Forbidding Odd-$K_{3,3}$ as a Graft Minor

by

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A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Master of Mathematics
in
Combinatorics and Optimization

Waterloo, Ontario, Canada, 2025

Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

A graph is $odd - K_5$ free if K_5 cannot be obtained by deleting edges and then contracting all edges in a cut. $odd - K_5$ free graphs play an important role in the study of multi-commodity flows. A graph is $odd - K_{3,3}$ free if $K_{3,3}$ cannot be obtained by contracting edges and then deleting all edges in an eulerian subgraph. A long-standing conjecture of Paul Seymour predicts that postman sets pack in $odd - K_{3,3}$ free graphs. We study $odd - K_{3,3}$ free graphs that are almost planar in this thesis and discuss the relation to Seymour's conjecture.

Acknowledgements

I would like to thank my supervisor, Bertrand Guenin for his patience and guidance during my completion of the Masters Program. I would also like to thank my readers Joseph Cheriyan and Levent Tunçel for generously providing their time and feedback. Thank you to Richard Long for setting me on this journey to begin with, and nurturing a love of mathematics. Finally, I am forever grateful for my parents' unending support, without which this would not have been possible.

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Chapter 1

Basic Classes of $odd - K_{3,3}$-free grafts

1.1 Background

The problem which asks when a graph G contains another graph H is the source of many deep and fascinating problems throughout the entirety of the field. A famous result states that planar graphs are exactly those which do not contain K_5 as a minor, and do not contain $K_{3,3}$ as a minor [1]. In fact, it is known that an analogous list exists for every possible surface Π. That is, for every surface Π, there exists a finite set of graphs $\mathcal{G}$ such that a graph G is Π embeddable, if and only if G contains no graph from $\mathcal{G}$ as a minor. In fact, this is a corollary of the Robertson-Seymour Theorem, which states that finite undirected graphs and graph minors form a well-quasi-ordering [2]. One might then ask what the structure is of graphs that only do not contain K_5 as a minor? What about those that do not contain $K_{3,3}$? Further, one may ask these same questions under different containment operations, such as what is the structure of graphs which do not contain K_5, as a minor, but the minor operation must preserve the parity of cycles? What is the structure of graphs which do not contain $K_{3,3}$ as a minor, but the minor operation must preserve the parity of cuts? Excluding certain subgraphs under containment operations that preserve the parities of cycles or cuts allows us to solve in polynomial time problems that are otherwise NP-complete. Insisting that the minor operation preserves the parity of cuts yields a problem which is a strict generalization of the four colour theorem.

In this section, we provide the main definitions, terminology and background results used in the thesis. A *Graph* $G = (V, E)$ consists of a finite set V of vertices, and a finite set E of pairs of vertices, which we call edges. An edge $e = \{u, v\}$ where $u, v \in V$, has u and v as *endpoints*. $[n]$ is the set $\{1, 2, ..., n\}$. We also use uv as shorthand for the edge

$e = \{u, v\}$. For any two distinct vertices u and v, if the edge uv exists, we say u and v are *adjacent*, and the edge e is *incident* to u and v. We will assume, for every edge $e = \{u, v\} \in E$, u and v are distinct. That is, G is loopless. The *degree* of a vertex, $\deg(v)$ is the number of edges of G which are incident to v.

Parallel edges are two edges which have the same endpoints, i.e. $e_1 = \{u, v\}$ and $e_2 = \{u, v\}$. For a graph G we call the set of its vertices $V(G)$ and edges $E(G)$. A *subgraph* of a graph G, is a graph H for which $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ where for any edge $uv \in E(H)$ we must have $\{u, v\} \subseteq V(H)$. A *path* in G is a subgraph of G, consisting of vertices $V = \{v_1, ..., v_n\}$ and edges $E = \{v_1v_2, v_2v_3, ..., v_{n-1}v_n\}$, where all v_i are distinct. The ends of the path are v_1 and v_n, and the length is the number of edges in the path. A cycle is a graph which is a path from v_1 to v_n plus the edge v_1v_n. The *complete graph* K_n is the graph on n vertices, where there exists an edge between every pair of vertices. The *complete bipartite* graph $K_{m,n}$ is a graph such that the vertex set can be partitioned into two sets V_1 and V_2, where $|V_1| = m, |V_2| = n$, and all edges are of the form $e = uv$ where $u \in V_1$ and $v \in V_2$ (or vice versa).

A cut $\delta_G(U)$, in G, induced by $U \subseteq V(G)$ is the set of edges in $E(G)$ with exactly one endpoint in U. At times when G is obvious, we will simply use $\delta(U)$. For $e = uv \in E(G)$, the deletion of the edge e, denoted as $G \setminus e$, is the graph $(V, E \setminus \{e\})$. The contraction of the edge e, denoted as G/e, is the graph G, with the edge e deleted, and the endpoints of e; u and v identified into a single vertex, where any edge that was incident to either u or v is now incident to the new vertex. We say a graph H is a minor of a graph G if and only if H can be obtained from G by a sequence of edge contractions and deletions. In this case we say $H \preccurlyeq G$. It can be shown that the order of deletions and contractions does not matter.

A foundational result for planar graphs is a result on forbidding graphs as minors:

Theorem 1.1 (Wagner's Theorem [1]). A graph G is planar if and only if it does not contain K_5 as a minor, and does not contain $K_{3,3}$ as a minor.

As a result of Wagner's theorem, one may be interested in the structure of those graphs that only do not contain K_5 as a minor, and graphs which only do not contain $K_{3,3}$ as a minor. We will also investigate forbidding these graphs under other containment operations, which are fascinating in their own right.

1.2 Forbidding K_5 as an Obstruction

In this section, we analyze graphs which do not contain K_5 as a minor. To begin with, we use the most common minor operation, which we will then restrict to minors that preserve the parity of cycles, which we will call cycle minors. The first theorem describes K_5 free graphs (that is, graphs with no K_5 minor). Given two graphs G_1 and G_2 that each have a k-clique X_1 and X_2, you can *glue* G_1 and G_2 along X_1 and X_2, to create a new graph G. To do this, first create a bijection f between $V(X_1)$ and $V(X_2)$, then G is obtained from $G_1 \cup G_2$ by identifying x and $f(x)$ for each $x \in X_1$, and deleting some (possibly none) edges with both ends in the new clique.

Theorem 1.2. A connected graph G does not contain K_5 as a minor if and only if G can be obtained from planar graphs and V_8 by a sequence of gluing along K_1, K_2, or K_3.

The graph V_8 is $([8], \{\{i, i+1 \pmod 8) | i \in [8]\} \cup \{i, i+4 \pmod 8) | i \in [8]\}\}$. A proof of Theorem 1.2 can be found in [1]

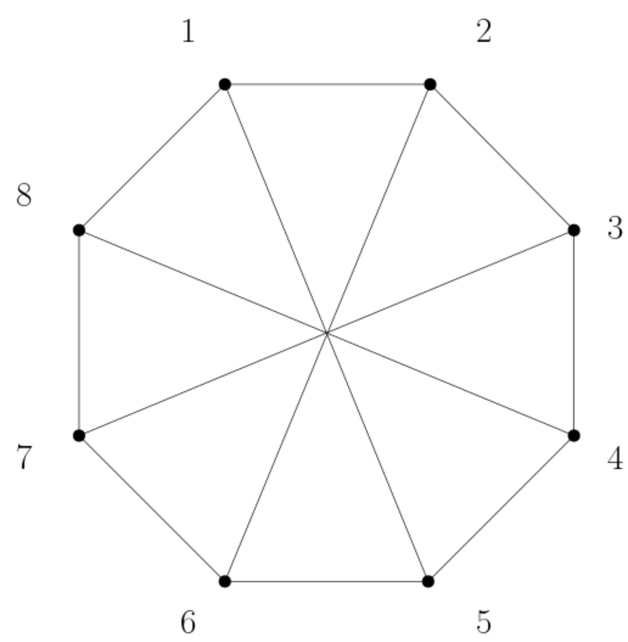

Figure 1.1: The Graph V_8

We will now consider graphs which do not contain K_5 as a minor, where the minor operation is restricted to preserve the parity (of the sizes) of cycles. This is a meaningful statement as any cycle in a graph $H \preccurlyeq G$, corresponds to a cycle of G. To ensure that even cycles of H are mapped into even cycles in G, and odd cycles are mapped into odd cycles, we will restrict the minor operation in the following way, first delete a set of edges F, then contract all the edges along a cut $\delta(U)$. Since it can be easily seen that cuts and cycles have an even intersection, by contracting along a cut, each cycle changes length by an even number. We will say for a graph H found in this way, from a graph G, that H is a *cycle minor* of G. Thus, cycle minors preserve the parity of cycles. This containment operation can be redefined as a minor operation on a more general class of graphs known as *signed graphs* we will omit a description of these here, but more information can be found in chapter 5 of [3]. Moreover, these signed graphs are of particular interest in the study of multicommodity flows [4]. We will say, for a graph H, that a graph G is cycle H free, if G does not contain H as a cycle minor.

1.2.1 Interesting Problems on Cycle K_5 Free Graphs

The problem of finding a minimum cut in a graph can be solved in polynomial time, see for instance [5]. Contrasting this, the problem of finding a maximum cut in a graph is NP-complete, see for instance [6]. However, if we restrict to cycle K_5 free graphs, the max-cut problem is also solvable in polynomial time. In fact, given any arbitrary graph G, there is a polynomial runtime algorithm to either find a maximum cut, or find K_5 as a cycle minor [7].

Another famous and difficult problem that has interesting results when forbidding K_5 as a cycle minor is the 4-Colour Theorem, which states that planar graphs are 4-colourable. That is, graphs that contain neither K_5, nor $K_{3,3}$ are 4-colourable. This begs the question, how does the chromatic number change if we forbid only one of these minors? Since planar graphs and V_8 are 4-colourable, and gluing 4-colourable graphs along a K_1, K_2, K_3 results in a 4-colourable graph, we know that any K_5 free graph is 4-colourable as well. However, what is not clear, since a graph may have K_5 as a minor, but not as a cycle minor, is the chromatic number of cycle K_5 free graphs. The following conjecture remains open:

Conjecture 1.3 (Strong 4-Colour Conjecture)**.** Cycle K_5 free graphs are 4-colourable.

There is a strengthening of the Strong 4-Colour Conjecture, which we will call the Odd Girth Conjecture, which makes a statement about the odd girth of cycle K_5 free graphs, this is due to Guenin in [8].

Conjecture 1.4 (Odd Girth Conjecture)**.** If a graph G is cycle K_5 free, then the *odd girth* (that is, the length of the shortest odd cycle) of G is equal to the maximum number of pairwise disjoint complements of cuts.

Here, cuts and their complements are viewed as subsets of the edge set. We have the following relationship between the Odd Girth Conjecture and the Strong 4-Colour Conjecture.

Proposition 1.5. The Odd Girth Conjecture implies the Strong 4-Colour Conjecture [8].

Proof. Assume the Odd Girth Conjecture. Then let G be a simple cycle K_5 free graph. Then, since G has no loops, the odd girth is at least 3. Since the Odd Girth Conjecture holds, there are at least 3 disjoint complements of cuts. In particular, there are at least 2 disjoint complements of cuts, say $E(G) \setminus \delta(U)$ and $E(G) \setminus \delta(W)$.

Then, we may partition $V(G)$ into four sets, $U \cap W$, $U \cap (V(G) \setminus W)$, $(V(G) \setminus U) \cap W$ and $(V(G) \setminus U) \cap (V(G) \setminus W)$. We claim that each of these sets is an independent set.

Say this is not the case, and that the edge uv has both endpoints in one of these sets, without loss of generality, say $u, v \in U \cap W$. Then u and v are both in U which means $uv \notin \delta(U)$. Similarly, u and v are both in W which means $uv \notin \delta(W)$. Hence, uv is in both $E(G) \setminus \delta(U)$ and $E(G) \setminus \delta(W)$. Which is a contradiction since $E(G) \setminus \delta(U)$ and $E(G) \setminus \delta(W)$ were assumed to be disjoint. So then each set described above is independent, and can be a colour class. So G is 4-colourable. □

It can be seen that the Odd Girth Conjecture is a vast generalization of the Odd Four Colour Conjecture, since it requires only the smallest case ($k = 3$) of the Odd Girth Conjecture.

1.2.2 Structural Results

As we have shown, forbidding K_5 as a cycle minor provides interesting results to previously well known problems. It is then natural to ask for the structure of cycle K_5 free graphs. This is currently an open problem in general, but some basic classes have been found. We will demonstrate some of these basic classes, and an in depth analysis can be found in [9].

The two main types of classes are those which, in some sense, have few odd cycles, and those which have topological obstructions.

An example of the first type is as follows: for a graph G, if there exists a pair of vertices which intersect all odd cycles, then G is cycle K_5 free. Graphs G with this obstruction can be constructed from any graph H, by fixing two vertices x and y, and replacing all edges not incident to x or y with a pair of consecutive edges, and replacing an arbitrary set of edges incident to x, y with a pair of consecutive edges to get G. Then any cycle in G which does not contain x or y can be mapped to a cycle of H which does not contain x or y. So any cycle of H of length k which does not contain x or y, results in a cycle of length $2k$, that is, every cycle which does not contain x or y is even, that is, all odd cycles contain x or y. A proof that the first example is cycle K_5 free can be found in [9].

There are also topological obstructions to containing K_5 as a cycle minor. The first is obvious, that being if G is planar. Then G does not contain K_5 as a minor, which means G certainly does not contain K_5 as a cycle minor. There is another less obvious topological obstruction. Graphs that have an even face embedding on the Klein bottle are cycle K_5 free. That is, a graph G which can be embedded on the Klein bottle, where every facial cycle is even. An example of this can be seen below, with G in green, and the labelled edges identified as they are labelled, and the Klein bottle, drawn in black, where the top and bottom sides are identified, and the left and right are identified in a particular way, where the top of the left is identified with the bottom of the right.

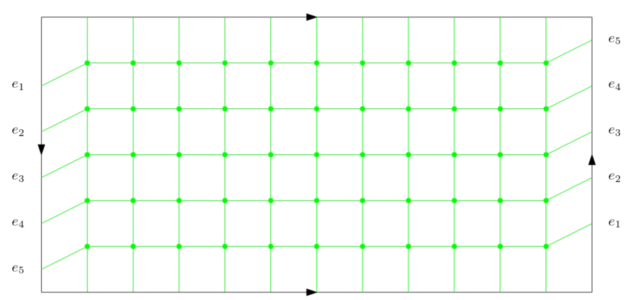

Figure 1.2: Even face embedding on the Klein bottle

This is clearly nonplanar in general, as the 2×3 grid induces $K_{3,3}$, as seen in Figure 1.3 with the bipartition indicated as circles and squares.

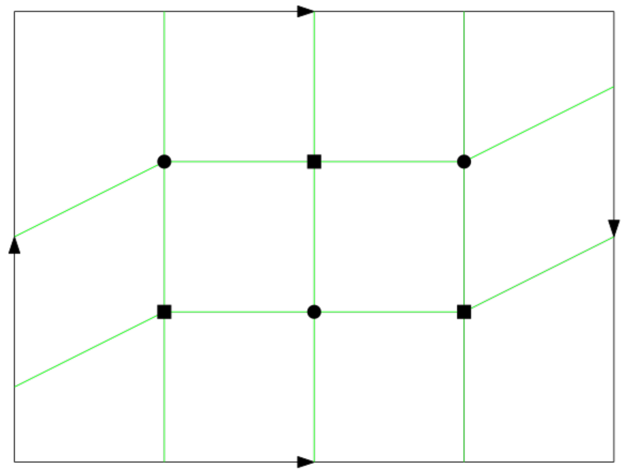

Figure 1.3: Embedding of $K_{3,3}$ on the Klein bottle

1.3 Forbidding $K_{3,3}$ as an obstruction

We will now discuss the counterpart to the problem of cycle minors, which will be a minor operation which preserves the parities of cuts, which we will call cut minors. We will also, instead of forbidding K_5, forbid $K_{3,3}$. To begin our investigation into forbidding $K_{3,3}$, we first consider the minor operation for graphs. As expected, there is a theorem analogous to Theorem 1.2, to forbid $K_{3,3}$, shown below:

Theorem 1.6. A connected graph G does not contain $K_{3,3}$ as a minor if and only if it can be obtained from K_5 and planar graphs by a sequence of gluing along K_1 and K_2 [1].

