An Affine Arithmetic-based Framework for Uncertain Power Flow and Optimal Power Flow Studies

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***Abstract*—This paper proposes a unified framework based on Affine Arithmetic (AA) for computing reliable enclosures of uncertain Power Flow (PF) and Optimal Power Flow (OPF) solutions. The main idea is to formulate a generic mathematical programming problem under uncertainty by means of equivalent deterministic problems, and to identify the affine forms describing the data uncertainty by means of a signal processing technique based on Principal Components Analysis (PCA). Compared to existing solution algorithms, this formulation presents greater flexibility, as it allows to find feasible solutions and inclusion of multiple equality and inequality constraints, and reduce the approximation errors to obtain better PF and OPF solution enclosures. Detailed numerical results are presented and discussed using a variety of realistic test systems, demonstrating the effectiveness of the proposed methodologies and comparing it to existing techniques for uncertain PF and OPF analysis.**

***Index Terms*—Big data, knowledge discovery, optimal power flow, power flow, uncertainty.**

I. Introduction

OPTIMAL power system operation requires intensive numerical analysis to study and improve system security and reliability. In this context, power system operators need to understand and reduce the impact of system uncertainties. To address this issue, Power Flow (PF) and Optimal Power Flow (OPF) analyses are some of the most important tools, since they represent the mathematical foundations of many power engineering applications such as state estimation, network optimization, voltage control, generation dispatch, etc.

For the most common formalization of the PF and OPF problems, all input data are specified using deterministic variables resulting either from a snapshot of the system, or defined by the analyst based on several assumptions about the system under study (e.g. expected/desired generation/load profiles). This approach allows to compute PF and OPF solutions for a single system state that is deemed representative of the limited set of system conditions corresponding to the data assumptions. Thus, when the input conditions are uncertain, numerous scenarios need to be analyzed (e.g. [1], [2]), and reliable solution paradigms incorporating the effect of data uncertainties are required.

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Conventional methodologies available in the literature propose the use of sampling, analytical, and approximate methods for PF and OPF analysis [3], [4], accounting for the variability and stochastic nature of the input data used. Uncertainty propagation studies based on sampling-based methods, such as Monte Carlo, require several model runs that sample various combinations of input values. In particular, the most popular Monte Carlo based algorithm adopted to solve PF and OPF problems is simple random sampling, in which a large number of samples are randomly generated from the probability distribution functions of the input uncertain variables. Although this technique may provide highly accurate results, it has the drawback of requiring large computational resources associated with the large number of repeated PF and OPF solutions [5]. This hinders the application of this solution algorithm, especially for large scale power system analysis, where the required number of simulations can be rather large making it computationally prohibitively expensive [6], [7].

Analytical methods are computationally more effective, but these approaches require some mathematical assumptions in order to simplify the problem and obtain an effective characterization of the output random variables [8]. These assumptions are typically based on model multi-linearization [9], convolution techniques [10], or fast Fourier transforms [11]. Analytical techniques present various shortcomings, as discussed in [12]–[14], such as the need to assume statistical independence of the input data and the problems associated with accurately identifying probability distributions for some input data.

In order to overcome some of the aforementioned limitations of sampling and analytical methods, the use of approximate methods have been proposed in the literature, such as the first-order second-moment method and point estimate methods [1], [8]. Rather than computing the exact PF/OPF solution, these methods aim at approximating the statistical proprieties of the output random variables by means of a probability distribution fitting algorithm. The application of these solution methods present several shortcomings; in particular, two-point estimate methods are not suitable to solve large scale problems, since they typically do not provide acceptable results in the presence of a large number of input random variables. Moreover, the identification of the most effective scheme that should be adopted to select the number of estimated points is still an open problem that requires further investigations [15].

Recent research has enriched the spectrum of available techniques to deal with uncertainty in PF and OPF by proposing

non-probabilistic formalisms, such as the theory of possibility [16], based on the theory of fuzzy sets, and the theory of evidence [17]. The application of fuzzy set theory to represent imprecise information, rather than using uncertainty associated with a frequency of occurrence, has been proposed in several papers [18], [19]. In this paradigm, the input data and the inequality constraints are modeled by fuzzy numbers, which are special instances of fuzzy sets [20], and the problem solution is computed by deploying efficient linear programming solution algorithms based on Dantzig-Wolfe decomposition and dual simplex [19].

Other studies reported in the literature have proposed the employment of self-validated computing for uncertainty representation in PF analysis. The main advantage of self-validated computation is that the algorithm itself keeps track of the accuracy of the computed quantities, as part of the process of computing them, without requiring information about the type of uncertainty in the variables [21]. The simplest and most popular of these models is Interval Mathematics (IM), which allows for numerical computation where each quantity is represented by an interval of real numbers without a probability structure [22]. The application of "standard" IM to PF analysis has been investigated by various authors [13], [23], [24]. However, the adoption of this solution technique presents many drawbacks derived mainly by the so called "dependency problem" and "wrapping effect" [21], [25]; as a consequence, the solution provided by an IM method for PF solution is not always informative.

To overcome the aforementioned limitations, in [26], the authors propose the employment of a more effective self validated paradigm based on Affine Arithmetic (AA) to represent the uncertainties of the PF state variables. The benefits have been confirmed in [27] and in [28], where the authors proposed AA-based methods to solve uncertain OPF problems. These methodologies have been recently recognized as a promising alternative for stochastic information management in bulk generation and transmission systems for smart grids [29], with several papers exploring the application of AA-based computing to power system analysis [30]–[32]. Although these papers offer considerable insight on the role that AA may play in power systems analysis, several open problems still remain unsolved, particularly:

- Further exploration of the application of AA-based techniques to uncertain OPF analysis.
- Rigorous methodologies aimed at selecting the noise symbols of the affine forms representing the power system state variables.
- More efficient paradigms aimed at reducing the overestimation errors of AA-based PF and OPF problems.

Hence, the current paper addresses these issues, thus providing the following contributions to the existing literature:

- Conceptualization of a unified AA-based computational paradigm aimed at solving more accurately both PF and OPF problems in the presence of data uncertainties, based on the idea of formulating a generic mathematical programming problem under uncertainty by means of equivalent deterministic problems, defining a coherent set of minimization, equality and inequality operators.
- Design of a knowledge-based process based on Principal Components Analysis (PCA) to extract from operation databases complex features, hidden relationships, and useful hypotheses to determine a formal connection between the principal components and the noise symbols of the uncertain variables.

The application of these techniques to PF and OPF analyses is explained in detail here, and several numerical results are presented and discussed, demonstrating the effectiveness of the proposed methodologies, especially in comparison to more traditional techniques.

The remaining of the paper is organized as follows: Section II briefly reviews the main AA definition and concepts, as well as defining the new AA operators in which the proposed unified AA-PF and AA-OPF approaches are based. Section III presents the formalization of the AA-PF and AA-OPF problems, and Section IV introduces the proposed PCA-based technique to reduce the affine form noise symbols. Finally, Section V presents and discusses the application of the proposed techniques to several realistic test systems, and Section VI summarizes the main conclusions and contribution of this paper.

II. Theoretical Framework

A. Elements of AA

AA, introduced in [21], is a method for range analysis to manipulate multiple uncertainty sources such as imprecise data, modeling errors, round off, and truncation errors. This paradigm is similar to standard IM, but, in addition, it keeps track of correlations between the input and the computed quantities, providing much tighter bounds in the computing process and reducing error explosion problems observed in IM computations [33]. In AA, each uncertain variable χ is represented by an affine form $\hat{\chi}$, which is a first-degree polynomial[1], as follows:

$$\hat{\chi} = \chi_0 + \chi_1\varepsilon_1 + \chi_2\varepsilon_2 + ... + \chi_p\varepsilon_p \tag{1}$$

The coefficients χ_h are defined finite floating-point numbers, and the ε_hs are p symbolic real variables whose quantity and values are problem dependent, and assumed to lie in the interval [-1,1]. The fundamental invariant of affine arithmetic states that, for every AA operations, there is a single assignment of values from the [-1,1] interval into each of the noise symbols that makes the value of every affine form equal to the true value of the corresponding variable [33]. Each noise symbol represent an independent component of the total uncertainty of the variable χ, which may be "external" (already present in the input data) or "internal" (due to arithmetic round-off or approximations in the computation of $\hat{\chi}$), and the corresponding coefficient χ_h gives the magnitude of that component. The same noise symbol may contribute to the uncertainty of two or more quantities arising in the evaluation of an expression; this sharing of noise symbols indicates some

[1]More sophisticated affine structures, which include higher uncertainty correlations, have been recently proposed in the literature (see for example the concept of generalized affine arithmetic proposed in [34]).

partial dependency between two underlying uncertain variables in affine form $\hat{\chi}$ and $\hat{\psi}$, determined by the corresponding coefficients χ_h and ψ_h.

In order to perform AA computations, it is necessary to replace each operation on real numbers by equivalent mappings between affine forms. In particular, given a generic mapping $\Gamma(\chi, \psi)$, the corresponding AA operation $\hat{\Gamma}(\hat{\chi}, \hat{\psi})$ is a procedure that computes an affine form $\hat{\zeta} = \hat{\Gamma}(\hat{\chi}, \hat{\psi})$, which is coherent with input affine forms $(\hat{\chi}, \hat{\psi})$. If the function Γ is a linear function of χ and ψ, the affine representation of $\hat{\zeta}$is obtained by expanding and rearranging the same "primitive" noise symbols ε_h $\forall h \in (1, \ldots, p)$. On the other hand, if Γ is a non-affine operation, $\hat{\zeta}$cannot be expressed exactly as an affine combination of the "primitive" noise symbols, and the problem boils down to the identification of an affine function:

$$\hat{\Gamma}^a(\hat{\chi}, \hat{\psi}) = \zeta_0 + \zeta_1\varepsilon_1 + \zeta_2\varepsilon_2 + ... + \zeta_p\varepsilon_p \tag{2}$$

that approximates the function $\hat{\Gamma}(\hat{\chi}, \hat{\psi})$ reasonably well over its domain, as follows:

$$\hat{\zeta} = \hat{\Gamma}^a(\hat{\chi}, \hat{\psi}) + \zeta_{p+1}\varepsilon_{p+1} \tag{3}$$

where the new noise symbol ε_{p+1} is introduced by the non-affine operation, hence describing an endogenous uncertainty, as opposed to the noise symbols $\varepsilon_1, ..., \varepsilon_p$, which describe the exogenous uncertainties. Consequently, the value of the partial deviation ζ_{p+1}, which depends on the approximation error introduced by the non-affine operation, must be an upper bound on the absolute magnitude of the approximation error $|\hat{\Gamma}(\hat{\chi}, \hat{\psi}) - \hat{\Gamma}^a(\hat{\chi}, \hat{\psi})|$.

