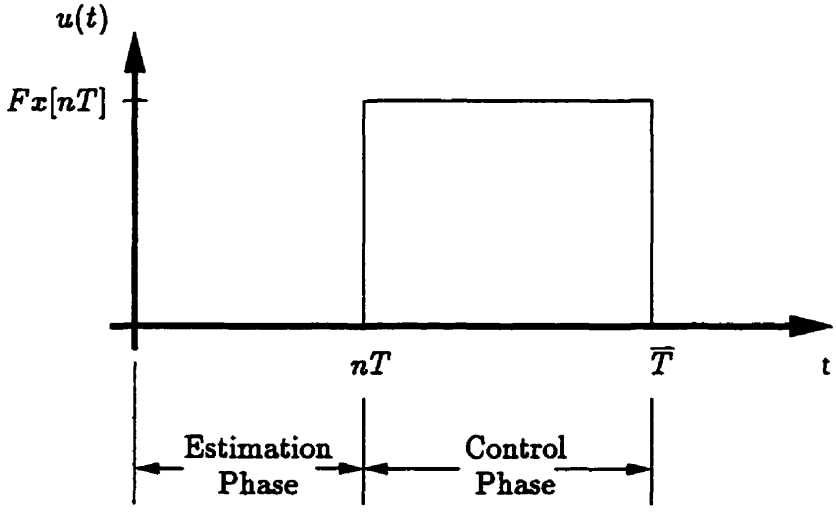


Figure 4.2: Estimation and control phase of $u(t)$.

Our first result, found in Section 4.3, will show that (4.3) can be implemented by an m^{th} order LPC of the form (4.2). Then the natural question is how do we choose F, T and p. Since we are assuming the plant is strictly proper, we know that the plant filters out the high frequency content of $u(t)$. Hence, for small T one might expect that (4.3) will have the same effect as

$$u(t) = \tfrac{p-n}{p} Fx(t)$$

(See Figure 4.3). If this is the case, then by choosing F appropriately, our low order LPC

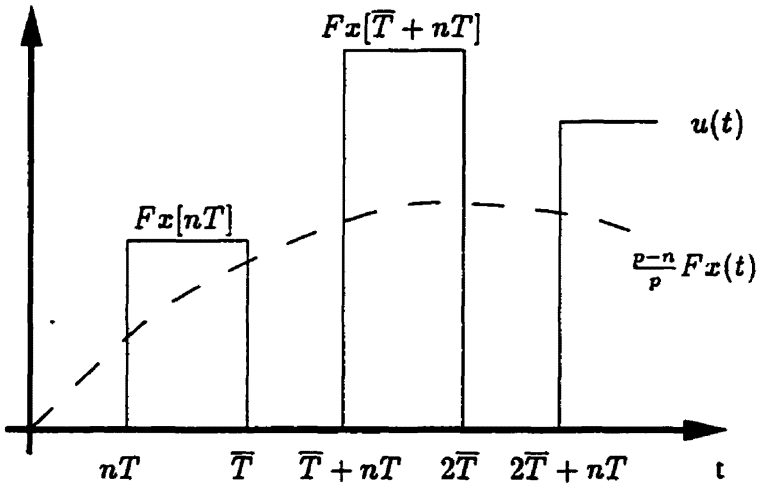


Figure 4.3: Control law.

should share some of the desirable properties provided by such a state feedback controller. In this chapter, F will be chosen using linear quadratic regulator (LQR) theory since it is well known that such controllers provide excellent gain and phase margins.

4.3 Control Law Implementation

In this section we will show that given a MIMO plant (4.1), for every fixed integer $p > n$ and almost all sampling periods T, there exists an m^{th} order LPC of the form (4.2) that will implement the control law (4.3).

Proposition 4.1 If $T>0$ is non-pathological, then for every $p>n$ and every $F\in\mathbf{R}^{m\times n}$, there exists an m^{th} order LPC (G,H,J,T) that will implement the control law (4.3).

Proof:

Let $p > n$, $\overline{T} = pT$ and $k \in \mathbf{Z}^+$. Recall from (4.3) that for $\tau \in [k\overline{T}, k\overline{T} + nT)$ we have $u(\tau) = 0$, so for $i = 0, 1, \cdots, n$, we have

$$x[k\overline{T} + iT] = (e^{AT})^i x[k\overline{T}]; \tag{4.4}$$

hence,

$$\underbrace{\begin{bmatrix} y[k\overline{T}] \\ \vdots \\ y[k\overline{T}+(n-1)T] \end{bmatrix}}_{=:\mathcal{Y}[k\overline{T}]} = \underbrace{\begin{bmatrix} C \\ \vdots \\ C(e^{AT})^{(n-1)} \end{bmatrix}}_{=:\mathcal{C}} x[k\overline{T}].$$

Since T is non-pathological and (C,A) is observable, it follows from Theorem 2.1 that the pair (C,e^{AT}) is observable, which implies that $\text{rank}\{\mathcal{C}\} = n$, so

$$x[k\overline{T}] = (\mathcal{C}^T\mathcal{C})^{-1}\mathcal{C}^T\mathcal{Y}[k\overline{T}]. \tag{4.5}$$

Using (4.4) and (4.5), it follows that

$$Fx[k\overline{T}+nT] = F(e^{AT})^n(\mathcal{C}^T\mathcal{C})^{-1}\mathcal{C}^T\mathcal{Y}[k\overline{T}],$$

so define $f_i \in \mathbf{R}^{m\times r}$ via

$$\begin{bmatrix} f_0 & \cdots & f_{n-1} \end{bmatrix} := F(e^{AT})^n(\mathcal{C}^T\mathcal{C})^{-1}\mathcal{C}^T.$$

Hence, we can form $Fx[k\overline{T}+nT]$ by sampling the output during the Estimation Phase while setting u to zero. During the Control Phase we simply keep u constant at $Fx[k\overline{T}+nT]$. To implement this, we choose $(G,H,J) \in \mathcal{S}(\mathbf{R}^{m\times m})\times\mathcal{S}(\mathbf{R}^{m\times r})\times\mathcal{S}(\mathbf{R}^{m\times m})$ to be

$$(G,H,J)(k) := \begin{cases} (0,f_0,0) & k=0 \\ (I,f_k,0) & k=1,\cdots,n-1 \\ (I,0,I) & k=n,\cdots,p-1, \end{cases}$$

and set

$$(G(k+p),H(k+p),J(k+p)) = (G(k),H(k),J(k)), \quad k \in \mathbf{Z}^+.$$

It is routine to verify that in closed loop, the response of (G,H,J,T) becomes (4.3).

■

4.4 Controller Design

Now that we know how to implement (4.3) as a low order LPC, the next step is to choose F. It is well known that a continuous-time LQR optimal state feedback controller provides a lower gain margin of $\frac{1}{2}$, an upper gain margin of ∞, and a 60 degree phase margin. Recall from Chapter 3 that in order to design the static GSHF controller that provided a lower gain margin of less than 1/2 and a phase margin greater than 60 degrees, we scaled the C matrix. For the same reason, we now scale the B matrix; with $\underline{\rho} \in (0,1]$, $\overline{\rho} \in [1,\infty)$, and $\overline{\phi} \in [0, \frac{\pi}{2})$, define

$$\alpha := 2\cos(\overline{\phi}) \tag{4.6}$$

and

$$\widehat{B} := \alpha\underline{\rho}B.$$

With $Q \in \mathbf{R}^{n\times n}$ and $R \in \mathbf{R}^{m\times m}$ symmetric and positive definite, we now consider the auxiliary system

$$\dot{w}(t) = Aw(t) + \widehat{B}v(t), \quad w(0) = w_0;$$

we would like to find the control law which, for each w_0, minimizes

$$\int_0^\infty [w^T(t)Qw(t) + v^T(t)Rv(t)]dt.$$

It is well known that since $(\sqrt{Q}, A)$ is observable and $(A, \widehat{B})$ is controllable, the optimal control law is of the form $v = F_0 w$; we can obtain the optimal gain F_0 by first solving the algebraic Riccati equation

$$P_0A + A^TP_0 - P_0\widehat{B}R^{-1}\widehat{B}^TP_0 + Q = 0 \tag{4.7}$$

for the unique positive definite symmetric solution P_0, and then setting

$$F_0 = -R^{-1}\widehat{B}^TP_0. \tag{4.8}$$

Not only is

$$\mathrm{sp}(A + \widehat{B}F_0) = \mathrm{sp}(A + \alpha\underline{\rho}B) \subset \mathbf{C}^-,$$

but it can also be shown that

$$\mathrm{sp}(A + \rho e^{j\phi} B F_0) \subset \mathbf{C}^-, \quad \rho \in [\underline{\rho}, \overline{\rho}], \phi \in [-\overline{\phi}, \overline{\phi}].$$

It would be nice if we could simply set F of (4.3) to F_0. However, the control is turned on for $\frac{p-n}{p}$ of the time, which means that we should scale this gain, i.e., set

$$F = \tfrac{p}{p-n} F_0.$$

While this may work when T is small, the controller will probably not even stabilize the nominal system when T is large. Hence, instead we will design F using discrete-time LQR theory so that we can always guarantee stability of the nominal system; since we would like the controller to inherit the gain and phase margin properties of the continuous-time LQR controller, we will incorporate the $\frac{p}{p-n}$ term in the design. To this end, with $p > n$ we define

$$A_d := e^{ApT}, \quad B_d := \int_0^{(p-n)T} e^{A\tau} B d\tau, \quad F_T := F e^{AnT}; \tag{4.9}$$

when we apply the control law (4.3) to the plant (4.1), we have

$$x[(k+1)\overline{T}] = (A_d + \gamma B_d F_T) x[k\overline{T}].$$

Hence, we can state our control objective as:

> Given the nominal system (A, B, C) and the set $\Gamma(\underline{\rho}, \overline{\rho}, \overline{\phi})$, find a $T > 0$, $p > n$, and an $F_T \in \mathbf{R}^{m \times n}$ such that for every $\gamma \in \Gamma(\underline{\rho}, \overline{\rho}, \overline{\phi})$, we have
> $\mathrm{sp}(A_d + \gamma B_d F_T) \subset \{z \in \mathbf{C} : |z| < 1\}$.

Now we will design the discrete-time control law. First, fix $p > n$; by Theorem 2.3, we can choose a $T_0 > 0$ so that (A_d, B_d) is controllable for $T \in (0, T_0)$. Let $T \in (0, T_0)$, define

$$\widehat{B_d} := \alpha \underline{\rho} B_d,$$

and consider the auxiliary system

$$w[(k+1)\overline{T}] = A_d w[k\overline{T}] + \widehat{B_d} v[k\overline{T}], \quad w[0] = w_0. \tag{4.10}$$

We would like to find the control law which, for each w_0, minimizes

$$\sum_{k=0}^{\infty} \left\{ w^T[k\overline{T}] Q w[k\overline{T}] + \left(\tfrac{p-n}{p}\right)^2 v^T[k\overline{T}] R v[k\overline{T}] \right\}. \tag{4.11}$$

It is well known that since $(\sqrt{Q}, A_d)$ is observable and $(A_d, \widehat{B_d})$ is controllable, the optimal control is of the form

$$v[k\overline{T}] = F_T w[k\overline{T}]; \tag{4.12}$$

furthermore, the following discrete-time algebraic Riccati equation

$$P_T - A_d^T P_T A_d + A_d^T P_T \widehat{B_d} \left[\widehat{B_d}^T P_T \widehat{B_d} + \left(\tfrac{p-n}{p}\right)^2 R \right]^{-1} \widehat{B_d}^T P_T A_d - Q = 0 \tag{4.13}$$

has a unique symmetric positive definite solution P_T and the optimal gain is

$$F_T = - \left[\widehat{B_d}^T P_T \widehat{B_d} + \left(\tfrac{p-n}{p}\right)^2 R \right]^{-1} \widehat{B_d}^T P_T A_d, \tag{4.14}$$

e.g. see the results of exercise 12.7 in [55]. If we set

$$F = F_T \left(e^{AnT}\right)^{-1}, \tag{4.15}$$

then we will be able to show that for sufficiently small T, our control objective can be attained.

4.4.1 Preliminary Results

We will require some preliminary results to prove that our new LPC provides good gain/phase margins with satisfactory intersample performance. Specifically, we would like to show that the solution of our discrete-time Riccati equation (4.13) approaches the solution of the continuous-time Riccati equation (4.7) (See Rosen and Wang [43] for a similar result). The approach that we take here will be analogous to the approach used in Appendix A, so let us begin with the following:

Lemma 4.1 For every $p > n$, we have $P_T = \mathcal{O}(\overline{T}^{-1})$.

Proof:

By our choice of F_T, it follows that when we apply the auxiliary control law (4.12) to the auxiliary system (4.10), the cost function (4.11) is minimized, and in fact, the associated optimal cost is $w_0^T P_T w_0$.

Choose $\widetilde{F} \in \mathbf{R}^{m\times n}$ so that $A + \widehat{B}\widetilde{F}$ is stable and consider the control law

$$v[k\overline{T}] = \tfrac{p}{p-n}\widetilde{F}w[k\overline{T}], \tag{4.16}$$

which means that the closed loop auxiliary system is given by

$$w[(k+1)\overline{T}] = (A_d + \tfrac{p}{p-n}\widehat{B_d}\widetilde{F})w[k\overline{T}], \quad w(0) = w_0.$$

Since

$$\begin{aligned} A_d + \tfrac{p}{p-n}\widehat{B_d}\widetilde{F} &= e^{A\overline{T}} + \tfrac{p}{p-n}\int_0^{(p-n)T} e^{A\tau}\widehat{B}d\tau\widetilde{F} \\ &= (I + A\overline{T} + \mathcal{O}(\overline{T}^2)) + \tfrac{p}{p-n}\int_0^{(p-n)T}[I + \mathcal{O}(\tau)]\widehat{B}d\tau\widetilde{F} \\ &= I + \overline{T}(A + \widehat{B}\widetilde{F}) + \mathcal{O}(\overline{T}^2), \end{aligned}$$

and $A + \widehat{B}\widetilde{F}$ is stable, it can be shown using a Lyapunov type argument that there exists a $T_0 > 0$, $\alpha > 0$ and $\lambda < 0$ so that

$$\|(A_d + \tfrac{p}{p-n}\widehat{B_d}\widetilde{F})^k\| \leq \alpha(e^{\lambda\overline{T}})^k, \quad \overline{T} \in (0, T_0).$$

Hence, for $\overline{T} \in (0, T_0)$, the cost associated with the control law (4.16), which is given by

$$w_0^T\left\{\sum_{k=0}^{\infty}[(A_d + \tfrac{p}{p-n}\widehat{B_d}\widetilde{F})^T]^k(Q + \widetilde{F}^T R\widetilde{F})(A_d + \tfrac{p}{p-n}\widehat{B_d}\widetilde{F})^k\right\}w_0,$$

is bounded above by

$$\sum_{k=0}^{\infty}\alpha^2(e^{2\lambda\overline{T}})^k\|Q + \widetilde{F}^T R\widetilde{F}\| \cdot \|w_0\|^2 = \frac{\alpha^2\|Q + \widetilde{F}^T R\widetilde{F}\|}{1 - e^{2\lambda\overline{T}}}\|w_0\|^2 = \mathcal{O}(\overline{T}^{-1})\|w_0\|^2;$$

this clearly is an upper bound on the optimal cost $w_0^T P_T w_0$. Hence, we have

$$\begin{aligned} &w_0^T P_T w_0 \leq \mathcal{O}(\overline{T}^{-1})\|w_0\|^2 \\ &\Rightarrow P_T = \mathcal{O}(\overline{T}^{-1}). \end{aligned}$$

■

We can now use this to prove the following:

Lemma 4.2 For every $p > n$, we have

$$\begin{aligned}\widehat{B_d} &= (p-n)T\widehat{B} + \mathcal{O}(T^2),\\ P_T &= \tfrac{1}{pT}P_0 + \mathcal{O}(1),\\ F_T &= \tfrac{p}{p-n}F_0 + \mathcal{O}(T).\end{aligned}$$

Proof:

Since

$$e^{A\tau} = I + A\tau + \mathcal{O}(\tau^2),$$

it follows that

$$\begin{aligned}\widehat{B_d} &= \alpha\underline{\rho}\int_0^{(p-n)T} e^{A\tau}B\,d\tau\\ &= (p-n)T\widehat{B} + \mathcal{O}(T^2).\end{aligned} \tag{4.17}$$

Define

$$P_\Delta := \overline{T}P_T.$$

From Lemma 4.1, it follows that $P_\Delta = \mathcal{O}(1)$. Substituting $P_T = P_\Delta/\overline{T}$ and

$$A_d = e^{A\overline{T}} = I + A\overline{T} + \mathcal{O}(\overline{T}^2), \tag{4.18}$$

into (4.13), we get

$$\begin{aligned}&\tfrac{P_\Delta}{\overline{T}} - \left(I + A\overline{T} + \mathcal{O}(\overline{T}^2)\right)^T \tfrac{P_\Delta}{\overline{T}}\left(I + A\overline{T} + \mathcal{O}(\overline{T}^2)\right)\\ &\quad + \left(I + A\overline{T} + \mathcal{O}(\overline{T}^2)\right)^T \tfrac{P_\Delta}{\overline{T}}\left((p-n)T\widehat{B} + \mathcal{O}(T^2)\right)\\ &\quad \times \left[\left((p-n)T\widehat{B} + \mathcal{O}(T^2)\right)^T \tfrac{P_\Delta}{\overline{T}}\left((p-n)T\widehat{B} + \mathcal{O}(T^2)\right) + \left(\tfrac{p-n}{p}\right)^2 R\right]^{-1}\\ &\quad \times \left((p-n)T\widehat{B} + \mathcal{O}(T^2)\right)^T \tfrac{P_\Delta}{\overline{T}}\left(I + A\overline{T} + \mathcal{O}(\overline{T}^2)\right) - Q = 0\\ \Rightarrow\ &\tfrac{P_\Delta}{\overline{T}} - \left(\tfrac{P_\Delta}{\overline{T}} + (A^TP_\Delta + P_\Delta A) + \mathcal{O}(\overline{T})\right) + \left(\tfrac{(p-n)}{p}P_\Delta\widehat{B} + \mathcal{O}(\overline{T})\right)\\ &\quad \times \left[\left(\tfrac{(p-n)^2}{p^2}\overline{T}\widehat{B}^TP_\Delta\widehat{B} + \mathcal{O}(\overline{T}^2)\right) + \tfrac{(p-n)^2}{p^2}R\right]^{-1}\left(\tfrac{(p-n)}{p}\widehat{B}^TP_\Delta + \mathcal{O}(\overline{T})\right) - Q = 0\\ \Rightarrow\ &-(A^TP_\Delta + P_\Delta A) + \mathcal{O}(\overline{T})\\ &\quad + \left(\tfrac{(p-n)}{p}P_\Delta\widehat{B} + \mathcal{O}(\overline{T})\right)\left(\tfrac{p^2}{(p-n)^2}R^{-1}\right)\left(\mathcal{O}(\overline{T}) + I\right)^{-1}\left(\tfrac{(p-n)}{p}\widehat{B}^TP_\Delta + \mathcal{O}(\overline{T})\right) - Q = 0.\end{aligned}$$

But by Lemma 2.1 (iii),

$$\left(I + \mathcal{O}(\overline{T})\right)^{-1} = \left(I + \mathcal{O}(\overline{T})\right),$$

so it follows that

$$A^T P_\Delta + P_\Delta A - P_\Delta \widehat{B} R^{-1} \widehat{B}^T P_\Delta + Q + \underbrace{\mathcal{O}(\overline{T})}_{=:\Delta_Q} = 0. \tag{4.19}$$

Since $\Delta_Q = \mathcal{O}(\overline{T})$, we know that there exists a $T_1 > 0$ such that for $\overline{T} \in (0, T_1)$ we have

$$Q + \Delta_Q > 0.$$

Furthermore, since $P_T = P_T^T > 0$, it follows that $P_\Delta = P_\Delta^T > 0$. Then together with the fact that Q and R are symmetric, it follows from (4.19) that $\Delta_Q = \Delta_Q^T$. Hence, we can apply Lemma 2.3 to show that for $\overline{T} \in (0, T_1)$, there exists a constant $c_1 > 0$ so that

$$\|P_\Delta - P_0\| \leq \max\left\{ \left(\frac{\|P_0\| + c_1\|\Delta_Q\|}{\lambda_{min}(Q) - \|\Delta_Q\|} \right) \|\Delta_Q\|, c_1\|\Delta_Q\| \right\}.$$

Since $\Delta_Q = \mathcal{O}(\overline{T})$, it follows that

$$\begin{aligned} &\|P_\Delta - P_0\| = \|\overline{T} P_T - P_0\| = \mathcal{O}(\overline{T}) \\ &\Rightarrow \overline{T} P_T - P_0 = \mathcal{O}(\overline{T}) \\ &\Rightarrow P_T = \tfrac{1}{p\overline{T}} P_0 + \mathcal{O}(1). \end{aligned} \tag{4.20}$$

Finally, if we substitute (4.17), (4.18), and (4.20) into (4.14), we get

$$\begin{aligned} F_T &= -\left[\left((p-n)T\widehat{B} + \mathcal{O}(T^2) \right)^T \left(\overline{T}^{-1} P_0 + \mathcal{O}(1) \right) \left((p-n)T\widehat{B} + \mathcal{O}(T^2) \right) + \tfrac{(p-n)^2}{p^2} R \right]^{-1} \\ &\quad \times \left((p-n)T\widehat{B} + \mathcal{O}(T^2) \right)^T \left(\overline{T}^{-1} P_0 + \mathcal{O}(1) \right) \left(I + A\overline{T} + \mathcal{O}(\overline{T}^2) \right) \\ &= -\left[\tfrac{(p-n)^2}{p^2} \overline{T} \widehat{B}^T P_0 \widehat{B} + \mathcal{O}(\overline{T}^2) + \tfrac{(p-n)^2}{p^2} R \right]^{-1} \left(\tfrac{(p-n)}{p} \widehat{B}^T P_0 + \mathcal{O}(\overline{T}) \right) \\ &= -\left(I + \mathcal{O}(\overline{T}) \right)^{-1} \left(\tfrac{p^2}{(p-n)^2} R^{-1} \right) \left(\tfrac{(p-n)}{p} \widehat{B}^T P_0 + \mathcal{O}(\overline{T}) \right) \\ &= -\left(I + \mathcal{O}(\overline{T}) \right) \left(\tfrac{p^2}{(p-n)^2} R^{-1} \right) \left(\tfrac{(p-n)}{p} \widehat{B}^T P_0 + \mathcal{O}(\overline{T}) \right) \qquad \text{(Using Lemma 2.1)} \\ &= \tfrac{p}{(p-n)} F_0 + \mathcal{O}(\overline{T}). \end{aligned}$$

■

In order to show that the intersample performance of our LPC is satisfactory, we will need the following result.

Lemma 4.3 For every $\gamma \in \Gamma(\underline{\rho}, \overline{\rho}, \overline{\phi})$, we have

$$\|e^{(A+\gamma BF_0)t}\|^2 \leq \|P_0\| \times \|P_0^{-1}\| \times e^{-\frac{1}{\|P_0\|\|Q^{-1}\|}t}, \ t \geq 0.$$

Proof:

Fix $\gamma \in \Gamma(\underline{\rho}, \overline{\rho}, \overline{\phi})$, write $\gamma = \rho e^{j\phi}$, let P_0 and F_0 satisfy (4.7)—(4.8), and consider

$$\dot{x}(t) = (A + \gamma BF_0)x(t), \quad x(0) = x_0.$$

Consider the candidate Lyapunov function $V : \mathbf{C}^n \to \mathbf{C}$:

$$V(x) := x^* P_0 x.$$

Since $P_0 = P_0^T$, we know that V is real valued, so

$$\dot{V}(x(t)) := \frac{\partial}{\partial t} V(x(t))$$

is also real-valued. Expanding $\dot{V}(x(t))$ and using (4.7) and (4.8) to simplify, it follows that

$$\begin{aligned} \dot{V}(x(t)) &= -x(t)^* Q x(t) + \alpha\underline{\rho}(\alpha\underline{\rho} - (\gamma^* + \gamma))x(t)^* P_0 B R^{-1} B^T P_0 x(t) \\ &= -x(t)^* Q x(t) + \alpha\underline{\rho}(\alpha\underline{\rho} - 2\rho\cos(\phi))x(t)^* P_0 B R^{-1} B^T P_0 x(t). \end{aligned}$$

But since $\cos(\phi) \geq \cos(\overline{\phi}) = \alpha/2$, and $\underline{\rho} \leq \rho$, we have

$$\alpha\underline{\rho} - 2\rho\cos(\phi) \leq 0,$$

so along with the fact that $R > 0$, we have

$$\begin{aligned} \dot{V}(x(t)) &\leq -x(t)^* Q x(t), \\ \Rightarrow \dot{V}(x(t)) &\leq -\lambda_{min}(Q)\|x(t)\|^2 = -\|Q^{-1}\|^{-1}\|x(t)\|^2, \\ \Rightarrow \|x(t)\|^2 &\leq -\|Q^{-1}\| \times \dot{V}(x(t)). \end{aligned}$$

Hence,

$$\begin{aligned} V(x(t)) &\leq \|P_0\| \times \|x(t)\|^2 \\ &\leq -\|P_0\| \times \|Q^{-1}\| \times \dot{V}(x(t)), \end{aligned}$$

so it follows from Lemma 2.2 that

$$V(x(t)) \leq e^{-\frac{1}{\|Q^{-1}\|\|P_0\|}t} V(x_0).$$

But

$$V(x(t)) \geq \lambda_{min}(P_0)\|x(t)\|^2 = \|P_0^{-1}\|^{-1}\|x(t)\|^2,$$

so it follows that

$$\begin{aligned} \|x(t)\|^2 &\leq \|P_0^{-1}\| \times e^{-\frac{1}{\|Q^{-1}\|\|P_0\|}t} V(x_0) \\ &\leq \|P_0^{-1}\| \times \|P_0\| \times e^{-\frac{1}{\|Q^{-1}\|\|P_0\|}t} \|x_0\|^2 \\ \Rightarrow \|e^{(A+\gamma BF_0)t}x_0\|^2 &\leq \|P_0^{-1}\| \times \|P_0\| \times e^{-\frac{1}{\|Q^{-1}\|\|P_0\|}t} \|x_0\|^2. \end{aligned} \tag{4.21}$$

Since this holds for every $x_0 \in \mathbf{R}^n$, the result follows.

■

4.4.2 Robust Stability

We can now use the results of Subsection 4.4.1 to show that if we choose F using (4.15), then the resulting LPC controller can provide the desired gain and phase margins.

Theorem 4.1 For every integer $p > n$, there exists a $T_{max} > 0$ so that for every $T \in (0, T_{max})$, there exists a m^{th} order pT periodic LPC (G, H, J, T) which stabilizes every system in

$$\{(A, \gamma B, C) : \gamma \in \Gamma(\underline{\rho}, \overline{\rho}, \overline{\phi})\}.$$

Proof:

Fix $p > n$. By Theorem 2.3, choose $T_0 > 0$ such that (A_d, B_d) is controllable for every $T \in (0, T_0)$. Recall that α, P_T, and F_T satisfy (4.6), (4.13), and (4.14). With $\widehat{\rho} \in \left[\frac{1}{\alpha}, \frac{\overline{\rho}}{\alpha\underline{\rho}}\right]$ and $\phi \in [-\overline{\phi}, \overline{\phi}]$, consider

$$w[k+1] = (A_d + \widehat{\rho}e^{j\phi}\widehat{B_d}F_T)w[k], \quad w[0] = w_0 \in \mathbf{R}^n.$$

Fix w_0 and consider the Lyapunov candidate function $V : \mathbf{C}^n \to \mathbf{C}$:

$$V(w) := w^* P_T w.$$

Since $P_T = P_T^T$ it follows that V is real-valued, so

$$\Delta V(w[k]) := V(w[k+1]) - V(w[k])$$

is also real-valued. Expanding $\Delta V(w[k])$ and using (4.13) and (4.14) to simplify, it follows that

$$\begin{aligned}\Delta V(w[k]) &= -w[k]^* Q w[k] + \widehat{\rho}^2 w[k]^* (F_T^T \widehat{B}_d^T P_T \widehat{B}_d F_T) w[k] \\ &\quad + (1 - 2\widehat{\rho}\cos(\phi)) w[k]^* F_T^T \left(\widehat{B}_d^T P_T \widehat{B}_d + \left(\tfrac{p-n}{p}\right)^2 R \right) F_T w[k];\end{aligned}$$

using the definition of α given in (4.6), the bounds on ϕ, and the upper bound on $\widehat{\rho}$, we see that

$$1 - 2\widehat{\rho}\cos(\phi) \leq 0,$$

so along with the fact that $P_T > 0$ and $R > 0$, we have

$$\Delta V(w[k]) \leq -w[k]^* Q w[k] + \left(\tfrac{\bar{\rho}}{\alpha \underline{\rho}}\right)^2 w[k]^* (F_T^T \widehat{B}_d^T P_T \widehat{B}_d F_T) w[k]. \tag{4.22}$$

We now go on to show that as $T \to 0$, the second term in the above inequality is dominated by the first term. From Lemma 4.2, we know that there exists positive constants c_1, c_2, c_3, and $T_1 \in (0, T_0)$ so that for $T \in (0, T_1)$ we have

$$\|\widehat{B}_d\| \leq c_1 T,$$

$$\|P_T\| \leq \frac{c_2}{T},$$

$$\|F_T\| \leq c_3.$$

Hence, there exists a constant $c_4 > 0$ so that for $T \in (0, T_1)$ we have

$$\left(\tfrac{\bar{\rho}}{\alpha \underline{\rho}}\right)^2 \left| w[k]^* (F_T^T \widehat{B}_d^T P_T \widehat{B}_d F_T) w[k] \right| \leq c_4 T \|w[k]\|^2.$$

If we let

$$T_{max} < \min\left\{ \tfrac{\lambda_{min}(Q)}{c_4}, T_1 \right\},$$

then it follows that for $T \in (0, T_{max})$ and $w[k] \neq 0$, we have

$$\begin{aligned}&\left(\tfrac{\bar{\rho}}{\alpha \underline{\rho}}\right)^2 \left| w[k]^* (F_T^T \widehat{B}_d^T P_T \widehat{B}_d F_T) w[k] \right| < \lambda_{min}(Q) \|w[k]\|^2 \leq w[k]^* Q w[k] \\ &\Rightarrow -w[k]^* Q w[k] + \left(\tfrac{\bar{\rho}}{\alpha \underline{\rho}}\right)^2 w[k]^* (F_T^T \widehat{B}_d^T P_T \widehat{B}_d F_T) w[k] < 0,\end{aligned}$$

so for $T \in (0, T_{max})$ and $w[k] \neq 0$, we have

$$\Delta V(w[k]) < 0.$$

This means that there exists a positive definite matrix $\overline{Q}$ so that

$$\Delta V(w[k]) \leq -w[k]^* \overline{Q} w[k], \quad T \in (0, T_{max}), \ k \geq 0.$$

Hence, for every w_0, $V(w[k])$ goes to zero as $k \to \infty$; since P_T is positive definite, it follows that $w[k]$ goes to zero as $k \to \infty$ as well, so using the definition of $\widehat{B_d}$, we have

$$\text{sp}(A_d + \widehat{\rho} e^{j\phi} \widehat{B_d} F_T) \subset \{z \in \mathbf{C} : |z| < 1\}$$

$$\Rightarrow \text{sp}(A_d + \alpha \underline{\rho} \widehat{\rho} e^{j\phi} B_d F_T) \subset \{z \in \mathbf{C} : |z| < 1\}.$$

But this holds for $\alpha \underline{\rho} \widehat{\rho} \in [\underline{\rho}, \overline{\rho}]$, so for $T \in (0, T_{max})$, $\rho \in [\underline{\rho}, \overline{\rho}]$ and $\phi \in [-\overline{\phi}, \overline{\phi}]$, we have

$$\text{sp}(A_d + \rho e^{j\phi} B_d F_T) \subset \{z \in \mathbf{C} : |z| < 1\}.$$

Finally, with $T \in (0, T_{max})$ and $F := F_T(e^{AnT})^{-1}$, apply Proposition 4.1 to construct the m^{th} order pT periodic LPC (G, H, J, T).