In a process analogous to the cycle K_5 free graphs, we define here a *cut minor*. We say a graph H is a cut minor of a graph G if and only if H can be obtained from G by contracting a set of edges F, and deleting some Eulerian subgraph. An Eulerian graph is one in which every vertex has even degree, and it can be seen that any Eulerian graph can be decomposed into a collection of edge disjoint cycles. Since cycles have even intersection with cuts, the cut minor containment operation preserves the parities of cuts in graphs.

Finding and forbidding graphs as cut minors will be the focus of this thesis, in particular forbidding $K_{3,3}$ as a cut minor. To this end, it will be useful to be able to delete single edges at a time instead of only Eulerian subgraphs. We will create a new containment operation on a new class of objects known as grafts that is equivalent to cut minors, which will allow us to do this.

Definition 1.7. A graft is an ordered pair (G, T) where G is a graph, and T is a subset of $V(G)$, such that for every connected component H of G, we have $|T \cap V(H)| \equiv 0 \pmod 2$. We will call T the set of terminals.

Note. This means necessarily that $|T|$ is even.

We say a cut $\delta(U)$ is odd (even) if and only if $|T \cap V(U)|$ is odd (even). In general, a subgraph H of G is odd (even) if it contains an odd (even) number of terminals.

To see that grafts generalize the concept of odd cuts, for a graph G, let (G, T) be the graft where T is exactly the set of vertices with odd degree in G. Then an odd cardinality cut of G is exactly an odd cut of (G, T). These grafts are called *Postman Sets* because these are the grafts that solve the Chinese Postman Problem [10]. Moreover, deleting an edge no longer changes the parity of a cut. For a graft (G, T), and an edge $e = uv \in E(G)$, we will define the contraction of (G, T) by e; $(G, T)/e$ as follows, let x_{uv} denote the new vertex created by contracting uv in G:

$$(G,T)/e = \begin{cases} (G/e, T \setminus \{u,v\}) & u \in T \text{ and } v \in T \\ (G/e, T) & \{u,v\} \cap T = \emptyset \\ (G/e, (T \setminus \{u,v\}) \cup \{x_{uv}\}) & \text{otherwise} \end{cases}$$

We say a cut-edge e of a graph is an edge that is a cut $\delta(U)$. This is equivalent to saying e is an edge whose deletion increases the number of components. Then an odd cut-edge e of a graft (G, T) is an edge which is an odd cut. In particular, this is an edge whose deletion creates an odd component.

For a graft (G, T), and an edge $e \in E(G)$, we define the deletion of e from (G, T) as

$$(G,T) \setminus e = \begin{cases} (G \setminus e, T) & e \text{ not a cut-edge} \\ (G \setminus e, \emptyset) & e \text{ a cut-edge} \end{cases}$$

Then, as for a graph, we say (H, S) is a minor of (G, T) if and only if (H, S) can be obtained from (G, T) by a sequence of edge deletions and contractions. Again, containment does not depend on the order of the sequence.

For any graft (H, S) we say a graft (G, T) is (H, S) free if and only if (G, T) does not contain (H, S) as a minor. Say $odd - K_{3,3}$ is the graft $(K_{3,3}, V(K_{3,3}))$, and $odd - P_{10}$ is the graft $(P_{10}, V(P_{10}))$ where P_{10} is the Petersen graph. The primary focus of this thesis will be describing the grafts which are $odd - K_{3,3}$ free. We say a graft (G, T) is *coherent* if and only if all odd cuts have even size, or all odd cuts have odd size. We can exactly characterize coherent grafts:

Proposition 1.8. A graft (G, T) is coherent if and only if either G is Eulerian or (G, T) is a postman set.

Proof. We first prove the reverse. Say G is Eulerian. Then all cuts are even, in particular, all odd cuts are even. So say (G, T) is a postman set. Then for every odd cut, $\delta(U)$, we have that $|\delta(U)|$ is odd, by definition. Now we will prove the forward direction. Assume that all odd cuts have the same parity. Since $|T| \neq 0$, there exists some odd cut say $\delta(U)$. Consider a vertex $v \notin T$. Then $\delta(U\Delta\{v\}) = \delta(U)\Delta\delta(u)$ is an odd cut. Now, since $\delta(U)$ and $\delta(U\Delta\{v\})$ have the same parity, we know that $\delta(v)$ must be even. So all non-terminals have even degree. If G has no vertex of odd degree, then G is Eulerian. So say some terminal w has odd degree. Now consider any vertex $u \in T$. Now $\delta(U\Delta\{u\}\Delta\{w\})$ is a T cut, so $\delta(U)$ and $\delta(U\Delta\{u\}\Delta\{w\}) = \delta(U)\Delta\delta(u)\Delta\delta(w)$ have the same parity, so $\delta(v)$ and $\delta(w)$ have the same parity, so any vertex in T has odd degree, so (G, T) is a postman set □

A T-join is a set of edges J such that in $G[J]$ (the subgraph induced by the edges J), the terminals are exactly the odd degree vertices. We say that a graft (G, T) *packs* if the minimum size of an odd cut is the maximum number of pairwise disjoint T-joins. Since any odd cut must intersect each T-join at least once, it is easily seen that the minimum size of an odd cut is *at least* the maximum number of pairwise disjoint T-joins.

We have the following conjecture due to Seymour, known as the Cycling Conjecture.

Conjecture 1.9 (Cycling Conjecture)**.** If (G, T) is coherent and $odd - P_{10}$ free, then the minimum size of an odd cut is equal to the maximum number of pairwise disjoint T-joins

The Cycling Conjecture is in fact a special case of a more general conjecture about packing binary clutters [4] [11]. An important special case of the Cycling Conjecture is proved in [12].

A *proper edge colouring* of a graph is an assignment of colours to edges such that no two edges incident to a vertex in common have the same colour. It is easy to see then that the minimum number of colours required is at least the maximum degree of a vertex Δ. In fact, the classic result due to Vizing states that (for simple graphs) the minimum number of colours required, that is, the chromatic index χ', is either Δ or $\Delta + 1$. In fact, for a graph where every vertex has degree r, that is, an r-regular graph, it is clear that the chromatic index is at least r. This however is not a tight bound. To easily see this, consider an odd length cycle, clearly these are 2-regular, but require 3 colours. Is there a subset of r-regular graphs that can be coloured using r colours? To this end, we can find a necessary condition for an r-regular graph to be r edge colourable. For all $U \subseteq V(G)$, where $|U|$ is odd, we must have $\delta(U) \geq r$. This is necessary because in an r-regular graph, the colour classes of an r edge colouring are perfect matchings [13, Section 17.4], and any perfect matching must intersect an odd cut. Therefore, we say that an $r - graph$ is an r-regular graph, where every odd cut has size at least r.

This however does not turn out to be a sufficient condition. Consider the Petersen graph. P_{10} is clearly 3-regular, and it can be seen that every odd cut has size at least 3, however it is not 3 edge colourable. P_{10} however, is not a planar graph, which brings the following conjecture.

Conjecture 1.10 (Planar r-graph Conjecture)**.** All planar r-graphs have chromatic index r.

This is known to be true for $r \leq 8$ [14] [15] [16] [17]. However, the problem becomes increasingly difficult even for small improvements on r.

We can see that Conjecture 1.9 implies Conjecture 1.10 as follows:

Let G be a planar r-graph. Then since all odd cuts have size at least r, $|V(G)|$ is even. This can be seen since if $|V(G)|$ is odd, then $\delta(V(G))$ is an odd cut, and so must have size at least r, which is obviously not the case. So let $T = V(G)$, then, (G, T) is a coherent graft. Since G is planar, (G, T) does not contain $odd - P_{10}$. Let $\delta(U)$ be any odd cut. Then $|U| = |U \cap T|$ is odd. So $|\delta(U)| \geq r$. So any odd cut of (G, T) has size at least r. Then, by assumption, there are at most r pairwise disjoint T-joins. Let u be an arbitrary vertex. Then $u \in T$ implies that $|\delta(u)| = d(U) \geq r$. Since G is r-regular each T-join is a perfect matching. Then each T-join is a colour class. So $\chi'(G) = r$

Another special case of the Cycling Conjecture is the following that was originally conjectured by Tutte [18]:

Conjecture 1.11. if G is a P_{10} free cubic graph with no cut-edge, then $\chi'(G) = 3$.

The following theorem is a special case of both Conjecture 1.11, since graphs which are planar are necessarily P_{10} free, and also a special case of 1.10, by setting $r = 3$.

Theorem 1.12. For any graph G, if G is cubic, has no cut-edges, and is planar, then $\chi'(G) = 3$.

Remark. Tait showed that this is equivalent to the 4-Colour Theorem.

We can see that these are equivalent as follows: Let G be a cubic planar graph with no cut-edge. Since G is planar, use the four colour map theorem to colour the faces of G with colours $\{00, 01, 10, 11\}$ via the map f. Then since G has no cut-edge, every edge e lies on the boundary of exactly two faces F_1 and F_2. So colour e with the colour $f(F_1) + f(F_2)$ (mod 2). Then it can be seen that this is a proper edge colouring. To show the reverse, consider the maps

$$L : 10 \leftrightarrow 00, 01 \leftrightarrow 11$$
$$R : 10 \leftrightarrow 11, 01 \leftrightarrow 00$$
$$F : 11 \leftrightarrow 00, 01 \leftrightarrow 10$$

Then each of these are bijective with no fixed point, the composition of any two gives the third, and the composition of all three is the identity. Then any maximal planar graph is a triangulation of the plane, so is a planar cubic graph with no cut-edge, so colour with three colours; L, R, F with the map f. Then choose any vertex, colour it any of $\{00, 01, 10, 11\}$ and iteratively colour a vertex that hasn't been coloured u which is adjacent to a vertex v that is coloured x. Say uv is coloured $f \in \{L, R, F\}$, then colour u the colour $f(v)$.

The class of graphs which are cubic with no cut-edges, and require at least 4 edge colours, are known as *snarks*. The smallest snark, that is, the snark with the fewest number of vertices is the Petersen graph. A natural generalization of Theorem 1.11 would be that snarks contain P_{10} as a minor. If this is the case, then the Theorem 1.11 is an immediate consequence, if G is a cubic P_{10} free graph with no cut-edge, then it would certainly not be a snark. This was originally conjectured by Tutte, and a proof of this conjecture has been announced, but remains to be published. For more information on Conjecture 1.11 see [16], [19], [20], and [21].

We can see that $odd - K_{3,3}$ is a graft minor of $odd - P_{10}$, as demonstrated in Figure 1.4 by deleting the dashed edges, and contracting the bold edges. As graphs, this can also be seen as contracting an arbitrary set of edges, (those in bold), then the dashed edges would be an Eulerian subgraph, so may be deleted.

It is then obvious that if a graft is $odd - K_{3,3}$ free, it is $odd - P_{10}$ free.

Then the following conjecture would be implied by the Cycling Conjecture, but is likely easier to prove:

Conjecture 1.13. If (G, T) is coherent, and $odd - K_{3,3}$ free, then the minimum size of an odd cut is equal to the maximum number of pairwise disjoint T-joins

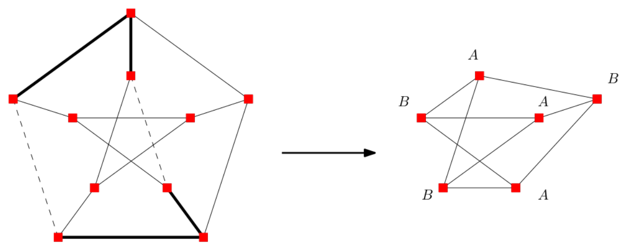

Figure 1.4: $odd - K_{3,3}$ as a minor of $odd - P_{10}$.

To prove the above conjecture, one must first describe the structure of those grafts that do not contain $odd - K_{3,3}$ as a minor. This will in general be the focus of this thesis, but the problem in its full strength remains open.

There seem to be 2 types of basic classes, those that obstruct $odd - K_{3,3}$ because there are too few terminals, such as $|T| < 6$, or all terminals are clustered together in some way such as having all terminals around a single face. The other type of basic class seems to be topological obstructions. The first obvious one being if G is planar. We will approach what seems to be the most difficult topological class, that being graphs where there exists a single edge e such that $G - e$ is planar, we call such graphs *near planar graphs*. Having more nonplanar edges seems to simply allow for more ways to find $odd - K_{3,3}$ as a minor.

To counteract the difficulty, we will create a specific method for counting terminals, and assume that G is highly connected. Under these conditions, we will show that (G, T) always has $odd - K_{3,3}$ as a minor. We will also demonstrate some classes that arise if you do not make these assumptions.

1.4 Organization of Thesis

Chapter 2 will construct some basic classes of graphs in general that are $odd - K_{3,3}$ free. We provide some classes based on topological properties of $K_{3,3}$, and some based on the number of terminals. We also give a method for constructing grafts with any number of terminals, and any topological properties. Chapter 3 will provide a process to reduce grafts, which preserves the ability for the graft to pack. In one of the basic classes, this reduces to the planar case. This then shows that it is not harder than the planar conjecture, even though the class is nonplanar. Chapter 4 will state the main theorem to be proved in this thesis, and will provide a description of how it will be done. Chapter 5 will provide the tools necessary to construct $odd - K_{3,3}$ as a minor, and Chapter 6 will use these tools to construct $odd - K_{3,3}$, in the case of the main theorem.

Chapter 2

Main Result and Extended Conjecture

2.1 List of some basic classes

There are two immediately obvious classes of grafts which do not contain $odd - K_{3,3}$. Since $|T|$ cannot be increased by deleting or contracting edges, we know that if $|T| < 6$, then $odd - K_{3,3} \not\leq (G, T)$ for any G. The second depends on topological properties of $K_{3,3}$. In particular, since $K_{3,3}$ is nonplanar, and the class of planar graphs is minor closed, we know that if G is planar, then $K_{3,3} \not\leq G$, and so $odd - K_{3,3} \not\leq (G, T)$ for any T. This outlines the two major types of basic classes that we discovered, those being obstructed by having too few terminals, and those having a topological obstruction. We will first investigate those with too few terminals.

2.1.1 Too Few Terminals

Simply saying '$|T| < 6$' is a class of $odd - K_{3,3}$ free grafts, is simplistic. Indeed, the graft in Figure 2.1 is $odd - K_{3,3}$ free, but can have any topological properties, and as many terminals as is desired. In this graft, terminals are indicates as red squares.

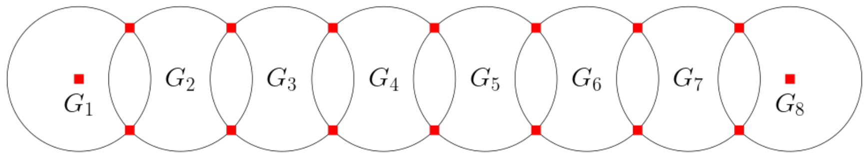

Figure 2.1: $odd - K_{3,3}$ free graft with 'many' terminals, indicated as red squares

To see why this graft is $odd - K_{3,3}$ free, we will need an operation known as a *Whitney Flip*, which preserves the containment of $odd - K_{3,3}$, but can change the number of terminals.

Definition 2.1 (Graph Whitney Flip)**.** Let G be a graph, and (X,Y) be a partition of the edges $E(G)$ where $V(G[X]) \cap V(G[Y]) = \{v_1, v_2\}$. Let G' be obtained from G by identifying v_i of $G[X]$ with v_{3-i} of $G[Y]$. Then we say G' is obtained from G via a *Whitney flip*.

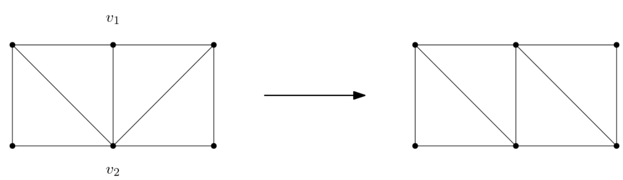

Figure 2.2: Whitney flip around $\{v_1, v_2\}$ in a graph

Since this will be applied to grafts, we need to determine how the set of terminals changes when performing a Whitney flip.

Remark. For a graft (G,T) and a T-join J of G, a cut $\delta(U)$ is odd if and only if $|\delta(U) \cap J|$ is odd [3].

Note that the previous remark does not depend on a specific choice of J.

Remark. For a graft (G,T), and a T-join J of G, and (H,R) where H is obtained via a Whitney flip of G, and R is the set of odd degree vertices of $H[J]$, then any cut of (G,T) has the same parity in (H,R).

Definition 2.2 (Graft Whitney Flip)**.** Let (G,T) be a graft. Let X and Y be a partition of $E(G)$, and say $V(G[X]) \cap V(G[Y] = \{v_1, v_2\}$. Then let H be obtained via a Whitney flip around X of G. Fix a T-join J. Then let R be the set of odd degree vertices of $H[J]$. Then we say (H,R) was obtained via a Whitney flip of (G,T) around X.

