Topics in Arithmetic Dynamics

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

This thesis comprises four papers completed during my doctoral studies at the University of Waterloo and is organized into three chapters.

The first chapter concerns preimage problems and dynamical cancellation, incorporating the papers *Dynamical Cancellation of Polynomials*, published in the *Bulletin of the London Mathematical Society*, and *Preimages Question for Surjective Endomorphisms on* $(\mathbb{P}^1)^n$, published in the *New York Journal of Mathematics*. We investigate stabilization phenomena for the set of rational points occurring in the preimages of invariant subvarieties under algebraic dynamical systems. In the special case in which the subvariety is the diagonal subvariety in $\mathbb{P}^1 \times \mathbb{P}^1$ and the dynamics is given by a pair of rational functions (f, f), the problem reduces to a dynamical cancellation question. We also obtain a generalization of dynamical cancellation to the setting in which the dynamics are generated by a semigroup of polynomials.

The second chapter contains the joint paper with Chatchai Noytaptim, *A Finiteness Result for Common Zeros of Iterates of Rational Functions*, published in *International Mathematics Research Notices*. Addressing a question posed by Hsia and Tucker concerning finiteness properties of greatest common divisors of polynomial iterates, we prove that if $f, g \in \mathbb{C}(X)$ are compositionally independent rational functions and $c \in \mathbb{C}(X)$, then—apart from a few explicit exceptional families with $f, g \in \mathrm{Aut}(\mathbb{P}^1_{\mathbb{C}})$—there exist only finitely many $\lambda \in \mathbb{C}$ for which there is an n satisfying

$$f^n(\lambda) = g^n(\lambda) = c(\lambda).$$

The final chapter presents my recent preprint, *Polynomial Endomorphisms of* $\mathbb{A}^2$ *with Many Periodic Curves*. We show that for any regular polynomial endomorphism of positive degree on $\mathbb{P}^2$, every family of curves containing a Zariski dense set of periodic curves must be invariant under some iterate of the map. This establishes a weaker form of the Relative Dynamical Manin–Mumford Conjecture of DeMarco and Mavraki in the setting where the endomorphism is fixed in the family of dynamical systems, and may also be viewed as a dynamical Manin–Mumford statement on the moduli space of divisors. As an application, we classify all regular polynomial endomorphisms of $\mathbb{P}^2$ that admit infinitely many periodic curves of bounded degree.

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Table of Contents

Examining Committee Membership ii

Author's Declaration iii

Abstract iv

Acknowledgements v

1 Preliminaries 1

1.0.1 Arithmetic geometry and the dynamical viewpoint 2

1.0.2 Rational points in the preimages 4

1.0.3 Unlikely intersection type questions 5

1.0.4 Dynamical Manin–Mumford type questions 6

2 Preimages Questions and dynamical cancellations 8

2.1 Introduction 8

2.1.1 Main result on the Preimages Questions on $(\mathbb{P}^1)^n$ 9

2.1.2 Improvement on the dynamical cancellation of a semigroup of polynomials 9

2.2 Preliminary lemmas 12

2.3 Proof of Theorem 2.1.6 15

2.3.1 Producing h_1 and an irreducible invariant curve 16

2.3.2 Reduction to cases (1) or (2) in Theorem 2.1.6 17
2.3.3 Sufficiency of (1), (2) in Theorem 2.1.6 22
2.4 Computing N in Theorem 2.1.6 23
2.5 Proof of Theorem 2.1.1 25

3 A finiteness result for common zeros of iterates of rational functions 41
3.1 Introduction 41
3.1.1 Historical background. 41
3.1.2 Statement of main results. 42
3.1.3 Proof strategy. 43
3.2 Preliminaries 44
3.2.1 Absolute values over global fields. 44
3.2.2 Dynamical height and preperiodic points. 45
3.2.3 Arakelov-Green's function. 46
3.2.4 Results from Diophantine geometry. 47
3.2.5 The Ping-Pong Lemma 48
3.2.6 Asymptotic notation 48
3.3 Proof of Theorem 3.1.3 49
3.4 Proof of Theorem 3.1.4 77

4 Polynomial endomorphisms of the affine plane with many periodic curves 82
4.1 Introduction 82
4.1.1 Relative Dynamical Manin-Mumford Conjecture 83
4.1.2 Main results 84
4.1.3 Some conditional results on degree stabilization 86
4.2 Outline of the chapter 88
4.3 Preliminaries 88
4.4 Proof of Theorem 4.1.3 92

4.5 Implications toward the Conjecture 4.1.2 101
4.6 Degree stabilization for families of endomorphisms 105
4.6.1 One-dimensional families of polynomial endomorphisms on $\mathbb{P}^2$ . . . 108
4.7 Endomorphisms with infinitely many periodic curves of bounded degree . . 115

References **124**

Chapter 1

Preliminaries

Arithmetic dynamics emerged as a distinct field in the 1990s, motivated by the interplay between dynamical systems and advances in arithmetic geometry. Since then, researchers have sought to reinterpret classical results—such as the Mordell–Lang conjecture—within dynamical frameworks.