■

4.4.3 Intersample Performance

We will now look at the performance of our LPC controller as $T \to 0$. Consider the following *desired* closed loop system:

$$\dot{\widehat{x}}(t) = (A + \gamma B F_0)\widehat{x}(t), \quad \widehat{x}(0) = x_0 \in \mathbf{R}^n. \tag{4.23}$$

Then our *desired* state is defined as

$$\widehat{x}(t) = e^{(A+\gamma B F_0)t} x_0.$$

If we apply the stabilizing LPC constructed in Theorem 4.1 to our plant (4.1), then the state of the plant satisfies

$$x[k\overline{T}] = (A_d + \gamma B_d F_T)^k x_0,$$

and for every $\gamma \in \Gamma(\underline{\rho}, \overline{\rho}, \overline{\phi})$, we have

$$\mathrm{sp}(A_d + \gamma B_d F_T) \subset \{z \in \mathbf{C} : |z| < 1\}.$$

Theorem 4.2 For every $\varepsilon > 0$, for sufficiently small $T > 0$ we have

$$\|x - \widehat{x}\|_\infty \leq \varepsilon \|x_0\|, \ \gamma \in \Gamma(\underline{\rho}, \overline{\rho}, \overline{\phi}).$$

Proof:

Fix $p > n$, and let $\gamma \in \Gamma(\underline{\rho}, \overline{\rho}, \overline{\phi})$, $T \in (0, T_{max})$, and $x_0 \in \mathbf{R}^n$ be arbitrary. Now consider

$$V(x) := x^T P_T x.$$

With $\overline{T} = pT$, it is routine to prove that if $T \in (0, T_{max})$, then $V(x[k\overline{T}])$ is monotonically decreasing and goes to zero (e.g. see the proof of Theorem 4.1). Hence,

$$\sup_{k \geq 0} \|x[k\overline{T}]\|^2 \leq \|P_T\| \times \|P_T^{-1}\| \times \|x_0\|^2.$$

Since P_0 is non-singular, and from Lemma 4.2 we have

$$P_T = \tfrac{P_0}{T} + \mathcal{O}(1),$$

it follows that there exists a constant $c_1 > 0$ and $T_1 \in (0, T_{max})$ so that

$$\sup_{k \geq 0} \|x[k\overline{T}]\| \leq c_1 \|x_0\|, \ T \in (0, T_1).$$

Also by Lemma 4.2 we have

$$F_T = \tfrac{p}{p-n} F_0 + \mathcal{O}(T),$$

which means that there exists a constant $c_2 > 0$ and $T_2 \in (0, T_1)$ so that

$$\|u\|_\infty \leq c_2 \|x_0\|, \ T \in (0, T_2).$$

Hence, it follows from the differential equation for x that there exists positive constants c_3 and c_4 so that

$$\|x\|_\infty \leq c_3 \|x_0\|, \ \ \|\dot{x}\|_\infty \leq c_4 \|x_0\|, \ \ T \in (0, T_2). \tag{4.24}$$

Now let's look at the difference equation for

$$\xi := x - \widehat{x}$$

at $t = 0, \overline{T}, 2\overline{T}, \ldots$. We have

$$\xi[(k+1)\overline{T}] = e^{(A+\gamma BF_0)\overline{T}}\xi[k\overline{T}] + [(A_d + \gamma B_d F_T) - e^{(A+\gamma BF_0)\overline{T}}]x[k\overline{T}].$$

But

$$e^{(A+\gamma BF_0)\overline{T}} = I + (A + \gamma BF_0)\overline{T} + \mathcal{O}(T^2),$$

and using Lemma 4.2, we have that

$$\begin{aligned} A_d + \gamma B_d F_T &= I + A\overline{T} + \mathcal{O}(T^2) + \gamma[(p-n)TB + \mathcal{O}(T^2)][\tfrac{p}{p-n}F_0 + \mathcal{O}(T)] \\ &= I + (A + \gamma BF_0)\overline{T} + \mathcal{O}(T^2). \end{aligned}$$

Hence, using our bound on $x[k\overline{T}]$ we have that

$$\xi[(k+1)\overline{T}] = e^{(A+\gamma BF_0)\overline{T}}\xi[k\overline{T}] + \mathcal{O}(T^2)\|x_0\|.$$

Now it follows from Lemma 4.3 that there exist constants $c_5 > 0$ and $\lambda < 0$, independent of γ, so that

$$\|e^{(A+\gamma BF_0)t}\| \leq c_5 e^{\lambda t}, \ t \geq 0. \tag{4.25}$$

Hence, there exist constants $c_6 > 0$ and $T_3 \in (0, T_2)$ so that

$$\|\xi[k\overline{T}]\| \leq \sum_{i=0}^{k-1} c_6 (e^{\lambda\overline{T}})^{k-1-i} T^2 \|x_0\|, \quad T \in (0, T_3),$$

$$\Rightarrow \sup_{k\geq 0} \|\xi[k\overline{T}]\| \leq c_6 \tfrac{T^2}{1-e^{\lambda\overline{T}}}\|x_0\|, \ T \in (0, T_3).$$

Since

$$\lim_{T\to 0} \tfrac{T^2}{1-e^{\lambda\overline{T}}} = 0,$$

we have that $x - \widehat{x}$ can be made small at $t = 0, \overline{T}, 2\overline{T}, \ldots$ by letting $T \to 0$.

Now let us examine what happens between the samples. For $T \in (0, T_3)$ and $\tau \in (0, \overline{T})$ we have

$$\begin{aligned} \|\xi(k\overline{T}+\tau)\| &\leq \|\xi[k\overline{T}]\| + \int_{k\overline{T}}^{k\overline{T}+\tau} \|\dot{x}(t) - \dot{\widehat{x}}(t)\| dt \\ &\leq c_6 \tfrac{T^2}{1-e^{\lambda\overline{T}}}\|x_0\| + \overline{T}(\|\dot{x}\|_\infty + \|\dot{\widehat{x}}\|_\infty). \end{aligned}$$

It follows from (4.24) that $\|\dot{x}\|_\infty$ is bounded in terms of $\|x_0\|$, while it follows from (4.23) and (4.25) that

$$\|\dot{\widehat{x}}\|_\infty \leq (\|A\| + \overline{\rho}\|B\| \times \|F_0\|)c_5\|x_0\|.$$

Hence, for $T \in (0, T_3)$ we have

$$\|\xi\|_\infty \leq \left[\frac{c_6T^2}{1-e^{\lambda T}} + c_4pT + c_5(\|A\| + \overline{\rho}\|B\| \times \|F_0\|)pT\right] \|x_0\|,$$

so the result follows. ∎

If we define the *desired* output as

$$\widehat{y}(t) := C\widehat{x}(t),$$

and note that the actual output is given by

$$y(t) = Cx(t),$$

then the following Corollary is a straightforward result of Theorem 4.2.

Corollary 4.1 For every $\epsilon > 0$, for sufficiently small $T > 0$ we have

$$\|y(t) - \widehat{y}(t)\|_\infty \leq \epsilon\|x_0\|, \quad \gamma \in \Gamma(\underline{\rho}, \overline{\rho}, \overline{\phi}).$$

Remark 4.1 With $\overline{\phi} \in \left[0, \frac{\pi}{2}\right)$, $\underline{\rho} \in (0, 1]$, and $\overline{\rho} \in [1, \infty)$, we now summarize an algorithm for constructing a low order LPC (G, H, J, T) that stabilizes every system in

$$\{(A, B, \gamma C) : \gamma \in \Gamma(\underline{\rho}, \overline{\rho}, \overline{\phi})\}.$$

i) Choose symmetric positive definite matrices Q and R, let

$$\alpha = 2\cos(\overline{\phi}),$$

and pick $p > n$.

ii) Find a T such that with

$$A_d := e^{ApT}, \quad B_d := \int_0^{(p-n)T} e^{A\tau}Bd\tau,$$

we have

a) (A_d, B_d) controllable,

b) (C, e^{AT}) observable,

c) P_T and F_T satisfying (4.13) and (4.14), and

d) $\overline{\rho}^2 \| F_T^T B_d^T P_T B_d F_T \| < \lambda_{min}(Q)$
(Note that this condition ensures ΔV in (4.22) is negative, which is sufficient to ensure the desired robustness).

iii) Let

$$\mathcal{C} := \begin{bmatrix} C \\ \vdots \\ C(e^{AT})^{n-1} \end{bmatrix}$$

and define $f_i \in \mathbf{R}^{m \times r}$ so that

$$\begin{bmatrix} f_0 & \cdots & f_{n-1} \end{bmatrix} = F_T(\mathcal{C}^T\mathcal{C})^{-1}\mathcal{C}^T.$$

iv) Set

$$(G, H, J)(k) := \begin{cases} (0, f_0, 0) & k = 0 \\ (I, f_k, 0) & k = 1, \cdots, n-1 \\ (I, 0, I) & k = n, \cdots, p-1. \end{cases}$$

□

Remark 4.2 The T obtained in step (ii) of the above algorithm is typically quite conservative. Hence, if we were to use this for design, it might be better to compute the combined gain margin and phase margin for various values of T, and obtain the largest one which achieves the desired robustness.

□

4.5 An Example

In this section, we will design a first order LPC for the second order SISO plant used in the examples found in Sections 3.3.3 and A.1.3. Suppose our nominal plant is

$$(A,B,C) = \left(\begin{bmatrix} -1 & 0 \\ 0 & 0.5 \end{bmatrix}, \begin{bmatrix} \frac{4}{3} \\ -\frac{1}{3} \end{bmatrix}, \begin{bmatrix} 1 & 1 \end{bmatrix} \right),$$

with associated transfer function

$$P_0(s) := C(sI - A)^{-1}B = \frac{s-1}{(s-0.5)(s+1)}.$$

Recall that this example was taken from Doyle et.al. [16, pp. 200–203], where it was shown that an LTI stabilizing compensator can provide at most a phase margin of 38.9^o, and the upper and lower gain margins must satisfy $\overline{\rho}/\underline{\rho} \leq 4$.

4.5.1 Controller Synthesis

Using Remark 4.1. we now construct a first order LPC to stabilize every system in

$$\{(A,B,\gamma C) : \gamma \in \Gamma(0.75, 6, 70^o)\}.$$

i) Choose $Q = I$, $R = 1$, let

$$\alpha = 2\cos(\overline{\phi}) = 2\cos(70^o) = 0.684,$$

and pick $p = 5 > n = 2$.

ii) Figure 4.4 shows a plot of $\overline{\rho}^2 \| F_T^T B_d^T P_T B_d F_T \|$ vs T. Since

$$\overline{\rho}^2 \| F_T^T B_d^T P_T B_d F_T \| < \lambda_{min}(Q) = 1, \quad T \in (0, 0.00002538),$$

we can choose $T = 0.00002$. However, as stated in Remark 4.2, this is typically conservative. So in a similar approach to that used in Sections 3.3.3 and A.1.3,

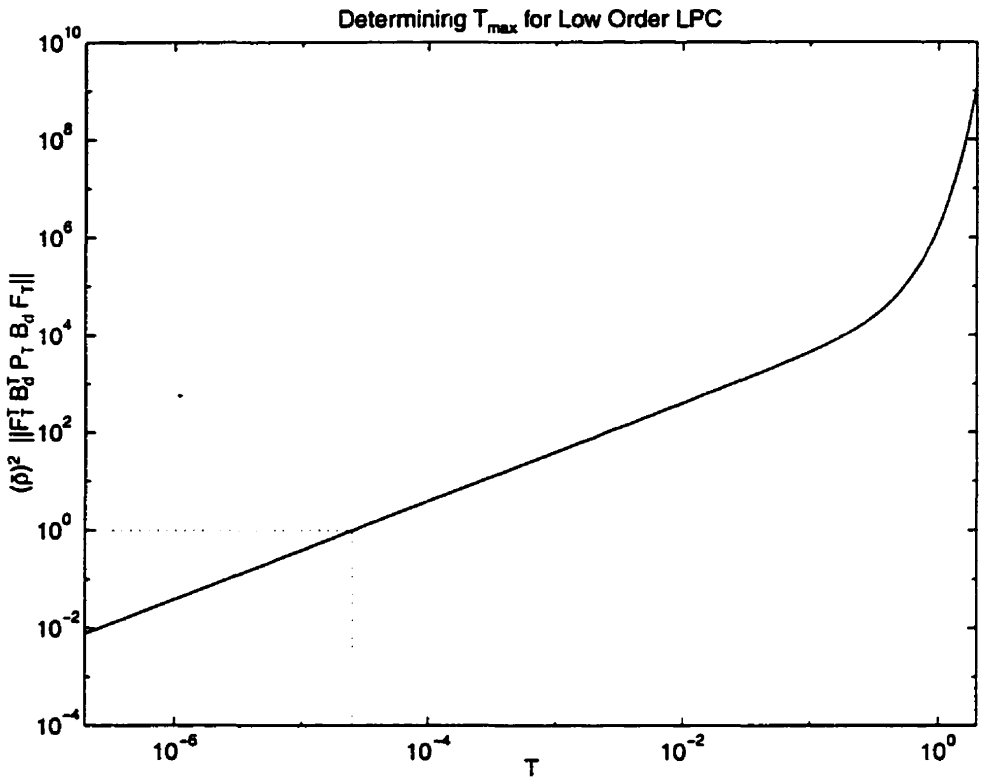


Figure 4.4: Determining T_{max} for the robust LPC using norm calculations.

we now propose a method to approximately determine a less conservative sampling period. Using a 2-dimensional search algorithm, determine

$$\begin{aligned}\underline{\rho}_{lpc}(T) := \min\{\,&\widehat{\rho} \in (0,1] : \mathrm{sp}(A_d + \rho e^{j\phi} B_d F_T) \subset \{z \in \mathbf{C} : |z| < 1\},\\ &\rho \in [\widehat{\rho}, 1], \phi \in [-\overline{\phi}, \overline{\phi}]\},\\ \overline{\rho}_{lpc}(T) := \max\{&\widehat{\rho} \in [1, \infty) : \mathrm{sp}(A_d + \rho e^{j\phi} B_d F_T) \subset \{z \in \mathbf{C} : |z| < 1\}\\ &\rho \in [1, \widehat{\rho}], \phi \in [-\overline{\phi}, \overline{\phi}]\}.\end{aligned}$$

To be consistent with Subsections 3.3.3 and A.1.3, we refer to $\underline{\rho}_{lpc}(T)$ ($\overline{\rho}_{lpc}(T)$) as the "combined lower (upper) gain/phase margin" provided by the LPC (G, H, J, T). A plot of $\underline{\rho}_{lpc}(T)$ and $\overline{\rho}_{lpc}(T)$ is provided in Figure 4.5. Observe that we can achieve the desired gain and phase margin for every $T \leq 0.00890$.

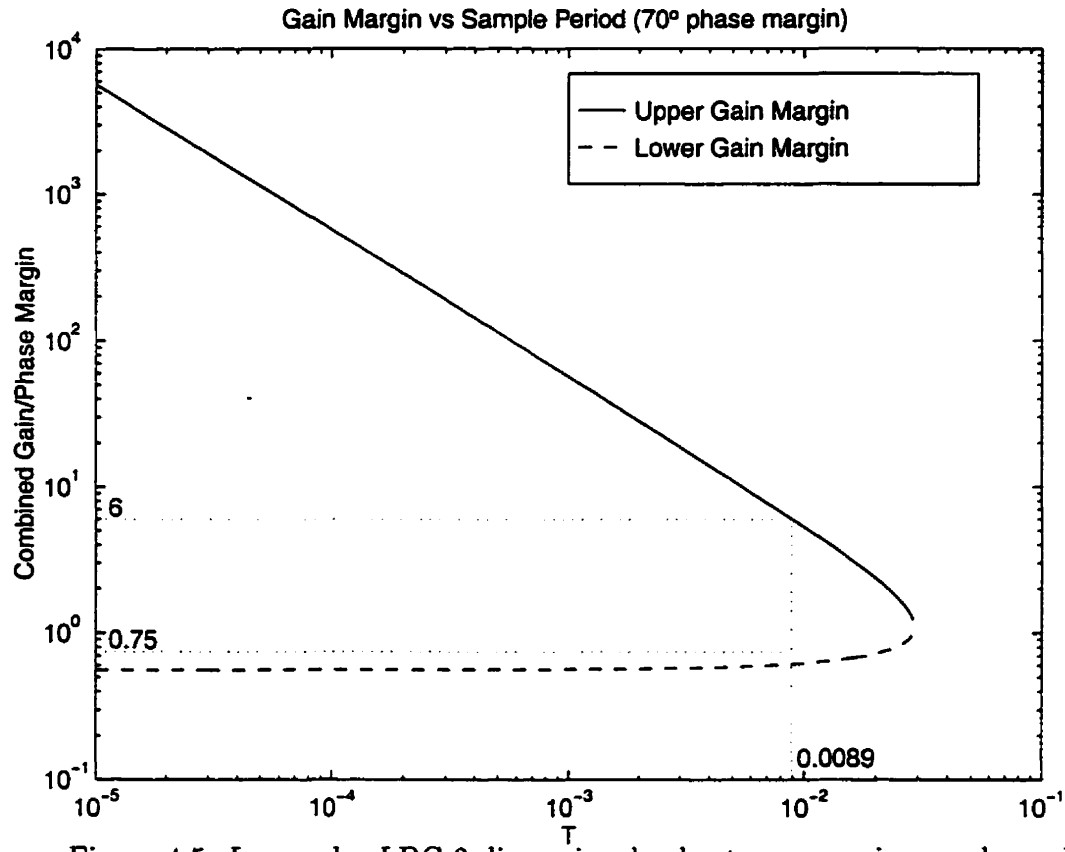


Figure 4.5: Low order LPC 2-dimensional robustness margin search result.

iii) If we pick $T = 0.0088$, it is easily verified that (A_d, B_d) is controllable and that (C, e^{AT}) is observable. With this choice of T, the solution P_T of (4.13) is

$$P_T = \begin{bmatrix} 11.60 & 31.46 \\ 31.46 & 1030 \end{bmatrix},$$

and the optimal gain F_T satisfying (4.14) is

$$F_T = \begin{bmatrix} -0.1623 & 11.23 \end{bmatrix}.$$

Hence, with the definition of C given in step iii) of Remark 4.1, it follows that

$$\begin{bmatrix} f_0 & \cdots & f_{n-1} \end{bmatrix} = F_T(C^TC)^{-1}C^T = \begin{bmatrix} -842.5 & 849.9 \end{bmatrix}.$$

iv) Set $(G, H, J)(k) := \begin{cases} (0, -842.5, 0) & k = 0 \\ (1, 849.9, 0) & k = 1, \\ (1, 0, 1) & k = 2, 3, 4. \end{cases}$

Figure 4.6 shows the output response for the case where $\gamma = 4$ and $x_0 = [1\ 0]^T$. It can be argued that the intersample performance of this first order LPC is much better than that of the static GSHF controller presented in the previous Chapter, since here we know that $y(t) \to \hat{y}(t)$ as $T \to 0$ and in the previous chapter we saw that the intersample performance degraded as $T \to 0$. (e.g. Compare Figure 4.6 to Figures 3.6 and A.4)

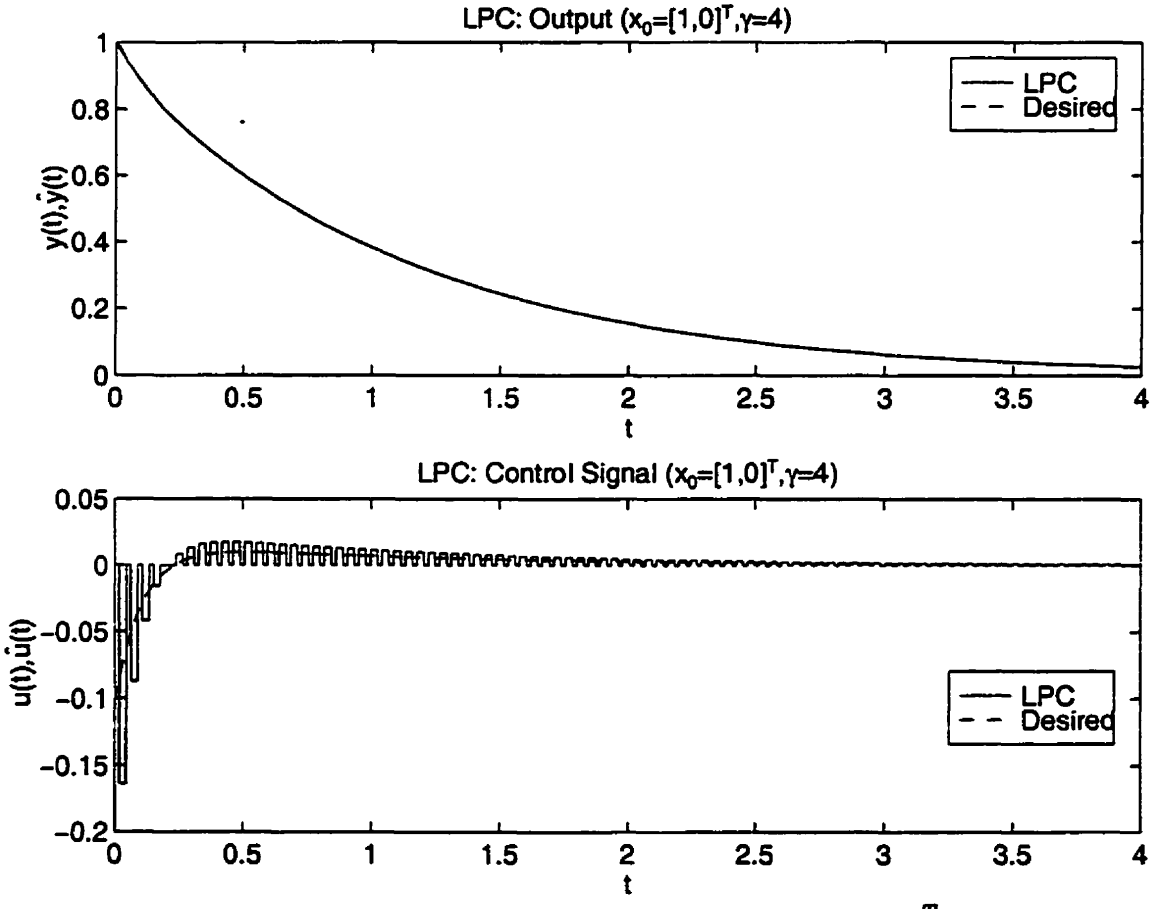


Figure 4.6: LPC simulation results: $\gamma = 4$, $x_0 = [1\ 0]^T$.

4.5.2 Properties of the Control Signal

In the previous subsection, we arbitrarily fixed $p = 5$ and then decreased T until our LPC controller provided the desired gain/phase margins. Since p was fixed, the LPC control signal was only nonzero for $(p - n)/p = 3/5$ of the time. Due to the approach adopted in the design of the LPC, we expected that during the Control Phase, the LPC control signal would be approximately 5/3 that of the desired continuous-time control signal. From the second plot in Figure 4.6, we see that this is indeed the case. In this

subsection, we will qualitatively determine what effect changing the ratio $(p-n)/p$ has on the LPC control signal.

We will perform simulations on the nominal model (i.e. $\gamma = 1$), and in order to make the trends more obvious in the simulation plots, we will fix $\overline{T}$ to be a relatively large value. We vary the ratio $(p-n)/p$ by decreasing the duration of the Estimation phase by increasing the value of p. Specifically, with $\overline{T} = 0.1$, we let $p \in \{3, 5, 12\}$, which means

$$\frac{p-n}{p} = \begin{cases} 0.3333 & p = 3 \\ 0.6000 & p = 5 \\ 0.8333 & p = 12. \end{cases}$$

and $T \in \{0.03333, 0.02, 0.008333\}$. The design specifications for our LPC will be $\underline{\rho} = 0.75$, $\overline{\rho} = 6$, $\overline{\phi} = 70^o$, $Q = I$, and $R = 1$. The LPC controllers can then be designed using steps (iii) and (iv) in Remark 4.1. In the following simulation results, we used $\gamma = 1$ and $x_0 = [1\ 0]^T$. Figure 4.7 shows the LPC control signal for each of the three LPCs, and as expected, the LPC control signal becomes smaller as $\frac{p-n}{p} \to 1$.

Furthermore, note that by fixing $\overline{T}$ and increasing p, we are effectively decreasing T. By Corollary 4.1 we expect that as T becomes smaller, the difference between the actual output $y(t)$ and the desired output $\widehat{y}(t)$ will also become smaller. This trend is verified in Figure 4.8.

4.6 Summary and Concluding Remarks

In this chapter, we have shown that it is possible to design a low order LPC that will solve an extended version of the combined gain/phase margin problem. The LPC can be implemented with a sampler, a zero-order-hold, and a m^{th} order p-periodic $(p > n)$ discrete-time compensator. Unlike the GSHF controllers of Chapter 3, this controller also provides satisfactory intersampler performance; in fact, for sufficiently small sampling periods, the LPC control signal has virtually the same effect as that of an ideal continuous-time state feedback control law. These results have been published and can be found in [45].

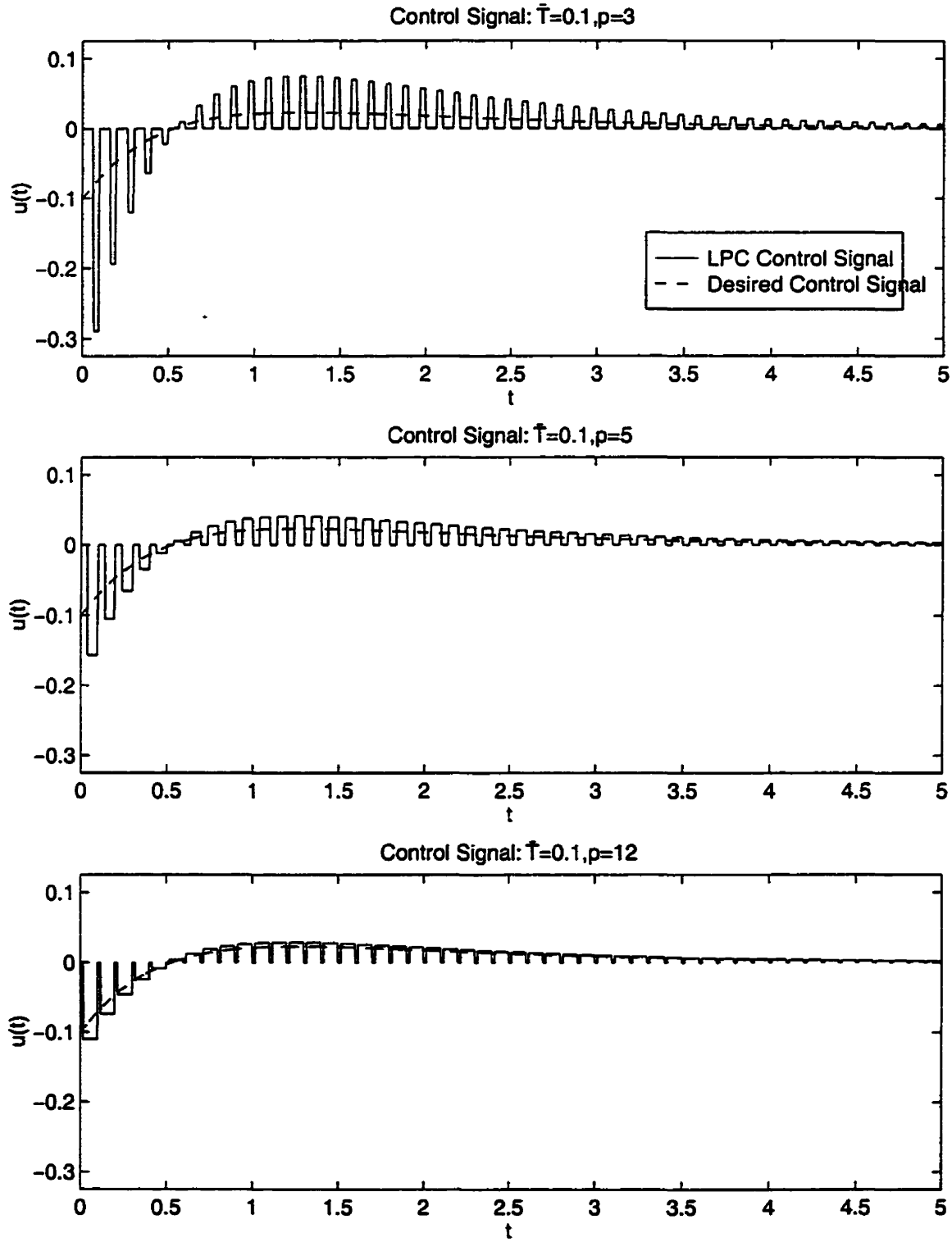


Figure 4.7: LPC control signal for $\overline{T} = 0.1$, $p \in \{3, 5, 12\}$, $\gamma = 1$, and $x_0 = [1\ 0]^T$.

