

On the relation among the maximum degree, the chromatic number and the clique number

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

We study the relation among the maximum degree $\Delta(G)$, the chromatic number $\chi(G)$ and the clique number $\omega(G)$ of a graph G by proving results related to the Borodin-Kostochka Conjecture [3] and Reed's Conjecture [56], which are two long-standing conjectures on this subject. The Borodin-Kostochka Conjecture states that a graph G with $\chi(G) = \Delta(G) \geq 9$ has a clique of size $\Delta(G)$. Reed's Conjecture states that any graph G satisfies $\chi(G) \leq \lceil \frac{1}{2}(\Delta(G) + \omega(G) + 1) \rceil$.

By combining and extending the approaches taken in Cranston and Rabern [19] and in Haxell and MacDonald [35], we use the technique of Mozhan partitions to prove a generalized Borodin-Kostochka-type result as follows. Given a nonnegative integer t , for every graph G with $\Delta(G) \geq 4t^2 + 11t + 7$ and $\chi(G) = \Delta(G) - t$, the graph G contains a clique of size $\Delta(G) - 2t^2 - 7t - 4$.

We further prove that both conjectures hold for graphs with at least one critical vertex that is not in an odd hole, by following similar proofs as in Aravind, Karthick and Subramanian [1] and in Chen, Lan, Lin and Zhou [9]. This result then motivates us to study Reed's Conjecture in the context of graphs in which all vertices are in some odd hole. By investigating blow-ups and powers of cycles, we enumerate several classes of graphs for which Reed's Conjecture holds with equality. They generalize many of the known tight examples. We also introduce a class of graphs constructed from blow-ups and powers of paths, which gives rise to a new family of irregular tight examples.

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Dedication

To the curious minds.

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Chapter 1

Introduction

In this thesis, we consider simple, finite and undirected graphs. For such a graph G , the **degree** $d(v)$ of a vertex v in G is the number of its neighbours. The **maximum degree** $\Delta(G)$ of G is the largest degree of a vertex in G . A **clique** in G is a subset of vertices in which any two distinct vertices are adjacent. The **clique number** $\omega(G)$ is the size of a maximum clique. The **chromatic number** $\chi(G)$ is the minimum number of colours needed to colour the vertices so that adjacent vertices receive different colours. Unless otherwise defined, we use standard graph theoretic notation and terminology found in textbooks, such as in Diestel [27].

1.1 Implications of a High Chromatic Number

One fundamental question that we may ask about graphs is what makes the chromatic number of a graph large. An initial guess might be the existence of a large clique, since the clique number always serves as a lower bound for the chromatic number. However, having a large clique number is not a necessary condition, as shown in the following theorem.

Theorem 1.1 (Tutte [21]). *For any $k \geq 1$, there exist triangle-free graphs with chromatic number greater than k .*

The first explicit construction that produces triangle-free graphs with arbitrarily large chromatic number is due to Tutte [21]. As nicely summarized by Scott and Seymour [59], there are many other constructions [54] [45] [50] [30] [64] [6] and non-constructive proofs

[29] [39]. Here we choose to prove Theorem 1.1 using Mycielski graphs [54], since they are simple to describe and easily motivate the rest of this thesis.

Suppose we have a graph G with n vertices, $v_1, v_2, \dots, v_n$. To construct the Mycielski graph of G , for all $i \in [n]$, we add a vertex u_i and join u_i to every neighbour of v_i in the original graph G , that is, the neighbourhood of u_i in G satisfies $N_G(u_i) = N_G(v_i)$. Then, we add a new vertex w and join w to all the u_i 's. The resulting graph G' is the Mycielski Graph of G . In Figure 1.1, we show as an example the Mycielski graph of the 4-cycle C_4 .

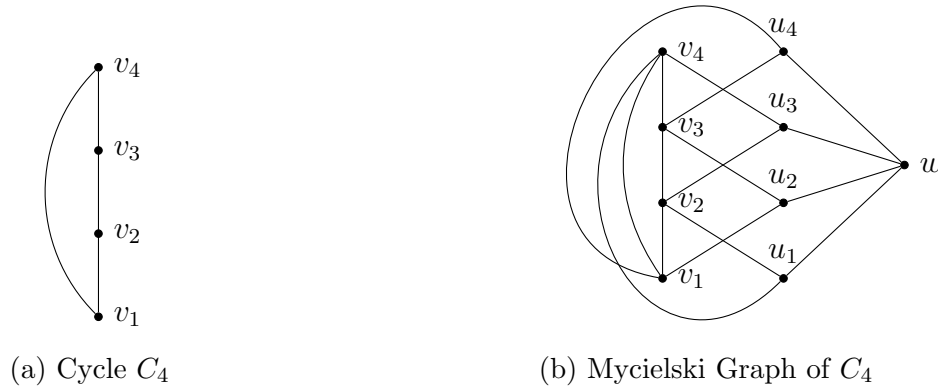


Figure 1.1: An example of a Mycielski graph

Define $U = \{u_1, \dots, u_n\}$. We observe that the Mycielski graph has the following properties.

1. If G is not edgeless, then $\omega(G') = \omega(G)$. This follows from the following argument. Since G is an induced subgraph of G' , we have $\omega(G') \geq \omega(G)$. Since U is an **independent set**, which means that no two vertices in the set are adjacent, a maximum clique in G' contains at most one vertex in U . Suppose for contradiction that $\omega(G') \geq \omega(G) + 1 \geq 3$. Since w is not adjacent to any vertex in $V(G)$, a maximum clique V' in G' does not contain w and contains exactly one vertex from U . That is, there exists some $i \in [n]$ such that $u_i \in V'$ and $V' \setminus \{u_i\} \subseteq V(G)$. By construction of G' , we have $V' \setminus \{u_i\} \subseteq N_G(u_i) = N_G(v_i)$. Hence $(V' \setminus \{u_i\}) \cup \{v_i\}$ is a clique in G of size $\omega(G') > \omega(G)$, which is a contradiction.
2. We have $\chi(G') = \chi(G) + 1$ because of the following argument. Let $\chi = \chi(G)$ and ϕ be a χ -colouring of G . For each $i \in [n]$, since $N_G(u_i) = N_G(v_i)$ and U is independent, we have that $N_{G'-w}(u_i) \subseteq N_{G'-w}(v_i)$. So we may extend ϕ to a χ -colouring of $G' - w$

by assigning u_i the colour of v_i . Then we have $\chi(G' - w) = \chi$. Since $N_{G'}(w) = U$, it remains to show that under any χ -colouring of $G' - w$, all χ colours appear in U . Suppose for contradiction that there exists a χ -colouring ϕ' of $G' - w$ such that a colour c appears only in $V(G)$. Then we construct a new colouring ϕ'' of G as follows. For all $i \in [n]$,

$$\phi''(v_i) := \begin{cases} \phi'(u_i) & \text{if } \phi'(v_i) = c \\ \phi'(v_i) & \text{otherwise.} \end{cases}$$

Since $N_G(u_i) = N_G(v_i)$ and any two vertices with colour c under ϕ' are non-adjacent, ϕ'' is a proper $(\chi - 1)$ -colouring of G , which contradicts that G is χ -chromatic.

This implies that by recursively applying Mycielski's construction, we can increase the chromatic number indefinitely while keeping the clique number unchanged. In particular, if we apply this construction repeatedly to a triangle-free graph, such as K_2 , then we obtain triangle-free graphs with arbitrarily high chromatic number. This proves Theorem 1.1 and implies that having a large clique is not a necessary condition for a graph to have a high chromatic number.

We further observe that the Mycielski graph has a larger maximum degree than the original graph. In particular, the degree of each v_i has doubled in G' , so $\Delta(G') \geq 2\Delta(G)$. Returning to the question that we posed at the beginning of this section, we shall update our answer to include the maximum degree as another factor that potentially correlates with the chromatic number. This motivates us to study the relation among the maximum degree, the chromatic number and the clique number.

1.2 Brooks' Theorem

The well-known Brooks' Theorem [4] from 1941 relates the maximum degree, the chromatic number and the clique number of a graph. It states that given a connected graph G , greedy coloring ensures that the chromatic number is upper bounded by $\Delta(G) + 1$. Moreover, if this upper bound is achieved, then G is either **complete**, that is, any two vertices in G are adjacent, or an odd cycle. We follow a similar proof as in West [62]. We first define the following terminologies and then proceed to a proof by cases.

Definition 1.2. A graph G is said to be **d -regular** (or simply **regular**) if every vertex of G has degree d . Note that if G is d -regular, then $\Delta(G) = d$.

Definition 1.3. Given a connected graph G , a vertex $v \in V(G)$ is a **cut vertex** if $G - v$ is disconnected. A subset $S \subseteq V(G)$ of vertices is a **separating set** if $G - S$ is disconnected. The graph G is said to be **k -connected** if G has at least $k + 1$ vertices and any separating set has size at least k . Note that a connected graph with a cut vertex is 1-connected. A **block** of G is a maximal connected subgraph with no cut vertices. A **block-cut tree** $T_{bc}(G)$ of G is a tree whose vertices are the blocks of G and two vertices in $T_{bc}(G)$ are adjacent if and only if the corresponding blocks in G share a cut vertex. A **leaf block** in G is a block that corresponds to a leaf in $T_{bc}(G)$. That is, a leaf block in G is a block that contains at most one cut vertex of G .

Theorem 1.4 (Brooks [4]). *Given a connected graph G , we have $\chi(G) \leq \Delta(G) + 1$. Equality holds if and only if G is complete or is an odd cycle.*

Proof. Let G be a connected graph. Define $\Delta := \Delta(G)$. If $\Delta \leq 2$, then the graph G is either a path or a cycle, and the statement holds. So we may assume that $\Delta \geq 3$. It is clear that if G is complete, then $\chi(G) = \omega(G) = \Delta + 1$. It then suffices to show that if $\Delta \geq 3$ and G is not complete, then $\chi(G) \leq \Delta$. Let $n = |V(G)|$. The idea of the remainder of this proof is to arrange the vertices of G into a list $L = v_1, v_2, \dots, v_n$ such that every vertex has at most $\Delta - 1$ preceding neighbours in L , with the possible exception of v_n , which may have Δ preceding neighbours among which v_1 and v_2 are allowed to have the same colour. Then a greedy coloring along the list yields a Δ -coloring. We consider the following four cases.

Case 1: The graph G is not regular.

Let $v_n \in V(G)$ be a vertex such that $d(v_n) < \Delta$. We grow a spanning tree T of G with v_n being the root and number the vertices we reach in decreasing order. Then every vertex other than v_n has its parent vertex in T appear later in L . Hence all vertices in G have at most $\Delta - 1$ neighbours that appear earlier in L .

Case 2: The graph G is regular and has a cut vertex u .

The cut vertex u together with any component H of $G - u$ form a graph $H' := H + u$ with $d_{H'}(u) < \Delta(G) = \Delta(H')$. By Case 1, $\chi(H') \leq \Delta(H') = \Delta$. That is, in a minimum colouring of H' , at most $\Delta - 1$ colours appear in $N_H(u)$. Since this holds for each component H of $G - u$, by Δ -colouring each $H + u$ and relabeling the colours if necessary, we obtain a Δ -colouring of $G - u$ with at most $\Delta - 1$ colours in $N_G(u)$, which leaves at least 1 colour available for u . Hence $\chi(G) \leq \Delta$.

Case 3: The graph G is regular and 2-connected and has a separating set $\{u, w\}$.

Consider the block-cut tree $T := T_{bc}(G - u)$ and the set V_c of all cut vertices in $G - u$. Since w is a cut vertex in $G - u$, $G - u$ has at least two blocks and hence at least two leaf blocks. Let H_1 and H_2 be two leaf blocks in $G - u$. Since G has no cut vertices, for $i = 1, 2$, vertex u has a neighbour $u_i \notin V_c$ in the leaf block H_i . Let $v_n = u$, $v_1 = u_1$ and $v_2 = u_2$. Then, $v_1 \sim v_n$, $v_2 \sim v_n$ and $v_1 \not\sim v_2$. The graph $G - \{u, v_1, v_2\}$ is connected since v_1, v_2 are not cut vertices of $G - u$ and are in distinct leaf blocks. Since G is regular, we have $d(u) = \Delta \geq 3$. So u has a neighbour in $G - \{v_1, v_2\}$, and hence $G - \{v_1, v_2\}$ is connected. Similar to Case 1, we grow a spanning tree of $G - \{v_1, v_2\}$ with v_n being the root and label the vertices in decreasing order. Then, by assigning the same colour to v_1 and v_2 and greedily colouring along the list, we obtain a Δ -colouring of G .

Case 4: The graph G is regular and 3-connected.

Since G is not complete, there exist $x, y \in V(G)$ such that $x \not\sim y$. By taking v_1, v_n, v_2 to be the first 3 vertices in a shortest xy -path in G , we have that $v_1 \sim v_n$, $v_n \sim v_2$ and $v_1 \not\sim v_2$. Since G is 3-connected, $G - \{v_1, v_2\}$ is connected. By the same argument as in Case 3, we have $\chi(G) \leq \Delta$. $\square$

In this thesis, we focus on the following restatement of Brooks' theorem and use this as a starting point for studying the interplay among the maximum degree, the chromatic number and the clique number.

Theorem 1.5 (Brooks [4]). *Given a graph G with $\Delta(G) \neq 2$, if $\chi(G) = \Delta(G) + 1$, then $\omega(G) = \Delta(G) + 1$.*

1.3 The Borodin-Kostochka Conjecture

Brooks' Theorem naturally leads to the following question: if a graph has a chromatic number equal or close to its maximum degree, what do we know about its clique number? In 1977, Borodin and Kostochka [3] made a first step toward addressing this question by conjecturing that, given a graph G with maximum degree at least 9, if the chromatic number equals $\Delta(G)$, then the clique number also equals $\Delta(G)$.

Conjecture 1.6 (Borodin and Kostochka [3]). *Given a graph G with $\Delta(G) \geq 9$, if $\chi(G) = \Delta(G)$, then $\omega(G) = \Delta(G)$.*

This seemingly small improvement to Brooks' Theorem turned out to be surprisingly challenging to resolve, and it remains an open problem. The Borodin-Kostochka Conjecture

has been studied extensively over the past nearly 50 years due to its importance. In addition to strengthening Brooks' theorem and providing further information on structural properties of graphs with a high chromatic number, Molloy and Reed [52] have shown that this conjecture is the strongest possible statement of this type in the following two senses.

First, the hypothesis of $\Delta \geq 9$ cannot be weakened. One counterexample with $\Delta = 8$ is the graph $G = C_5[K_3]$ [48], which is obtained by blowing up each vertex of a 5-cycle into a triangle, as shown in Figure 1.2. This example was known to Borodin and Kostochka but was not mentioned in the paper in which they first proposed the conjecture. Since G has 15 vertices and the **independence number** of G , which is the size of a maximum independent set, is $\alpha(G) = 2$, we have $\chi(G) \geq \lceil |V(G)|/\alpha(G) \rceil = 8$. It is straightforward to verify that $\chi(G) = 8 = \Delta(G)$. However, the clique number of G is 6. So, $\Delta \geq 9$ is a necessary condition for the conjecture.

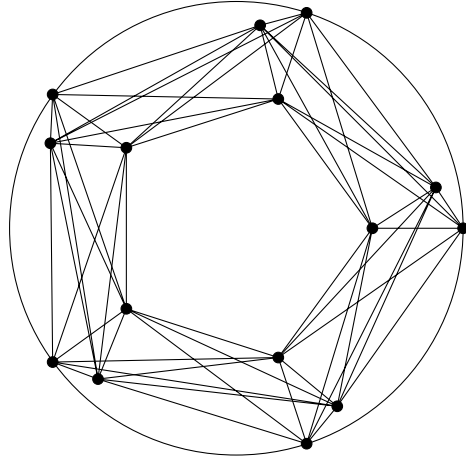


Figure 1.2: Graph $G = C_5[K_3]$, which has $\chi(G) = \Delta(G) = 8$ and $\omega(G) = 6$

Second, we cannot further strengthen the statement by decreasing the chromatic number to $\Delta - 1$ and claiming that $\omega = \Delta - 1$. A family of counterexamples were described in [52]. For integers $t \geq 5$, define the graph G_t to be the join of the complete graph K_{t-4} and the cycle C_5 , where the **join** $G \vee H$ of two graphs G and H is the **union** $G \cup H$, (that is, $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$) together with all edges that joins a vertex in G and a vertex in H . Each G_t satisfies $\Delta(G_t) = t$, $\chi(G_t) = t - 1$ and $\omega(G_t) = t - 2$.

The progress made toward the Borodin-Kostochka Conjecture mainly falls into two categories. First, as shown in Table 1.1, mathematicians have proved various weaker versions of the conjecture by showing that the clique number satisfies some weaker lower

bound. The best known result is due to Reed [57] in 1999. He proved that the conjecture holds for graphs with maximum degree at least 10^{14} and remarked that by modifying his proof, this bound could be made as low as 10^6 , or possibly even 10^3 . However, a large gap remains between Reed's result and the condition $\Delta \geq 9$ proposed by Borodin and Kostochka.

Table 1.1: Borodin-Kostochka-type results for graphs with $\chi = \Delta$

Results	Author(s)	Year
$\Delta \geq 7 \implies \omega \geq \lfloor \frac{1}{2}(\Delta + 1) \rfloor$	Borodin and Kostochka [3]	1977
$\Delta \geq 10 \implies \omega \geq \lfloor \frac{1}{3}(2\Delta + 1) \rfloor$	Mozhan [53]	1983
$\Delta \geq 9 \implies \omega \geq \Delta - 28$	Kostochka [47]	1980
$\Delta \geq 31 \implies \omega \geq \Delta - 3$	Mozhan (see [57])	1987
$\Delta \geq 13 \implies \omega \geq \Delta - 3$	Cranston and Rabern [19]	2015
$\Delta \geq 10^{14} \implies \omega \geq \Delta$	Reed [57]	1999

For graphs with relatively small maximum degrees, the best result to date is by Cranston and Rabern [19] in 2015. They proved the existence of a clique with size at least $\Delta - 3$. This result has been generalized by Haxell and MacDonald [35] in 2023 to address graphs with chromatic number close to the maximum degree.

Theorem 1.7 (Haxell and MacDonald [35]). *Let t be a nonnegative integer. Then for every graph G with $\chi(G) = \Delta(G) - t$ and $\Delta(G) \geq 6t^2 + 20t + 16$, we have $\omega(G) \geq \Delta(G) - 2t^2 - 6t - 3$.*

Both [19] and [35] used the technique of Mozhan partitions. In Chapter 2 of this thesis, we investigate their methods in more detail and, by combining ideas from both papers, prove the following theorem.

Theorem 1.8. *Let t be a nonnegative integer. Then for every graph G with $\chi(G) = \Delta(G) - t$ and $\Delta(G) \geq 4t^2 + 11t + 7$, we have $\omega(G) \geq \Delta(G) - 2t^2 - 7t - 4$.*

Another type of approach toward the Borodin-Kostochka Conjecture is to prove the conjecture for certain classes of graphs. These classes are often defined by forbidden subgraphs. Table 1.2 provides a noncomprehensive list of results of this type. There also exists a multitude of unpublished work in this category, as shown in Table 1.3. In Section

3.2, we focus on the theorem by Chen, Lan, Lin and Zhou [9] in 2024. They showed that the Borodin-Kostochka Conjecture holds for odd-hole-free graphs. An **odd hole** is an induced odd cycle of length at least five, and **odd-hole-free** graphs are graphs that do not contain an odd hole. Odd-hole-free graphs constitute a class of graphs that have been heavily studied in many other contexts, which makes it intriguing to investigate how the odd-hole-free constraint allows us to prove the Borodin-Kostochka Conjecture as well as Reed's Conjecture.

Table 1.2: The Borodin-Kostochka Conjecture for graphs with forbidden subgraphs

Family of Graphs	Author(s)	Year
A subclass of $(K_{1,3}, K_5 - e)$ -free graphs	Dhurandhar [22]	1982
Claw-free	Cranston and Rabern [18]	2012
(P_5, C_4) -free	Gupta and Pradhan [33]	2020
(P_5, gem) -free	Cranston, Lafayette and Rabern [17]	2022
$(P_2 \cup P_3, \text{house})$ -free	Chen, Lan and Zhou [10]	2024
Odd-hole-free	Chen, Lan, Lin and Zhou [9]	2024
Hammer-free	Chen, Lan and Lin [8]	2024
(P_{10}, paw) -free; (P_6, C_4, bull) -free; $(P_6, C_4, \text{diamond})$ -free; (P_5, C_5) -free	Gupta and Pradhan [34]	2024
$\overline{K_{1,3}}$ -free	Lan and Lin [49]	2024

Table 1.3: The Borodin-Kostochka Conjecture for graphs with forbidden subgraphs (unpublished)

Family of Graphs	Author(s)	Year
$2K_2$ -free	Dhurandhar [23]	2017
$3K_1$ -free (i.e. $\alpha \leq 2$)	Dhurandhar [24]	2017
$(P_3 \cup K_1)$ -free; $(K_2 \cup \overline{K_2})$ -free	Dhurandhar [25]	2017
$(P_4 \cup K_1)$ -free; P_5 -free; Chair-free	Dhurandhar [26]	2021
$(P_6, \text{apple}, \text{torch})$ -free	Wu and Wu [63]	2023
$(P_2 \cup P_3, \text{house})$ -free; $(P_2 \cup P_3, \text{kite})$ -free; $(P_2 \cup P_3, \text{dart})$ -free	Wang [61]	2023

1.4 Reed's Conjecture

In 1998, Reed [56] made another conjecture involving the three parameters of interest. It states that the chromatic number of any graph is upper bounded in terms of the maximum degree and the clique number, as follows.

Conjecture 1.9 (Reed [56]). *For any graph G , we have*

$$\chi(G) \leq \left\lceil \frac{\Delta(G) + 1 + \omega(G)}{2} \right\rceil.$$

Comparing to the Borodin-Kostochka Conjecture, Reed's Conjecture is a more general statement that addresses all graphs. A more detailed comparison between the two conjectures was made by Cranston, Lafayette and Rabern [17]. When $\chi(G) = \Delta(G) + 1$, Reed's Conjecture holds by Brooks' Theorem. When $\chi(G) = \Delta(G)$, Reed's Conjecture reduces to $\omega(G) \geq \Delta(G) - 2$. This is weaker than the Borodin-Kostochka Conjecture, which claims $\omega(G) = \Delta(G)$ if $\Delta(G) \geq 9$. For graphs with $\chi(G) \leq \Delta(G) - 1$, which are not of interest to the Borodin-Kostochka Conjecture, Reed's Conjecture provides a nontrivial lower bound on $\omega(G)$.

Reed's Conjecture is also an attempt at finding the optimal ϵ value in Reed's Theorem.

Theorem 1.10 (Reed [56]). *There exists $\epsilon > 0$ such that for every graph G ,*

$$\chi(G) \leq \lceil (1 - \epsilon)(\Delta(G) + 1) + \epsilon\omega(G) \rceil. \quad (1.1)$$

For any graph G , if G is complete, that is, $\omega(G) = \Delta(G) + 1$, then the bound above reduces to Brooks' Theorem. Otherwise, we can rearrange Inequality 1.1 and use the fact that $\omega(G) < \Delta(G) + 1$ to obtain the following upper bound on ϵ .

$$\begin{aligned} \chi(G) &\leq \lceil \Delta(G) + 1 - \epsilon(\Delta(G) + 1 - \omega(G)) \rceil \\ \implies \chi(G) &< \Delta(G) + 2 - \epsilon(\Delta(G) + 1 - \omega(G)) \\ \implies \epsilon &< \frac{\Delta(G) + 2 - \chi(G)}{\Delta(G) + 1 - \omega(G)}. \end{aligned}$$

That is, any graph G with $\omega(G) < \Delta(G) + 1$ provides an upper bound on the value of ϵ . In particular, a concrete upper bound is given by the family of blow-ups of C_5 . For every integer $t > 0$, define G_t to be the graph $C_5[K_t]$. Similar to how we obtained the maximum

degree, the clique number and the chromatic number of $C_5[K_3]$ in the previous section, we have that

$$\Delta(G) = 3t - 1, \quad \omega(G) = 2t, \quad \chi(G) = \left\lceil \frac{|V(G)|}{\alpha(G)} \right\rceil = \left\lceil \frac{5t}{2} \right\rceil.$$

Thus,

$$\epsilon < \frac{(3t - 1) + 2 - \lceil \frac{5t}{2} \rceil}{(3t - 1) + 1 - 2t} = \frac{\lfloor \frac{t}{2} \rfloor + 1}{t}.$$

The right-hand side approaches $\frac{1}{2}$ as $t \rightarrow \infty$, and hence we have that $\epsilon \leq \frac{1}{2}$. Since $\Delta(G) + 1 \geq \omega(G)$ for every graph G , the bound in Reed's Theorem is tighter when ϵ is larger. It is therefore appealing to find the maximum $0 < \epsilon_0 \leq \frac{1}{2}$ such that Inequality 1.1 holds for all $\epsilon \leq \epsilon_0$. A lot of research has been done on increasing the lower bound on this ϵ_0 for graphs with sufficiently large maximum degree. A summary is shown in Table 1.4. Additionally, Molloy and Reed started the study of Reed's Theorem for sparse graphs [51], and their work is followed by [5] [2] [37].

Table 1.4: Lower bound on the maximum value of ϵ in Reed's Theorem

Lower bound on ϵ_0	Author(s)	Year
1.4×10^{-8}	Reed [56]	1998
$\frac{1}{320e^6}$	King and Reed [43]	2016
$\frac{1}{26}$	Bonamy, Perrett and Postle [2]	2022
$\frac{1}{13}$	Delcourt and Postle [20]	2017
0.119	Hurley, de Joannis de Verclos and Kang [37]	2022

To prove the lower bound of 1.4×10^{-8} , Reed showed the following result.

Theorem 1.11 (Reed [56]). *Let $M = 7 \times 10^7$. There exists a constant Δ_0 such that for any graph G , if $\Delta(G) \geq \Delta_0$, then*

$$\chi(G) \leq \frac{\Delta(G) + 1 + k}{2},$$

where $k = \max\{\omega(G), (1 - \frac{1}{M})\Delta(G)\}$.

Note that when $\omega(G) \geq (1 - \frac{1}{M})\Delta(G)$, this theorem implies that Reed's Conjecture holds for G . This then allows us to relate Reed's Conjecture to graphs with $\chi = \Delta - t$ for some nonnegative integer t . For sufficiently large t , Reed's Conjecture holds for any graph with a maximum degree that is large in terms of t .

Corollary 1.12. *Let $M = 7 \times 10^7$. There exists a positive integer t_0 such that for all $t \geq t_0$ and any graph G with $\chi(G) = \Delta(G) - t$, if $\Delta(G) \geq (2t + 1)M + 1$, then Reed's Conjecture holds for G .*

Proof. Let Δ_0 be the constant from Theorem 1.11. We fix t_0 to be some large integer such that $(2t_0 + 1)M + 1 \geq \Delta_0$. Take any integer $t \geq t_0$ and any graph G with $\chi(G) = \Delta(G) - t$ and $\Delta(G) \geq (2t + 1)M + 1 \geq \Delta_0$. Define $\omega_0 = \left(1 - \frac{1}{M}\right) \Delta(G)$.

Suppose for contradiction that $\omega(G) < \omega_0$. Then by Theorem 1.11, we have

$$\begin{aligned} \chi(G) &\leq \frac{\Delta(G) + 1 + \omega_0}{2} \\ \implies \Delta(G) - t &\leq \frac{1}{2} \left(\Delta(G) + 1 + \left(1 - \frac{1}{M}\right) \Delta(G) \right) \\ \implies \Delta(G) &\leq (2t + 1)M, \end{aligned}$$

which contradicts the assumption that $\Delta(G) \geq (2t + 1)M + 1$. So we have $\omega(G) \geq \omega_0$. By Theorem 1.11, Reed's Conjecture holds for G . $\square$

In 2011, Aravind, Karthick and Subramanian [1] proved that the conjecture for odd-hole-free graphs. In fact, they proved a stronger variant where the ceiling operator is replaced by the floor operator and the maximum degree is replaced by the **degeneracy**, which is the maximum of the minimum degree in any subgraph.

Reed's conjecture has been proven for graphs with a maximum degree that is large in terms of its number of vertices and its independence number [55] [46]. The conjecture has also been verified for certain classes of graphs, including claw-free graphs, circular interval graphs, line graphs [40], graphs with $\alpha = 2$ [46], triangle-free graphs with $\Delta \geq 524$ [38] and many others [44] [55] [31] [41] [1] [11]. We notice that the independence number seems to have an important role.