Given a graft (G, T) we will want to consider the 'parity' of a subgraph. So let H be a subgraph of G. Then $P(H) \equiv |T \cap V(H)| \pmod 2$. We provide the three different 'types' of Whitney flips in grafts one may encounter

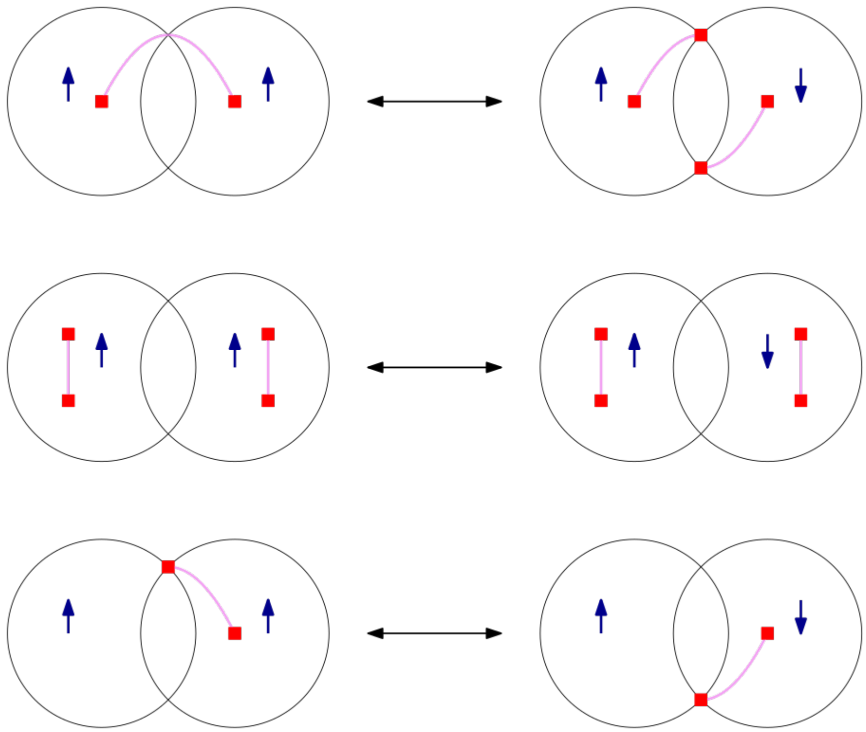

Figure 2.3: 3 types of Whitney Flips. Orientation of subgraphs is indicated with an arrow, terminals are indicated by red squares, and a T-join is indicated in purple.

We will provide some intuition for Definition 2.2 here. Essentially, if $P(G[X]) = 0$, then the set of terminals is unchanged. If $G[X]$ is odd and the terminals that form the two separation, say u and v are both terminals, or both not terminals, then the new set of terminals is $T \Delta \{u, v\}$. If $G[X]$ is odd, and one vertex is a terminal and the other is not, then, keeping the labels of $\{v_1, v_2\}$ consistent with $G[X]$, the parity of v_1 and v_2 remains unchanged.

Proposition 2.3. For any *odd* $- K_{3,3}$ free graft (G, T), if a graft (H, R) can be obtained from (G, T) via a sequence of Whitney flips, then (G', T') is *odd* $- K_{3,3}$ free.

Proof. Indeed, let (G, T) be a graft that contains *odd* $- K_{3,3}$, and say (G', T') is a Whitney flip of (G, T) around the edge set X. Say $H = G[X]$ and $H' = G[V(G) \setminus X]$. So then, since order of edge contractions and deletions is irrelevant, say $((G,T)/C) \setminus D$ consists of $odd - K_{3,3}$ and a collection of isolated vertices. Let C_i be the components of $G[C]$. Call the corresponding subgraphs $G[C_i] = H_i$. Then six subgraphs $H_1, ..., H_6$ correspond to the six vertices of $odd - K_{3,3}$, and $P(H_i) = 1$ if and only if H_i corresponds to a vertex of $odd - K_{3,3}$. Say (for $i \leq 6$) H_i corresponds to v_i. First, we claim that:

Claim. Either, all H_i that do not contain any of $\{x, y\}$ lie in H, or they all lie in H'.

Proof of Claim: Say there was some such H_i in H, and some such H_j in H'. Then, clearly H_i and H_j do not correspond to adjacent vertices in $K_{3,3}$, since $\{x, y\}$ is a two separation. So there exist three elements $v_{k_1} v_{k_2}, v_{k_3}$, at most two of which correspond to subgraphs containing x or y. So there is some v_{k_l} such that $v_i v_{k_l} \in E(K_{3,3})$, $v_j v_{k_l} \in E(K_{3,3})$, and H_{k_l} does not contain x and does not contain y. So since $\{x, y\}$ is a two separation, H_{k_l} lies in H or H'. But $\{x, y\}$ is a two separation so v_{k_l} cannot be incident to both v_i and v_j, a contradiction. $\Diamond$

Then at most two of the H_i ($i \leq 6$) contain x or y. Let A be the H_i that contains x and B be the H_i that contains y. It is possible that $A = B$. Let K consist of all H_i ($i \leq 6$) that do not contain x and do not contain y, along with all corresponding edges between them. Then without loss of generality say K is a subgraph of H. Then after performing the Whitney flip, keep the labelling of x and y consistent with the subgraph H (containing K).

$$P(H) = P(A_1) + P(B_1) + P(K)$$

Then, since $P(K)$ is odd if and only if exactly one of A or B is odd we get, since if both A and B, or neither A nor B correspond to a vertex of K, then there are an even number of vertices of K remaining, and all other H_i are even. Otherwise if exactly one of A or B correspond to vertices of K, then there are an odd number of odd H_i that form K:

$$P(K) = P(A_1) + P(A_2) + P(B_1) + P(B_2)$$

Let $A_1 = A \cap H$ and $A_2 = A \cap \bar{H}$, and say when considering parity that $x \in A_1$, and $x \notin A_2$. Similarly, say $y \in B_1$ and $y \notin B_2$. Then say $P'(D)$ is the parity of a subgraph D in (G', T').

$$
\begin{aligned}
P'(A) &= P'(A_1 \cup B_2) \\
&= P'(A_1) + P'(B_2) \\
&= P(A_1) + P(H) + P(B_2) \\
&= P(A_1) + P(A_1) + P(B_1) + P(K) + P(B_2) \\
&= P(B_1) + P(K) + P(B_2) \\
&= P(B) + P(K).
\end{aligned}
$$

Similarly,

$$P'(B) = P(A) + P(K),$$

For simplicity we will consider all possible cases.

Case 1: Say both x and y lie in the same H_i. Then $P'(H_i) = P(H_i)$ since $P'(\{x, y\}) = P(\{x, y\})$. Moreover, if $i \leq 6$, then in G, there are 3 edges incident to K and H_i, and this remains unchanged in G' since the only incidences that change are in $\bar{H}$, and of those, only those incident to $\{x, y\}$.

Case 2: Say $x \in H_i,\ i \leq 6$ and $y \in H_j,\ j > 6$. Then $P(A) = 1$ and $P(B) = 0$.

Then there are 3 edges incident to K and to H_i. In particular, these edges lie in H, which means these edges are incident to A_1, which remains unchanged in G'.

Then we must ensure that $P'(A) = 1$ and $P'(B) = 0$. K has 5 of 6 vertices of $odd - K_{3,3}$, which means $P(K) = 1$.

$$P'(A) = P(B) + P(K) = 0 + 1 = 1$$

and

$$P'(B) = P(A) + P(K) = 1 + 1 = 0.$$

As desired

Case 3: $x \in H_i$ and $y \in H_j$ for $i, j > 6$. Then, $P(K) = 0$ since all of $H_k, k \leq 6$ are the only odd H_k and there are an even number of them.

So then $P(A) = P(B) = 0$ and $P'(A) = P(B) + P(K) = 0 + 0 = 0$ and $P'(B) = P(A) + P(K) = 0 + 0 = 0$.

Case 4: $x \in H_i$ and $y \in H_j$ for $i, j \leq 6,\ i \neq j$. Then $P(K) = 0$ since there are an even number, 4, of the H_i's contained in K. Then $P(A) = 1$ and $P(B) = 1$.

Then $P'(A) = P(B) + P(K) = 1 + 0 = 1$ and $P'(B) = P(A) + P(K) = 1 + 0 = 1$, moreover, any edges between K and A remain edges between K and A, and edges between K and B remain edges between K and B. □

Corollary 2.4. Say (G, T) is a graft, if there exists a sequence of Whitney flips such that the resulting graft (H, R) has $|R| < 6$ then (G, T) does not contain $odd - K_{3,3}$ as a minor.

Proof. Since Whitney flips preserve containing $odd - K_{3,3}$ as minor, then since the resulting graft does not contain $odd - K_{3,3}$ as a minor, since $odd - K_{3,3}$ has 6 terminals, and the resulting graft does not, we get that (G, T) does not contain $odd - K_{3,3}$ as a minor. □

Now, we can return to Figure 2.1 to demonstrate why it, in fact, does not contain sufficient terminals. To see this, consider the T join drawn in purple, and notice that iteratively flipping over each $\cup_{j \leq i} G_j$ for each i, results in a graft with only two terminals. So since the second graft has only 2 terminals, that it is $odd - K_{3,3}$ free. This means, see by Proposition 2.3 the original graft is $odd - K_{3,3}$ free.

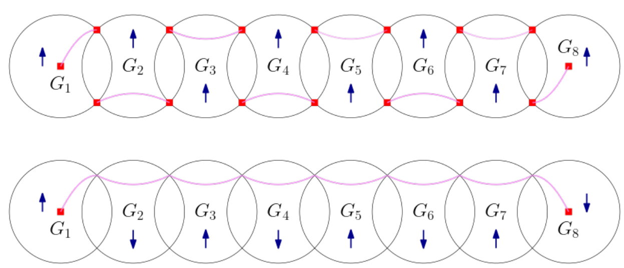

Figure 2.4: Whitney Flips to Reduce T

2.1.2 Topological Obstructions

We will now discuss expanding the class of grafts which are planar. The main subject of this thesis will be discussing grafts which are *near planar*. That is, grafts which have G nonplanar, but contain an edge e such that $G \setminus e$ is planar. Such an edge is called a *critical edge*. In this case, we will also be assuming that the graphs are highly connected, however we will also provide a construction for $odd - K_{3,3}$ free grafts that are not highly connected. Originally, we investigated grafts that were embeddable on other surfaces, akin to finding an even face embedding on the Klein bottle obstruction. To do so, this always ended up placing terminals in a specific way such that the graft was only degenerately embeddable on other surfaces. That is, there was only a single edge that was causing it to be nonplanar. It is possible that these examples do exist, but the focus of this thesis will be to analyze those grafts that are near planar.

We will now list some of the basic classes that were found, beginning with some that are easily constructed and shown to not contain $odd - K_{3,3}$, and others which are more complex.

In a plane graph, we denote the boundary of a face F by $\mathrm{bd}(F)$. In the case of 2-connected graphs, $\mathrm{bd}(F)$ is a cycle for any face F. 3-connected planar graphs have unique plane embeddings, hence *facial cycles* are unique [22].

For a nonplanar graft (G, T), if after removing an edge, the graph is planar where all terminals lie around the boundary of a single face, then (G, T) is $odd - K_{3,3}$ free.

Proposition 2.5. If G has a critical edge e such that in $G \setminus e$, there exists a face F such that $T \subseteq V(\mathrm{bd}(F))$, then (G, T) is $odd - K_{3,3}$ free.

Proof. Say (G, T) does contain $odd - K_{3,3}$ as a minor. Then $(G \setminus e, T)$ contains $odd - K_{3,3} \setminus e$ as a minor, for any edge e of $odd - K_{3,3}$. Then consider the subgraphs that contract to each vertex of $K_{3,3} \setminus e$. Then each one of these must contain a terminal. But then since all terminals lie around one face, after contracting the edges, all the corresponding terminals lie around one face of $odd - K_{3,3} \setminus e$, but for any face in $odd - K_{3,3} \setminus e$, there exists a vertex that is not covered by it. So there exists a vertex of $odd - K_{3,3}$ such that the corresponding subgraph of (G, T) does not contain a terminal. □

We also get a similar class, based on the fact that in $odd - K_{3,3} \setminus e$ one of the remaining cycles of length 4 contains a terminal in the interior and a terminal in the exterior. To make sense of this fact, we first must define what 'interior' and 'exterior' mean.

Let G be a plane graph. That is, a planar graph with a fixed embedding. Then any given cycle C corresponds to a closed curve on the plane, that is, a Jordan curve. Then, by the Jordan curve theorem [23], C separates the plane into two regions, exactly one of which is infinite. Then we call the subgraph which lies in the finite region the interior of C, $\mathrm{int}(C)$. We call the exterior of C $\mathrm{ext}(C)$ the subgraph that lies in the infinite region.

Proposition 2.6. Given any two vertices x and y that are not on the same face of a planar 3 connected graph, there exists a cycle C such that C separates the plane into two regions, one containing x and the other containing y.

Proof. Indeed, you can simply consider the cycle C with the containment wise minimal $\text{int}(C)$ that contains $x \in \text{int}(C)$. C can be constructed by considering G as a plane graph, and deleting all edges incident to x. Then x remains embedded in a face of G. Then since G was 3 connected, $G \setminus x$ is 2 connected, so this face is bounded by a cycle. So then x is the only vertex embedded in $\text{int}(C)$. If $y \in V(C)$, then we could include the edge xy and G would remain planar. But then this means that x and y lie on the boundary of a single face. □

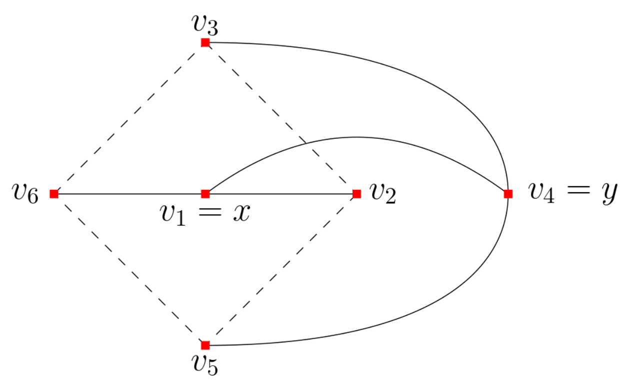

Figure 2.5: Labelled $odd - K_{3,3}$

Call the smallest cycle containing x as in the proof of Lemma 2.6, $\hat{C}_x$.

Lemma 2.7. If (G, T) has a critical edge $e = xy$ and contains $odd - K_{3,3}$ as a minor, then there exists a cycle that contains x and a terminal in the interior, and a y and a terminal in the exterior.

Proof. Indeed, consider the 4 subgraphs that get mapped to vertices of the 4 cycle of $odd - K_{3,3} \setminus \{x, y\}$. Then these are connected, so find a path between the endpoints of the edges that are incident to said subgraph. Then the collection of these paths, plus the collection of 4 edges of the 4 cycle of $odd - K_{3,3}$ forms a cycle of (G, T) that contains x and a terminal in the interior, and y and a terminal in the exterior. □

Then it is easy to see that we get the following class:

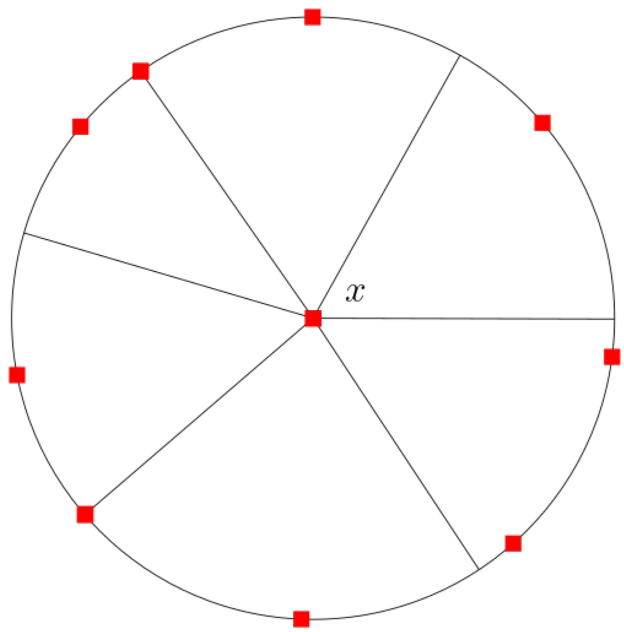

Figure 2.6: Visualization of Proposition 2.8

Proposition 2.8. If (G, T) has a critical edge $e = xy$ and for every terminal $t \in T$, there exists a face F of $G \setminus e$ such that $V(\text{bd}(F))$ contains both x and t, then (G, T) is $odd - K_{3,3}$ free.