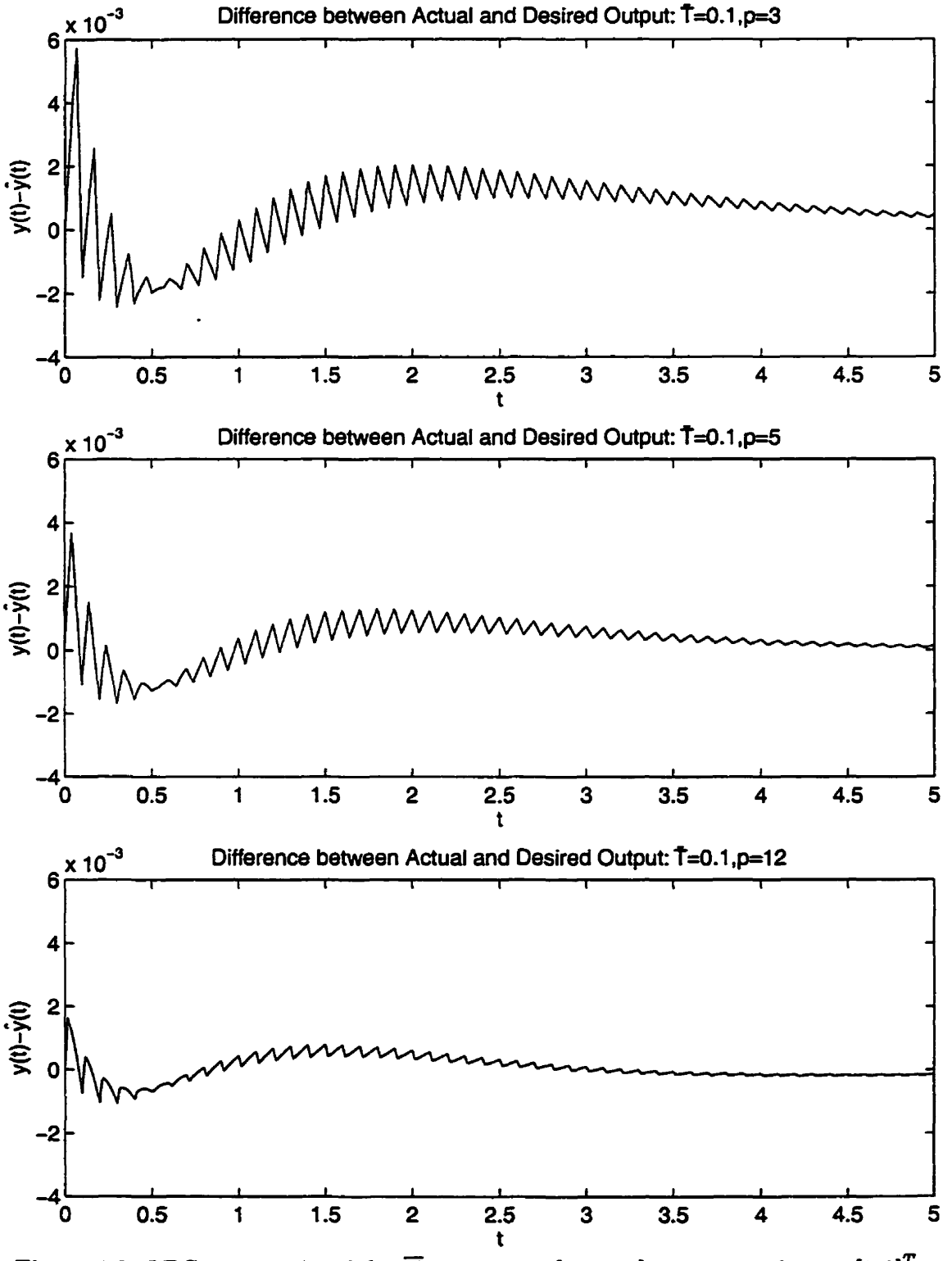


Figure 4.8: LPC output signal for $\overline{T} = 0.1$, $p \in \{3, 5, 12\}$, $\gamma = 1$, and $x_0 = [1\ 0]^T$.

Chapter 5

Low Order Model Reference Control

5.1 Introduction

Up to this point, our objective has been to design a low order LPC that provides robust stabilization. In this chapter, we turn our attention to the tracking and disturbance rejection problem, where our goal is to make the plant output track a prespecified reference signal from a class of reference signals, and to attenuate disturbances from another class of signals. The class of reference signals that we consider is the set of all possible outputs of a prespecified stable LTI *reference model* in response to a signal in $\mathscr{L}_\infty$. The reference model is chosen by the control system designer to embody a desired behaviour, so by forcing the plant output to track the reference model output, we are making the plant behave like the reference model. This problem is commonly referred to as the Model Reference Control Problem (MRCP).

Specifically, we will consider the use of a static GSHF controller to provide near exact tracking and disturbance rejection when the strictly proper SISO LTI plant is minimum phase and the strictly proper SISO LTI reference model is stable. In the literature, static GSHF controllers have been used to ensure that the plant output tracks the output of a discrete-time stable LTI reference model [26, 41] but nothing is proven about the

intersample behaviour. As was seen in Chapter 3, good performance at the sample points does not imply that the intersample performance is satisfactory. Furthermore, if the plant is non-minimum phase, then there are limits to the best achievable closed loop performance [38]. Here we consider the case where the plant is minimum phase since it is well known that LTI compensators can be used to obtain near exact tracking and disturbance rejection. We will show that the same is true for the static GSHF controller.

This chapter is organized in the following manner. In Section 5.2 we formulate the MRCP and describe the set of GSHF controllers that we will consider. A brief outline of the approach that will be adopted is also presented where the overall problem is broken down into two parts. In Section 5.3 we provide the first preliminary result that deals with an "ideal" LTI system. The second preliminary result that relates the actual LTV closed loop system to the ideal LTI system is presented in Section 5.4. These preliminary results are then used in Section 5.5 where we prove that there exists a static GSHF controller that provides near exact tracking and disturbance rejection. We address some computational issues in 5.6 and provide an example in Subsection 5.6.1. Finally, in Section 5.7 we provide a summary and make some concluding remarks.

5.2 Problem Formulation

Our SISO plant P is described by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + Ew(t),\ x(0) = 0, \\ y(t) &= Cx(t), \end{aligned} \tag{5.1}$$

with $x(t) \in \mathbf{R}^n$ the plant state, $u(t) \in \mathbf{R}$ the plant input, $w(t) \in \mathbf{R}$ the disturbance, and $y(t) \in \mathbf{R}$ the plant output. The associated transfer function from u to y is given by

$$P(s) := C(sI - A)^{-1}B.$$

We assume that (A, B) is controllable, (C, A) is observable, and $P(s)$ is minimum phase; we denote the relative degree of $P(s)$ by q.

Our stable SISO *reference model* P_m is described by

$$\begin{aligned} \dot{x}_m(t) &= A_m x_m(t) + B_m u_m(t),\ x_m(0) = 0, \\ y_m(t) &= C_m x_m(t), \qquad . \end{aligned} \tag{5.2}$$

with $x_m(t) \in \mathbf{R}^{n_m}$ the reference model state, $u_m(t) \in \mathbf{R}$ the reference model input, and $y_m(t) \in \mathbf{R}$ the reference model output. The model is chosen to embody the desired behaviour of the closed loop system.

We now define the set of controllers that will be considered. With

$$e(t) := y_m(t) - y(t) \tag{5.3}$$

the tracking error, $T > 0$ the sampling period of the generalized hold, and $\overline{F} : \mathbf{R} \to \mathbf{R}$ piecewise continuous and periodic of period T, we consider the static GSHF controller

$$u(t) = \overline{F}(t)\ e(kT),\quad t \in [kT, (k+1)T),\quad k \in \mathbf{Z}^+; \tag{5.4}$$

we represent the controller by the pair $(\overline{F}, T)$.

The structure of our closed loop system is illustrated in Figure 5.1, and we denote the closed loop map from $[u_m\ w]^T$ to e as G_{LTV}.

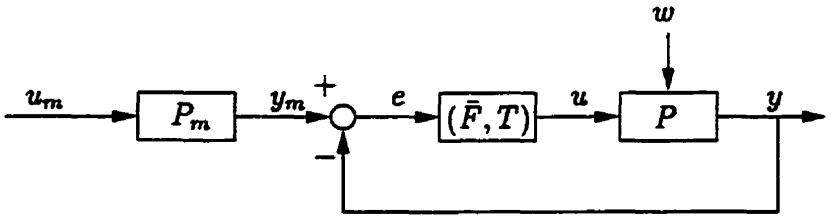


Figure 5.1: MRCP closed loop system.

Our objective can be stated as the following:

> Given the minimum phase plant P, the stable reference model P_m, and an $\varepsilon > 0$, find a static GHSF controller $(\overline{F}, T)$ so that
> $$\|G_{LTV}\|_\infty \leq \varepsilon.$$

Before continuing, let us provide some motivation for the choice of how the disturbances enter into (5.1). In practice, there is always some input noise d_1 and output

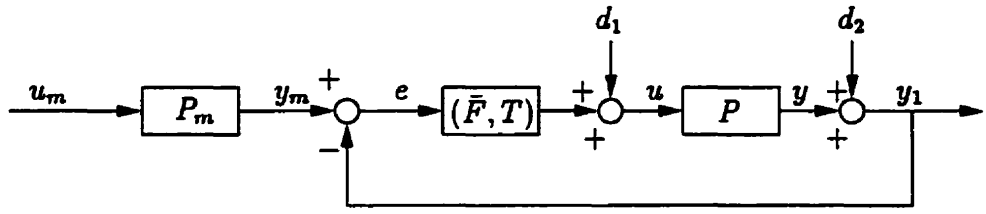


Figure 5.2: MRCP closed loop system with input and output noise.

noise d_2 entering into the closed loop system as displayed in Figure 5.2. However, since our plant is strictly proper, it is clearly impossible to reject a disturbance d_2 with step changes. Hence, we will assume that d_2 is the output of a strictly proper stable LTI system W (see Figure 5.3).

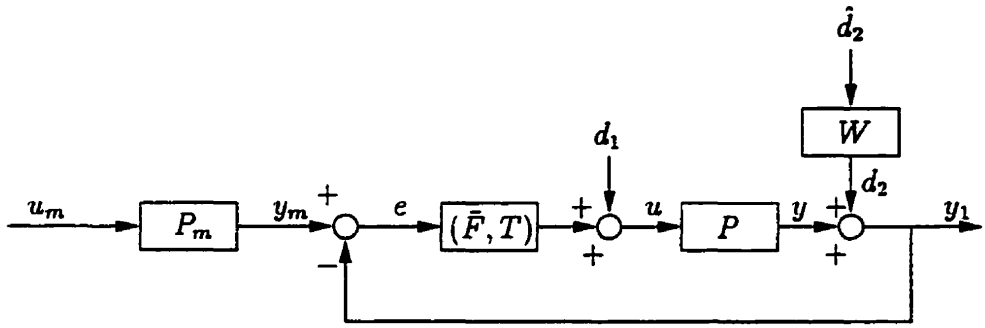


Figure 5.3: MRCP closed loop system with assumption on output noise.

If we let

$$\begin{aligned} \dot{x}_w(t) &= A_w x_w(t) + B_w \hat{d}_2(t), \\ d_2(t) &= C_w x_w(t) \end{aligned}$$

be a state space representation for W, and transform the block diagram in Figure 5.3 to the block diagram shown in Figure 5.4,

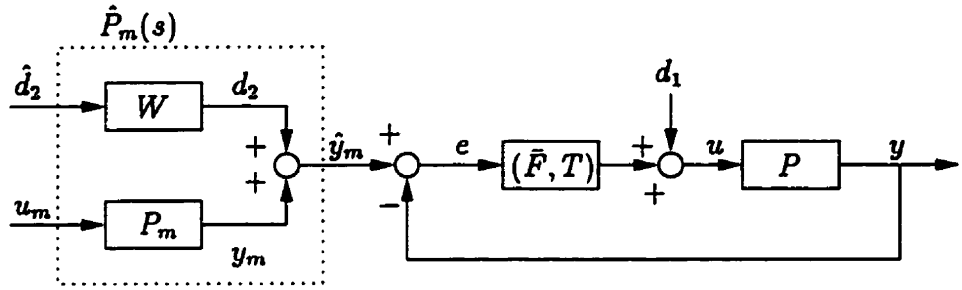


Figure 5.4: Transformed MRCP closed loop system with assumption on output noise.

then we can redefine our reference model to be

$$\begin{bmatrix} \dot{x}_m(t) \\ \dot{x}_w(t) \end{bmatrix} = \begin{bmatrix} A_m & 0 \\ 0 & A_w \end{bmatrix} \begin{bmatrix} x_m(t) \\ x_w(t) \end{bmatrix} + \begin{bmatrix} B_m & 0 \\ 0 & B_w \end{bmatrix} \begin{bmatrix} u_m(t) \\ \hat{d}_2(t) \end{bmatrix},$$

$$\hat{y}_m(t) = \begin{bmatrix} C_m & C_w \end{bmatrix} \begin{bmatrix} x_m(t) \\ x_w(t) \end{bmatrix},$$

and let

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + Id_1(t) \\ y(t) &= Cx(t). \end{aligned}$$

Hence, except for the fact that the new reference model has two inputs, our reformulated problem is of the required form. The fact that the reference model has two inputs does not present any problem in any of the results presented in this chapter, but to simplify notation we will remain with the original setup.

Before continuing, recall the following fact from [39, Lemma 1]: the minimum phase system (5.1) of relative degree q admits a state space model of the form

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} A_1 & b_1c_2 \\ b_2c_1 & A_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ b_2g \end{bmatrix} u(t) + \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} w(t),$$

$$y(t) = \begin{bmatrix} 0 & c_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{5.5}$$

with A_1 stable with eigenvalues at the zeros of $P(s)$, and with A_2, b_2, and c_2 of the form

$$\begin{aligned} c_2 &= \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} \in \mathbf{R}^{1\times q}, \\ b_2 &= \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbf{R}^q, \\ A_2 &= \begin{bmatrix} 0 & \cdots & 0 & -a_0 \\ 1 & & 0 & -a_1 \\ & \ddots & & \cdots \\ 0 & & 1 & -a_{q-1} \end{bmatrix} \in \mathbf{R}^{q\times q}. \end{aligned} \tag{5.6}$$

Without loss of generality, we will assume that our plant model is already in this form, so the plant and reference model combined are given by

$$\begin{aligned} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_m(t) \end{bmatrix} &= \begin{bmatrix} A_1 & b_1c_2 & 0 \\ b_2c_1 & A_2 & 0 \\ 0 & 0 & A_m \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_m(t) \end{bmatrix} + \begin{bmatrix} 0 \\ gb_2 \\ 0 \end{bmatrix} u(t) \\ &\quad + \begin{bmatrix} 0 \\ 0 \\ B_m \end{bmatrix} u_m(t) + \begin{bmatrix} E_1 \\ E_2 \\ 0 \end{bmatrix} w(t), \end{aligned} \tag{5.7}$$

$$e(t) = \begin{bmatrix} 0 & -c_2 & C_m \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_m(t) \end{bmatrix}. \tag{5.8}$$

To simplify the controller synthesis and analysis, we will first transform our combined system to make the error one of the state variables. To this end, define

$$\bar{b}_2 := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \mathbf{R}^q,$$

and apply the transformation

$$\begin{bmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & -\bar{b}_2 C_m \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_m(t) \end{bmatrix}$$

to yield

$$\begin{bmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \\ \dot{z}_3(t) \end{bmatrix} = \overbrace{\begin{bmatrix} A_1 & b_1 c_2 & b_1 C_m \\ b_2 c_1 & A_2 & A_2 \bar{b}_2 C_m - \bar{b}_2 C_m A_m \\ 0 & 0 & A_m \end{bmatrix}}^{=: \overline{A}} \overbrace{\begin{bmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \end{bmatrix}}^{=: z(t)} + \overbrace{\begin{bmatrix} 0 \\ g b_2 \\ 0 \end{bmatrix}}^{=: \overline{B}_1} u(t)$$

$$+ \underbrace{\begin{bmatrix} 0 \\ -\bar{b}_2 C_m B_m \\ B_m \end{bmatrix}}_{=: \overline{B}_2} u_m(t) + \underbrace{\begin{bmatrix} E_1 \\ E_2 \\ 0 \end{bmatrix}}_{=: \overline{E}} w(t). \tag{5.9}$$

$$e(t) = \underbrace{\begin{bmatrix} 0 & -c_2 & 0 \end{bmatrix}}_{=: \overline{C}} z(t). \tag{5.10}$$

Notice that from the form of c_2 that $-e$ is one of the state variables.

Before delving into the details of our results, let us first outline the approach that will be taken. The GSHF controller synthesis will be accomplished in two steps. In the first step (Section 5.3), we will define an ideal LTI system $G_{LTI} : [u_m \ w]^T \mapsto \widetilde{e}$ that is parametrized by a scalar gain k, and then show that as $k \to \infty$, the $\mathscr{L}_\infty$ induced gain of G_{LTI} goes to zero. In the second step (Section 5.4), we fix k and show that it is possible to design a static GSHF controller parametrized by T, so that when it is applied to the plant (5.9)–(5.10), the resulting LTV closed loop system $[u_m \ w]^T \mapsto e$ (see Figure 5.1) is close to the corresponding ideal LTI system when T is small.

5.3 Ideal Closed Loop System

In this section, we will define the *ideal LTI closed loop system* G_{LTI} (parametrized by the scalar k) based on (5.9)-(5.10), and show that the $\mathscr{L}_\infty$ induced gain of this ideal LTI system approaches zero as $k \to \infty$.

If we apply the control law (5.4) to the system (5.9) and let

$$F = T^{-1} \int_0^T e^{\overline{A}\tau} \overline{B}_1 \overline{F}(\tau) d\tau \in \mathbf{R}^{n+n_m}, \tag{5.11}$$

then the closed loop system state satisfies

$$z[(k+1)T] = (e^{\overline{A}T} + TF\overline{C})z[kT] + \int_0^T e^{\overline{A}(T-\tau)} [\overline{B}_2 \;\; \overline{E}] \begin{bmatrix} u_m(\tau) \\ w(\tau) \end{bmatrix} d\tau.$$

Since $(\overline{A}, \overline{B}_1)$ is not controllable, there exists an $F \in \mathbf{R}^{n+n_m}$ and $T > 0$ for which no $\overline{F}(t)$ satisfying (5.11) exists. However,

$$(A, B) = \left(\begin{bmatrix} A_1 & b_1 c_2 \\ b_2 c_1 & A_2 \end{bmatrix}, \begin{bmatrix} 0 \\ g b_2 \end{bmatrix} \right)$$

is controllable, so if we let

$$F = T^{-1} \int_0^T e^{A\tau} B \overline{F}(\tau) d\tau \in \mathbf{R}^n, \tag{5.12}$$

then it follows from the structure of $\overline{A}$, $\overline{B}_1$ and $\overline{C}$ that the closed loop system state satisfies

$$z[(k+1)T] = \left(e^{\overline{A}T} + T \begin{bmatrix} -FC & 0 \\ 0 & 0 \end{bmatrix} \right) z[kT] + \int_0^T e^{\overline{A}(T-\tau)} [\overline{B}_2 \;\; \overline{E}] \begin{bmatrix} u_m(\tau) \\ w(\tau) \end{bmatrix} d\tau;$$

since (A, B) is controllable, for every $F \in \mathbf{R}^n$ and $T > 0$ there exists an $\overline{F}(t)$ satisfying (5.12). Since A_1 is stable, we will set the first $n - q$ elements of F to zero and only use the last q elements: with $F = -[0 \;\; \widetilde{F}^T]^T$, we have

$$\begin{bmatrix} -FC & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \widetilde{F} c_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

With this in mind, we define the ideal LTI closed loop system $G_{LTI}(\widetilde{F}) : (u_m, w) \mapsto \widetilde{e}$ by (5.9)–(5.10) with the (2,2) element adjusted:

$$\begin{aligned}\dot{\widetilde{z}}(t) &= \overbrace{\begin{bmatrix} A_1 & b_1c_2 & b_1C_m \\ b_2c_1 & A_2+\widetilde{F}c_2 & A_2\bar{b}_2C_m - \bar{b}_2C_mA_m \\ 0 & 0 & A_m \end{bmatrix}}^{=:\widetilde{A}} \widetilde{z}(t) \\ &\quad + \underbrace{\begin{bmatrix} 0 \\ -\bar{b}_2C_mB_m \\ B_m \end{bmatrix}}_{=\overline{B}_2} u_m(t) + \underbrace{\begin{bmatrix} E_1 \\ E_2 \\ 0 \end{bmatrix}}_{=\overline{E}} w(t), \qquad (5.13) \\ \widetilde{e}(t) &= \underbrace{\begin{bmatrix} 0 & -c_2 & 0 \end{bmatrix}}_{=\overline{C}} \widetilde{z}(t); \qquad (5.14)\end{aligned}$$

we label the output as $\widetilde{e}$ to distinguish it from the error signal produced when the GSHF controller is applied.

Since the plant is minimum phase and the reference model is stable, we know that both A_1 and A_m are stable. So let us turn our attention to the (2,2) element in $\widetilde{A}$. Specifically, we will show that we can make $\|G_{LTI}(\widetilde{F})\|_\infty$ small by choosing $\widetilde{F}$ so that the (2,2) element in $\widetilde{A}$ has very "fast" eigenvalues. To this end, define

$$\eta(k,s) := (s+k)(s+2k)\cdots(s+qk) = \sum_{i=0}^{q} \eta_i k^{q-i} s^i,$$

and set

$$F_k := -\begin{bmatrix} \eta_0 k^q - a_0 \\ \vdots \\ \eta_{q-1}k - a_{q-1} \end{bmatrix}. \qquad (5.15)$$

Then

$$A_2 + F_k c_2 = \begin{bmatrix} 0 & & 0 & -\eta_0 k^q \\ 1 & & 0 & -\eta_1 k^{q-1} \\ 0 & \ddots & \vdots & \vdots \\ 0 & & 1 & -\eta_{q-1} k \end{bmatrix}$$

$$\Rightarrow \mathrm{sp}(A_2 + F_k c_2) = \{-k, -2k, \cdots, -qk\}.$$

We are now ready to present the result needed to perform the first main step in our GHSF controller synthesis.

Proposition 5.1 For every $\epsilon > 0$, there exists a $k_{min} > 0$ so that for every $k \geq k_{min}$, there exists a $F_k \in \mathbf{R}^q$ so that $\|G_{LTI}(F_k)\|_\infty \leq \epsilon$.

Proof:

We first apply a similarity transformation to our ideal LTI system (5.13)–(5.14). Let

$$M_k := \begin{bmatrix} \frac{1}{k^{q-1}} & & & \\ & \ddots & & \\ & & \frac{1}{k} & \\ & & & 1 \end{bmatrix}, \quad M := \begin{bmatrix} I_{n-q} & 0 & 0 \\ 0 & M_k & 0 \\ 0 & 0 & I_{n_m} \end{bmatrix},$$

and set

$$\widehat{z}(t) = M\widetilde{z}(t).$$

Then with the notation

$$\Lambda := \begin{bmatrix} 0 & & 0 & -\eta_0 \\ 1 & & 0 & -\eta_1 \\ 0 & \ddots & \cdots & \vdots \\ 0 & & 1 & -\eta_{q-1} \end{bmatrix},$$

it can be shown that

$$\begin{aligned}\dot{\widehat{z}}(t) &= \begin{bmatrix} A_1 & b_1c_2 & b_1C_m \\ \frac{1}{k^{q-1}}b_2c_1 & k\Lambda & M_kA_2\bar{b}_2C_m - \bar{b}_2C_mA_m \\ 0 & 0 & A_m \end{bmatrix}\widehat{z}(t) \\ &\quad + \begin{bmatrix} 0 \\ -\bar{b}_2C_mB_m \\ B_m \end{bmatrix} u_m(t) + \begin{bmatrix} E_1 \\ M_kE_2 \\ 0 \end{bmatrix} w(t), \qquad (5.16)\\ \widetilde{e}(t) &= \begin{bmatrix} 0 & -c_2 & 0 \end{bmatrix}\widehat{z}(t). \qquad (5.17)\end{aligned}$$

Since Λ, A_1, and A_m are stable, it is easy to prove that the above system is stable for large $k > 0$ (a Lyapunov-type argument would work). Proving error regulation is not as straightforward. We begin by partitioning $\widehat{z}(t)$ into $[\widehat{z}_1(t)^T\ \widehat{z}_2(t)^T\ \widehat{z}_3(t)^T]^T$ and taking the Laplace transform of (5.16)–(5.17) to get

$$\begin{aligned}\widehat{z}_1(s) &= (sI - A_1)^{-1}(b_1c_2\widehat{z}_2(s) + b_1C_m\widehat{z}_3(s) + E_1w(s)), \\ \widehat{z}_2(s) &= (sI - k\Lambda)^{-1}\left[\tfrac{1}{k^{q-1}}b_2c_1\widehat{z}_1(s) + (M_kA_2\bar{b}_2C_m - \bar{b}_2C_mA_m)\widehat{z}_3(s)\right. \\ &\quad \left. -\bar{b}_2C_mB_mu_m(s) + M_kE_2w(s)\right], \\ \widehat{z}_3(s) &= (sI - A_m)^{-1}B_mu_m(s), \\ \widetilde{e}(s) &= -c_2\widehat{z}_2(s).\end{aligned}$$

From these equations, we can construct the block diagram illustrated in Figure 5.5.

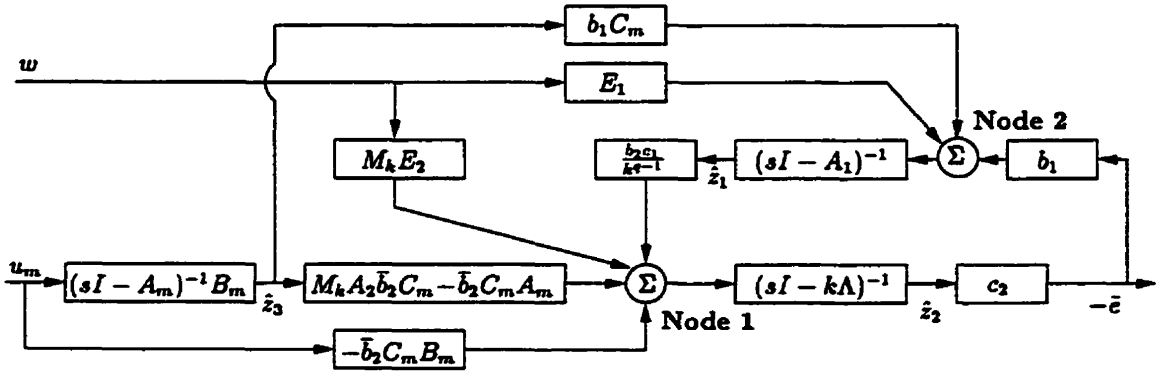


Figure 5.5: Ideal LTI closed loop system block diagram.

By "shifting" **Node 2** to **Node 1**, we can redraw the block diagram in Figure 5.5 so that we have a feedforward part and a feedback part, as illustrated in Figure 5.6.

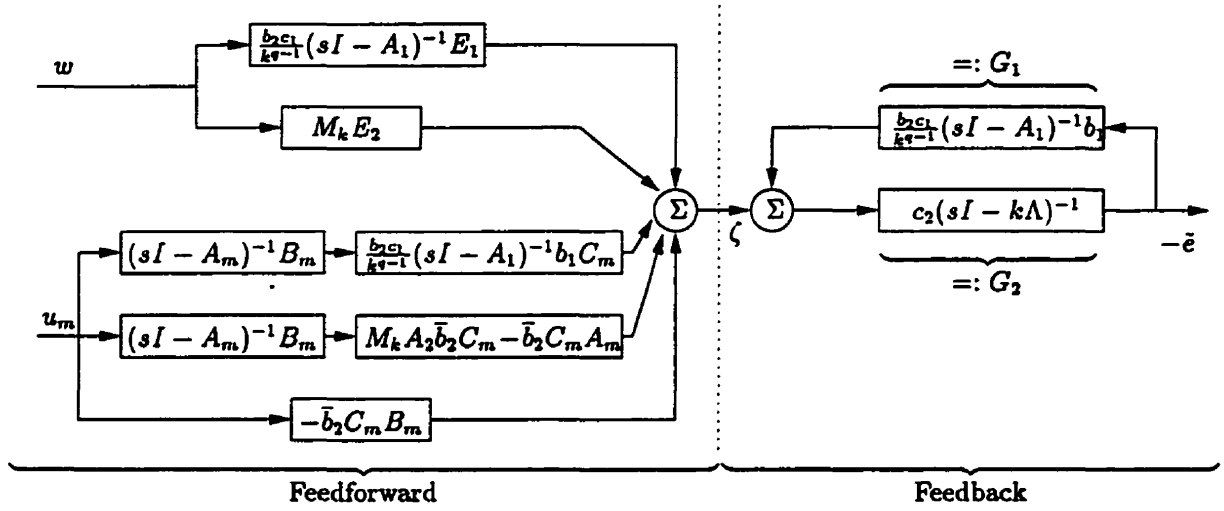


Figure 5.6: Transformed ideal LTI closed loop system block diagram.

Let us look at the feedforward part first. Define the intermediate signal ζ as illustrated in Figure 5.6. Using the fact that A_m and A_1 are stable, and the definition of M_k, it can be shown that there exists positive constants β_0 and k_0 so that for $k \geq k_0$, we have

$$\|\zeta\|_\infty \leq \beta_0 \|[u_m \; w]^T\|_\infty. \tag{5.18}$$

For the feedback part, we begin by defining G_1 and G_2 as illustrated in Figure 5.6. Since A_1 is stable, we know that there exists a $\beta_1 > 0$ so that

$$\|G_1\|_\infty \leq \tfrac{1}{k^{q-1}}\beta_1. \tag{5.19}$$

We now show that there exists a constant $\beta_2 > 0$ so that

$$\|G_2\|_\infty \leq \tfrac{1}{k}\beta_2. \tag{5.20}$$

To see this, first observe that $g_2 := \mathcal{L}^{-1}\{G_2\}$ is given by

$$g_2(t) = c_2 e^{k\Lambda t}.$$

Now there exists an $\alpha > 0$ and $\lambda < 0$ such that

$$\|e^{\Lambda t}\| \leq \alpha e^{\lambda t}.$$

Hence,

$$\begin{aligned}\|G_2\|_\infty &= \|g_2\|_1 \\ &= \int_0^\infty \|c_2 e^{kAt}\| dt \\ &\leq \int_0^\infty \|c_2\| \alpha e^{\lambda k t} dt \\ &\leq \frac{\alpha\|c_2\|}{|\lambda|}\frac{1}{k}.\end{aligned}$$

Therefore, from Figure 5.6 and (5.19)–(5.20), it follows that

$$\begin{aligned}\|\widetilde{e}\|_\infty &\leq \|G_2\|_\infty (\|G_1\|_\infty \times \|\widetilde{e}\|_\infty + \|\zeta\|_\infty) \\ &\leq \frac{\beta_2}{k}\left(\frac{\beta_1}{k^{q-1}}\|\widetilde{e}\|_\infty + \|\zeta\|_\infty\right), \\ \Rightarrow \left(1 - \frac{\beta_1\beta_2}{k^q}\right)\|\widetilde{e}\|_\infty &\leq \frac{\beta_2}{k}\|\zeta\|_\infty,\end{aligned}$$

so for $k > (\beta_1\beta_2)^{1/q} =: k_1$, we have

$$\|\widetilde{e}\|_\infty \leq \frac{\beta_2}{k}\frac{1}{\left(1 - \frac{\beta_1\beta_2}{k^q}\right)}\|\zeta\|_\infty. \tag{5.21}$$

Finally, by combining (5.21) and (5.18), it follows that for $k > \max\{k_0, k_1\}$ we have

$$\|\widetilde{e}\|_\infty \leq \left(\frac{\beta_2}{k}\frac{1}{\left(1 - \frac{\beta_1\beta_2}{k^q}\right)}\right)\beta_0\|[u_m \; w]^T\|_\infty,$$

so our result follows. ∎

5.4 Low Order GSH Controller

In this section, we will fix k large enough so that the ideal LTI system is stable with an acceptable level of performance, and then show that it is possible to design a static GSHF controller $(\overline{F}, T)$ so that when it is applied to (5.9)–(5.10), the resulting LTV closed loop system performance approaches that of the ideal LTI closed loop system as $T \to 0$.

Let us begin by designing the hold function. We assume that k has been chosen large enough so that with F_k satisfying (5.15), the matrix $\widetilde{A}$ defined in (5.13) is stable; by

Proposition 5.1, we know such a k exists. Now choose an $\overline{F}(t)$ satisfying

$$\int_0^T e^{A(T-\tau)} B\overline{F}(\tau)d\tau = \begin{bmatrix} 0 \\ -TF_k \end{bmatrix}. \tag{5.22}$$

We know that since (A, B) is controllable, such an $\overline{F}(t)$ exists. Specifically, we will choose $\overline{F}(t)$ to be a piecewise constant function taking on n values over each sample period. Namely, with $\overline{T} = T/n$, we consider hold functions of the form

$$\overline{F}(t) = f_i, t \in [i\overline{T}, (i+1)\overline{T}); \tag{5.23}$$

we represent this specific GSHF controller by the pair (f_i, T). Recall that the corresponding map $(u_m, w) \mapsto e$ was denoted by G_{LTV}; to make the dependence of the map on T and F_k explicit, we write $G_{LTV}(T, F_k)$. From Section 3.4, we know that for every $F_k \in \mathbf{R}^q$ and for sufficiently small T, there exists constants f_i satisfying (5.23), and that the resulting GSHF controller can be implemented with a sampler, a first order discrete-time n-periodic compensator, and a zero-order-hold. When examining the intersample behaviour of our GSHF controller later on, it will be necessary to know what happens to the magnitude of the gains f_i as the sampling period T tends to zero. Hence the following result:

Lemma 5.1 Given the plant (5.5) and a matrix $F_k \in \mathbf{R}^q$, if $\overline{F}(t)$ of the form (5.23) satisfies (5.22), then $f_i = \mathcal{O}(T^{1-q})$.