Furthermore, the affine approximation function $\hat{\Gamma}^a$ in (2) could assume different structures, depending on the desired degree of accuracy and the available computational resources. A good trade off between these goals could be obtained by assuming the following approximation structure:

$$\hat{\Gamma}^a(\hat{\chi}, \hat{\psi}) = \alpha\hat{\chi} + \beta\hat{\psi} + \xi \tag{4}$$

Note that (4) is not itself the approximation of the non-affine operation ζ, since that approximation is given by (3). Hence, to find adequate values to approximate the non-affine operation, an additional noise symbol is needed, and proper values for α, β, ξ, and ζ_{p+1} need to be computed, e.g. by applying the Chebyshev approximation theorem for univariate functions [33].

Finally, it is important to observe that AA is a particular instance of self-validated computing, since it uses ranges, rather than distributions, to describe the data uncertainty. Specifically the affine form $\hat{\zeta} = \hat{\Gamma}(\hat{\chi}) = \zeta_0 + \sum_{h=1}^{p+1} \zeta_h\varepsilon_h$ defines a range $\left[\zeta_0 - \sum_{h=1}^{p+1} |\zeta_h|, \zeta_0 + \sum_{h=1}^{p+1} |\zeta_h|\right]$ for the variable ζ, which is guaranteed to contain the true value of ζ, provided that the input variable χ lies in the range $[\chi_0 - \sum_{h=1}^{p} |\chi_h|, \chi_0 + \sum_{h=1}^{p} |\chi_h|]$. The corresponding over-estimation error, and hence the conservatism of the range determined by the affine form $\hat{\zeta}$, depends on the computing paradigm adopted to approximate $\hat{\Gamma}$, which is selected on the basis of a proper trade-off between accuracy and computational burden. [33].

B. Unified Framework for Solving AA-based Constrained Optimization Problems

In this section, a new theoretical framework is presented aimed at effectively solving linear and nonlinear systems of equations for constrained optimizations problems, under multiple and heterogeneous source of data uncertainty, based on a unified AA-based formalism. Thus, the main aim is to solve the following non-linear constrained optimization problem in the presence of data uncertainties represented as affine forms:

$$\begin{aligned} \min_{\hat{\mathbf{z}}} \quad & \hat{f}(\hat{\mathbf{z}}) \\ \text{s.t.} \quad & \hat{g}_j(\hat{\mathbf{z}}) = 0 \quad \forall j \in (1, \ldots, n) \\ & \hat{h}_k(\hat{\mathbf{z}}) < 0 \quad \forall k \in (1, \ldots, m) \end{aligned} \tag{5}$$

where $\hat{\mathbf{z}} = (\hat{z}_1, ..., \hat{z}_{N_z})$ represents the unknown affine form of the state variables, which include both the dependent and the control variables, $\hat{f}$is the affine, continuous and differentiable function describing the problem objectives, and $\hat{g}_j$and $\hat{h}_k$are continuous and differentiable affinefunctions describing the j^{th} equality and k^{th} inequality constraints, respectively.

To solve (5), novel AA-based mathematical operators are defined here to extend the minimization operator and the main comparison operators $<$, $>$, $\leq$, $\geq$, and $=$, into the affine domain. To accomplish this, starting from the definition of these operators and according to the Invariance Theorem of Affine Arithmetic [33], it is shown that (5) can be recasted as a dual deterministic problem, which can be solved employing a traditional numerical programming technique. In particular, the mathematical definitions introduced in Section II-A allow stating the following similarity propriety of affine forms, which is based on the equality of the partial deviations of the noise symbols ε_h $\forall h \in (1, \ldots, p)$, and on the definition of an approximation degree based on the radius of the uncertainties generated by the approximation of the non-affine operations, denoted here as ε_h $\forall h \in (p+1, \ldots, p+p_{na})$:

Definition 1: (Similarity operator for affine forms $\overset{A}{\approx}$): Two affine forms $\hat{\chi} = \chi_0 + \sum_{h=1}^{p+p_{na}} \chi_h\varepsilon_h$ and $\hat{\psi} = \psi_0 + \sum_{h=1}^{p+p_{na}} \psi_h\varepsilon_h$ are similar with an approximation degree $Ł_{\chi,\psi}$, i.e. $\hat{\chi} \overset{A}{\approx} \hat{\psi}$, if and only if:

$$\{\chi_h = \psi_h \; \forall h \in (0, \ldots, p)\} \wedge \left\{ Ł_{\chi,\psi} = \sum_{h=p+1}^{p+p_{na}} (|\chi_h| + |\psi_h|) \right\} \tag{6}$$

Note that the noise symbols $\varepsilon_{p+1}, ..., \varepsilon_{p+n_a}$ describe the endogenous uncertainties, which are generated by the approximation of n_a non-affine functions such as multiplications, trigonometric functions, and other nonlinear operations, while the partial deviations $\chi_{p+1}, ..., \chi_{p+n_a}$ are the upper-bound of the corresponding approximation errors, which can be computed by using the expressions derived by the Chebyshev Theorem, as explained in Section II-A. The adoption of this operator is particularly useful in solving optimization problems in the presence of interval uncertainties, where the bounds of the uncertain variables are the only available information.

By following the same approach, it is possible to define an inequality operator for affine forms as follows:

Definition 2: (Inequality operator for affine forms $\overset{A}{<}$): Given two affine forms $\hat{\chi} = \chi_0 + \sum_{h=1}^{p_\chi} \chi_h \varepsilon_h^\chi$ and $\hat{\psi} = \psi_0 + \sum_{h=1}^{p_\psi} \psi_h \varepsilon_h^\psi$, then $\hat{\chi} \overset{A}{<} \hat{\psi}$ if and only if:

$$\chi_0 + \sum_{h=1}^{p_\chi} |\chi_h| < \psi_0 - \sum_{h=1}^{p_\psi} |\psi_h| \tag{7}$$

This definition directly follows from the basic theory of interval analysis, since this states that the upper bound of $\hat{\chi}$ is less than the lower bound of $\hat{\psi}$.

Finally, the problem of the minimization of a scalar and non-linear affine function could be effectively addressed by defining the following operator:

Definition 3: (Minimization operator for functions of affine forms): Given a differentiable, non-linear function $f : \Re \to \Re$, and the affine form $\hat{\chi} = \chi_0 + \sum_{h=1}^{p} \chi_h \varepsilon_h$, then the following AA-based minimization problem:

$$\min_{\substack{A \\ \hat{\chi}}} \quad \hat{f}(\hat{\chi}) = f_0(\hat{\chi}) + \sum_{h=1}^{p} f_h(\hat{\chi})\varepsilon_h + \sum_{h=p+1}^{p+p_{na}} f_h(\hat{\chi})\varepsilon_h \tag{8}$$

is equivalent to the following deterministic multi-objective programming problem:

$$\min_{(\chi_0,\chi_1,...,\chi_p)} \quad \{f_0(\chi_0,\chi_1,...,\chi_p), \sum_{h=1}^{p+p_{na}} |f_h(\chi_0,\chi_1,...,\chi_p)|\} \tag{9}$$

This definition follows from the AA-based robust circuit design approach proposed in [35], and the principles of risk-based programming theory, since the minimization of the affine central value $f_0(\chi_0,\chi_1,...,\chi_p)$ aims at identifying the most effective solutions, without considering the uncertainty represented by the noise symbols, while the minimization of the affine radius $\sum_{h=1}^{p+p_{na}} |f_h(\chi_0,\chi_1,...,\chi_p)|$ aims at identifying the most reliable solutions that exhibit the lowest tolerance to data uncertainty. The trade-off between these two conflicting objectives basically represents the decision maker's risk. Based on this, the minimization of an affine function should be equivalent to finding an affine form which minimizes both its central value and its radius. More details on the derivation of (6), (7) and (9) can be found in [36].

Based on (6), (7) and (9), problem (5) can be solved by solving the following deterministic multi-objective constrained optimization problem:

$$\begin{aligned} \min_{\hat{\mathbf{z}}} \quad & \{f_0(\hat{\mathbf{z}}), \sum_{h=1}^{p+p_{na}} |f_h(\hat{\mathbf{z}})|\} \\ \text{s.t.} \quad & \hat{g}_j(\hat{\mathbf{z}}) \overset{A}{\approx} 0 \qquad \forall j \in (1,\ldots,n) \\ & \hat{h}_k(\hat{\mathbf{z}}) \overset{A}{<} 0 \qquad \forall k \in (1,\ldots,m) \end{aligned} \tag{10}$$

The rigorous solution of the problem formalized in (5) requires the solution of the multi-objective problem (10), where both the central values and the partial deviations of the affine forms describing the state and the control problem variables are simultaneously determined. To solve this problem it is necessary to design weighting methodologies aimed at measuring the overall quality of the feasible OPF solutions by defining the risk that the analyst is willing to take in solving the dichotomy between the minimization of the central value versus the minimization of the radius of the objective function, or, in other words, between the determinism and the robustness of the OPF solution. A possible approach to address this issue is to decompose the original problem in two subsequent stages. In the first stage, the main idea is to identify the central values of the unknown state vector by first considering the system operating at its nominal condition. In this case, which is referred here as the "nominal state", uncertainties are not considered and thus the corresponding solution can be computed by solving the following deterministic optimization problem:

$$\begin{aligned} \min_{(z_{1_0},...,z_{N_{z0}})} \quad & f_0(z_{1_0},...,z_{N_{z0}}) \\ \text{s.t.} \quad & g_j(z_{1_0},...,z_{N_{z0}}) = 0 \qquad \forall j \in (1,\ldots,n) \\ & h_k(z_{1_0},...,z_{N_{z0}}) < 0 \qquad \forall k \in (1,\ldots,m) \end{aligned} \tag{11}$$