The approach in [9] is very similar to the one in [1], and both are discussed in Chapter 3. Moreover, by a similar proof as [1], we may show that Reed's Conjecture holds for graphs in which at least one **critical vertex** (that is, a vertex whose deletion reduces the chromatic number) is not in any odd hole. This indicates that to obtain further results on Reed's Conjecture, one direction to explore is to construct graphs in which every vertex is in an odd cycle. This generates many interesting graphs for which Reed's Conjecture holds with equality, including the graph $C_5[K_3]$ shown in Figure 1.2. Such tight examples of Reed's Conjecture are studied in Chapter 4, in which the independence number again plays an important role.

1.5 Overview

The structure of the remainder of this thesis is as follows. In Chapter 2, we analyze the approaches taken by [19] and [35] and prove Theorem 1.8, which may be viewed as a weaker but more general variant of the Borodin-Kostochka Conjecture. In Section 2.1, we prove some facts regarding critical vertices under a coloring, which are used again in Chapter 3. The main technique used in Chapter 2 is Mozhan partitions, whose definition and properties are discussed in detail in Section 2.2. Section 2.3 defines and studies a move sequence that exploits the power of Mozhan partitions, which then allows us to prove the theorem in Section 2.4.

In Chapter 3, we follow similar approaches as in [9] and [1], while generalizing some steps of their proofs, to prove that the Borodin-Kostochka Conjecture and Reed's Conjecture hold for graphs with at least one critical vertex that is not in an odd hole. Some interesting subclasses of odd-hole-free graphs are discussed in Section 3.1. The two main results of the chapter are proved in Sections 3.2 and 3.3. In Section 3.4, we pose some open questions regarding graphs closely related to odd-hole-free graphs.

In Chapter 4, we enumerate several classes of graphs for which Reed's Conjecture holds with equality. Section 4.1 discusses such tight examples that are known in the literature. In Sections 4.2 and 4.3, we study some broader classes of graphs involving blow-ups and powers of cycles. From these, we derive a list of tight examples that generalize many of the known examples. In Section 4.4, we study a closely related class of graphs constructed from blow-ups and powers of paths. This gives rise to a new family of irregular tight examples. Finally in Section 4.5, we outline some unanswered questions related to the results in this chapter.

In Chapter 5, we summarize the main results that are proved and briefly discuss some related interesting problems for future research.

Chapter 2

Large Cliques in Graphs with High Chromatic Number

As seen in Section 1.3, the Borodin-Kostochka Conjecture is an attempt at strengthening Brooks' Theorem.

Theorem 2.1 (Brooks [4]). *Given a graph G with $\Delta(G) \neq 2$, if $\chi(G) = \Delta(G) + 1$, then $\omega(G) = \Delta(G) + 1$.*

Conjecture 2.2 (Borodin and Kostochka [3]). *Given a graph G with $\Delta(G) \geq 9$, if $\chi(G) = \Delta(G)$, then $\omega(G) = \Delta(G)$.*

In this chapter, we prove a weaker but more general result of this form using Mozhan partitions. The technique of Mozhan partitions was first introduced by Mozhan in [53]. Since then, it has been used to make significant progress toward the Borodin-Kostochka Conjecture, such as in [18], [19] and [35], including the best known result due to Cranston and Rabern [19], which is stated in the following theorem.

Theorem 2.3 (Cranston and Rabern [19]). *Given a graph G with $\Delta(G) \geq 13$, if $\chi(G) = \Delta(G)$, then $\omega(G) \geq \Delta(G) - 3$.*

This theorem builds upon Theorem 2.4 below, which we will generalize to cases where $\chi \leq \Delta$. It also exploits the assumption that $\chi = \Delta$. In particular, the lower bound 13 comes from the existence of a **hitting set**, which is an independent set that intersects every maximum clique, in Δ -critical graphs with $\Delta \geq 14$ and $\omega = \Delta - 4$. A graph G is said to be **k -critical** (or simply **critical**) for some integer $k \geq 0$ if $\chi(G) = k$ and every vertex in

G is a critical vertex. Results on hitting sets are commonly used in proving results related to the Borodin-Kostochka Conjecture, such as in [3], [7], [53] and [35].

Theorem 2.4 (Cranston and Rabern [19]). *If G is a graph with $\chi(G) = \Delta(G) =: \Delta$, then $\omega(G) \geq \Delta - 3$ if $\Delta \equiv 1 \pmod{3}$ and $\omega(G) \geq \Delta - 4$ otherwise.*

Haxell and MacDonald [35] generalized Theorem 2.3 by introducing a nonnegative parameter t such that $\chi = \Delta - t$ and obtained the following theorem, which we also prove in this chapter.

Theorem 2.5 (Haxell and MacDonald [35]). *Let t be a nonnegative integer. Then for every graph G with $\chi(G) = \Delta(G) - t$ and $\Delta(G) \geq 6t^2 + 20t + 16$, we have $\omega(G) \geq \Delta(G) - 2t^2 - 6t - 3$.*

This result is derived from another theorem on hitting sets due to King [42] and the following theorem, which we will generalize to all sufficiently large Δ values.

Theorem 2.6 (Haxell and MacDonald [35]). *Let q and t be nonnegative integers and let $\Delta = (2t + 3)q + t + 1$. Let G be a graph with $\Delta(G) = \Delta$ and $\chi(G) = \Delta - t$, and suppose that G has a critical vertex. Then there exists a clique in G of size $\Delta - 2t^2 - 6t - 3$.*

The idea behind the technique of Mozhan partitions is the following. We partition the vertices according to groups of colour classes. With certain constraints, a small clique is guaranteed to exist in some part. Given a graph G , there are many Mozhan partitions of G . We define a move sequence to explore a subset of them. This allows us to identify small cliques in different parts that are complete to each other. By taking the union of such small cliques, we obtain a large clique of the desired size.

In proving Theorems 2.4 and 2.6, the two papers differ in two main aspects. First, balanced Mozhan partitions, in which all parts contain the same number of colour classes, are considered in [35], whereas [19] considered general Mozhan partitions where the parts may have different numbers of colour classes. Second, the two papers used different move sequences, with the one in [35] being much simpler. In this chapter, our approach is a combination of the two above. We consider general Mozhan partitions as in [19], which allow more flexibility on the value of $\Delta(G)$, and use the same move sequence as in [35]. This allows us to prove the following theorem.

Theorem 2.7. *Let t be a nonnegative integer and let $\Delta \geq (2t + 3)(2t + 2) + t + 1$. Let G be a graph with $\Delta(G) = \Delta$ and $\chi(G) = \Delta - t$, and suppose that G has a critical vertex. Define $m := \Delta - t - 1 \pmod{2t + 3}$. Then,*

$$\omega(G) \geq \begin{cases} \Delta - 2t^2 - 6t - 3 - m & \text{if } 0 \leq m \leq t + 1 \\ \Delta - 2t^2 - 7t - 4 & \text{if } t + 2 \leq m \leq 2t + 2. \end{cases}$$

Note that for graphs with a critical vertex and a sufficiently large maximum degree, this theorem generalizes both Theorem 2.4, which corresponds to the case $t = 0$, and Theorem 2.6, which corresponds to the case $m = 0$. In fact, in the special cases of $t = 0$ or $m = 0$, it is straightforward to modify our proof of Theorem 2.7 to prove the exact statements of both Theorems 2.4 and 2.6. As corollaries of Theorem 2.7, we prove both Theorem 2.5 and the following theorem, which is the main result of this chapter.

Theorem 2.8. *Let t be a nonnegative integer. Then for every graph G with $\chi(G) = \Delta(G) - t$ and $\Delta(G) \geq 4t^2 + 11t + 7$, we have $\omega(G) \geq \Delta(G) - 2t^2 - 7t - 4$.*

Comparing to Theorem 2.5, we have reduced the lower bound on the maximum degree at the cost of obtaining a slightly smaller clique. Moreover, our proof of Theorem 2.8 is simpler in the sense that it does not rely on any results involving hitting sets. Our arguments broadly follow the steps of [35]. We point out in the text the main places where additional ideas and arguments are used.

2.1 Critical Vertices

We start by focusing on graphs with a critical vertex, since this assumption is required for Mozhan partitions to exist. To better describe the properties of the neighbourhood of a critical vertex under a colouring, we introduce the following definitions, as in [35].

Definition 2.9. Let ϕ be a (partial) colouring of a graph G . A colour c is said to be a **singleton colour** for a vertex v in G if there is a unique neighbour u of v that has colour c under ϕ . The vertex u is said to be a **singleton neighbour** of v . A colour c is said to be a **multiple colour** for v if it appears more than once in $N(v)$.

We now prove the following lemma, which is a slightly stronger version of Lemma 3 in [35]. We replace the maximum degree in their results by the degree of particular vertices in order to obtain a possibly tighter upper bound on the number of multiple colours. Part 2 of this lemma is also similar to Lemma 3.0.2 in [1].

Lemma 2.10. *Let G be a graph with a critical vertex v . Define $\chi := \chi(G)$. Let $\phi : V(G - v) \rightarrow [\chi - 1]$ be a minimum colouring of $G - v$. Then the following holds.*

1. *For all $i \in [\chi - 1]$, define V_i to be the set of neighbours of v with colour i under ϕ , that is, $V_i := \{u \in N(v) \mid \phi(u) = i\}$, then $V_i \neq \emptyset$.*

2. There are at most $d(v) - \chi + 1$ multiple colours for v and at least $2\chi - d(v) - 2$ singleton neighbours of v .
3. There are at most $d(u) - \chi + 1$ multiple colours for any singleton neighbour u of v .
4. Let $w \notin N(v)$ be a singleton neighbour of a singleton neighbour u of v . There are at most $d(w) - \chi + 2$ multiple colours for w , including $\phi(u)$.

Proof. Since v is a critical vertex, we have $\chi(G - v) = \chi(G) - 1$. So ϕ exists.

We first prove Part 1. Suppose for contradiction that there exists some colour $i \in [\chi - 1]$ such that $V_i = \emptyset$. Then the colour i does not appear in $N(v)$. By assigning i to v , we extend ϕ to a $(\chi - 1)$ -colouring of G , which contradicts the fact that G is χ -chromatic.

Then we prove Part 2. By Part 1, v has at most $d(v) - (\chi - 1) = d(v) - \chi + 1$ multiple colours. So v has at least $(\chi - 1) - (d(v) - \chi + 1) = 2\chi - d(v) - 2$ singleton colours and hence at least $2\chi - d(v) - 2$ singleton neighbours.

To prove Part 3, let u be a singleton neighbour of v . If a colour $i \in [\chi - 1] \setminus \{\phi(u)\}$ does not appear in $N(u) \setminus \{v\}$, then by assigning colour i to u and colour $\phi(u)$ to v , we obtain a $(\chi - 1)$ -colouring of G , which again contradicts the fact that G is χ -chromatic. Thus, u has at most $(d(u) - 1) - (\chi - 2) = d(u) - \chi + 1$ multiple colours.

Lastly we prove Part 4. Let u and w be as defined. If a colour $i \in [\chi - 1] \setminus \{\phi(w)\}$ does not appear in $N(w) \setminus \{u\}$, then by assigning colours i , $\phi(w)$ and $\phi(u)$ to vertices w , u and v respectively, we obtain a $(\chi - 1)$ -colouring of G , which contradicts the fact that G is χ -chromatic. This implies that $\phi(u)$ is a multiple colour for w , and there are at most $d(w) - (\chi - 2) = d(w) - \chi + 2$ multiple colours for w . $\square$

We now consider the adjacency relation between a critical vertex and a clique. We show that when a critical vertex has enough neighbours in a clique, we obtain an upper bound on the number of non-neighbours in the same clique. The scenario described in the following lemma and its proof can be visualized in Figure 2.1.

Lemma 2.11. *Let G be a graph with a critical vertex v . Let $K \subseteq V(G)$ be a clique in G . Let $t := \Delta(G) - \chi(G)$. If $|K \cap N(v)| \geq 2t + 4$, then $|K \setminus N(v)| \leq t + 1$.*

Proof. For simplicity, we write $\chi = \chi(G)$ and $\Delta = \Delta(G)$. Fix a $(\chi - 1)$ -colouring ϕ of $G - v$. By Lemma 2.10(2), v has at most $d(v) - \chi + 1 \leq \Delta - \chi + 1 = t + 1$ multiple colours. Since $d(v) \geq |K \cap N(v)| \geq 2t + 4$ and all vertices in K have different colours, v has at least

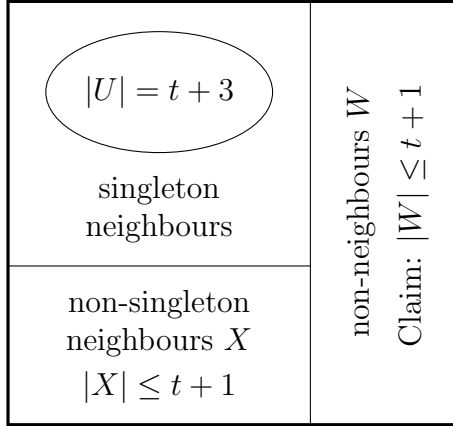


Figure 2.1: Partition of a clique based on relation with a critical vertex

$t + 3$ singleton neighbours in K . Let U be a set of $t + 3$ singleton neighbours of v in K . Let W be the set of non-neighbours of v in K . Suppose for contradiction that $|W| \geq t + 2$.

Note that $U \cup W \subseteq K$, so every vertex in $U \cup W$ has a distinct colour. We consider the complete bipartite subgraph of K with (U, W) being the bipartition. For any pair of vertices $u \in U$ and $w \in W$, if u (or w) is a singleton neighbour of w (or u) under ϕ . Then we direct the edge uw from u (or w) to w (or u). If u and w are singleton neighbours of each other, then uw is a bidirected edge.

Claim 2.11.1. *For all $u \in U$, u has positive indegree.*

Proof. Every $u \in U$ is a singleton neighbour of v . By Lemma 2.10(3), u has at most $d(u) - \chi + 1 \leq t + 1$ multiple colours. Since W contains at least $t + 2$ vertices of distinct colours, at least one vertex in W is a singleton neighbour of u . So u has a positive indegree. $\square$

Claim 2.11.2. *For all $w \in W$, if w has positive outdegree, then w has positive indegree.*

Proof. Take any $w \in W$ with a positive outdegree, then w is a singleton neighbour of some vertex $u \in U$, where u is a singleton neighbour of v . By Lemma 2.10(4), w has at most $d(w) - \chi + 2 \leq t + 2$ multiple colours. Since U contains $t + 3$ vertices of distinct colours, at least one vertex in U is a singleton neighbour of w . So w also has positive indegree. $\square$

Let $W' = \{w \in W \mid w \text{ has positive indegree}\}$, then by Claim 2.11.2, all vertices in W with a positive outdegree are in W' . By Claim 2.11.1, $W' \neq \emptyset$. Note that (U, W') induces

a bipartite graph K' in which every vertex has positive indegree. Since K is finite, by starting from an arbitrary vertex in $U \cup W'$ and following the incoming edges backwards, we obtain a directed cycle C in K' . By recolouring every vertex in C with the colour of its corresponding singleton neighbour, we obtain a $(\chi - 1)$ -colouring of $G - v$ in which the colours $\{\phi(u) | u \in C \cap U\}$ do not appear in $N(v)$. This contradicts Lemma 2.10(1). $\square$

2.2 Mozhan Partitions

We define the notion of a Mozhan partition and later that of a minimum Mozhan partition. All definitions and results in this section are generalizations of the ones in [35], which only considered balanced (minimum) Mozhan partitions. Lemma 2.18 is an important result that demonstrates the strong connection between minimum Mozhan partitions and cliques. A similar result has been proven in [19].

Throughout this section, we assume that q is a positive integer and $R = (r_1, r_2, \dots, r_q)$ is a sequence of q positive integers. Moreover, we only consider graphs G that have a critical vertex and satisfy $\chi(G) = \Delta(G) - t$, where t is a nonnegative integer. Note that the graphs for which such t is negative have been studied in Brooks' Theorem.

For a subset of vertices $P \subseteq V(G)$, when there is no confusion, we write the induced subgraph $G[P]$ as P . For example, we write $\chi(G[P])$ as $\chi(P)$, similarly for $\Delta(\cdot)$, $\omega(\cdot)$, $\deg(\cdot)$, $E(\cdot)$, etc. Given a vertex v , we write $G[P \cup \{v\}]$ (or $G[P \setminus \{v\}]$) as $P + v$ (or $P - v$). For a subgraph H of G and a vertex $v \in V(G)$, we use $\deg(H, v)$ to denote the degree of v into H , that is $\deg(H, v) = |N(v) \cap V(H)|$. Given two subsets of vertices $A, B \subseteq V(G)$, we say that A is **complete** to B if $G[A \cup B] = G[A] \vee G[B]$, that is, every vertex in A is joined to every vertex in B .

Definition 2.12 (Mozhan [53]). Let q be a positive integer and $R = (r_1, r_2, \dots, r_q)$ be a q -tuple of positive integers. Let G be a graph. An **R -Mozhan partition (R -MP)** of G is a triple $M = (\mathcal{P}, v, s)$ satisfying the following properties.

1. $\mathcal{P} = (P_1, P_2, \dots, P_q)$ is a partition of $V(G)$,
2. $s \in [q]$ and $v \in P_s$ is critical for P_s ,
3. $\chi(P_i) = r_i$ for each $i \neq s$ and $\chi(P_s) = r_s + 1$,
4. $\chi(G) = \sum_{i=1}^q \chi(P_i) = \sum_{i=1}^q r_i + 1$.

The component in P_s that contains v is said to be the **active component** of M , denoted by $act(M)$. Sometimes, to emphasize the critical vertex being considered, we may write M as M^v .

We note that when $\chi(G) = \sum_{i=1}^q r_i + 1$ and G has a critical vertex v , an R -MP of G exists. For example, an R -MP can be obtained by partitioning the vertices according to groups of colour classes in a $(\chi(G) - 1)$ -colouring of $G - v$ and then adding v to any part. We now prove a series of properties of Mozhan partitions. Most lemmas in this section are similar to the ones in [19] and [35], but we provide simpler proofs for some of them. We also attempt to make the assumptions as weak as possible, so the statements are in fact stronger than what we need for the proof of Theorem 2.7, in which R is chosen to be a sequence of $2t + 3$'s and $2t + 4$'s and q satisfies $q \geq 2t + 2$.

Lemma 2.13. *Let $M = (\mathcal{P}, v, s)$ be an R -MP of a graph G . Then the following holds.*

1. *Let u be a critical vertex in its part $P_j \in \mathcal{P}$. Then u is critical in G , and for all $k \in [q]$, u is critical in $P_k + u$.*
2. *Let u be a critical vertex in the part P_s . Then for all $s \neq k \in [q]$, moving u to P_k results in a new R -MP $M' = (\mathcal{P}', u, k)$ where $P'_s = P_s \setminus \{u\}$, $P'_k = P_k \cup \{u\}$ and all other parts of $\mathcal{P}'$ remain the same as in $\mathcal{P}$.*

Proof. We first prove Part 1. Since u is critical in P_j , we have $\chi(P_j - u) = \chi(P_j) - 1$. So

$$\chi(G - u) \leq \sum_{i \neq j} \chi(P_i) + \chi(P_j - u) = \sum_{i=1}^q \chi(P_i) - 1 = \chi(G) - 1.$$

Fix $k \in [q] \setminus \{j\}$. Then

$$\begin{aligned} \sum_{i=1}^q \chi(P_i) &= \chi(G) \leq \sum_{i \neq j, k} \chi(P_i) + \chi(P_j - u) + \chi(P_k + u) = \sum_{i \neq k} \chi(P_i) - 1 + \chi(P_k + u) \\ \implies \chi(P_k + u) &\geq \chi(P_k) + 1. \end{aligned}$$

So u is critical in G , and u is critical in $P_k + u$ for all $k \in [q]$.

Part 2 follows immediately from Part 1 and the definition of an R -MP. We will later refer to such a vertex u as a traveller from M to M' . $\square$

In the following lemma, we study the neighbourhood of critical vertices with respect to a Mozhan partition.

Lemma 2.14. *Let $M = (\mathcal{P}, v, s)$ be an R-MP of a graph G . Let $t := \Delta(G) - \chi(G)$. If u is a critical vertex in its part, then the following holds.*

1. *For each $i \in [q]$, there exists a component $C \subseteq P_i$ such that $\deg(C, u) \geq r_i$.*
2. *There exist at least $q - t - 1$ parts $P_i \in \mathcal{P}$ such that $\deg(P_i, u) = r_i$.*
3. *Let $W \subseteq N(u)$ be a set of $t + 2$ neighbours of u . Then there exists a vertex $w \in W$ and a component C in the part P_k of w such that $w \in C$ and $\deg(C, u) \geq r_k$.*

Proof. We first prove Part 1. Fix $i \in [q]$. Suppose for contradiction that every component of P_i contains at most $r_i - 1$ neighbours of u . Since $\chi(P_i) = r_i$, there is an r_i -colouring ϕ of P_i . By relabeling the colors in each component if necessary, we may assume that at most $r_i - 1$ colours appear in $N(u) \cap P_i$. By assigning any of the remaining colours to u , we obtain an r_i -colouring of $P_i + u$. So u is not critical in $P_i + u$. This contradicts Lemma 2.13(1).

Next we prove Part 2. Define $X = \{i \in [q] : \deg(P_i, u) = r_i\}$ and $x = |X|$. By Part 1, for all $i \in [q]$, we have $\deg(P_i, u) \geq r_i$. Consider the following lower and upper bounds of $d(u)$,

$$d(u) = \sum_{i=1}^q \deg(P_i, u) \geq \sum_{i \in X} r_i + \sum_{i \in [q] \setminus X} (r_i + 1) = \sum_{i=1}^q r_i + (q - x), \text{ and}$$

$$d(u) \leq \Delta(G) = \chi(G) + t = \sum_{i=1}^q r_i + 1 + t.$$

Simplifying gives $x \geq q - t - 1$.

Lastly we prove Part 3. The vertex u has at most

$$d(u) - \sum_{i=1}^q r_i \leq \Delta(G) - (\chi(G) - 1) = t + 1$$

neighbours that do not belong to any of the components described in Part 1. Since $|W| > t + 1$, the result follows. $\square$

Next we show that under certain assumptions, if a vertex has many neighbours into a clique that is a component, then it is in fact complete to that clique.

Lemma 2.15. *Let $M = (\mathcal{P}, v, s)$ be an R -MP of a graph G and u be a critical vertex in P_s . Let $t := \Delta(G) - \chi(G)$. Suppose K is an r_k -clique that is a component in its part $P_k \in \mathcal{P}$ and $\deg(K, u) \geq t + 2$. Then the following holds.*

1. *The vertex u is complete to K . Additionally, let $M' = (\mathcal{P}', u, k)$ be the R -MP obtained by moving u to P_k . Then $K \subseteq \text{act}(M')$ and each vertex $z \in K$ is critical in its part P'_k .*
2. *Let $A = \text{act}(M)$. Suppose that $\deg(A, z) \geq t + 3$ for some $z \in K$. Then $\deg(A, z) \geq r_s + 1$.*

Proof. Since $\text{act}(M)$ is $(r_s + 1)$ -chromatic, and K is an r_k -clique, which is r_k -chromatic, we have that $s \neq k$.

First we prove Part 1. By Lemma 2.13(2), M' is indeed an R -MP. Since u is in the same component as K in P'_k , we have $K \subseteq \text{act}(M')$. Since $\deg(K, u) \geq t + 2$ and $|K| = r_k$, by Lemma 2.14(3) applied to M , u and K , we have $t + 2 \leq \deg(K, u) = r_k$. So u is complete to K . By the same reasoning as in the proof of Lemma 2.14(3), u has at most $t + 1 \leq r_k - 1$ neighbours in $P'_k \setminus K$. Fix $z \in K$, then u has $r_k - 1$ neighbours in $K - z$. By the same argument as in the proof of Lemma 2.14(1), since $P'_k - \{u, z\}$ is r_k -colourable, $P'_k - z$ is also r_k -colourable. Hence z is critical in P'_k .

Then we prove Part 2. By Part 1, moving u to P_k forms an R -MP $M' = (\mathcal{P}', u, k)$, and z is critical in P'_k . Since $\deg(A \setminus \{u\}, z) \geq t + 2$, by Lemma 2.14(3) applied to M' , z and $A \setminus \{u\}$, we have $\deg(A \setminus \{u\}, z) \geq r_s$. Since u is complete to K , u is adjacent to z . So $\deg(A, z) \geq r_s + 1$. $\square$

We introduce minimum Mozhan partitions in the following definition. The criterion for minimality is chosen so that we are guaranteed to find cliques in some parts. In particular, the active component itself will be a clique.

Definition 2.16. Let q be a positive integer and $R = (r_1, r_2, \dots, r_q)$ be a q -tuple of positive integers. Given a graph G , let $M = (\mathcal{P}, v, s)$ be an R -MP of G . We say that M is **minimum** if M minimizes the sum

$$\sigma(M) := \sum_{i \neq s} |E(P_i)| + |E(P_s - v)|$$

and subject to that, minimizes $\deg(P_s, v) - r_s$.

To facilitate our discussion of cliques in Mozhan partitions, we make the following definition.

Definition 2.17. Let $M = (\mathcal{P}, v, s)$ be an R -MP of a graph G . Let K be a clique that is a component in its part P_j , then we say that K is a **full clique component** if $K \setminus \{v\}$ is an r_j -clique. Note that if K is a full clique component, then either K is an r_j -clique or $K = \text{act}(M)$ is an $(r_s + 1)$ -clique.

In the following lemma, we show that two full clique components that are both complete to the active component are complete to each other. We also use Brooks' Theorem to show that in a minimum Mozhan partition, the active component is always a clique and hence a full clique component. This lemma allows us to construct larger cliques.

Lemma 2.18. Let $M = (\mathcal{P}, v, s)$ be an R -MP of G and $A = \text{act}(M)$. Let $t := \Delta(G) - \chi(G)$. Then the following holds.

1. Suppose A , B and K are full clique components in parts P_s , P_j and P_k respectively. Suppose further that $r_s \geq 2t + 3$, $\max\{r_j, r_k\} \geq 2t + 3$ and $\min\{r_j, r_k\} \geq t + 2$. If A is complete to both B and K , then B is complete to K .
2. Suppose $q \geq t + 2$ and $r_s \neq 2$. If M is minimum, then A is an $(r_s + 1)$ -clique.

Proof. We first prove Part 1. Without loss of generality, we may assume that $r_j \geq 2t + 3$ and $r_k \geq t + 2$. Since A is complete to K , $\deg(K, v) = r_k \geq t + 2$. Fix $z \in K$. By Lemma 2.15(1), after moving v to P_k , $M' = (\mathcal{P}', v, k)$ is an R -MP, and z is critical in P'_k . By Lemma 2.13(1), z is also critical in G . Since A is complete to K , $\deg(A, z) = r_s + 1$. So z has $r_s + 1 \geq 2t + 4$ neighbours in the clique $A \cup B$. By Lemma 2.11, z has at most $t + 1$ non-neighbours in B and hence at least $r_j - (t + 1) \geq t + 2$ neighbours in B . By Lemma 2.15(1), z is complete to B . Since this holds for all $z \in K$, the statement follows.

We now prove Part 2 using an argument that follows the exposition of [19]. Since $\chi(A) = \chi(P_s) = r_s + 1 \neq 3$, by Brooks' Theorem, it suffices to show that $\Delta(A) \leq r_s$. Suppose for contradiction that $\Delta(A) \geq r_s + 1$. Let $u \in A$ be a vertex such that $\deg(A, u) \geq r_s + 1$ and the distance $d(v, u)$ between v and u is minimized.

We claim that u is critical in P_s . Let $v = x_1 x_2 \dots x_n = u$ be a shortest vu -path in A and denote it by Q . Since v is critical in P_s , there exists an r_s -colouring of $P_s - Q$. By minimality of $d(v, u)$, for all $i \in [n - 1]$, every vertex x_i in $Q - u$ has at most r_s neighbours including x_{i+1} , which is uncoloured. By greedily coloring along $Q - u$ starting from v , we obtain an r_s -colouring of $P_s - u$. So u is critical in P_s .