Proof:

Since the piecewise constant $\overline{F}(t)$ satisfies (5.22), it follows from Proposition 3.3 that with Ψ given by (3.25), we have

$$\begin{bmatrix} f_0 \\ \vdots \\ f_{n-1} \end{bmatrix} = n\Psi^{-1} \begin{bmatrix} \frac{1}{T^{n-1}} & & \\ & \ddots & \\ & & \frac{1}{T} \\ & & & 1 \end{bmatrix} \begin{bmatrix} A^{n-1}B & \cdots & B \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -F_k \end{bmatrix} + \mathcal{O}(T). \tag{5.24}$$

But here A and B have the specific form given in (5.5)–(5.6), so it follows that with

$$J := \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix} \in \mathbf{R}^{q \times q}$$

there exists a $\Lambda_1 \in \mathbf{R}^{(n-q)\times(n-q)}$ and $\Lambda_2 \in \mathbf{R}^{q\times(n-q)}$ so that

$$\begin{bmatrix} A^{n-1}B & \cdots & B \end{bmatrix} = \begin{bmatrix} \Lambda_1 & 0 \\ \Lambda_2 & gJ \end{bmatrix}.$$

Since (A, B) is controllable, we know that $[A^{n-1}B \ \cdots \ B]$ is invertible, so it follows that

$$\begin{bmatrix} A^{n-1}B & \cdots & B \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -F_k \end{bmatrix} = \begin{bmatrix} 0 \\ -g^{-1}JF_k \end{bmatrix} \quad \text{(Note } J = J^{-1}\text{)}.$$

Substituting this into (5.24), it follows that

$$\begin{bmatrix} f_0 \\ \vdots \\ f_{n-1} \end{bmatrix} = -n\Psi^{-1} \begin{bmatrix} & 0 & & \\ \frac{1}{T^{q-1}} & & & \\ & \ddots & & \\ & & \frac{1}{T} & \\ & & & 1 \end{bmatrix} g^{-1}JF_k + \mathcal{O}(T),$$

and our result follows. ∎

Let us now turn to the second main step in the synthesis of our GSHF controller, namely that of showing that the closed loop LTV system behaves (from an input-output point of view) like the ideal LTI closed loop system for small T. If we apply our GSHF controller to the actual system (5.9)–(5.10), then it can be shown that

$$z[(k+1)T] = \overbrace{\left(e^{\overline{A}T} + T \begin{bmatrix} 0 \\ -F_k \\ 0 \end{bmatrix} \overline{C} \right)}^{=: \Phi} z[kT] + \int_{kT}^{(k+1)T} e^{\overline{A}[(k+1)T-\tau]} [\overline{B}_2 \ \overline{E}] \begin{bmatrix} u_m(\tau) \\ w(\tau) \end{bmatrix} d\tau, \tag{5.25}$$

while from (5.13)-(5.14) it follows that

$$\widetilde{z}[(k+1)T] = \mathrm{e}^{\widetilde{A}T}\widetilde{z}[kT] + \int_{kT}^{(k+1)T} \mathrm{e}^{\widetilde{A}[(k+1)T-\tau]}[\overline{B}_2 \ \overline{E}]\begin{bmatrix} u_m(\tau) \\ w(\tau) \end{bmatrix} d\tau. \tag{5.26}$$

Our key observation here is that for small T, we have

$$\begin{aligned}
\Phi &\approx (I + T\overline{A}) + \begin{bmatrix} 0 \\ -TF_k \\ 0 \end{bmatrix} \overline{C} \\
&= I + T\left(\begin{bmatrix} A_1 & b_1c_2 & b_1C_m \\ b_2c_1 & A_2 & A_2\bar{b}_2C_m - \bar{b}_2C_mA_m \\ 0 & 0 & A_m \end{bmatrix} + \begin{bmatrix} 0 \\ -F_k \\ 0 \end{bmatrix} \begin{bmatrix} 0 & -c_2 & 0 \end{bmatrix} \right) \\
&= I + T\widetilde{A} \\
&\approx \mathrm{e}^{\widetilde{A}T}.
\end{aligned}$$

Hence, one might expect that for small T, $z[kT]$ is close to $\widetilde{z}[kT]$, which means that for small T, $e[kT]$ is close to $\widetilde{e}[kT]$. Showing that the intersample response is also well behaved is more complicated and will require the following preliminary result.

Lemma 5.2 If $\widetilde{A}$ is stable, then there exists an $\alpha > 0$, $\lambda < 0$ and a $T_0 > 0$ so that

$$\|\Phi^k\| \leq \alpha \left(\mathrm{e}^{\lambda T}\right)^k, \quad T \in (0, T_0), \ k \in \mathbf{Z}^+.$$

Proof:

Let

$$\Delta_1(T) := \Phi - (I + \widetilde{A}T),$$

and consider the difference equation

$$w(k+1) = \Phi w(k) = (I + \widetilde{A}T + \Delta_1(T))w(k), \quad w(0) = w_0.$$

Since $\widetilde{A}$ is stable, it follows from [55, Lemma 12.1] that there exists a unique positive definite matrix P satisfying

$$\widetilde{A}^T P + P\widetilde{A} + I = 0, \tag{5.27}$$

so consider the Lyapunov candidate function

$$V[w(k)] := w(k)^T P w(k).$$

Using the fact that $\Delta_1(T) = \mathcal{O}(T^2)$, it can easily be shown that there exists a $\Delta_2(T) = \mathcal{O}(T)$ so that

$$V[w(k+1)] - V[w(k)] = Tw(k)^T[(\widetilde{A}^T P + P\widetilde{A}) + \Delta_2(T)]w(k),$$

so from (5.27) it follows that

$$V[w(k+1)] - V[w(k)] = Tw(k)^T(-I + \Delta_2(T))w(k).$$

Since $\Delta_2(T) = \mathcal{O}(T)$, it follows that there exists a $T_0 > 0$ so that

$$\|\Delta_2(T)\| \leq \tfrac{1}{2}, \quad T \in (0, T_0),$$

which means that for $T \in (0, T_0)$ we have

$$V[w(k+1)] - V[w(k)] \leq -\tfrac{T}{2} w(k)^T w(k).$$

But

$$V[w(k)] \leq \lambda_{max}(P) w(k)^T w(k), \quad \Rightarrow \quad -w(k)^T w(k) \leq -\tfrac{1}{\lambda_{max}(P)} V[w(k)]$$

so for $T \in (0, T_0)$ we have

$$V[w(k+1)] - V[w(k)] \leq -T \underbrace{\frac{1}{2\lambda_{max}(P)}}_{=: -2\lambda} V[w(k)]$$

$$\Rightarrow V[w(k+1)] \leq (1 + 2\lambda T) V[w(k)]$$

$$\Rightarrow V[w(k+1)] \leq \mathrm{e}^{2\lambda T} V[w(k)]$$

$$\Rightarrow V[w(k)] \leq \left(\mathrm{e}^{2\lambda T}\right)^k V[w_0]$$

$$\Rightarrow \|w(k)\|^2 \leq \underbrace{\frac{\lambda_{max}(P)}{\lambda_{min}(P)}}_{=: \alpha^2} \left(\mathrm{e}^{2\lambda T}\right)^k \|w_0\|^2$$

$$\Rightarrow \|\Phi^k w_0\|^2 \leq \alpha^2 \left(\mathrm{e}^{2\lambda T}\right)^k \|w_0\|^2$$

$$\Rightarrow \|\Phi^k\| \leq \alpha \left(\mathrm{e}^{\lambda T}\right)^k.$$

■

We can now use Lemma 5.2 to show the following result.

Proposition 5.2 If k is chosen such that the ideal LTI system (5.13)–(5.14) is stable, then for every $\varepsilon > 0$, there exists a $T_{max} > 0$ so that for every $T \in (0, T_{max})$, the GSHF controller (f_i, T) ensures that

$$\|G_{LTI}(F_k) - G_{LTV}(T, F_k)\|_\infty \leq \varepsilon.$$

Proof:

This proof will be broken down into three parts. In the first part, we use the fact that $\widetilde{A}$ is stable and $\Phi \approx e^{\widetilde{A}T}$ for small T to show $z[kT]$ is bounded for small T. Then this is used in the second part to show that $z[kT]$ approaches $\widetilde{z}[kT]$ uniformly as $T \to 0$, which means that $e[kT] \to \widetilde{e}[kT]$ uniformly as $T \to 0$. Finally, we will combine this with Lemma 5.1 to show that for sufficiently small T, $\widetilde{e} - e$ is small between samples as well.

Since $\widetilde{A}$ is stable, it follows from (5.13) that there exists an $\alpha_0 > 0$ so that

$$\|\widetilde{z}\|_\infty \leq \alpha_0 \|[u_m \; w]^T\|_\infty. \tag{5.28}$$

Furthermore, using Lemma 5.2, it follows that there exists an $\alpha_1 > 0$, $\lambda_0 < 0$, and a $T_0 > 0$ so that for every $T \in (0, T_0)$ we have

$$\|\bar{\Phi}^k\| \leq \alpha_1 \left(e^{\lambda_0 T}\right)^k.$$

Combining this with the fact that

$$\int_{kT}^{(k+1)T} e^{\overline{A}[(k+1)T-\tau]}[\overline{B}_2 \; \overline{E}] \begin{bmatrix} u_m(\tau) \\ w(\tau) \end{bmatrix} d\tau = \mathcal{O}(T) \left\| \begin{bmatrix} u_m \\ w \end{bmatrix} \right\|_\infty,$$

it follows from (5.25) that there exists an $\alpha_2 > 0$ and $T_1 \in (0, T_0)$ so that for every $T \in (0, T_1)$ we have

$$\|z[kT]\| \leq \sum_{i=0}^{k-1} \alpha_2 T \left(e^{\lambda_0 T}\right)^{k-1-i} \|[u_m \; w]^T\|_\infty,$$

$$\Rightarrow \sup_{k \in \mathbb{Z}^+} \|z[kT]\| \leq \alpha_2 \frac{T}{1 - e^{\lambda_0 T}} \|[u_m \; w]^T\|_\infty.$$

Since

$$\lim_{T \to 0} \frac{T}{1 - e^{\lambda_0 T}} = \frac{1}{|\lambda_0|},$$

we know that there exists a $T_2 \in (0, T_1)$ and an $\alpha_3 > 0$ so that

$$\sup_{k\in\mathbb{Z}^+} \|z[kT]\| \leq \alpha_3 \|[u_m \; w]^T\|_\infty, \quad T \in (0, T_2). \tag{5.29}$$

Now let's look at the difference equation for

$$\xi := \widetilde{z} - z$$

at $t = 0, T, 2T, \cdots$. Using (5.25) and (5.26), it follows that

$$\begin{aligned}\xi[(k+1)T] &= e^{\tilde{A}T}\xi[kT] + \left(e^{\tilde{A}T} - \Phi\right) z[kT] \\ &\quad + \int_{kT}^{(k+1)T} \left(e^{\tilde{A}[(k+1)T-\tau]} - e^{\overline{A}[(k+1)T-\tau]}\right) [\overline{B}_2 \; \overline{E}] \begin{bmatrix} u_m(\tau) \\ w(\tau) \end{bmatrix} d\tau. \end{aligned} \tag{5.30}$$

Since $\widetilde{A}$ is stable, there exists an $\alpha_4 > 0$ and a $\lambda_1 < 0$ so that

$$\|(e^{\tilde{A}T})^k\| \leq \alpha_4 \left(e^{\lambda_1 T}\right)^k.$$

If we combine this with

$$\int_{kT}^{(k+1)T} \left(e^{\tilde{A}[(k+1)T-\tau]} - e^{\overline{A}[(k+1)T-\tau]}\right) [\overline{B}_2 \; \overline{E}] \begin{bmatrix} u_m(\tau) \\ w(\tau) \end{bmatrix} d\tau = \mathcal{O}(T^2) \left\| \begin{bmatrix} u_m \\ w \end{bmatrix} \right\|_\infty,$$

use (5.29) to obtain a bound on $z[kT]$, and use the fact that $e^{\tilde{A}T} - \Phi = \mathcal{O}(T^2)$, it follows that there exists a $T_3 \in (0, T_2)$ and an $\alpha_5 > 0$ so that for every $T \in (0, T_3)$ we have

$$\begin{aligned} &\|\xi[kT]\| \leq \sum_{i=0}^{k-1} (\alpha_5 T^2)\alpha_4 \left(e^{\lambda_1 T}\right)^{k-1-i} \|[u_m \; w]^T\|_\infty \\ &\Rightarrow \sup_{k\in\mathbb{Z}^+} \|\xi[kT]\| \leq \alpha_4\alpha_5 \tfrac{T^2}{1-e^{\lambda_1 T}} \|[u_m \; w]^T\|_\infty \\ &\Rightarrow \sup_{k\in\mathbb{Z}^+} \|\widetilde{e}[kT] - e[kT]\| \leq \alpha_4\alpha_5 \tfrac{T^2}{1-e^{\lambda_1 T}} \|\overline{C}\| \times \|[u_m \; w]^T\|_\infty. \end{aligned} \tag{5.31}$$

Since

$$\lim_{T\to 0} \tfrac{T^2}{1-e^{\lambda_1 T}} = 0,$$

we have that $\widetilde{e} - e$ can be made small at $t = 0, T, 2T, \cdots$ by letting $T \to 0$.

We now begin the final step. With $\tau \in [0, T)$, it follows from (5.13)–(5.14) that

$$\widetilde{e}(kT + \tau) = \overline{C}e^{\tilde{A}\tau}\widetilde{z}[kT] + \int_{kT}^{kT+\tau} \overline{C}e^{\tilde{A}(kT+\tau-\theta)} [\overline{B}_2 \; \overline{E}] \begin{bmatrix} u_m(\theta) \\ w(\theta) \end{bmatrix} d\theta. \tag{5.32}$$

But $e^{\widetilde{A}\tau} = I + \mathcal{O}(\tau)$, which means

$$\int_{kT}^{kT+\tau} \overline{C} e^{\widetilde{A}(kT+\tau-\theta)} [\overline{B}_2 \ \overline{E}] \begin{bmatrix} u_m(\theta) \\ w(\theta) \end{bmatrix} d\theta = \mathcal{O}(T) \left\| \begin{bmatrix} u_m \\ w \end{bmatrix} \right\|_\infty,$$

so along with the bound on $\widetilde{z}$ given in (5.28), it can be shown that there exists an $\alpha_6 > 0$ and a $T_4 \in (0, T_3)$ so that for $T \in (0, T_4)$, we have

$$\begin{aligned} &|\widetilde{e}(kT+\tau) - \widetilde{e}(kT)| \\ &\leq |\overline{C}(e^{\widetilde{A}\tau} - I)\widetilde{z}[kT]| + \left| \int_{kT}^{kT+\tau} \overline{C} e^{\widetilde{A}(kT+\tau-\theta)} [\overline{B}_2 \ \overline{E}] \begin{bmatrix} u_m(\theta) \\ w(\theta) \end{bmatrix} d\theta \right| \\ &\leq \alpha_6 T \| [u_m \ w]^T \|_\infty, \quad \tau \in [0, T), \ k \in \mathbf{Z}^+. \end{aligned} \tag{5.33}$$

When we apply the GSHF controller to the plant (5.9)-(5.10), we have

$$\begin{aligned} e(kT+\tau) &= \overline{C} e^{\overline{A}\tau} z[kT] + \int_{kT}^{kT+\tau} \overline{C} e^{\overline{A}(kT+\tau-\theta)} [\overline{B}_2 \ \overline{E}] \begin{bmatrix} u_m(\theta) \\ w(\theta) \end{bmatrix} d\theta \\ &\quad + \left[\int_{kT}^{kT+\tau} \overline{C} e^{\overline{A}(kT+\tau-\theta)} \overline{B}_1 \overline{F}(\theta) d\theta \right] \overline{C} z[kT]. \end{aligned} \tag{5.34}$$

But since the plant has relative degree q, it follows that

$$\begin{aligned} CA^i B &= 0 \qquad\qquad i = 0, 1, \cdots, q-2 \\ \Rightarrow \ \overline{C} \, \overline{A}^i \, \overline{B}_1 &= 0 \qquad\qquad i = 0, 1, \cdots, q-2 \\ \Rightarrow \ \overline{C} e^{\overline{A}\theta} \overline{B}_1 &= \textstyle\sum_{i=q-1}^{\infty} \overline{C} \, \overline{A}^i \, \overline{B}_1 \frac{\theta^i}{i!} = \mathcal{O}(\theta^{q-1}), \end{aligned}$$

which means that, along with Lemma 5.1, we have

$$\begin{aligned} \left\| \int_{kT}^{kT+\tau} \overline{C} e^{\overline{A}(kT+\tau-\theta)} \overline{B}_1 \overline{F}(\theta) d\theta \right\| &\leq \int_0^T \| \overline{C} e^{\overline{A}\theta} \overline{B}_1 \| d\theta \ \ \| \overline{F} \|_\infty \\ &= \mathcal{O}(T^q) \| \overline{F} \|_\infty \\ &= \mathcal{O}(T^q) \mathcal{O}(T^{1-q}) = \mathcal{O}(T). \end{aligned} \tag{5.35}$$

Combining this with the bound on $z[kT]$ given in (5.29), the fact that $e^{\widetilde{A}\tau} = I + \mathcal{O}(\tau)$, and

$$\int_{kT}^{kT+\tau} \overline{C} e^{\overline{A}(kT+\tau-\theta)} [\overline{B}_2 \ \overline{E}] \begin{bmatrix} u_m(\theta) \\ w(\theta) \end{bmatrix} d\theta = \mathcal{O}(T) \left\| \begin{bmatrix} u_m \\ w \end{bmatrix} \right\|_\infty,$$

it follows that there exists an $\alpha_7 > 0$ and a $T_5 \in (0, T_4)$ so that for $T \in (0, T_5)$, we have

$$|e(kT+\tau) - e(kT)| \leq \alpha_7 T \| [u_m \ w]^T \|_\infty, \quad \tau \in [0, T), k \in \mathbf{Z}^+. \tag{5.36}$$

Finally, combining (5.31), (5.33) and (5.36), it follows that for every $T \in (0, T_5)$, $\tau \in [0, T)$, and $k \in \mathbf{Z}^+$, we have

$$|\widetilde{e}(kT+\tau) - e(kT+\tau)| \leq \left(\alpha_4\alpha_5\|\overline{C}\|\frac{T^2}{1-e^{\lambda_1 T}} + \alpha_6 T + \alpha_7 T\right)\|[u_m \; w]^T\|_\infty,$$

so our desired result follows. ∎

5.5 Main Result

We can now summarize our results in the following Theorem.

Theorem 5.1 Given a minimum phase plant P and a stable reference model P_m, for every $\epsilon > 0$, there exists a static GSHF controller that ensures

$$\|G_{LTV}(T, F_k)\|_\infty \leq \epsilon.$$

Proof:

By Proposition 5.1, we know that there exists a k so that with F_k satisfying (5.15), we have

$$\|\widetilde{e}\|_\infty \leq \tfrac{\epsilon}{2}\|[u_m \; w]^T\|_\infty.$$

Using the results of Section 3.4, we know that there exists a $T_0 > 0$ so that for every $T \in (0, T_0)$, there exists a $\overline{F}(t)$ of the form (5.23) that satisfies (5.22); this hold function is associated with the GSHF controller (f_i, T). From Proposition 5.2, we know that there exists a $T_{max} \in (0, T_0)$ so that for every $T \in (0, T_{max})$, the GSHF controller (f_i, T) ensures

$$\|\widetilde{e} - e\|_\infty \leq \tfrac{\epsilon}{2}\|[u_m \; w]^T\|_\infty.$$

Hence, for every $T \in (0, T_{max})$, the GSHF controller (f_i, T) ensures that

$$\|e\|_\infty \leq \|\widetilde{e} - e\|_\infty + \|\widetilde{e}\|_\infty \leq \epsilon\|[u_m \; w]^T\|_\infty.$$

∎

5.6 Computational Issues

In this section, we address some computational issues related to the synthesis of our controller and provide an example to demonstrate the design procedure.

Suppose that with $\epsilon > 0$, we wish to design a GSHF controller so that

$$\|G_{LTV}(T, F_k)\|_\infty \leq \epsilon.$$

We will construct the controller in two stages. In the first stage, we determine a large enough value of k, and the corresponding gain F_k, so that the $\mathscr{L}_\infty$ induced gain of the ideal LTI system

$$\|G_{LTI}(F_k)\|_\infty < \tfrac{\epsilon}{2}.$$

Since $G_{LTI}(F_k)$ is an LTI continuous-time system, this step is not difficult. We then fix F_k and note from Proposition 5.2 that

$$\|G_{LTI}(F_k) - G_{LTV}(T, F_k)\|_\infty \to 0 \tag{5.37}$$

as $T \to 0$. Hence, in the second stage we find a sufficiently small T so that with the fixed gain F_k, we have

$$\|G_{LTV}(T, F_k)\|_\infty \leq \epsilon.$$

Computing $\|G_{LTV}(T, F_k)\|_\infty$ is not trivial. Results related to computing the $\mathscr{L}_\infty$ induced gain for sampled-data systems have been obtained for the case where the plant is a continuous-time LTI system and the controller is composed of a sampler, a zero-order-hold, and a discrete-time LTI system. For example, in [8, 17, 6] a continuous-time lifting technique is used to calculate an approximation of the $\mathscr{L}_\infty$ induced gain, while in [47] an explicit formula for the induced gain is given. Since the discrete-time component of our controller is time-varying, we can not directly apply these results, but by modifying the argument of [47], we will be able to derive a formula for $\|G_{LTV}(T, F_k)\|_\infty$.

To this end, if we define the LTI map

$$\widehat{G}_{LTI}: \left[\begin{bmatrix} u_m \\ w \end{bmatrix} \right] \mapsto e,$$

then it is easily shown that a state space representation for $\widehat{G}_{LTI}$ is given by

$$\begin{bmatrix} \dot{x}(t) \\ \dot{x}_m(t) \end{bmatrix} = \overbrace{\begin{bmatrix} A & 0 \\ 0 & A_m \end{bmatrix}}^{=:\widehat{A}} \overbrace{\begin{bmatrix} x(t) \\ x_m(t) \end{bmatrix}}^{=:\widehat{x}(t)} + \overbrace{\begin{bmatrix} 0 & E \\ B_m & 0 \end{bmatrix}}^{=:\widehat{B}_1} \overbrace{\begin{bmatrix} u_m(t) \\ w(t) \end{bmatrix}}^{=:\widehat{w}(t)} + \overbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}^{=:\widehat{B}_2} u(t), \tag{5.38}$$

$$e(t) = \underbrace{\begin{bmatrix} -C & C_m \end{bmatrix}}_{=:\widehat{C}} \begin{bmatrix} x(t) \\ x_m(t) \end{bmatrix},$$

and that the closed loop time-varying map $G_{LTV}(T, F_k)$ can be represented by the block diagram illustrated in Figure 5.7.

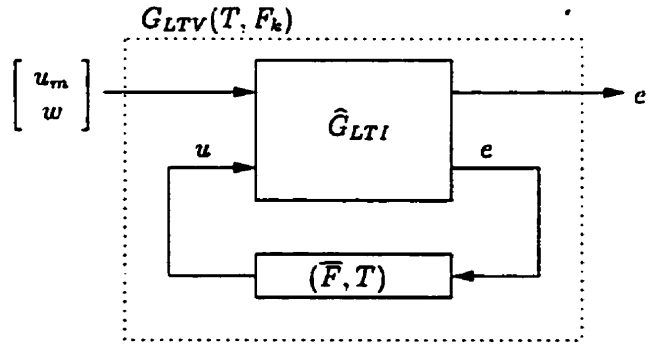


Figure 5.7: General framework for GSHF controlled system.

There are two perspectives from which to view our controller $(\overline{F}, T)$. First, since the hold $\overline{F}(t)$ has been chosen to be a piecewise constant function, it follows from Section 3.4 that there exists a controller of the form

$$\begin{aligned} \nu[k+1] &= G(k)\nu[k] + H(k)e(k\overline{T}), \quad \nu[0] = \nu_0 \in \mathbf{R} \\ u(k\overline{T}+\tau) &= J(k)\nu[k] + K(k)e(k\overline{T}), \quad \tau \in [0, \overline{T}), \end{aligned}$$

that implements $(\overline{F}, T)$ and that this controller consists of a sampler, a zero-order-hold, and a discrete-time n-periodic linear system (see Figure 5.8(a)). If the discrete-time

component of the controller were LTI, then we could directly apply the results obtained in [8, 17, 6, 47], but this is not the case.

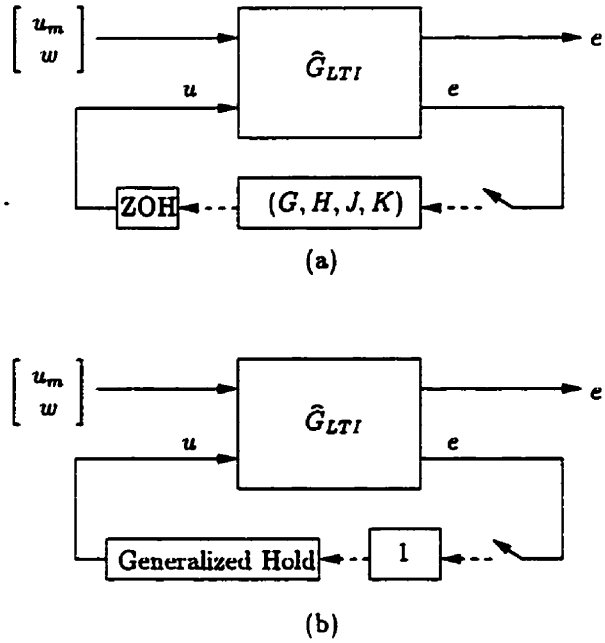


Figure 5.8: (a) $(\overline{F}, T)$ as a LPC. (b) $(\overline{F}, T)$ as a GSHF controller.

The second way of looking at $(\overline{F}, T)$ is from the GSHF perspective, where the controller consists of a sampler, a generalized-hold, and a unity gain discrete-time component (see Figure 5.8(b)). By looking at the controller this way, the results of [47] can more easily be extended to come up with an equation for $\|G_{LTV}(T, F_k)\|_\infty$. Hence, we proceed with this approach, but since the argument is virtually identical to that given in [47], most of the details are omitted and only a brief derivation of the formula is provided.

We begin by noting from (5.38) that

$$\widehat{x}[(k+1)T] = e^{\widehat{A}T}\widehat{x}[kT] + \int_0^T e^{\widehat{A}(T-\tau)}\widehat{B}_1\widehat{w}(kT+\tau)d\tau + \int_0^T e^{\widehat{A}(T-\tau)}\widehat{B}_2 u(kT+\tau)d\tau,$$

but since

$$u(kT+\tau) = \overline{F}(kT+\tau)e[kT] = \overline{F}(\tau)\widehat{C}\widehat{x}[kT],$$

it follows from the definition of $\widehat{A}$, $\widehat{B}_2$, and (5.22) that

$$\int_0^T \mathrm{e}^{\widehat{A}(T-\tau)}\widehat{B}_2 u(kT+\tau)d\tau = \begin{bmatrix} 0 \\ -TF_k \\ 0 \end{bmatrix} \widehat{C}\widehat{x}[kT].$$

Hence, with

$$\widehat{\Phi} := \mathrm{e}^{\widehat{A}T} + \begin{bmatrix} 0 \\ -TF_k \\ 0 \end{bmatrix} \widehat{C},$$

we have

$$\widehat{x}[kT] = \sum_{i=0}^{k-1} \widehat{\Phi}^{k-1-i} \int_0^T \mathrm{e}^{\widehat{A}(T-\tau)}\widehat{B}_1 \widehat{w}(iT+\tau)d\tau. \tag{5.39}$$

Now, for the intersample behaviour we note that

$$\begin{aligned} \widehat{x}(kT+t) &= \mathrm{e}^{\widehat{A}t}\widehat{x}[kT] + \int_0^t \mathrm{e}^{\widehat{A}(t-\theta)}\widehat{B}_1\widehat{w}(kT+\theta)d\theta + \int_0^t \mathrm{e}^{\widehat{A}(t-\theta)}\widehat{B}_2 u(kT+\theta)d\theta \\ &= \left(\mathrm{e}^{\widehat{A}t} + \int_0^t \mathrm{e}^{\widehat{A}(t-\theta)}\widehat{B}_2\overline{F}(\theta)\widehat{C}d\theta\right)\widehat{x}[kT] \\ &\quad + \int_0^t \mathrm{e}^{\widehat{A}(t-\theta)}\widehat{B}_1\widehat{w}(kT+\theta)d\theta, \quad t \in [0,T), \end{aligned}$$

so combining this with (5.39), it follows that

$$\begin{aligned} e(kT+t) &= \widehat{C}\widehat{x}(kT+t) \\ &= \widehat{C}\left(\mathrm{e}^{\widehat{A}t} + \int_0^t \mathrm{e}^{\widehat{A}(t-\theta)}\widehat{B}_2\overline{F}(\theta)\widehat{C}d\theta\right)\sum_{i=0}^{k-1}\widehat{\Phi}^{k-1-i}\int_0^T \mathrm{e}^{\widehat{A}(T-\tau)}\widehat{B}_1\widehat{w}(iT+\tau)d\tau \\ &\quad + \int_0^t \widehat{C}\mathrm{e}^{\widehat{A}(t-\theta)}\widehat{B}_1\widehat{w}(kT+\theta)d\theta, \quad t \in [0,T), \end{aligned}$$

which means[a]

$$\sup_{k\in\mathbb{Z}^+} |e(kT+t)| \le \sum_{j=1}^{2}\left[\int_0^T \left|\left[\widehat{C}\left(\mathrm{e}^{\widehat{A}t} + \int_0^t \mathrm{e}^{\widehat{A}(t-\theta)}\widehat{B}_2\overline{F}(\theta)\widehat{C}d\theta\right)\sum_{i=0}^{\infty}\widehat{\Phi}^i\mathrm{e}^{\widehat{A}(T-\tau)}\widehat{B}_1\right]_{1j}\right| d\tau \right.$$
$$\left. + \int_0^t \left|[\widehat{C}\mathrm{e}^{\widehat{A}(t-\theta)}\widehat{B}_1]_{1j}\right| d\theta\right] \|\widehat{w}\|_\infty, \quad t \in [0,T).$$

[a]We use the notation $[Q]_{ij}$ to represent the (i,j) entry of Q.