Proof. Then it can be seen that $T \subseteq \hat{C}_x$. So then $T \subseteq \text{int}(C)$ for any cycle C that contains x in the interior. In particular, any cycle C which separates x from y contains all the terminal either on C, or in $\text{int}(C)$. Which contradicts Lemma 2.7. $\square$

We can generalize one of the classes above slightly, from requiring all of T to be around a single face, to requiring all of $T \setminus \{x, y\}$ to be around a single face of $G \setminus xy$.

Proposition 2.9. If (G, T) has a critical edge xy and there exists a face F of $G \setminus xy$ such that $T \setminus \{x, y\} \subseteq V(\text{bd}(F))$ then (G, T) is $odd - K_{3,3}$ free.

Proof. Say for a contradiction that $odd - K_{3,3} \leq (G, T)$. Then we know that G contains a subdivision H_0 of $K_{3,3}$ [24], labelled as in Figure 2.5.

Then since $G \setminus e$ is planar, we know that e is contained in this subdivision. This means that x and y are on the path in H_0 between two degree 3 vertices say v_1 and v_4. Then in

H_0, if we remove v_1, v_4 and the path between v_1 and v_4, we get a subdivision of K_4, with degree 3 vertices v_2, v_3, v_5, v_6. Then we know there are 4 terminals, and 4 paths with these terminals as one endpoint, and the degree 3 vertices of the K_4 subdivision as the other. Call these terminals t_2, t_3, t_5, t_6 and the paths P_2, P_3, P_5, P_6. Then these four terminals all bound a single face. Then we may create a new graph, by adding a vertex v, and 4 edges from v to t_2, t_3, t_5, t_6. Then since $G \setminus e$ was planar, this new graph must be planar since v may be embedded in the face which is bounded by $T \setminus \{x, y\}$. However, then since we get paths from v to all degree 3 vertices of K_4, we get a subdivision of K_5 as a subgraph, which is a contradiction. So $odd - K_{3,3} \not\leq (G, T)$ □

Here we will describe two more classes that are significantly more complex, but the proof that they are $odd - K_{3,3}$ free will be omitted.

For these constructions, one needs the concept of a *bridge* of a subgraph.

Definition 2.10 (Bridge)**.** Consider a graph G and a subgraph H. A *trivial* $H-$*bridge* in G is an edge $e = uv$ where $e \notin E(H)$ but $\{u, v\} \subseteq V(H)$. A *nontrivial* H-*bridge* is a connected component of $G \setminus V(H)$, along with all edges with exactly one endpoint in H.

Let B be an H-bridge. Then the vertices in common to both B and H are the *attachments* of B.

Proposition 2.11. If (G, T) is a graft where G has a critical edge $e = xy$, and $(G, T) \setminus e$ contains 3 faces, F_1, F_2, F_3, with the following properties:

P1: $\mathrm{bd}(F_1) \cap \mathrm{bd}(F_2)$ forms a path P_1 and $\mathrm{bd}(F_2) \cap \mathrm{bd}(F_3)$ forms a path P_2, where P_1 and P_2 are vertex disjoint.

P2: y lies on $\mathrm{bd}(F_1) \setminus P_1$

P3: $\mathrm{bd}(F_2) \setminus (P_1 \cup P_2)$ has two components, say Q and R.

P4: x lies on P or Q, say without loss of generality, $x \in V(P)$.

P5: All the terminals lie on P_1, P_2, Q and $\mathrm{bd}(F_3)$.

Then (G, T) is $odd - K_{3,3}$ free.

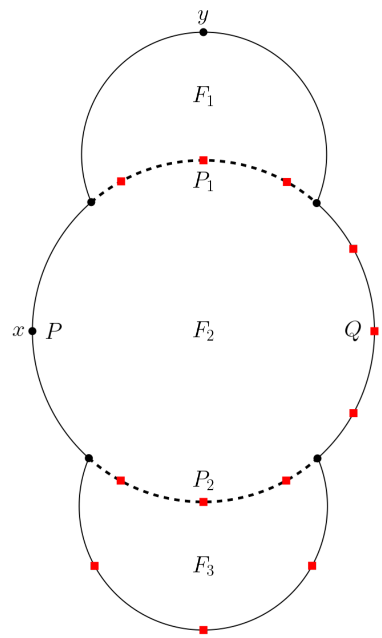

Figure 2.7: Complex Class 1. Dashed lines form the paths P_1 and P_2. Terminals are indicated as red squares.

One can gain an intuition for why this class does not contain $odd - K_{3,3}$. This is because any cycle which contains x and a terminal in the interior, which must exist, since it does in $odd - K_{3,3}$, must contain the paths P_1 and P_2. But this means that the bridge of C containing y and the bridge of C containing the exterior terminals are not the same. So this cycle does not work.

A second complex class whose proof will be omitted is as follows:

Proposition 2.12. Say (G, T) is a graft where G has a critical edge $e = xy$. Suppose $(G, T) \setminus e$ has a cycle C, and a fixed embedding with the following properties:

P1: C has exactly 1 bridge (G_1) in int(C), and exactly 2 bridges (G_2 and G_3) in ext(C).

P2: G_1 contains x, and a set of terminals T_1, and G_2 contains y, and G_3 contains $T \setminus T_1$.

P3: y and $T \setminus T_1$ lie on the boundary of the infinite face F_∞.

P4: x lies on a face which contains a subset of C as part of its boundary, call this face F_x, where $\text{bd}(F_x) \cap \text{bd}(F_\infty) \neq \emptyset$

P5: There exists a face F_1 that contains all of T_1 on its boundary, and $\mathrm{bd}(F_1) \cap \mathrm{bd}(F_\infty) \neq \emptyset$.

P6: Let P be the shortest subpath of C that contains all the attachments G_2, but no attachments of G_3. Let Q be the shortest subpath of C that contains all B_2 attachments, and no B_y attachments. This is well-defined, since $G \setminus xy$ is planar, so the bridges in the exterior of C do not overlap.

P7: $\mathrm{bd}(F_x)$ intersects one of P or Q, and $\mathrm{bd}(F_1)$ intersects the other.

Then (G, T) is $odd - K_{3,3}$ free.

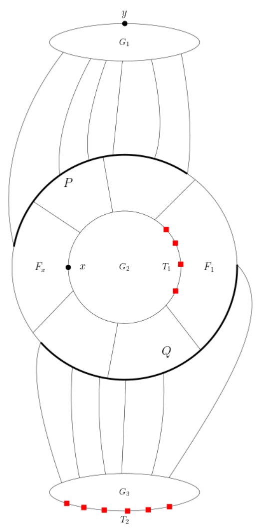

Figure 2.8: Complex Class 2. P and Q are indicated in bold.

This can be seen intuitively to not contain $odd - K_{3,3}$ for the same reason. Any cycle which separates x and one terminal from y and another terminal, results in the bridge containing y to have no terminals. Moreover, in this case, all the terminals lie on the boundary of two faces.

2.1.3 Gluing Grafts to forbid $odd - K_{3,3}$

Our analysis into forbidding $K_{3,3}$ as a minor will rely on high connectivity. To lower the connectivity, one would be interested in gluing higher connected pieces over small separations. We will describe one such gluing operation here.

$G \oplus_\Delta H$ is defined to be the graph created by identifying a triangle in G with a triangle in H, and possible removing some edges of said triangle. Even if H_1 and H_2 are highly connected, $H_1 \oplus_\Delta H_2$ is at most 3 connected.

Proposition 2.13. Consider two graphs H_1 and H_2 where H_1 is planar with at least 6 terminals, and H_2 has no terminals, but is nonplanar. Then $odd - K_{3,3} \not\preceq H_1 \oplus_\Delta H_2$.

Proof. This can be seen by considering how $odd - K_{3,3}$ may be constructed in $H_1 \oplus_\Delta H_2$. Say $H_1 \oplus_\Delta H_2$ does contain $odd - K_{3,3}$ as a minor. Then consider the connected subgraphs that get mapped to the vertices of $K_{3,3}$. Then we know that at must 3 of these subgraphs must contain vertices of Δ. This means there must be one of these subgraphs contained entirely within H_1, otherwise H_2 would be nonplanar. But this is impossible because then this subgraph does not contain a terminal, since all terminal lie in H_2. □

Chapter 3

Packing T-joins in a basic class

If we would like to claim that a characterization of the basic classes of grafts that are $odd - K_{3,3}$ free will be of use in proving Conjecture 1.9, then we should ensure that in the above described basic classes, we have that the minimum size of an odd cut is equal to the maximum number of pairwise disjoint T-joins. This problem remains open for now, but we will provide evidence that this should be the case. Indeed, even for planar graphs, the problem of packing T-joins remains unsolved. We will show that even in some cases where G is nonplanar, the problem reduces to a special case of the Planar r-graph Conjecture (1.10), and should in fact be strictly easier.

Consider the following basic class; let (G, T) be a graft where G is nonplanar, and there exists some critical edge h, and in $(G \setminus h, T)$ all terminals lie around a single face. To reduce this to the case of planar r-graphs, we will iteratively use a *splitting* operation, that preserves the parity condition, and the size of the smallest odd cut. It is possible to perform this operation on graphs that are nonplanar, but since we will only be applying it to planar graphs we will describe the operation for planar graphs, and will make a note of the changes to be made if applying to nonplanar graphs.

The next section will be dedicated to proving the following proposition:

Proposition 3.1. If (G, T) is a graft where:

H1: All cuts have the same parity

H2: G has a critical edge h

H3: $G \setminus h$ has a face F such that all terminals T lie on $\mathrm{bd}(F)$.

Then, if the Planar r-graph Conjecture (1.10) holds, the T-joins of (G, T) pack.

3.1 Minimum Odd Cuts

We first give some results about minimum cuts that will prove useful in our analysis of packing T-joins. First we define an operation that we will call *shrinking*. Let $\delta(U)$ be a minimum odd cut. Then, *shrinking* (G,T) over $\delta(U)$ to get 2 grafts (G_1,T_1) and (G_2,T_2) consists of shrinking all of U into a single vertex u, to get a graft (G_1,T_1), where $T_1 = (T \setminus U) \cup \{u\}$, and also of shrinking $\bar{U} = V(G) \setminus U$ into a single vertex $\bar{u}$ to get (G_2,T_2), where $T_2 = (T \setminus \bar{U}) \cup \{\bar{u}\}$. First we have the following claim:

Proposition 3.2. If G is connected, and $\delta(U)$ is a minimum odd cut, then both $G[U]$ (that is the graph with vertices U, and edge $E(G)$ restricted to edges with both ends in U) and $G[\bar{U}]$ are connected.

Proof. Say for a contradiction that $G[U]$ is not connected. Then it has at least 2 components, at least one of which is odd. Let U_1 be an odd component, and let $U_2 = U \setminus U_1$. Note that U_2 is not necessarily a connected component. Then, since G is connected, there must be an edge of $\delta(U)$ with an endpoint in U_2. So then $|\delta(U_1)| < \delta(U)$, contradicting the minimality of $\delta(U)$. Similarly, $G[\bar{U}]$ is connected. □

This tells us that the shrinking operation arises directly from contracting both sides of a minimal odd cut. In particular, if G is planar, then both resulting graphs of shrinking (G,T), are planar. If G is nonplanar, but there exists an edge h such that $G \setminus h$ is planar, then we say h is a critical edge, then shrinking being exactly contracting U tells us that if G has a critical edge h that both resulting graphs are either planar or have a critical edge. We can see this since the minor operation preserve planarity, and if G had a critical edge h, then certainly $(G \setminus h)/U$ is planar, so $(G/U) \setminus h$ is planar, so either G/U is planar, or h is a critical edge of G/U.

We also have the following proposition about packing T joins after shrinking a graft.

Proposition 3.3 ([14]). Let $\delta(U)$ be a minimum odd cut. Then if (G_1,T_1) and (G_2,T_2) are the grafts obtained by shrinking over U, then if the T-joins of (G_1,T_1) pack, and the T-joins of (G_2,T_2) pack, then the T-joins of (G,T) pack

Proof. Let u be the vertex corresponding to U in $(G,T)/U$ and $\bar{u}$ be the vertex corresponding to $\bar{U}$ in $(G,T)/\bar{U}$. Note that u and $\bar{u}$ are terminals.

Then say that $|\delta(U)| = \tau$, being minimal. This means that $|\delta(U)| = |\delta(\bar{u})| = \tau$.

Then let $J_1, ..., J_\tau$ be a disjoint set of T_1-joins of (G_1,T_1), which exist since they pack in (G_1,T_1). Then since every T_1-join intersects every odd cut, each edge of $\delta(u)$ belongs to a different T_1 joins. Similarly get τ disjoint T_2 joins $(J'_1, ..., J'_\tau)$ of (G_2, T_2). Say (up to reordering) that J_i and J'_i use the same edge of $\delta(U)$. Then $J_i \cup J'_i$ is a T joins of (G, T), so we can find τ T disjoint T joins, so the T-joins pack in (G, T). □

3.2 Splitting Operation

We first describe a splitting operation. Let (G, T) be a graft, and let $v \in V(G)$ be given. Then let e and f be two non-parallel edges adjacent to v. In the planar case the edges incident to v are in some cyclical order around v as defined by the embedding, choose e and f so they are consecutive in this order. Then *splitting* v over $e = u_1v$ and $f = u_2v$, results in the graft (G', T), where G' is obtained by deleting e and f, and adding the edge u_1u_2. Then we have the following proposition about splitting a vertex which appeared in [14]

Proposition 3.4. Let (G, T) be a graft such that:

H1: All odd cuts have the same parity

H2: All minimum odd cuts are trivial cuts

H3: G has a critical edge h.

Suppose there exists some $v \notin T$, and non-parallel edges e and f incident to v. Then, after splitting v over e and f into g, we get a graft (H, T) such that:

(a) All T cuts have the same parity

(b) The smallest size of an odd cut in H is the smallest size of an odd cut in G.

(c) H is either planar or one of g or h is a critical edge.

Proof. If $h \notin \{e, f\}$ then certainly $H \setminus h$ is planar, so either h is critical or $H + h := (V(H), E(H) \cup \{h\})$ is planar. Otherwise, $h \in \{e, f\}$ so then again, $H \setminus h$ is planar, so that either $H + g$ is planar, or $H + g$ is nonplanar, so that g is a critical edge.

Say $e = uv$ and $f = wv$.

Any cut that contains neither e nor f, has u, v and w all on one side of the cut. So those cuts also will not contain ef, so they are unchanged. If a cut contains exactly one of

e or f, say e, then v and w are on one side of the cut, and u is on the other, which means e will be removed from the cut, and ef will be added to the cut, so the size of this cut is unchanged. Now if the cut contains both e and f, then v is on one side of the cut, and both u and w is on the other. So then e and f is removed from the cut, and the new edge ef is between u and w which are on the same side of the cut, so ef is not added. So the size of the cut goes down by exactly two. In any case, the parity of all cuts are unchanged, so if the parity condition held before splitting, it holds after as well.

Since all minimum cuts are trivial, then since u is not a terminal, it is not a trivial odd cut, so it is not a minimum odd cut, so it has size at least τ. Then consider any odd cut that changes in size. Then it must contain u. So then it has size at least $\tau + 2$, so all cuts of H have size at least τ. □

3.3 Proof of Proposition 3.1

Proof of Proposition 3.1. Assume that H_1, H_2, H_3 hold, and that the Planar r-graph Conjecture (1.10) holds. Suppose for a contradiction that the T-joins of (G, T) do not pack. Let (G, T) be a counterexample that minimizes $|V(G)| + |E(G)|$.

Claim. Minimum cuts are trivial cuts.

Proof of Claim: Then, let $\delta(U)$ be a minimum odd cut that is not a trivial cut, then let (G_1, T_1) and (G_2, T_2) be the grafts obtained by shrinking G over $\delta(U)$.

Then all odd cuts of (G_1, T_1) and (G_2, T_2) have the same parity, notably, the parity of all odd cuts in (G, T). To see this, let $\delta(W)$ be an odd cut of, without loss of generality, (G_1, T_1). Then without loss of generality, say $u \notin W$ (otherwise, take $V(G) \setminus W$). Then $\delta_{G_1}(W) = \delta_G(W)$. So since (G, T) is coherent, then (G_1, T_1) and (G_2, T_2) are coherent.

If G has a critical edge, then G_1 and G_2 are either both planar, or both have a critical edge. If they are planar, then the T-joins pack.