By Lemma 2.14(2), since $q \geq t + 2$, there exists $k \in [q]$ such that $\deg(P_k, u) = r_k$. Since $\deg(P_s, u) \geq r_s + 1$, we have $k \neq s$. By Lemma 2.13(2), moving u to P_k results in a new R -MP $M' = (\mathcal{P}', u, k)$. If $u = v$, then $\sigma(M') = \sigma(M)$, but

$$\deg(P'_k, u) - r_k = 0 < 1 \leq \deg(P_s, v) - r_s,$$

which contradicts the minimality of M . If $u \neq v$, then

$$\begin{aligned} \sigma(M') &= \sum_{i \neq k, s} |E(P'_i)| + |E(P'_k - u)| + |E(P'_s)| \\ &= \sum_{i \neq k, s} |E(P_i)| + |E(P_k)| + |E(P_s - u)| \\ &= \sum_{i \neq s} |E(P_i)| + |E(P_s - v)| - \deg(P_s, u) + \deg(P_s, v) \\ &\leq \sigma(M) - (r_s + 1) + r_s = \sigma(M) - 1, \end{aligned} \tag{2.1}$$

which again contradicts the minimality of M and hence completes the proof. $\square$

2.3 Move Sequences

In this section, we define a move sequence that allows us to explore multiple minimum Mozhan partitions. We assume that t and q are nonnegative integers and that $R = (r_1, r_2, \dots, r_q)$ is a sequence of q positive integers such that each r_i satisfies $r_i \geq t + 3$. The constraint $r_i \geq t + 3$ is required for Lemma 2.21, which implies that if two cliques are complete to each other, then this property is preserved by the move sequence. We further assume that G is a graph with a critical vertex and satisfies

$$\chi(G) = \Delta(G) - t = \sum_{i=1}^q r_i + 1.$$

As noted in the previous section, such a graph G admits an R -MP and hence a minimum R -MP $M = (\mathcal{P}, v, s)$. From Lemma 2.18(2), we know that the active component $A := \text{act}(M)$ is an $(r_s + 1)$ -clique. So for every vertex u in A , the triple $(\mathcal{P}, u, s)$ is also a minimum Mozhan partition. To find more cliques, it is then natural to look for more minimum R -MP's with a different partition $\mathcal{P}' \neq \mathcal{P}$ and possibly a different active component. We define a move and prove that a move relates two minimum Mozhan partitions with different active components.

Definition 2.19. Let $M = (\mathcal{P}, v, s)$ be a minimum R -MP of G and u be a vertex in $act(M)$. Then $M^u := (\mathcal{P}, u, s)$ is also a minimum R -MP. We say that M and M^u are **similar** and denote it by $M \sim M^u$. Note that the relation $\sim$ is transitive. For all $j \in [q]$, a full clique component K in its part P_j is said to be **accessible** to u in M if $\deg(P_j, u) = \deg(K, u) = r_j$. Note that in particular, $act(M)$ is accessible to u .

Suppose there exists $k \neq s$ such that a full clique component in P_k is accessible to u in M . Let $M' = (\mathcal{P}', u, k)$ be an R -MP where

$$P'_i = \begin{cases} P_s \setminus \{u\} & \text{if } i = s \\ P_k \cup \{u\} & \text{if } i = k \\ P_i & \text{otherwise.} \end{cases}$$

Then the pair (M, M') is called a **move**, and the vertex u is called the **traveller** of the move, denoted by $tr(M, M')$.

Note that from the definition, since $k \neq s$, M and M' have different active components. In the following lemma, we show that this definition is nontrivial, since starting from any minimum R -MP M , we can always move to another minimum R -MP M' .

Lemma 2.20. *Let $M = (P, v, s)$ be a minimum R -MP of G . If $q \geq t + 3$, then there exists an R -MP M' such that (M, M') is a move. Moreover, whenever (M, M'') is a move, M'' is also a minimum R -MP of G .*

Proof. By Lemma 2.18(2), $act(M)$ is an $(r_s + 1)$ -clique. Take any $u \in act(M)$, then u is critical in P_s . Since $q - t - 1 \geq 2$, by Lemma 2.14(2), there exist 2 parts P_j and P_k such that $\deg(P_j, u) = r_j$ and $\deg(P_k, u) = r_k$. Without loss of generality, we may assume that $k \neq s$. By Lemma 2.14(1), there exists a component K in P_k such that $\deg(P_k, u) = \deg(K, u) = r_k$. By Lemma 2.13(2), moving u to P_k gives us a new R -MP $M' = (\mathcal{P}', u, k)$. Note that by the same argument as Equation 2.1 in the proof of Lemma 2.18(2), we have

$$\sigma(M') = \sum_{i \neq s} |E(P_i)| + |E(P_s - v)| - \deg(P_s, u) + \deg(P_s, v) = \sigma(M). \quad (2.2)$$

Additionally, we have

$$\deg(P'_k, u) - r_k = 0 = \deg(P_s, v) - r_s. \quad (2.3)$$

Since M is a minimum R -MP of G , M' is also a minimum R -MP of G . By Lemma 2.18(2), $act(M')$ is an $(r_k + 1)$ -clique, so K is an r_k -clique and hence is a full clique component that is accessible to u in M . By definition, (M, M') is a move.

Finally, let (M, M'') be any move from M , then M'' is obtained from M by moving u to a part with an accessible component. So M'' is an R -MP by Lemma 2.13(2). Since Equations 2.2 and 2.3 still hold (with M' replaced by M''), M'' is a minimum R -MP. $\square$

We observe that for any minimum R -MP M and vertex u in $act(M)$, the pair (M, M') is a move if and only if (M^u, M') is a move. So the choice of a critical vertex in the active component is often irrelevant when we want to prove that two minimum R -MP's form a move. Moreover, a move is reversible in the sense that (M, M') is a move with traveller u if and only if (M', M^u) is a move.

Another important property of a move is that it preserves completeness between full clique components in the following sense.

Lemma 2.21. *Suppose all elements in R are at least $t + 3$. Let $M = (\mathcal{P}, v, s)$ and $M' = (\mathcal{P}', u, k)$ be two minimum R -MP's of G such that (M, M') is a move. Let K_1 and K_2 be full clique components in their respective parts P_j and P_k in M . Let K'_1 and K'_2 be the corresponding full clique components in P'_j and P'_k in M' . Then, K_1 is complete to K_2 if and only if K'_1 is complete to K'_2 .*

Proof. Since K_1, K_2, K'_1, K'_2 are all cliques, we observe that if $K_1 \cup K_2 = K'_1 \cup K'_2$, then the lemma holds. So it remains to consider the scenario $K_1 \cup K_2 \neq K'_1 \cup K'_2$. From the definition of a move, this occurs if and only if $\{u\} = (K_1 \cup K_2) \setminus (K'_1 \cup K'_2)$ or $\{u\} = (K'_1 \cup K'_2) \setminus (K_1 \cup K_2)$. By reversibility of a move, we note that (M', M^u) is a move and K_1 and K_2 are full clique components in M^u . So without loss of generality, we may assume that $\{u\} = (K_1 \cup K_2) \setminus (K'_1 \cup K'_2)$ and $K_1 = act(M)$. Then we have $j = s$, $K'_1 = K_1 \setminus \{u\}$ and $K'_2 = K_2$.

The forward direction is immediate since $K'_1 \cup K'_2 \subseteq K_1 \cup K_2$. To prove the backward direction, suppose that K'_1 is complete to K'_2 . Then for any vertex $w \in K'_1 = K_1 \setminus \{u\} \subseteq act(M)$, w is a critical vertex in P_s and $\deg(K_2, w) = \deg(K'_2, w) = r_k > t + 2$ and for any vertex $z \in K_2 = K'_2$, we have $\deg(K_1, z) \geq \deg(K'_1, z) = r_s \geq t + 3$. By Lemma 2.15(2), we have that $\deg(K_1, z) \geq r_s + 1$. So z is complete to K_1 for all $z \in K_2$. Hence K_1 is complete to K_2 . $\square$

We now consider a sequence of moves among minimum R -MP's and study how components and especially full clique components evolve in such a sequence.

Definition 2.22. A **move sequence** $\mathcal{S}$ is a finite list of minimum R -MP's $(M_0, M_1, \dots, M_J)$ such that for all $j \in \{0, 1, \dots, J - 1\}$, the pair (M_j, M_{j+1}) is a move. The index j is called the **timestep** of this move. Let $u_j = tr(M_j, M_{j+1})$ be the traveller at timestep j . A **club**

is a sequence $C = (C_0, C_1, \dots, C_J)$ such that C_0 is a component in its part of M_0 , and for all $j \in [J]$,

$$C_{j+1} = \begin{cases} C_j - u_j & \text{if } C_j = \text{act}(M_j) \\ C_j + u_j & \text{if } C_j = \text{act}(M_{j+1}) \\ C_j & \text{otherwise.} \end{cases}$$

For all $x \in V(G)$, we say that C is the **home** of x if $x \in C_0$ and denote it by $H(x) = C$. A club C is said to be **accessible** to a vertex u at some timestep j if C_j is accessible to u in M_j . We say that C is **active** in M_j (or **active** at timestep j) if $C_j = \text{act}(M_j)$. A club C that is active at some timestep j in the move sequence is called a **house**. Given a house H , let j be the smallest timestep such that $H_j = \text{act}(M_j)$. We say that H is **awakened** at j by the traveller u_{j-1} (if $j \geq 1$) and refer to j as the **awakening time** of H , denoted by $a(H) = j$. We define a function r from the set of all houses to $\{r_1, r_2, \dots, r_q\}$ such that for each house H in its part P_k , we have $r(H) = r_k$. We refer to $r(H)$ as the **index** of H .

For simplicity, we may omit the timestep when it is implicitly clear and refer to a component by its club. We make the following observations on clubs and houses.

1. Any component at any timestep uniquely identifies a club, and a club at any timestep is a component.
2. If a component changes in a move sequence, then it must be the active component at some timestep, that is, its club is a house.
3. A club that ever contains a traveller is a house.
4. Suppose H is a house, then at some timestep j , H in its part P_k is active in M_j . Since M_j is a minimum R -MP, H_j is an $(r_k + 1)$ -clique. So H is a full clique component at all timesteps.
5. At any timestep j , from the proof of Lemma 2.20, we see that a vertex $u \in \text{act}(M_j)$ has at least $q - t - 1$ accessible clubs, and there is at most one accessible club for u in each part.

Using Lemma 2.21, we have the following corollary, which allows us to generalize the notion of completeness between components to completeness between houses. A similar result was proven in [19] but not in [35], since in proving Theorem 2.6, [35] focused on one particular timestep at which it is convenient to show that sufficiently many full clique components are complete to each other. Corollary 2.23 is essential for proving Theorem

2.7, since our proof considers the completeness relation among all houses discovered in the move sequence.

Corollary 2.23. *Given two houses H and H' in a move sequence, if there exists a timestep j' such that $H_{j'}$ is complete to $H'_{j'}$, then at any timestep j , H_j is complete to H'_j .*

Proof. The statement follows directly from Lemma 2.21 and induction on the timestep. $\square$

Definition 2.24. Given two houses H and H' in a move sequence, we say that H is **complete** to H' if there exists a timestep j such that H_j is complete to H'_j . We denote this by $H \equiv_c H'$.

We now define a particular move sequence, which we refer to as the **depth-first move sequence (DFMS)**. We will eventually view $\equiv_c$ as an equivalence relation on the set of all houses in some DFMS. Then each equivalence class potentially forms a large clique in the graph. As suggested by its name, in a DFMS, we prioritize moving vertices in the most recently awakened house to their accessible components. To achieve this, we keep a last-in-first-out list of travellers throughout the procedure. This list keeps track of the order of travellers for subsequent moves, and we refer to it as the **priority list PL** . For each traveller u , when added to the PL , we initialize an **itinerary $I(u)$** , which is a list of all accessible clubs that u has not yet visited. Every house H keeps a **guest book $B(H)$** , which is the set of all travellers that have visited the house (including all vertices whose home is H). For simplicity, when we are not concerned with the order of the elements, we may view PL and $I(u)$ as sets and use set notation. We provide the DFMS procedure below and then prove its termination and validity.

Given nonnegative integers t and q , let $R = (r_1, r_2, \dots, r_q)$ be a sequence of q integers that are at least $t + 3$. Let G be a graph with a critical vertex and such that

$$\chi(G) = \Delta(G) - t = \sum_{i=1}^q r_i + 1.$$

Let M be a minimum R -MP of G . We define a DFMS to be the following procedure.

1. Initialize a priority list PL by setting $PL := \emptyset$.
2. Set $j := 0$.
3. Set $M_j := M$.

4. For each $x \in \text{act}(M_j) \setminus PL$, initialize its itinerary $I(x)$ by setting $I(x) := I^0(x)$ where $I^0(x)$ is a list of all clubs that are accessible to x in M_j , in arbitrary fixed order.
5. Set $I(x) := I(x) \setminus \{H(x)\} = I(x) \setminus \{\text{act}(M_j)\}$.
6. Add all vertices in $\text{act}(M_j) \setminus PL$ to the top of PL , in arbitrary fixed order.
7. For the active house H in M_j , initialize its guest book $B(H)$ by setting $B(H) := \text{act}(M_j)$.
8. (a) If $PL = \emptyset$, then stop and output $\mathcal{S} = (M_0, M_1, \dots, M_j)$.
 (b) Otherwise, set u to be the vertex at the top of PL . Set H' to be the house that contains u in M_j .
9. If $H' \neq H(u)$, then set M_{j+1} to be the R -MP resulting from moving u back to $H(u)$. Set $I(u) := I(u) \setminus \{H'\}$. Set $j := j + 1$.
10. If $I(u) = \emptyset$, then set $PL := PL \setminus \{u\}$.
11. Otherwise (that is, $H' = H(u)$ and $I(u) \neq \emptyset$), set H to be the club at the top of $I(u)$. Set M_{j+1} to be the R -MP resulting from moving u to H . Set $j := j + 1$.
 (a) If H has never been awakened at $j' < j$, then go to Step 4.
 (b) Otherwise, set $B(H) := B(H) \cup \{u\}$.
12. Go to Step 8.

Definition 2.25. Whenever Line 11 is executed, we say that u **visits** H . Whenever Line 11a is executed, we say that u **awakens** H . For the house H' that is active in M_0 , we pick an arbitrary vertex x in H'_0 and say that x **awakens** H' . For each traveller u , we define its **awakening time** $a(u)$ to be the awakening time of its home, that is $a(u) = a(H(u))$ and its **bedtime** $b(u)$ to be the timestep at which $I(u)$ becomes empty. For each house H in $I(u)$, we denote by $M_{a(u)}(H)$ the R -MP obtained from $M_{a(u)}$ by moving u to H . Note that $(M_{a(u)}, M_{a(u)}(H))$ is a move. Moreover, $M_{a(u)}(H(u)) = M_{a(u)}$.

Lemma 2.26. *The DFMS procedure terminates.*

Proof. Let $\mathcal{S}$ be a DFMS. Let n be the number of vertices in G . Let $\mathcal{H}$ be the set of all houses in $\mathcal{S}$. Then $|\mathcal{H}| \leq n$, and for any house H , we have $|B(H)| \leq n$ at any timestep. Thus $\sum_{H \in \mathcal{H}} |B(H)| \leq n^2$. Since each traveller visits each house H at most once (Lines 9

and 11), this quantity increases each time Line 11 is executed. So Line 11 is executed at most n^2 times.

Let U be the set of travellers, then $|U| \leq n$. For each traveller u , its itinerary $I(u)$ is initialized in Line 4, and no elements are added to $I(u)$ at any other time. So $|I(u)| \leq n$. Thus $\sum_{u \in U} |I(u)| < n^2$. Since this quantity decreases each time Line 9 is executed, Line 9 is executed at most n^2 times.

Every traveller is added to PL exactly once, which happens at its awakening time. So every traveller is removed from PL exactly once. Thus Line 10 is executed at most n times.

Since one of Lines 9, 10 and 11 is executed in each iteration, the procedure terminates after at most $2n^2 + n$ iterations. $\square$

We now show that the DFMS procedure indeed defines a move sequence, that is, every pair (M_j, M_{j+1}) is a move. We do so by proving that the traveller u_j at a timestep j always moves to an accessible club. In the following lemma, for any given timestep j , we consider the value of each variable at the start of that iteration.

Lemma 2.27. *For every timestep j and every traveller u , the following holds.*

1. *The following are equivalent.*
 - (a) $a(u) < j \leq b(u)$,
 - (b) $I(u) \neq \emptyset$ at timestep j , and
 - (c) u is in PL at timestep j .
2. *If $I(u) = \emptyset$ at timestep j , then $u \in H(u)$ in M_j .*
3. *Let u be at the top of PL at j and H be the house that contains u in M_j , then $M_j \sim M_{a(u)}(H)$.*
4. *(M_j, M_{j+1}) (Line 9 or 11) is a move.*

Proof. We first prove Part 1. The itinerary $I(u)$ is made nonempty exactly once at timestep $a(u)$ (Lines 11a and 4). After $a(u)$, no elements are added to $I(u)$. So $I(u)$ is only empty again when the last element is removed (Line 9). So $I(u) \neq \emptyset$ at j if and only if $a(u) < j \leq b(u)$. Since u is added to PL at $a(u)$ (Lines 11a and 6) and removed from PL at $b(u)$ (Line 10), we have that u is in PL at j if and only if $a(u) < j \leq b(u)$.

Next we prove Part 2. Suppose $I(u) = \emptyset$ at timestep j . By Part 1, u is not in PL at j , which implies that u does not move at j , and either $j \leq a(u)$ or $j > b(u)$. So it suffices to

show that $u \in H(u)$ at timesteps 0 and $b(u) + 1$. By definition, $u \in H(u)$ in M_0 . At $b(u)$, u moves back to $H(u)$ (Line 9), so $u \in H(u)$ in $M_{b(u)+1}$.

To prove Part 3, we consider an arbitrary traveller $x \neq u$ and show that x is in the same house in M_j as it is in $M_{a(u)}$. Note that vertices that are not travellers do not move in a DFMS.

First suppose $x \in PL$ at j . Since u is at the top of PL at j , x is below u . So x entered PL (Line 6) at some timestep $a(x) \leq a(u)$. Hence, at each timestep $a(u) \leq j' \leq j$, u is above x and x never moves. So, x is in the same house in M_j as it is in $M_{a(u)}$.

Now suppose $x \notin PL$ at j . Suppose for contradiction that $x \in PL$ in $M_{a(u)}$, then, again, at each timestep $a(x) \leq j' \leq j$, u is above x and x never moves. In particular, $x \in PL$ at j , which is a contradiction. So by Part 1, $I(x) = \emptyset$ at j and $a(u)$. By Part 2, $x \in H(x)$ in both M_j and $M_{a(u)}$.

Finally we prove Part 4. Let x and u be at the top of PL at j and $j+1$ respectively. Note that $x = tr(M_{j-1}, M_j)$ and $u = tr(M_j, M_{j+1})$. Let H' and H be the houses that contain x and u in M_j and M_{j+1} respectively. By Part 3, $M_j \sim M_{a(x)}(H')$ and $M_{j+1} \sim M_{a(u)}(H)$.

Suppose $u = x$. Since a move only occurs in Lines 9 and 11, exactly one of H and H' is $H(u)$. By reversibility of a move, we may assume that $H' = H(u)$ and $H \neq H(u)$ is accessible to u in $M_{a(u)}$. So $M_j \sim M_{a(u)}(H') \sim M_{a(u)}^u$, and hence (M_j, M_{j+1}) is a move.

Otherwise, suppose $u \neq x$. Then either x awakens $H(u)$ or x is immediately above u in PL until its bedtime $b(x) = j - 1$. If x awakens u , then $M_j = M_{a(u)}$, hence (M_j, M_{j+1}) is a move. It remains to consider the case that x is immediately above u in PL , so either $H(u) = H(x)$ or u is the traveller that awakened $H(x)$. Since $b(x) = j - 1$, by Parts 1 and 2, we have $x \in H(x)$ in M_j . So $H' = H(x)$. If $H(u) = H(x)$, then $M_j \sim M_{a(x)}(H') \sim M_{a(x)} = M_{a(u)}$. So (M_j, M_{j+1}) is a move. Otherwise, u awakened $H(x)$ (Line 11a), so $H' \in I^0(u) \setminus \{H(u)\}$ is accessible to u in $M_{a(u)}$. This implies that (M_j, M_{j+1}) occurs in Line 9, that is, $H = H(u)$. So $M_j \sim M_{a(u)}(H')$ and $M_{j+1} \sim M_{a(u)}(H) \sim M_{a(u)}$. Therefore, (M_{j+1}, M_j^u) is a move. By reversibility, (M_j, M_{j+1}) is a move. $\square$

2.4 Existence of Large Cliques

We are now ready to use the DFMS and properties of minimum Mozhan partitions to find large cliques. In [35], since balanced Mozhan partitions are used, all houses in a DFMS have the same index. This no longer holds in our approach, so we make another generalization of their work. Instead of studying the Mozhan partition at a carefully chosen timestep, we

make use of all R-MP's in a DFMS. We construct multiple large cliques using all houses and show that at least one of them has the desired size.

In the following lemma, we show that every house $\hat{H}$ in a DFMS belongs to a larger clique consisting of all houses that are complete to $\hat{H}$. The size of such a clique can be lower bounded in terms of the size of the guest book of $\hat{H}$.

Lemma 2.28. *Let q and t be nonnegative integers and $R = (r_1, r_2, \dots, r_q)$ be a q -tuple of positive integers such that each r_i satisfies $r_i \geq 2t + 3$. Let $\Delta = \sum_{i=1}^q r_i + t + 1$. Let G be a graph with $\Delta(G) = \Delta$ and $\chi(G) = \Delta - t$, and suppose that G has a critical vertex. Let $\mathcal{S}$ be a DFMS on G . Let $\mathcal{H}$ be the set of all houses in $\mathcal{S}$ and $\hat{H}$ be a house in $\mathcal{H}$. Let $M = M_{a(\hat{H})} = (\mathcal{P}, v, s)$ and $A = \text{act}(M)$. Define*

$$Z = \{z \in V(G) \mid z \text{ is in a full clique component } C^z \text{ in } M \text{ and } \deg(A, z) \geq r_s\}, \text{ and}$$

$$Z' = \{z \in Z \mid \text{the full clique component } C^z \ni z \text{ satisfies } |C^z \cap Z| \leq t + 1\}.$$

Let $\mathcal{K}_{\hat{H}}$ be the set of full clique components that each contain at least $t + 2$ vertices of Z . Let $\mathcal{H}_{\hat{H}}$ be the set of houses corresponding to the components in $\mathcal{K}_{\hat{H}}$. That is,

$$\mathcal{H}_{\hat{H}} = \{H \in \mathcal{H} \mid H_{a(\hat{H})} \in \mathcal{K}_{\hat{H}}\}.$$

Then the following holds.

1. *Any two full clique components in $\mathcal{K}_{\hat{H}}$ are complete to each other.*
2. *We have $\mathcal{H}_{\hat{H}} = \{H \in \mathcal{H} \mid H \equiv_c \hat{H}\}$.*
3. *We have*

$$|B(\hat{H})| \leq \sum_{H \in \mathcal{H}_{\hat{H}}} r(H) + \left\lfloor \frac{r(\hat{H})(t + 2)}{r(\hat{H}) - t - 1} \right\rfloor.$$

Proof. We first prove Part 1. Take any full clique component $K \in \mathcal{K}_{\hat{H}} \setminus \{A\}$, then $|K \cap Z| \geq t + 2$. Let K' be a $(t + 2)$ -subset of $K \cap Z$. For all $z \in K' \subseteq Z$, since $\deg(A, z) \geq r_s \geq 2t + 3$, there is at most one vertex in A that is not adjacent to z . So at least $|A| - |K'| = (r_s + 1) - (t + 2) > 1$ vertex in A is adjacent to all $t + 2$ vertices in K' and is critical in P_s . By Lemma 2.15(2), $\deg(A, z) \geq r_s + 1$. Hence for all $u \in A$, u is critical in P_s , and $\deg(K, u) \geq |K'| = t + 2$. By Lemma 2.15(1), u is complete to K . Hence A is complete to K . By Lemma 2.18(2), all cliques in $\mathcal{K}_{\hat{H}}$ are complete to each other.

Next we prove Part 2. By Part 1, all houses in $\mathcal{H}_{\hat{H}}$ are complete to each other. So $\mathcal{H}_{\hat{H}} \subseteq \{H \in \mathcal{H} | H \equiv_c \hat{H}\}$. Let $H \in \mathcal{H}$ be any house such that $H \equiv_c \hat{H}$. Let $K = H_{a(\hat{H})}$, then K is complete to A . So $|K \cap Z| = |K| \geq 2t+3$. We have $K \in \mathcal{K}_{\hat{H}}$ and hence $H \in \mathcal{H}_{\hat{H}}$.

Now we prove Part 3. By Part 1, every $K \in \mathcal{K}_{\hat{H}}$ is complete to A , so $K \subseteq Z$ and we have

$$Z = \left(\bigcup_{K \in \mathcal{K}_{\hat{H}}} K \right) \cup Z' \implies |Z| = \sum_{K \in \mathcal{K}_{\hat{H}}} |K| + |Z'| = \sum_{H \in \mathcal{H}_{\hat{H}}} r(H) + 1 + |Z'|, \quad (2.4)$$

where the term 1 accounts for the critical vertex v in M .

We now prove an upper bound for $|Z'|$. For every edge $uz \in E(G)$ with $u \in A$ and $z \in Z'$, color uz red if u is complete to C^z and pink otherwise. Let

$$Z_0 = \{z \in Z' | \deg(A, z) = r_s\} \quad \text{and} \quad Z_1 = \{z \in Z' | \deg(A, z) = r_s + 1\}$$

so that $Z' = Z_0 \cup Z_1$. Fix $z \in Z'$ with z being in some part P_k .

If $z \in Z_0$, then z is not complete to A . Suppose for contradiction that some $u \in A$ is complete to C^z , that is, $\deg(C^z, u) = r_k \geq t+2$. Then by Lemma 2.15(2), since $\deg(A, z) = r_s \geq t+3$, we have $\deg(A, z) \geq r_s + 1$, which contradicts the assumption that $z \in Z_0$. So z is incident with r_s pink edges.

Otherwise, $z \in Z_1$. Suppose for contradiction that at least $t+3$ vertices of A are complete to C^z . Fix $y \in C^z$. Then $\deg(A, y) \geq t+3$ and $\deg(C^z, u) = r_k \geq t+2$ for some $u \in A$. By Lemma 2.15(2), $\deg(A, y) \geq r_s + 1$. Since this holds for all $y \in C^z$, C^z is complete to A . Hence $C^z \subseteq Z$ and $|C^z \cap Z| = |C^z| = r_k \geq t+2$, which contradicts the assumption that $z \in Z'$. So $z \in Z_1$ is incident with $r_s + 1$ edges, at most $t+2$ of which are red.

For all $u \in A$, since u is critical in its part, by Lemma 2.14(3), there exist at most $t+1$ vertices $z \in Z$ such that $\deg(C^z, u) < r_j$. So u is incident with at most $t+1$ pink edges.

Counting the number of pink edges, we obtain

$$\begin{aligned} (r_s + 1)(t + 1) &= |A|(t + 1) \geq \# \text{pink edges} \geq |Z_0|r_s + |Z_1|(r_s + 1 - (t + 2)) \\ \implies (r_s + 1)(t + 1) &\geq (|Z_0| + |Z_1|)(r_s - t - 1) + |Z_0|(t + 1) \geq |Z'| (r_s - t - 1) \\ \implies |Z'| &\leq (t + 1) \frac{r_s + 1}{r_s - t - 1}. \end{aligned}$$

We observe that $|Z|$ is an integer and $r_s = r(\hat{H})$. So by combining this result with Equation 2.4 and simplifying, we obtain

$$|Z| \leq \sum_{H \in \mathcal{H}_{\hat{H}}} r(H) + 1 + \left\lfloor (t+1) \frac{r(\hat{H}) + 1}{r(\hat{H}) - t - 1} \right\rfloor = \sum_{H \in \mathcal{H}_{\hat{H}}} r(H) + \left\lfloor \frac{r(\hat{H})(t+2)}{r(\hat{H}) - t - 1} \right\rfloor.$$

It remains to show that $|B(\hat{H})| \leq |Z|$. Take any vertex $u \in B(\hat{H})$, it suffices to show that $\deg(A, u) \geq r_s$. If $u \in A$, then $\deg(A, u) = r_s$. Otherwise, u visits $\hat{H}$ at some time $j \geq a(\hat{H})$. By Lemma 2.27(4), $\hat{H}$ is accessible to u at time j , so $\deg(\hat{H}_j, u) = r_s$. Let $x \in A$ be the vertex that awakens $\hat{H}$. By construction of PL , we observe that every vertex except x that visits $\hat{H}$ stays at the top of PL and hence leaves immediately at the next timestep. So for every vertex in $\hat{H}_j \setminus \{x\}$, $\hat{H}$ is its home. Hence $\hat{H}_j \subseteq A$. So $\deg(A, u) \geq \deg(\hat{H}_j, u) = r_s$. $\square$

We are now ready to prove Theorem 2.7. The sequence R we use is a generalization of the one in [19]. In particular, they used a sequence of 3's and 4's, and their lower bound for the clique number depends on the value of $(\Delta - 1) \bmod 3$. Here we use a sequence of $2t + 3$'s and $2t + 4$'s, and our lower bound on the clique number depends on the value of $(\Delta - t - 1) \bmod 2t + 3$, which we refer to as m . We show that the completeness relation $\equiv_c$ is an equivalence relation on the set of houses in a DFMS and then prove the existence of a large equivalence class. Using Lemma 2.28(2), we know that all vertices in such an equivalence class form a clique at any timestep.