Hence

$$\alpha_{opt} := \max_{t\in[0,T)} \sum_{j=1}^{2} \left[\int_0^T \left| \left[\widehat{C} \left(e^{\widehat{A}t} + \int_0^t e^{\widehat{A}(t-\theta)} \widehat{B}_2 \overline{F}(\theta) \widehat{C} d\theta \right) \sum_{i=0}^{\infty} \widehat{\Phi}^i e^{\widehat{A}(T-\tau)} \widehat{B}_1 \right]_{1j} \right| d\tau \right.$$

$$\left. + \int_0^t \left| [\widehat{C} e^{\widehat{A}(t-\theta)} \widehat{B}_1]_{1j} \right| d\theta \right] \tag{5.40}$$

is an upper bound on $\|G_{LTV}(T, F_k)\|_\infty$. Using an argument identical to that found in [47], we can construct a $\widehat{w}(t)$ that shows that α_{opt} is the least upper bound for $\|G_{LTV}(T, F_k)\|_\infty$, which means

$$\|G_{LTV}(T, F_k)\|_\infty = \alpha_{opt}.$$

5.6.1 An Example

We now demonstrate the design method with an example. Consider the plant (5.5) with

$$A_1 = -1, \quad b_1 = 1, \quad c_1 = 18,$$

$$A_2 = \begin{bmatrix} 0 & 6 \\ 1 & -4 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad c_2 = \begin{bmatrix} 0 & 1 \end{bmatrix},$$

$$E_1 = 1, \quad E_2 = 1;$$

it can easily be shown that the transfer function from u to y is given by

$$P(s) = \frac{s+1}{(s-2)(s+3)(s+4)}.$$

We will choose our reference model to be

$$\begin{aligned} \dot{x}_m(t) &= -5x_m(t) + 5u_m(t), \; x_m(0) = 0, \\ y_m(t) &= x_m(t), \end{aligned}$$

and the associated transfer function from u_m to y_m is given by

$$\frac{5}{s+5}.$$

We choose $\varepsilon = 0.1$, and our objective will be to design a GSHF controller (f_i, T) so that

$$\|G_{LTV}(T, F_k)\|_\infty \leq \varepsilon = 0.1$$

Step 1

Here we choose a large enough value of k so that $\|G_{LTI}(F_k)\|_\infty \leq \varepsilon/2 = 0.05$. If we choose $k = 60$, let

$$\eta(k, s) := (s + 60)(s + 120) = s^2 + 180s + 7200,$$

and set

$$F_k := -\begin{bmatrix} 7200 + 6 \\ 180 - 4 \end{bmatrix} = -\begin{bmatrix} 7206 \\ 176 \end{bmatrix},$$

then it can easily be verified that

$$\mathrm{sp}(A_2 + F_k c_2) = \{-60, -120\},$$

and

$$\|G_{LTI}(F_k)\|_\infty = \int_0^\infty |\overline{C}\mathrm{e}^{\tilde{A}t}\overline{B}_2| + |\overline{C}\mathrm{e}^{\tilde{A}t}\overline{E}|dt = 0.04968 < 0.05.$$

Step 2

We now fix F_k and find a sufficiently small T so that

$$\|G_{LTV}(T, F_k)\|_\infty \leq 0.1.$$

To do this, we use (5.40) to compute $\alpha_{opt} = \|G_{LTV}(T, F_k)\|_\infty$; as was done in [47], we use a finite sum approximation for the infinite series in the formula for α_{opt}, i.e. we truncate the sum after the first 10000 terms. It can be verified that for $T = 0.009$, we have

$$\|G_{LTV}(T, F_k)\|_\infty = \alpha_{opt} = 0.09281.$$

For this value of T, we have $\overline{T} = 0.009/3 = 0.003$, and it can be verified that

$$(A_d, B_d) = \left(\mathrm{e}^{A\overline{T}}, \int_0^{\overline{T}} \mathrm{e}^{A\tau} B d\tau\right)$$

is controllable. Hence, the gains f_i satisfy

$$\begin{bmatrix} f_0 \\ \vdots \\ f_{n-1} \end{bmatrix} = T \begin{bmatrix} A_d^2 B_d & A_d B_d & B_d \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ F_k \end{bmatrix} = \begin{bmatrix} -1.711472199(10)^5 \\ 5.074883224(10)^5 \\ -3.147230015(10)^5 \end{bmatrix}.$$

We now provide simulation results for the case where

$$u_m(t) = 1, \quad t \geq 0,$$

and

$$w(t) = \sin(2t), \quad t \geq 0,$$

in Figures 5.9 and 5.10. Note in Figure 5.9 that $\|e\|_\infty \leq 0.1$.

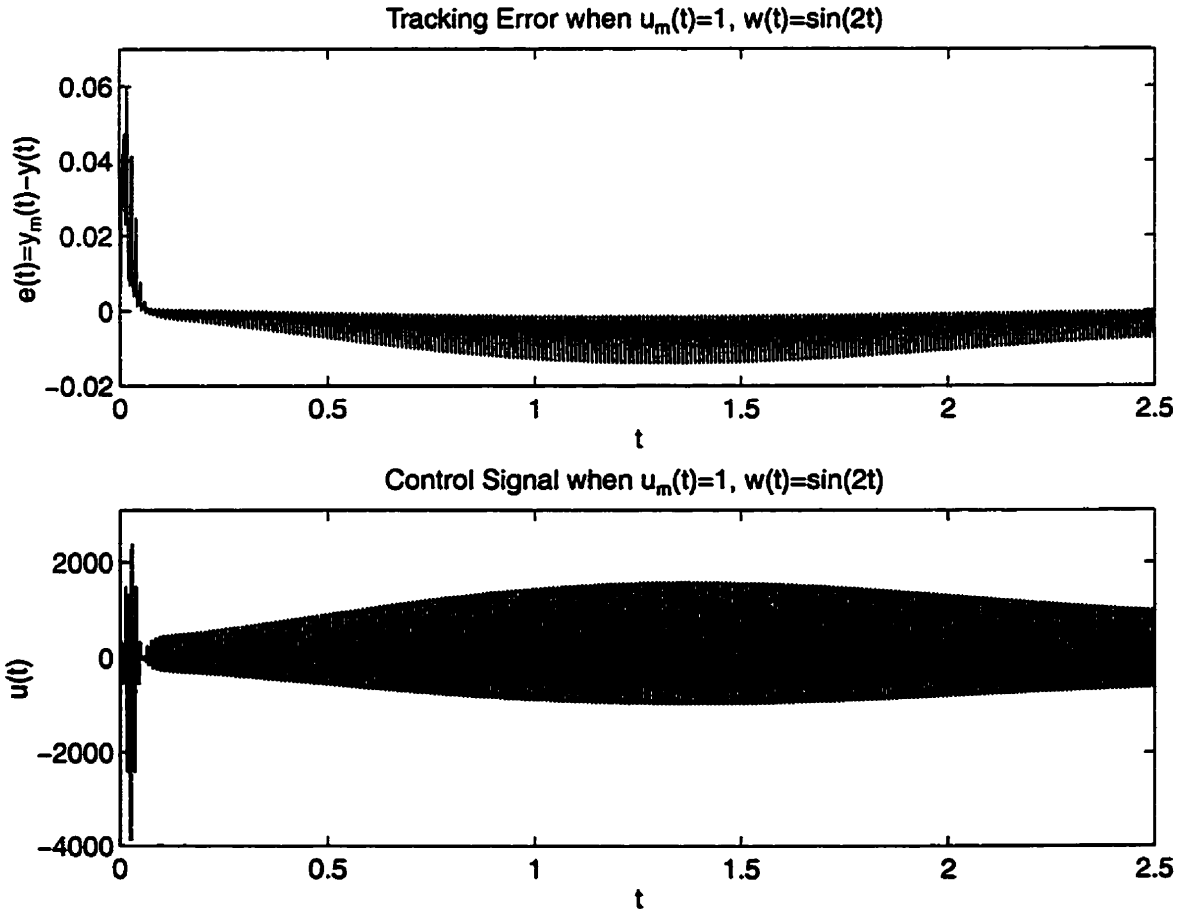


Figure 5.9: Model reference control tracking error and control signal.

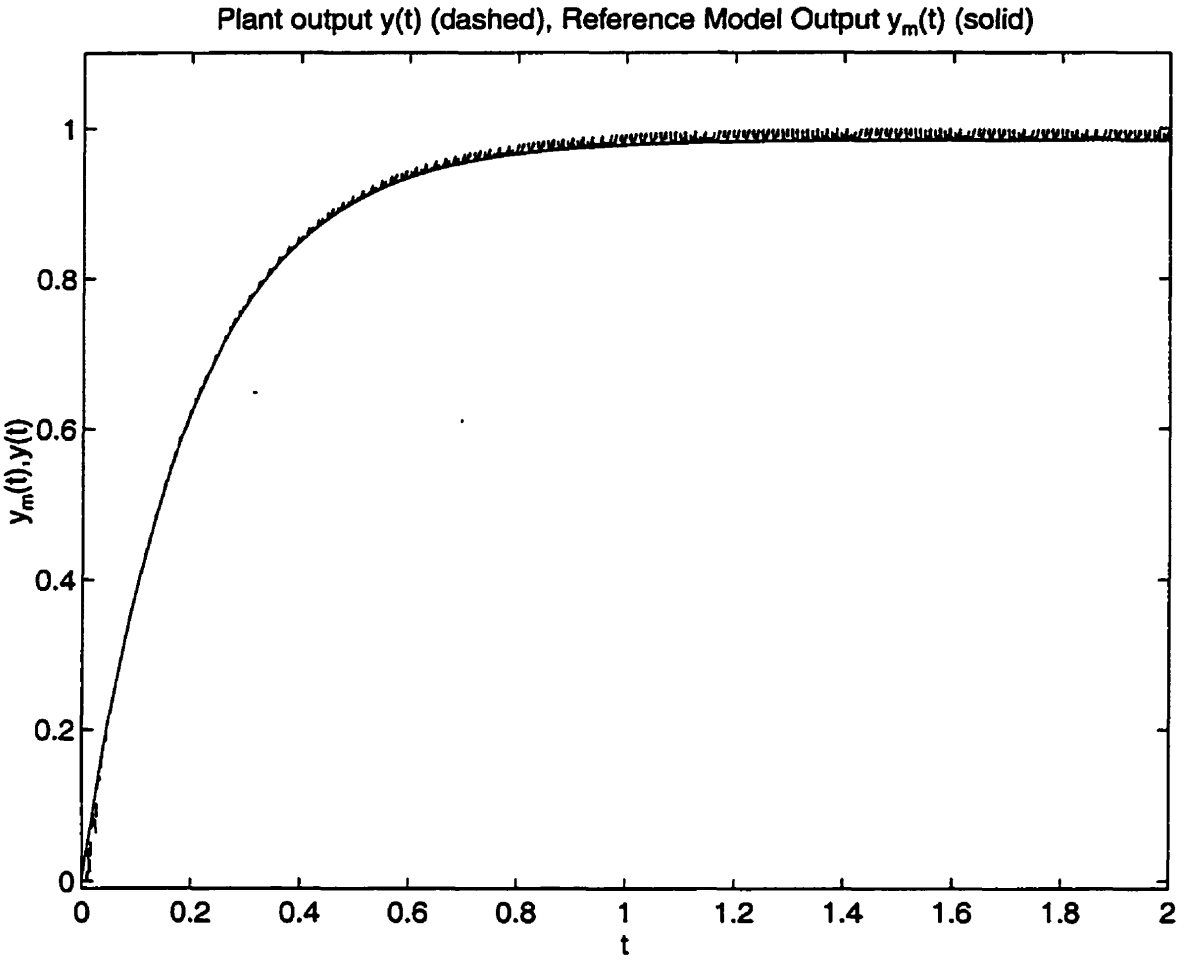


Figure 5.10: Plant output and reference model output.

Using the results of Section 3.4, we know that we can implement the GSHF controller (f_i, T) with a first order LPC of the form

$$\begin{aligned} \nu[k+1] &= G(k)\nu[k] + H(k)e(k\overline{T}), \quad \nu[0] = \nu_0 \in \mathbf{R} \\ u(k\overline{T}+\tau) &= J(k)\nu[k] + K(k)e(k\overline{T}), \quad \tau \in [0, \overline{T}), \end{aligned}$$

by choosing

$$(G(k), H(k), J(k), K(k)) := \begin{cases} (0, 1, 0, -1.711472199(10)^5) & k = 0 \\ (1, 0, 5.074883224(10)^5, 0) & k = 1 \\ (1, 0, -3.147230015(10)^5, 0) & k = 2, \end{cases}$$

and

$$(G(k+3), H(k+3), J(k+3), K(k+3)) = (G(k), H(k), J(k), K(k)), \quad k \in \mathbf{Z}^+;$$

this LPC can be implemented with a sampler, a linear first order discrete-time 3-periodic compensator, and a zero-order-hold.

5.7 Summary and Concluding Remarks

In this chapter, we have shown that it is possible to design a static GSHF controller that provides near exact tracking and disturbance rejection when the strictly proper SISO LTI plant is minimum phase and the strictly proper SISO LTI reference model is stable. As far as we are aware, this is the first time that a GSHF controller has been designed which guarantees nice intersample behaviour in the model matching setting. This GSHF controller is easy to design and can be implemented with a sampler, a first order n-periodic discrete-time compensator. and a zero-order hold. Unfortunately, the controller synthesis is based on high gain control, so if we want the plant output to closely track the model reference output, then we will typically have large control signals.

Chapter 6

Performance

6.1 Introduction

In the previous chapter, we showed how to use a GSHF controller to provide model matching for a minimum phase plant. This is inapplicable to non-minimum phase plants since exact model matching is typically impossible. However, a common problem in industry is that of step tracking. Hence, here we will use the ideas of Chapter 4 to design a stable low order sampled-data controller which provides near optimal step tracking in an LQR sense, *even when the plant state cannot be measured.* Furthermore, the controller will be able to recover the gain and phase margin characteristics of the optimal state feedback control law.

We first pose the optimal step tracking problem and convert it to a standard LQR problem. Since the optimal LQR control law is linear state feedback, we proceed by considering a more general problem, in which we are given an arbitrary stabilizing state feedback control law that provides a given performance, and we would like to find a low order sampled-data controller that not only provides a performance arbitrarily close to that provided by the state feedback control law, but also recovers its gain and phase margin characteristics.

This chapter is organized in the following manner. In Section 6.2, we will pose the optimal step tracking problem and show that it can be converted to the standard LQR problem. In Section 6.3 we will formulate our general problem, for which our optimal step tracking problem is a special case, and provide an outline of the approach that we will take to achieve our objective. Then some preliminary results are proven in Sections 6.4.1 and 6.4.2 relating the state and control signal of the LPC controlled system to an "ideal" LTI system. These preliminary results are then used in Section 6.5 to prove the main result. Finally, an illustrative example is provided in Section 6.6 and concluding remarks are made in Section 6.7.

6.2 The Optimal Step Tracking Problem

In this section, we will pose an optimal step tracking problem and show that it can be converted to a standard LQR problem.

Assume that the actual m-input, r-output plant is described by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + \gamma Bu(t),\ x(0) = 0,\ \gamma \in \mathbf{C} \\ y(t) &= Cx(t). \end{aligned} \tag{6.1}$$

We assume that (A, B) is controllable, (C, A) is observable, the nominal plant (i.e. $\gamma = 1$) has no transmission zeros at the origin, and that there are at least as many inputs as there are outputs ($m \geq r$).

With $y_r \in \mathbf{R}^r$, we would like to track reference inputs of the form

$$y_{ref}(t) = y_r,\ t \geq 0,$$

in an optimal fashion. There are a number of ways to define optimality; with the tracking error defined by

$$e(t) := y(t) - y_{ref}(t),$$

$\gamma = 1$ and $R \in \mathbf{R}^{m\times m}$ positive definite and symmetric here, we would like to find the control signal which for each y_r, minimizes the quadratic cost function

$$\int_0^\infty e(t)^T e(t) + \dot{u}(t)^T R\dot{u}(t)dt; \tag{6.2}$$

note that we are not considering the cost function

$$\int_0^\infty e(t)^T e(t) + u(t)^T Ru(t)dt$$

since the control signal will not typically go to zero when tracking a step reference signal. To proceed, we consider the augmented state

$$\eta(t) := \begin{bmatrix} \dot{x}(t) \\ e(t) \end{bmatrix},$$

so that with $\nu(t) = \dot{u}(t)$, the augmented plant can be defined as

$$\begin{aligned} \dot{\eta}(t) &= \overbrace{\begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}}^{=:\overline{A}} \eta(t) + \gamma \overbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}^{=:\overline{B}} \nu(t), \quad \eta(0) = \eta_0 = \begin{bmatrix} 0 \\ y_r \end{bmatrix}, \\ e(t) &= \underbrace{\begin{bmatrix} 0 & I \end{bmatrix}}_{=:\overline{C}} \eta(t). \end{aligned} \tag{6.3}$$

Since (C, A) is observable, it follows that $(\overline{C}, \overline{A})$ is observable. Since (C, A) is observable, (A, B) is controllable, $m \geq r$, and the plant has no transmission zeros at the origin, it can be shown that $(\overline{A}, \overline{B})$ is controllable. Furthermore, (6.2) becomes

$$\begin{aligned} J(\eta_0) &:= \int_0^\infty e(t)^T e(t) + \dot{u}(t)^T R\dot{u}(t)dt \\ &= \int_0^\infty \eta(t)^T \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \eta(t) + \nu(t)^T R\nu(t)dt. \end{aligned}$$

Since the state weighting matrix is only positive semidefinite and not positive definite, we will slightly modify the cost function. To this end, with $\delta > 0$, we consider the cost

function

$$\begin{aligned} J_\delta(\eta_0) &= \int_0^\infty e(t)^T e(t) + \delta \dot{x}(t)^T \dot{x}(t) + \dot{u}(t)^T R \dot{u}(t) dt \\ &= \int_0^\infty \eta(t)^T \underbrace{\begin{bmatrix} \delta I & 0 \\ 0 & I \end{bmatrix}}_{=: Q_\delta} \eta(t) + \nu(t)^T R \nu(t) dt. \end{aligned} \tag{6.4}$$

It is easy to prove that when $\gamma = 1$, the optimal J_δ converges uniformly to the optimal J as $\delta \to 0$.

When $\gamma = 1$, the control law that minimizes (6.4) for every η_0 is of the form

$$\nu(t) = F_\delta \eta(t); \tag{6.5}$$

we can obtain the optimal gain F_δ by first solving the algebraic Riccati equation

$$P_\delta \overline{A} + \overline{A}^T P_\delta - P_\delta \overline{B} R^{-1} \overline{B}^T P_\delta + Q_\delta = 0$$

for the unique positive definite symmetric solution P_δ, and then setting

$$F_\delta = -R^{-1} \overline{B}^T P_\delta.$$

Notice that this controller is a proportional-integral state feedback control law and that when it is applied to (6.1) we have

$$\dot{\eta}(t) = (\overline{A} + \gamma \overline{B} F_\delta) \eta(t), \ \eta(0) = \eta_0.$$

Now, not only will this control law minimize the cost function (6.4) when $\gamma = 1$, but it is well known that it will also provide an infinite upper gain margin, a lower gain margin of at least 1/2, and a phase margin of at least 60 degrees. Indeed,

$$\overline{\Gamma} := \{\gamma \in \mathbf{C} : \overline{A} + \gamma \overline{B} F_\delta \text{ is stable}\} \supset \{\rho e^{j\phi} : \rho \in [1/2, \infty), \phi \in [-\pi/3, \pi/3]\}.$$

While we should not expect that a sampled-data controller will be capable of providing an infinite gain margin, for every compact subset Γ of $\overline{\Gamma}$, we will show that there exists a low order LPC that provides near optimal performance for the nominal plant and provides closed loop stability and step tracking for every $\gamma \in \Gamma$.

To summarize, we assume that with $(\overline{A}, \overline{B})$ controllable and $(\overline{C}, \overline{A})$ observable, we have determined F_δ and chosen Γ a compact subset of $\overline{\Gamma}$, and our objective is to find a low order sampled-data controller which measures only e and provides performance arbitrarily close that provided by the control law (6.5) when $\gamma = 1$, as well as closed loop stability and step tracking for every $\gamma \in \Gamma$.

Before leaving this section, let us comment on the implementation of the LPC. Since we are designing the LPC for the augmented plant (6.3), it follows that the output of the controller will be $\nu(t)$. To obtain $u(t)$ from $\nu(t)$, we can simply use an integrator - see Figure 6.1.

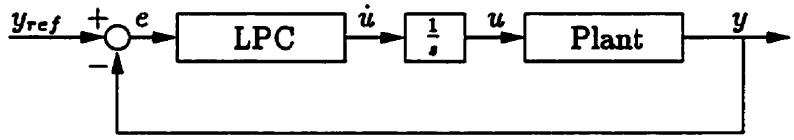


Figure 6.1: Implementation of the LPC for optimal step tracking.

6.3 Problem Formulation

In this section, we formulate our general problem, for which the optimal step tracking problem of Section 6.2 is a special case. Suppose our nominal model is

$$\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u(t), \quad x(0) = x_0, \\ y(t) &= C\,x(t), \end{aligned} \tag{6.6}$$

with $x(t) \in \mathbf{R}^n$ the state, $u(t) \in \mathbf{R}^m$ the control input, and $y(t) \in \mathbf{R}^r$ the plant output. We assume that (A, B) is controllable and (C, A) is observable. We capture uncertainty in the model by assuming that for some $\gamma \in \mathbf{C}$, our actual plant is given by

$$\begin{aligned} \dot{x}(t) &= A\,x(t) + \gamma B\,u(t), \quad x(0) = x_0, \\ y(t) &= C\,x(t); \end{aligned} \tag{6.7}$$

we associate the actual plant with the triple $(A, \gamma B, C)$. Suppose we are also given a stabilizing state feedback matrix $F \in \mathbf{R}^{m \times n}$, and then we define the open set

$$\overline{\Gamma} := \{\gamma \in \mathbf{C} : A + \gamma BF \text{ is stable}\}.$$

To provide a measure of the performance of the closed loop system, with $Q \in \mathbf{R}^{n\times n}$ and $R \in \mathbf{R}^{m\times m}$ positive definite and symmetric, we define the cost function as

$$J(x_0) := \int_0^\infty [x(t)^* Q x(t) + u(t)^* R u(t)]dt; \tag{6.8}$$

since Q and R are symmetric, $J(x_0)$ is real-valued. Let Γ be a compact subset of $\overline{\Gamma}$ and let $\gamma \in \Gamma$.

With $T > 0$ the sampling rate, we will consider LPCs of the form

$$\begin{aligned} z[k+1] &= G(k)z[k] + H(k)y(kT), \quad z[0] = z_0 \in \mathbf{R}^l, \\ u(kT+\tau) &= J(k)z[k], \qquad\qquad\qquad \tau \in [0,T); \end{aligned} \tag{6.9}$$

we associate this system with the 4-tuple (G,H,J,T). Here we let p denote the period of the controller parameters G, H, and J, so that pT is the period of the controller (G,H,J,T). Note that (6.9) can be implemented with a sampler, a zero-order-hold, and an l^{th} order periodically time-varying discrete-time system of period p (See Figure 4.1).

Since our LPC will be emulating a state feedback control law of the form "$u(t) = Fx(t)$", it will be useful to define the following ideal LTI closed loop system:

$$\dot{\widehat{x}}(t) = (A + \gamma BF)\widehat{x}(t), \quad \widehat{x}(0) = x_0 \in \mathbf{R}^n. \tag{6.10}$$

Then the ideal state is defined as

$$\widehat{x}(t) = e^{(A+\gamma BF)t}x_0,$$

the ideal control law is defined as

$$\widehat{u}(t) = F\widehat{x}(t), \tag{6.11}$$

and the ideal cost function is defined as

$$\begin{aligned} \widehat{J}(x_0) &= \int_0^\infty [\widehat{x}(t)^* Q \widehat{x}(t) + \widehat{u}(t)^* R \widehat{u}(t)]dt \\ &= x_0^T \int_0^\infty e^{(A+\gamma BF)^* t}(Q + F_\delta^T R F_\delta)e^{(A+\gamma BF)t}dt\, x_0; \end{aligned}$$

since Q and R are symmetric, $\widehat{J}(x_0)$ is real-valued.

Hence our objective be stated as:

> Given a nominal system (A, B, C), a state feedback gain F, and a set $\overline{\Gamma}$, for every compact subset Γ of $\overline{\Gamma}$ and every $\epsilon > 0$, find a low order LPC (G, H, J, T) so that
> $$|J(x_0) - \widehat{J}(x_0)| \leq \epsilon \|x_0\|^2, \quad \gamma \in \Gamma.$$

Before presenting any results, let us first provide some motivation for the approach that we take in this chapter. In a similar approach to that adopted in Chapter 4, with $F \in \mathbf{R}^{m \times n}$, $p > n$, and $\overline{T} := pT$, we consider the control law

$$u(t) = \begin{cases} 0 & t \in [k\overline{T}, k\overline{T}+nT) \\ \frac{p}{p-n} F x[k\overline{T}+nT] & t \in [k\overline{T}+nT, (k+1)\overline{T}). \end{cases} \tag{6.12}$$

From Proposition 4.1, it follows that this control law can be implemented by an m^{th} order controller of the form (6.9). We then want to show that if we apply (6.12) to the actual plant, then $|J(x_0) - \widehat{J}(x_0)|$ can be made small by choosing a sufficiently large p and a sufficiently small T. The fact that we need T to be small is not unexpected given the discussion and analysis provided in Chapter 4. To see why it is that we need p to be large here, but not in Chapter 4, consider the following argument. In Chapter 4 we relied on the fact that for arbitrary p and small T, the average of u over $[k\overline{T}, (k+1)\overline{T})$ is approximately equal to the average of $\widehat{u}$ over $[k\overline{T}, (k+1)\overline{T})$. Then since the plant is strictly proper, it turns out that u has approximately the same effect on x as $\widehat{u}$ has on $\widehat{x}$, which was sufficient to show that u could recover the gain/phase margin provided by $\widehat{u}$. Note however that u enters into $J(x_0)$ in a quadratic fashion, so

$$\begin{aligned} \int_0^{\overline{T}} u(t)^T R u(t) dt &= \int_{nT}^{\overline{T}} u(t)^T R u(t) dt \\ &\approx \left(\tfrac{p}{p-n}\right)^2 \int_{nT}^{\overline{T}} \widehat{u}(t)^T R \widehat{u}(t) dt \\ &\approx \left(\tfrac{p}{p-n}\right) \int_0^{\overline{T}} \widehat{u}(t)^T R \widehat{u}(t) dt, \end{aligned}$$

which means we are off by a factor of $p/(p-n)$. Hence, by choosing p sufficiently large, $p/(p-n)$ can be made arbitrarily close to unity, so we should be able to make $J(x_0)$ arbitrarily close to $\widehat{J}(x_0)$.

6.4 Preliminary Results

To prove our main result, we will break the problem down into two parts. In the first part, we will show that for an arbitrary $p > n$, we can make $\|x - \widehat{x}\|_2$ small by choosing T small. In the second part, we show that by picking p sufficiently large and T sufficiently small, we can make $\|u - \widehat{u}\|_2$ small. Finally these two results can be combined to prove our desired result.

Since we wish to implement the control law (6.12) with an LPC of the form (6.9), we will need Proposition 4.1, which we repeat here:

Proposition 6.1 If $T > 0$ is non-pathological, then for every $p > n$ and every $F \in \mathbf{R}^{m \times n}$, there exists an m^{th} order LPC (G, H, J, T) that will implement the control law (6.12).

Remark 6.1 The LPC (G, H, J, T) that is constructed in Proposition 6.1 is stable, in the sense that when the input to the controller is zero, for every controller initial condition $z_0 \in \mathbf{R}^m$, we have

$$\lim_{k \to \infty} z[k] = 0.$$

To see this, note that $G(0) = 0$, so it follows that for every $z_0 \in \mathbf{R}^m$, we have

$$z[p] = G(p-1) \cdots G(0) z_0 = 0,$$

which means $z[p + k] = 0$ for $k \geq 0$. □

In most of the analysis that will follow, we typically first determine the behaviour at discrete points in time, and then go on to analyze the intersample behaviour. Hence, we now introduce the following discrete-time equations. From (6.10), it follows that

$$\widehat{x}[(k+1)\overline{T}] = \mathrm{e}^{(A + \gamma BF)\overline{T}} \widehat{x}[k\overline{T}], \tag{6.13}$$

and if we apply the control law (6.12) to the actual plant (6.7), then with the notation

$$A_d := \mathrm{e}^{ApT}, \quad B_d := \tfrac{p}{p-n} \int_0^{(p-n)T} \mathrm{e}^{A\tau} B \, d\tau, \quad F_d := F \mathrm{e}^{AnT},$$

it can be shown that

$$x[(k+1)\overline{T}] = (A_d + \gamma B_d F_d)x[k\overline{T}]. \tag{6.14}$$

We will also require the following result to show that the closed loop system behaves like the ideal system uniformly over Γ.