If $G \setminus h$ has a face F such that all terminals lie on $\text{bd}(F)$, then either G_i ($i \in [2]$) is planar, or has a critical edge h, such that all terminals lie around a single face of $G_i \setminus h$. To see this, note that since all terminals lie around a single face in $G \setminus h$, we get that all terminals lie around a single face of $(G \setminus h)/U = (G/U) \setminus h$. To see this, note that in contracting or deleting an edge not contained in $\text{bd}(F)$, $\text{bd}(F)$ is unchanged. If deleting an edge on $\text{bd}(F)$ then we create a new face F' whose boundary contains $\text{bd}(F)$ as a subset. If contracting an edge on $\text{bd}(F)$, then the new vertex still lies on $\text{bd}(F)$, and all other vertices are unchanged.

Then, we get that either (G_1, T_1) and (G_2, T_2) both pack, in which case (G, T) packs, or (G_i, T_i) does not pack, which, since U was nontrivial, shows that (G, T) was not a minimal counter example. $\Diamond$

There must be a vertex that doesn't lie on $\mathrm{bd}(F)$. Otherwise, if all vertices lie on F, then $G \setminus h$ is planar, where all vertices lie on $\mathrm{bd}(F)$, in particular, G is outerplanar, but then since the endpoints of h lie on $\mathrm{bd}(F)$, h can be embedded through F, and we can see that in fact, G is planar. In particular, the T-joins pack in this case.

Then, say there is some vertex $v \notin \mathrm{bd}(F)$. Since G has a critical edge h, and all odd cuts have the same parity, and now, all minimum cuts are trivial cuts, we can use Proposition 3.4 to split v, and get a new graft (G', T'). Then if the T'-joins pack, since minimum cuts still have size τ, we can find τ disjoint T'-joins. If any T'-join uses the edge g, since v is not in T we can add the edges e, f to this T'-join to get a T-join of G, and keep all other T'-joins unchanged to get τ T-joins of G, to see that the T-joins pack.

Otherwise, if the T'-joins do not pack, then (G, T) was not a minimal counter example. $\square$

This shows that packing T-joins in this basic class is not any harder than packing T-joins in the planar case, despite G not being planar. Moreover, it reduces to packing T-joins in the 'near outerplanar' case, i.e. a non outerplanar graph, with an edge whose deletion results in an outerplanar graph. It is likely possible to leverage the additional structure of outerplanar graphs and pack the T-joins directly in this case without relying on the planar conjecture, but this remains open for now.

It is also likely possible to show a similar result for the remaining basic classes using the splitting argument again, this too remains open.

Chapter 4

Main Result and Extended Conjecture

As seen in Chapter 2, (G, T) must have many terminals, but simply counting $|T|$ is insufficient, as Whitney flips may change the number of terminals, but will not affect containment of $odd - K_{3,3}$. We also saw that if all terminals are around one face, T may be arbitrarily large, but (G, T) is still $odd - K_{3,3}$ free. Moreover, if all terminals share a face with x, then the graft is $odd - K_{3,3}$ free. In this case, the minimum number of faces required to contain all terminals on their boundaries, may be arbitrarily large. For this reason we will describe a method for counting terminals that encapsulates all of these, and more, possible cases, to allow us to find $odd - K_{3,3}$.

Given a set of terminals, say T', if there exist connected subgraphs $H_1, ..., H_k$, such that each H_i is outerplanar, and $T' \subseteq V(\cup H_i)$, when each H_i is viewed as a subgraph of G, then we say that T' is *covered* by k outerplanar graphs. So then, the *covering number* of a subset of the terminals, denoted by $\theta(G, T')$, is the smallest possible k such that T' is covered by k outerplanar graphs.

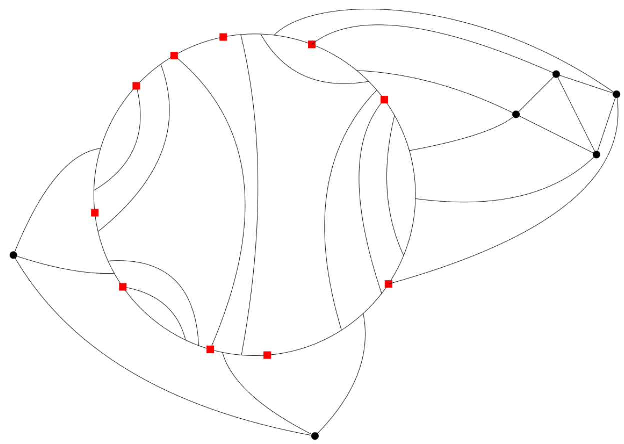

Figure 4.1: $\theta(G,T) = 1$

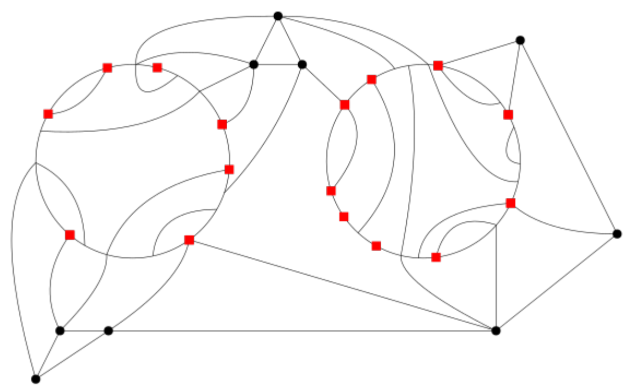

Figure 4.2: $\theta(G,T) = 2$

4.1 Connectivity

We will also require high connectivity. As planar graphs cannot be 6 vertex connected, a fact which can be derived arithmetically from Euler's Formula [24], we will enforce large connectivity only for certain subgraphs. For a graph G, and a partition of the edges X, Y, where both X and Y are nonempty, we say the 'boundary' of the partition is $\partial(X) = V(G[X]) \cap V(G[Y])$.

Then we say that (G,T) is oddly k-connected if for any $X \subseteq E(G)$ with $V(G[X]) \cap T \neq \emptyset$ and $(V(G) \setminus V(G[X]) \cap T \neq \emptyset$ then $|\partial(X)| \geq k$. In particular, this means that between any two terminals, there is no separating set of size less than $\bar{k}$. By Menger's Theorem [25] there are $\bar{k}$ internally disjoint paths from $G[X]$ to $G[E(G) \setminus X]$.

This allows to state the main theorem that is the focus of this thesis:

Theorem 4.1. There exists some $\bar{k}$ and $\bar{h}$ such that for any graft (H, T) with a critical edge xy, if $H \setminus xy$ is 4-connected, and

- $H \setminus xy$ is oddly $\bar{k}$ connected, and
- $\theta(H \setminus xy, T) \geq \bar{h}$

Then (H, T) contains $odd - K_{3,3}$ as a minor.

We conjecture here a strengthening of the main theorem, where G need not have a critical edge. Since in an outerplanar graph all vertices lie on the boundary of a single face, instead of covering terminals with outerplanar graphs, we will cover with the non-planar analogue of facial cycles, that is, nonseparating cycles. A *nonseparating* cycle of a graph G, is a cycle C such that the number of components of $G \setminus V(C)$ is the same as the number of components of G.

Conjecture 4.2. For any graft (G, T), if G is 4-connected, and for any 3 nonseparating cycles C_1, C_2, C_3, we have $T \nsubseteq V(C_1) \cup V(C_2) \cup V(C_3)$, then $odd - K_{3,3} \preccurlyeq (G, T)$.

4.2 Outline

The remainder of this thesis will be dedicated to proving Theorem 4.1. To do so, let (H, T) be a graft that satisfies the hypotheses of Theorem 4.1. Then $G := H \setminus xy$ is planar. Let (G, T) be a plane graft with a fixed embedding where y lies on the infinite face. Since any Jordan curve separates the plane into two regions, exactly one of which contains the infinite face, and (G, T) has a fixed embedding, any cycle C of G is embedded into the plane via the fixed embedding, and so induces a Jordan curve, and splits the plane into two regions. We call the subgraph that lies in the infinite region of the plane $\mathrm{ext}(C)$ and the subgraph that lies in the finite region $\mathrm{int}(C)$.

To find $odd - K_{3,3}$ as a minor of (H, T) we will construct $(odd - K_{3,3}) \setminus xy$, known as the *target-graft* as a minor of (G, T). $(odd - K_{3,3}) \setminus \{x, y\}$ is a 4-cycle, which we call C.

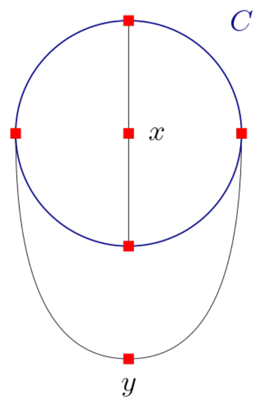

Figure 4.3: target-graft: $(odd - K_{3,3}) \setminus xy$

The strategy for constructing the target-graft as a minor will be to construct a cycle that will correspond to C which has certain properties that the cycle C has in the target graft, such as a cycle which contains $x \in \text{int}(C)$ and $y \in \text{ext}(C)$. Indeed, by Lemma 2.6, since x and y must not be on the same cycle (otherwise $G + xy = H$ would be planar by embedding xy in the face which contains x and y), we find a cycle which separates x and y, and since y lies on the infinite face, this cycle necessarily has $y \in \text{ext}(C)$ and $x \in \text{int}(C)$.

We will then repeatedly modify C to get additional properties until we construct the target graft. We will call this 'rerouting', when we take a path P with two endpoints on C, but is internally disjoint from C, and replacing part of C with P.

Since G is planar, for any C-bridge H, all of H must be embedded in $C \cup \text{int}(C)$, or all of H must be embedded in $C \cup \text{ext}(C)$. So then the following is well-defined:

Definition 4.3. A C-bridge H is an *in-bridge* if H is embedded in $\text{int}(C)$ and an *out-bridge* if H is embedded in $\text{ext}(C)$.

Given a cycle C with $x \in \text{int}(C)$, and $y \in \text{ext}(C)$, we will reroute C to obtain 3 properties, the first 2 of which we will describe here, and the third will be defined later, but is a specific way of counting the number of terminals on C, and we will force this number to be large.

P1: There exists a unique C in-bridge and there exists a unique C out-bridge.

P2: There exist terminals in $\text{int}(C)$ and there exist terminals in $\text{ext}(C)$.

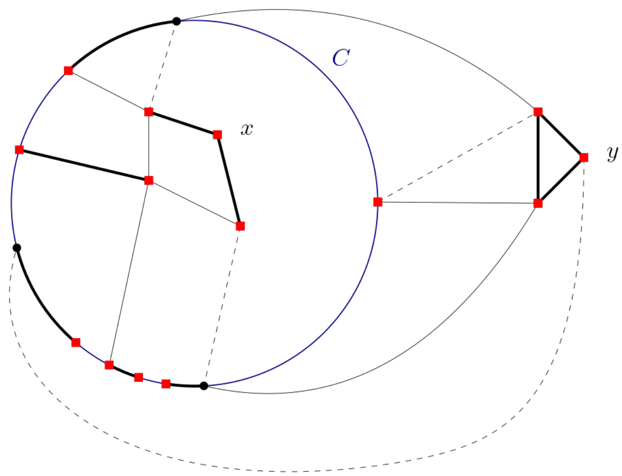

Figure 4.4: Target graft as a minor

We will also need a third property **P3**, which is a measure of the number of terminals contained on the cycle, which we will ensure is large. Chapter 5 will be dedicated to finding a cycle such that P1, P2, and P3 hold, and Chapter 6 will find the target graft as a minor. An example of a graft where P1, P2, and P3 hold is given in Figure 4.4. It contains the target graft as a minor, which can be seen by contracting bold edges, and then deleting dashed edges. For the remainder of this thesis we will let (H, T) satisfy the hypotheses of Theorem 4.1.

Chapter 5

Bridges in Grafts

This chapter will be focused on developing the necessary tools to be able to choose a suitable cycle, by considering bridges, and their properties.

For a graph G, a path P of G, and a subgraph H of G, where $E(H) \cap E(P) = \emptyset$, then we say the *shadow* of H on P, $\text{shad}_P(H)$ is the smallest subpath of P which contains all of $V(P) \cap V(H)$. In the case where the path is clear, we will omit the subscript P.

For any path Q with endpoints u and v we say that the *internal* vertices are $V(Q) \setminus \{u, v\}$. Then, we say a vertex $v \in V(P)$ is *covered* by a bridge B when v is an internal vertex of $\text{shad}_P(B)$. A path $Q \subseteq B$ is *extremal* if $\text{shad}_P(Q) = \text{shad}_P(B)$. Note that extremal paths always exist.

For the remainder of this thesis, we will make the following assumptions:

$H_1 :$ (H, T) is a graft

$H_2 :$ H has a critical edge $\hat{e} = xy$, and $G = H \setminus xy$

$H_3 :$ For some large $\bar{k}$, (G, T) is oddly $\bar{k}$-connected

$H_4 :$ For some large $\bar{h}$, $\theta(G, T) \geq \bar{h}$

$H_5 :$ G has a fixed embedding on the plane where y lies on the infinite face

$H_6 :$ C is a cycle in C with $x \in \text{int}(C)$, and $y \in \text{ext}(C)$

Note that H_6 is a valid assumption by Proposition 2.6, and H_5.

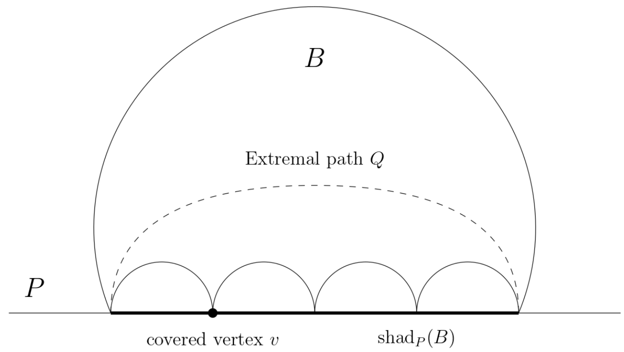

Figure 5.1: Example of a bridge B of a path P, with an extremal path Q indicated as a dashed line, a covered vertex v, and the shadow of B on P indicated as a bold line.

5.1 Cycles with only two bridges

In this section, we will construct a cycle C such that C has exactly two bridges, one bridge in $\text{int}(C)$ and one in $\text{ext}(C)$.

We will first extend the definition of shadow from paths to cycles. To do so, we fix another bridge that the shadow is relative to.

Definition 5.1. Let C be a cycle of a 3-connected planar graph G. Let B_1 and B_2 be two C-bridges that are either both in $\text{int}(C)$ or both in $\text{ext}(C)$. Since G is planar, all attachments of B_1 lie in a single segment P of $C \setminus V(B_2)$ [13, Theorem 10.26]. Then $\text{shad}_C(B_1, B_2)$ is $\text{shad}_P(B_2)$.

Remark. This definition is well-defined, because G is 3 connected so no bridge of C has only 2 attachments.

If hypotheses $H_1 - H_6$ hold, then for C-bridges $B \notin \{B_x, B_y\}$ we will omit the first argument of the shadow as follows:

$$\text{shad}(B) = \begin{cases} \text{shad}(B_x, B) & B \in \text{int}(C) \\ \text{shad}(B_y, B) & B \in \text{ext}(C) \end{cases}$$

In this section we will prove:

Proposition 5.2. Assume hypotheses H_1, H_2, H_3, H_4, then there exists a cycle C, such that P1 holds.

To do so, we will first define a partial ordering on the set of possible cycles.

Definition 5.3 (Level)**.** Given any cycle C, we say the C-bridge that contains x, B_x, and the C-bridge containing y, B_y, both have *level* 0. Then for any C-bridge B that contains neither x nor y, we define the level of B as follows: B has level $i > 1$, if B covers any attachment of a bridge of level $i-1$, but no attachment of any bridge of level $j < i-1$.

First we show that this is well-defined:

Proposition 5.4. In a 3-connected graph G, for any cycle C, all bridges of C have a level.

Proof. Define the following directed graph $\hat{G}$. Let $V(\hat{G}) = \{v_B : B \text{ a bridge of } C\}$. Then define $E(\hat{G})$ as: $(v_A, v_B) \in E(\hat{G})$ if and only if the bridge A covers some attachment of the bridge B.

Then say for a contradiction that there is some bridge B of C that does not have a level. In $\hat{G}$ this corresponds to there being no directed path from v_B to v_{B_x}, and no directed path to v_{B_y}. This means there exists some directed cut $\delta(U)$, where v_{B_x} and v_{B_y} both do not lie in U. Moreover, all edges in $\delta(U)$ are directed *into* U. Among all such directed cuts, choose one which is minimal with respect to containment. Say $U = \{v_{B_i}\}$. Then $\text{bd}(\cup\,\text{shad}(B_i))$ is a 2 separation of G.