Arithmetic dynamics studies the arithmetic and geometry of pairs (X, f) consisting of an algebraic variety X together with a self-map $f : X \to X$. At a formal level this is a simple idea: one starts with a map and iterates it. At a conceptual level, however, it provides a flexible language in which many questions from arithmetic geometry can be reformulated, compared, and generalized. In particular, several classical arithmetic phenomena that are naturally expressed in terms of torsion points, finitely generated groups, or special subvarieties admit dynamical counterparts in terms of preperiodic points, semigroup orbits, and invariant or preperiodic subvarieties.

A recurring principle in algebraic geometry is that one often understands a variety most effectively by studying the maps it carries: morphisms to other varieties, endomorphisms, correspondences, and symmetries. Arithmetic dynamics fits naturally into this philosophy. Rather than studying a variety in isolation, one studies how arithmetic information is organized by iteration of a map. The self-map produces orbits, periodicity, and invariant subvarieties; it also gives a natural notion of special points and special geometric configurations. From this point of view, arithmetic dynamics is not merely an analogy with arithmetic geometry. In many important cases, it contains the classical arithmetic-geometric picture as a special case, while at the same time suggesting broader conjectures that continue to make sense beyond the setting of algebraic groups.

1.0.1 Arithmetic geometry and the dynamical viewpoint

Formally, a dynamical system consists of a set S together with a self-map $f : S \to S$. The most basic objects attached to such a system are the forward orbits of points.

Definition 1.0.1. For a point $\alpha \in S$, the *orbit* of α under f is the set

$$\mathrm{Orb}_f(\alpha) = \{f^n(\alpha) : n \in \mathbb{N}^+\}.$$

Definition 1.0.2. The sets of *periodic* and *preperiodic* points of f are defined respectively by

$$\mathrm{Per}(f, S) = \{\alpha \in S : f^n(\alpha) = \alpha \text{ for some } n \in \mathbb{N}^+\},$$

and

$$\mathrm{Prep}(f, S) = \{\alpha \in S : f^{m+n}(\alpha) = f^m(\alpha) \text{ for some } m \geqslant 0,\ n \geqslant 1\}.$$

When one studies several maps at once, semigroup orbits play the same role as forward orbits for a single map.

Definition 1.0.3. Let $\mathcal{F}$ be a collection of self-maps of S, and let $\langle \mathcal{F} \rangle$ denote the semigroup generated by $\mathcal{F}$ under composition. For $\alpha \in S$, the *semigroup orbit* of α under $\mathcal{F}$ is

$$\mathrm{Orb}_{\mathcal{F}}(\alpha) = \{g(\alpha) : g \in \langle \mathcal{F} \rangle\}.$$

These elementary notions already reveal the connection with arithmetic geometry. Let E be an elliptic curve over $\mathbb{C}$, and let $[m] : E \to E$ be the multiplication-by-m map for some integer $m \geqslant 2$. The identity point $\mathcal{O}$ is fixed by $[m]$, and a point $P \in E(\mathbb{C})$ is torsion if and only if it is preperiodic for $[m]$. Indeed,

$$\mathrm{Tor}(E, \mathbb{C}) = \mathrm{Prep}([m], E(\mathbb{C})).$$

Thus torsion points, which are fundamental objects in arithmetic geometry, appear dynamically as preperiodic points of a natural self-map.

This example extends beyond individual points. On an abelian variety A, the map $[m] : A \to A$ preserves algebraic subgroups and sends torsion translates to preperiodic subvarieties. In this way, the special subvarieties that play a central role in classical arithmetic geometry become dynamical objects defined by invariance or eventual periodicity under iteration. Said differently, the group structure on A produces distinguished self-maps, and once those maps are placed at the center of the discussion, familiar arithmetic geometric notions are naturally reinterpreted in dynamical language.