Lemma 6.1

i) There exists an $\alpha_1 > 0$ and a $\lambda_1 < 0$ so that

$$\|e^{(A+\gamma BF)t}\| \leq \alpha_1 e^{\lambda_1 t}, \quad \gamma \in \Gamma, \ t \geq 0.$$

ii) For every $p > n$, there exists an $\alpha_2 > 0$, $\lambda_2 < 0$, and a $T_1 > 0$ so that

$$\|(A_d + \gamma B_d F_d)^k\| \leq \alpha_2 \left(e^{\lambda_2 \overline{T}}\right)^k, \quad \overline{T} \in (0, T_1), \ \gamma \in \Gamma, \ k \in \mathbf{Z}^+.$$

Proof:

We will prove the first result and then use it in a discrete-time Lyapunov argument to prove the second result. Let

$$A_\gamma := A + \gamma BF.$$

For every $\gamma \in \Gamma$, A_γ is stable, so there exists an $\alpha_\gamma > 0$ and $\lambda_\gamma < 0$ such that

$$\|e^{(A+\gamma BF)t}\| \leq \alpha_\gamma e^{\lambda_\gamma t}, \quad t \geq 0.$$

In fact, there exists a neighborhood of γ, say N_γ, so that

$$\|e^{(A+\gamma BF)t}\| \leq 2\alpha_\gamma e^{\frac{\lambda_\gamma}{2} t}, \quad \gamma \in N_\gamma.$$

Now $\{N_\gamma : \gamma \in \Gamma\}$ is an open cover of Γ, so by compactness, it has a finite subcover, say

$$\{N_{\gamma_1}, \cdots, N_{\gamma_q}\}.$$

Define

$$\alpha_1 := \max\{2\alpha_{\gamma_i}, i = 1, 2, \cdots, q\},$$

$$\lambda_1 := \max\{\lambda_{\gamma_i}/2, i = 1, 2, \cdots, q\}.$$

Therefore,

$$\|e^{(A+\gamma BF)t}\| \leq \alpha_1 e^{\lambda_1 t}, \quad \gamma \in \Gamma.$$

Now let us use this to prove our second result. For every $\gamma \in \Gamma$, A_γ is stable, so there exists a unique $P_\gamma = P_\gamma^* > 0$ satisfying

$$A_\gamma^* P_\gamma + P_\gamma A_\gamma + I = 0; \tag{6.15}$$

in fact

$$P_\gamma = \int_0^\infty e^{A_\gamma^* t} e^{A_\gamma t} dt,$$

e.g. see [55, Lemma 12.1]. By applying the first result, it follows that

$$\|P_\gamma\| \leq \int_0^\infty \alpha_1^2 e^{2\lambda_1 t} dt = \frac{\alpha_1^2}{2|\lambda_1|} =: \beta_0, \quad \gamma \in \Gamma. \tag{6.16}$$

Now since γ enters (6.15) in a linear fashion, it follows that P_γ is a continuous function of γ. Then since Γ is compact and $\lambda_{min}(P_\gamma) > 0$ for every $\gamma \in \Gamma$, it follows that there exists a constant $\beta_1 > 0$ so that

$$\lambda_{min}(P_\gamma) \geq \beta_1, \quad \gamma \in \Gamma. \tag{6.17}$$

We now use these bounds on P_γ in a discrete-time Lyapunov argument to prove the second result. Fix $p > n$ and let $T := \overline{T}/p$. If we define

$$\Delta_1(T) := (A_d + \gamma B_d F_d) - (I + A_\gamma \overline{T}),$$

then using the fact that γ is bounded, it can be shown that $\Delta_1(T) = \mathcal{O}(T^2)$. Now consider the discrete-time equation

$$\begin{aligned} w(k+1) &= (A_d + \gamma B_d F_d) w(k) \\ &= [I + A_\gamma \overline{T} + \Delta_1(T)] w(k), \quad w(0) = w_0, \end{aligned} \tag{6.18}$$

and define the Lyapunov candidate function $V : \mathbf{C}^n \to \mathbf{C}$:

$$V(w) := w^* P_\gamma w.$$

Since $P_\gamma = P_\gamma^*$, it follows that V is real-valued, which means

$$V[w(k+1)] - V[w(k)]$$

is also real-valued. Using (6.18) and the facts that $\Delta_1(T) = \mathcal{O}(T^2)$ and P_γ and γ are bounded, it can be shown that there exists a $\Delta_2(T) = \mathcal{O}(T)$ such that

$$V[w(k+1)] - V[w(k)] = \overline{T} w(k)^* [(A_\gamma^* P_\gamma + P_\gamma A_\gamma) + \Delta_2(T)] w(k).$$

Using (6.15), it follows that there exists a $T_1 > 0$ so that

$$V[w(k+1)] - V[w(k)] \leq -\frac{\overline{T}}{2} w(k)^* w(k), \quad \overline{T} \in (0, T_1).$$

But

$$V[w(k)] \leq \|P_\gamma\| w(k)^* w(k) \quad \Rightarrow \quad -w(k)^* w(k) \leq -\frac{1}{\|P_\gamma\|} V[w(k)],$$

so for $\overline{T} \in (0, T_1)$ we have

$$V[w(k+1)] \leq \left(1 - \frac{1}{2\|P_\gamma\|}\overline{T}\right) V[w(k)] \leq e^{-\frac{1}{2\|P_\gamma\|}\overline{T}} V[w(k)]$$

$$\Rightarrow V[w(k)] \leq \left(e^{-\frac{1}{2\|P_\gamma\|}\overline{T}}\right)^k V[w_0]$$

$$\Rightarrow \|w(k)\|^2 \leq \frac{\|P_\gamma\|}{\lambda_{min}(P_\gamma)} \left(e^{-\frac{1}{2\|P_\gamma\|}\overline{T}}\right)^k \|w_0\|^2.$$

Using the bounds given in (6.16) and (6.17) along with (6.18), it follows that

$$\|(A_d + \gamma B_d F_d)^k w_0\|^2 \leq \frac{\beta_0}{\beta_1} \left(e^{-\frac{1}{2\beta_0}\overline{T}}\right)^k \|w_0\|^2, \quad \gamma \in \Gamma, \ \overline{T} \in (0, T_1),$$

$$\Rightarrow \|(A_d + \gamma B_d F_d)^k\| \leq \sqrt{\frac{\beta_0}{\beta_1}} \left(e^{-\frac{1}{4\beta_0}\overline{T}}\right)^k, \quad \gamma \in \Gamma, \ \overline{T} \in (0, T_1).$$

■

6.4.1 State Equations

We are now ready to prove our first preliminary result:

Lemma 6.2 For every $\varepsilon > 0$ and $p > n$, there exists an $\alpha > 0$, $T_{max} > 0$ and $\lambda < 0$ so that for every $\gamma \in \Gamma$ and $\overline{T} \in (0, T_{max})$ we have

i) $\|x - \widehat{x}\|_2 \leq \varepsilon \|x_0\|$,

ii) $\displaystyle\int_{k\overline{T}}^{(k+1)\overline{T}} \|\dot{\widehat{x}}(\theta)\| d\theta \leq \alpha \overline{T} (e^{\lambda \overline{T}})^k \|x_0\|$,

iii) $\|x(k\overline{T} + \tau) - \widehat{x}(k\overline{T} + \tau)\| \leq \alpha \overline{T} (e^{\lambda \overline{T}})^k, \ \tau \in [0, \overline{T})$.

Proof:

Fix $p > n$ and define

$$\xi := x - \widehat{x}.$$

Observe that

$$\|\xi\|_2^2 = \int_0^\infty \|\xi(t)\|^2 dt = \sum_{k=0}^{\infty} \int_0^{\overline{T}} \|\xi(k\overline{T}+\tau)\|^2 d\tau, \tag{6.19}$$

and that for $\tau \in [0, \overline{T})$,

$$\|\xi(k\overline{T}+\tau)\| \leq \|\xi(k\overline{T})\| + \int_{k\overline{T}}^{(k+1)\overline{T}} \left[\|\dot{x}(\theta)\| + \|\dot{\widehat{x}}(\theta)\|\right] d\theta. \tag{6.20}$$

We begin by looking at the first term in (6.20). From (6.13), (6.14) and the definition of ξ, it follows that

$$\xi[(k+1)\overline{T}] = \mathrm{e}^{(A+\gamma BF)\overline{T}}\xi(k\overline{T}) + \left[(A_d + \gamma B_d F_d) - \mathrm{e}^{(A+\gamma BF)\overline{T}}\right] x(k\overline{T}).$$

From Lemma 6.1, we know there exists a $\alpha_1 > 0$, $\lambda_1 < 0$, and $T_1 > 0$ so that

$$\|x[k\overline{T}]\| \leq \|(A_d + \gamma B_d F_d)^k\| \times \|x_0\| \leq \alpha_1(\mathrm{e}^{\lambda_1 \overline{T}})^k \|x_0\|, \quad \gamma \in \Gamma, \ \overline{T} \in (0, T_1). \tag{6.21}$$

But since γ is bounded, it can be shown that $[(A_d + \gamma B_d F_d) - \mathrm{e}^{(A+\gamma BF)\overline{T}}] = \mathcal{O}(T^2)$, so along with our bound on $x[k\overline{T}]$, it follows that there exists a $T_2 \in (0, T_1)$ and an $\alpha_2 > 0$ so that

$$\|\xi[k\overline{T}]\| \leq \alpha_2 \overline{T}^2 \sum_{i=0}^{k-1} \|(\mathrm{e}^{(A+\gamma BF)\overline{T}})^{k-1-i}\| \times (\mathrm{e}^{\lambda_1 \overline{T}})^i \times \|x_0\|, \quad \gamma \in \Gamma, \ \overline{T} \in (0, T_2).$$

But also from Lemma 6.1 we know there exists an $\alpha_3 > 0$ and a $\lambda_2 < 0$ so that

$$\|(\mathrm{e}^{(A+\gamma BF)\overline{T}})^{k-1-i}\| \leq \alpha_3(\mathrm{e}^{\lambda_2 \overline{T}})^{k-1-i}, \quad \gamma \in \Gamma,$$

so

$$\|\xi[k\overline{T}]\| \leq \alpha_2\alpha_3 \overline{T}^2 \mathrm{e}^{\lambda_2 \overline{T}(k-1)} \sum_{i=0}^{k-1} \left(\mathrm{e}^{(\lambda_1-\lambda_2)\overline{T}}\right)^i \|x_0\|, \quad \gamma \in \Gamma, \ \overline{T} \in (0, T_2).$$

Without loss of generality, assume that $\lambda_2 \in (\lambda_1, 0)$, which means that $\lambda_1 - \lambda_2 < 0$, so

$$\|\xi[k\overline{T}]\| \leq \alpha_2\alpha_3 \mathrm{e}^{\lambda_2 \overline{T}(k-1)} \frac{\overline{T}^2}{1-\mathrm{e}^{(\lambda_1-\lambda_2)\overline{T}}} \|x_0\|, \quad \gamma \in \Gamma, \ \overline{T} \in (0, T_2).$$

Then since

$$\lim_{\overline{T}\to 0} \frac{\overline{T}^2}{1-e^{(\lambda_1-\lambda_2)\overline{T}}} = 0,$$

it follows that there exists an $\alpha_4 > 0$ and a $T_3 \in (0, T_2)$ so that

$$\|\xi[k\overline{T}]\| \leq \alpha_4 T (e^{\lambda_2 \overline{T}})^k \|x_0\|, \quad \gamma \in \Gamma, \ \overline{T} \in (0, T_3). \tag{6.22}$$

Now let us turn to the second term in (6.20). Since

$$\begin{aligned} \|\dot{\hat{x}}(\theta)\| &\leq \|(A+\gamma BF)\| \times \|e^{(A+\gamma BF)\theta}\| \times \|x_0\| \\ &\leq \alpha_3 e^{\lambda_2 \theta} \|(A+\gamma BF)\| \times \|x_0\|, \quad \gamma \in \Gamma, \end{aligned}$$

and γ is bounded, it follows that there exists an $\alpha_5 > 0$ and a $T_4 \in (0, T_3)$ so that

$$\int_{k\overline{T}}^{(k+1)\overline{T}} \|\dot{\hat{x}}(\theta)\| d\theta \leq \alpha_5 \overline{T} (e^{\lambda_2 \overline{T}})^k \|x_0\|, \quad \gamma \in \Gamma, \ \overline{T} \in (0, T_4). \tag{6.23}$$

Using the definition of $u(t)$ given in (6.12) it follows that

$$\|u(k\overline{T}+\tau)\| \leq \tfrac{p}{p-n} \|F e^{An\tau}\| \times \|x[k\overline{T}]\|, \quad \tau \in [0, \overline{T}),$$

so along with the bound on $x[k\overline{T}]$ given in (6.21), it follows that there exists a $T_5 \in (0, T_4)$ and $\alpha_6 > 0$ so that

$$\|u(k\overline{T}+\tau)\| \leq \alpha_6 (e^{\lambda_1 \overline{T}})^k \|x_0\|, \quad \gamma \in \Gamma, \ \overline{T} \in (0, T_5), \ \tau \in [0, \overline{T}).$$

Combining this with the fact that

$$x(k\overline{T}+\tau) = e^{A\tau} x(k\overline{T}) + \gamma \int_{k\overline{T}}^{k\overline{T}+\tau} e^{A(k\overline{T}+\tau-\nu)} B u(\nu) d\nu,$$

it follows that there exists a $T_6 \in (0, T_5)$ and an $\alpha_7 > 0$ so that

$$\|x(k\overline{T}+\tau)\| \leq \alpha_7 (e^{\lambda_1 \overline{T}})^k \|x_0\|, \quad \gamma \in \Gamma, \ \overline{T} \in (0, T_6).$$

Then it follows from the differential equation for x given in (6.7) and the fact that γ is bounded that there exists a $\alpha_8 > 0$ so that

$$\begin{aligned} &\|\dot{x}(\theta)\| \leq \alpha_8 (e^{\lambda_1 \overline{T}})^k \|x_0\|, \quad \gamma \in \Gamma, \ \overline{T} \in (0, T_6), \ \theta \in [k\overline{T}, (k+1)\overline{T}), \\ &\Rightarrow \int_{k\overline{T}}^{(k+1)\overline{T}} \|\dot{x}(\theta)\| d\theta \leq \alpha_8 \overline{T} (e^{\lambda_1 \overline{T}})^k \|x_0\|, \quad \gamma \in \Gamma, \ \overline{T} \in (0, T_6). \end{aligned} \tag{6.24}$$

By combining (6.22), (6.23), (6.24) and (6.20), and recalling that $\lambda_2 \in (0, \lambda_1, 0)$, it follows that there exists an $\alpha_9 > 0$ so that

$$\|\xi(k\overline{T}+\tau)\| \leq \alpha_9 \overline{T}(e^{\lambda_2 \overline{T}})^k \|x_0\|, \quad \gamma \in \Gamma, \ \overline{T} \in (0, T_7), \ \tau \in [0, \overline{T}). \tag{6.25}$$

Hence, for every $\gamma \in \Gamma$ and $\overline{T} \in (0, T_7)$,

$$\begin{aligned} \|\xi\|_2^2 &= \sum_{k=0}^{\infty} \int_0^{\overline{T}} \|\xi(k\overline{T}+\tau)\|^2 d\tau \\ &\leq \alpha_9^2 \overline{T}^3 \sum_{k=0}^{\infty} (e^{2\lambda_2 \overline{T}})^k \|x_0\|^2 \\ &= \alpha_9^2 \frac{\overline{T}^3}{1-e^{2\lambda_2 \overline{T}}} \|x_0\|^2. \end{aligned} \tag{6.26}$$

Since

$$\lim_{\overline{T} \to 0} \frac{\overline{T}^3}{1-e^{2\lambda_2 \overline{T}}} = 0,$$

our results follow from (6.26), (6.23), and (6.25). ∎

6.4.2 Control Law Equations

We now turn to our second preliminary result.

Lemma 6.3 For every $\varepsilon > 0$, there exists a $T_{max} > 0$ and an integer $p > n$ such that

$$\|u - \widehat{u}\|_2 \leq \varepsilon \|x_0\|, \quad \gamma \in \Gamma, \ \overline{T} \in (0, T_{max}).$$

Proof:

From the definition of u given in (6.12) and $\widehat{u}$ given in (6.11), it follows that

$$\begin{aligned} \|u - \widehat{u}\|_2^2 &= \int_0^{\infty} \|u(t) - \widehat{u}(t)\|^2 dt \\ &= \underbrace{\sum_{k=0}^{\infty} \int_{k\overline{T}}^{k\overline{T}+nT} \|F\widehat{x}(t)\|^2 dt}_{=: \Lambda_1(p, \overline{T})} + \underbrace{\sum_{k=0}^{\infty} \int_{k\overline{T}+nT}^{(k+1)\overline{T}} \|\tfrac{p}{p-n} Fx[k\overline{T}+nT] - F\widehat{x}(t)\|^2 dt}_{=: \Lambda_2(p, \overline{T})}. \end{aligned} \tag{6.27}$$

We begin by looking at $\Lambda_1(p,\overline{T})$. Using Lemma 6.1(i), we know there exists an $\alpha_1 > 0$ and $\lambda_1 < 0$ so that

$$\|\widehat{x}(t)\| \leq \alpha_1 e^{\lambda_1 t}\|x_0\|, \quad \gamma \in \Gamma, \tag{6.28}$$

so for every $\gamma \in \Gamma$, we have

$$\begin{aligned}\Lambda_1(p,\overline{T}) &\leq \|F\|^2 \sum_{k=0}^{\infty} \int_{k\overline{T}}^{k\overline{T}+nT} \alpha_1^2 e^{2\lambda_1 t} dt \|x_0\|^2 \\ &= \frac{\alpha_1^2\|F\|^2}{2\lambda_1}(e^{2\lambda_1 nT} - 1)\sum_{k=0}^{\infty} e^{2\lambda_1 k\overline{T}}\|x_0\|^2 \\ &= \frac{\alpha_1^2\|F\|^2}{2|\lambda_1|}\frac{(1-e^{2\lambda_1 \frac{n}{p}\overline{T}})}{(1-e^{2\lambda_1\overline{T}})}\|x_0\|^2.\end{aligned}$$

But

$$\lim_{\overline{T}\to 0} \frac{\alpha_1^2\|F\|^2}{2|\lambda_1|}\frac{(1-e^{2\lambda_1\frac{n}{p}\overline{T}})}{(1-e^{2\lambda_1\overline{T}})} = \frac{\alpha_1^2\|F\|^2}{2|\lambda_1|}\cdot\frac{n}{p},$$

so there exists a $p_1 > n$ so that

$$\frac{\alpha_1^2\|F\|^2}{2|\lambda_1|}\cdot\frac{n}{p} < \frac{\epsilon^2}{4}, \quad p > p_1,$$

which means that there exists a $T_0 > 0$ so that

$$\Lambda_1(p,\overline{T}) \leq \frac{\epsilon^2}{4}\|x_0\|^2, \quad \gamma \in \Gamma, \ \overline{T} \in (0,T_0), \ p > p_1. \tag{6.29}$$

We now turn to $\Lambda_2(p,\overline{T})$. With $\xi := x - \widehat{x}$, $\gamma \in \Gamma$, and $t \in [k\overline{T}+nT, (k+1)\overline{T})$ we have

$$\begin{aligned}&\|\tfrac{p}{p-n}Fx[k\overline{T}+nT] - F\widehat{x}(t)\| \\ &\leq \|\tfrac{p}{p-n}Fx[k\overline{T}+nT] - F\widehat{x}[k\overline{T}+nT]\| + \int_{k\overline{T}+nT}^{t}\|\dot{\widehat{x}}(\theta)\|d\theta \\ &\leq \tfrac{p}{p-n}\|F\| \times \|\xi[k\overline{T}+nT]\| + \left(\tfrac{p}{p-n}-1\right)\|Fe^{(A+\gamma BF)\frac{n}{p}\overline{T}}\| \times \|\widehat{x}[k\overline{T}]\| \\ &\quad + \int_{k\overline{T}}^{(k+1)\overline{T}}\|\dot{\widehat{x}}(\theta)\|d\theta.\end{aligned} \tag{6.30}$$

Let us first look at the second term on the RHS. Clearly there exists a $T_1 \in (0,T_0)$ and an $\alpha_2 > 0$ so that

$$\|Fe^{(A+\gamma BF)\frac{n}{p}\overline{T}}\| \leq \alpha_2, \quad \overline{T} \in (0,T_1), \ p > p_1, \ \gamma \in \Gamma,$$

so combining this with our bound on $\widehat{x}$ given in (6.28), it follows that

$$\|Fe^{(A+\gamma BF)\frac{n}{p}\overline{T}}\| \times \|\widehat{x}[k\overline{T}]\| \leq \alpha_1\alpha_2(e^{\lambda_1\overline{T}})^k\|x_0\|, \quad \overline{T} \in (0,T_1), \ p > p_1, \gamma \in \Gamma.$$

Pick $p > p_1$ so that

$$\left(\tfrac{p}{p-n}-1\right) = \tfrac{n}{p-n} \le \varepsilon\sqrt{\tfrac{|\lambda_1|}{2}}\cdot\tfrac{1}{\alpha_1\alpha_2},$$

which means that the second term on the RHS of (6.30) satisfies

$$\left(\tfrac{p}{p-n}-1\right)\|Fe^{(A+\gamma BF)\frac{n}{p}\overline{T}}\|\times\|\widehat{x}[k\overline{T}]\| \le \varepsilon\sqrt{\tfrac{|\lambda_1|}{2}}(e^{\lambda_1\overline{T}})^k\|x_0\|,\ \overline{T}\in(0,T_1),\gamma\in\Gamma. \quad (6.31)$$

Now that we have fixed p, we turn to the two remaining terms on the RHS of inequality (6.30). Using Lemma 6.2 (ii) and (iii), it can easily be shown that there exists a $T_2 \in (0,T_1)$, $\alpha_3 > 0$, and $\lambda_2 < 0$ so that for every $\gamma\in\Gamma$ and $\overline{T}\in(0,T_2)$, we have

$$\int_{k\overline{T}}^{(k+1)\overline{T}}\|\dot{\widehat{x}}(\theta)\|d\theta \le \alpha_3\overline{T}(e^{\lambda_2\overline{T}})^k\|x_0\|, \quad (6.32)$$

$$\tfrac{p}{p-n}\|F\|\times\|\xi[k\overline{T}+nT]\| \le \alpha_3\overline{T}(e^{\lambda_2\overline{T}})^k\|x_0\|. \quad (6.33)$$

By combining (6.30), (6.31), (6.32), and (6.33), it follows that for every $\gamma\in\Gamma$, $\overline{T}\in(0,T_2)$, and $t\in[k\overline{T}+nT,(k+1)\overline{T})$ we have

$$\begin{aligned}
&\|\tfrac{p}{p-n}Fx[k\overline{T}+nT]-F\widehat{x}(t)\| \\
&\quad\le \left(\alpha_3\overline{T}(e^{\lambda_2\overline{T}})^k+\varepsilon\sqrt{\tfrac{|\lambda_1|}{2}}(e^{\lambda_1\overline{T}})^k+\alpha_3\overline{T}(e^{\lambda_2\overline{T}})^k\right)\|x_0\| \\
&\quad= \left(2\alpha_3\overline{T}(e^{\lambda_2\overline{T}})^k+\varepsilon\sqrt{\tfrac{|\lambda_1|}{2}}(e^{\lambda_1\overline{T}})^k\right)\|x_0\| \\
&\Rightarrow \|\tfrac{p}{p-n}Fx[k\overline{T}+nT]-F\widehat{x}(t)\|^2 \\
&\quad\le \left(4\alpha_3^2\overline{T}^2(e^{2\lambda_2\overline{T}})^k+4\alpha_3\varepsilon\sqrt{\tfrac{|\lambda_1|}{2}}\overline{T}(e^{(\lambda_1+\lambda_2)\overline{T}})^k+\tfrac{\varepsilon^2|\lambda_1|}{2}(e^{2\lambda_1\overline{T}})^k\right)\|x_0\|^2.
\end{aligned}$$

Hence, for every $\gamma\in\Gamma$ and $\overline{T}\in(0,T_2)$ we have

$$\begin{aligned}
\Lambda_2(p,\overline{T}) &\le \sum_{k=0}^{\infty}\left(4\alpha_3^2\overline{T}^3(e^{2\lambda_2\overline{T}})^k+4\alpha_3\varepsilon\sqrt{\tfrac{|\lambda_1|}{2}}\overline{T}^2(e^{(\lambda_1+\lambda_2)\overline{T}})^k+\tfrac{\varepsilon^2|\lambda_1|}{2}\overline{T}(e^{2\lambda_1\overline{T}})^k\right)\|x_0\|^2 \\
&= \left(4\alpha_3^2\tfrac{\overline{T}^3}{1-e^{2\lambda_2\overline{T}}}+4\alpha_3\varepsilon\sqrt{\tfrac{|\lambda_1|}{2}}\tfrac{\overline{T}^2}{1-e^{(\lambda_1+\lambda_2)\overline{T}}}+\tfrac{\varepsilon^2}{4}\tfrac{2|\lambda_1|\overline{T}}{1-e^{2\lambda_1\overline{T}}}\right)\|x_0\|^2. \quad (6.34)
\end{aligned}$$

Since

$$\lim_{\overline{T}\to 0}\left(4\alpha_3^2\tfrac{\overline{T}^3}{1-e^{2\lambda_2\overline{T}}}+4\alpha_3\varepsilon\sqrt{\tfrac{|\lambda_1|}{2}}\tfrac{\overline{T}^2}{1-e^{(\lambda_1+\lambda_2)\overline{T}}}+\tfrac{\varepsilon^2}{4}\tfrac{2|\lambda_1|\overline{T}}{1-e^{2\lambda_1\overline{T}}}\right)=\tfrac{\varepsilon^2}{4},$$

it follows that there exists a $T_3\in(0,T_2)$ so that

$$\Lambda_2(p,\overline{T})\le\tfrac{\varepsilon^2}{2},\ \ \gamma\in\Gamma,\overline{T}\in(0,T_3).$$

By combining this with (6.29) we have

$$\|u - \widehat{u}\|_2^2 \leq \varepsilon^2 \|x_0\|^2 \quad \gamma \in \Gamma, \ \overline{T} \in (0, T_3).$$

■

6.5 Main Result

We can now use the results of Section 6.4 to prove our main result.

Theorem 6.1 For every $\varepsilon > 0$, there exists a m^{th} order, p periodic LPC (G, H, J, T) that ensures

$$|J(x_0) - \widehat{J}(x_0)| \leq \varepsilon \|x_0\|^2, \ \gamma \in \Gamma.$$

Proof:

With $\xi := x - \widehat{x}$, observe that

$$\begin{aligned}
J(x_0) &= \int_0^\infty \left[x(t)^T Q x(t) + u(t)^T R u(t)\right] dt \\
&= \|Q^{1/2} x\|_2^2 + \|R^{1/2} u\|_2^2 \\
&\leq \left(\|Q^{1/2}\xi\|_2 + \|Q^{1/2}\widehat{x}\|_2\right)^2 + \left(\|R^{1/2}(u - \widehat{u})\|_2 + \|R^{1/2}\widehat{u}\|_2\right)^2 \\
&= \|Q^{1/2}\widehat{x}\|_2^2 + \|R^{1/2}\widehat{u}\|_2^2 + 2\|Q^{1/2}\xi\|_2 \|Q^{1/2}\widehat{x}\|_2 \\
&\quad + \|Q^{1/2}\xi\|_2^2 + 2\|R^{1/2}(u - \widehat{u})\|_2 \|R^{1/2}\widehat{u}\|_2 + \|R^{1/2}(u - \widehat{u})\|_2^2 \\
&= \widehat{J}(x_0) + 2\|Q^{1/2}\xi\|_2 \|Q^{1/2}\widehat{x}\|_2 + \|Q^{1/2}\xi\|_2^2 \\
&\quad + 2\|R^{1/2}(u - \widehat{u})\|_2 \|R^{1/2}\widehat{u}\|_2 + \|R^{1/2}(u - \widehat{u})\|_2^2. \qquad (6.35)
\end{aligned}$$

It follows from Lemma 6.1 that there exists positive constants β_x and β_u so that for every $\gamma \in \Gamma$, we have

$$\|Q^{1/2}\widehat{x}\|_2 \leq \beta_x \|x_0\|,$$

$$\|R^{1/2}\widehat{u}\|_2 \leq \beta_u \|x_0\|.$$

Now let $\varepsilon_1 > 0$. It follows from Lemma 6.3 that there exists a $T_1 > 0$ and an integer $p > n$ such that

$$\|u - \widehat{u}\|_2 \leq \varepsilon_1 \|x_0\|, \ \overline{T} \in (0, T_1).$$

It follows from Lemma 6.2 that there exists a $T_2 \in (0, T_1)$ such that

$$\|\xi\|_2 \leq \varepsilon_1 \|x_0\|, \ \overline{T} \in (0, T_2).$$

Combining this with (6.35) yields

$$\begin{aligned} |J(x_0) - \widehat{J}(x_0)| \ &\leq \ 2\|Q^{1/2}\|\varepsilon_1\beta_x\|x_0\|^2 + \|Q^{1/2}\|^2\varepsilon_1^2\|x_0\|^2 \\ &\quad +2\|R^{1/2}\|\varepsilon_1\beta_u\|x_0\|^2 + \|R^{1/2}\|^2\varepsilon_1^2\|x_0\|^2. \end{aligned}$$

Since ε_1 was arbitrary, it follows that

$$|J(x_0) - \widehat{J}(x_0)| \leq \varepsilon\|x_0\|^2, \ \ \gamma \in \Gamma, \overline{T} \in (0, T_2).$$

Choose $\overline{T} \in (0, T_2)$ so that $T := \overline{T}/p$ is non-pathological. Hence, we can construct our m^{th} order, p periodic LPC (G, H, J, T) using Proposition 6.1.

■

6.6 An Example

In this section, we provide an example that illustrates the design method. Suppose our nominal plant is

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & 1 \\ 6 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \; x(0) = 0, \\ y(t) &= \begin{bmatrix} -1 & 1 \end{bmatrix} x(t), \end{aligned}$$

so the associated transfer function is

$$\frac{s-1}{(s-2)(s+3)}.$$

We begin by designing our continuous-time state feedback control law using the method described in Section 6.2. The augmented plant (6.3) is given by

$$\begin{aligned} \dot{\eta}(t) &= \overbrace{\begin{bmatrix} 0 & 1 & 0 \\ 6 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix}}^{=\overline{A}} \eta(t) + \gamma \overbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}^{=\overline{B}} \nu(t), \quad \eta(0) = \eta_0 = \begin{bmatrix} 0 \\ 0 \\ y_r \end{bmatrix}, \\ e(t) &= \underbrace{\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}}_{=\overline{C}} \eta(t). \end{aligned}$$

Since the plant is SISO, (A, B) is controllable and (C, A) is observable, we have $(\overline{A}, \overline{B})$ controllable and $(\overline{C}, \overline{A})$ observable. With $R = 0.005$, $\delta = 0.01$[a] and

$$Q_\delta = \begin{bmatrix} \delta & 0 & 0 \\ 0 & \delta & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

we find the control law that minimizes

$$J_\delta(\eta_0) := \int_0^\infty \eta(t)^T Q_\delta \eta(t) + \nu(t)^T R \nu(t) dt \tag{6.36}$$

[a]This value of R was chosen so that the resulting step response rise time was less than $t = 10$ and the value of δ was chosen so that $|J - J_\delta|/J \leq 0.001$.

by finding the unique positive definite solution P_δ of

$$P_\delta \overline{A} + \overline{A}^T P_\delta - P_\delta \overline{B} R^{-1} \overline{B}^T P_\delta + Q_\delta = 0,$$

setting

$$F_\delta = -R^{-1}\overline{B}^T P_\delta,$$

and letting

$$\nu(t) = F_\delta \eta(t).$$

Here

$$P_\delta = \begin{bmatrix} 7.2909147 & 0.23292288 & -4.02276959 \\ 0.23292288 & 0.076529884 & -0.070710678 \\ -4.02276959 & -0.070710678 & 2.8697629 \end{bmatrix},$$

$$F_\delta = \begin{bmatrix} -46.583287 & -15.317632 & 14.143276 \end{bmatrix},$$

so the optimal cost provided by the continuous-time state feedback is

$$J_\delta(\eta_0) = \eta_0^T P_\delta \eta_0 = 2.8697629 \|y_r\|^2.$$

Let us now design a low order LPC so that when $y_r = 1$, the resulting actual cost is within 1% of the optimal cost. To do so, we chose $\overline{T}$ sufficiently small and p sufficiently large so that when $y_r = 1$, the actual cost was less than $1.01 * (2.8697629) = 2.8984605$. Specifically with $p = 6$, $\overline{T} = 0.1$, and $T := \overline{T}/p$, the actual cost can be evaluated to be 2.8980162. We design the LPC using the proof of Proposition 6.1 (i.e. proof of Proposition 4.1); we define

$$\begin{bmatrix} f_0 & f_1 & f_2 \end{bmatrix} := \frac{p}{p-n} F_\delta e^{\overline{A}nT} \begin{bmatrix} \overline{C} \\ \overline{C}(e^{\overline{A}T}) \\ \overline{C}(e^{\overline{A}T})^2 \end{bmatrix}$$

$$= \begin{bmatrix} -8.4215737(10)^4 & 1.7273468(10)^5 & -8.8497730(10)^4 \end{bmatrix}$$

and construct the LPC (6.9) by choosing

$$(G,H,J)(k) := \begin{cases} (0,-8.4215737(10)^4,0) & k=0 \\ (1,1.7273468(10)^5,0) & k=1, \\ (1,-8.8497730(10)^4,0) & k=2, \\ (1,0,1) & k=3,4,5. \end{cases}$$

Figure 6.2 compares the actual output $y(t)$ to the ideal output $\widehat{y}(t)$ and the actual control signal $u(t)$ to the ideal control signal $\widehat{u}(t)$.