First, we show $|\,\text{bd}(\cup\,\text{shad}(B_i))| = 2$. Obviously the boundary is at least 2, since the cut is non-empty, and the bridges cannot cover all of C, since they specifically do not cover B_x or B_y. The boundary is not more than 2, since the cut was chosen to be minimal. Moreover, this is a 2 separation since there is no bridge with attachments both in this segment, and outside this segment. If there were, if there was such a bridge, that was not a member of U, then $\delta(U)$ was not a directed cut. □

Definition 5.5. Let C be a cycle in a 3-connected graph. Then, by Proposition 5.4, each bridge has a level i. Then let a_i be the number of edges in all bridges of level i. Then the *trace* of a cycle C is the sequence $(a_0, a_1, ...)$

Then we define a partial order on traces as follows:

$$(a_0, a_1, ...) \gtrdot (a'_0, a'_1, ..) \iff \exists i \geq 0 : a_i > a'_i \textit{ and } a_j = a'_j \;\forall j < i$$

This induces a partial order '$\gtrdot$' on cycles C as defined above. We call a cycle C_1 *most preferred* when there is no C_2 with $C_2 \gtrdot C_1$. This method was inspired by a process found in [26].

Definition 5.6 (Rerouting)**.** Given a graph G, a path of G, P_0, and a subgraph H of G, where $E(H) \cap E(P_0) = \emptyset$, but $|V(H) \cap V(P_0)| \geq 2$. Then *rerouting* P_0 through H consists of choosing a path Q in H with endpoints on P_0, but internally disjoint from P_0, and constructing a new path P_1, which is P_0 with $\text{shad}_{P_0}(Q)$ removed, and Q added.

We will also be rerouting cycles through their bridges to create cycles with more preferred properties.

Definition 5.7. For a graph in which hypotheses $H_1 - H_6$ hold, we define rerouting C through a C-bridge that contains neither x nor y. Indeed, let B be such a bridge. Then let P be a C-path contained in B, that is a path P that has endpoints on C but is internally disjoint from C. Then create a new cycle by removing $\text{shad}_C(P)$, and adding P.

Given a subgraph L of G, we say L is an in-subgraph (out-subgraph respectively) if $E(L) \cap E(C) = \emptyset$ and $L \subseteq \text{int}(C)$ (ext respectively). An *in-bridge* is a bridge of C which is an in-subgraph, and an *out-bridge* is a bridge of C which is an out-subgraph.

Proposition 5.8. Assuming $H_1 - H_6$, given a C-path P that is internally disjoint from both B_x and B_y, rerouting through P results in a new cycle D which has $y \in \text{ext}(D)$ and $x \in \text{int}(D)$.

Proof. Obviously such a path P must be embedded in either $\text{int}(C)$ or $\text{ext}(C)$. Say without loss of generality that $P \subseteq \text{ext}(C)$. Then clearly since $\text{int}(D) \supseteq \text{int}(C)$, $x \in \text{int}(D)$. Moreover, since by definition, there are no B_y attachments in $\text{shad}(P)$ it must be the case that $y \in \text{int}(D)$ □

Proposition 5.9. Assuming $H_1 - H_6$, if C is a most preferred cycle, C has only two bridges, B_x and B_y.

Proof. For the sake of contradiction, say there exist other bridges. Then, if there exists another bridge, it must have a level, but is not level 0. So let B be a bridge of level 1. Say without loss of generality B is an in-bridge. Then B covers an attachment of B_y. Let P be an extremal path of B. Then reroute through P to get a new cycle D. Then, by Proposition 5.8 $D \in \mathcal{C}$.

Claim. $D \gtrdot C$

Proof of Claim: $|E(B_x)|$ is unchanged since P is disjoint from B_x. $|E(B_y)|$ increases, since the C edges incident to the B_y attachment covered by B are moved into the y bridge. So the first coordinate of the trace of D is larger than that of C, so $D \gtrdot C$ ◊

This is a contradiction to C being a most preferred cycle. □

We can therefore include the following assumption:

H_7 : C has exactly two bridges, one in the interior which contains x, and one in the exterior which contains y.

5.2 Cycles with terminals in both bridges

In this section, we will construct a cycle that satisfies both P1 and P2 (see section 4.2).

Any connected graph can be decomposed into a collection of 2 connected components, called blocks (note that these blocks may consist of only an edge and its endpoints). Then create an auxiliary graph called the block graph, where each block has a corresponding vertex in the block graph, and two vertices of the block graph are adjacent if and only if the corresponding two blocks have a vertex in common. Then it is easy to see that the block graph is a tree. Let S be the collection of 1-separations of this graph. Then, for any block A, let $\partial(A) = V(A) \cap S$.

Given a graft in which hypotheses $H_1 - H_7$ hold, we know that C separates the plane into two regions [23]. Given a C-path P, which contains neither x nor y, the graph formed by $C \cup P$ separates the plane into 3 regions, one of which contains x, one of which contains y. We know this, because they do not lie on C or P, and they do not lie in the same region of C. Let $\text{span}(P)$ be the third of these regions. Then we say P is *minimal* if and only if there is no other C path Q, with the same endpoints as P, where $\text{span}(Q) \subsetneq \text{span}(P)$. If P is a minimal C-path, there are no $(C \cup P)$ bridges that are embedded in $\text{span}(P)$ such that two of the attachments lie on P. This can be seen by considering a bridge B_P in $\text{span}(P)$ with two attachments on P. Then there exists a path Q in B_P. Then rerouting P through Q results in a new path P' where $\text{span}(P') \subsetneq \text{span}(P)$, a contradiction.

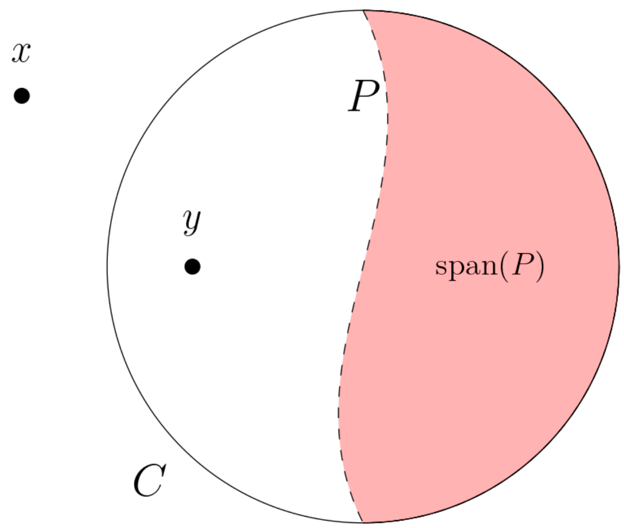

Figure 5.2: Example of span(P) indicated in red, with P indicated as a dashed line.

Let u be a vertex in $V(C)$, and let P be a minimal C-path with endpoints u_s, u_t, the internal vertices of P lie in ext(C). Let s (resp. t) be the vertex of P adjacent to u_s (resp. u_t). Let $Q = \text{span}(P) \cap C$. Assume that:

1: Every edge of B_y with an end in Q has an end in $\{s, t, u\}$.

2: If such an e of B_y has an end in s or t, then $e \in \text{span}(P)$

Such a P is called an *outside u-bite*. An inside u-bite is defined similarly, replace 'y' with 'x' and 'outside' with 'inside'

Notice that for every $u \in V(C)$, there exists both an inside, and an outside u-bite.

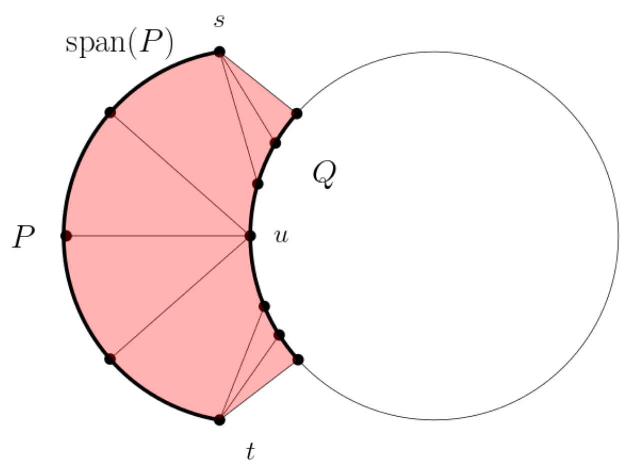

Figure 5.3: An Outside u-bite

We say a vertex $u \in V(C)$ is a *tasty* attachment of B_y, if the outside u-bite does not intersect the set of 1-separations of B_y.

Lemma 5.10 (Nibbling)**.** Let u be a tasty attachment of B_x. Let P be an outside u-bite. Then after rerouting along P to get a new cycle C', C' still has a unique in-bridge and a unique out-bridge

Proof. Say there is a bridge B' of C' that contains neither x nor y.

Case 1: B' is a trivial bridge embedded in int(C'). Clearly B' is embedded in ext(C), since C contains exactly 2 bridges, neither of which are trivial. Moreover, it cannot be the case that both endpoints lie on P, since P is a minimal path. Therefore, B' must have one endpoint on P and the other on C. But then there is a path, namely span$(P) \cap V(C)$, which contains both u, and an attachment of B'. So B' is contained in the x-bridge of C'.

Case 2: B' is a nontrivial bridge embedded in int(C'). Again, B' is not embedded in int(C), since C has a unique in-bridge. So B' is embedded in span(P). Again, B' has at most one attachment on P, say w. B' does not have both s and t as attachments, since P was assumed to be minimal. Say B' does not contain s. Then $\{t, u, w\}$ is a 3 separation, by the definition of a u-bite.

Case 3: B' is a trivial bridge embedded in ext(C'). Clearly B' has an endpoint in P, since B' is not a bridge of C. Moreover, B' does not have an endpoint in Q, since it is not contained in span(P). Then if B' has both ends in P, since G has no parallel edges, there must be a vertex w in P, between these endpoints. But then the endpoints of B', along with u, form a 3-separation, by planarity, since B' is embedded in ext(C'). So then B' has an endpoint not in P. Then B' must have the other endpoint on $C \cap C'$. Then since P does not contain any 1-separations of B_y, $V(P) \setminus V(C)$ is contained in a single 2-connected block A of B_y. Either A consists only of 2 vertices, or A is bounded by a cycle.

If A consists of only 2 vertices, then A must be a leaf, since P must contain some vertices of A that aren't 1-separations of B_y. Then P has only 3 vertices, u_s, u_2 and $s = t$. But then, since B' has an endpoint in P, $s = t$ must be an endpoint. But then, as in the definition of a u-bite, B' must be embedded in $R(P)$, a contradiction.

Then we know that A is bounded by a cycle. So the endpoint of B' that does not lie on C must be on the boundary of this cycle, by planarity. But since B' is not embedded in span(P), it must be that one end of B' is either s or t. But then this contradicts the definition of a u-bite

Case 4: B' is a nontrivial bridge embedded in ext(C'). Then let B'' be the bridge of $P \cup C$ containing y. Then if shad$_P(B')$ and shad$_P(B'')$ are internally disjoint, then there is a vertex between them on P, say w. Then w is a 1-separation of B_y, since there is no bridge embedded in span(P) with two attachments on P, but this is a contradiction to P being a u-bite.

So then we know that shad$_P(B')$ and shad$_P(B'')$ are not internally disjoint. Then, if one is not a subset of the other, one of them must be embedded in span(P), which was already covered in Case 2. So then without loss of generality shad$_P(B')$ is a subset of shad$_P(B'')$ and they are both embedded in ext(span(P)). But then, bd(shad$_P(B')$) and u forms a 3 separation of G.

Therefore, for any tasty attachment u, rerouting through the corresponding u-bite results in a cycle that still satisfies $H_1 - H_7$ □

We next show that the number of outer planar graphs it takes to cover the boundary vertices of a bridge scales linearly with the number of leaves of the block tree of said bridge.

Proposition 5.11. Let G be a 4-connected plane graph that consists only of a cycle C and a single bridge B. Then let bd(B) be the vertices that lie on the boundary of the infinite face of $B \setminus C$. Then B can be viewed as a tree of two connected blocks. Say this tree has n leaves.

Then $\theta((\text{bd}(B) \cup C) \cap T) \leq n + 1$

Proof. We will prove this by induction.

First, say this tree has at most two leaves, then it must be a path. Then choose two edges from B to C, so that each leaf has an endpoint of an edge, and these edges are not adjacent.

Then the C endpoints of these edges partition C into two paths, Q_1 and Q_2. Then we will construct two new paths P_1 and P_2 so that $G[Q_1 \cup P_1]$, and $G[Q_2 \cup P_2]$ are outerplanar.

Indeed, consider any non leaf block A along this path. Then either this block is a single edge, in which case, we may add this edge, and both endpoints to both P_1 and P_2, or it is bounded by a cycle D. In which case, $\partial(A)$ partitions D into two paths D_1 and D_2. Since G is planar, it is not possible that both D_1 and D_2 have edges incident to C_1, and it is not possible that both D_1 and D_2 have edges incident to C_2. Without loss of generality, say D_1 has no edges incident to C_2 and D_2 has no edges incident to C_1. So then add D_1 to P_1, and D_2 to P_2. Then $G[C_i \cup P_i]$ is outerplanar for $i \in \{1, 2\}$, and $P_1 \cup P_2 = \text{bd}(B)$.

Now assume the lemma for a tree with $n-1$ leaves. Consider a bridge B whose block tree has $n > 2$ leaves. Then choose some leaf. Then there is a unique shortest path in the tree from the leaf to a vertex of degree at least 3. Then removing this path results in a tree with $n-1$ leaves. Cover this tree with n outerplanar graphs. Then the path of tree blocks is embedded within one of these outerplanar graphs, H_0. Then we can use an attachment of the leaf, and the 1-separation of where the path attaches to the rest of the tree, then as in the base case, construct two outerplanar graph to cover this path, and the boundary of H_0, then replace H_0 with these two outerplanar graphs, these increasing θ by 1. □

Proposition 5.12. In a graft that satisfies hypotheses $H_1 - H_5$, there exists a cycle C that satisfies H_6, H_7, and each C bridge contains a terminal.

Proof. First, we show that at least one of the bridges contains a terminal.

Claim. There exists a cycle C that satisfies H_6, H_7, and $T \not\subseteq V(C)$.

Proof of Claim: Say this is not the case. Then, among all cycles that satisfy H_6, H_7, minimize $\text{ext}(C)$. Note that $|V(\text{ext}(C))| \geq 3$, otherwise we would have that $\theta(G, T) \leq 2$.

Case 1: There exists a leaf block A of the y-bridge, which does not contain y. Then, since G is 4 connected, there exists 3 pairwise non-adjacent edges from $A \setminus \partial(A)$ to C. Then by planarity, there is a subpath of C which contains all attachments of A, and no attachments of any other block of B_y. Then there must be an attachment of x, say u contained in the interior of this path. Then u is a tasty x attachment. Then the outside u-bite does not contain y, so we may reroute through the outside u-bite to make $\text{ext}(C)$ smaller. So then $\text{ext}(C)$ was not minimal.

Case 2: There is no leaf block which doesn't contain y. This means there is a single leaf block, which means the y-bridge is 2 connected. Then, there are at least 3 non-adjacent edges between the y-bridge and C, which do not contain y. Again, these must cover an x attachment, and so there exists a tasty x attachment, where the outside bite does not contain y, again contradicting minimality of $\text{ext}(C)$.

So then at least one of the x-bridge or y-bridge contains a terminal ◊

Claim. There exists a cycle C that satisfies H_6, H_7 where $T \cap \text{int}(C)$ is nonempty, and $T \cap \text{ext}(C)$ is nonempty.

Proof of Claim: Say this is not the case. Then we may assume that $T \cap \text{ext}(C)$ is nonempty, and $T \cap \text{int}(C)$ is empty. Then, among these, minimize $\text{ext}(C)$.

Case 1: The y-bridge is 2-connected. Then we may assume that not all terminals lie on the boundary of the infinite face of the y-bridge. Otherwise, we could cover T with 2 outerplanar graphs. So then, as before, there must be at least 3 pairwise non-adjacent edges which do not contain y, and this must cover an x attachment, which means there is a tasty x attachment whose bite does not contain y, and does not contain all the terminals, since the outside u-bite is contained in the boundary of the infinite face of the block of the y-bridge which does not contain all the terminals in $\text{ext}(C)$.