In the second stage, referred here as the "perturbed state", the effect of data uncertainty is considered, computing the partial deviations of the unknown state vector by solving the following deterministic optimization problem:

$$\begin{aligned} \min_{(z_{1_1},..,z_{N_{zp}})} \quad & \sum_{h=1}^{p+p_{na}} |f_h(z_{1_1},..,z_{N_{zp}})| \\ \text{s.t.} \quad & g_j(z_{1_1},..,z_{N_{zp}}) \overset{A}{\approx} 0 \qquad \forall j \in (1,\ldots,n) \\ & h_k(z_{1_1},..,z_{N_{zp}}) \overset{A}{<} 0 \qquad \forall k \in (1,\ldots,m) \end{aligned} \tag{12}$$

This two stage decomposition algorithm, which could be considered as a tolerance analysis, is based on the concept that in AA-based computing the central values of the computed quantities depends on the central values of the input variables, while the corresponding partial deviations depends on both the central values and the partial deviations of all the input variables [37]. This result, which can be generalized for any continuous and differentiable function, allows approximating the solution of the uncertain OPF problem in the AA domain by the described decomposition algorithm, which is widely adopted in uncertain circuit theory for analyzing the impact of data uncertainty on the "nominal" operation of a circuit, where the "nominal" behavior is analyzed by considering all the uncertain circuit parameters assuming their nominal values, while the tolerances of the circuit variables are assessed by computing the corresponding deviations from the "nominal" values due to the effect of all the parameter uncertainties [38].

The aforementioned decomposition approach, which is an approximation of the rigorous multi-objective solution approach (10), is expected to work properly for OPF problems for which solutions can be obtained. Hence, for OPF problems that present convergence difficulties, robust optimization solvers should be used to obtain solutions, as discussed in [39], in which case the proposed approach should yield adequate results even for systems operating close to critical conditions."

Note that the solution of the AA-based optimization problem for the "perturbed state" requires the identification of a larger number of state variables, i.e., $p \times N_z$, making

(12) a large optimization problem, with the corresponding computational challenges.

Finally, it is worth noting that the different solution strategy discussed in the paper is based on the use of range arithmetic to compute proprieties of sets rather than points, thus assessing the proprieties of functions on uncertain domains by deploying a finite number of interval-based operations. The main idea of this approach is to define for every continuous and differentiable function f, a corresponding interval extension F, which is characterized by the so-called inclusion isotonicity [40], [41]:

$$x \in X \Rightarrow f(x) \in F(X) \wedge X \subseteq Y \Rightarrow F(X) \subseteq F(Y) \quad (13)$$

This fundamental propriety of range-based computing, which guarantees that the interval extension $F(X)$ includes all possible values of $f(x)$ for all $x \in X$, has been widely exploited in solving uncertain optimization problems. In particular, several papers proposed the application of interval-based computing to find global optimum of uncertain optimization problems (e.g. []). The main idea in these papers is to exploit the capability of range arithmetic to cover finite partitions of the search space, thus overcoming the intrinsic limitations of optimization algorithms based on standard real arithmetic, which can only sample single points and thus cannot find the global optimum in a deterministic way. The isotonicity property guarantees that, if the original problem is feasible for certain intervals, an interval-based optimization approach will obtain a "conservative" estimation of the solution domain. Hence, the optimization problem (10) in the paper, is guaranteed to have a solution if the considered intervals are feasible. However, whether an actual numerical solution can be found really depends on the algorithmic implementation, as in the case of conventional, deterministic PFs and OPFs. This is also the case of the proposed bi-level solution approach corresponding to (11) and (12), which is similar to an approach proposed in [40] to solve a robust design optimization based on interval analysis using a bi-level mathematical model, where during the first stage, the algorithm tries to optimize the objective function by considering proper deterministic constraints, and then the obtained solution is transferred to the lower-level subproblem to estimate the value of the middle vector and radius vector of the intervals in the uncertainty space. Therefore, one cannot theoretically guarantee convergence of the proposed solution methodology, even if the original interval optimization problem is feasible. However, as in the case of solution techniques for deterministic PFs and OPFs, the robustness for the proposed solution algorithm can be numerically tested by stressing the base system and testing if the method still yields a solution, which in the case of (10), corresponds to solving the optimization problem with increasing interval ranges.

C. Differences with Previous AA-based Approaches

The definition of the AA-based operators allows solving a generic uncertain optimization problem, such as the OPF problem, according to a different and more effective computing paradigm compared to previously proposed AA-based methods in [26]–[28]. In particular, previous techniques try to solve uncertain optimization problems using a "domain contraction" technique, which starting from an outer estimation of the problem solution, tries to refine the outer solution domain by contracting the noise symbols of the affine forms describing the state and the decision variables. Although this solution paradigm is computationally efficient, since the domain contraction technique can be implemented by solving a limited number of deterministic linear optimization problems, and reliable, since it starts from a very conservative estimation of the real solution domain, it presents the following limitations:

- The initial estimation of the outer solution, which influences the performance of the overall solution framework, is not a trivial task, since it should be conservative to include the real solution domain, but it cannot be too large to avoid convergence problems. To solve this problem, sensitivity based approaches and noise amplification techniques were proposed in [26]–[28], which allows to obtain a rough estimation of the affine forms describing the problem solution, using a simple trial-and-error approach tuned on the basis of the specific problem under investigation. None of these papers propose a formal method as the one proposed here that guarantees the inclusion of the real solution domain in the intervals being used, while keeping these intervals reasonably sized.
- The conservatism of the problem solution strictly depends on the conservatism of the initial outer estimation, since the bounds of the endogenous uncertainties, which are induced by the approximation of the non-affine operations, such as multiplications and trigonometric function approximations, depend on the outer solution bounds, and cannot be reduced by applying a domain contraction technique. This could lead to inaccurate solution bounds, which is not the case for the approach proposed here. Therefore, the AA-based framework conceptualized in this paper presents the following advantages over the techniques described in [26]–[28]:
- It does not use a domain contraction technique; hence, it does not require an initial outer estimation of the problem solution.
- The solution is accurate, since both the exogenous and the endogenous uncertainties are minimized during the computational process.
- The same framework can be adopted to solve various types optimization problems, without requiring special tuning of the algorithm parameters.

The proposed technique, however, has the disadvantage of yielding a number of variables directly associated with the number of noise symbols, which may lead to an increase in computational complexity. To address this problem, a data-driven methodology is proposed in Section IV to determine an appropriate number of noise symbols, instead of using a the simple sensitivity approach proposed in [26]–[28]. This results in a rigorous approach for selecting the most relevant noise symbols in the required affine forms, which better represent the uncertainty intervals and also reduce the number of variables associated with the proposed approach, improving computational performance."

III. AA-BASED PF AND OPF ANALYSES

A. PF Problem

The uncertain PF problem can be effectively solved by applying the proposed framework, since, as proposed in [42], the PF can be stated as a particular instance of the optimal power flow problem (5), as follows:

$$\begin{aligned}
\min_{(\hat{V_j},\hat{\delta_k},\hat{V_{i_a}},\hat{V_{i_b}},\hat{Q_i})} \quad & \sum_{k\in\mathcal{N_P}} (\hat{P_k}^{SP} - \hat{P_k})^2 + \sum_{j\in\mathcal{N_Q}} (\hat{Q_j^{SP}} - \hat{Q_j})^2 \\
\text{s.t.} \quad & \hat{P_k} = \hat{V_k}\sum_{t=1}^{N}\hat{V_t}Y_{kt}\cos\left(\hat{\delta}_k - \hat{\delta}_t - \theta_{kt}\right) \\
& \hat{Q_j} = \hat{V_j}\sum_{t=1}^{N}\hat{V_t}Y_{jt}\sin\left(\hat{\delta}_j - \hat{\delta}_t - \theta_{jt}\right) \\
& \hat{Q_i} = \hat{V_i}\sum_{t=1}^{N}\hat{V_t}Y_{it}\sin\left(\hat{\delta}_i - \hat{\delta}_t - \theta_{it}\right) \\
& \hat{V_i} = V_{i_0} + \hat{V}_{i_a} - \hat{V}_{i_b} \\
& 0 \overset{A}{\leq} (\hat{Q_i} - \hat{Q}_{i,min})\perp\hat{V}_{i_a} \overset{A}{\geq} 0 \\
& 0 \overset{A}{\leq} (\hat{Q}_{i,max} - \hat{Q_i})\perp\hat{V}_{i_b} \overset{A}{\geq} 0 \\
& V_{i_0}, \hat{V}_{i_a}, \hat{V}_{i_b} \overset{A}{\geq} 0 \\
& \forall k \in \mathcal{N_P},\ j \in \mathcal{N_Q},\ i \in \mathcal{N_{PV}}
\end{aligned} \tag{14}$$

where N is the total number of buses; p is the number of noise symbols representing the source of uncertainties affecting the power system operation; $\mathcal{N_P}$ and $\mathcal{N_Q}$ are the set of the buses at which the active and reactive power is specified; $\mathcal{N_{PV}}$ is the set of voltage controlled buses, where the injected active power and the voltage magnitude are specified; $\hat{P}_k^{SP}$ and $\hat{Q}_j^{SP}$ are the affine forms of the real and reactive power injections specified at k^{th} and j^{th} bus; $Y_{kt}\angle\theta_{kt}$ is the kt^{th} element of the bus admittance matrix; $\hat{V_j} = V_{j_0} + \sum_{h=1}^{p} V_{j_h}\epsilon_h$ is the affine form describing the voltage magnitude of the j^{th} load bus; $\hat{V_i} = V_{i_0} + V_{i_{a_0}} + V_{i_{b_0}} + \sum_{h=1}^{p}(V_{i_{a_h}} + V_{i_{b_h}})\epsilon_h$ and $\hat{Q_i} = Q_{i_0} + \sum_{h=1}^{p} Q_{i_h}\epsilon_h$ are the affine forms describing the voltage magnitude and the reactive power of the i^{th} generation bus; and $\hat{\delta_k} = \delta_{k_0} + \sum_{h=1}^{p}\delta_{k_h}\epsilon_h$ is the affine form describing the voltage angle for the k^{th} bus (different from the angle-reference bus). Consequently, the dependent variables of the AA-based PF problem are $(V_{j_0}, V_{j_1}, ..., V_{j_p})$for each load bus; $(V_{i_0}, V_{i_{a_0}}, V_{i_{a_1}}, ..., V_{i_{a_p}}, V_{i_{b_0}}, V_{i_{b_1}}, ..., V_{i_{b_p}})$for each generation bus; and $(\delta_{k_0}, \delta_{k_1}, ..., \delta_{k_p})$for each bus except the angle-reference bus.