Proof of Theorem 2.7. Define

$$q := \frac{\Delta - t - m - 1}{2t + 3}.$$

Note that $\Delta = (2t + 3)q + t + 1 + m$ and $q \geq 2t + 2$. Let $R = (r_1, r_2, \dots, r_q)$ be the q -tuple such that $r_1 = r_2 = \dots = r_{q-m} = 2t + 3$ and $r_{q-m+1} = r_{q-m+2} = \dots = r_q = 2t + 4$. Then

$$\chi(G) = \Delta - t = (2t + 3)q + m + 1 = (2t + 3)(q - m) + (2t + 4)m + 1 = \sum_{i=1}^q r_i + 1.$$

Since G has a critical vertex, there exists a DFMS $\mathcal{S}$ consisting of minimum R -MP's of G . Let U be the set of all travellers in $\mathcal{S}$ and $\mathcal{H}$ be the set of all houses in $\mathcal{S}$. Let L be the number of houses in $\mathcal{H}$ with index $2t + 4$. That is,

$$L = |\{\hat{H} \in \mathcal{H} | r(\hat{H}) = 2t + 4\}|.$$

Let Γ be the bipartite graph with $(U, \mathcal{H})$ being the two parts and such that each $u \in U$ and $\hat{H} \in \mathcal{H}$ are adjacent if u has visited $\hat{H}$ or if $H(u) = \hat{H}$. Then, $d_\Gamma(\hat{H}) = |B(\hat{H})|$ for all $\hat{H} \in \mathcal{H}$ and $d_\Gamma(u) = |I^0(u)|$ for all $u \in U$. Note that each vertex in a house is a traveller, and the home of each traveller is a house. So we can write the size of U as the sum of the sizes of all houses at time 0.

$$|U| = \sum_{\hat{H} \in \mathcal{H}} |\hat{H}_0| = \sum_{\hat{H} \in \mathcal{H}} r(\hat{H}) + 1 = (2t + 3)|\mathcal{H}| + (L + 1).$$

As we have observed, every traveller has at least $q - t - 1$ accessible components, so we have $|I^0(u)| \geq q - t - 1$. Thus

$$\begin{aligned} \sum_{\hat{H} \in \mathcal{H}} d_\Gamma(\hat{H}) &= |E(J)| = \sum_{u \in U} d_\Gamma(u) \geq (q - t - 1)|U| \\ \implies \sum_{\hat{H} \in \mathcal{H}} |B(\hat{H})| &\geq (q - t - 1)(2t + 3)|\mathcal{H}| + (q - t - 1)(L + 1). \end{aligned} \quad (2.5)$$

This gives us a lower bound on the total size of all guest books. We now look for an upper bound.

For all $\hat{H} \in \mathcal{H}$, let $\mathcal{H}_{\hat{H}}$ be as defined in Lemma 2.28. Let $l_{\hat{H}}$ be the number of houses in $\mathcal{H}_{\hat{H}}$ with index $2t + 4$. That is,

$$l_{\hat{H}} = |\{H \in \mathcal{H}_{\hat{H}} | r(H) = 2t + 4\}|.$$

We now prove the following upper bound on each $|B(\hat{H})|$.

Claim 2.29.1. *For all $\hat{H} \in \mathcal{H}$, we have*

$$|B(\hat{H})| \leq (|\mathcal{H}_{\hat{H}}| + 2)(2t + 3) + l_{\hat{H}} - r(\hat{H}).$$

Proof. Take any house $\hat{H} \in \mathcal{H}$. By definition of $l_{\hat{H}}$, we have

$$\sum_{H \in \mathcal{H}_{\hat{H}}} r(H) = |\mathcal{H}_{\hat{H}}|(2t + 3) + l_{\hat{H}}.$$

If $r(\hat{H}) = 2t + 3$, then

$$\left\lfloor \frac{r(\hat{H})(t + 2)}{r(\hat{H}) - t - 1} \right\rfloor = \left\lfloor \frac{(2t + 3)(t + 2)}{t + 2} \right\rfloor = 2t + 3 = 2(2t + 3) - r(\hat{H}).$$

If $r(\hat{H}) = 2t + 4$, then

$$\left\lfloor \frac{r(\hat{H})(t+2)}{r(\hat{H}) - t - 1} \right\rfloor = \left\lfloor \frac{(2t+4)(t+2)}{t+3} \right\rfloor = \left\lfloor \frac{(2t+2)(t+3) + 2}{t+3} \right\rfloor = 2t+2 = 2(2t+3) - r(\hat{H}).$$

The claim then follows directly from Lemma 2.28(3). $\square$

We now show that the completeness relation $\equiv_c$ is an equivalence relation on $\mathcal{H}$. Reflexivity and symmetry are obvious. So it remains to prove that the relation $\equiv_c$ is transitive. Let H_1, H_2, H_3 be three houses in $\mathcal{H}$ such that $H_1 \equiv_c H_2$ and $H_2 \equiv_c H_3$. By Corollary 2.23, we have that in particular, H_1 and H_3 are complete to H_2 at its awakening time $a(H_2)$. Since H_2 is active in $M_{a(H_2)}$, by Lemma 2.18(1), we have that H_1 is complete to H_3 at $a(H_2)$, hence $H_1 \equiv_c H_3$.

By Lemma 2.28(2), the set of equivalence classes in $\mathcal{H}$ with respect to the relation $\equiv_c$ is $\mathcal{C} := \{\mathcal{H}_{\hat{H}} | \hat{H} \in \mathcal{H}\}$. We define a mapping $[\cdot]_c : \mathcal{H} \rightarrow \mathcal{C}$ that sends any house $\hat{H} \in \mathcal{H}$ to its equivalence class $[\hat{H}]_c \in \mathcal{C}$.

Now, we are ready to use the upper and lower bounds on the sizes of the guest books to prove the existence of some large $\mathcal{H}_{\hat{H}}$.

Claim 2.29.2. *There exists $\hat{H} \in \mathcal{H}$ such that $|\mathcal{H}_{\hat{H}}| \geq q - t - 1$.*

Proof. Suppose for contradiction that for all $\hat{H} \in \mathcal{H}$, we have $|\mathcal{H}_{\hat{H}}| \leq q - t - 2$. Note that the parameter l is invariant with respect to $\equiv_c$, so we can extend its definition to elements of $\mathcal{C}$. For each $C \in \mathcal{C}$, we define

$$l_C = |\{H \in C | r(H) = 2t + 4\}|.$$

Then we have

$$\sum_{\hat{H} \in \mathcal{H}} l_{\hat{H}} = \sum_{\hat{H} \in \mathcal{H}} l_{[\hat{H}]_c} = \sum_{C \in \mathcal{C}} |C| l_C \leq (q - t - 2) \sum_{C \in \mathcal{C}} l_C = (q - t - 2)L.$$

We derive an upper bound on $\sum_{\hat{H} \in \mathcal{H}} |B(\hat{H})|$ as follows using Claim 2.29.1.

$$\begin{aligned} \sum_{\hat{H} \in \mathcal{H}} |B(\hat{H})| &\leq \sum_{\hat{H} \in \mathcal{H}} \left((|\mathcal{H}_{\hat{H}}| + 2)(2t + 3) + l_{\hat{H}} - r(\hat{H}) \right) \\ &\leq \sum_{\hat{H} \in \mathcal{H}} ((q - t - 2) + 2)(2t + 3) + \sum_{\hat{H} \in \mathcal{H}} l_{\hat{H}} - \sum_{\hat{H} \in \mathcal{H}} r(\hat{H}) \\ &\leq (q - t)(2t + 3)|\mathcal{H}| + (q - t - 2)L - ((2t + 3)|\mathcal{H}| + L) \\ &= (q - t - 1)(2t + 3)|\mathcal{H}| + (q - t - 3)L. \end{aligned}$$

Combining this upper bound with the lower bound in Inequality 2.5, we have

$$(q - t - 1)(l + 1) \leq (q - t - 3)l,$$

which is a contradiction. $\square$

Claim 2.29.3. *If there exists $\hat{H} \in \mathcal{H}$ such that $|\mathcal{H}_{\hat{H}}| \geq q - t - 1$, then*

$$\omega(G) \geq \begin{cases} \Delta - 2t^2 - 6t - 3 - m & \text{if } 0 \leq m \leq t + 1 \\ \Delta - 2t^2 - 7t - 4 & \text{if } t + 2 \leq m \leq 2t + 2. \end{cases}$$

Proof. By Lemma 2.28(1), all houses in $\mathcal{H}_{\hat{H}}$ are complete to each other at $a(\hat{H})$ and hence form a large clique. When $0 \leq m \leq t + 1$, we have

$$\begin{aligned} \omega(G) &\geq \sum_{H \in \mathcal{H}_{\hat{H}}} |H_{a(\hat{H})}| = \sum_{H \in \mathcal{H}_{\hat{H}}} r(H) + 1 \\ &\geq (q - t - 1)(2t + 3) + 1 \\ &= ((2t + 3)q + t + 1 + m) - (t + 1)(2t + 3) - t - m \\ &= \Delta - 2t^2 - 6t - 3 - m. \end{aligned}$$

When $t + 1 \leq m \leq 2t + 2$, since each of the $q - t - 1$ houses in $\mathcal{H}_{\hat{H}}$ belongs to a different part, at least $(q - t - 1) - (q - m) = m - t - 1$ of them have index $2t + 4$. So

$$\begin{aligned} \omega(G) &\geq \sum_{H \in \mathcal{H}_{\hat{H}}} |H_{a(\hat{H})}| = \sum_{H \in \mathcal{H}_{\hat{H}}} r(H) + 1 \\ &\geq (q - t - 1)(2t + 3) + (m - t - 1) + 1 \\ &= ((2t + 3)q + t + 1 + m) - (t + 1)(2t + 3) - 2t - 1 \\ &= \Delta - 2t^2 - 7t - 4. \end{aligned}$$

$\square$

The theorem follows directly from Claims 2.29.2 and 2.29.3. $\square$

We can now prove Theorem 2.8 as a corollary by showing that a minimum counterexample is a critical graph, to which we can directly apply Theorem 2.7.

Proof of Theorem 2.8. Suppose for contradiction that the statement does not hold. Take the smallest t for which the statement fails, and let G be a vertex-minimal counterexample for this t . Then we have

$$\begin{aligned}\chi(G) &= \Delta(G) - t, \\ \Delta(G) &\geq 4t^2 + 11t + 7 = (2t + 3)(2t + 2) + t + 1, \text{ and} \\ \omega(G) &< 2t^2 - 7t - 4.\end{aligned}$$

We claim that G is critical, then the result follows immediately from Theorem 2.7.

Claim 2.30.1. *The graph G is critical.*

Proof. Suppose for contradiction that G is not critical. Then there exists a vertex $v \in V(G)$ such that $\chi(G - v) = \chi(G)$. We consider the following cases.

If $\Delta(G - v) = \Delta(G) \geq 4t^2 + 11t + 7$, then $\chi(G - v) = \chi(G) = \Delta(G) - t = \Delta(G - v) - t$. By minimality of G , we have

$$\omega(G) \geq \omega(G - v) \geq \Delta(G - v) - 2t^2 - 7t - 4 = \Delta(G) - 2t^2 - 7t - 4,$$

contradicting that G is a counterexample. So we may assume that $\Delta(G - v) = \Delta(G) - 1$.

If $t \geq 1$, then

$$\begin{aligned}\chi(G - v) &= \chi(G) = \Delta(G) - t = \Delta(G - v) - (t - 1), \text{ and} \\ \Delta(G - v) &= \Delta(G) - 1 \geq 4t^2 + 11t + 6 > 4(t - 1)^2 + 11(t - 1) + 7.\end{aligned}$$

By minimality of t ,

$$\omega(G) \geq \omega(G - v) \geq \Delta(G - v) - 2(t - 1)^2 - 7(t - 1) - 4 > \Delta(G) - 2t^2 - 7t - 4,$$

contradicting that G is a counterexample.

It remains to consider the case $t = 0$. We have $\chi(G - v) = \chi(G) = \Delta(G) = \Delta(G - v) + 1$. By Brooks' Theorem, $\omega(G) \geq \omega(G - v) = \Delta(G - v) + 1 = \Delta(G)$, contradicting that G is a counterexample. $\square$

$\square$

We can also use Theorem 2.7 to prove Theorem 2.5 by following the same approach as in [35]. We will again consider a minimum counterexample, which is critical. Then we attempt to remove a hitting set from G . The following lemma by King [42] guarantees the existence of a hitting set for graphs with a large clique.

Theorem 2.31 (King [42]). *If a graph G satisfies $\omega(G) > \frac{2}{3}(\Delta(G) + 1)$, then G contains a hitting set.*

Proof of Theorem 2.5. Suppose for contradiction that the statement does not hold. Take the smallest t for which the statement fails, and let G be a vertex-minimal counterexample for this t . Then we have $\chi(G) = \Delta(G) - t$, $\Delta(G) \geq 6t^2 + 20t + 16$ and $\omega(G) < 2t^2 - 6t - 3$. By the same argument as in the proof of Claim 2.30.1 with the bounds of $\Delta(G)$ and $\omega(G)$ being replaced by the corresponding expressions in the current statement, we have that G is critical.

Note that if $\Delta(G) = 6t^2 + 20t + 16 = (2t + 3)(3t + 5) + t + 1$, then by Theorem 2.7, we have $\omega(G) \geq \Delta(G) - 2t^2 - 6t - 3$, which contradicts that G is a counterexample. Thus, we may assume $\Delta(G) \geq 6t^2 + 20t + 17$. We will remove a maximal independent set I from G and apply the theorem to the subgraph $G - I$. We first prove the following property of $G - I$.

Claim 2.32.1. *Let I be a maximal independent set in G . Then there exists $1 \leq x \leq t + 2$ such that*

$$\begin{aligned}\chi(G - I) &= \Delta(G - I) - (t + 1 - x), \text{ and} \\ \Delta(G - I) &\geq 6(t + 1 - x)^2 + 20(t + 1 - x) + 16.\end{aligned}$$

Proof. Let I be a maximal independent set in G . Note that given a $\chi(G - I)$ -colouring of $G - I$, colouring all vertices in I with an extra colour yields a $(\chi(G - I) + 1)$ -colouring of G , so $\chi(G - I) \leq \chi(G) \leq \chi(G - I) + 1$. Since G is critical, we have $\chi(G - I) = \chi(G) - 1 = \Delta(G) - t - 1$. By Brooks' Theorem, $\Delta(G - I) \geq \chi(G - I) - 1 = \Delta(G) - (t + 2)$. By maximality of I , every vertex in $G - I$ is adjacent to some vertex in I , so $\Delta(G - I) \leq \Delta(G) - 1$. Thus, $\Delta(G - I) = \Delta(G) - x$ for some $1 \leq x \leq t + 2$. Hence,

$$\begin{aligned}\chi(G - I) &= \Delta(G) - t - 1 = \Delta(G - I) + x - t - 1 = \Delta(G - I) - (t + 1 - x), \text{ and} \\ \Delta(G - I) &= \Delta(G) - x \geq 6t^2 + 20t + 17 - x \geq 6(t + 1 - x)^2 + 20(t + 1 - x) + 16.\end{aligned}$$

The last inequality is obvious for $1 \leq x \leq t + 1$, which implies $0 \leq t + 1 - x \leq t$. When $x = t + 2$, the inequality becomes $6t^2 + 19t + 15 \geq 6 - 20 + 16$, which is true. $\square$

To apply Theorem 2.31, we consider the following two cases.

Case 1: $\omega(G) \leq \frac{2}{3}(\Delta(G) + 1)$.

Let I be a maximal independent set in G . Define x as in the claim. Note that when $x = t + 2$, Brooks' Theorem applies, and we have

$$\omega(G) \geq \omega(G - I) = \Delta(G - I) + 1 = \Delta(G) - x + 1 = \Delta(G) - t - 1,$$

which contradicts G being a counterexample.

Otherwise, $1 \leq x \leq t + 1$. By minimality of t and G , we have that

$$\begin{aligned} \frac{2}{3}(\Delta(G) + 1) &\geq \omega(G) \geq \omega(G - I) \\ &\geq \Delta(G - I) - 2(t + 1 - x)^2 - 6(t + 1 - x) - 3 \\ &\geq (\Delta(G) - x) - 2t^2 - 6(t + 1 - x) - 3 \\ &= \Delta(G) + 5x - 2t^2 - 6t - 9 \\ &\geq \Delta(G) - 2t^2 - 6t - 4, \end{aligned}$$

which implies $\Delta(G) \leq 6t^2 + 18t + 14$. This contradicts the assumption that $\Delta(G) \geq 6t^2 + 20t + 17$.

Case 2: $\omega(G) > \frac{2}{3}(\Delta(G) + 1)$.

By Theorem 2.31, there exists a hitting set I_0 , which satisfies $\omega(G - I_0) = \omega(G) - 1$. By adding vertices if necessary, we construct a maximal independent set $I \supseteq I_0$. Every maximum clique in G has exactly one vertex in I_0 and hence cannot have any additional vertices added to I . So, $\omega(G - I) = \omega(G - I_0) = \omega(G) - 1$. We again use the claim and the minimality of G and obtain through similar steps as in Case 1 that

$$\omega(G) = \omega(G - I) + 1 \geq \Delta(G) - 2t^2 - 6t - 3.$$

□

Chapter 3

Odd-hole-free Graphs

There has been a lot of research on the class of odd-hole-free graphs and its subclasses, such as perfect graphs. The structure of odd-hole-free graphs is also very well studied. For example, Conforti, Cornuéjols and Vušković [16] gave a complete characterization of all odd-hole-free graphs using double star cutsets, 2-joins and graphs related to bipartite graphs. According to Chudnovsky, Scott, Seymour and Spirkl [14], there exists a polynomial time algorithm with running time $O(|V(G)|^9)$ for detecting the existence of odd holes in a graph G . All these results make it appealing to study the Borodin-Kostochka Conjecture and Reed's Conjecture in the context of odd-hole-free graphs. The theorems in this chapter tie into the existing results and give us an even better understanding of such graphs. We restate the two conjectures here.

Conjecture 3.1 (Borodin and Kostochka [3]). *Given a graph G with $\Delta(G) \geq 9$, if $\chi(G) = \Delta(G)$, then $\omega(G) = \Delta(G)$.*

Conjecture 3.2 (Reed [56]). *For any graph G , we have*

$$\chi(G) \leq \left\lceil \frac{\Delta(G) + \omega(G) + 1}{2} \right\rceil.$$

In Section 3.1, we discuss some subclasses of odd-hole-free graphs, with a focus on perfect graphs. Very recently in 2024, Chen, Lan, Lin and Zhou [9] proved the Borodin-Kostochka Conjecture for all odd-hole-free graphs. In Section 3.2, we present their proof while generalizing some of their claims in preparation for proving a similar result for Reed's Conjecture. We show that their result can be strengthened so that the hypothesis only involves local constraints. In section 3.3, we prove Reed's Conjecture for odd-hole-free

graphs. Avarind, Karthick and Subramanian [1] have proven this result in 2011 using a similar technique. We then discuss some stronger variants of Reed’s Conjecture, which also involve some of their work. At the end of these discussions, we outline some remaining questions in Section 3.4.

3.1 Subclasses of Odd-hole-free Graphs

We start by defining the class of odd-hole-free graphs, which is the main object studied in this chapter.

Definition 3.3. A **hole** in a graph is an induced cycle of length at least 4. An **odd hole** is a hole with an odd length. An **odd-hole-free** graph is a graph that does not contain an odd hole.

Complementarily, we also define an odd-antihole-free graph.

Definition 3.4. An **antihole** is the complement of a hole. An **odd antihole** is an antihole with an odd number of vertices. An **odd-antihole-free** graph is a graph that does not contain an odd-antihole.

Another interesting class of graphs is perfect graphs. We will soon see that they form a subclass of odd-hole-free graphs.

Definition 3.5. A graph G is **perfect** if for every induced subgraph H of G , $\chi(H) = \omega(H)$.

In particular, this definition implies that for a perfect graph G , we have $\chi(G) = \omega(G)$. All together, these definitions allow us to state the Strong Perfect Graph Theorem by Chudnovsky, Robertson, Seymour and Thomas [12], which characterizes all perfect graphs.

Theorem 3.6 (Chudnovsky, Robertson, Seymour, Thomas [12]). *A graph is perfect if and only if it is odd-hole-free and odd-antihole-free.*

It is straightforward to show that the Borodin-Kostochka Conjecture and Reed’s Conjecture hold for perfect graphs, which is a subclass of odd-hole-free graphs.

Corollary 3.7. *The Borodin-Kostochka Conjecture holds for graphs that are both odd-hole-free and odd-antihole-free.*

Proof. This follows from the Strong Perfect Graph Theorem and the definition of a perfect graph. $\square$

Corollary 3.8. *Let G be a graph that is both odd-hole-free and odd-antihole-free, then*

$$\chi(G) \leq \left\lfloor \frac{\Delta(G) + \omega(G) + 1}{2} \right\rfloor.$$

Hence Reed's Conjecture holds.

Proof. By the Strong Perfect Graph Theorem, G is a perfect graph. Then by definition, $\chi(G) = \omega(G)$. By Brooks' Theorem, $\chi(G) \leq \Delta(G) + 1$. So we have

$$\chi(G) = \frac{\chi(G)}{2} + \frac{\chi(G)}{2} \leq \frac{\Delta(G) + 1}{2} + \frac{\omega(G)}{2}.$$

The result then follows from the fact that $\chi(G)$ is an integer. $\square$

In the next two sections, we prove that the two corollaries above can be generalized to all odd-hole-free graphs; that is, we remove the constraint of being odd-antihole-free.

Another subclass of odd-hole-free graphs for which we can prove the two conjectures immediately using existing results is $(K_4, \text{odd hole})$ -free graphs. Chudnovsky, Robertson, Seymour and Thomas [13] showed the following result on the chromatic number of these graphs.

Theorem 3.9 (Chudnovsky, Robertson, Seymour and Thomas [13]). *K_4 -free graphs with no odd holes are 4-colourable.*

Using this theorem, we see that such graphs trivially satisfy the Borodin-Kostochka Conjecture, and Reed's Conjecture also holds with a simple proof.

Corollary 3.10. *The Borodin-Kostochka Conjecture holds for $(K_4, \text{odd hole})$ -free graphs.*

Corollary 3.11. *Reed's Conjecture holds for $(K_4, \text{odd hole})$ -free graphs.*

Proof. Let G be a $(K_4, \text{odd hole})$ -free graph, then $\omega(G) \leq 3$. We consider the following cases.

If $\omega(G) \leq 1$, then G has no edges, and hence $\Delta(G) = 0$ and $\chi(G) \leq 1$.

If $\omega(G) = 2$ and $\Delta(G) \leq 2$, then every component of G is a path or an even cycle. So $\chi(G) = 2$.

If $\omega(G) = 3$ and $\Delta(G) = 2$, then by Brooks' Theorem, $\omega(G) \leq \chi(G) \leq \Delta(G) + 1$, and hence $\chi(G) = 3$.

If $\omega(G) = 2$ and $\Delta(G) = 3$, then we first prove that G is bipartite by showing that G has no odd cycles. Suppose not. Let C be a shortest odd cycle in G . Since $\omega(G) = 2$, C is not a triangle and hence has length at least 5. Since G is odd-hole-free, C has a chord e , which splits C into an even cycle and an odd cycle with a shorter length. This contradicts the minimality of C . Thus, G is a bipartite graph, and hence $\chi(G) = 2$.

Otherwise, either $\omega(G) = 3$ and $\Delta(G) \geq 3$ or $\omega(G) = 2$ and $\Delta(G) \geq 4$, and we have $\chi(G) \leq 4$ by Theorem 3.9. In all cases, it is straightforward to verify that Reed's Conjecture holds. $\square$

Dhurandhar proved the following theorem in an unpublished work [23]. This theorem immediately implies that the Borodin-Kostochka Conjecture holds for $2K_2$ -free graphs, which are the graphs that do not contain an independent pair of edges as an induced subgraph.

Theorem 3.12 (Dhurandhar [23]). *If G is a $2K_2$ -free graph with $\Delta(G) \geq 5$, then $\chi(G) \leq \max\{\omega(G), \Delta(G) - 1\}$.*

We note that C_5 is the only odd hole that may exist in a $2K_2$ -free graph. So we have that the Borodin-Kostochka Conjecture also holds for the following subclass of odd-hole-free graphs.

Corollary 3.13. *The Borodin-Kostochka Conjecture holds for $(2K_2, C_5)$ -free graphs.*

Although this is a non-comprehensive list of subclasses of odd-hole-free graphs for which the Borodin-Kostochka Conjecture and/or Reed's Conjecture holds, it suggests that these conjectures are true for a broader family of graphs. Indeed, in Sections 3.2 and 3.3, we prove that both conjectures hold for all odd-hole-free graphs. This generalizes all the corollaries above. Moreover, the upper bound $f(\omega)$ on the chromatic number of any odd-hole-free graph in terms of its clique number ω has been studied, and it remains an open problem to find a tight upper bound. Scott and Seymour [58] proved the following super-exponential $f(\omega)$. They remarked that this bound is likely far from optimal and that the best known lower bound on $f(\omega)$ is polynomial.

Theorem 3.14 (Scott and Seymour [58]). *Let G be a graph with no odd hole. Then*

$$\chi(G) \leq 2^{2^{\omega(G)+2}}.$$

Reed's Conjecture provides an upper bound that is linear in the sum of the clique number and the maximum degree. This is a significantly tighter bound than Theorem 3.14 when the maximum degree is small.

3.2 The Borodin-Kostochka Conjecture for Odd-hole-free Graphs

In 2024, Chen, Lan, Lin and Zhou [9] proved the following theorem, which immediately implies that the Borodin-Kostochka Conjecture is true for odd-hole-free graphs.

Theorem 3.15 (Chen, Lan, Lin and Zhou [9]). *Let G be an odd-hole-free graph. Then $\chi(G) \leq \max\{6, \Delta(G) - 1, \omega(G)\}$.*

In this section, we prove the following main theorem, which is a stronger result since it only requires one vertex in a graph G to not belong to an odd hole. In return, the extra assumption we make is that this vertex is critical.

Theorem 3.16. *Let G be a graph with $\Delta(G) \geq 7$. Suppose G contains at least one critical vertex that is not in an odd hole. If $\chi(G) = \Delta(G)$, then $\omega(G) = \Delta(G)$.*

Recall that in Section 1.3, we showed that the bound $\Delta \geq 9$ in the Borodin-Kostochka Conjecture is necessary using the graph $C_5[K_3]$, in which every vertex belongs to an odd holes. So it is unsurprising that in the theorem above, the lower bound on the maximum degree is reduced. Near the end of the proof of Theorem 3.16, it will become clear why this lower bound is 7 and how this condition is used. Roughly speaking, this provides us with enough neighbours of a critical vertex in order to build the desired contradiction.

We then prove the following statement as a corollary of Theorem 3.16. Theorem 3.17 is equivalent to Theorem 3.15 but better aligns with the form of the Borodin-Kostochka Conjecture that we have been using in this thesis.

Theorem 3.17. *Let G be an odd-hole-free graph with $\Delta(G) \geq 7$. If $\chi(G) = \Delta(G)$, then $\omega(G) = \Delta(G)$.*

Let us first show that the two theorems are indeed equivalent.