Case 2: There are at least 2 leafs, and there is some leaf that contains neither y nor all of $T \cap \text{ext}(C)$. Then, choosing this leaf, as before, it must cover an x attachment, which therefore must be tasty, then we may reroute through the outside u-bite, since it is contained in the leaf block so contains neither y nor all the terminals, contradicting minimality.

Case 3: There are exactly 2 leafs, and one of these contains y, and the other contains all of $T \cap \text{ext}(C)$. Then if there is a terminal that doesn't lie on the boundary of the infinite face of this bridge, then we can choose the leaf which does not contain y. Then, again, this leaf must cover an x attachment u. Then the u-bite does not contain all the terminals, and does not contain y. So we may reroute through this path. Then, we know that all of $T \cap \text{ext}(C)$ lie on the boundary of the infinite face of the y-bridge, which means we can cover with 2 outerplanar graphs. ◊

So then there are terminals both in $\text{int}(C)$ and in $\text{ext}(C)$. □

So we add the hypothesis

$$H_8 : T \cap \text{int}(C) \neq \emptyset \text{ and } T \cap \text{ext}(C) \neq \emptyset$$

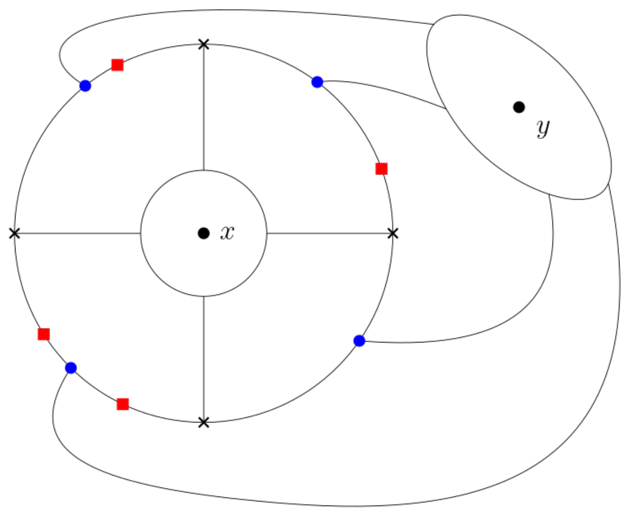

Figure 5.4: Example of L_y indicated by blue dots, with x attachments indicated by crosses, and terminals indicated by red squares.

5.3 Cycles with many terminals

For this section, we first describe exactly how we count the number of terminals on a cycle C. Given a cycle with an out-bridge and an in-bridge, we define the following structure that we call a *magic set*.

Consider a set of attachments, L_y of the y bridge. We see that $C \setminus L_y$ is a disjoint union of paths. If each of these paths contains an x attachment and a terminal, we say L_y is a magic set with order $|L_y|$. Then for a cycle C to satisfy **P3** means there exists a magic set on C of order at least n for large n.

Proposition 5.13. For any n, there exists some $\bar{h}$ and some $\bar{k}$, such that any graft that satisfies $H_1 - H_6$, there exists a cycle C that satisfies H_7 and H_8 that has a magic set of order at least n.

Proof. Let C be a cycle such that P1 and P2 hold. Then assume that P3 does not hold. Then we may assume that $\theta(G, \text{ext}(C) \cap T) \geq \theta(G, \text{int}(C) \cap T)$. Among such C choose one that first maximizes $|V(C) \cap T|$, and second, subject to $|V(C) \cap T|$ being maximized, minimizes $\text{ext}(C)$.

Claim. If there exists a tasty x attachment u, then there is a terminal in $\text{span}(P) \cap C$, where P is the outside u-bite, or $y \in V(P)$.

Proof of Claim: Say this is not the case. Then we reroute through the outside u-bite P to get C'. If there is a terminal in $\text{int}(P)$, then this is a contradiction to C maximizing $|V(C) \cap T|$. So then $V(C) \cap T = V(C') \cap T$, and $\text{ext}(C) \cap T - \text{ext}(C') \cap T$, so we still have that $\theta(G, \text{ext}(C') \cap T) \geq \theta(G, \text{int}(C') \cap T)$. But $\text{ext}(C') \subsetneq \text{ext}(C)$ contradicting the minimality of $\text{ext}(C)$. $\Diamond$

Then the C-bridge containing y forms a tree of 2-connected blocks.

Case 1: There are $n + 2$ leaves of this tree.

Then there is a set $\mathcal{L}$ of n leaves such that no leaf $L \in \mathcal{L}$ contains y, and there is some leaf $L \notin \mathcal{L}$ that contains a terminal.

Then for each leaf $H \in \mathcal{L}$, since G has no 3 separations, $G \setminus \partial(L)$ has no 2 separations.

Then in $G \setminus \partial(L)$, $\text{shad}_C(L \setminus \partial(L))$ is well-defined. Then the endpoints of $\text{shad}_C(L \setminus \partial(L)$ must not be a 2-separation. That means there must be, in particular, an x attachment u in the interior vertices of $\text{shad}_C(L \setminus \partial(L)$. So then there exists at least one outside bite for each leaf in L, so there exists at least $|L| = n$ outside bites.

Case 2: There are at most $n + 1$ leaves of the tree of 2-connected blocks of the C-bridge containing y.

Then if every block B contains all of $T \cap V(B)$ on the boundary of the infinite face of B, then all of $T \cap V(B_y)$ lies on the boundary of the infinite face of $B \setminus C$, which means by Proposition 5.11:

$$\begin{aligned}
&\theta(G, T) \\
&\leq \theta(G, T \cap (\text{int}(C) \cup C)) + \theta(G, T \cap (\text{ext}(C) \cup C)) \\
&\leq 2\theta(G, T \cap (\text{int}(C) \cup C)) \\
&= 2\theta((B_y \cup C) \cap T) \\
&= 2\theta((\text{bd}(B_y) \cup C) \cap T) \\
&\leq 2(n + 2) \\
&< \bar{h}
\end{aligned}$$

Which is a contradiction. So there exists some block B that contains a terminal in the interior of this block. Then, since there are at most $n + 1$ leaves, we know $|\partial(B)| \leq n+1$. Then assume for a contradiction that there are fewer than n internally disjoint outside u-bites.

Let U be the set of x attachments that forms the largest set of pairwise internally disjoint outside u-bites. Let P_U be the corresponding set of outside u-bites. Then, removing $\text{shad}_C(P)$ from C for each $P \in P_U$, results in $|P_U|$ segments of C. For each of these segments Γ, there are at most two (B_x, B) paths that contain vertices on Γ, otherwise we may increase the size of P_U.

So then with this we may find a B_x, B separating set of size $3|P_U|$, which, since G is oddly $\bar{k}$-connected, we have that $|P_U| \geq \bar{k}/3$. At most $n+1$ paths of P_U contain vertices of $\partial(B)$. This means there are at least $\frac{\bar{k}}{3} - n - 1$ disjoint outside u-bites that do not contain any of $\partial(B)$, which means there are at least $\frac{\bar{k}}{3} - n - 2$ outside u-bites which contain neither y, nor any vertex of $\partial(B)$. This means, for any $\bar{k} \geq 6n+6$, we guarantee that there are at least n disjoint outside u-bites that contain neither y, nor any of $\partial(B)$, nor all the terminals in $\text{ext}(C)$. In particular, there are at least n tasty attachments. This means there must be a terminal in the shadow of the outside u-bite. This means there is a magic set of order n.

Therefore, provided $\bar{k} \geq 6n+6$, there exists a cycle that has a magic set of order n. □

We now include the hypothesis $H_9 : \bar{k} \geq 6n+6$, so that C has a magic set of order n.

Chapter 6

Finding $odd - K_{3,3}$

We now assume that we have a cycle C with a unique in-bridge, a unique out-bridge, a terminal in both int(C) and ext(C), and a large magic set. We will now find a pair of alternating paths in B_x and B_y, so that we can find the target graft as a minor.

Definition 6.1. e and f are $x - compatible$ if:

1: They are edges adjacent to $V(C)$ and $V(B_x) \setminus V(C)$.

2: There exists a C-path obtained by deleting and contracting edges in the x-bridge, such that the C-path contains edges e and f, and contains x which is a terminal, and no other terminals, and at most one vertex of $V(C)$, is changed between being a terminal or not.

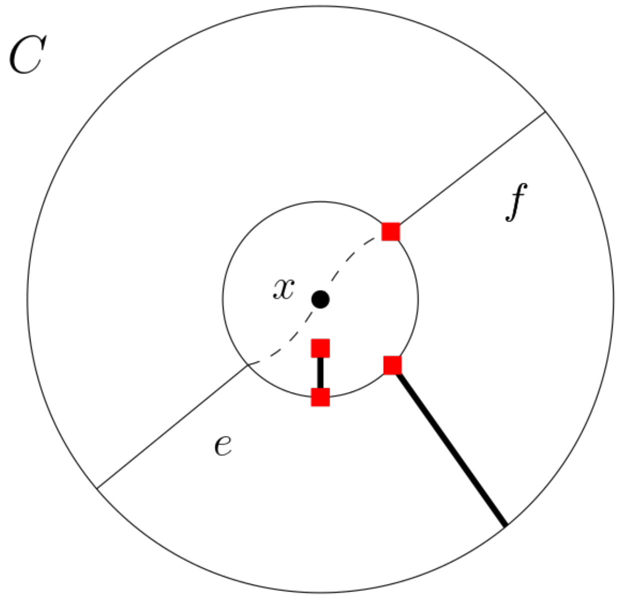

Figure 6.1: Example of compatible edges e and f. Contract the bolded edges, then the dashed path along with edges e and f form the C-path of Definition 6.1.

Proposition 6.2. There exists a partition R, Y of $\delta(V(\text{int}(C)))$, where all the C ends of edges in R appear in consecutive order, and then all the C ends of Y appear in consecutive order, such that, two edges e and f are x-compatible unless $e, f \in R$.

Proof. The proof of this will be shown in Section 6.3 $\square$

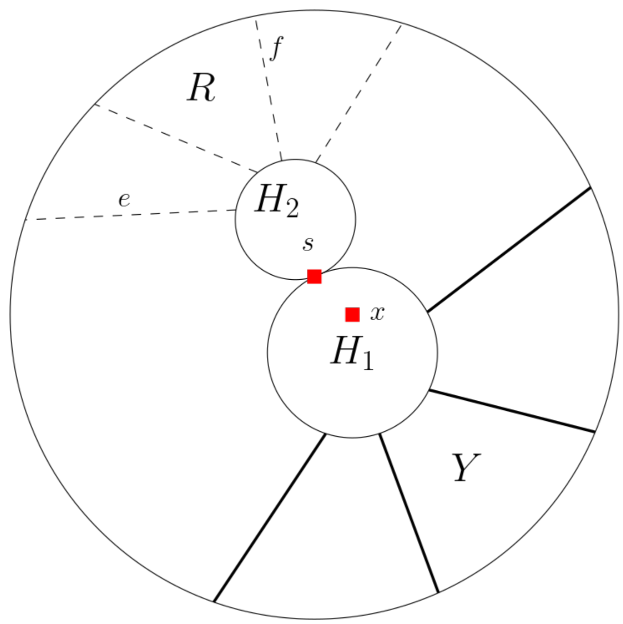

Figure 6.2: Example of compatible partition, where the set R is formed by dashed edges, and the set Y is formed by the bold edges. The bridge containing x consists of two 2-connected graphs H_1 and H_2

In Figure 6.2, it can be seen that no two edges e and f in R are compatible, since there is no path that includes e and f and x, since s is a single vertex separation of the bridge containing x. So for there to exist such a path after contraction, C must contain all edges of H_1, which means x is identified with s, which reduces the number of terminals to 0. On the other hand, if at least one edges lies in Y, then there exists a path containing e, f and x.

6.1 Target graft subdivision

Let $(\tilde{G}, \tilde{T})$ be a graft where $\tilde{G}$ is a subdivision of the underlying graph of the target graft (Figure 4.3). Let P_y be the C-path containing y, and P_x be the C-path containing x. Then say v_1 and v_3 are the endpoints of P_x, and v_2 and v_4 are the endpoints of P_y. It can be seen that $C \setminus \{v_1, v_2, v_3, v_3\}$ forms 4 disjoint paths Q'_i, where the endpoints of Q'_i are adjacent in C to vertices v_{i-1}, v_i. Then form Q_i by adjoining v_{i-1} and v_i to the endpoints of Q'_i. Note that in this case the $i-1$ arithmetic is performed modulo 4. Then let $R_i = Q_i \cup Q_{i+1}$. Paths Q_i are pictured in Figure 6.3.

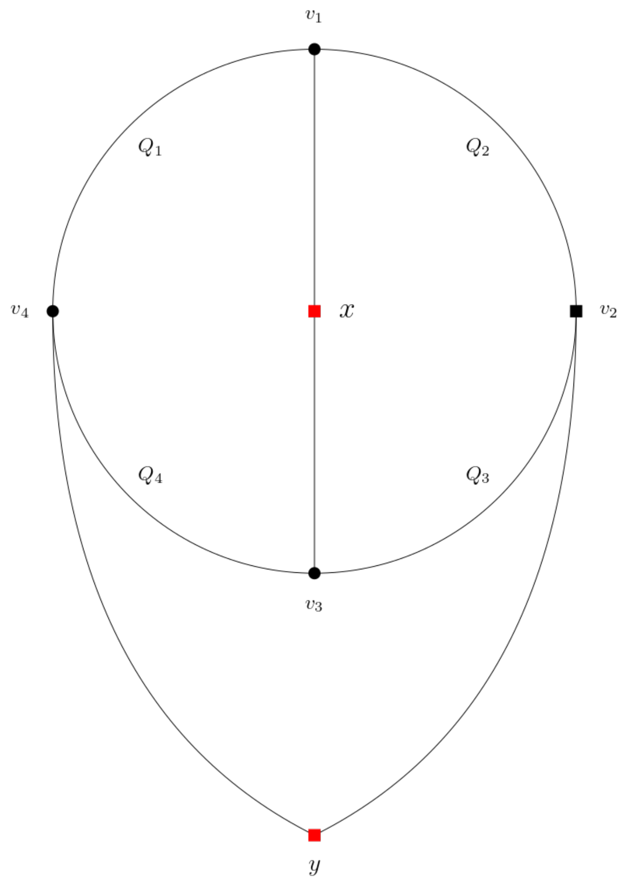

Figure 6.3: Labelled graft $(\tilde{G}, \tilde{T})$

We say $(\tilde{G}, \tilde{T})$ is *clustered* if $\tilde{T} \subseteq V(P_i)$ for any $i \in [4]$. Now note that $C \setminus (\text{bd}(P_y) \cup \text{bd}(P_x))$ forms 4 disjoint paths Q_i, where the endpoints of Q_i are adjacent to one endpoint v_1 of P_x, and one endpoint v_2 of P_y. Then let $Q'_i = Q_i \cup v_1 \cup v_2$. We say (G, T) is *skewed* if $P(Q'_i) = P(Q'_{i+2}) = 1$ and $Q_{i+1} \cap T = \emptyset = Q_{i+3} \cap T$. Note that the arithmetic $i + 2, i + 2$ is performed modulo 4, and will be for the remainder of the section.

Proposition 6.3. For a graph G that is a subdivision of the underlying graph of the target graft, where both P_x and P_y contain an odd number of terminals, not counting the endpoints on C, given any Q_i that has an even nonzero number of terminals, then there exists a set of edges $\hat{E} \subseteq E(Q_i)$ to contract so that the set of terminals in $(G, T)/\hat{E}$ is $(T \setminus Q_i)\Delta\{v_i, v_{i+3}\}$.

Proof. Consider a Q_i with a non-zero even number of terminals. Then traversing from

v_i to v_{i+3} through Q_i, there is a first terminal encountered, say s, and a last terminal encountered, say t. Then contract the edges that lie in the subpath between v_i and s, and those on the subpath between t and v_{i+3}. Then clearly the endpoints have changed between being a terminal or not, also Q_i still contains an even number of terminals. If there are no terminals in Q_i, then we are done. So say there are at least 2. Then there exists a subpath whose endpoints are terminals, and no internal vertices are terminals, so contract the edges of this path, and the number of terminals has decreased, so by induction we are done. □

Also note that if Q_i has an even number of terminals, and we contract all edges of Q_i that are incident to neither v_i nor v_{i+3} then Q_i has no terminals, and the endpoints remain unchanged.

Lemma 6.4. For a graph G that is a subdivision of the underlying graph of the target graft, where P_x and P_y both contain an odd number of terminals, not counting the endpoints on C. Given any Q_i that has an odd number of terminals, then there exists a set of edges $\hat{E}_i$ to contract so that the set of terminals in $(G,T)/\hat{E}_i$ is $(T \setminus Q_i)\Delta\{v_i\}$, and there exists a set of edges $\hat{E}_{i+3}$ to contract so that the set of terminals in $(G,T)/\hat{E}_{i+3}$ is $(T \setminus Q_i)\Delta\{v_{i+3}\}$.