The same point of view also reinterprets the Mordell–Weil theorem. If $E(\mathbb{Q})$ is finitely generated, then after choosing generators of its free part and of its finite torsion subgroup, every rational point can be reached from $\mathcal{O}$ by composing finitely many translations and inverse translations. More precisely, if $\mathcal{F}$ is the finite collection of translations by those chosen generators and their inverses, then

$$\mathrm{Orb}_{\mathcal{F}}(\mathcal{O}) = E(\mathbb{Q}).$$

Hence the arithmetic problem of describing $E(\mathbb{Q})$ may be viewed as a problem about a finitely generated dynamical semigroup orbit. From this perspective, finitely generated groups in arithmetic geometry are modeled dynamically by orbits under finitely generated semigroups of morphisms.

These examples suggest a useful dictionary.

- torsion points correspond to preperiodic points for multiplication maps;
- finitely generated groups are modeled by semigroup orbits generated by finitely many morphisms;
- special subvarieties such as algebraic subgroups and torsion translates correspond to invariant or preperiodic subvarieties;
- questions about the distribution or intersection of special points become questions about orbits, preperiodic sets, and periodic subvarieties.

This dictionary is not merely rhetorical. Several major conjectures and theorems of arithmetic geometry have dynamical analogies or extensions. The Mordell–Lang conjecture has the dynamical Mordell–Lang conjecture as its counterpart. Similarly, the Manin–Mumford conjecture has the dynamical Manin–Mumford conjecture as its counterpart. More broadly, unlikely intersection phenomena in arithmetic geometry suggest analogous rigidity statements for intersections of orbits, preperiodic points, or other dynamically defined objects. In many situations the dynamical setting is genuinely more flexible: the language of self-maps still makes sense on varieties that do not carry a group law.

This map-centered viewpoint is one of the main motivations of the present thesis. The three directions developed here should be understood as instances of the same general principle: each begins with a classical phenomenon from arithmetic geometry or number theory and asks to what extent it persists, or must be reformulated, once it is placed in a genuinely dynamical setting. All three topics have received substantial attention in

recent years and have seen important progress by many authors, reflecting their importance in the broader program of understanding arithmetic dynamics as a natural extension of arithmetic geometry. The results presented in this thesis contribute further to that program by pushing beyond previously known cases, sharpening the existing analogies between arithmetic geometry and dynamics, and enlarging the range of situations in which the dynamical framework not only recovers classical results as special cases, but also suggests new and more general phenomena.

1.0.2 Rational points in the preimages

One of the most basic questions in arithmetic geometry is to understand the distribution of rational points on a variety and, more specifically, on its distinguished subvarieties. On an abelian variety, algebraic subgroups and their translates form the natural special subvarieties. As noted above, these objects are closely related to invariant and preperiodic subvarieties for multiplication maps.

One of the projects I worked on during my Ph.D. concerns the following dynamical version of rational points distribution question: given a self-map $f : X \to X$ and a subvariety $Y \subseteq X$ satisfying $f(Y) \subseteq Y$, what can one say about the rational points appearing in the backward orbit of Y?

If Y is invariant under f, then one obtains an increasing tower

$$Y \subseteq f^{-1}(Y) \subseteq f^{-2}(Y) \subseteq \cdots \subseteq f^{-n}(Y) \subseteq \cdots .$$

Geometrically, the preimages typically become more complicated as n increases. Arithmetically, however, one expects a form of stabilization over a fixed number field. The intuition is that although the geometry becomes richer under repeated pullback, genuinely new rational points should not continue to appear indefinitely without a structural reason as more complicated geometry usually imposes stronger restrictions on the set of rational points admitted. This expectation is formalized in the Preimages Question of Matsuzawa, Meng, Shibata, and Zhang [59, Question 8.4(1)].

There is also a close relation with cancellation problems. If one considers a map of the form $f = (g, g)$ on $X \times X$ and takes $Y = \Delta$ to be the diagonal, then the question whether the tower of K-points on $f^{-n}(\Delta)$ eventually stabilizes becomes a uniform dynamical cancellation question: if two K-points become equal after a long iterate of g, must they already become equal after a bounded number of iterates? Viewed this way, equality of late iterates is a special relation created by the map, and arithmetic stabilization predicts that such coincidences are uniformly controlled.

The first chapter of this thesis studies this circle of ideas in two concrete directions. First, I prove a positive solution to the Preimages Question for surjective endomorphisms on $(\mathbb{P}^1)^n$. In that setting, the result shows that although the inverse images of an invariant subvariety may continue to change geometrically, their rational points over a fixed number field eventually stop producing anything new. Second, I study an improved dynamical cancellation problem for finitely generated semigroups of polynomial self-maps on $\mathbb{P}^1$. There one can go further and characterize precisely the exceptional configurations that obstruct uniform cancellation.