B. OPF Problem

The application of the proposed AA-based framework for OPF analysis is straightforward, since the dependent variables are the affine forms of the voltage magnitudes at the load buses and the voltage angles at all buses except the angle-reference bus, namely $(V_{0_j}, V_{1_j}, ..., V_{p_j})\ \forall j \in \mathcal{N_{PQ}}$ and $(\delta_{0_k}, \delta_{1_k}, ..., \delta_{p_k})\ \forall k \in \mathcal{N_P}$. On the other hand the control variables depend on the particular application domain; thus, for example, in optimal economic dispatch analysis they include the active power generated by the dispatchable generators $\mathcal{N_G}$, namely, $(P_{0_i}, P_{1_i}, ..., P_{p_i})\ \ \forall i \in \mathcal{N_G}$.

The AA-OPF equality constraints correspond to the PF equations, while the inequality constraints typically include in practice the maximum allowable apparent power flow P_l at each line and other operational limits, as follows:

$$\begin{aligned}
\min_{(\hat{V_k},\hat{\delta_k},\hat{P_i},\hat{Q_i})} \quad & \hat{f}(\hat{V_j}, \hat{\delta_k}, \hat{P_i}) \\
\text{s.t.} \quad & \hat{P_k}^{SP} - \hat{V_k}\sum_{t=1}^{N}\hat{V_t}Y_{kt}\cos\left(\hat{\delta}_k - \hat{\delta}_t - \theta_{kt}\right) \overset{A}{\approx} 0 \\
& \hat{Q_j^{SP}} - \hat{V_j}\sum_{t=1}^{N}\hat{V_t}Y_{jt}\sin\left(\hat{\delta}_j - \hat{\delta}_t - \theta_{jt}\right) \overset{A}{\approx} 0 \\
& \hat{Q_i} - \hat{V_i}\sum_{t=1}^{N}\hat{V_t}Y_{it}\sin\left(\hat{\delta}_i - \hat{\delta}_t - \theta_{it}\right) \overset{A}{\approx} 0 \\
& V_j^{min} \overset{A}{\leq} \hat{V_j} \overset{A}{\leq} V_j^{max} \\
& P_i^{min} \overset{A}{\leq} \hat{P_i} \overset{A}{\leq} P_i^{max} \\
& Q_i^{min} \overset{A}{\leq} \hat{Q_i} \overset{A}{\leq} Q_i^{max} \\
& P_l^{min} \overset{A}{\leq} \hat{P}_l(\hat{V}_1, ..., \hat{V}_N, \hat{\delta}_1, ..., \hat{\delta}_N) \overset{A}{\leq} P_l^{max} \\
& \forall k \in \mathcal{N_P},\ j \in \mathcal{N_{PQ}},\ i \in \mathcal{N_{PV}},\ l \in \mathcal{N_L}
\end{aligned} \tag{15}$$

where $\mathcal{N_L}$ is the set of the constrained lines, and $\mathcal{N_{PQ}}$ is the set of buses at which the active and reactive power are specified.

Compared to, for example, [43], which uses a sensitivity based method to compute the bounds of the input data uncertainty, and employs a boundary OPF model to compute the solution ranges, the proposed AA-based operators are independent from the number and the semantics of the "primitive" noise symbols, hence making AA-based computing particularly suitable for power system analysis in the presence of multiple and heterogeneous uncertainties. From a conceptual point of view, affine forms can model any kind of data uncertainty, including line parameters and limit uncertainties, by defining proper noise symbols; thus, AA has been extensively applied to analyze uncertain electronic circuits, where the uncertainty sources typically affect the circuit parameters values (i.e. resistors, capacitors). In the present paper, and given the current relevance of analyzing the impact of uncertain power injections from wind and solar power sources, it has been concentrated only on studying these particular sources of uncertainties.

IV. NOISE SYMBOLS REDUCTION

The computing framework proposed in this paper is based on the definition of rigorous relational and minimization operators, which differ from the approximated minimization operators adopted in traditional AA-based PF and OPF analyses (i.e. domain contraction). However, as previously mentioned, this approach requires solving the large optimization problem (12). Nevertheless, the number of the relevant noise symbols describing the affine forms of the state vector, which sensibly influences the problem cardinality, can be significantly reduced

by exploring statistical correlations between the uncertainty sources. To address this issue, a knowledge-based paradigm based on PCA is proposed here to extract from operation databases complex features and hidden relationships, which allow to define a formal connection between the principal components and the affine forms used to describe the uncertain variables.

A. PCA Overview

A knowledge-base formed by $\overline{T}$ observations of N_x variables is represented by the $\overline{T} \times N_x$ matrix $\mathbf{X} = \left[\mathbf{x}(1) - \frac{1}{\overline{T}}\sum_{K=1}^{\overline{T}} \mathbf{x}(K) \ \ldots \ \mathbf{x}(\overline{T}) - \frac{1}{\overline{T}}\sum_{K=1}^{\overline{T}} \mathbf{x}(K)\right]^T$, which has rank $L \leq \min(\overline{T}, Nx)$. This matrix can be decomposed using the following singular value decomposition [44]:

$$\mathbf{X} = \mathbf{A}\mathbf{\Sigma}\mathbf{\Omega}^T \tag{16}$$

where $\mathbf{A}$ is the $\overline{T} \times L$ matrix of left singular vectors, $\mathbf{\Omega}$ is the $N_x \times L$ matrix of right singular vectors, and $\mathbf{\Sigma}$ is the diagonal matrix of singular values. Note that $\mathbf{\Sigma}^2$ is a diagonal matrix of the eigenvalues of $\mathbf{X}\mathbf{X}^T$ and $\mathbf{X}^T\mathbf{X}$ [44].

Starting from this decomposition, the $\overline{T} \times L$ matrix of the factor scores can be obtained as:

$$\mathbf{S} = \mathbf{A}\mathbf{\Sigma} = \mathbf{A}\mathbf{\Sigma}\mathbf{\Omega}^T\mathbf{\Omega} = \mathbf{X}\mathbf{\Omega} \tag{17}$$

where the matrix $\mathbf{\Omega}$ can be considered as a projection matrix, since the product $\mathbf{X}\mathbf{\Omega}$ represents the projections of the observations on the principal components. The only information available from the principal components computed by (17) is a measure of the relative importance of the observed variables. In this context, the $\overline{T}$ observations are considered to be the population of interest, and conclusions are limited to these specific observations.

PCA represents the observed data by using a fixed regressive model, which is identified by using only the first N_{PC} principal components of the knowledge-base $\mathbf{X}$. In particular, let $\hat{\mathbf{X}}_{N_{PC}}$ the estimation of the matrix $\mathbf{X}$ computed by using only N_{PC} principal components, then it follows that:

$$\hat{\mathbf{X}}_{N_{PC}} = \mathbf{A}_{N_{PC}}\mathbf{\Sigma}_{N_{PC}}\mathbf{\Omega}_{N_{PC}}^T = \mathbf{S}_{N_{PC}}\mathbf{\Omega}_{N_{PC}}^T \tag{18}$$

where $\mathbf{A}_{N_{PC}}$, $\mathbf{\Sigma}_{N_{PC}}$, and $\mathbf{\Omega}_{N_{PC}}$ correspond to the first N_{PC} columns of the corresponding matrices $\mathbf{A}$, $\mathbf{\Sigma}$, and $\mathbf{\Omega}$. A different way to represent (18), is to define the error matrix $\mathbf{E} = \mathbf{X} - \hat{\mathbf{X}}_{N_{PC}}$, and to express the matrix $\mathbf{X}$ as follows:

$$\mathbf{X} = \hat{\mathbf{X}}_{N_{PC}} + \mathbf{E} = \mathbf{S}_{N_{PC}}\mathbf{\Omega}_{N_{PC}}^T + \mathbf{E} \tag{19}$$