Proof. $\hat{E}_i$ is all edges of $E(Q_i)$ except for the edge which is incident to v_{i+3}, similarly $\hat{E}_{i+3}$ is all edges except the edge incident to v_i □

Lemma 6.5. Let (G,T) be a graft where G is a subdivision of the underlying graph of the target graph, and P_x and P_y both contain an odd number of terminals, not counting the endpoints on C. If C is not clustered or skewed, we can find the target graft as a minor.

Proof. The first possibility is that $\Lambda_T = \{v_1, v_2, v_3, v_4\}$.

We will proceed by induction. Clearly, if all Q_i contain no terminals then we have a subdivision of the underlying graph of the target graft, where the only terminals are v_i, and $P(P_x) = P(P_y) = 1$.

Now, if all Q_i contain an even number of terminals, then contract all edges not incident to any v_i, to get a subdivision of the underlying graph of the target graft, where the only terminals are v_i, and $P(P_x) = P(P_y) = 1$.

So say there is some Q_i with an odd number of terminals. Then, since $|T|$ is even, there must be some other Q_j that has an odd number of terminals.

If $j = i+1$, then as in Lemma 6.4 move all terminals from Q_i and Q_j to $v_j = v_{i+1}$, to get (H,R) where H is a subdivision of the underlying graph of the target graft, and

$v_i \in R$ for each i, and $|R| < |T|$. Moreover, we claim that R is not clustered, and is not skewed. Clearly R is not clustered since each P_i contains $v_i \in T$. Moreover, R is not skewed, since that would mean that one of Q'_i or Q'_{i+1} contains an odd number of terminals, but since each v_i is a terminal that means either Q_i or Q_{i+1} contains an odd number of terminals, but we know they both contain no terminals.

So then say $j = i + 2$. Since T is not skewed, we know one of Q_{i+1} or Q_{i+3} contain some terminals. Say Q_{i+1} contains some terminals. If Q_{i+1} contains an odd number of terminals, then this is simply the case covered above. So Q_{i+1} contains a non-zero even number of terminals. So then as in Lemma 6.3, change the parity of v_i and v_{i+1}. Then as in Lemma 6.4, move all terminals of Q_i to v_i, and all terminals of Q_{i+2} to v_{i+1}. So then this results in a graph (H, R) where each v_i is a terminal, none of Q_i, Q_{i+1} or Q_{i+2} contain terminals, so R is neither clustered nor skewed, and $|R| < |T|$, so we are done by induction.

The next possibility is that $|\Lambda_T \cap \{v_1, v_2, v_3, v_4\}| = 3$. Say $\{v_i, v_{i+1}, v_{i+2}\} \subseteq T$. Then, since T is not clustered, we know that P_{i+3} contains a terminal. So, one of Q_i or Q_{i+3} contains a terminal. If one of these contains an odd number of terminals, say Q_i does, then as in Lemma 6.4 move all terminals to v_{i+3}. Then clearly the terminals are not clustered. So say the resulting terminals are skewed. Then this means Q_{i+1} has an odd number of terminals, and Q_{i+2} contains no terminals, since Q_i has no terminals. So in the original graft, move all of Q_i and Q_{i+1} to v_i. So v_i remains a terminal. Then, since $|T|$ is even, we see that Q_{i+3} contains an odd number of terminals. So, as in Lemma 6.4, move all terminals to v_{i+3}. Then we have a desired subdivision.

So say both Q_i and Q_{i+3} contain an even number of terminals. Then say Q_i contains a non-zero even number of terminals. Then move one to v_{i+3}. Then, if the remaining terminals are skewed, this means that Q_{i+2} contains an odd number of terminals, and Q_{i+3} and Q_{i+1} contain no terminals. But this is impossible since T is not skewed.

The next possibility is that $|\Lambda_T \cap \{v_1, v_2, v_3, v_4\}| = 2$.

Say v_i and v_j are terminals. Then since T is not skewed, either Q_k, Q_{k+1}, Q_{k+2} contain terminals for some k, or Q'_k, Q'_{k+2} contain an even number of terminals, and Q'_{k+1} and Q'_{k+3} contain no terminals.

Say Q_k, Q_{k+1}, Q_{k+2} each contain some terminals. So then one of Q'_k or Q'_{k+2} don't contain both v_i and v_j. Say Q'_k does not contain v_i. Then via Lemma 6.4 or Lemma 6.3 send a terminal from Q_k to an endpoint which is not v_j, call this v_l. Then 3 of v_1, v_2, v_3, v_4 are terminals. Since not both Q_{k+1} and Q_{k+2} can be incident to v_l, so the terminals are not clustered. Moreover, both Q_{k+1} and Q_{k+2} contain terminals, so the terminals are not skewed. So this reduces to the previous case.

The next possibility is that $|\Lambda_T \cap \{v_1, v_2, v_3, v_4\}| = 1$. Say v_i is a terminal. Then, since T is not clustered, one of Q_{i+2} or Q_{i+3} contains a terminal. Say Q_{i+2} has a terminal. Then one of Q_i or Q_{i+3} must have a terminal.

Say Q_i has a terminal, then if Q_{i+3} and Q_{i+1} have no terminals, Q_{i+2} must have an even number of terminals, so send one to v_{i+1} and one to v_{i+2}. Then since $|T|$ is even, Q_i must have a terminal. So the terminals are not clustered, and are not skewed, so this reduces to a previous case. So then one of Q_{i+3} or Q_{i+1} has a terminal. If Q_{i+1} has a terminal, send a terminal from Q_{i+2} to v_{i+2}. Then the terminals remain not clustered. Moreover, Q_i and Q_{i+1} both have terminals, so the terminals are not skewed. If Q_{i+3} has a terminal, send one from Q_{i+2} to v_{i+1}. Then the terminals again are neither clustered nor skewed, but more than one of v_j is a terminal, which is a previous case.

The final possibility is that $|\Lambda_T \cap \{v_1, v_2, v_3, v_4\}| = 0$.

If all Q_i contain an odd number of terminals, then send all terminals in Q_i to v_i, and we are done.

If all are even, then since T is not clustered, at least Q_i and Q_{i+2} must have a non-zero number of terminals, so send these to the endpoints, and reduce all others to no terminals, and we are done.

So then two of Q_i are odd, and two are even. If Q_i and Q_{i+1} contain an odd number of terminals, then one of Q_{i+2} and Q_{i+3} must have a non-zero number of terminals, because T is not clustered. Say Q_{i+1} contains a non-zero number of terminals. Then send Q_i to v_{i+3}, Q_{i+1} to v_{i+2}, and Q_{i+1} to v_i and v_{i+1}, and reduce Q_{i+3} to no terminals, and we are done.

So then if Q_i and Q_{i+2} contain an odd number of terminals, one of Q_{i+1} or Q_{i+3} must have a terminal, since T is not skewed. So say Q_{i+1} has a non-zero number of terminals. So send Q_{i+1} to v_i and V_{i+1} and Q_i to v_{i+3} and Q_{i+2} to v_{i+2}, and reduce Q_{i+3} to no terminals, and we are done.

□

This means that given a graft (G,T), if we can find a cycle C with one in bridge that contains x and one out bridge that contains y, and a set of edges $\hat{E}$ to contract so that $(G,T)/\hat{E}$ is a subdivision of the target graft with an odd number of terminals on P_x and

P_y, where the terminals are neither clustered nor skewed, then (G, T) contains $odd - K_{3,3}$ as a minor.

6.2 Target graft as a minor

Let C be a cycle with R_x and Y_x as the partition of x-compatible edges, and R_y and Y_y the partition of y-compatible edges, and a large magic set. Then let e_1 be an edge in Y_x, and e_2 an edge in Y_y, so that e_1 and e_2 are not incident to each other. Then $e_1 \cap V(C)$ and $e_2 \cap V(C)$ separates C into two paths. Since there exists a magic set of order n, one of the two paths, say P must contain a magic set of order at least $n/4$ (divide by 4 to ignore any edge cases). Then traverse from $e_1 \cap V(C)$ through P to $e_2 \cap V(C)$, until we have encountered a magic set of order $(n/4)/6 = n/24$ (again to ignore edge cases). Then choose the next x attachment to be f_1, then continue along C until we have encountered a magic set of order $n/24$. Then choose the next y attachment to be f_2. Then there must be a magic set of order at least $n/24$ between f_2 and e_2. These edges partition C into 4 segments, where 3 of them contain at least $n/24 \geq 5$ terminals. Then, no matter which two vertices (if any) are flipped between being a terminal and not being a terminal, there are 3 segments which contain at least 3 terminals. In particular, the cycle is neither clustered nor skewed. So, by Lemma 6.5, we can find the target graft as a minor. This means if we have a large magic set, as long as we can construct R_x, Y_x, R_y, Y_y, then we can find $odd - K_{3,3}$ as a minor. This means, $n \geq 200$ is easily seen to be sufficient. There are some obvious areas in which this bound can be improved, by analyzing the edge cases carefully. Thus to prove Theorem 4.1, it suffices to prove Proposition 6.2, which we shall do in the next section.

6.3 Finding a Compatible Partition

We first show a technical lemma which will be used to move a single vertex in a bridge to one of its attachments.

Lemma 6.6. Say H is a graph with $S \subseteq V(H)$, $|S| \geq 2$, and $V(H) \setminus S \neq \emptyset$. Then assume $H \setminus S$ is connected and there is no vertex v such that there exists a vertex u, where any $u - S$ path must contain v. Assume also that there exists some $T \subseteq V(H) \setminus S$ with $|T|$ even.

Then for any $a, b \in S$ there exists some $(H', \{a, b\}) \preccurlyeq (H, T)$ where S remains distinct vertices of H'.

Proof. This proof is left as an exercise. To begin, consider 2 paths that minimize the number of bridges that only have attachments on one of the 2 paths. □

Consider the x bridge of C as a tree of two connected blocks. We say a leaf is *simple*, if all vertices except its attachment to the rest of the tree are neither x nor a terminal.

Proof of Proposition 6.2.

Claim 1: We may assume the x bridge is even

Proof of Claim 1: Say the x-bridge is odd. Then since any pair of edges incident to C of the x-bridge are compatible, indeed, contract the entire x-bridge into a single odd vertex, we may arbitrarily choose any consecutive pair of B_x edges with one end in C to be R_x and all other such edges to be Y_x. Then clearly this partition satisfies the conditions of Definition 6.2. ♢

Claim 2: We may assume there are no simple leaves.

Proof of Claim 2: if we had one, then e and f are compatible if and only if they are compatible after contracting the simple leaf block. Clearly if e and f are compatible after contracting the simple leaf, they are compatible beforehand, simply add all the edges of the leaf to the contracting edges.

If e and f are compatible in G, then since there is a single branch vertex, at most one path can use this vertex. This path still exists after contracting the leaf.

So we may contract all edges in simple leafs, and the sets of compatible edges remain unchanged. ♢

Claim 3: There do not exist 2 leaves where neither contains x.

Proof of Claim 3: Otherwise, there exist leaves L_1 and L_2 which are not simple, and don't contain x. So they have a terminal. Then we prove any pair is compatible. Then we pick arbitrary consecutive edges to form R_x.

Choose any e and f. Say L_1 does not contain both e and f, without loss of generality, say e is contained in L_2. Then if $\partial(L_1)$ contains an odd number of terminals, contract all edges of L_1 with no endpoint in C, and no endpoints in $\partial(L_1)$. Then this turns into a single vertex. Then if f is contained in L_1, contract f, otherwise, contract any edge of L_1 with an endpoint in C. Then the remainder of B_x is odd, so we may use a previous claim, to see that all edges are compatible, so choose any consecutive (along C) edges to be R_x, and all other edges to be Y_x. Notice that at most one vertex of C has changed parity. ♢

So this means that there are at most 2 leaves, and if we have exactly 2, then one of them contains x, and the other a terminal.

Say there are two leaves. Then the x bridge forms a path of two connected blocks. *Claim 4:* R_x is all the edges with one endpoint in C contained in the leaf not containing x.

Proof of Claim 4: Let L be the leaf that does not contain x. If $|(L \setminus \partial(L)) \cap T|$ is odd, then contract all edges not incident to $\partial(L)$ and not incident to C, and if e (or f) is contained in L, contract e (or f) as well, otherwise contract any L edge with an endpoint in C. Then at most one vertex of C has changed parity, and since $|T \cap V(B_x)|$ is even, the remainder of B_x must have an odd number of terminals, so be a previous claim, we are done.

So then $|(L \setminus \partial(L)) \cap T|$ is even. So then as in Lemma 6.6 let L be H, and if e (or f) is contained in L, let v be the non C endpoint of e (or f), otherwise, let v be the non C endpoint of any edge of L incident to C. Let S be $\partial(L) \cup \{v\}$. Then this satisfies the conditions of Lemma 6.6. So then contract the edges so that $\partial(L)$ is a terminal and v is a terminal. Then contract the edge between v and C, so that the remainder of B_x is odd, and at most one vertex on C has changed parity, as desired. $\Diamond$

So now say the bridge is 2 connected.

Claim 5: If e and f are not compatible, then e and f are adjacent.

Proof of Claim 5: Let e and f be non-adjacent edges. Then, since the bridge is 2-connected, find paths from the non C endpoints of e and f to x. Then using Lemma 6.6, we can move terminals to this path, and since e and f are non-adjacent, we may contract the shortest subpath which contains e or f, and does not contain x, and has the non e endpoint in T. Then the remainder of the x bridge has an odd number of terminals, and at most one vertex on C has changed parity. $\Diamond$

Claim 6: If e and f are not compatible, then any path between the common end in $\text{int}(C)$, and x, separates all the terminals from all the attachments.

Proof of Claim 6: Say this is not the case. Then by definition there is a non separating path from the mutual endpoint of e and f to x, say P. Then consider the bridges in B_x of P. Since P was non separating, one of these bridges contains a C attachment, and contains a terminal. If this bridge is odd, then contract all edges in this bridge that are incident to neither C nor P. Then contract a single edge incident to C, so that the remainder of B_x is odd, and a single vertex on C has changed parity.

So now all bridges of P are even. For each bridge except for one that contains a terminal and has some C attachments, contract all edges incident to neither C nor P.

Then consider this bridge as a tree of blocks. Then, by connectivity, all leaf blocks contain edges incident to C. Then contract all blocks that contain no terminals. Then, if there are any blocks with an odd number of terminals, contract all blocks with an even number of terminals. Then there exists some leaf block with an odd number of terminals and an edge incident to C, so contract all edges of this block, except those incident to C, or incident to a 1-separation of the bridge. Then contract one edge incident to C. Then the remainder of the P bridge is odd. So contract onto P. Then the remainder of B_x is odd, and one vertex on C has changed parity.

So say all blocks of the P bridge have an even number of terminals. So contract all blocks except for a single block with a non-zero even number of terminals. Then the remaining P bridge is even, and two connected, with a P attachment and a C attachment. So let the non P endpoint, and non C endpoint of these edges form S, and this two connected block form H as in Lemma 6.6. Then move one terminal to C, and the other to P, so that the remainder of B_x is odd, and one vertex on C has changed parity.

So then e and f were compatible, a contradiction. $\diamondsuit$

Claim 7: There is at most one common vertex to form incompatible pairs.

Proof of Claim 7: Say there are two, v_1 and v_2. Then any $x - v_1$ path separates all the terminals from all the attachments, and so does any $x - v_2$ path. Then by Menger's theorem, choose 2 paths from x to $\{v_1, v_2\}$ that only intersect in x, say P_1 and P_2. Then P_2 separates the x-bridge into two regions, one of which contains all the attachment, and one of which contains all the terminals. Then, since v_2 is an attachment, it must lie in the region that contains no terminals. But then since G is planar, and P_1 and P_2 do not intersect, this must mean that v_1 is embedded in the region that contains the terminals of the x-bridge after removing P_2. Which means there is a path from a terminal to an attachment, which means v_2 was not a vertex that forms incompatible pairs. $\diamondsuit$

Then, if there are any incompatible pairs, we know they are all the edges incident to a vertex v. So let R_x be all the edges with one endpoint on C, and the other endpoint v. Then let Y_x be all other edges. $\square$

Then we can always find a partition R_x, Y_x and a partition R_y, Y_y as in Proposition 6.2. This means we can always find the target graft as a minor, which means the original graft, with the edge xy has $odd - K_{3,3}$ as a minor.

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