Lemma 3.18. *Theorem 3.15 and Theorem 3.17 are equivalent.*

Proof. Let G be an odd-hole-free graph with $\chi(G) = \Delta(G) \geq 7$, and suppose Theorem 3.15 holds. Then,

$$7 \leq \Delta(G) = \chi(G) \leq \max\{6, \Delta(G) - 1, \omega(G)\},$$

which implies that $\Delta(G) = \chi(G) \leq \omega(G)$. Since $\chi(G) \geq \omega(G)$ always holds, we have $\omega(G) = \chi(G) = \Delta(G)$.

Conversely, let G be an odd-hole-free graph and suppose Theorem 3.17 holds. Suppose for contradiction that $\chi(G) > \max\{6, \Delta(G) - 1, \omega(G)\}$. That is, $\chi(G) \geq 7$, $\chi(G) \geq \Delta(G)$, and $\chi(G) > \omega(G)$. Since $\chi(G) \geq 7$, by Theorem 3.17, if $\chi(G) = \Delta(G)$, then $\omega(G) = \Delta(G) = \chi(G)$, which contradicts the assumption that $\chi(G) > \omega(G)$. By Brooks' Theorem, it remains to consider the case $\chi(G) = \Delta(G) + 1$, but then $\omega(G) = \Delta(G) + 1 = \chi(G)$, which leads to the same contradiction. $\square$

We generalize the proof in [9] to prove Theorem 3.16. Here we summarize the main idea of their proof, which consists of a series of four claims. As with many other results related to the Borodin-Kostochka Conjecture, they used a proof by contradiction and considered a vertex-minimal counterexample G , which by their Claim 2.1, turns out to be critical. They then focused on the neighbourhood of a vertex v in G that achieves the maximum degree. They showed using their Claim 2.2 that the vertex v together with its neighbourhood contains a $(\Delta(G) - 1)$ -clique. Using the assumption that G does not contain a $\Delta(G)$ -clique, they proved in Claim 2.3 that a minimum colouring of $G - v$ heavily restricts the number and locations of multiple colours in $N(v)$. Finally, using their Claim 2.4, they constructed a minimum colouring that does not satisfy those restrictions, which leads to the eventual contradiction.

We observe that the proof in [9] exploits various properties of the neighbourhood of a vertex v that achieves maximum degree in a critical odd-hole-free graph G . In order to modify this result to suit our purpose, we consider the neighbourhood of a critical vertex v that does not belong to any odd hole in a graph G . Since v is critical, we recall Lemma 2.10. Using the extra assumption that v is not in an odd-hole, we prove the following corollary, which is a generalization of both Claim 2.2 in [9] and Lemma 3.0.1 in [1]. This corollary will be used again in the next section to prove Reed's Conjecture for odd-hole-free graphs. It states that given a minimum colouring of $G - v$, different colour classes in $N(v)$ are highly connected. This allows us to construct large cliques in G using singleton neighbours of v .

Corollary 3.19. *Let G be a graph with a critical vertex v that does not belong to any odd hole. Let $\chi := \chi(G)$ and $\Delta := \Delta(G)$. Let $\phi : V(G - v) \rightarrow [\chi - 1]$ be a minimum colouring of $G - v$. For all $i \in [\chi - 1]$, define $V_i := \{u \in N(v) \mid \phi(u) = i\}$. Then the following holds.*

1. For all $1 \leq i < j \leq \chi - 1$, we have that $E(V_i, V_j) \neq \emptyset$.
2. Let S be the set of all singleton neighbours of v , then $S \cup \{v\}$ is a clique of size at least $2\chi - d(v) - 1$.

Proof. By Lemma 2.10(1), such a colouring ϕ exists, and $V_i \neq \emptyset$ for all $i \in [\chi - 1]$.

We first prove Part 1. Suppose for contradiction that there exist $1 \leq i < j \leq \chi - 1$ such that $E(V_i, V_j) = \emptyset$. Let H be the subgraph of G induced by all vertices with colours i or j under ϕ . Let H' be a component of H such that $V(H') \cap V_i \neq \emptyset$ and $V(H') \cap V_j \neq \emptyset$. If such H' does not exist, then all components of H that contain a vertex in V_i do not contain any vertex in V_j . For readers who are familiar with the concept, these components are simply (i, j) -Kempe chains that contain vertices in V_i . By swapping the colours i and j in these components, we obtain a $(\chi - 1)$ -colouring of $G - v$ in which the colour i does not appear in $N(v)$. This contradicts Lemma 2.10(1). So H' exists.

Let P be a shortest $u_i u_j$ -path in H' such that $u_i \in V(H') \cap V_i$ and $u_j \in V(H') \cap V_j$. Note that P is an induced subgraph of G ; in particular, $P = G[V(P)]$. By minimality of P , no internal vertices of P are in $N(v)$, so $N(v) \cap V(P) = \{u_i, u_j\}$. This implies that $C := G[V(P) \cup \{v\}]$ is an induced cycle in G . Since the vertices in P are alternately coloured i and j by ϕ with $\phi(u_i) = i$ and $\phi(u_j) = j$, P has an odd length. Since $E(V_i, V_j) = \emptyset$, the vertices u_i and u_j are not adjacent. So P has length at least 3. Thus, C is an induced odd cycle in G with length at least 5; that is, C is an odd hole in G that contains v . This contradicts the choice of v .

Now we prove Part 2. By Lemma 2.10(2), $|S| \geq 2\chi - d(v) - 2$. By Part 1, S is a clique. Since $S \subseteq N(v)$, we conclude that $S \cup \{v\}$ is a clique of size at least $2\chi - d(v) - 1$. $\square$

Corollary 3.19(2) implies that G has a clique of size at least $2\chi(G) - \Delta(G) - 1$. We will see in the next section that Reed's Conjecture on odd-hole-free graphs follows easily from this. This inequality is trivial when the chromatic number is small, in particular, when $\chi(G) \leq \frac{\Delta(G)+2}{2}$. However, since we will use this result in the context of the Borodin-Kostochka Conjecture, we usually have that $\chi(G)$ is equal or close to $\Delta(G)$. In this case, we obtain a clique of a comparable size.

We are now ready to prove Theorem 3.16. Our proof differs from the one in [9] in the following ways. Claim 2.1 in [9] is made obsolete by the assumption that v is critical. Claim 2.2 in [9] is replaced by Corollary 3.19. We also take a simpler approach to Claim 3.20.2, which is Claim 2.4 in [9].

Proof of Theorem 3.16. Suppose for contradiction that the theorem does not hold. Let G be a counterexample. That is, G is a graph with a critical vertex v that is not in an odd-hole, and we have $\Delta := \chi(G) = \Delta(G) \geq 7$ and $\omega(G) \neq \Delta(G) = \Delta$. Since $\Delta = \chi(G) \geq \omega(G)$ always holds, we have $\omega(G) < \Delta$.

Define $G' = G - v$ and $N := N_G(v)$. Since v is critical, we have $\chi(G') = \chi(G) - 1 = \Delta - 1$. Let $\phi : V(G') \rightarrow [\Delta - 1]$ be a $(\Delta - 1)$ -colouring of G' . By Lemma 2.10(1), all $\Delta - 1$ colours appear in N . So $\Delta - 1 \leq d(v) \leq \Delta$.

If $d(v) = \Delta - 1$, then by Corollary 3.19(2), $\omega(G) \geq 2\Delta - (\Delta - 1) - 1 = \Delta$, which contradicts the assumption of $\omega(G) < \Delta$. Thus, $d(v) = \Delta$. We denote the neighbours of v by $N = \{u_1, u_2, \dots, u_\Delta\}$. Without loss of generality, we may assume that for $i \in [\Delta - 2]$, $\phi(u_i) = i$ and that $\phi(u_{\Delta-1}) = \phi(u_\Delta) = \Delta - 1$. Let V_i , for $i \in [\Delta - 1]$, and S be as defined in Corollary 3.19. That is, $V_i := \{u_i\}$ for all $i \in [\Delta - 2]$, $V_{\Delta-1} := \{u_{\Delta-1}, u_\Delta\}$ and $S = \{u_1, u_2, \dots, u_{\Delta-2}\}$. By Corollary 3.19(2), $S \cup \{v\}$ is a $(\Delta - 1)$ -clique.

Claim 3.20.1. *For all $(\Delta - 1)$ -colourings ϕ' of G' , we have $\phi'(u_{\Delta-1}) = \phi'(u_\Delta)$.*

Proof. Suppose not. By Lemma 2.10(1), v has exactly two non-singleton neighbours. So there exists a $(\Delta - 1)$ -colouring ϕ' of G' such that $\phi'(x) = \phi'(y)$ for distinct vertices $x, y \in N$ such that $\{x, y\} \neq \{u_{\Delta-1}, u_\Delta\}$. Since $\phi'(x) = \phi'(y)$, we have $x \not\sim y$. Since S is a clique, we have that $\{x, y\} \not\subseteq S$. That is, $|\{x, y\} \cap \{u_{\Delta-1}, u_\Delta\}| = 1$. Without loss of generality, we may assume that $x = u_\Delta$ and $y \in S$. Since $y \not\sim x$, by Corollary 3.19(1) applied to the colouring ϕ and $j = \Delta - 1$, we have that $y \sim u_{\Delta-1}$. By Lemma 2.10(1), the $\Delta - 2$ vertices in $N \setminus \{x, y\}$ are distinctly coloured by ϕ' . By applying Corollary 3.19(2) to ϕ' instead of ϕ , we have that $\{v\} \cup N \setminus \{x, y\}$ is a $(\Delta - 1)$ -clique and contains $u_{\Delta-1}$. So $u_{\Delta-1}$ is adjacent to all vertices in $\{v\} \cup N \setminus \{x, u_{\Delta-1}\} = \{v\} \cup S$. So $S \cup \{v, u_{\Delta-1}\}$ is a Δ -clique. This contradicts the assumption that $\omega(G) < \Delta$. $\square$

Next, we prove that a result similar to Lemma 2.10(3) holds for all vertices in N .

Claim 3.20.2. *Under ϕ , for all $i \in [\Delta]$, u_i has at most 2 non-singleton neighbours in G' . Moreover, the 2 non-singleton neighbours, if exist, have the same colour.*

Proof. For all $i \in [\Delta]$, since $v \in N(u_i)$, we have $d_{G'}(u_i) \leq \Delta - 1$. It suffices to prove that all $\Delta - 2$ colours, that is, the colours $[\Delta - 1] \setminus \{\phi(u_i)\}$, appear in $N_{G'}(u_i)$. We consider the following two cases. For all $i \in [\Delta - 2]$, we have $V_i = \{u_i\}$. By Corollary 3.19(1), u_i is adjacent to at least one vertex in V_j for all $j \in [\Delta - 1] \setminus \{i\}$. By definition of the V_j 's, the claim follows. For $i \in \{\Delta - 1, \Delta\}$, suppose for contradiction that there exists a colour $j \in [\Delta - 2]$ such that j does not appear in $N_{G'}(u_i)$. Then, we may recolour u_i by j , which contradicts Claim 3.20.1. $\square$

Now, we use Claim 3.20.2 to construct a $(\Delta - 1)$ -colouring of G' that contradicts Claim 3.20.1. By Corollary 3.19(1), every vertex in S is adjacent to at least one of $u_{\Delta-1}$ and u_{Δ} . We use the assumption of $\Delta \geq 7$ to obtain the following inequality.

$$\left\lceil \frac{|S|}{2} \right\rceil = \left\lceil \frac{\Delta - 2}{2} \right\rceil \geq 3.$$

We may assume without loss of generality that $\{u_1, u_2, u_3\} \subseteq N_G(u_{\Delta})$. On the other hand, since $S \cup \{v\}$ is a $(\Delta - 1)$ -clique, and $\omega(G) < \Delta$, u_{Δ} is not adjacent to all vertices in S . We may assume without loss of generality that $u_4 \notin N(u_{\Delta})$. This situation can be visualized as in Figure 3.1. Note that $\{u_1, u_2, u_3\} \subseteq N_{G'}(u_4) \cap N_{G'}(u_{\Delta})$, and u_1, u_2 and u_3 have distinct colours. By Claim 3.20.2, at least one vertex in $\{u_1, u_2, u_3\}$ is a singleton neighbour (under ϕ) of both u_4 and u_{Δ} . Without loss of generality, we assume that u_1 is such a vertex. That is, u_1 is the only vertex in $N_{G'}(u_4) \cup N_{G'}(u_{\Delta})$ with colour 1. Since $\phi(u_4) \neq \phi(u_{\Delta})$, applying Claim 3.20.2 to u_1 , we have that u_4 (or u_{Δ}) is a singleton neighbour of u_1 . By recolouring u_1 with colour 4 (or $\Delta - 1$) and recolouring u_4 and u_{Δ} with colour 1, we obtain a $(\Delta - 1)$ -colouring of G' in which $u_{\Delta-1}$ and u_{Δ} are coloured differently. This contradicts Claim 3.20.1 and hence completes the proof of Theorem 3.16. $\square$

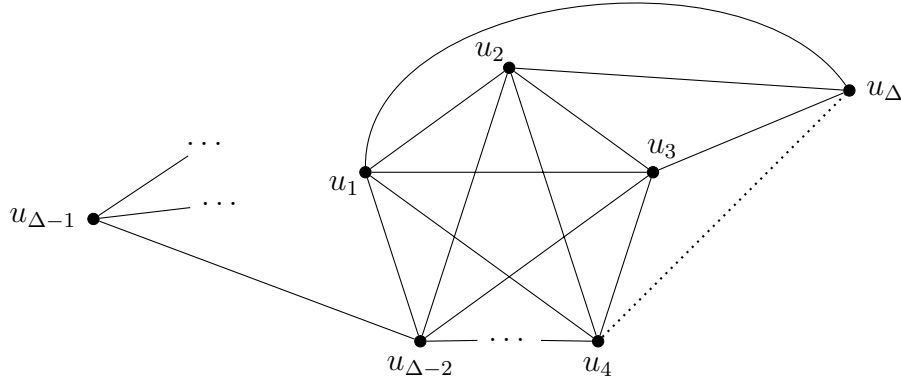


Figure 3.1: A sketch of N , where the dotted edge indicates non-adjacency

Using Theorem 3.16, we now prove Theorem 3.17 by showing that a vertex-minimal counterexample is critical.

Proof of Theorem 3.17. Suppose for contradiction that the theorem does not hold. Let G be a vertex-minimal counterexample. That is, $\Delta := \chi(G) = \Delta(G) \geq 7$ and $\omega(G) \neq$

$\Delta(G) = \Delta$. Since $\Delta = \chi(G) \geq \omega(G)$ always holds, we have $\omega(G) < \Delta$. It suffices to prove that G is critical. Then we obtain from Theorem 3.16 that G cannot be a counterexample.

Take any vertex $v \in V(G)$ and let $G' = G - v$. We need to show that $\chi(G') < \chi(G) = \Delta$. Since G is odd-hole-free and any induced subgraph of G' is also an induced subgraph in G , we have that G' is also odd-hole-free. If $\chi(G') < \Delta(G')$, then

$$\chi(G') < \Delta(G') \leq \Delta(G) = \Delta.$$

Otherwise, $\chi(G') \geq \Delta(G')$. By Brooks' Theorem, we have $\Delta(G') \leq \chi(G') \leq \Delta(G') + 1$. Moreover, if $\chi(G') = \Delta(G') + 1$, then $\omega(G') = \Delta(G') + 1$. Hence,

$$\chi(G') = \omega(G') \leq \omega(G) < \Delta.$$

So it remains to consider the case $\chi(G') = \Delta(G')$. If $\Delta(G') \leq 6$, then

$$\chi(G') = \Delta(G') \leq 6 < \Delta.$$

Otherwise, we have $\chi(G') = \Delta(G') \geq 7$. By minimality of G , the theorem holds for G' , so $\omega(G') = \Delta(G')$. Thus, we have

$$\chi(G') = \Delta(G') = \omega(G') \leq \omega(G) < \Delta.$$

□

3.3 Reed's Conjecture for Odd-hole-free Graphs

In this section, we make use of Corollary 3.19 to prove a series of partial results on Reed's Conjecture, some of which have been proven by Aravind, Karthick and Subraminian [1].

We first prove a theorem that is analogous to Theorem 3.16.

Theorem 3.21. *Let G be a graph with at least one critical vertex that is not in an odd-hole, then*

$$\chi(G) \leq \left\lfloor \frac{\Delta(G) + \omega(G) + 1}{2} \right\rfloor.$$

Proof. By Corollary 3.19(2), $\omega(G) \geq 2\chi(G) - \Delta(G) - 1$. We obtain the theorem by rearranging and using the fact that $\chi(G)$ is an integer. □

We now show that Reed's Conjecture holds for odd-hole-free graphs. This has been proven by Aravind, Karthick and Subraminian [1] in 2011. Our proof is similar to theirs. In fact, the following theorem in [1] is a stronger result than Reed's Conjecture since the ceiling operator is replaced by the floor operator and the maximum degree is replaced by the degeneracy, a quantity that never exceeds the maximum degree. The following theorem again follows from Corollary 3.19(2).

Definition 3.22. The **degeneracy** of a graph G is defined to be the maximum of the minimum degree of all subgraphs of G . That is,

$$d(G) := \max_{H \subseteq G} \delta(H).$$

It follows directly from the above definition that for a graph G , $\delta(G) \leq d(G) \leq \Delta(G)$.

Theorem 3.23 (Aravind, Karthick and Subraminian [1]). *For any odd-hole-free graph G , we have*

$$\chi(G) \leq \left\lfloor \frac{d(G) + \omega(G) + 1}{2} \right\rfloor.$$

Hence, Reed's Conjecture holds.

Proof. Suppose not. Let G be a vertex-minimal counterexample. That is, G is odd-hole-free and

$$\chi(G) > \left\lfloor \frac{d(G) + \omega(G) + 1}{2} \right\rfloor.$$

Consider any vertex $v \in V(G)$. Since the degeneracy and the clique number are non-increasing upon deletion of any vertex, by minimality of G , we have

$$\chi(G - v) \leq \left\lfloor \frac{d(G - v) + \omega(G - v) + 1}{2} \right\rfloor \leq \left\lfloor \frac{d(G) + \omega(G) + 1}{2} \right\rfloor < \chi(G).$$

So G is critical. Let v be a vertex in G with degree $\delta(G)$. By Corollary 3.19(2), we have

$$\omega(G) \geq 2\chi(G) - \delta(G) - 1 \geq 2\chi(G) - d(G) - 1.$$

Rearranging and using the fact that $\chi(G)$ is an integer, we have

$$\chi(G) \leq \left\lfloor \frac{d(G) + \omega(G) + 1}{2} \right\rfloor.$$

This contradicts the choice of G . □

It is tempting to replace the degeneracy in the above theorem by the minimum degree. However, this cannot be done trivially since the minimum degree of a subgraph is not necessarily smaller than that of the original graph. This prevents us from showing that the vertex-minimal counterexample is critical.

We now look at some local variations of Reed's Conjecture. Instead of the global parameters $\Delta(G)$ and $\omega(G)$, we work with local parameters $d(v)$ and $\omega(v)$ for vertices v in G , where $\omega(v)$ is the size of a largest clique in G that contains v .

Theorem 3.24. *Let G be a graph with at least one critical vertex u that is not in an odd hole, then*

$$\chi(G) \leq \max_{v \in V(G)} \left\lfloor \frac{d(v) + \omega(v) + 1}{2} \right\rfloor.$$

Proof. By Corollary 3.19(2), $\omega(u) \geq 2\chi(G) - d(u) - 1$. We obtain the theorem by rearranging and using the fact that $\chi(G)$ is an integer. $\square$

Strengthenings of Reed's Conjecture of this sort first appeared in King's thesis [40], where King conjectured a Local Strengthening of Reed's Conjecture.

Conjecture 3.25 (King [40]). *For any graph G ,*

$$\chi(G) \leq \max_{v \in V(G)} \left\lceil \frac{d(v) + \omega(v) + 1}{2} \right\rceil.$$

This local strengthening has been proven for odd-hole-free graphs in [1].

Theorem 3.26 (Aravind, Karthick and Subramanian [1]). *For any odd-hole-free graph G , we have*

$$\chi(G) \leq \max_{v \in V(G)} \left\lfloor \frac{d(v) + \omega(v) + 1}{2} \right\rfloor.$$

Proof. Suppose not. Let G be a vertex-minimal counterexample. Given any vertex v of G , the degree of v and the size of the largest clique that contains v are non-increasing upon deletion of any other vertex. Using a similar reasoning as in the proof of Theorem 3.23, by minimality of G , G is critical. By Theorem 3.24, G cannot be a counterexample. $\square$

3.4 Further Questions

As corollaries of the Strong Perfect Graph Theorem, we know that the Borodin-Kostochka Conjecture and Reed's Conjecture hold for graphs that are both odd-hole-free and odd-antihole-free. In this chapter, we showed that we can discard the condition of being odd-antihole-free. This leads to the following question.

Question 3.27. Do the Borodin-Kostochka Conjecture and Reed's Conjecture hold for odd-antihole-free graphs?

Having proved Theorem 3.21, it is then natural to consider what happens when every vertex in a graph is in an odd hole. In particular, we have the following theorem and question.

Theorem 3.28 (Aravind, Karthick and Subramanian [1]). *If G is a minimum counterexample to Reed's Conjecture, then every vertex of G belongs to some odd hole in G .*

Question 3.29. Does Reed's Conjecture hold for graphs in which every vertex belongs to some odd-hole?

This leads to the next chapter, where we consider some classes of graphs where every vertex belongs to an odd hole and their superclasses. We have not found any counterexamples, but we are able to identify multiple families of graphs for which Reed's Conjecture holds with equality. We refer to them as tight examples.

Chapter 4

Tight Examples for Reed's Conjecture

In this chapter, we focus mainly on Reed's Conjecture, which, as we have seen, is a simple inequality that involves the chromatic number, the maximum degree and the clique number of a graph.

Conjecture 4.1 (Reed [56]). *For any graph G , we have*

$$\chi(G) \leq \left\lceil \frac{\Delta(G) + \omega(G) + 1}{2} \right\rceil.$$

In particular, we investigate the graphs G for which Reed's Conjecture holds with equality; that is,

$$\chi(G) = \left\lceil \frac{\Delta(G) + \omega(G) + 1}{2} \right\rceil.$$

Various classes of such graphs and some sporadic examples are found in the literature. In this chapter, we generalize some of the known examples. This provides us with some intuition as to why Reed's Conjecture might be true as well as directions to search for counterexamples should they exist.

In section 4.1, we provide an overview of the known examples for which Reed's Conjecture holds with equality. In section 4.2, we show that Reed's Conjecture holds for all powers of cycles and their complements. These results have been proven earlier by Serrato [60]. We provide a more streamlined proof and characterize all tight examples that are complements of cycle powers. In section 4.3, we analyze a more general class of graphs obtained

from powers and blow-ups of cycles. This includes most of the known tight examples. We also briefly discuss how these graphs relate to the Borodin-Kostochka Conjecture.

Conjecture 4.2 (Borodin and Kostochka [3]). *Given a graph G with $\Delta(G) \geq 9$, if $\chi(G) = \Delta(G)$, then $\omega(G) = \Delta(G)$.*

Then in section 4.4, we move on to another closely related class of graphs obtained from powers and blow-ups of paths, which gives rise to a new family of irregular tight examples. We also prove that both conjectures hold for this newly defined class of graphs. Section 4.5 outlines some questions that follow naturally from the results and discussions in this chapter.

4.1 Known Tight Examples

There are infinitely many tight examples for Reed's Conjecture. Two families of such examples are the two extreme cases of Brooks' Theorem, that is, the complete graphs K_t , for $t \in \mathbb{Z}_{\geq 1}$, which have $\Delta(K_t) + 1 = \chi(K_t) = \omega(K_t) = t$, and the odd cycles C , which have $\Delta(C) + 1 = \chi(C) = 3$ and $2 \leq \omega(C) \leq 3$. Most known tight examples are closely related to cycles. For example, Kostochka [48] gave another infinite family of tight examples, $G = C_5[K_t]$ for $t \in \mathbb{Z}_{\geq 1}$. The graph $G = C_5[K_t]$ is obtained by blowing up each vertex of C_5 into a K_t . Such graphs satisfy $\Delta(G) = 3t - 1$, $\chi(G) = \lceil \frac{5t}{2} \rceil$ and $\omega(G) = 2t$. In particular, the graph $C_5[K_t]$ with odd t shows that the ceiling operator in Reed's Conjecture is necessary. More generally, Haxell and Naserasr [36] showed that the class of graphs $G = C_{3r+2}^r[K_t]$, for $r, t \in \mathbb{Z}_{\geq 1}$, which has $\Delta(G) = (2r + 1)t - 1$, $\chi(G) = (r + 1)t + \lceil \frac{rt}{2} \rceil$ and $\omega(G) = (r + 1)t$, is an infinite family of tight examples. We will see that this is a subclass of the graphs discussed in Section 4.3.

There also exist tight examples that do not arise from cycles. For example, as noted by Reed [56], Chvatal's graph [15], which appeared 25 years before the conjecture was published, also shows that the ceiling operator is necessary. As shown in Figure 4.1a, Chvatal's graph is triangle-free, 4-regular and 4-chromatic. As remarked by Edwards and King [28], tight examples of Reed's Conjecture are usually vertex-transitive or, more generally, regular. However, tight examples are not always regular. For example, Grötzsch's graph [32] is an irregular tight example. As shown in Figure 4.1b, Grötzsch's graph is also triangle-free and 4-chromatic, but it is irregular and has maximum degree 5. More irregular tight examples are explored in Section 4.4.

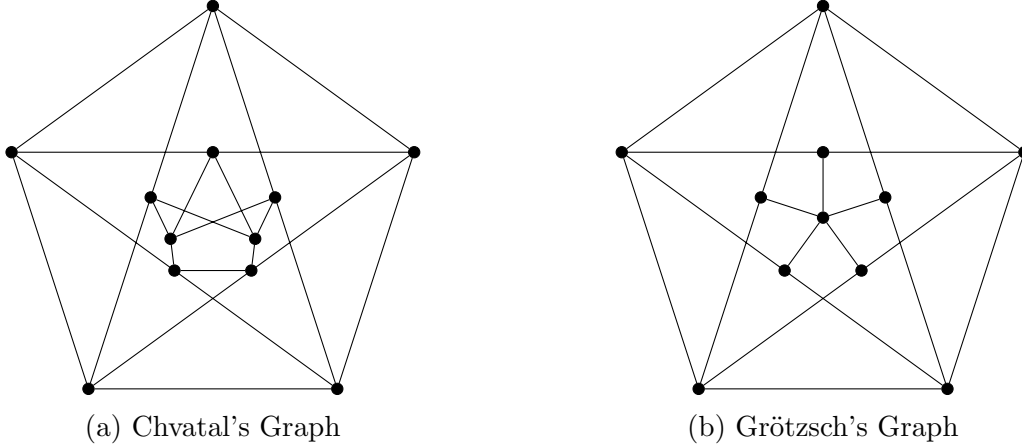


Figure 4.1: Two tight examples of Reed's Conjecture that do not arise from cycles

4.2 Powers of Cycles and their Complements

As we have seen in the previous chapter and as shown by Aravind, Karthick and Subramanian [1], minimum counterexamples to Reed's Conjecture would have the property that every vertex belongs to some odd hole. This motivates us to consider graphs in which every vertex is in an odd hole. One way to construct such graphs is to start with an odd cycle and consider its power and/or blow-up. More generally, we define the class of expanded cycles.

Definition 4.3. Let r, s and t be positive integers such that $s \geq 2r + 2$. The **expanded cycle (EC)** $G = C_s^r[K_t]$ is constructed as follows.

1. C_s is a cycle of length s . We denote its vertices by $v_1, v_2, \dots, v_s$ along the cycle. For the remainder of this section, all additions and subtractions in the subscripts are carried out in $\mathbb{Z}_s$.
2. C_s^r is the r th power of C_s . That is, for $i, j \in [s]$, the vertices v_i and v_j are adjacent if and only if $1 \leq |i - j| \leq r$. Note that $C_s^1 = C_s$.
3. $C_s^r[K_t]$ is the graph obtained by blowing up each vertex of C_s^r into a copy of K_t . For $i \in [s]$, we denote the copy of K_t corresponding to v_i by G_i .

We make some observations on expanded cycles. The notation in Definition 4.3 and in the following observations will be used throughout this section and the next.