B. PCA-based Affine Forms

PCA is adopted here to discover the potential relationships among the active and reactive power injections, from the following set of historical observations:

$$\begin{gathered}\left(P_k^{SP}(K), Q_j^{SP}(K)\right) \\ \forall k \in \mathcal{N}_{\mathcal{P}},\ j \in \mathcal{N}_{\mathcal{PQ}},\ K \in (1, \ldots, \overline{T})\end{gathered} \tag{20}$$

where $(1, \ldots, \overline{T})$ defines the integer sample time interval of available data. The application of PCA to the aforementioned data set allows representing the injected active and reactive powers with a linear combination of a proper number of orthogonal and uncorrelated principal components, as follows:

$$\begin{gathered}P_k^{SP}(K) = \mathbf{\Omega}^P\ \mathbf{s}(K) + P_{k,med}^{SP} \\ Q_j^{SP}(K) = \mathbf{\Omega}^Q\ \mathbf{s}(K) + Q_{j,med}^{SP} \\ P_{k,med}^{SP} = \frac{1}{\overline{T}}\sum_{t=1}^{\overline{T}} P_k^{SP}(t) \\ Q_{j,med}^{SP} = \frac{1}{\overline{T}}\sum_{t=1}^{\overline{T}} Q_j^{SP}(t) \\ \forall k \in \mathcal{N}_{\mathcal{P}},\ j \in \mathcal{N}_{\mathcal{PQ}},\ K \in (1, \ldots, \overline{T})\end{gathered} \tag{21}$$

where $\mathbf{s}(K)$ is the principal component vector, and $\mathbf{\Omega}^P$ and $\mathbf{\Omega}^Q$ are matrices of dimensions $N_P \times N_{PC}$ and $N_{PQ} \times N_{PC}$, respectively, with N_{PC} being the number of principal components, and N_P and N_{PQ} being the number of buses in which the active and reactive power is specified. Since each principal component carries different and uncorrelated information to other components, only a limited number of these "primitive" variables are necessary to accurately represent the active and reactive power injections for highly correlated datasets [36], [45]. Hence, the number of noise symbols describing the injected power uncertainties can be set to N_{PC}, and the corresponding affine forms can be defined as follows:

$$\begin{gathered}\hat{P}_k^{SP} = P_{k_0}^{SP} + \sum_{h=1}^{N_{PC}} P_{k_h}\varepsilon_h \\ \hat{Q}_j^{SP} = Q_{j_0}^{SP} + \sum_{h=1}^{N_{PC}} Q_{j_h}\varepsilon_h \\ \forall k \in \mathcal{N}_{\mathcal{P}}\ \forall j \in \mathcal{N}_{\mathcal{PQ}}\end{gathered} \tag{22}$$

where the noise symbols $\varepsilon_h\ \forall h \in (1, \ldots, N_{PC})$ represent the uncertainty affecting the principal components. Compared to the traditional AA-based formulation, which are based on the definition of a separate noise symbol for each active and reactive power injection [26], the proposed technique should yield a significant reduction in the number of noise symbols.

The unknown parameters of the affine forms defined in (22) can be identified by observing that the injected active and reactive powers at a fixed sample time K are related to the principal components $s_1(K), ..., s_{N_{PC}}(K)$ by (21). Hence, these principal components become uncertain, and thus these can be described by the affine forms $\hat{s_h} = s_{h_0} + \Delta s_h \varepsilon_h\ \forall h \in (1, \ldots, N_{PC})$, whose parameters can be computed by solving the following equations, once the range of the injected powers are known:

$$\begin{gathered}P_{k,min}^{SP} = \sum_{h=1}^{N_{PC}} \Omega_{k,h}^P(s_{h_0} - \Delta s_h) + P_{k,med}^{SP} \\ P_{k,max}^{SP} = \sum_{h=1}^{N_{PC}} \Omega_{k,h}^P(s_{h_0} + \Delta s_h) + P_{k,med}^{SP} \\ Q_{j,min}^{SP} = \sum_{h=1}^{N_{PC}} \Omega_{j,h}^Q(s_{h_0} - \Delta s_h) + Q_{j,med}^{SP} \\ Q_{j,max}^{SP} = \sum_{h=1}^{N_{PC}} \Omega_{j,h}^Q(s_{h_0} + \Delta s_h) + Q_{j,med}^{SP} \\ \forall k \in \mathcal{N}_{\mathcal{P}}\ \forall j \in \mathcal{N}_{\mathcal{PQ}}\end{gathered} \tag{23}$$

Once the affine forms of the principal components have been computed, it is possible to obtain the following relations between the affine forms of the injected powers and the

principal components, which directly follow from (21):

$$\begin{aligned}\hat{P}_k^{SP} &= P_{k_0}^{SP} + \sum_{h=1}^{N_{PC}} P_{k_h}\varepsilon_h = \sum_{h=1}^{N_{PC}} \Omega_{k,h}^{P}\hat{s_h} + P_{k,med}^{SP} = \\ &= \sum_{h=1}^{N_{PC}} \Omega_{k,h}^{P} s_{h_0} + \sum_{h=1}^{N_{PC}} \Omega_{k,h}^{P}\Delta s_h\varepsilon_h + P_{k,med}^{SP} \\ \hat{Q}_j^{SP} &= Q_{j_0}^{SP} + \sum_{h=1}^{N_{PC}} Q_{j_h}\varepsilon_h = \sum_{h=1}^{N_{PC}} \Omega_{j,h}^{Q}\hat{s_h} + Q_{j,med}^{SP} = \\ &= \sum_{h=1}^{N_{PC}} \Omega_{j,h}^{Q} s_{h_0} + \sum_{h=1}^{N_{PC}} \Omega_{j,h}^{Q}\Delta s_h\varepsilon_h + Q_{j,med}^{SP} \\ &\forall k \in \mathcal{N}_{\mathcal{P}}\ \forall j \in \mathcal{N}_{\mathcal{PQ}}\end{aligned} \tag{24}$$

This allows computing the parameters of the affine forms describing the injected powers as follows:

$$\begin{aligned}P_{k_0}^{SP} &= \sum_{h=1}^{N_{PC}} \Omega_{k,h}^{P} s_{h_0} + P_{k,med}^{SP} \\ P_{k_h} &= \Omega_{k,h}^{P}\Delta s_h \\ Q_{j_0}^{SP} &= \sum_{h=1}^{N_{PC}} \Omega_{j,h}^{Q} s_{h_0} + Q_{j,med}^{SP} \\ Q_{j_h} &= \Omega_{j,h}^{Q}\Delta s_h \\ &\forall k \in \mathcal{N}_{\mathcal{P}}\ \forall j \in \mathcal{N}_{\mathcal{PQ}}\end{aligned} \tag{25}$$

Based on these equations, it is possible to identify the proper number of noise symbols, and the optimal parameters of the affine forms describing the uncertain PF and OPF variables, which is a significant issue in AA applications.

V. Numerical Results

A. Unified AA-based Method

To assess the benefits of the unified AA method for solving both uncertain PF and OPF problems various case studies are analyzed here, and PF and OPF solution bounds are compared to those calculated using the AA-based PF method proposed in [26], and the range artihmetic-based OPF proposed in [27], respectively. In all the analyzed case studies, a Monte Carlo simulation with a uniform distribution is assumed to yield the "correct" solution intervals. For the latter, 5000 different values of the input variables within the assumed input bounds were randomly selected, and a conventional solution was obtained for each one; this procedure yielded the desired interval solutions defined by the largest and the smallest values of the bus voltage magnitudes and angles as well as line flows. It should be noted that increasing the number of Monte Carlo simulations beyond 5000 did not yield any significant changes to the solution intervals.

Without loss of generality, a $\pm 20\%$ (40%) tolerance on load and generator powers was assumed, since this would define an interval wide enough to represent, for example, uncertain wind and solar generation. Based on the assumed load and generator power bounds to represent input data uncertainty, the unified AA-based method was applied to estimate the bounds of the PF and OPF solution by assuming the independence of all the input uncertainties, i.e. by considering a different noise symbol for each injected active and reactive power. It is important to note that the application of AA requires that all the uncertainty sources be described by noise symbols, representing independent sources of data uncertainty; hence, by sharing noise symbols, it is possible to correlate uncertain input variables. A different and more rigorous representation of statistically correlated uncertainty sources could be obtained by applying the generalized AA-based methodology proposed in [46]. All the simulation studies were carried out on a PC workstation equipped with a 3 GHz Intel Core Duo CPU with 3 GB RAM. The obtained results are presented and discussed next.

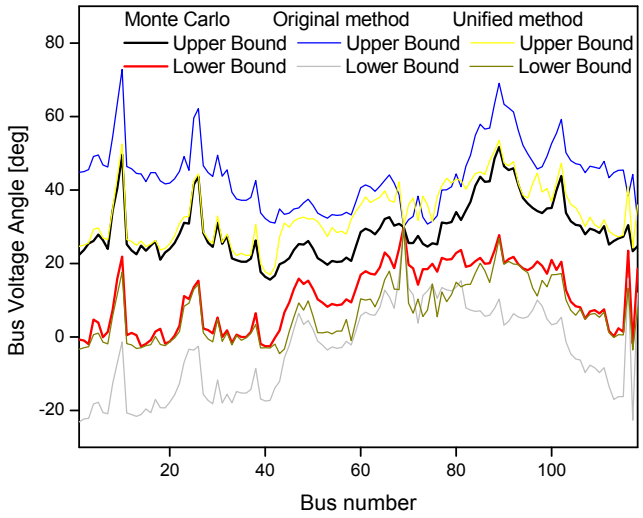


Fig. 1. Bus voltage angle bounds obtained for the IEEE 118-bus test system for both AA-PF methods.

1) PF Analysis: The uncertain power flow problem for three IEEE test systems are solved here using the unified AA formulation presented in (14). The obtained results are summarized in Tabs. I and II, which report, for each case study, the average errors in the upper and lower bounds for the bus voltage magnitudes and angles computed as follows:

$$\begin{aligned}eV_j^{UB} &= \left(V_{j_0} + \sum_{h=1}^{p} |V_{j_h}|\right) - \max_m V_{j,m}^{MC} \\ eV_j^{LB} &= \min_m V_{j,m}^{MC} - \left(V_{j_0} - \sum_{h=1}^{p} |V_{j_h}|\right) \\ e\delta_k^{UB} &= \left(\delta_{k_0} + \sum_{h=1}^{p} |\delta_{k_h}|\right) - \max_m \delta_{k,m}^{MC} \\ e\delta_k^{LB} &= \min_m \delta_{k,m}^{MC} - \left(\delta_{k_0} - \sum_{h=1}^{p} |_{k_h}|\right) \\ &\forall k \in \mathcal{N}_{\mathcal{P}}\ \forall j \in \mathcal{N}_{\mathcal{PQ}}\end{aligned} \tag{26}$$

where $V_{j,m}^{MC}$ and $\delta_{k,m}^{MC}$ are the voltage magnitude and angle at the corresponding buses, computed at the m^{th} Monte Carlo simulation, respectively. Table III depicts the execution times registered for the simulations, and Fig.1 illustrates the bus voltage angle bounds obtained for the IEEE 118-bus test system.