The chapter incorporates two papers: *Dynamical Cancellation of Polynomials* [93] and *Preimages Question for Surjective Endomorphisms on* $(\mathbb{P}^1)^n$ [95].

1.0.3 Unlikely intersection type questions

Another major theme in arithmetic geometry is the study of *unlikely intersections*. Roughly speaking, one is interested in situations where two sets having little reason to meet nevertheless intersect in a way that is too large to be explained by naive dimension heuristics. In the arithmetic setting, the sets involved are often special subvarieties, torsion sets, or other distinguished loci, and the conclusion is that an unexpectedly large intersection forces hidden algebraic structure. We refer to Zannier [90] for a broad account of this theme.

The dynamical setting naturally produces its own version of unlikely intersection phenomena. Here the special sets are created by iteration. Two forward orbits may meet too often, two sets of preperiodic points may have an unexpectedly large intersection, or two sequences generated by iteration may satisfy far more algebraic relations than one would predict at random. When this happens, the expected conclusion is not accidental coincidence but a strong dynamical relation between the maps themselves, such as shared iterate behavior, semiconjugacy, common exceptional structure, or another rigid link.

For the purposes of this thesis, unlikely intersection-type questions provide a good example of how arithmetic dynamics reinterprets classical phenomena. In the classical theory, a large intersection often signals the presence of a hidden subgroup or a multiplicative dependence. In the dynamical theory, the corresponding signal is a hidden relation between self-maps.

The second chapter studies a problem of this kind motivated by a theorem of Bugeaud, Corvaja, and Zannier [18] on the greatest common divisors of integer sequences. Their result can be interpreted as saying that repeated arithmetic coincidence is rare unless there is a special multiplicative dependence between the generators. In the dynamical setting, an analogous question asks how often two sequences produced by iterating rational functions

can have common zeros. Since the two corresponding dynamical level sets are countable sets of points, an abundance of common zeros is an unlikely intersection phenomenon in a genuinely dynamical sense.

Chapter two develops this viewpoint by proving a finiteness result for common zeros of iterates of rational functions. The statement belongs to the same general family as many rigidity results in arithmetic dynamics: if a special intersection occurs too often, then the maps cannot be independent in any meaningful sense. The chapter therefore fits naturally into the broader program of understanding how classical intersection principles in arithmetic geometry generalize to statements about the interaction of dynamical systems. This chapter consists of the joint paper with Chatchai Noytaptim, *A Finiteness Result for Common Zeros of Iterates of Rational Functions* [64].

1.0.4 Dynamical Manin–Mumford type questions

A further guiding principle in arithmetic geometry is that a subvariety containing "many" special points should itself be special. The classical Manin–Mumford conjecture, proved by Raynaud [72, 73], is the prototypical example: if V is a subvariety of an abelian variety A over $\mathbb{C}$, then V contains a Zariski dense set of torsion points if and only if V is a torsion translate of an abelian subvariety. In other words, the abundance of special points forces special geometry.

As discussed earlier, torsion points are dynamically interpreted as preperiodic points for multiplication maps. This suggests that Manin–Mumford should admit a dynamical analogue in which torsion points are replaced with preperiodic points and torsion translates are replaced with dynamically special subvarieties. The resulting picture is subtler than a literal translation, because general self-maps may admit additional symmetries or commuting maps that also contribute to the structure of special subvarieties. Nevertheless, the basic principle remains the same: a subvariety supporting a Zariski dense set of dynamically special points should itself be special for dynamical reasons.

One formulation, developed through work of Ghioca, Tucker, and Zhang [91, 92, 43], is the following conjecture.

Conjecture 1.0.4. *Let $f : X \to X$ be a polarized endomorphism of a smooth projective variety over a field of characteristic zero, and let $Z \subset X$ be a subvariety containing a Zariski dense set of preperiodic points. Then either Z is preperiodic, or Z is special in the sense that it is contained in a subvariety Y which is both f^n- and ψ-invariant for some $n \geqslant 1$, where ψ is another polarized endomorphism commuting with f^n on Y, and Z is preperiodic under ψ.*

This conjecture remains widely open, with only a limited number of cases known [41, 42, 28]. Even so, it provides an important organizing principle: rather than viewing dense sets of preperiodic points as isolated accidents, one interprets them as evidence of an underlying dynamical structure. Once again, the self-map is the central object. The specialness of the subvariety is measured not in an absolute sense, but relative to the dynamics that act on it.