1. The condition $s \geq 2r + 2$ ensures that C_s^r is not the complete graph. For example, v_1 and $v_{\lceil \frac{s+1}{2} \rceil}$ are non-adjacent since

$$\left\lceil \frac{s+1}{2} \right\rceil - 1 = \left\lceil \frac{s-1}{2} \right\rceil > \frac{s-2}{2} \geq r, \text{ and}$$

$$1 - \left\lceil \frac{s+1}{2} \right\rceil = s+1 - \left\lceil \frac{s+1}{2} \right\rceil = \left\lfloor \frac{s+1}{2} \right\rfloor > \frac{s-2}{2} \geq r.$$

Consequently, G is not complete either.

2. The vertex sets of G_i , for $i \in [s]$, form a partition of the vertex set of G . That is,

$$V(G) = \bigcup_{i \in [s]} V(G_i).$$

Hence

$$n := |V(G)| = \sum_{i=1}^s |V(G_i)| = st.$$

3. For distinct $i, j \in [s]$, if $1 \leq |i - j| \leq r$, then G_i is complete to G_j ; otherwise, $E(G_i, G_j) = \emptyset$. So for all $i \in [s]$ and every vertex $v \in V(G_i)$, v is adjacent to every vertex in $\bigcup_{i-r \leq j \leq i+r} V(G_j) \setminus \{v\}$. Hence, G is a $((2r+1)t - 1)$ -regular graph and

$$\Delta := \Delta(G) = (2r+1)t - 1 = (r+1)t + rt - 1.$$

4. By construction, $\bigcup_{1 \leq i \leq r+1} V(G_i)$ is a clique of size $(r+1)t$. Let $V' \subseteq V(G)$ be a set of $(r+1)t + 1$ vertices. Since every G_i has exactly t vertices for $i \in [s]$, there exist vertices $u, v \in V'$ and indices $i, j \in [s]$ such that $u \in V(G_i), v \in V(G_j)$ and $|i - j| > r$. So u and v are non-adjacent and V' is not a clique. Thus, the clique number of G is

$$\omega := \omega(G) = (r+1)t.$$

5. Since $E(G_i, G_j) = \emptyset$ for all $i, j \in [s]$ such that $|i - j| \geq r+1$, we have that the independence number is

$$\alpha := \alpha(G) = \left\lfloor \frac{s}{r+1} \right\rfloor \geq 2.$$

6. It then follows that the chromatic number satisfies

$$\chi := \chi(G) \geq \left\lceil \frac{n}{\alpha} \right\rceil = \left\lceil \frac{st}{\alpha} \right\rceil. \quad (4.1)$$

We will see in Lemma 4.18 that this basic lower bound on χ is consistent with Reed's Conjecture and hence does not lead to any immediate counterexamples. To identify the expanded cycles for which Reed's Conjecture holds with equality, it remains to determine the value of the chromatic number. For the rest of this section and the next one, we focus on particular subclasses of expanded cycles for which we are able to find a meaningful upper bound on the chromatic number.

In this section, we prove the following two theorems, which state that Reed's Conjecture is true for all powers of cycles and their complements.

Theorem 4.4. *Reed's Conjecture holds for powers of cycles.*

Theorem 4.5. *Reed's Conjecture holds for complements of powers of cycles.*

We first prove the following result, which will soon be used to bound the chromatic number of powers of cycles. In number theoretic terms, Lemma 4.6 states that for any positive integer k , the set $\{k, k+1, \dots, k + \lfloor \frac{k}{2} \rfloor\}$ generates the numerical semigroup $\{0, k, k+1, \dots, k + \lfloor \frac{k}{2} \rfloor\} \cup \mathbb{Z}_{\geq 2k}$.

Lemma 4.6. *Let k be a positive integer. Then for all integer $s \geq 2k$, there exist nonnegative integers $a_0, a_1, \dots, a_{\lfloor \frac{k}{2} \rfloor}$ such that*

$$\sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} a_j(k+j) = s.$$

Proof. By division algorithm, there exist nonnegative integers d and e such that $s = dk + e$ and $0 \leq e \leq k-1$. Since $s \geq 2k$, we have $d \geq 2$. Note that

$$0 \leq e \leq k-1 \leq \left\lfloor \frac{k}{2} \right\rfloor + \left\lfloor \frac{k}{2} \right\rfloor.$$

So there exist two integers $0 \leq j_1, j_2 \leq \lfloor \frac{k}{2} \rfloor$ such that $j_1 + j_2 = e$. Then

$$s = dk + e = (d-2)k + 2k + (j_1 + j_2) = (d-2)k + (k + j_1) + (k + j_2).$$

□

This lemma allows us to bound the chromatic number of powers of cycles. Intuitively, we use the result above to partition the vertices of G into subsets of consecutive vertices with sizes between $r + 1$ and $r + 1 + \lfloor \frac{r+1}{2} \rfloor$. In each subset, we colour the vertices along the cycle C_s using the same sequence of colours.

To better understand the seemingly cumbersome subscripts in Lemma 4.7, we first consider as an example a colouring of the graph $G = C_{19}^4$. Applying Lemma 4.6 with $k := r + 1 = 5$ and $s = 19$, we can write 19 as a sum of integers from $\{5, 6, 7\}$. One way to do so is $19 = 6 + 6 + 7$. (Note that this is not unique, for example, we also have $19 = 5 + 7 + 7$.) We can rewrite this with coefficients a_j for $j \in \{0, 1, 2\}$. That is, $a_0(5) + a_1(5 + 1) + a_2(5 + 2) = 19$, where $a_0 = 0, a_1 = 2$ and $a_2 = 1$. We then partition G into 3 segments, each with 6, 6 and 7 consecutive vertices. We keep track of the segments using two indices J and K , where J indicates that the length of the segment is $r + 1 + J$ and $K \in [a_J]$ sequentially numbers each segment of length J . So the three segments of G are denoted by $(1, 1)$, $(1, 2)$ and $(2, 1)$, as in Figure 4.2.

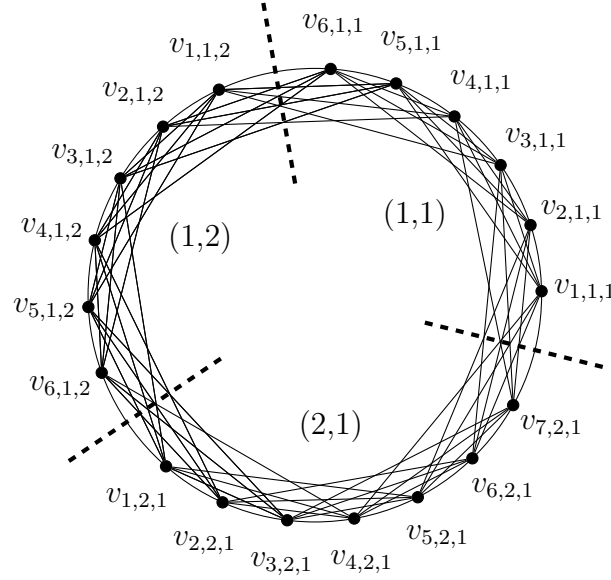


Figure 4.2: Colouring C_{19}^4

Within each segment (J, K) , we introduce another index $I \in [5 + J]$ to sequentially number the vertices. Every vertex in G is then relabeled using indices I, J, K , as in Figure 4.2. By construction, any two vertices with the same index I have distance at least 5 in the underlying 19-cycle and hence are non-adjacent. This allows us to construct a 7-colouring of G by assigning colour I to the vertex $v_{I,J,K}$.

The following lemma is a formalization of the procedure carried out above, generalized for all powers of cycles.

Lemma 4.7. *Let r and s be positive integers such that $s \geq 2r + 2$. Let $G = C_s^r$ be a cycle power, then $\chi(G) \leq r + 1 + \left\lceil \frac{r}{2} \right\rceil$.*

Proof. Define $R = \left\lfloor \frac{r+1}{2} \right\rfloor$. By Lemma 4.6, since $s \geq 2r + 2$, there exist nonnegative integers $a_0, a_1, \dots, a_R$ such that

$$\sum_{j=0}^R a_j(r + 1 + j) = s.$$

Our goal is to construct a colouring of G using colours $[r + 1 + R]$. We relabel the vertices as follows. For $i \in [s]$, there exists a unique $J \in [R]$ such that

$$\sum_{j=0}^{J-1} a_j(r + 1 + j) < i \leq \sum_{j=0}^J a_j(r + 1 + j).$$

By division algorithm, there exist unique positive integers $K \leq a_J$ and $I \leq r + 1 + J$ such that

$$i - \sum_{j=0}^{J-1} a_j(r + 1 + j) = (K - 1)(r + 1 + J) + I.$$

We relabel the vertex v_i as $v_{I,J,K}$. For any $I \in [r + 1 + R]$, if there exist two distinct vertices $v_{i_1} = v_{I,J_1,K_1}$ and $v_{i_2} = v_{I,J_2,K_2}$, then by construction,

$$|i_1 - i_2| \geq \min\{r + 1 + J_1, r + 1 + J_2\} \geq r + 1.$$

So v_{i_1} and v_{i_2} are non-adjacent. Hence we have the following colouring of G . For $J \in [R]$, $K \in [a_J]$ and $I = [r + 1 + J]$, if $v_{I,J,K}$ is a vertex in G , then assign colour I to $v_{I,J,K}$. We conclude that

$$\chi(G) \leq r + 1 + R = r + 1 + \left\lfloor \frac{r + 1}{2} \right\rfloor = r + 1 + \left\lceil \frac{r}{2} \right\rceil.$$

□

Theorem 4.4 follows immediately from this lemma.

Proof of Theorem 4.4. From the observations, we have $\Delta = 2r$ and $\omega = r + 1$. By Lemma 4.7, $\chi \leq r + 1 + \lceil \frac{r}{2} \rceil$. Therefore,

$$\chi \leq r + 1 + \lceil \frac{r}{2} \rceil = \lceil \frac{3r + 2}{2} \rceil = \lceil \frac{2r + (r + 1) + 1}{2} \rceil = \lceil \frac{\Delta + \omega + 1}{2} \rceil.$$

□

We now study the complement of a cycle power. Theorem 4.5 follows immediately from the following lemma.

Lemma 4.8. *Let r and s be positive integers such that $s \geq 2r + 2$ so that $G = C_s^r$ is a cycle power. Let $\overline{G}$ be the complement of G . Then $\overline{G}$ satisfies $\Delta(\overline{G}) = s - 2r - 1$, $\omega(\overline{G}) = \lfloor \frac{s}{r+1} \rfloor$ and $\chi(\overline{G}) = \lceil \frac{s}{r+1} \rceil$. Moreover, Reed's Conjecture holds for $\overline{G}$.*

Proof. Since G has s vertices and is a regular graph where each vertex has degree $2r$, we have that

$$\Delta(\overline{G}) = s - 2r - 1.$$

By properties of a complement graph, we have that

$$\omega(\overline{G}) = \alpha(G) = \lfloor \frac{s}{r+1} \rfloor \quad \text{and} \quad \alpha(\overline{G}) = \omega(G) = r + 1.$$

This gives the following lower bound on the chromatic number,

$$\chi(\overline{G}) \geq \left\lceil \frac{|V(\overline{G})|}{\alpha(\overline{G})} \right\rceil = \left\lceil \frac{s}{r+1} \right\rceil.$$

Since every $r+1$ consecutive vertices form a clique in G , we have that every $r+1$ consecutive vertices form an independent set in $\overline{G}$ and thus can be coloured the same. So $\chi(\overline{G})$ achieves the above lower bound; that is,

$$\chi(\overline{G}) = \left\lceil \frac{s}{r+1} \right\rceil.$$

To show that Reed's Conjecture holds for $\overline{G}$, we consider the following cases.

When $s = 2r + 2$, we have

$$\Delta(\overline{G}) = s - 2r - 1 = 1, \quad \omega(\overline{G}) = \left\lfloor \frac{s}{r+1} \right\rfloor = 2, \quad \chi(\overline{G}) = \left\lceil \frac{s}{r+1} \right\rceil = 2.$$

When $2r + 3 \leq s \leq 2r + 4$, we have

$$\Delta(\overline{G}) = s - 2r - 1 \geq 2, \quad \omega(\overline{G}) = \left\lfloor \frac{s}{r+1} \right\rfloor \geq 2, \quad \chi(\overline{G}) = \left\lceil \frac{s}{r+1} \right\rceil = 3.$$

It is straightforward to verify that Reed's Conjecture holds for the above cases.

It remains to consider the case $s \geq 2r + 5$. Note that the floor and ceiling operators have the following property,

$$\frac{s-r}{r+1} \leq \left\lfloor \frac{s}{r+1} \right\rfloor \leq \left\lceil \frac{s}{r+1} \right\rceil \leq \frac{s+r}{r+1}. \quad (4.2)$$

We have

$$s - \left\lfloor \frac{s}{r+1} \right\rfloor (r+1) \leq r \leq r(s - 2r - 4) = rs - 2r^2 - 4r.$$

Rearranging, we have

$$2s + 2r \leq sr + s + \left\lfloor \frac{s}{r+1} \right\rfloor (r+1) - 2r^2 - 2r.$$

Dividing both sides by $2(r+1)$, we have

$$\begin{aligned} \frac{s+r}{r+1} &\leq \frac{s}{2} + \frac{\left\lfloor \frac{s}{r+1} \right\rfloor}{2} - r \leq \left\lceil \frac{s + \left\lfloor \frac{s}{r+1} \right\rfloor - 2r - 1 + 1}{2} \right\rceil = \left\lceil \frac{\Delta(\overline{G}) + \omega(\overline{G}) + 1}{2} \right\rceil \\ \Rightarrow \chi(\overline{G}) = \left\lceil \frac{s}{r+1} \right\rceil &\leq \left\lceil \frac{\Delta(\overline{G}) + \omega(\overline{G}) + 1}{2} \right\rceil \quad \text{by Inequality 4.2.} \end{aligned}$$

Therefore, we conclude that Reed's Conjecture holds for complements of cycle powers. $\square$

The two results above, that is, Theorems 4.4 and 4.5, have been proven by Serrato [60] in 2014. The idea behind the proofs above are similar to what is presented in their work, but the number of cases was reduced by using Inequality 4.2.

Let $\overline{G}$ be as defined in Lemma 4.8. Then this lemma allows us to find all tight examples of Reed's Conjecture that are complements of powers of cycles. In this scenario, Reed's Conjecture with equality is equivalent to the following.

$$\begin{aligned} \chi(\overline{G}) &= \left\lceil \frac{\Delta(\overline{G}) + \omega(\overline{G}) + 1}{2} \right\rceil \\ \Leftrightarrow \left\lceil \frac{s}{r+1} \right\rceil &= \left\lceil \frac{s - 2r - 1 + \left\lfloor \frac{s}{r+1} \right\rfloor + 1}{2} \right\rceil = \left\lceil \frac{s + \left\lfloor \frac{s}{r+1} \right\rfloor}{2} \right\rceil - r. \end{aligned} \quad (4.3)$$

We consider the following cases.

First suppose that $s + \lfloor \frac{s}{r+1} \rfloor$ is even, then we have the following two subcases.

If $(r+1)|s$, then Equation 4.3 can be rewritten and simplified as

$$\frac{s}{r+1} = \frac{s + \frac{s}{r+1}}{2} - r \iff s = 2r + 2.$$

In fact, we see directly from the proof of Lemma 4.8 that when $s = 2r + 2$, $\overline{G}$ with

$$\Delta(\overline{G}) = 1, \quad \omega(\overline{G}) = 2, \quad \chi(\overline{G}) = 2$$

is a tight example for Reed's Conjecture. We note that for all $i \in [s]$, the vertex $v_i \in V(\overline{G})$ is only adjacent to the vertex v_{i+r+1} and vice versa. Hence $\overline{G}$ is a perfect matching.

Otherwise, assume $(r+1) \nmid s$. Then Equation 4.3 can be rewritten as

$$\begin{aligned} \left\lfloor \frac{s}{r+1} \right\rfloor &= \frac{s + \left\lceil \frac{s}{r+1} \right\rceil - 1}{2} - r \\ \iff rs + s - \left\lceil \frac{s}{r+1} \right\rceil (r+1) &= 2r^2 + 3r + 1 \\ \implies rs - r &\leq 2r^2 + 3r + 1 && \text{by Inequality 4.2} \\ \iff s &\leq 2r + 4 + \frac{1}{r} \\ \iff s &= \begin{cases} 2r + 4 & \text{for } r \geq 2 \\ 2r + 5 & \text{for } r = 1. \end{cases} \end{aligned}$$

That is, the graph $\overline{C}_7$ with

$$\Delta(\overline{C}_7) = 4, \quad \omega(\overline{C}_7) = 3, \quad \chi(\overline{C}_7) = 4$$

is a tight example. This is an odd antihole. Moreover, we obtain that the graphs $\overline{G}$ with $s = 2r + 4$ for $r \geq 2$ form another family of tight examples. They satisfy

$$\Delta(\overline{G}) = 3, \quad \omega(\overline{G}) = 2, \quad \chi(\overline{G}) = 3.$$

In such a graph $\overline{G}$, each vertex v_i , for $i \in [s]$, is adjacent to three vertices, v_{i+r+1} , v_{i+r+2} and v_{i+r+3} . Let us describe the structure of $\overline{G}$. Let $R := \langle r+1 \rangle$ be the subgroup of $\mathbb{Z}_s = \mathbb{Z}_{2r+4}$ generated by the element $r+1$. We prove the following result on the index of R in $\mathbb{Z}_s$.

Lemma 4.9. *Let $r \geq 2$ be a positive integer. Define $s = 2r + 4$. If r is odd, then there are two cosets of $\langle r + 1 \rangle$ in $\mathbb{Z}_s$, one containing all odd elements and the other containing all even elements. If r is even, then there is one coset of $\langle r + 1 \rangle$ in $\mathbb{Z}_s$; that is, $\langle r + 1 \rangle = \mathbb{Z}_s$.*

Proof. Let $p = \gcd(r + 1, s) = \gcd(r + 1, 2r + 4)$. Then there exist integers k and l such that $r + 1 = kp$ and $2r + 4 = lp$. So we have

$$\begin{aligned} lp &= 2r + 4 = 2(r + 1) + 2 = 2kp + 2 \\ \implies (l - 2k)p &= 2 \\ \implies p &\in \{1, 2\}. \end{aligned}$$

If r is odd, then $r + 1$ is even. So $\gcd(r + 1, 2r + 4) = 2$, which implies that $\langle r + 1 \rangle = \langle 2 \rangle$ and $1 + \langle r + 1 \rangle = 1 + \langle 2 \rangle$ are the only two cosets of $\langle r + 1 \rangle$ in $\mathbb{Z}_s$. If r is even, then $r + 1$ is odd. So $\gcd(r + 1, 2r + 4) = 1$, which implies that $\langle r + 1 \rangle = \mathbb{Z}_s$. $\square$

Using Lemma 4.9, we can now describe $\overline{G}$ based on the parity of r . First consider the case where r is even, then $R = \mathbb{Z}_s$. This means that by arranging the vertices in $\overline{G}$ in a way such that the vertex v_i , $i \in [s]$, is followed by v_{i+r+1} , we obtain a single s -cycle. Note that in $\mathbb{Z}_s$, $(i + r + 3) + r + 1 = i$, so it remains to address the edge $v_i v_{i+r+2}$. Since r is even, observe that

$$i + r + 2 = i + \frac{2r + 4}{2} + \frac{r}{2}s = i + \frac{s}{2}(r + 1).$$

That is, the vertex v_{i+r+2} is directly opposite to v_i in the cycle described above. So the edges $v_i v_{i+r+2}$ join each pair of opposite vertices in the cycle. The smallest example $\overline{C}_8^2$ is shown in Figure 4.3a.

Next we consider the case where r is odd, a similar arrangement results in two disjoint $\frac{s}{2}$ -cycles, one with vertex set $\{v_i | i \in R\} = \{v_i | i \in \mathbb{Z}_s \text{ is even}\}$ and the other with vertex set $\{v_i | i \in 1 + R\} = \{v_i | i \in \mathbb{Z}_s \text{ is odd}\}$. For all $i \in [s]$, since i and $i + r + 2$ have different parities, the edge $v_i v_{i+r+2}$ joins two vertices from different cycles. The smallest example $\overline{C}_{10}^3$, is shown in Figure 4.3b.

Now we consider the case where $s + \lfloor \frac{s}{r+1} \rfloor$ is odd. We again have two subcases.

If $(r + 1) | s$, then Equation 4.3 can be rewritten and simplified as

$$\frac{s}{r + 1} = \frac{s + \frac{s}{r+1} + 1}{2} - r \iff s = 2r + 1 - \frac{1}{r},$$

which does not lead to a valid solution.

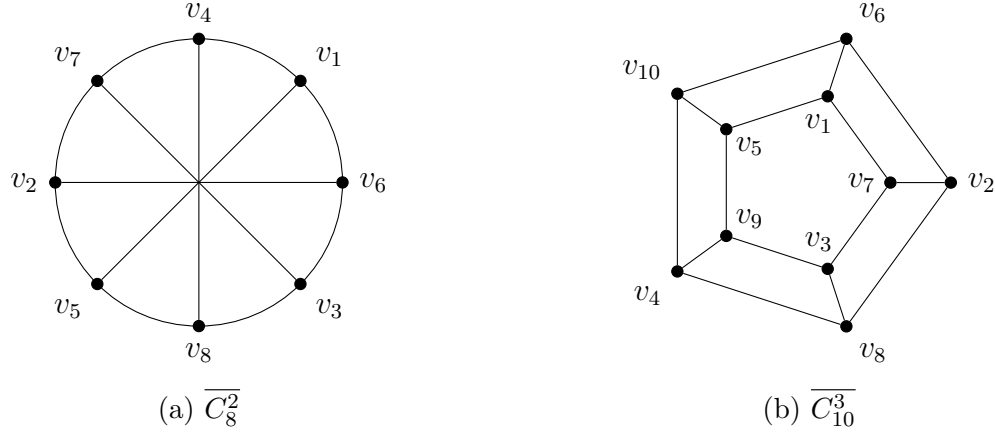


Figure 4.3: Small examples of $\overline{G}$ with $s = 2r + 4$

Lastly, assume $(r + 1) \nmid s$. Then Equation 4.3 can be rewritten as

$$\begin{aligned}
 \left\lfloor \frac{s}{r+1} \right\rfloor &= \frac{s + \left(\left\lceil \frac{s}{r+1} \right\rceil - 1 \right) + 1}{2} - r \\
 \iff rs + s - \left\lfloor \frac{s}{r+1} \right\rfloor (r+1) &= 2r^2 + 2r \\
 \implies rs - r &\leq 2r^2 + 2r && \text{by Inequality 4.2} \\
 \iff s = 2r + 3
 \end{aligned}$$

Hence the graphs $\overline{G}$ with $s = 2r + 3$, which satisfy

$$\Delta(\overline{G}) = 2, \quad \omega(\overline{G}) = 2, \quad \chi(\overline{G}) = 3$$

form another family of tight examples. In such a graph $\overline{G}$, each vertex v_i , for $i \in [s]$, is adjacent to two vertices v_{i+r+1} and $v_{i+r+2} = v_{i-r-1}$. We once again define the subgroup $R := \langle r + 1 \rangle$ of $\mathbb{Z}_s = \mathbb{Z}_{2r+3}$. By a similar argument as the proof of Lemma 4.9, we obtain that $\gcd(r + 1, s) = 1$ and hence $R = \mathbb{Z}_s$. Thus, by arranging the vertices in $\overline{G}$ in a way such that the vertex v_i , $i \in [s]$, is followed by v_{i+r+1} , we obtain a single s -cycle, where $s = 2r + 3$ is always odd. This implies that $\overline{G}$ with $s = 2r + 3$ is exactly the class of odd cycles with length at least 5.

We summarize all tight examples obtained above in Table 4.1. We note that $\overline{C_{2r+4}^r}$, for $r \geq 2$, is an interesting new result.

Table 4.1: Complements of cycle powers for which Reed's Conjecture holds with equality

Graph	r	Δ	χ	ω	Description
$\overline{C_{2r+2}^r}$	$\mathbb{Z}_{\geq 1}$	1	2	2	perfect matching
$\overline{C_{2r+3}^r}$	$\mathbb{Z}_{\geq 1}$	2	3	2	odd hole
$\overline{C_{2r+4}^r}$	$\mathbb{Z}_{\geq 2}, \text{ odd}$	3	3	2	two disjoint cycles with edges between corresponding vertices
	$\mathbb{Z}_{\geq 2}, \text{ even}$	3	3	2	one cycle with chords between opposite vertices
$\overline{C_7}$	1	4	4	3	odd antihole of size 7

4.3 General Expanded Cycles

We now look at some other subclasses of expanded cycles. In particular, we focus on the following.

Definition 4.10. Let $G = C_s^r[K_t]$ be an expanded cycle.

1. If $r = 1$, $t = 2$ and s is odd, then we say that G is of **type I**.
2. If $\alpha | (st)$, then we say that G is of **type II**.
3. Define $a = s \bmod \alpha$ and $b = t \bmod \alpha$. If $\alpha \nmid (st)$ and $\frac{a}{\alpha} + \frac{1}{b} > 1$, then we say that G is of **type III**.

Type I expanded cycles are exactly the graphs obtained by blowing up each vertex in odd cycles with length at least 5 into a K_2 . We note that type III expanded cycles are disjoint from both type I and type II. However, type I and type II expanded cycles are not disjoint, and the three subclasses defined above do not fully categorize all expanded cycles. For example, $C_5[K_2]$ is of both type I and type II, and $C_9[K_3]$ does not belong to any of the above subclasses.

In this section, we prove the following theorem, which shows that the definition of type I, II and III expanded cycles provide sufficient conditions so that the chromatic number is equal to the lower bound given in Inequality 4.1. This allows us to find more tight examples for Reed's Conjecture as well as some Δ -chromatic graphs, which are of interest for the Borodin-Kostochka Conjecture. It is worth remarking that when searching for tight or counter-examples of Reed's Conjecture, we strive to increase the chromatic number and

lower the maximum degree and the clique number. Thus, among the expanded cycles whose chromatic number exceeds this lower bound, we expect to find much more families of tight examples.

Theorem 4.11. *If G is an expanded cycle of type I, II or III, then $\chi(G) = \left\lceil \frac{|V(G)|}{\alpha(G)} \right\rceil$.*

The converse of the above theorem does not hold. For example, $C_9[K_3]$ is a counterexample as it satisfies

$$\Delta(C_9[K_3]) = 8, \quad \omega(C_9[K_3]) = 6, \quad \chi(C_9[K_3]) = 7, \quad \alpha(C_9[K_3]) = 4.$$

We now show that the lower bound $\chi \geq \left\lceil \frac{n}{\alpha} \right\rceil$ is achieved for type I, II and III expanded cycles by providing explicit constructions of colourings of G using at most $\left\lceil \frac{n}{\alpha} \right\rceil$ colours.

Lemma 4.12. *Let $G = C_s[K_2]$ be a type I expanded cycle, then $\chi = \left\lceil \frac{2s}{\alpha} \right\rceil = 5$.*

Proof. By definition of a type I expanded cycle, $r = 1$ and s is odd, so the independence number of G is

$$\alpha = \left\lfloor \frac{s}{r+1} \right\rfloor = \frac{s-1}{2}.$$

Note that $s \geq 2r + 2 = 4$. Since s is odd, we have $s \geq 5$ and hence

$$\left\lceil \frac{2s}{\alpha} \right\rceil = \left\lceil \frac{2s \cdot 2}{s-1} \right\rceil = \left\lceil \frac{4s-4+4}{s-1} \right\rceil = 4 + \left\lceil \frac{4}{s-1} \right\rceil = 5.$$

Thus, to prove the lemma, it suffices to construct a 5-colouring of G . Let $[5]$ be the set of colours. We colour G_1, G_2 and G_3 with $\{1, 5\}$, $\{2, 3\}$ and $\{4, 5\}$ respectively. For $i = 4, 5, \dots, s$, if i is even, we colour G_i with $\{1, 2\}$; and if i is odd, we colour G_i with $\{3, 4\}$. It is then straightforward to verify that for all $i \in [s]$, every vertex in G_i and G_{i+1} is distinctly coloured. Therefore, G has a 5-colouring. $\square$

Lemma 4.13. *Let $G = C_s^r[K_t]$ be a type II expanded cycle, that is $\alpha|(st)$, then $\chi = \left\lceil \frac{st}{\alpha} \right\rceil = \frac{st}{\alpha}$.*

Proof. Since $\alpha|(st)$, there exist positive integers k and l such that $\alpha = kl$ and $k|s, l|t$.