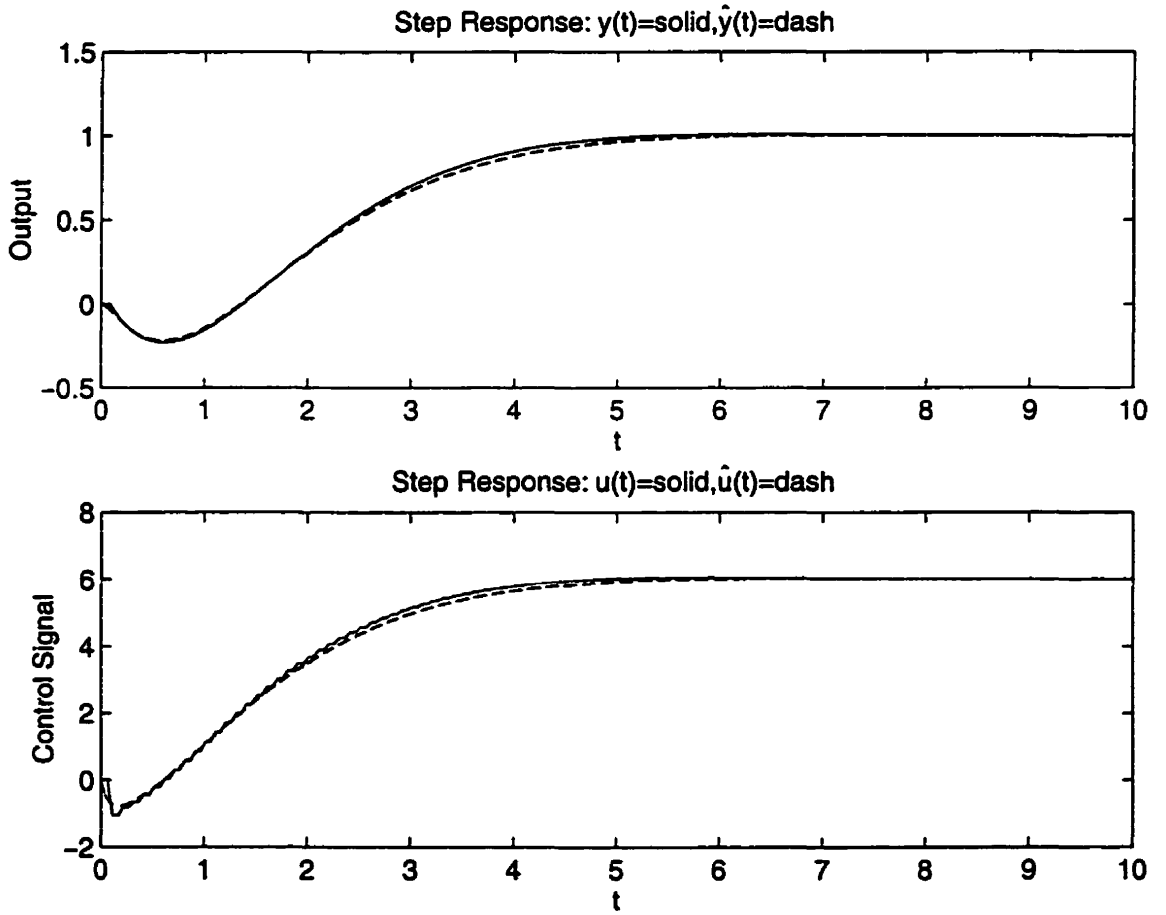


Figure 6.2: Optimal step tracking simulation.

To evaluate the robustness properties of this controller, we use a 2-dimensional search to approximately determine

$$\Gamma = \{\gamma \in \mathbf{C} : (G, H, J, T) \text{ stabilizes } (A, \gamma B, C)\},$$

and illustrate it in Figure 6.3. For comparison, we also approximately determine the set

$$\overline{\Gamma} = \{\gamma \in \mathbf{C} : \mathrm{sp}(\overline{A} + \gamma \overline{B} F_\delta) \subset \mathbf{C}^-\},$$

illustrate it in Figure 6.3, and note that $\Gamma \subset \overline{\Gamma}$.

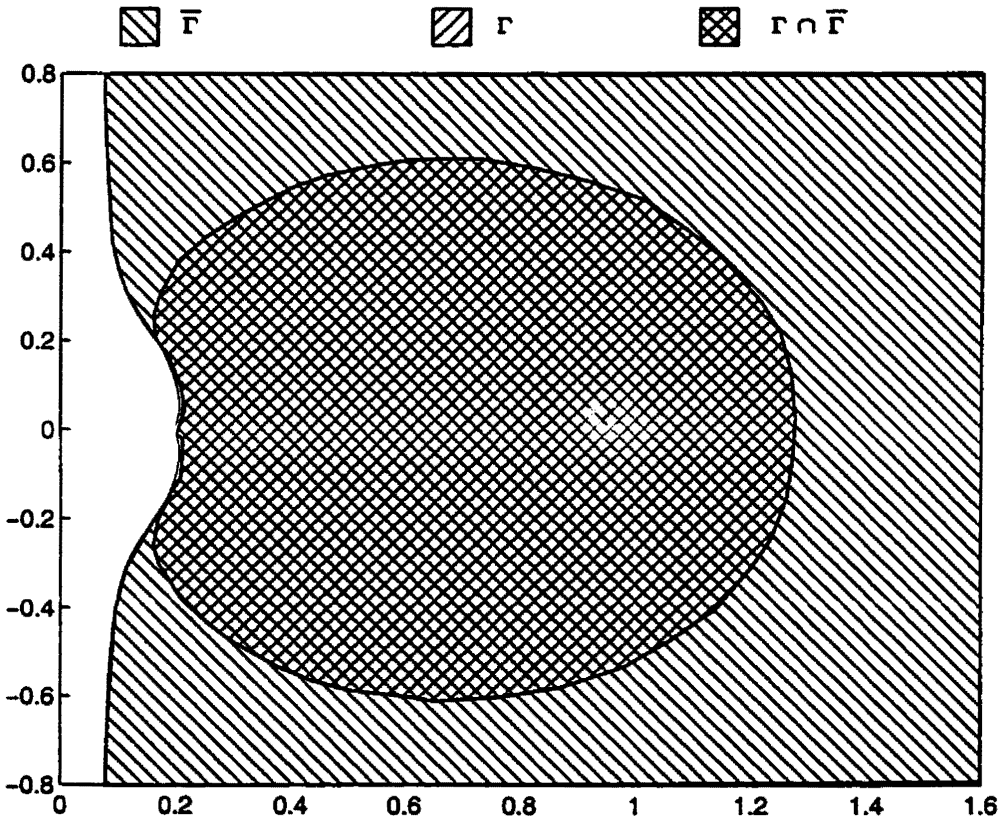


Figure 6.3: Comparing $\overline{\Gamma}$ to Γ ($\Gamma \subset \overline{\Gamma}$).

Let us now qualitatively investigate the noise properties of this controller. Suppose that with $\omega_n \in \{10, 20, 30\}$, we introduce the output noise signal

$$w(t) = 0.01 \sin(\omega_n t), \ t \geq 0,$$

to the closed loop system as illustrated in Figure 6.4:

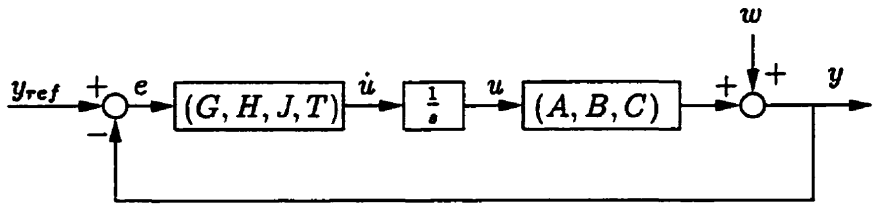


Figure 6.4: Optimal step tracking closed loop system with output noise.

The resulting ideal state feedback controlled system response is compared to the actual LPC controlled system response in Figure 6.5. Note that the tracking performance degrades as the frequency of the noise becomes large.

6.7 Summary and Concluding Remarks

In this chapter, we posed an optimal step tracking problem for MIMO LTI plants that are possibly non-minimum phase, and we showed that we could design a stable low order sampled-data controller that could provide near optimal step tracking in an LQR sense, even when the state of the plant cannot be measured. This controller is superior to an LTI controller since it is capable of recovering the gain phase margins of the optimal LQR continuous-time state feedback control law and the resulting LPC is stable. Unlike the GSHF controller discussed in Chapter 5, the control signal generated by this LPC does not become large when the sampling period tends to zero, but approaches the optimal state feedback control law. Unfortunately, this controller is more sensitive to noise than the ideal optimal state feedback control law.

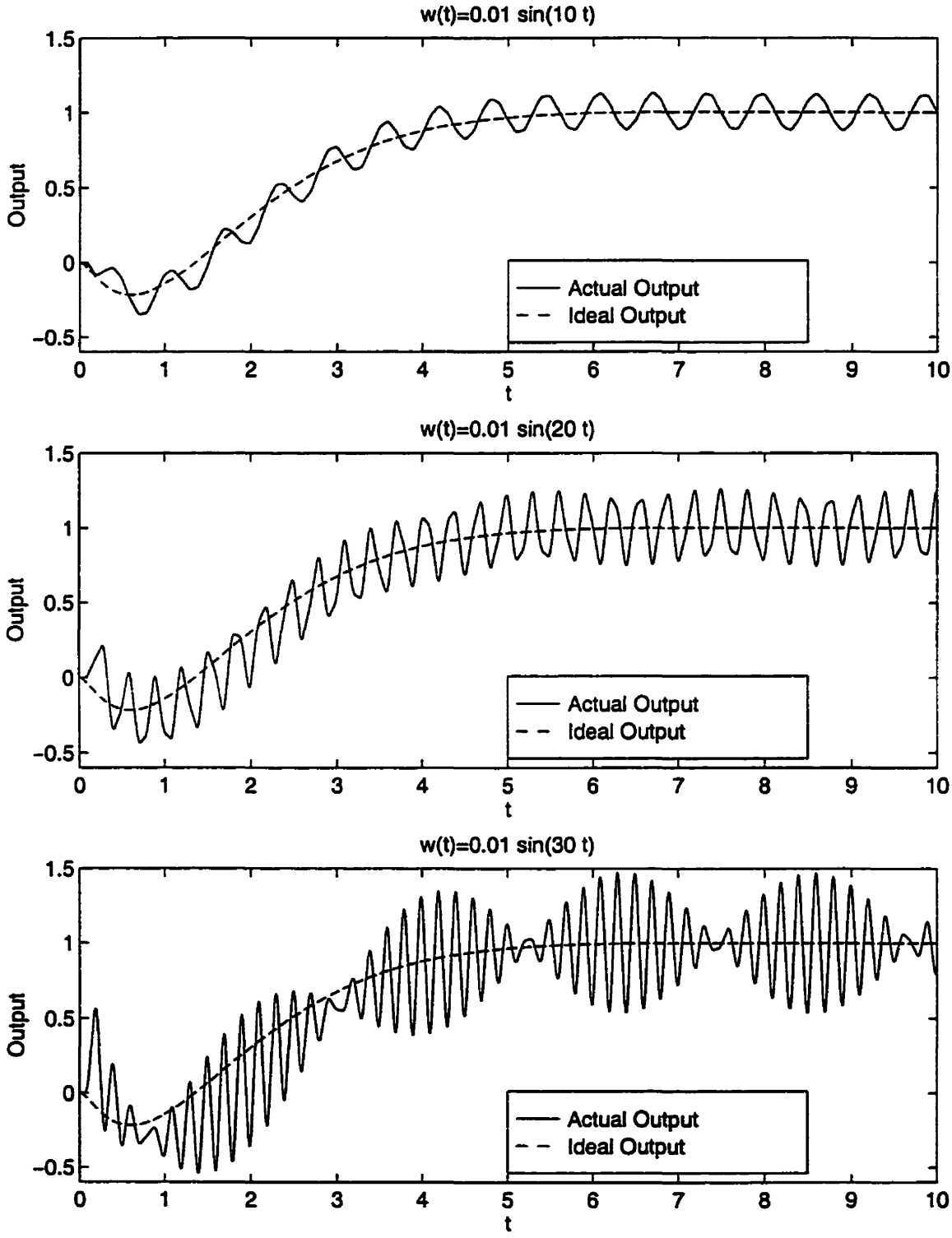


Figure 6.5: Optimal step tracking simulation with output noise.

Chapter 7

Conclusions

7.1 A Summary

We have shown that given a multi-input, multi-output (MIMO) linear time-invariant (LTI) strictly proper plant, it is possible to design a low order linear periodic controller (LPC) that can provide any desired gain margin and any desired phase margin up to 90 degrees. If we use a static generalized sampled-data hold function (GSHF) controller to accomplish this, then the intersample performance is typically poor, but the controller can tolerate dynamic additive perturbations to the nominal model. If we use the first order LPC presented in Chapter 4, then the intersample performance is satisfactory, but the tolerance to unstructured uncertainty in the nominal model deteriorates as the sampling period tends to zero.

We have also shown that given a single-input, single-output (SISO) LTI strictly proper plant that is minimum phase together with a SISO LTI strictly proper reference model, it is possible to design a static GSHF controller that solves the sampled-data model reference control problem (MRCP). While the GHSF controller can provide arbitrarily good tracking and disturbance rejection, the control signal can become be very large.

Finally, we have shown that given an arbitrary MIMO LTI strictly proper plant where measurements of the state are not available to the controller, it is possible to design a low order LPC that will track step inputs in a near optimal fashion. This controller has a moderate control signal even as the sample period tends to zero, and can be designed to provide a gain and phase margin close to that of an optimal state feedback control law.

All the controllers presented are stable in the sense that if the input to the controller is zero, then the state of the controller will approach zero as time approaches infinity. In fact, all the controllers presented are deadbeat.

7.2 Future Research

In Section 3.5, we provided an analysis that showed that the GSHF controller designed to provide a desired gain and phase margin also tolerated stable additive perturbations to the nominal plant. First, we are quite certain that it would be possible to show that the additive perturbation need not be stable, but we would only require that the nominal plant and the perturbed plant have the same number of unstable poles. Secondly, there are many other uncertainty models that can be found in the literature, and the tolerance of the GSHF controller to other types of unstructured uncertainty could be determined. A complete analysis of the tolerance of the LPC presented in Chapter 4 to unstructured uncertainty in the nominal model would also be useful.

Khargonekar et al. [31] provided a solution to the problem of simultaneously stabilizing a finite family of discrete-time LTI bicausal plants using periodic control, while Miller [37] showed that it is possible to find a single low order discrete-time LPC that simultaneously stabilizes a finite set of LTI plants. It may be possible to use some of the ideas presented in Chapter 6 to come up with a low order LPC that simultaneously stabilizes a finite set of continuous-time strictly proper LTI plants, while providing performance

arbitrarily close to that provided by an ideal state feedback control law.

For the MRCP, we assumed that the plant was SISO. The result should be generalizable to the case where the plant is MIMO. Furthermore, recall that the static GSHF controller that solves the MRCP suffers from large control signals. However, it can be shown that there exists an ideal continuous-time state feedback control law that solves the MRCP, and that the magnitude of this control signal is typically smaller than the GSHF control signal. Hence, using some of the ideas presented in Chapters 4 and 6, we could perhaps design a low order LPC that emulates this ideal continuous-time control law. However, those low order LPC's do not tolerate disturbances well, so the noise tolerance may be poor.

In Chapter 4 we showed that as the sampling period of the LPC tends to zero, the output signal of the LPC controlled system approached the output of a desired continuous-time state-feedback controlled system in the ∞-norm. In Chapter 6 we showed that as the sampling period of the LPC tends to zero and the periodicity of the LPC parameters tends to infinity, both the output and control signal of the LPC controlled system approaches the output and control signal of a desired continuous-time state-feedback controlled system in the 2-norm. It would be interesting to determine if similar results can be shown for the general p-norm case. An approach that might be fruitful might be to show that similar results holds for the 1-norm and ∞-norm, which would imply that our desired result holds for the p-norm [15, pg. 17, Fact 7].

Extending the optimal step tracking problem to that of tracking a more general class of reference signals, e.g. sinusoids, in an optimal fashion would be a natural extension of the results of Chapter 6.

Appendix A

Discrete-Time Approach to Robust Stability: Static GSHF Controllers

A.1 Discrete-Time Approach

In this Appendix, we improve on the design of the static GSHF controller that was presented in Chapter 3. The controller that we synthesize here is based on a discrete-time approach, and like the controller in Chapter 3, it will be capable of simultaneously providing any desired gain margin and any phase margin of up to 90 degrees, as long as the sampling period is sufficiently small. However, unlike the controller of Chapter 3 where we typically needed the sampling period to be small to even stabilize the nominal plant, this GSHF controller will be capable of stabilizing the nominal plant for almost all sampling periods.

Before presenting any results, let us first provide some motivation for this approach. Recall that

$$F = \int_0^T e^{A(T-\tau)} B \, \overline{F}(\tau) \, d\tau \in \mathbf{R}^{n \times r},$$

and the closed loop discrete-time system satisfies

$$x[(k+1)T] = \left(e^{AT} + \gamma FC\right) x[kT],\ k \in \mathbf{Z}^+.$$

Following a similar approach to that found in Section 3.3, define

$$F_T := T^{-1}F = T^{-1}\int_0^T e^{A(T-\tau)}\, B\, \overline{F}(\tau)\, d\tau.$$

Then it follows that the GSHF controller (3.3) stabilizes (3.2) if and only if

$$\mathrm{sp}(e^{AT} + \gamma F_T(TC)) \subset \{z \in \mathbf{C} : |z| < 1\}.$$

We use the notation F_T here and F_0 in Chapter 3, since in Chapter 3, F_0 was chosen independent of T while here F_T will depend on the sampling period T.

Given the observable pair (C, A) and a large, but non-pathological T, we know that (TC, e^{AT}) is observable, and thus we can directly design F_T such that

$$\mathrm{sp}(e^{AT} + F_T(TC)) \subset \{z \in \mathbf{C} : |z| < 1\}$$

using any one of a number of **discrete-time** state-feedback controller design techniques; hence, the motivation for the name of this approach. Thus, we will no longer require T to be small in order to stabilize the nominal plant. However, we must also ensure that our choice of F_T will result in a controller that will be capable of providing good gain/phase margins. Recall from Section 3.3 that F_0 was defined as the continuous-time LQR optimal gain, and that by choosing

$$F = TF_0$$

the resulting GSHF controller provided desirable gain/phase margin properties as T tended to zero. Here, we have chosen

$$F = TF_T,$$

so if we design F_T so that for small T we have $F_T \approx F_0$, then we might expect the resulting GSHF controller will share the desirable gain/phase margin properties of the GSHF controller designed using the continuous-time LQR approach. Designing F_T in

this way is accomplished by solving a modified discrete-time LQR problem, which we discuss in the next subsection.

A.1.1 Preliminary Discrete-Time LQR Results

In this subsection, we first formulate the modified discrete-time LQR problem and then prove the preliminary result that as $T \to 0$, we have

$$F_T \to F_0.$$

Since we want F_T to approach F_0, we begin by first modifying the C matrix as was done in Subsection 3.3.1; namely, with $\overline{\phi} \in \left[0, \frac{\pi}{2}\right)$, $\underline{\rho} \in (0, 1]$, and

$$\alpha = 2\cos(\overline{\phi}),$$

we define

$$\widehat{C} := \alpha\underline{\rho}C. \tag{A.1}$$

To improve the readability of the following equations, we also introduce the notation

$$A_d := \mathrm{e}^{AT}.$$

Now consider the auxiliary system

$$w[k+1] = A_d^T w[k] + T\widehat{C}^T v[k], \quad w(0) = w_0, \tag{A.2}$$

and using the same positive definite symmetric weighting matrices Q and R as those in Section 3.3.1, we wish to find the control law which, for each w_0, minimizes

$$\sum_{k=0}^{\infty} w[k]^T Q w[k] + v[k]^T R v[k]. \tag{A.3}$$

Using Lemma 12.1′ and Theorem 12.2′ found in [55, Exercise 12.7], it follows that for every non-pathological T, the optimal control law is of the form

$$v[k] = F_T^T w[k]. \tag{A.4}$$

We can obtain the optimal gain F_T by first solving the discrete-time algebraic Riccati equation

$$P_T - A_d P_T A_d^T + A_d P_T (T\widehat{C})^T [(T\widehat{C}) P_T (T\widehat{C})^T + R]^{-1} (T\widehat{C}) P_T A_d^T - Q = 0 \qquad \text{(A.5)}$$

for the unique positive definite symmetric solution P_T, and then setting

$$F_T = -A_d P_T (T\widehat{C})^T [(T\widehat{C}) P_T (T\widehat{C})^T + R]^{-1}. \qquad \text{(A.6)}$$

For convenience, let us rewrite (3.13) and (3.14) here using the above notation:

$$P_0 A^T + A P_0 - P_0 \widehat{C}^T R^{-1} \widehat{C} P_0 + Q = 0, \qquad \text{(A.7)}$$

$$F_0 = -P_0 \widehat{C}^T R^{-1}. \qquad \text{(A.8)}$$

The remainder of this subsection is devoted to proving

$$\lim_{T \to 0} F_T = F_0.$$

We begin by showing that as $T \to 0$, the solution P_T of the discrete-time Riccati equation (A.5) is related to to the solution P_0 of the continuous-time Riccati equation (A.7) via

$$\lim_{T \to 0} T P_T = P_0.$$

Roughly speaking, this is done by by substituting

$$A_d = e^{AT} = I + AT + \mathcal{O}(T^2),$$

into (A.5), and showing that the resulting equation can be written as

$$(TP_T) A^T + A(TP_T) - (TP_T) \widehat{C}^T R^{-1} \widehat{C} (TP_T) + \underbrace{(Q + \mathcal{O}(T))}_{=: \overline{Q}} = 0. \qquad \text{(A.9)}$$

Then by comparing (A.9) and (A.7), we expect that since $\overline{Q} \to Q$ as $T \to 0$, we have $TP_T \to P_0$ as $T \to 0$ as well. Once we have this result, it is straightforward to show that $F_T \to F_0$ as $T \to 0$ by simply using the definitions of F_0 and F_T.

Lemma A.1 For every $\varepsilon > 0$, there exists a $T_{max} > 0$ such that for every $T \in (0, T_{max})$, we have

$$\begin{aligned} \|TP_T - P_0\| &< \varepsilon, \\ \|F_T - F_0\| &< \varepsilon. \end{aligned}$$

Proof:

Let T be nonpathological, P_T be the unique positive definite solution of (A.5), and define

$$P_\Delta := TP_T.$$

We start by showing that $P_\Delta = \mathcal{O}(1)$. Recall that by our choice of F_T, when the control law (A.4) is applied to (A.2), the cost function (A.3) is minimized, and in fact the associated optimal cost is $w_0^T P_T w_0$. Now choose $\widetilde{F} \in \mathbf{R}^n$ so that $A^T + \widehat{C}^T\widetilde{F}^T$ is stable; it is easy to see that $A_d^T + T\widehat{C}^T\widetilde{F}^T$ is stable for small enough T. Next, consider the control law

$$v[k] = \widetilde{F}^T w[k],$$

so that the closed loop system is

$$w[k+1] = (A_d^T + T\widehat{C}^T\widetilde{F}^T)w[k],\ w(0) = w_0,$$

and the associated cost is

$$w_0^T \left[\sum_{k=0}^{\infty}(A_d + T\widetilde{F}\widehat{C})^k(Q + \widetilde{F}R\widetilde{F}^T)(A_d^T + T\widehat{C}^T\widetilde{F}^T)^k\right] w_0.$$

Using a Lyapunov argument, it can be shown that there exists a $T_0 > 0$, $\alpha > 0$ and $\lambda < 0$ so that

$$\|(A_d + T\widetilde{F}\widehat{C})^k\| \leq \alpha(\mathrm{e}^{\lambda T})^k,\ T \in (0, T_0),$$

so for every $T \in (0, T_0)$, the cost is bounded above by

$$\sum_{k=0}^{\infty}\alpha^2(\mathrm{e}^{2\lambda T})^k\|Q + \widetilde{F}R\widetilde{F}^T\| \times \|w_0\|^2 = \frac{\alpha^2\|Q + \widetilde{F}R\widetilde{F}^T\|}{1 - \mathrm{e}^{2\lambda T}}\|w_0\|^2 = \mathcal{O}(T^{-1})\|w_0\|^2;$$

this also is an upper bound on the optimal cost $w_0^T P_T w_0$. Hence, we have

$$\begin{aligned}
&w_0^T P_T w_0 \leq \mathcal{O}(T^{-1})\|w_0\|^2 \\
&\Rightarrow Tw_0^T P_T w_0 = w_0^T P_\Delta w_0 \leq \mathcal{O}(1)\|w_0\|^2 \\
&\Rightarrow P_\Delta = \mathcal{O}(1).
\end{aligned}$$

We now go on to show our desired result. If we substitute $P_T = P_\Delta/T$ and

$$A_d = \mathrm{e}^{AT} = I + AT + \mathcal{O}(T^2)$$

into (A.5), we get

$$\begin{aligned}
&\tfrac{P_\Delta}{T} - (I + AT + \mathcal{O}(T^2))\tfrac{P_\Delta}{T}(I + A^TT + \mathcal{O}(T^2)) \\
&\quad +(I+AT+\mathcal{O}(T^2))P_\Delta\widehat{C}^T(T\widehat{C}P_\Delta\widehat{C}^T+R)^{-1}\widehat{C}P_\Delta(I+A^TT+\mathcal{O}(T^2))-Q=0 \\
\Rightarrow\ &\tfrac{P_\Delta}{T} - (\tfrac{P_\Delta}{T} + AP_\Delta + P_\Delta A^T + \mathcal{O}(T)) \\
&\qquad +(P_\Delta\widehat{C}^T + \mathcal{O}(T))(\mathcal{O}(T) + R)^{-1}(\widehat{C}P_\Delta + \mathcal{O}(T)) - Q = 0 \\
\Rightarrow\ &-AP_\Delta - P_\Delta A^T - \mathcal{O}(T) \\
&\qquad +(P_\Delta\widehat{C}^T + \mathcal{O}(T))R^{-1}(I + \mathcal{O}(T))^{-1}(\widehat{C}P_\Delta + \mathcal{O}(T)) - Q = 0.
\end{aligned}$$

But $(I + \mathcal{O}(T))^{-1} = I + \mathcal{O}(T)$, so

$$AP_\Delta + P_\Delta A^T - P_\Delta\widehat{C}^TR^{-1}\widehat{C}P_\Delta + Q + \underbrace{\mathcal{O}(T)}_{=:\Delta_Q} = 0.$$

Since $P_T = P_T^T > 0$, it follows that $P_\Delta = P_\Delta^T > 0$. Then, together with the definition of Δ_Q and the fact that $Q = Q^T$ and $R = R^T$, it also follows that $\Delta_Q = \Delta_Q^T$. Furthermore, since $\Delta_Q = \mathcal{O}(T)$, we can choose $T_1 < T_0$ such that for $T \in (0, T_1)$, we have

$$Q + \Delta_Q > 0.$$

Hence, from Lemma 2.3, there exists a $c_1 > 0$ such that for every $T \in (0, T_1)$, we have

$$\|P_\Delta - P_0\| = \|TP_T - P_0\| \le \max\left\{\left(\frac{\|P_0\| + c_1\|\Delta_Q\|}{\lambda_{min}(Q) - \|\Delta_Q\|}\right)\|\Delta_Q\|, c_1\|\Delta_Q\|\right\}.$$

Since Δ_Q is $\mathcal{O}(T)$, it follows that $P_\Delta - P_0$ is as well, so our first result follows.

Using (A.6) and (A.8) we have

$$\begin{aligned}
F_T - F_0 &= -A_dP_T(T\widehat{C})^T[(T\widehat{C})P_T(T\widehat{C})^T + R]^{-1} + P_0\widehat{C}^TR^{-1} \\
&= -(I + AT + \mathcal{O}(T))P_\Delta\widehat{C}^T[T\widehat{C}P_\Delta\widehat{C}^T + R]^{-1} + P_0\widehat{C}^TR^{-1} \\
&= -(P_\Delta\widehat{C}^T + \mathcal{O}(T))(\mathcal{O}(T) + R)^{-1} + P_0\widehat{C}^TR^{-1} \\
&= -(P_\Delta\widehat{C}^T + \mathcal{O}(T))R^{-1}(I + \mathcal{O}(T)) + P_0\widehat{C}^TR^{-1} \\
&= -(P_\Delta - P_0)\widehat{C}^TR^{-1} + \mathcal{O}(T).
\end{aligned}$$

Since $P_\Delta - P_0$ is $\mathcal{O}(T)$ from above, our last result follows.

■

A.1.2 Controller Design

In this subsection we will use the results of Subsection A.1.1 to provide an alternate proof to Theorem 3.1. Furthermore, we will show that we do not necessarily need T to be less than some $T_{max} > 0$ in order to stabilize the nominal plant.

Proof:(Alternate proof of Theorem 3.1)

This proof is based on a discrete-time Lyapunov approach. Given $\overline{\phi} \in [0, \frac{\pi}{2})$, $\underline{\rho} \in (0,1]$, and $\overline{\rho} \in [1, \infty)$, define $\widehat{C}$ via (A.1), and choose positive definite symmetric matrices Q and R. Using Theorem 2.2, we can choose a $T_0 > 0$ so that $(\widehat{C}, A_d)$ is observable for every $T \in (0, T_0)$. It follows that for $T \in (0, T_0)$, there exists a unique positive definite symmetric solution P_T for the discrete-time Riccati equation (A.5), and the optimal discrete-time LQR gain F_T is given by (A.6). Define

$$A_n := A_d^T, \quad B_n := T\widehat{C}^T, \quad F_n := F_T^T,$$

and with $\widehat{\rho} \in [\frac{1}{\alpha}, \frac{\overline{\rho}}{\alpha\underline{\rho}}]$ and $\phi \in [-\overline{\phi}, \overline{\phi}]$, consider

$$w[k+1] = (A_n + \widehat{\rho}e^{j\phi}B_nF_n)w[k], \quad w[0] = w_0 \in \mathbf{C}^n.$$

Fix w_0 and consider the Lyapunov candidate function $V : \mathbf{C}^n \to \mathbf{C}$:

$$V(w) := w^*P_Tw. \tag{A.10}$$

Since $P_T = P_T^T$ we know that V is real-valued, so

$$\Delta V(w[k]) := V(w[k+1]) - V(w[k])$$

is also real-valued. Expanding $\Delta V(w[k])$ and simplifying, we have

$$\begin{aligned}\Delta V(w[k]) &= w^*[k](A_n^TP_TA_n - P_T)w[k] + \widehat{\rho}^2w^*[k](F_n^TB_n^TP_TB_nF_n)w[k] \\ &\quad +\widehat{\rho}w^*[k](e^{-j\phi}F_n^TB_n^TP_TA_n + e^{j\phi}A_n^TP_TB_nF_n)w[k].\end{aligned}$$

Using (A.5) and (A.6), this can be written as

$$\begin{aligned}\Delta V(w[k]) &= -w^*[k]Qw[k] + (1 - 2\widehat{\rho}\cos(\phi))w^*[k](F_n^T(B_n^TP_TB_n + R)F_n)w[k] \\ &\quad +\widehat{\rho}^2w^*[k](F_n^TB_n^TP_TB_nF_n)w[k].\end{aligned}$$

From the definition of α given in (3.11), the bounds on ϕ, and the upper bound on $\widehat{\rho}$, we see that

$$1 - 2\widehat{\rho}\cos(\phi) \leq 0,$$

so since $P_T > 0$ and $R > 0$, it follows that

$$\begin{aligned}\Delta V(w[k]) &\leq -w^*[k]Qw[k] + \frac{\overline{\rho}^2}{\alpha^2\underline{\rho}^2}w^*[k](F_n^T B_n^T P_T B_n F_n)w[k] \\ &= -w^*[k]Qw[k] + \overline{\rho}^2 w^*[k](T^2 F_T C P_T C^T F_T^T)w[k].\end{aligned}$$

We now go on to show that as $T \to 0$, the second term in the above inequality is dominated by the first term. Using Lemma A.1, it follows that there exists positive constants c_1, c_2, and $T_1 \in (0, T_0)$ so that for $T \in (0, T_1)$ we have

$$\|F_T\| < c_1,$$

$$\|P_T\| < \frac{c_2}{T}.$$

Hence, for $T \in (0, T_1)$ and $w[k] \neq 0$, we have

$$\begin{aligned}\left|\overline{\rho}^2 w^*[k](T^2 F_T C P_T C^T F_T^T)w[k]\right| &< \overline{\rho}^2 T^2 \|F_T\|^2 \times \|P_T\| \times \|C\|^2 \times \|w[k]\|^2 \\ &\leq \overline{\rho}^2 T c_1^2 c_2 \|C\|^2 \times \|w[k]\|^2.\end{aligned}$$

If we let

$$T_{max} < \min\left\{T_1, \frac{\lambda_{min}(Q)}{\overline{\rho}^2 c_1^2 c_2 \|C\|^2}\right\},$$

then it follows that there exists a positive definite matrix U so that for $T \in (0, T_{max})$

$$\Delta V(w[k]) \leq -w[k]^* U w[k], \quad k \in \mathbf{Z}^+.$$

Hence, for every w_0, $V(w[k])$ goes to zero as $k \to \infty$; since P_T is positive definite, it follows that $w[k]$ goes to zero as $k \to \infty$ as well, so

$$\begin{aligned}&\mathrm{sp}(A_n + \widehat{\rho}e^{j\phi}B_n F_n) \subset \{z \in \mathbf{C} : |z| < 1\} \\ \Leftrightarrow \quad &\mathrm{sp}(A_d + T\widehat{\rho}e^{j\phi}F_T\widehat{C}) \subset \{z \in \mathbf{C} : |z| < 1\} \\ \Leftrightarrow \quad &\mathrm{sp}(A_d + T\alpha\underline{\rho}\widehat{\rho}e^{j\phi}F_T C) \subset \{z \in \mathbf{C} : |z| < 1\}.\end{aligned}$$

But this holds for $\alpha\underline{\rho}\widehat{\rho} \in [\underline{\rho}, \overline{\rho}]$, so for $\rho \in [\underline{\rho}, \overline{\rho}]$, and $\phi \in [-\overline{\phi}, \overline{\phi}]$, we have

$$\mathrm{sp}(A_d + \rho e^{j\phi} F_T(TC)) \subset \{z \in \mathbf{C} : |z| < 1\}.$$

Finally, with $T \in (0, T_{max})$ and $F = TF_0$, we find an $\overline{F}(t)$ which satisfies (3.4).