TABLE I
AA-PF AVERAGE ERRORS IN BUS VOLTAGE MAGNITUDE BOUNDS

	30 bus		57 bus		118 bus	
	Up	Low	Up	Low	Up	Low
Original method [$10^{-4} p.u.$]	55	46	90	88	102	101
Proposed method [$10^{-4} p.u.$]	20	30	47	71	62	65

TABLE II
AA-PF AVERAGE ERRORS IN BUS VOLTAGE ANGLE BOUNDS

	30 bus		57 bus		118 bus	
	Up	Low	Up	Low	Up	Low
Original method [deg]	0.65	0.97	3.32	3.33	3.18	3.16
Proposed method [deg]	0.26	0.10	0.96	0.98	0.99	0.01

2) Optimal Reactive Power Dispatch: To confirm the benefits deriving from the application of the unified AA method to solve a complex OPF problem, the uncertain optimal reactive power dispatch for the IEEE 118-bus test system is analyzed here. To solve this problem, the objective function in the solution paradigm formalized in (15) is defined as follows:

$$\min_{(\hat{V}_k, \hat{\delta}_k, \hat{Q}_i)} \quad \hat{f}(\hat{V}_k, \hat{\delta}_k, \hat{Q}_i) = \sum_j \left(\hat{V}_j - V_n \right) \qquad (27)$$

The solution of this problem, which required about 398 s, yields the results summarized in Figs. 2 and 3, which depict the bounds of the bus voltage magnitudes and angles, respectively. In these figures the OPF solution bounds computed by a Monte Carlo model with 5,000 simulations, which required about 2650 s, are also shown.

3) Result Discussions: From the analysis of the obtained results, it can be observed that, compared to the Monte Carlo model, the PF and OPF solution bounds computed by the unified AA technique are slightly conservative, which is due to the fact that the AA-based methodology yields "worst case" bounds that take into account any uncertainties in input data. This can be argued to be an advantage of AA, as widely discussed in the reliable computing literature, since probabilistic methods that deal with non probabilistic uncertainty could neglect solutions that present a very low probability, but whose possibility of realization is greater than zero. Moreover, it is worth observing that the unified AA method gives better approximations of the AA-PF and AA-OPF solution bounds, when compared to the benchmark intervals obtained with the traditional AA-based solution methods, mainly due to the intrinsic characteristic of the proposed AA-based formalization, which employs rigorous relational and minimization operators, rather than approximated minimization operators based on domain contraction techniques. However, it is the costliest approach in terms of computational times, since this is mainly influenced by the large number of control variables, which depend on the number of noise symbols characterizing the parameter uncertainties. This poses computational difficulties in applying the proposed unified AA-PF and AA-OPF techniques to large scale power systems.

TABLE III
AA-PF EXECUTION TIMES

	30 bus	57 bus	118 bus
Monte Carlo (5000 trials) [s]	149.9	211.8	603.1
Original method [s]	1.7	2.5	5.7
Proposed method [s]	20.59	75.84	192.81

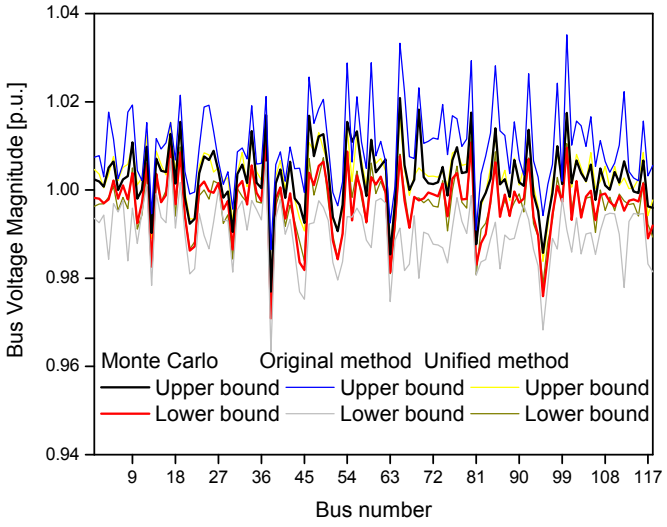


Fig. 2. Voltage magnitude bounds of the computed reactive power dispatch AA-OPF solutions for the 118-bus test system.

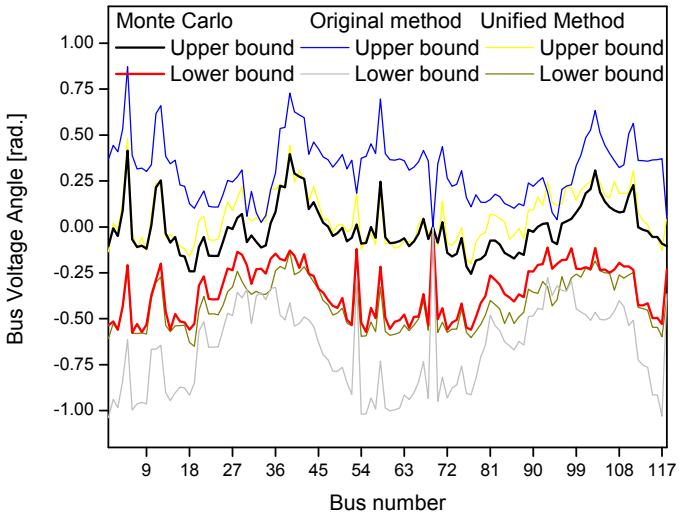


Fig. 3. Voltage angle bounds for all buses of the computed reactive power dispatch AA-OPF solutions for the 118-bus test system.

B. Noise Symbols Reduction

To reduce the computational burden of the unified AA method, the proposed PCA-based technique is used to identify the optimal number of independent uncertainties, i.e. the minimum number of feasible noise symbols, in the input variables for the uncertain economic dispatch analysis of the IEEE 118-bus test system, thus discovering main correlations among historical power system operation data, which was generated by clustering the load buses in 10 different classes characterized by the bi-weekly 15 min profiles depicted in Fig. 4, and using similar load profiles as the ones proposed in

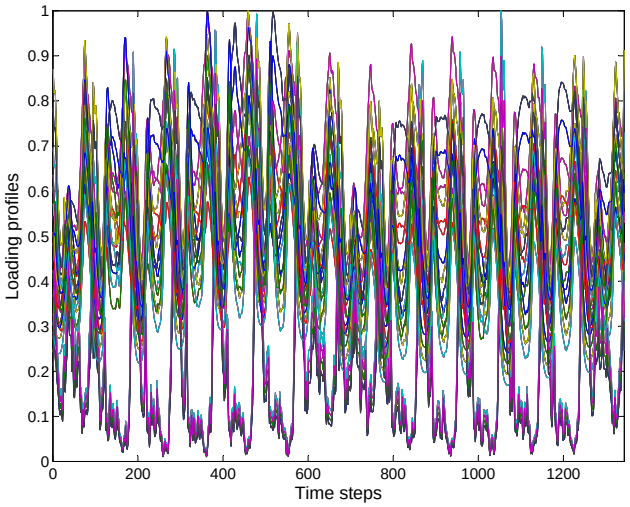


Fig. 4. Loading profiles used for the AA-PCA OPF dispatch analysis of the 118-bus IEEE test system.

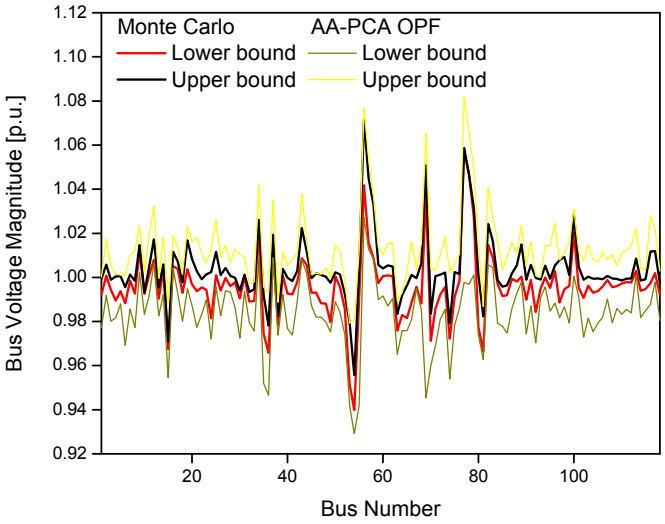


Fig. 5. Voltage magnitude bounds of the AA-PCA OPF dispatch solutions for the 118-bus test system.

[47]. From these data, the knowledge extraction process was implemented for various numbers of principal components in the interval $N_{PC} \in [3, 50]$, and for an approximation error tolerance of 10^{-3}; this yielded 12 principal components that can be extracted from the knowledge base. Therefore, the injected active and reactive powers can be properly represented by the following affine forms:

$$\begin{aligned} \hat{P}_i^{SP} &= P_{i_0}^{SP} + \textstyle\sum_{h=1}^{12} P_{i_h} \varepsilon_h \quad \forall i \in \mathcal{N}_{\mathcal{P}} \\ \hat{Q}_j^{SP} &= Q_{j_0}^{SP} + \textstyle\sum_{h=1}^{12} Q_{j_h} \varepsilon_h \quad \forall j \in \mathcal{N}_{\mathcal{Q}} \end{aligned} \tag{28}$$

Comparing (28) with the previously assumed affine forms, which had 117 noise symbols, demonstrates the sensible complexity reduction obtained by the application of the proposed PCA-based framework.

To check the consistency of the PCA-based affine forms, the OPF problem was solved by applying the unified AA method, assuming a $\pm 20\%$ (40%) variation of the active and reactive power injections around the corresponding mean values. The obtained results are reported in Figs. 5 and 6, which depict the obtained bounds of the bus voltage magnitudes and angles. In these figures, the corresponding bounds computed by a Monte Carlo-based approach are depicted in order to check the accuracy of the obtained solutions, demonstrating that the affine forms obtained using the PCA-based approach yield adequate results, allowing to represent the data uncertainties with a reduced number of noise symbols, which lead to a sensible reduction of the computational burden. Thus, compared to the assumption of considering all the uncertainties to be independent, an improvement of the convergence times of about 80%, namely 83 s vs 420 s, was observed when solving this case study.