First we show that G is a subgraph of the expanded cycle $C_{st}^{r'} \left[K_{\frac{t}{l}} \right]$, where $r' := (r + 1)l - 1$. For all $i \in [s]$, partition G_i into l copies of $K_{\frac{t}{l}}$, denoted by $H_{(i-1)l+j}$, for $j \in [l]$. Take any two adjacent vertices $u, v \in V(G)$. Let $i, i' \in [s]$ be such that $u \in V(G_i)$ and

$v \in V(G_{i'})$. Then, $|i - i'| \leq r$ and there exist $j, j' \in [l]$, such that $u \in H_{(i-1)l+j}$ and $v \in H_{(i'-1)l+j'}$. Note that

$$|((i-1)l+j) - ((i'-1)l+j')| \leq |i - i'|l + |j - j'| \leq rl + (l-1) = (r+1)l - 1 = r'.$$

Since $s \geq 2r + 2$, we have that

$$sl \geq (2r+2)l - 2 + 2 = 2r' + 2.$$

Hence, we can view G as a subgraph of the expanded cycle $H = C_{sl}^{r'} \left[K_{\frac{t}{l}} \right]$. Then for $j \in [sl]$, H_j cyclically denotes the copies of $K_{\frac{t}{l}}$ in H obtained by blowing up the vertices in $C_{sl}^{r'}$.

Now we verify that $\chi(H) \leq \frac{st}{\alpha}$, and hence $\chi(G) \leq \frac{st}{\alpha}$. Since $\alpha | (sl)$, $\frac{sl}{\alpha}$ is a positive integer. For all $j \in [\frac{sl}{\alpha}]$, we colour H_j with disjoint sets of $\frac{t}{l}$ colours. Since

$$\alpha = \left\lfloor \frac{s}{r+1} \right\rfloor \leq \frac{s}{r+1} \implies s \geq \alpha(r+1),$$

we have that

$$\frac{sl}{\alpha} \geq \frac{\alpha(r+1)l}{\alpha} = (r+1)l > r' \text{ and } \frac{sl}{\alpha} \leq \frac{sl}{2}.$$

Hence for all $j \in [sl]$, $E(H_j, H_{j-\frac{sl}{\alpha}}) = \emptyset$. For all $j \in [sl] \setminus [\frac{sl}{\alpha}]$, we colour H_j with the same set of $\frac{t}{l}$ colours as $H_{j \bmod \frac{sl}{\alpha}}$. Then we obtain a colouring of H . The number of colours used is

$$\frac{sl}{\alpha} \frac{t}{l} = \frac{st}{\alpha} = \left\lceil \frac{st}{\alpha} \right\rceil.$$

Since G is a subgraph of H , we have that $\chi(G) \leq \left\lceil \frac{n}{\alpha} \right\rceil$. □

The following lemma illustrates the motivation behind the definition of type III expanded cycles. This is exactly the condition needed for us to obtain the desired chromatic number in Lemma 4.15.

Lemma 4.14. *Let a, b and α be positive integers such that $1 \leq a, b \leq \alpha - 1$, then*

$$\frac{a}{\alpha} + \frac{1}{b} > 1 \iff \left\lceil \frac{a}{\alpha} \right\rceil b = \left\lceil \frac{ab}{\alpha} \right\rceil.$$

Proof. Since $1 \leq a \leq \alpha - 1 < \alpha$, we have

$$\left\lceil \frac{a}{\alpha} \right\rceil b = b \quad \text{and} \quad \frac{ab}{\alpha} < b.$$

Thus,

$$\frac{a}{\alpha} + \frac{1}{b} > 1 \iff b - 1 < \frac{ab}{\alpha} < b \iff \left\lceil \frac{ab}{\alpha} \right\rceil = b = \left\lceil \frac{a}{\alpha} \right\rceil b.$$

□

We are now ready to find the chromatic number of a type III expanded cycle G . To do so, we partition G into a type II expanded cycle G_1 , whose chromatic number is known by Lemma 4.13, and a subgraph G_2 induced by the remaining vertices, whose chromatic number can be bounded using Lemma 4.14. The idea of the latter is similar to that in the proof of Lemma 4.7. That is, we partition G_2 into segments of certain lengths in order to identify independent sets. To make the subscripts in the proof of Lemma 4.15 easier to understand, we start with an example where G_1 is the empty graph to illustrate the main idea.

Consider the graph $G = C_{11}^2[K_2]$, which has parameters

$$r = 2, \quad s = 11, \quad t = 2, \quad \alpha = \left\lfloor \frac{s}{r+1} \right\rfloor = \left\lfloor \frac{11}{3} \right\rfloor = 3.$$

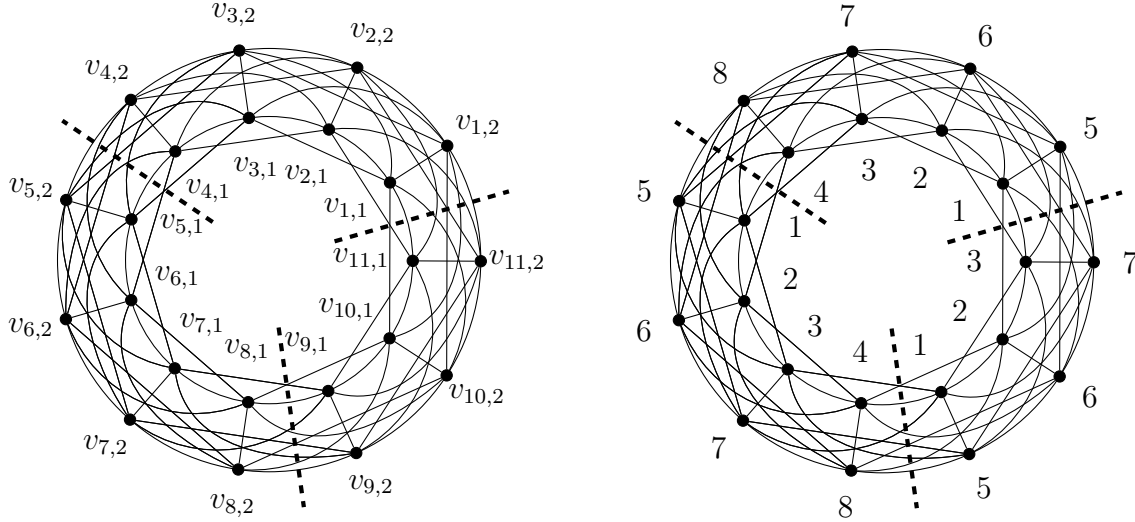
Note that $\alpha \nmid (st)$. Define $a = s \bmod \alpha = 2$ and $b = t \bmod \alpha = 2$. We have

$$\frac{a}{\alpha} + \frac{1}{b} = \frac{2}{3} + \frac{1}{2} > 1.$$

So G is indeed a type III expanded cycle.

For all $i \in [s]$, we denote the two vertices in G_i by $v_{i,1}$ and $v_{i,2}$, as in Figure 4.4a. Define $q = \frac{s-a}{\alpha} = 3$. We can then rewrite s as $s = \alpha q + a = a(q+1) + (\alpha-a)q = 2 \times 4 + 1 \times 3$. We partition G into 3 subgraphs, each containing 4, 4 and 3 consecutive copies of K_2 . Note that $q = 3 \geq r+1$, so by picking one vertex (if it exists) “in the same position” of each subgraph, we can partition G into qb independent sets of size α and b independent sets of size a as follows. For all $i \in [q] = \{1, 2, 3\}$ and $j \in [b] = \{1, 2\}$, $\{v_{i,j}, v_{i+4,j}, v_{i+2 \times 4,j}\}$ is an independent set of size $\alpha = 3$. For all $j \in [b] = \{1, 2\}$, $\{v_{4,j}, v_{8,j}\}$ is an independent set of size $a = 2$. Hence G can be partitioned into $qb + b = 3 \times 2 + 2 = 8$ independent sets. By assigning a distinct colour to each independent set described above, we obtain an 8-colouring of G , as in Figure 4.4b, where each number is a colour.

We now generalize this idea to all type III expanded cycles.



(a) The graph $C_{11}^2[K_2]$ with all vertices labelled

(b) An 8-colouring of $C_{11}^2[K_2]$

Figure 4.4: Colouring $C_{11}^2[K_2]$

Lemma 4.15. *Let $G = C_s^r[K_t]$ be a type III expanded cycle, then $\chi = \lceil \frac{st}{\alpha} \rceil$.*

Proof. Define $a = s \bmod \alpha$ and $b = t \bmod \alpha$. Since G is a type III expanded cycle, we have $\alpha \nmid (st)$ and $\frac{a}{\alpha} + \frac{1}{b} > 1$.

For all $i \in [s]$, take b vertices $v_{i,1}, v_{i,2}, \dots, v_{i,b}$ from G_i . Define $V_b = \{v_{i,j} | i \in [s], j \in [b]\}$. Then $G - V_b = C_s^r[K_{t-b}]$, where $\alpha | (t-b)$ by definition of b . Hence $\alpha | (s(t-b))$. By Lemma 4.13,

$$\chi(G - V_b) = \frac{s(t-b)}{\alpha}.$$

We now construct a colouring of $G[V_b]$. Since $a = s \bmod \alpha$, the number $q := \frac{s-a}{\alpha}$ is a nonnegative integer. We can rewrite s as $s = \alpha q + a$. Since $s \geq \alpha(r+1)$, we have $q \geq r+1$. Since $\alpha \nmid (st)$, we have $1 \leq a \leq \alpha - 1$. So

$$s = \alpha q + a \geq 2q + 1 \geq (r+1) + (q+1).$$

Thus, for all $i \in [s]$, $E(G_i, G_{i+q}) = \emptyset$ and $E(G_i, G_{i+q+1}) = \emptyset$. We can also rewrite s as $s = a(q+1) + (\alpha - a)q$. So for all $i \in [q]$ and $j \in [b]$,

$$V_{i,j} := \{v_{(m-1)(q+1)+i,j} | m \in [a]\} \cup \{v_{a(q+1)+(m-1)q+i,j} | m \in [\alpha - a]\}$$

is an independent set of size α .

For all $j \in [b]$,

$$V'_j := \{v_{m(q+1),j} | m \in [a]\}$$

is an independent set of size a .

Further note that

$$V_b = \left(\bigcup_{i \in [q], j \in [b]} V_{i,j} \right) \cup \left(\bigcup_{j \in [b]} V'_j \right).$$

So

$$\chi(G[V_b]) \leq qb + b = \frac{s-a}{\alpha}b + \left\lceil \frac{a}{\alpha} \right\rceil b = \frac{sb-ab}{\alpha} + \left\lceil \frac{ab}{\alpha} \right\rceil = \left\lceil \frac{sb}{\alpha} \right\rceil$$

where the second equality follows from Lemma 4.14.

Since the graph G can be partitioned into $G - V_b$ and $G[V_b]$, we have

$$\chi(G) \leq \chi(G - V_b) + \chi(G[V_b]) = \frac{s(t-b)}{\alpha} + \left\lceil \frac{sb}{\alpha} \right\rceil = \left\lceil \frac{st}{\alpha} \right\rceil.$$

□

Theorem 4.11 follows immediately from Lemmas 4.12, 4.13 and 4.15. Next we discuss how the graphs addressed by Theorem 4.11 relate to the Borodin-Kostochka Conjecture as well as Reed's Conjecture. We define $s' = s - \alpha(r+1)$, then $0 \leq s' \leq r$. Note that we have

$$\left\lceil \frac{n}{\alpha} \right\rceil = \left\lceil \frac{st}{\alpha} \right\rceil = \left\lceil \frac{(\alpha(r+1) + s')t}{\alpha} \right\rceil = (r+1)t + \left\lceil \frac{s't}{\alpha} \right\rceil. \quad (4.4)$$

We are now ready to show that the Borodin-Kostochka Conjecture is vacuously true for any expanded cycles with $\chi = \left\lceil \frac{n}{\alpha} \right\rceil$.

Lemma 4.16. *For any expanded cycle that satisfies $\chi = \left\lceil \frac{n}{\alpha} \right\rceil$, the Borodin-Kostochka Conjecture holds.*

Proof. Recall that $s' = s - \alpha(r+1) \leq r$. We prove this lemma by showing that there does not exist an expanded cycle with $\chi = \left\lceil \frac{n}{\alpha} \right\rceil$ and $\chi = \Delta \geq 9$. Suppose for contradiction that

there exists such an expanded cycle G , then using Equation 4.4,

$$\begin{aligned}\chi = \Delta &\iff (r+1)t + \left\lceil \frac{s't}{\alpha} \right\rceil = (r+1)t + rt - 1 \\ &\implies rt = \left\lceil \frac{s't}{\alpha} \right\rceil + 1 \leq \left\lceil \frac{rt}{\alpha} \right\rceil + 1 < \frac{rt}{\alpha} + 2 \\ &\implies rt < 2\frac{\alpha}{\alpha-1} \leq 4.\end{aligned}$$

That is, $rt \leq 3$, so $\Delta = (2r+1)t - 1 = 2rt + t - 1 \leq 3rt - 1 < 9$.

This contradicts the assumption that $\Delta \geq 9$. Therefore, such G does not exist, and the Borodin-Kostochka conjecture is vacuously true. $\square$

By computing χ and Δ for all expanded cycles with $rt \leq 3$, we find that the only expanded cycles that satisfy $\chi = \left\lceil \frac{n}{\alpha} \right\rceil = \Delta$ are as listed in Table 4.2.

Table 4.2: Expanded cycles for which $\chi = \left\lceil \frac{n}{\alpha} \right\rceil = \Delta$

Graph	Δ	χ	ω	α	a	b	Class
$C_{2\alpha}$	2	2	2	$\mathbb{Z}_{\geq 2}$			even cycle / type II EC
C_8^2	4	4	3	2			type II EC
$C_{3\alpha+2}^2$				$\mathbb{Z}_{\geq 3}$	2	1	type III EC
$C_{3\alpha+1}^2$				$\mathbb{Z}_{\geq 2}$	1	1	
$C_{2\alpha+1}[K_2]$	5	5	4	$\mathbb{Z}_{\geq 2}$			type I EC
C_{11}^3	6	6	4	2	1	1	type III EC
$C_5[K_3]$	8	8	6				

Theorem 4.11 and Lemma 4.16 immediately imply the following corollary.

Corollary 4.17. *The Borodin-Kostochka Conjecture holds for expanded cycles of type I, II or III.*

Similarly, we can show that Reed's Conjecture holds for the same class of graph.

Lemma 4.18. *For any expanded cycle that satisfies $\chi = \left\lceil \frac{n}{\alpha} \right\rceil$, Reed's Conjecture holds.*

Proof. Recall that $s' = s - \alpha(r + 1) \leq r$. Using Equation 4.4,

$$\begin{aligned}\chi &= (r + 1)t + \left\lceil \frac{s't}{\alpha} \right\rceil \\ &\leq (r + 1)t + \left\lceil \frac{rt}{2} \right\rceil = \left\lceil \frac{(r + 1)t + rt - 1 + (r + 1)t + 1}{2} \right\rceil = \left\lceil \frac{\Delta + \omega + 1}{2} \right\rceil.\end{aligned}\quad (4.5)$$

□

Theorem 4.11 and Lemma 4.18 immediately imply the following corollary.

Corollary 4.19. *Reed's Conjecture holds for expanded cycles of type I, II or III.*

To classify all expanded cycles with $\chi = \left\lceil \frac{st}{\alpha} \right\rceil$ that constitute tight examples for Reed's Conjecture, we first note that using $s' = s - \alpha(r + 1) \leq r$ and Inequality 4.5, Reed's Conjecture with equality can be rewritten as follows.

$$\begin{aligned}\left\lceil \frac{st}{\alpha} \right\rceil = \chi = \left\lceil \frac{\Delta + \omega + 1}{2} \right\rceil &\iff \left\lceil \frac{s't}{\alpha} \right\rceil = \left\lceil \frac{rt}{2} \right\rceil \\ &\implies \frac{rt}{2} \leq \left\lceil \frac{rt}{2} \right\rceil = \left\lceil \frac{s't}{\alpha} \right\rceil \leq \left\lceil \frac{rt}{\alpha} \right\rceil < \frac{rt}{\alpha} + 1 \\ &\implies \frac{\alpha - 2}{2}rt < \alpha.\end{aligned}$$

All expanded cycles with $\chi = \left\lceil \frac{n}{\alpha} \right\rceil$ for which Reed's Conjecture is tight are listed in Table 4.3. Among them, $C_{3r+2}^r[K_t]$ and $C_{2\alpha+1}$ are known examples as mentioned in Section 4.1. Even though types I, II and III do not cover all expanded cycles with $\chi = \left\lceil \frac{n}{\alpha} \right\rceil$, they contain all tight examples with $\chi = \left\lceil \frac{n}{\alpha} \right\rceil$.

It is worth mentioning that all expanded cycles are highly symmetric. They are regular graphs. Any permutation of vertices within G_i , for $i \in [r]$, and any cyclic permutation of the G_i 's along the underlying cycle is a graph isomorphism. To show that tight examples for Reed's Conjecture do not need to have such symmetry, we now consider a simple asymmetric example. This also serves as a motivation for the definition of expanded paths, which are explored in the next section.

Consider the graph C_7^2 , which is the smallest expanded cycle of the form $C_{3\alpha+1}^2$. We know from Table 4.3 that C_7^2 is a tight example for Reed's Conjecture. We label its vertices as $v_1, v_2, \dots, v_7$ along the cycle, as shown in Figure 4.5a. Remove the edges v_1v_6 , v_2v_4 and

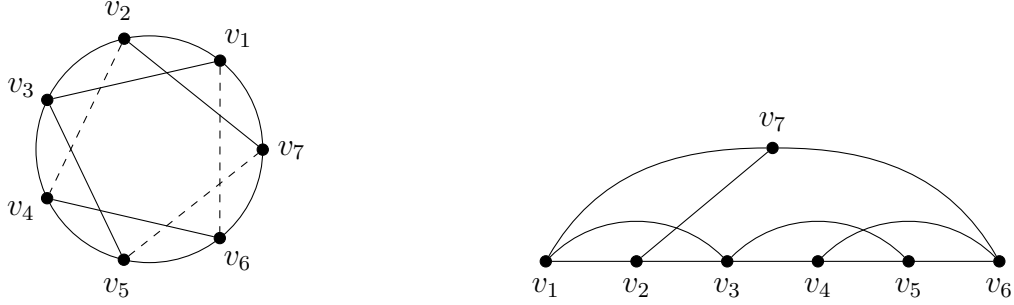
Table 4.3: Expanded cycles for which $\chi = \left\lceil \frac{n}{\alpha} \right\rceil = \left\lceil \frac{\Delta + \omega + 1}{2} \right\rceil$

Graph	Δ	χ	ω	α	a	b	Class
$C_{3r+2}^r[K_t]$							
t even	$(r+1)t + rt - 1$	$(r+1)t + \left\lceil \frac{rt}{2} \right\rceil$	$(r+1)t$	2			type II EC
t odd, r even							
t odd, r odd					1	1	type III EC
C_{3r+1}^r , r even	$2r$	$\frac{3r+2}{2}$	$r+1$	2	1	1	
$C_7[K_4]$	11	10	8	3	1	1	
$C_{11}^2[K_2]$	9	8	6	3	2	2	
C_{19}^4	8	7	5	3	1	1	
$C_{3\alpha+1}^2$	4	4	3	$\mathbb{Z}_{\geq 2}$	1	1	
$C_{3\alpha+2}^2$				$\mathbb{Z}_{\geq 3}$	2	1	
$C_{2\alpha+1}$	2	3	2	$\mathbb{Z}_{\geq 2}$	1	1	
$C_{2\alpha+1}[K_2]$	5	5	4	$\mathbb{Z}_{\geq 2}$			type I EC

v_5v_7 , and call the resulting graph H . We now show that H is a 4-critical subgraph of C_7^2 . It is straightforward to verify that any proper subgraph of H is 3-colourable. So it suffices to show that $\chi(H) = 4$. Suppose for contradiction that H is 3-colourable using colours $\{1, 2, 3\}$. We colour the vertices $v_1, v_2, \dots, v_6$ from left to right as drawn in Figure 4.5b. Without loss of generality, the vertices v_1, v_2 and v_3 , which form a 3-clique, have colours 1, 2 and 3 respectively. Then v_4 and v_5 have colours 1 and 2, in either order, since they are adjacent to each other and to v_3 . This forces v_6 to be coloured by 3. Since v_7 is adjacent to v_1, v_2 and v_6 , which are coloured 1, 2 and 3 respectively, there is no colour left for v_7 . This contradicts the assumption that H is 3-colourable. So we see that

$$\left\lceil \frac{\Delta(H) + \omega(H) + 1}{2} \right\rceil = \left\lceil \frac{4 + 3 + 1}{2} \right\rceil = 4 = \chi(H).$$

Hence H is also a tight example for Reed's Conjecture. Taking the perspective as in Figure 4.5b allows us to make the following key observation. Up to permutation of the colours, in any 3-colouring of $H - v_7$, the colours of v_1, v_2 and v_6 are fixed and they each have a different colour. Thus, by connecting these three vertices to another vertex v_7 , we increase the chromatic number by 1. Using this observation, we construct a new class of



(a) Remove the dashed edges in C_7^2 to obtain H

(b) Another way of drawing H

Figure 4.5: A 4-critical subgraph of C_7^2

graphs for which Reed's Conjecture holds with equality by first constructing a graph that has a unique minimum colouring and then introducing an extra vertex that has a neighbour in each of the existing colour classes.

4.4 Expanded Paths

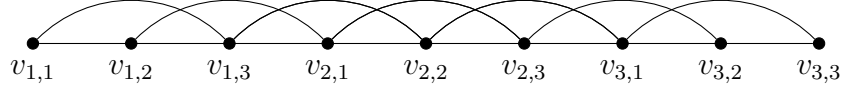
As motivated by the 4-critical subgraph of C_7^2 at the end of the previous section, in this section, we consider another class of graphs, which will be referred to as expanded paths. Each expanded path is in fact a class of graphs for which the parameters that we are concerned with share the same properties. For concreteness, prior to stating the definitions, we give the construction of one possible graph G in the class of expanded path P_9^{2+} .

We start with P_9^2 , which is the square of the path P_9 . That is, two distinct vertices are adjacent in P_9^2 if and only if they have distance at most 2 in P_9 . We label the vertices from left to right as $v_{1,1}, v_{1,2}, v_{1,3}, v_{2,1}, v_{2,2}, v_{2,3}, v_{3,1}, v_{3,2}, v_{3,3}$, as shown in Figure 4.6a. Note that we have

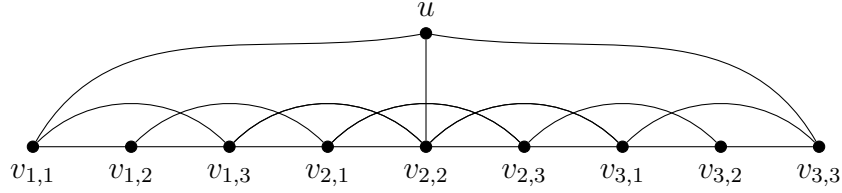
$$\Delta(P_9^2) = 4, \quad \omega(P_9^2) = 3, \quad \chi(P_9^2) = 3.$$

Since $v_{1,1}, v_{1,2}, v_{1,3}$ form a 3-clique and 9 is a multiple of 3, it is straightforward to verify by colouring the vertices from left to right that up to permutations of colours, P_9^2 has a unique 3-colouring, where the second index of each vertex is its colour.

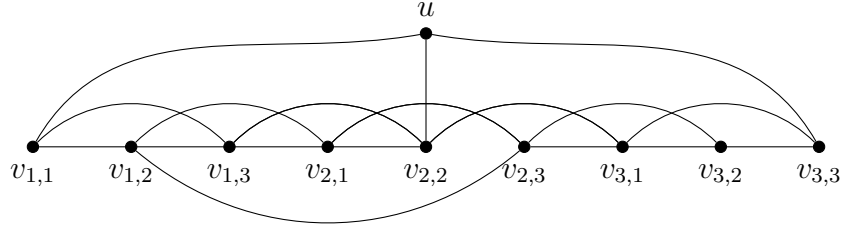
To generate tight examples or potential counterexamples for Reed's Conjecture, we aim to increase the chromatic number without increasing the maximum degree or the clique number. To do so, we first introduce a new vertex u and connect it to one vertex from



(a) A drawing of P_9^2 . Up to permutation of colours, P_9^2 has a unique colouring where every vertex is coloured by its second index.



(b) Add a vertex u and connect it to one vertex from each colour class.



(c) One construction of the expanded path P_9^{2+}

Figure 4.6: An example of constructing P_9^{2+} from P_9^2

each colour class so as to increase the chromatic number to 4. Note that to keep the clique number constant, we need to ensure that the neighbours of u are not consecutive vertices, that is, they do not form a clique in P_9^2 . For example, we may connect u to vertices $v_{1,1}$, $v_{2,2}$ and $v_{3,3}$ to obtain Figure 4.6b. The chromatic number, clique number and maximum degree of this graph are 4, 3 and 5, respectively.

The only undesired consequence of this operation is that the degree of $v_{2,2}$ becomes 5, which is higher than $\Delta(P_9^2) = 4$. We would like to bring the degree of $v_{2,2}$ back down to 4 while making sure that if u is removed, the remaining subgraph still has the same unique 3-colouring. We further note that the degree of $v_{1,2}$ is 3, which is strictly less than 4. So we remove the edge $v_{2,2}v_{2,3}$ and replace it by a new edge $v_{1,2}v_{2,3}$, as in Figure 4.6c. Since $v_{1,2}$ has the same colour as $v_{2,2}$, and is also on the left of $v_{2,3}$, $v_{2,3}$ is forced to have the same

colour after this edge replacement. We call the resulting graph G , and G is an example of the expanded path P_9^{2+} . Note that G has parameters

$$\Delta(G) = 4 = \Delta(P_9^2), \quad \omega(G) = 3 = \omega(P_9^2), \quad \chi(G) = 4 > \chi(P_9^2),$$

as desired.

We first generalize and formalize the notion of a unique colouring, which helps justify the notation used in the rest of this section.

Lemma 4.20. *Let $\alpha \geq 2$ and r be positive integers. Let $G = P_{\alpha(r+1)}^r$ be a path power, that is, any two distinct vertices in G are adjacent if and only if their distance in $P_{\alpha(r+1)}$ is at most r . Then $\chi(G) = r + 1$. Moreover, up to relabeling of the colors, G has a unique $(r + 1)$ -colouring.*

Proof. We denote the vertices by $v_{1,1}, v_{1,2}, \dots, v_{1,r+1}, v_{2,1}, v_{2,2}, \dots, v_{2,r+1}, \dots, v_{\alpha,1}, v_{\alpha,2}, \dots, v_{\alpha,r+1}$ along the path. We can interpret this as partitioning G into α segments of length $r + 1$. Then the first index numbers the segments, and the second index numbers each vertex within its respective segment.

By definition of a path power, $\{v_{1,j} : j \in [r + 1]\}$ is an $(r + 1)$ -clique. So $\chi(G) \geq r + 1$. We now construct an $(r + 1)$ -colouring of G using colours $[r + 1]$. Without loss of generality, we may assume that for all $j \in [r + 1]$, the vertex $v_{1,j}$ has colour j . We now show that for all $i \in [\alpha]$ and for all $j \in [r + 1]$, the vertex $v_{i,j}$ has colour j . We proceed by induction on $i(r + 1) + j$, that is, “from left to right” along the path. The base cases with $i = 1$ hold. For any $i = 2, 3, \dots, \alpha$ and $j \in [r + 1]$, we have

$$\{v_{i-1,j'} | j' = j + 1, j + 2, \dots, r + 1\} \cup \{v_{i,j'} | j' = 1, 2, \dots, j - 1\} \subseteq N(v_{i,j}),$$

and

$$\{v_{i',j} | i' = 1, 2, \dots, i - 1\} \cap N(v_{i,j}) = \emptyset.$$

By induction, the only color available for $v_{i,j}$ is the colour j . Therefore, the colouring we constructed is unique and $\chi(G) = r + 1$. $\square$

We now define expanded paths by providing the general algorithm for constructing such graphs. Note that all complications are captured in Step 3, which is designed so that the maximum degree and the clique number both stay constant after the extra vertex is introduced. This is later proven in Corollaries 4.28 and 4.29.