The third chapter of this thesis studies a problem that belongs to this same circle of ideas, but from a relative and moduli theoretic perspective. Instead of fixing a single subvariety and asking whether it contains a dense set of preperiodic points, I consider families of subvarieties for a regular endomorphism of $\mathbb{A}^2$ and ask what happens when the family contains a Zariski dense set of periodic members. This shifts the emphasis from one special subvariety to a parameter space of subvarieties on which the ambient map induces additional dynamical structure. In that sense, the chapter pushes the Manin–Mumford philosophy one level higher: one studies not only special points inside a fixed geometric object, but also special subvarieties moving in families under the action induced by a self-map.

This perspective is closely related to recent relative dynamical Manin–Mumford questions studied by DeMarco and Mavraki [27]. It also illustrates the broader philosophy of the thesis. Arithmetic dynamics encourages us to study maps first and then ask how they act not only on points, but also on subvarieties, moduli spaces, and families. Once this viewpoint is adopted, many familiar arithmetic geometric questions naturally reappear in a more flexible dynamical form. Chapter three consists of my recent preprint, *Polynomial Endomorphisms of $\mathbb{A}^2$ with Many Periodic Curves* [94].

The three topics treated in this thesis are therefore not separate stories but different manifestations of the same guiding idea. Chapter one concerns backward orbits of invariant subvarieties and the eventual stabilization of their rational points. Chapter two concerns unexpectedly large intersections between dynamically defined sets. Chapter three concerns the geometric structure forced by an abundance of periodic subvarieties in a family. In each case, the central question is how a classical arithmetic geometric rigidity phenomenon should look after one replaces arithmetic geometry specialness with dynamical specialness. The results of this thesis provide concrete evidence that the dynamical framework is broad enough both to recover familiar arithmetic geometry and to point toward more general phenomena that become visible only when maps and iteration are placed at the center of the picture.

Chapter 2

Preimages Questions and dynamical cancellations

2.1 Introduction

Let X be a projective variety and let $f : X \to X$ be a surjective self-map such that both X and f are defined over a number field K. To study the dynamics of (X, f), it is important to identify the closed subvarieties $Y \subseteq X$ that are invariant under f; i.e., subvarieties with $f(Y) \subseteq Y$. For an invariant subvariety Y for the map f, it is natural to study its preimages under iterates of f. An important principle within arithmetic dynamics is that the underlying geometric structure should exert significant influence on the arithmetic structure, which gives rise to the expectation that the tower of K-points:

$$Y(K) \subseteq (f^{-1}(Y))(K) \subseteq (f^{-2}(Y))(K) \subseteq \cdots$$

should eventually stabilize. This expectation has been made precise in the form of the Preimages Question of Matsuzawa, Meng, Shibata, and Zhang [59, Question 8.4(1)].

There are recent works dealing with some special cases of the Preimages Question. Notably, it has been solved with affirmative answer when X is a smooth variety with non-negative Kodaira dimension and f is étale [13, Theorem 1.2]. Additionally, if we consider the case when $f = (g, g) : X \times X \to X \times X$ is a diagonal map, with $g : X \to X$ a surjective morphism, then the diagonal subvariety $\Delta \subseteq X \times X$ is an invariant subvariety under f, and in this special case the Preimages Question becomes a cancellation problem, which asks whether there exists a natural number s with the property that, for all $x, y \in X(K)$,

if $g^n(x) = g^n(y)$ for some natural number n then we must in fact have $g^s(x) = g^s(y)$. This special form is a dynamical cancellation problem considered in the work of Bell, Matsuzawa, and Satriano [13], and this question is again answered affirmatively when X is a curve [13, Theorem 1.3] by applying p-adic uniformization techniques of Rivera-Letelier [76].

2.1.1 Main result on the Preimages Questions on $(\mathbb{P}^1)^n$

Our main result is to give a positive solution to the Preimages Question for surjective endomorphisms on $(\mathbb{P}^1)^n$. It is well known that a surjective endomorphism of $(\mathbb{P}^1)^n$ has some iterate that becomes a split rational map $(f_1, \ldots, f_n)$. Since answering the Preimages question for an iterate of a self-map gives one the answer for the original map, our main theorem, stated below, solves the Preimages Question for surjective endomorphisms on $(\mathbb{P}^1)^n$.