C. Practical Application

In order to prove the effectiveness of the proposed framework, a real case study for a large-scale power system has been considered. Thus, the network under investigation is

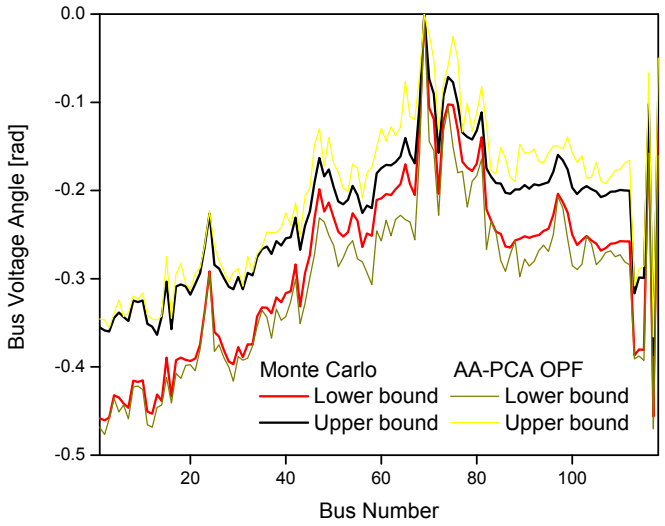


Fig. 6. Voltage angle bounds of the AA-PCA OPF dispatch solutions for the 118-bus test system.

a portion of a real power distribution system located in the south of Italy composed of 1310 buses, 12 substations, 1318 lines, and 7 large and 3 small wind power plants, whose main characteristics are summarized in Table IV. An advanced Distribution Management System, which includes smart grid functions such as wind power forecasting, pervasive load monitoring, and proactive voltage regulation, has been recently installed in this network. In this context, one of the more relevant issues that needs to be addressed is how to determine, one-day ahead, the reactive power flows imported from the 12 substations, which is important information for the Transmission System Operator to optimize ancillary service scheduling on the electricity market. To solve this problem, the Distribution System Operator (DSO) needs to define a robust voltage regulation strategy, which should minimize the deviations of the bus voltage magnitude from the nominal

values, while complying with the network constraints, and considering the load and generation forecasted errors. To address this complex issue, the proposed AA-based framework was used and the results are discussed next.

The main objective in the present study is to identify the deterministic values of the voltage controllers set points, the affine forms of the reactive power imported form each substation, and the affine form of the network voltage magnitudes and angles, so that the upper and lower deviations from a nominal voltage (e.g. 1 p.u.) of all the voltage magnitude affine forms are minimized, while satisfying, in the AA-domain, the power flow equations, the allowable voltage magnitude bounds, and the maximum substation reactive power limits. This problem can be formalized as the following AA-based OPF model:

$$\begin{aligned}
\min_{(V_k,\hat{\delta_k},\hat{Q}_k,\hat{V}_j,\hat{\delta_j})} \quad & \sum_{j\in\mathcal{N}_{\mathcal{PQ}}}\left(V_{j0}+\sum_{h=1}^{p}|V_{jh}|-V_n\right)^2+ \\
& \sum_{j\in\mathcal{N}_{\mathcal{PQ}}}\left(V_{j0}-\sum_{h=1}^{p}|V_{jh}|-V_n\right)^2 \\
\text{s.t.} \quad & \hat{P}_j^{\,SP}-\hat{V}_j\sum_{t=1}^{N}\hat{V}_tY_{jt}\cos\left(\hat{\delta}_j-\hat{\delta}_t-\theta_{jt}\right)\overset{A}{\approx}0 \\
& \hat{Q}_j^{SP}-\hat{V}_j\sum_{t=1}^{N}\hat{V}_tY_{jt}\sin\left(\hat{\delta}_j-\hat{\delta}_t-\theta_{jt}\right)\overset{A}{\approx}0 \\
& \hat{Q}_k-V_k\sum_{t=1}^{N}\hat{V}_tY_{kt}\sin\left(\hat{\delta}_k-\hat{\delta}_t-\theta_{kt}\right)\overset{A}{\approx}0 \\
& V_j^{min}\overset{A}{\leq}\hat{V}_j\overset{A}{\leq}V_j^{max} \\
& Q_k^{min}\overset{A}{\leq}\hat{Q}_k\overset{A}{\leq}Q_k^{max} \\
& \forall k\in\mathcal{N}_{sub},\ j\in\mathcal{N}_{\mathcal{PQ}}
\end{aligned} \tag{29}$$

where $\mathcal{N}_{sub}$ is the set of the substation buses.

The more relevant data uncertainties in (29) are related to the load and wind power forecasting errors. Hence, a preliminary analysis aimed at characterizing the statistical properties of these random variables is required to identify the parameters of the corresponding affine forms, namely the noise symbols and their partial deviations. To this aim, the histogram of the one-day forecasting errors observed on each wind power plant over a 5-month time scenario were analyzed, resulting in the tolerance bounds shown in Table V for each random variable, which guarantee that 97% of the measurements are within these bounds; the corresponding variables were assumed to be statistically independent, and uniformly distributed in these intervals. For the load forecasting errors, the DSO has identified three tolerance bounds, namely $\pm5\%$, $\pm7\%$ and $\pm10\%$, for industrial, commercial, and residential users, respectively. Hence, based on this data analysis, the following noise symbols can be defined:

- ε_1, ε_2, ε_3 to describe the uncertainty of the industrial, commercial, and residential power demands, respectively, which results in the following affine forms for the load powers in (29):

$$\begin{aligned}
\hat{P}_j^{SP} =& P_{j_0}^{SP}+(P_{j_0}^{SP}\ I_j\ 0.05)\varepsilon_1+(P_{j_0}^{SP}\ C_j\ 0.07)\varepsilon_2+ \\
&+(P_{j_0}^{SP}\ R_j\ 0.1)\varepsilon_3 \\
\hat{Q}_j^{SP} =& Q_{j_0}^{SP}+(Q_{j_0}^{SP}\ I_j\ 0.05)\varepsilon_1+(Q_{j_0}^{SP}\ C_j\ 0.07)\varepsilon_2+ \\
&+(Q_{j_0}^{SP}\ R_j\ 0.1)\varepsilon_3
\end{aligned} \tag{30}$$

  where I_j,C_j, and R_j, represent the percentage of industrial, commercial and residential loads served by the j^{th} bus, respectively.
- $\varepsilon_4, ..., \varepsilon_{10}$ to describe the power generation uncertainty of the 7 large wind power plants, respectively, resulting in the power generated by these plants being described by the following affine forms, modeling the ranges defined in Table V:

$$\begin{aligned}
\hat{P}_{514}^{SP} =& 0.27+0.0729\varepsilon_4=[0.1971,0.3429]MW \\
\hat{P}_{763}^{SP} =& 0.4243+0.1315\varepsilon_5=[0.2928,0.5558]MW \\
\hat{P}_{966}^{SP} =& 0.997+0.3290\varepsilon_6=[0.6680,1.3260]MW \\
\hat{P}_{1006}^{SP} =& 1.7+0.51\varepsilon_7=[1.19,2.21]MW \\
\hat{P}_{1124}^{SP} =& 0.1194+0.0382\varepsilon_8=[0.0812,0.1576]MW \\
\hat{P}_{1133}^{SP} =& 0.66496+0.179\varepsilon_9=[0.4855,0.8445]MW \\
\hat{P}_{1310}^{SP} =& 9.922+3.4727\varepsilon_{10}=[6.4493,13.3947]MW
\end{aligned} \tag{31}$$

Based on (30) and (31), and assuming that the voltage controllers at the substations operate at their nominal value, i.e. 1 p.u., an initial AA-based power flow analysis can be carried out in order to define the base case scenario. The corresponding results are shown in Fig. 7, where the AA-based voltage magnitude bounds obtained are benchmarked with respect to those computed using a Monte Carlo approach

TABLE IV
MAIN CHARACTERISTIC OF RENEWABLE POWER GENERATION FOR THE PRACTICAL TEST SYSTEM

Bus	Capacity [MW]	Forecasted Output [MW]	Power Factor
514	0.5	0.27	0.9
763	0.7	0.4243	0.9
966	2.4	0.997	0.9
1006	2.4	1.7	1
1070	0.3	NA	0.9
1124	0.25	0.1194	0.9
1126	0.03	NA	0.9
1128	0.03	NA	0.9
1133	1.2	0.66496	0.9
1310	12	9.922	0.9

TABLE V
TOLERANCE BOUNDS OF THE WIND POWER FORECASTING ERRORS FOR THE PRACTICAL TEST SYSTEM

Bus	Tolerance bound
514	$\pm27\%$
763	$\pm31\%$
966	$\pm33\%$
1006	$\pm30\%$
1124	$\pm32\%$
1133	$\pm27\%$
1310	$\pm35\%$

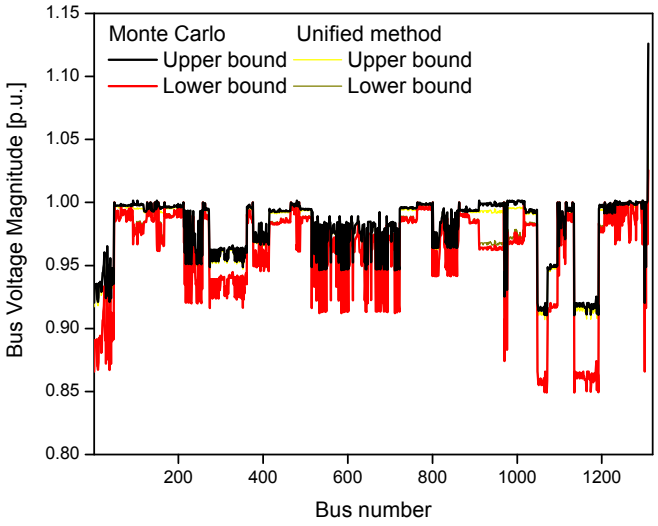


Fig. 7. Base-case bus voltage magnitudes of the practical 1310-bus system.

with 5,000 simulations. Observe the good accuracy of the results obtained with the proposed framework, in only 12 s of CPU time, versus the 232 s required for the Monte Carlo simulations.