Definition 4.21. Let r, s, t be positive integers such that $s = \alpha(r + 1)$ for some integer $\alpha \geq 2$. We define an **expanded path (EP)** $G = P_s^{r+}[K_t]$ to be the class of graphs that can be constructed as follows.

1. P_s is a path of length s . As in Lemma 4.20, we denote the vertices by $v_{1,1}, v_{1,2}, \dots, v_{1,r+1}, v_{2,1}, v_{2,2}, \dots, v_{2,r+1}, \dots, v_{\alpha,1}, v_{\alpha,2}, \dots, v_{\alpha,r+1}$ along the path. Note that the second index of each vertex may be interpreted as its colour. We also use the lexicographic order of this sequence to define a total order on $V(P_s)$. That is, for all $i, i' \in [\alpha]$ and $j, j' \in [r + 1]$, we have $v_{i,j} \leq v_{i',j'}$ if either $i = i'$ and $j \leq j'$ or $i < i'$. Note that this is also the natural ordering along the path with the vertex on the left being the smallest.
2. P_s^r is the r th power of P_s . That is, for $i, i' \in [\alpha]$ and $j, j' \in [r + 1]$, $v_{i,j}$ and $v_{i',j'}$ are adjacent if and only if

$$1 \leq |((i - 1)(r + 1) + j) - ((i' - 1)(r + 1) + j')| \leq r.$$

Note that $P_s^1 = P_s$.

3. We construct P_s^{r+} as follows. We first add a new vertex u to P_s^r . Then for each colour $j \in [r]$, we pick one vertex from the colour class $\{v_{i,j} | i \in [\alpha]\}$, such that the r vertices chosen are not all in the “rightmost” clique $\{v_{\alpha,j} | j \in [r + 1]\}$. Together with the vertex $v_{\alpha,r+1}$, these $r + 1$ vertices constitute the neighbourhood of the new vertex u . More formally, we define $i_{r+1} = \alpha$ and pick some $i_j \in [\alpha]$ for all $j \in [r]$ such that

$$|\{i_j : j \in [r + 1]\}| \geq 2.$$

We add an edge joining u and $v_{i_j,j}$ for all $j \in [r + 1]$. Lastly, for all $j \in [r]$, we delete the edge $v_{i_j,j}v_{i_j,j+1}$ and add the edge $v_{1,j}v_{i_j,j+1}$. Note that the edge is unchanged if $i_j = 1$.

4. $P_s^{r+}[K_t]$ is the graph obtained by blowing up each vertex of P_s^{r+} into a copy of K_t . For $i \in [\alpha]$ and $j \in [r + 1]$, we denote the copy of K_t corresponding to $v_{i,j}$ by $G_{i,j}$. We denote the copy of K_t corresponding to u by G_0 .

Expanded paths and expanded cycles are closely related. For example, we have the following equivalence when $r = 1$.

Lemma 4.22. $P_{2\alpha}^+[K_t] = C_{2\alpha+1}[K_t]$.

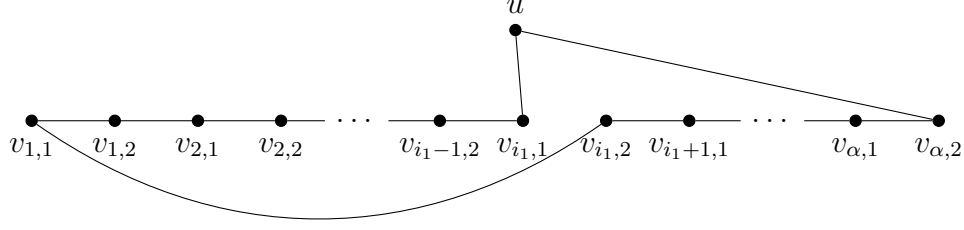


Figure 4.7: $P_{2\alpha}^+ = C_{2\alpha+1}$

Proof. It suffices to show that $P_{2\alpha}^+ = C_{2\alpha+1}$. Consider $P_{2\alpha}^+$, as shown in Figure 4.7. We have that $v_{1,1}, v_{i_1,2}, v_{i_1+1,1}, \dots, v_{\alpha,1}, v_{\alpha,2}, u, v_{i_1,1}, v_{i_1-1,2}, \dots, v_{1,2}, v_{1,1}$ is an odd cycle of length $2\alpha + 1$. $\square$

This equivalence turns out to be useful when we prove the following two main theorems of this section, which state that both the Borodin-Kostochka Conjecture and Reed's Conjecture hold for expanded paths.

Theorem 4.23. *The Borodin-Kostochka Conjecture holds for all expanded paths.*

Theorem 4.24. *Reed's Conjecture holds for all expanded paths.*

We make the following observations on expanded paths. The notation in Definition 4.21 and in the following observations will be used throughout this section.

1. As previously mentioned, each expanded path $P_s^{r+}[K_t]$ is a class of graphs. Whenever needed, we can specify a unique graph by fixing the value of i_j for each $j \in [r]$. Every i_j can take any value in $[\alpha]$. Since $i_{r+1} = \alpha$ and we require $|\{i_j : j \in [r+1]\}| \geq 2$, we cannot have $i_1 = i_2 = \dots = i_r = \alpha$. Therefore, each expanded path $P_s^{r+}[K_t]$ contains exactly $\alpha^r - 1$ graphs.
2. $P_{\alpha(r+1)}^r$ can be viewed as the expanded cycle $C_{\alpha(r+1)+1}^r$ with one vertex removed. Thus, using the observations in Section 4.2 we have

$$\begin{aligned}\Delta(P_{\alpha(r+1)}^r) &= 2r \\ \omega(P_{\alpha(r+1)}^r) &= r + 1 \\ \alpha(P_{\alpha(r+1)}^r) &= \alpha.\end{aligned}$$

3. The vertex sets of G_0 and $G_{i,j}$, for $i \in [\alpha]$ and $j \in [r+1]$, form a partition of the vertex set of G . That is,

$$V(G) = V(G_0) \cup \left(\bigcup_{i \in [\alpha], j \in [r+1]} V(G_{i,j}) \right).$$

Hence

$$n := |V(G)| = |V(G_0)| + \sum_{i=1}^{\alpha} \sum_{j=1}^{r+1} |V(G_{i,j})| = (\alpha(r+1) + 1)t.$$

4. By the construction of P_s^{r+} from P_s^r , no edge is added between any two vertices in $\{v_{i,r+1} : i \in [\alpha]\}$. Since $\{v_{i,r+1} : i \in [\alpha]\}$ is an independent set in P_s^r , it remains an independent set in P_s^{r+} . Hence by picking one vertex from $G_{i,r+1}$ for each $i \in [\alpha]$, we obtain an independent set of size α in G . So

$$\alpha(G) \geq \alpha.$$

Equality does not always hold. For example, consider the expanded path P_{12}^{3+} with $\alpha = 3, r = 3$ and $i_1 = 1, i_2 = 2, i_3 = i_4 = 3$, as shown in Figure 4.8. The set $\{v_{1,1}, v_{2,2}, v_{2,3}, v_{3,3}\}$ is independent and has size 4, which is strictly larger than $\alpha = 3$.

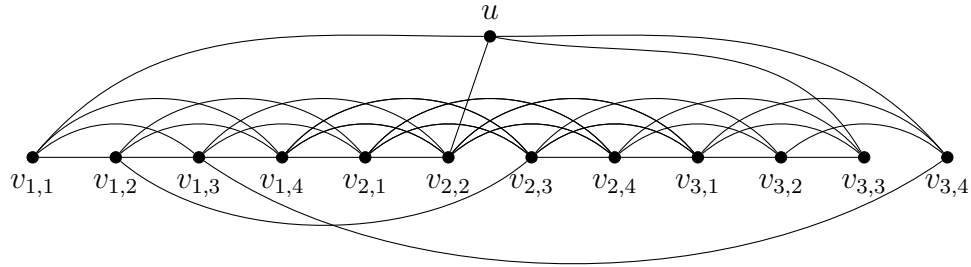


Figure 4.8: Expanded path P_{12}^{3+} with $\alpha = 3, r = 3$ and $i_1 = 1, i_2 = 2, i_3 = i_4 = 3$.

In this section, we mainly study the case $t = 1$. So we make the following definition.

Definition 4.25. Let $G = P_s^{r+}[K_t]$ be an expanded path. If $t = 1$, then we say that G is **simple**.

We primarily focus on the simple expanded paths, that is $G = P_{\alpha(r+1)}^r{}^+$, since we can determine the chromatic number of such graphs exactly. For the non-simple expanded paths, we provide upper and lower bounds on the chromatic number.

In the following lemmas, we show that the maximum degree and the clique number of a simple expanded path $P_s^r{}^+$ is the same as those of P_s^r . The values of these parameters for a general expanded path then follow as an immediate corollary.

Lemma 4.26. *Let $G = P_{\alpha(r+1)}^r{}^+$ be a simple expanded path, then $\Delta := \Delta(G) = 2r$. Moreover, every vertex in $G - u$ has at most r neighbours that are smaller in the lexicographic order.*

Proof. First note that u has $r + 1 \leq 2r$ neighbours by construction. So it remains to consider the vertices in $G - u$. Every vertex in P_s^r has degree at most $2r$ and has at most r smaller neighbours. It suffices to prove that Step 3 in Definition 4.21 preserves this property. Consider the smallest (i.e. leftmost) $r+1$ vertices and the largest (i.e. rightmost) vertex $v_{\alpha, r+1}$. Note that for $j \in [r]$,

$$N_{P_s^r}(v_{1,j}) = \{v_{1,j'} | j' \in [r+1] \setminus \{j\}\} \cup \{v_{2,j'} | j' \in [j-1]\} \implies d_{P_s^r}(v_{1,j}) = r + (j-1) \leq 2r-1.$$

Similarly,

$$\begin{aligned} N_{P_s^r}(v_{1,r+1}) &= \{v_{i,j} | i \in \{1, 2\}, j \in [r]\} \implies d_{P_s^r}(v_{1,r+1}) = 2r, \\ N_{P_s^r}(v_{\alpha, r+1}) &= \{v_{\alpha, j} | j \in [r]\} \implies d_{P_s^r}(v_{\alpha, r+1}) = r \leq 2r-1. \end{aligned}$$

Following Step 3 in Definition 4.21, we first add a vertex u to P_s^r and connect it to vertices $v_{i,j}$ for $j \in [r+1]$. We call this intermediate graph G' . We have that for all $i \in [\alpha]$ and $j \in [r+1]$,

$$d_{G'}(v_{i,j}) = \begin{cases} d_{P_s^r}(v_{i,j}) + 1 & \text{if } i = i_j \\ d_{P_s^r}(v_{i,j}) & \text{otherwise.} \end{cases}$$

To obtain G , for every $j \in [r]$, we remove the edge $v_{i_j, j} v_{i_j, j+1}$ and add the edge $v_{1,j} v_{i_j, j+1}$. If $i_j = 1$, this operation does not change the graph. Otherwise, there is no net change in the degree of $v_{i_j, j+1}$, but the degree of $v_{i_j, j}$ is decreased by 1 and the degree of $v_{1,j}$ is increased by 1. So we have that for all $i \in [\alpha]$ and $j \in [r+1]$,

$$d_G(v_{i,j}) = \begin{cases} d_{G'}(v_{1,j}) + 1 & \text{if } i = 1, j \neq r+1 \text{ and } i_j \neq 1 \\ d_{G'}(v_{i_j, j}) - 1 & \text{if } i = i_j, j \neq r+1 \text{ and } i_j \neq 1 \\ d_{G'}(v_{i,j}) & \text{otherwise} \end{cases}$$

$$\implies d_G(v_{i,j}) \leq \begin{cases} d_{P_s^r}(v_{1,j}) + 1 & \text{if } i = 1 \text{ and } j \neq r + 1 \\ d_{P_s^r}(v_{\alpha,r+1}) + 1 & \text{if } i = \alpha \text{ and } j = r + 1 \\ d_{P_s^r}(v_{i,j}) & \text{otherwise} \end{cases} \leq 2r.$$

So $\Delta(G) \leq 2r$. Note that no vertex has a net increase in the number of smaller neighbours.

Additionally, we have $d_G(v_{1,r+1}) = d_{P_s^r}(v_{1,r+1}) = 2r$. Therefore, $\Delta(G) = 2r$. $\square$

Lemma 4.27. *Let $G = P_{\alpha(r+1)}^r +$ be a simple expanded path, then $\omega := \omega(G) = r + 1$.*

Proof. It suffices to prove that Step 3 in Definition 4.21 does not affect the clique number. We note that for distinct indices $j, j' \in [r + 1]$, $v_{1,j}$ and $v_{1,j'}$ are adjacent in P_s^r . The edge $v_{1,j}v_{1,j'}$ is either left intact or deleted and readded in P_s^{r+} . Thus, the vertices $\{v_{1,j} | j \in [r + 1]\}$ still form a clique of size $r + 1$ in G . So $\omega \geq r + 1$.

Next we show that $\omega \leq r + 1$ by considering the following 2 cases. If there is a largest clique V' in G that does not contain u , then there exists a largest vertex $v \in V'$. We know from Lemma 4.26 that v has at most r neighbour smaller than itself, so $|V'| \leq r + 1$. Otherwise, a largest clique V' in G contains u . By construction, u has $r + 1$ neighbours and there exists some $j \in [r]$ such that $i_j \neq \alpha$. Note that $v_{i_j,j}, v_{\alpha,r+1} \in N(u)$ and they are non-adjacent. So $|V'| \leq r + 1$. $\square$

We can now determine the maximum degree and the clique number of any expanded path.

Corollary 4.28. *Let $G = P_{\alpha(r+1)}^r + [K_t]$ be an expanded path, then $\Delta := \Delta(G) = (2r + 1)t - 1$.*

Proof. For any two distinct $G', G'' \in \{G_0\} \cup \{G_{i,j} | i \in [\alpha], j \in [r + 1]\}$, obtained by blowing up vertices $v', v'' \in V(P_s^{r+})$ respectively, we have that G' is complete to G'' if v' and v'' are adjacent; and $E(G', G'') = \emptyset$ if v' and v'' are non-adjacent. Using Lemma 4.26, we have

$$\Delta = \Delta(P_s^{r+})t + (t - 1) = 2rt + t - 1 = (2r + 1)t - 1.$$

$\square$

Corollary 4.29. *Let $G = P_{\alpha(r+1)}^r + [K_t]$ be an expanded path, then $\omega := \omega(G) = (r + 1)t$.*

Proof. Similar to Corollary 4.28, using Lemma 4.27, we have

$$\omega = \omega(P_s^{r+})t = (r+1)t.$$

□

From Corollary 4.29, we obtain a lower bound for the chromatic number of G , that is,

$$\chi := \chi(G) \geq \omega = (r+1)t.$$

To upper bound the chromatic number of the expanded path G , we first consider the chromatic number of $G - V(G_0)$, and we show that it has a unique minimum colouring in the following sense. Note that this is a generalization of Lemma 4.20.

Lemma 4.30. *Let $G = P_s^{r+}[K_t]$ be an expanded path, then $\chi(G - V(G_0)) = (r+1)t$. Up to relabeling of the colors and up to swapping colours within each subgraph $G_{i,j}$, for $i \in [\alpha]$ and $j \in [r+1]$, the subgraph $G - V(G_0)$ has a unique $(r+1)t$ -colouring.*

Proof. We know from the proof of Lemma 4.27 that $\cup_{j \in [r+1]} V(G_{1,j})$ is a $(r+1)t$ -clique. For all $j \in [r+1]$, let C_j be disjoint sets of size t , so that $C = \cup_{j \in [r+1]} C_j$ is a set of $(r+1)t$ colours. We colour the vertices in $G_{1,j}$ with the colours from C_j . For $i \in [\alpha]$ and $j \in [r+1]$, we show by induction on $i(r+1) + j$ that we can only colour $G_{i,j}$ with C_j . The base cases with $i = 1$ hold. For any $i = 2, 3, \dots, \alpha$ and $j \in [r+1]$, if $j \geq 2$ and $i_{j-1} = i$, then $v_{i,j-1}$ is a neighbour of u in P_s^{r+} , and hence the edge $v_{i,j-1}v_{i,j}$ has been replaced by the edge $v_{1,j-1}v_{i,j}$. So

$$V(G_{1,j-1}) \cup \left(\bigcup_{j'=j+1}^{r+1} V(G_{i-1,j'}) \right) \cup \left(\bigcup_{j'=1}^{j-2} V(G_{i,j'}) \right) \subseteq N(V(G_{i,j})).$$

Otherwise, we have $j = 1$ or $i_{j-1} \neq i$. Then

$$\left(\bigcup_{j'=j+1}^{r+1} V(G_{i-1,j'}) \right) \cup \left(\bigcup_{j'=1}^{j-1} V(G_{i,j'}) \right) \subseteq N(V(G_{i,j})).$$

In either case, by induction, we have that the only colors available for $G_{i,j}$ are C_j . Therefore, the colouring we have constructed is unique, and $\chi(G - V(G_0)) = (r+1)t$. □

We can now find upper and lower bounds on the chromatic number of expanded paths. Note that when the expanded path is simple, the lower and upper bounds agree.

Corollary 4.31. *Let $G = P_s^{r+}[K_t]$ be an expanded path, then*

$$(r+1)t+1 \leq \chi \leq (r+2)t.$$

Proof. The upper bound follows immediately from Lemma 4.30, since

$$\chi \leq \chi(G - V(G_0)) + \chi(G_0) = (r+1)t + t = (r+2)t.$$

To show the lower bound, suppose for contradiction that G is $(r+1)t$ -colourable. We use the same notation as Lemma 4.30. We have shown that up to permutations of C , for all $i \in [\alpha]$ and $j \in [r+1]$, vertices in $G_{i,j}$ have colours C_j . By definition of an expanded path, for all $j \in [r+1]$, G_0 is complete to $G_{i_j,j}$, which has colour C_j . So there is no colour in C left for vertices in G_0 . This contradicts the assumption that G is $(r+1)t$ -colourable. Therefore, $\chi \geq (r+1)t+1$. $\square$

Corollary 4.32. *Let $G = P_s^{r+}$ be a simple expanded path, then $\chi = r+2$.*

In general, the upper bound in Lemma 4.31 is not tight. For example, the expanded path $P_9^{2+}[K_2]$ with $i_1 = 1, i_2 = 2$ and $i_3 = 3$ has chromatic number 7, as shown in Figure 4.9. However, this upper bound is sufficient for us to prove the Borodin-Kostochka Conjecture and Reed's Conjecture for all expanded paths with $r \neq 1$.

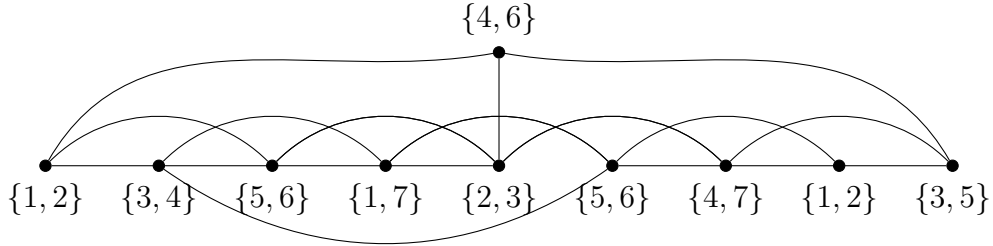


Figure 4.9: A 7-colourable graph in $P_9^{2+}[K_2]$, where every node is a K_2 and the numbers are colours

Lemma 4.33. *Let $G = P_s^{r+}[K_t]$ be an expanded path with $r \geq 2$, then the Borodin-Kostochka Conjecture holds for G .*

Proof. Similar to Lemma 4.16, we prove this lemma by showing that there does not exist an expanded path with $r \geq 2$ and $\chi = \Delta \geq 9$. Suppose for contradiction that there exists

such an expanded path G , then by Corollary 4.28 and Lemma 4.31, we have

$$\begin{aligned} \chi = \Delta &= (2r + 1)t - 1 \quad \text{and} \quad (r + 1)t + 1 \leq \chi \leq (r + 2)t \\ \implies (r + 1)t + 1 &\leq (2r + 1)t - 1 \leq (r + 2)t \\ \implies 2 &\leq rt \leq t + 1 \end{aligned}$$

Since $r \geq 2$, the only solution is $r = 2$ and $t = 1$. So $\chi = \Delta = 4$, which contradicts the assumption that $\Delta \geq 9$. Therefore, such G does not exist, and the Borodin-Kostochka Conjecture is vacuously true. $\square$

Lemma 4.34. *Let $G = P_s^{r+}[K_t]$ be an expanded path with $r \geq 2$, then Reed's Conjecture holds for G .*

Proof. By Corollaries 4.28, 4.29 and Lemma 4.31, we have

$$\begin{aligned} r \geq 2 &\implies t \leq \left\lceil \frac{rt}{2} \right\rceil \\ \implies \chi &\leq (r + 2)t \leq (r + 1)t + \left\lceil \frac{rt}{2} \right\rceil = \left\lceil \frac{(2r + 1)t - 1 + (r + 1)t + 1}{2} \right\rceil \\ \implies \chi &\leq \left\lceil \frac{\Delta + \omega + 1}{2} \right\rceil. \end{aligned}$$

Hence Reed's Conjecture holds. $\square$

To prove the two conjectures for expanded paths with $r = 1$, we recall Lemma 4.22 and appeal to existing results. It is more intuitive to work with blow-ups of odd holes, that is, $C_{2\alpha+1}[K_t]$. One well-known result by Cranston and Rabern [18] on the Borodin-Kostochka Conjecture is that the conjecture holds for claw-free graphs, where a **claw** is the complete bipartite graph $K_{1,3}$.

Theorem 4.35 (Cranston and Rabern [18]). *Every claw-free graph G satisfying $\chi(G) \geq \Delta(G) \geq 9$ contains a clique of size $\Delta(G)$.*

With this result, it suffices to prove that $C_{2\alpha+1}[K_t]$ is claw-free.

Lemma 4.36. *Let $\alpha \geq 2$ and t be positive integers, then $C_{2\alpha+1}[K_t]$ is claw-free.*

Proof. We adopt the same notation as in the discussion of expanded cycles in the previous sections. Suppose for contradiction that $C_{2\alpha+1}[K_t]$ contains a claw with vertices x, y_1, y_2, y_3 and edges xy_1, xy_2 and xy_3 . By symmetry of $C_{2\alpha+1}[K_t]$, we may assume that $x \in V(G_2)$. Then

$$\{y_1, y_2, y_3\} \subseteq N(x) = V(G_1) \cup (V(G_2) \setminus \{x\}) \cup V(G_3).$$

Since $\{y_1, y_2, y_3\}$ is an independent set, no two vertices in this set are in the same G_i . So without loss of generality, we may assume that $y_i \in V(G_i)$ for $i = 1, 2, 3$. However, this implies that y_2 is adjacent to y_1 and y_3 , which is a contradiction. $\square$

Theorem 4.23 follows immediately from Lemmas 4.33, 4.36 and Theorem 4.35.

To show that Reed's Conjecture also holds for blow-ups of odd holes, we appeal to the following result by King [40] on circular interval graphs, which are defined below.

Definition 4.37. A **circular interval representation** of a graph G consists of a set of points on a circle, each corresponding to a vertex of G , and a set of closed subintervals of the circle such that two points are in the same interval if and only if the corresponding vertices are adjacent in G . A **circular interval graph** is a graph that admits a circular interval representation.

Theorem 4.38 (King [40]). *For any circular interval graph G ,*

$$\chi(G) \leq \max_{v \in V(G)} \left\lceil \frac{d(v) + \omega(v) + 1}{2} \right\rceil.$$

As noted by Aravind, Karthick and Subramanian [1], the graph $C_{2\alpha+1}[K_t]$, which they refer to as a complete expansion of an odd hole, is a circular interval graph, which itself is a subclass of claw-free graphs. To obtain a circular interval representation of $G := C_{2\alpha+1}[K_t]$, we may assign an arbitrary order to the t vertices in each K_t and place all vertices of G on a circle by following the same order as in the cycle $C_{2\alpha+1}$. Then each closed interval in the representation corresponds to a $2t$ -clique in G . For example, the circular interval representation of $C_5[K_2]$ is shown in Figure 4.10. We remark that, in fact, all expanded cycles are circular interval graphs. Theorem 4.24 follows immediately from Lemma 4.34 and Theorem 4.38.

Since we have full information on Δ , ω , and χ of a simple expanded path, we can

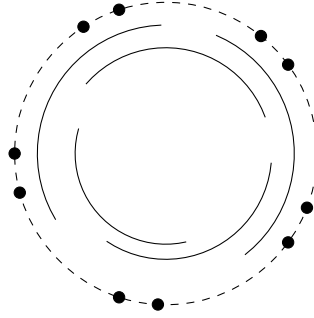


Figure 4.10: The circular interval representation of $C_5[K_2]$

categorize all such graphs for which Reed's Conjecture holds with equality. Note that

$$\begin{aligned} \chi = \left\lceil \frac{\Delta + \omega + 1}{2} \right\rceil &\iff r + 2 = \left\lceil \frac{2r + (r + 1) + 1}{2} \right\rceil \\ &\iff \left\lceil \frac{r}{2} \right\rceil = 1 \\ &\iff r \in \{1, 2\}. \end{aligned}$$

We obtain that $P_{2\alpha}^+$ and $P_{3\alpha}^{2+}$ are two classes of tight examples for Reed's Conjecture. The graph $P_{2\alpha}^+$ is simply an odd cycle, as shown in Lemma 4.22, whereas $P_{3\alpha}^{2+}$ is an infinite family of irregular tight examples with $\Delta = 4$, $\omega = 3$ and $\chi = 4$.

4.5 Further Questions

It is very likely that the results in this chapter may be strengthened, which leads to the following questions.

Question 4.39. Theorem 4.11 provides sufficient conditions for an expanded cycle G to satisfy $\chi(G) = \left\lceil \frac{|V(G)|}{\alpha(G)} \right\rceil$. What is the necessary condition?

Question 4.40. What is a tight upper bound for the chromatic number of expanded cycles that do not belong to type I, II or III?

Question 4.41. What is a tight upper bound for the chromatic number of non-simple expanded paths?

Following Lemmas 4.16 and 4.18, we have the following questions.

Question 4.42. Does the Borodin-Kostochka Conjecture hold for all expanded cycles? Does Reed's Conjecture hold for all expanded cycles?

Many families of tight examples for Reed's Conjecture have been found by analyzing expanded cycles and expanded paths. One may wonder if more such examples exist and if there is a larger class of graphs that generalizes the ones discussed.

Question 4.43. How to construct more infinite families of tight examples of Reed's Conjecture? Is there a class of graphs that naturally generalizes expanded cycles and/or expanded paths?

Chapter 5

Conclusion

In this thesis, we focused on the Borodin-Kostochka Conjecture, which is a natural next step in strengthening Brooks' Theorem, and Reed's Conjecture, which is a more general statement that attempts to upper bound the chromatic number of a graph using a linear function of the maximum degree and the clique number.

We improved upon a recent result [35] related to the Borodin-Kostochka Conjecture, by proving the existence of cliques of size $\Delta - 2t^2 - 7t - 4$ in graphs with $\Delta \geq 4t^2 + 11t + 7$ and $\chi = \Delta - t$. The main technique used in our proof is Mozhan partitions. We partitioned the graph according to groups of colour classes in order to identify small cliques in the parts and to recombine them into a large clique of the desired size. There is still much to learn and to exploit regarding the method of Mozhan partitions. For example, it is unknown whether the move sequence that we have defined explores all possible minimum R -MP's of a graph, for a given tuple R . We speculate that by working with a different definition of minimality or a different move sequence, this method may be used to prove even stronger results of this form.

We then followed the work in [9] and [1] to prove the Borodin-Kostochka Conjecture and Reed's Conjecture for odd-hole-free graphs, and more generally, for graphs that contain at least one critical vertex that is not in an odd hole. This is both a progress toward the two conjectures and a result that ties into the well studied structure of odd-hole-free graphs. It is worth investigating the counterparts of these results, that is, whether the two conjectures hold for odd-antihole-free graphs.

Finally, we made some new contributions to examples of graphs for which Reed's Conjecture holds with equality. By studying the properties of Δ , ω , χ and α of expanded cycles and working with the definitions of type I, II and III expanded cycles, we classified

numerous families of tight examples, which generalized many of the tight examples known in the literature. Moreover, we studied expanded paths, which led to a new family of irregular tight examples. All these tight examples are essential to expanding the scope of our understanding of Reed's Conjecture. It remains an open problem to find tighter upper bounds on the chromatic number of expanded cycles and paths as well as to construct a more general superclass of tight examples.

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