■

Remark A.1 With $\overline{\phi} \in [0, \frac{\pi}{2})$, $\underline{\rho} \in (0, 1]$, and $\overline{\rho} \in [1, \infty)$, we now summarize an alternate algorithm for constructing a GSHF controller $(\overline{F}, T)$ that stabilizes every system in

$$\{(A, B, \gamma C) : \gamma \in \Gamma(\underline{\rho}, \overline{\rho}, \overline{\phi})\}.$$

i) Choose symmetric positive definite matrices Q and R. Let $\alpha = 2\cos(\overline{\phi})$ and

$$\widehat{C} := \alpha\underline{\rho}C.$$

ii) Determine a value for T so that when $A_d = e^{AT}$, P_T satisfies

$$P_T - A_d P_T A_d^T + A_d P_T (T\widehat{C})^T [(T\widehat{C}) P_T (T\widehat{C})^T + R]^{-1} (T\widehat{C}) P_T A_d^T - Q = 0,$$

and F_T satisfies

$$F_T = -A_d P_T (T\widehat{C})^T [(T\widehat{C}) P_T (T\widehat{C})^T + R]^{-1},$$

we have

$$\overline{\rho}^2 T^2 \| F_T C P_T C^T F_T^T \| < \lambda_{min}(Q).$$

iii) Let $F = TF_T$ and use a method to find an $\overline{F}(t)$ to satisfy (3.4), e.g. use (3.6).

□

Remark A.2 The T obtained in the above algorithm is based on a Lyapunov approach, and hence is typically quite conservative; perhaps a better choice for the Lyapunov candidate function (A.10) would result in a less conservative value of T. However, from a practical point of view, it might be better to compute the combined gain margin and phase margin for various values of T, and obtain the largest one which achieves the desired robustness.

□

A.1.3 An Example

Let us consider the same example mentioned in Section 3.3.3. Namely, suppose our nominal plant is

$$(A,B,C) = \left(\begin{bmatrix} -1 & 0 \\ 0 & 0.5 \end{bmatrix}, \begin{bmatrix} \frac{4}{3} \\ -\frac{1}{3} \end{bmatrix}, \begin{bmatrix} 1 & 1 \end{bmatrix} \right),$$

with associated transfer function

$$P_0(s) := C(sI - A)^{-1}B = \frac{s-1}{(s-0.5)(s+1)}.$$

Using the algorithm outline in Remark A.1, we can construct a static GSHF controller to stabilize every system in

$$\{(A,B,\gamma C) : \gamma \in \Gamma(0.75, 6, 70^\circ)\}.$$

i) Let $Q = I$ and $R = 1$. Here $\alpha = 2\cos(70^\circ) = 0.6840403$, and

$$\widehat{C} = \alpha \underline{\rho} C = \begin{bmatrix} 0.6840403 & 0.6840403 \end{bmatrix}.$$

ii) We determine T by plotting $\overline{\rho}^2 T^2 \| F_T C P_T C^T F_T^T \|$ vs T. From Figure A.1 we see that for $T \leq 0.0008774830$ we have

$$\overline{\rho}^2 T^2 \| F_T C P_T C^T F_T^T \| \leq 0.9833175 < \lambda_{min}(Q) = 1.$$

Recall from Remark A.2 that choosing $T = 0.0008774830$ is quite conservative. As was done in Subsection 3.3.3, we now propose a method to approximately determine a less conservative sampling period. Using a 2-dimensional search algorithm, we determine

$$\begin{aligned} \underline{\rho}_{d.t.}(T) := \min\{ \widehat{\rho} \in (0,1] : \mathrm{sp}(e^{AT} + \rho e^{j\phi} T F_T C) \subset \{z \in \mathbf{C} : |z| < 1\}, \\ \rho \in [\widehat{\rho}, 1], \phi \in [-\overline{\phi}, \overline{\phi}]\}, \\ \overline{\rho}_{d.t.}(T) := \max\{\widehat{\rho} \in [1, \infty) : \mathrm{sp}(e^{AT} + \rho e^{j\phi} T F_T C) \subset \{z \in \mathbf{C} : |z| < 1\}, \\ \rho \in [1, \widehat{\rho}], \phi \in [-\overline{\phi}, \overline{\phi}]\}; \end{aligned}$$

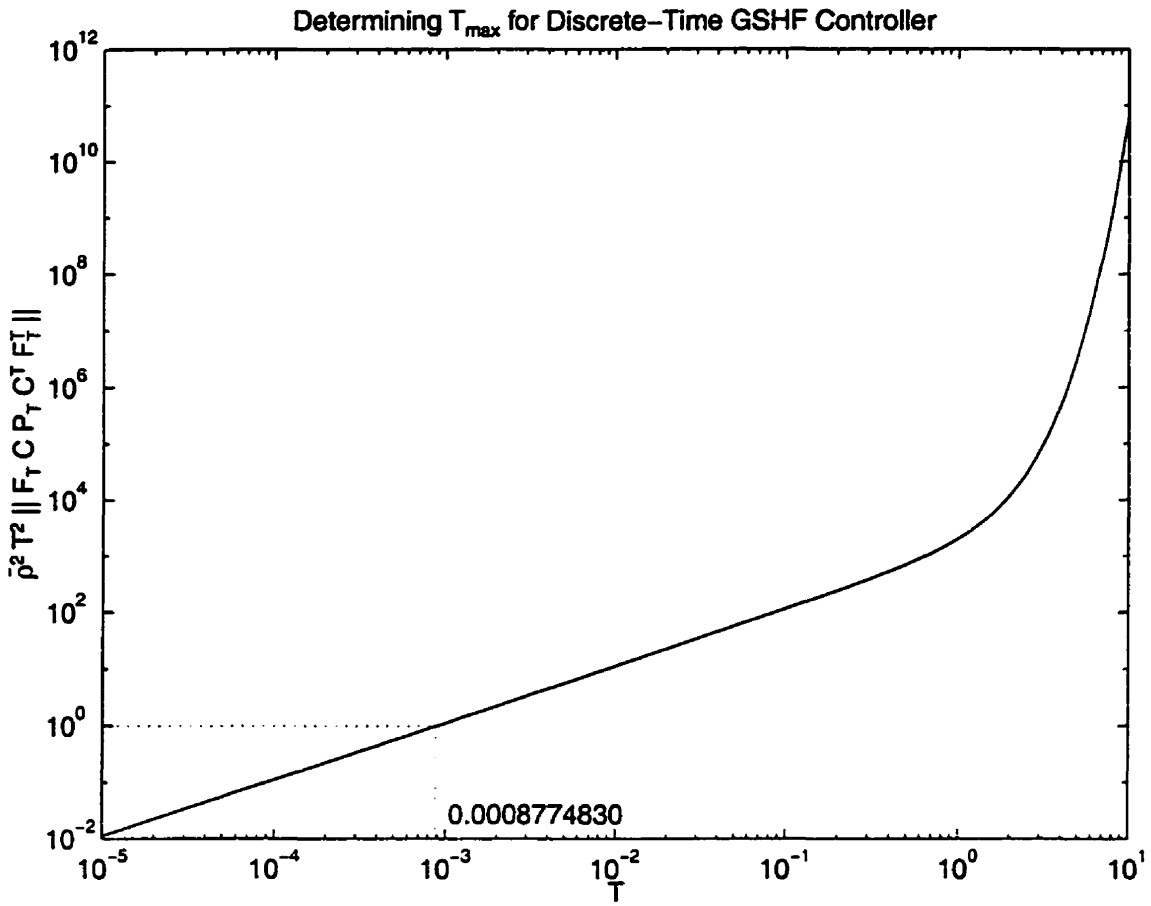


Figure A.1: Plot of $\overline{\rho}^2 T^2 \| F_T C P_T C^T F_T^T \|$ vs T.

to be consistent with Subsection 3.3.3, we refer to $\underline{\rho}_{d.t.}(T)$ ($\overline{\rho}_{d.t.}(T)$) as the "combined lower (upper) gain/phase margin" provided by the GSHF controller $(\overline{F}, T)$ using the "discrete-time" approach. A plot of $\underline{\rho}_{d.t.}(T)$ and $\overline{\rho}_{d.t.}(T)$ is provided in Figure A.2. Observe that for $T = 0.0416000$, we achieve the desired gain and phase margin. For $T = 0.04160$, we have

$$P_T = \begin{bmatrix} 12.63665 & -8.988131 \\ -8.988131 & 130.7065 \end{bmatrix}, \quad F_T = \begin{bmatrix} -0.06979002 \\ -2.475947 \end{bmatrix},$$

so we set

$$F = T F_T = \begin{bmatrix} -0.002861391 \\ -0.1015138 \end{bmatrix},$$

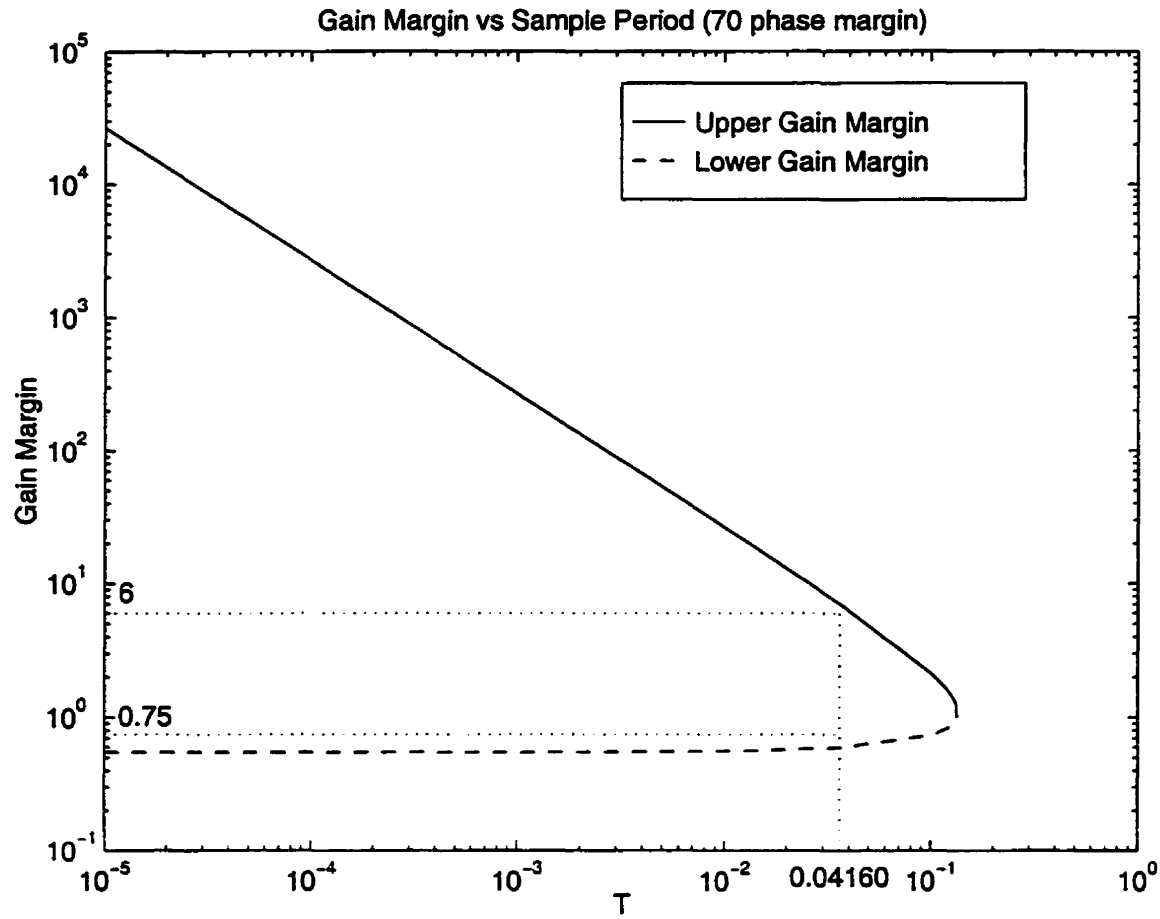


Figure A.2: Combined upper and lower gain and phase margin as a function of T (d.t. approach).

and from (3.6), we can choose

$$\overline{F}(t) = -2.397830(10)^4 e^{-(0.0416-t)} + 2.325921(10)^4 e^{0.5*(0.0416-t)}.$$

Figure A.3 illustrates the response of the closed loop system at the GSHF sample points when the initial condition $x_0 = [\ 1 \quad 0\ ]^T$ and the scalar gain uncertainty $\gamma = 4$. While the GSHF controller provides adequate performance at the sample points, it can be seen in Figure A.4 that the intersample performance is quite poor. This is primarily due to the fact that the generalized hold has large gains.

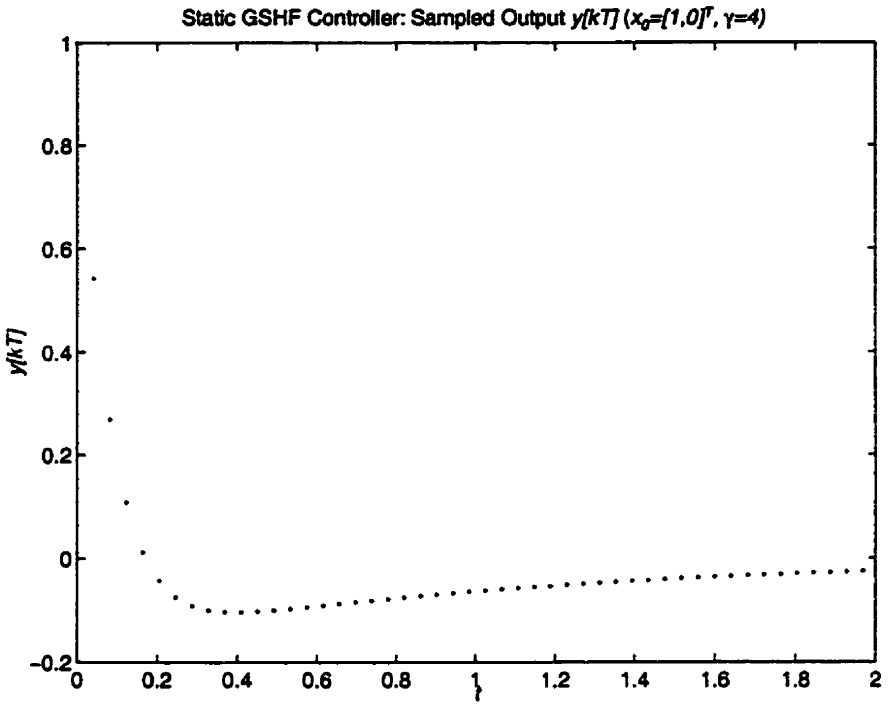


Figure A.3: Sampled output $y[kT]$ when $x_0 = [1\ 0]^T$ and $\gamma = 4$ (d.t. approach).

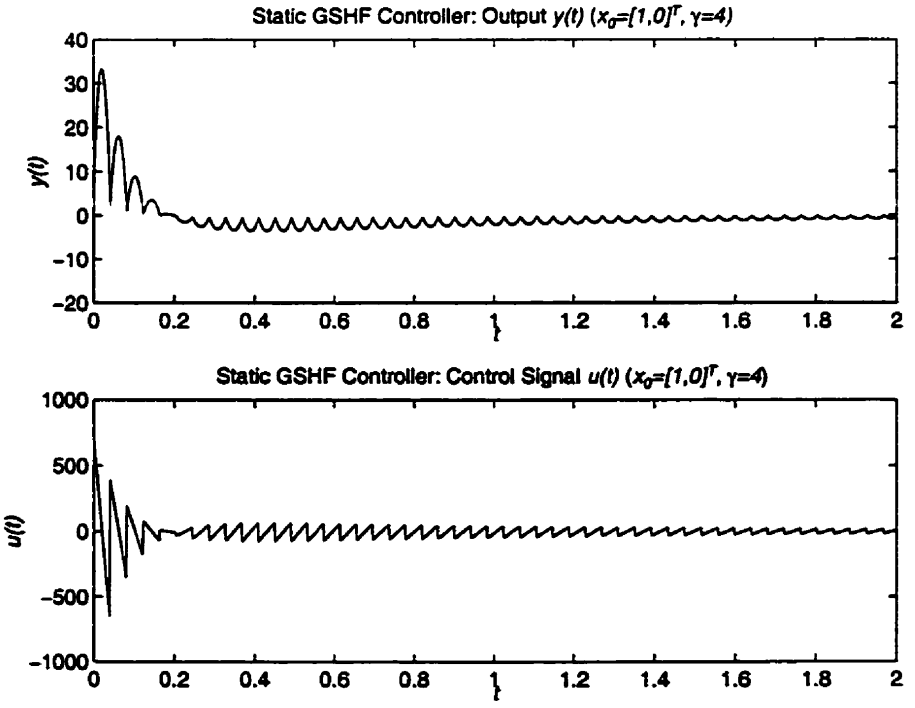


Figure A.4: Output $y(t)$ and control signal $u(t)$: $x_0 = [1\ 0]^T$ and $\gamma = 4$ (d.t. approach).

A.2 Comparing the Continuous-time and Discrete-time Approaches

In this section we will verify, through example, the theoretical results discussed in the previous sections and compare the GSHF controllers obtained by the continuous-time and discrete-time approaches. The nominal plant will be the same one used in the examples found in Sections 3.3.3 and A.1.3:

$$(A,B,C) = \left(\begin{bmatrix} -1 & 0 \\ 0 & 0.5 \end{bmatrix}, \begin{bmatrix} \frac{4}{3} \\ -\frac{1}{3} \end{bmatrix}, \begin{bmatrix} 1 & 1 \end{bmatrix} \right).$$

Suppose we have the same design objective as that found in the previous examples. Namely, we wish to design a static GSHF controller that will stabilize every system in

$$\{(A,B,\gamma C) : \gamma \in \Gamma(0.75, 6, 70°)\}.$$

A.2.1 Verification of Lemma A.1

To verify Lemma A.1, we do the following:

i) Solve for P_0 and F_0 in (3.13) and (3.14).

ii) Pick a value for T so that (C, e^{AT}) is observable.

iii) Solve for P_T and F_T in (A.5) and (A.6).

iii) Evaluate $\|F_T - F_0\|$ and $\|TP_T - P_0\|$. Then pick another smaller T such that (C, e^{AT}) is observable and go back to step iii).

Figures A.5 and A.6 show plots of $\|F_T - F_0\|$ vs T and $\|TP_T - P_0\|$ vs T, respectively, and as expected,

$$\lim_{T \to 0} \|F_T - F_0\| = 0,$$

$$\lim_{T \to 0} \|TP_T - P_0\| = 0.$$

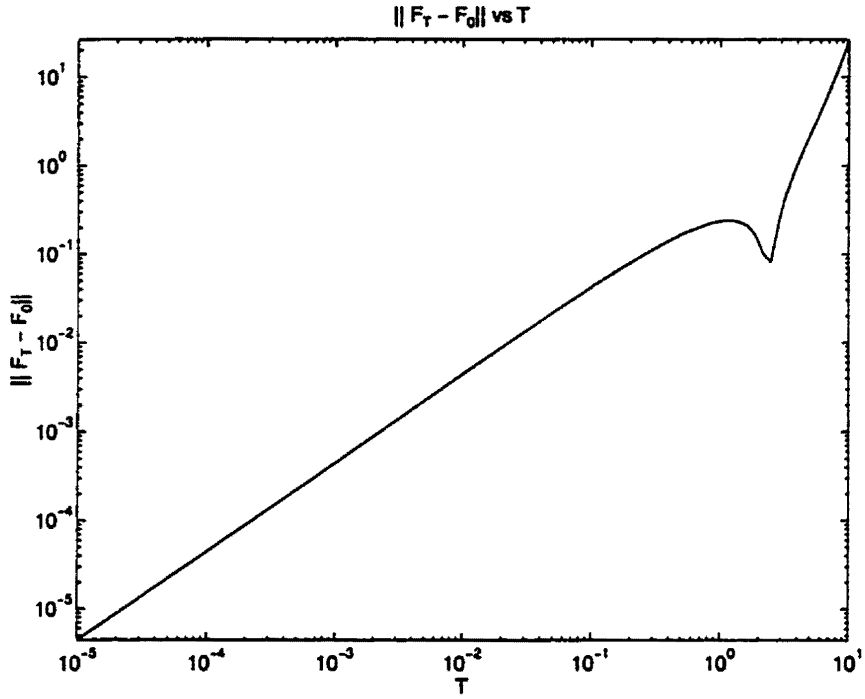


Figure A.5: Plot of $\|F_T - F_0\|$ vs T.

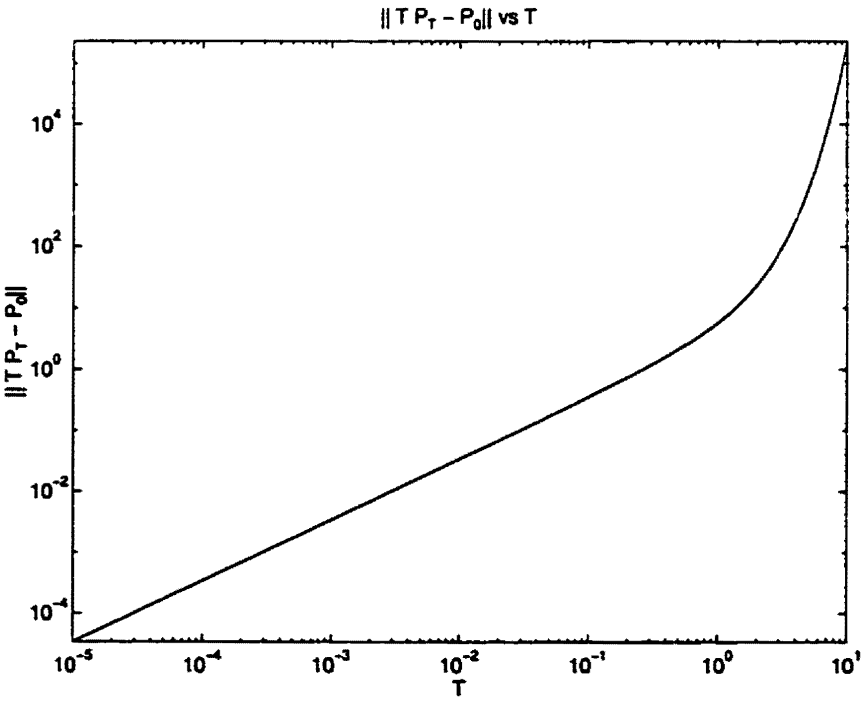


Figure A.6: Plot of $\|TP_T - P_0\|$ vs T.

A.2.2 Advantages of the Discrete-Time Approach

Note that the final GSHF controllers obtained in Section 3.3.3 (Continuous-time approach) and Section A.1.3 (Discrete-time approach) were very similar. In fact, if we compare the simulation results of the two GSHF controllers illustrated in Figure 3.6 and A.4, we see that they are virtually identical.

However, we will now verify that the discrete-time based GSHF controller does provide better gain margins when the sampling period T is large. We will denote $(\overline{F},T)_{c.t.}$ and $(\overline{F},T)_{d.t.}$ as the GSHF controllers designed based on the continuous-time and discrete-time approaches, respectively. For both GSHF controllers, we choose $Q = I$, $R = 1$, and $\overline{\phi} = 70^o$ (i.e. $\alpha = 0.684$) for our GSHF design parameters.

STEP 1: **Gain margin for $(\overline{F},T)_{c.t.}$**

Set $\widehat{C} = [0.684\ 0.684]$. Determine P_0 and F_0 satisfying (3.13) and (3.14). Then for various values of T we determine

$$\begin{aligned}\underline{\rho}_{c.t}(T) &:= \min\{\widehat{\rho} \in (0,1] : \mathrm{sp}(\mathrm{e}^{AT} + \rho T F_0 C) \subset \{z \in \mathbf{C} : |z| < 1\}, \rho \in [\widehat{\rho}, 1]\},\\ \overline{\rho}_{c.t}(T) &:= \max\{\widehat{\rho} \in [1,\infty) : \mathrm{sp}(\mathrm{e}^{AT} + \rho T F_0 C) \subset \{z \in \mathbf{C} : |z| < 1\}, \rho \in [1, \widehat{\rho}]\}.\end{aligned}$$

STEP 2: **Gain margin for $(\overline{F},T)_{d.t.}$**

Set $\widehat{C} = [0.684\ 0.684]$. Then for various values of T, let $A_d = \mathrm{e}^{AT}$, determine P_T and F_T satisfying (A.5) and (A.6), and find

$$\begin{aligned}\underline{\rho}_{d.t}(T) &:= \min\{\widehat{\rho} \in (0,1] : \mathrm{sp}(\mathrm{e}^{AT} + \rho T F_T C) \subset \{z \in \mathbf{C} : |z| < 1\}, \rho \in [\widehat{\rho}, 1]\},\\ \overline{\rho}_{d.t}(T) &:= \max\{\widehat{\rho} \in [1,\infty) : \mathrm{sp}(\mathrm{e}^{AT} + \rho T F_T C) \subset \{z \in \mathbf{C} : |z| < 1\}, \rho \in [1, \widehat{\rho}]\}.\end{aligned}$$

STEP 3: **Compare gain margins**

We plot $\overline{\rho}_{c.t}(T)/\underline{\rho}_{c.t}(T)$ and $\overline{\rho}_{d.t}(T)/\underline{\rho}_{d.t}(T)$ in Figure A.7.

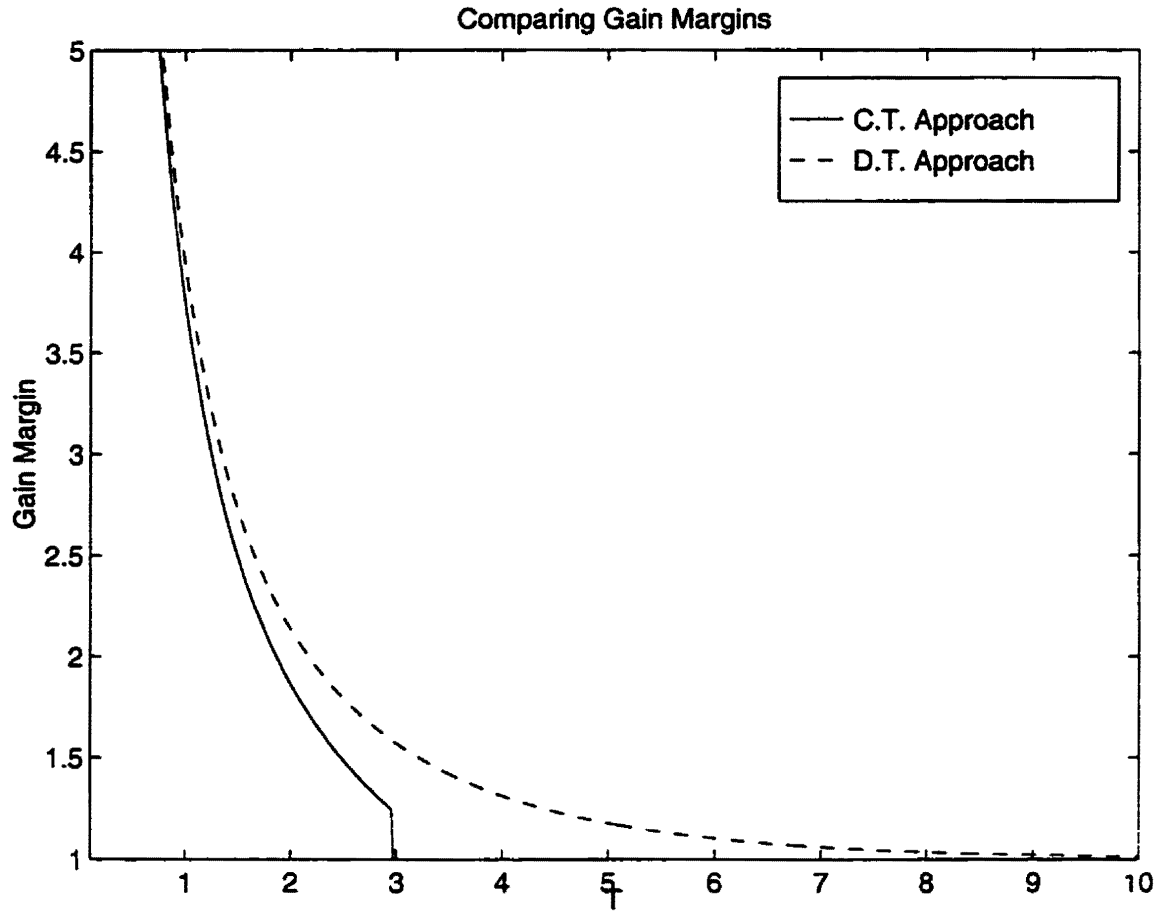


Figure A.7: Comparing the gain margins obtained from the continuous-time and discrete-time approaches.

Note that for $T > 2.98$ the GSHF controller based on the continuous-time approach does not stabilize the nominal plant. On the other hand, the gain margin provided by the GSHF controller that was designed using the discrete-time approach is always greater than one. Note however that as $T \to \infty$, the gain margin approaches unity, which means that the GSHF controller cannot tolerate much uncertainty in the gain of the plant when T is large.

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