Note in Fig. 7 that, for the base case scenario, the voltage magnitude on several buses are outside the allowable interval of [0.9, 1.1] p.u., and the deviations of the upper and lower bounds of the voltage magnitudes from the nominal 1 p.u. value are rather large at various buses. To address this problem, the proposed AA-based OPF (29) can be employed, yielding in 359 s the following set points for the substation voltage controllers V_k:

$$V_{opt} = [0.945\ 1.022\ 0.991\ 0.991\ 1.076\ 1.055\ 1.100\ 0.918\ 1.021\ 1.024\ 0.972\ 1.060]^T \tag{32}$$

The corresponding bounds of the bus voltage magnitudes, and the reactive power imported from the transmission network, which were constrained to ± 10 MVar, are depicted in Figs. 8 and 9, respectively. In these figures, the corresponding bounds obtained from solving the uncertain OPF problem (29) using 1,000 Monte Carlo simulations are also depicted to benchmark the AA solution, which takes about 10 h. From these results, the following can be observed:

- The proposed AA-based framework allows to accurately and reliably estimate the bound of the state variables. Note that the estimation errors are small, and always positive (negative) when estimating the upper (lower) bounds of the state variables. Consequently, the AA-based framework computes reliable enclosures of the state variable bounds, which, based on the Fundamental Invariant of Range Analysis property [33], are guaranteed to include the real solution domains.
- The voltage controller set points identified by the AA-based method are "robust", since these values guarantee that the bus voltage magnitudes and the reactive power imported by each substation are within the given operating constraints, for the assumed load and wind power forecasting error ranges.
- The deviations from the nominal value of the bus voltage magnitude is minimized by considering the combined effect of all the data uncertainties, without requiring complex and time-consuming multi-scenario analysis. In this context, it is important to note that the application of a stochastic programming approach for solving the problem under study would require the analysis of M^{10} scenarios, given the 10 sources of uncertainty considered, where M is the number of values used to discretize the random variables; this could become intractable for $M > 3$.
- The computational times required to solve the AA-based framework are quite reasonable, and certainly much less than the solution times required to study the problem using standard Monte Carlo simulations.
- Since the solution computed by the AA-based framework results in state variables expressed in affine form, these parameters could be used to infer the impact of each source of uncertainty on each state variable, so as to characterize the uncertainty propagation effects. Thus, from the affine forms of the reactive power imported from the transmission system shown in Table VI, which is an output of the AA-based OPF model (29), one can obtain the bounds of the reactive power injected in each substation, which can be used by the TSO to define the reactive power reserve that should be acquired on the ancillary services market, and the effect of the forecasting errors on these variables. For example, based on the values in Table VI, the reactive power imported from the third substation can be shown to be in the range 9.054 ± 0.9459 MVar, considering that the noise symbols ε vary from -1 to +1, and the forecasting error on the seventh wind power plant represented by $\varepsilon_{10} = [-1, 1]$ induces an uncertainty of ± 0.0331 MVar, which is very low compared to the ± 0.4185 MVar uncertainty induced on this variable by the residential load forecasting error represented by ε_3; thus, the forecasting uncertainty and the corresponding costs could be estimated for each source of uncertainty.

Moreover, it is important to note that for this case study, the application of the PCA-based methodology for reducing the noise symbols of the wind power generators does not work properly, since the corresponding forecasting profiles are not strongly correlated. This is clearly shown in Fig. 10, where the norm of the approximation error between the forecasted power profiles for the 7 large wind power plants and the corresponding profiles computed using PCA for 1 month is depicted for a number of principal components ranging from 1 to 7; observe that a minimum number of 6 principal components are required to obtain an accurate representation of the 7 uncertain variables, thus resulting in a very moderate reduction in the number of primitive noise symbols.

Finally, to test the robustness of the proposed bi-level solution approach, (29) was solved increasing all wind forecasting errors in Table V to $\pm 38\%$, and all load forecasting errors

TABLE VI
AFFINE FORMS OF THE REACTIVE POWER IMPORTED FROM THE TRANSMISSION SYSTEM

Sub	Cent. Val.	ε_1	ε_2	ε_3	ε_4	ε_5	ε_6	ε_7	ε_8	ε_9	ε_{10}
1	-2.60	0.23	0.334	0.484	0.0005	0.0006	0.0021	0.0019	0.0001	0.0003	0.0751
2	0.178	0.011	0.0161	0.0233	0	0	0	0	0	0	0
3	9.054	0.203	0.2877	0.4185	0.0002	0.0001	0.0013	0.0014	0.0001	0.0001	0.0331
4	8.3693	0.354	0.5014	0.7293	0.0001	0.0006	0.0035	0.0041	0.0001	0.0004	0.0364
5	1.4721	0.087	0.1237	0.1794	0	0	0	0	0	0	0
6	1.0522	0.048	0.0684	0.0989	0	0	0	0	0	0	0
7	0.7091	0.035	0.0492	0.0712	0	0	0	0	0	0	0
8	-5.14	0.164	0.2314	0.3368	0.0001	0.0002	0.0002	0.0014	0	0.0001	0.1145
9	0.4105	0.028	0.0399	0.0580	0	0	0	0	0	0	0
10	0.7567	0.047	0.0666	0.0967	0	0	0	0	0	0	0
11	9.0147	0.202	0.2855	0.4153	0.0002	0.0013	0.0015	0.0009	0.0001	0.0003	0.0782
12	1.2781	0.059	0.0846	0.1225	0	0	0	0	0	0	0

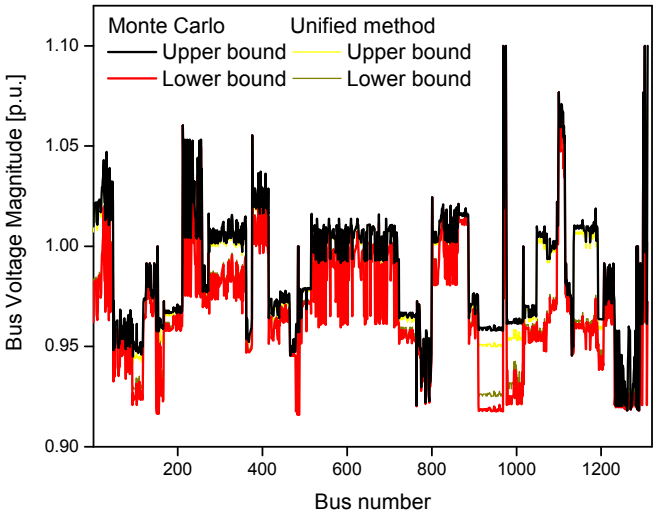


Fig. 8. Optimal bus voltage magnitudes of the practical 1310-bus system.

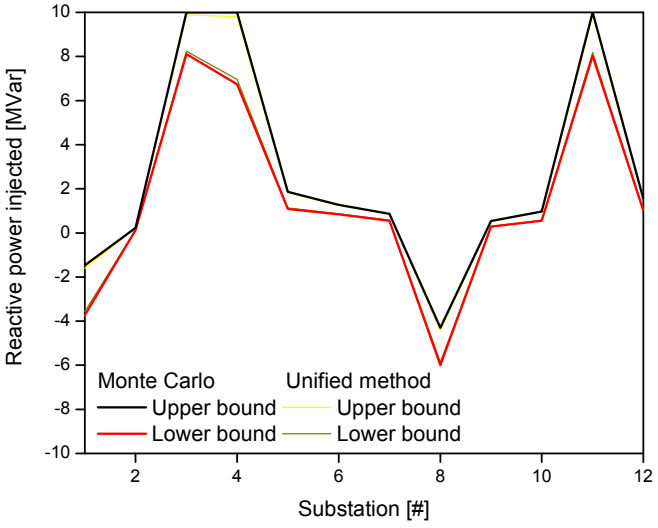


Fig. 9. Optimal reactive power imported from the transmission system for the practical 1310-bus system.

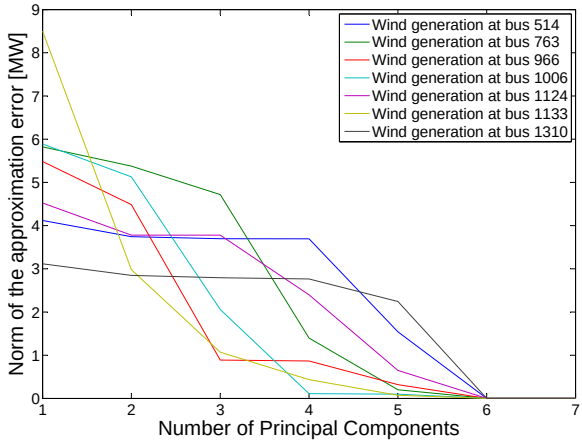


Fig. 10. Approximation error norm of the PCA-based methodology with respect to the number of principal components.

to $\pm 22\%$. In this case, a feasible solution containing the benchmark intervals obtained by Monte Carlo simulations was obtained, as expected.

VI. CONCLUSION

This paper proposed a novel AA-based computing framework for solving uncertain PF and OPF problems based on formulating a generic mathematical programming problem under uncertainty by means of equivalent deterministic problems, defining a coherent set of minimization, equality, and inequality operators. Compared to existing AA-based solution paradigms, this formulation was shown to have greater flexibility, as it allows to find feasible solutions and include multiple equality and inequality constraints, reducing the approximation errors. However, it resulted in higher computational costs, mainly due to the large number of control variables required to solve the "perturbed state" problem. To address this problem, a PCA-based paradigm for knowledge discovery from historical operation data-sets was proposed to identify the minimum feasible affine forms describing the input uncertain data.

On the basis of the obtained results, it could be argued that using AA-based techniques for solving uncertain PF and

OPF problems presents an accuracy/complexity trade-off. On one hand, AA techniques based on domain contraction can be used to obtain a rough qualitative insight of the solution in a very short time, compared to the time required with simulations-based methods like Monte Carlo. On the other hand, solution methods based on the definition of formal AA operators can be used to obtain a better description of the uncertainty evolution at the cost of higher simulation times. In both cases, the use of PCA can contribute to sensibly reduce the problem cardinality, and to better identify the affine forms describing the data uncertainty. The results presented in this paper should help to make the choice between the accuracy and computational costs, based on information about the application of the simulation outputs, and the existing computational resources.

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