Theorem 2.1.1. *Let K be a number field, let $n \geqslant 1$, and let $f = (f_1, \ldots, f_n) : (\mathbb{P}^1_K)^n \to (\mathbb{P}^1_K)^n$ be a split rational map defined over a number field K with at least one f_i of degree greater than 1. If $V \subseteq (\mathbb{P}^1)^n$ is a subvariety defined over K that is invariant under f then there exists a non-negative integer s_0 such that*

$$(f^{-s-1}(V) \backslash f^{-s}(V))(K) = \varnothing$$

for all $s \geqslant s_0$.

It is also well known that Theorem 2.1.1 holds when $n = 1$.

2.1.2 Improvement on the dynamical cancellation of a semigroup of polynomials

Henceforth we let $P_r(x)$ and $T_r(x)$ denote respectively the cyclic polynomial, x^r, of degree r and the Chebyshev polynomial of degree r.

Definition 2.1.2. We define the Chebyshev polynomial of degree r to be the unique polynomial T_r of degree r such that

$$T_r((x + x^{-1})/2) = (x^r + x^{-r})/2$$

for any $x \in \mathbb{C}^*$.

Notice that the set of Chebyshev polynomials forms a basis of the vector space of polynomials; i.e. for any polynomial $f(x)$ defined over $\mathbb{C}$ of degree d there exists a unique expansion

$$f(x) = \sum_{i=0}^{d} a_i T_i(x)$$

for some constant $a_i \in \mathbb{C}$.

Definition 2.1.3. For a polynomial $f(x)$ over an algebraically closed field F, if we can decompose $f(x)$ as

$$f(x) = f_n(x) \circ \cdots \circ f_1(x)$$

for some polynomials $f_i(x)$ over F of degree greater than 1, $i = 1, \ldots, n$ such that $f_i(x)$ cannot be written as a composition of two polynomials of degree greater than 1, then we call each of the $f_i(x)$'s the indecomposable factor of $f(x)$.

Definition 2.1.4. We say that two polynomials $f(x)$ and $g(x)$ over an algebraically closed field F are linearly related (or linearly conjugated) if there exists a linear polynomial $l(x)$ defined over F such that $f(x) = l(x) \circ g(x) \circ l^{-1}(x)$.

About the Dynamical Cancellation problems, considerably more is known for polynomial maps on $\mathbb{P}^1$, and a more general dynamical cancellation results allowing multiple polynomial self-maps on $\mathbb{P}^1$ is also proved along these lines in [13, Theorem 1.7]. The statement is as follows:

Theorem 2.1.5. *[13, Theorem 1.7] Let K be a number field and let $\phi_1, \ldots, \phi_r$ be polynomial maps on $\mathbb{P}^1_K$ of degree at least two. Suppose that none of the indecomposable factors of $(\phi_i)_{\overline{K}}$ are linearly related to a T_d with d odd or a P_m for some m. Then there is a finite set $Z \subset (\mathbb{P}^1_K \times \mathbb{P}^1_K)(K)$ with the following property: for any point a, $b \in \mathbb{P}^1_K$ such that $(a, b) \notin Z$, if*

$$\phi_{i_n} \circ \cdots \circ \phi_{i_1}(a) = \phi_{i_n} \circ \cdots \circ \phi_{i_1}(b)$$

for some $n \geqslant 0$ and $i_1, \ldots, i_n \in \{1, \ldots, r\}$, then

$$\phi_{i_2} \circ \phi_{i_1}(a) = \phi_{i_2} \circ \phi_{i_1}(b).$$

This theorem, however, is not optimal. For example this theorem does not say anything when considering the polynomials $\phi_1 = T_5$, $\phi_2 = P_5$, although a cancellation result can be shown to hold in this case. In this paper, we completely characterize when we have a dynamical cancellation result involving multiple polynomials by proving a necessary and sufficient statement in terms of the polynomials in a finite set S.

Theorem 2.1.6. *Let S be a finite set of polynomials of degree at least 2 defined over a number field K and let $\langle S\rangle$ be the monoid generated by S under composition. Then there exists a $N \in \mathbb{N}^+$ and a finite set of points $Z \subset \mathbb{P}^1_K \times \mathbb{P}^1_K$ such that if*

$$\phi_k \circ \cdots \circ \phi_1(a) = \phi_k \circ \cdots \circ \phi_1(b) \tag{2.1.1}$$

with $\phi_j \in S$, $k > N$ and $(a,b) \in (\mathbb{P}^1_K \times \mathbb{P}^1_K)\backslash Z$ then we have

$$\phi_N \circ \cdots \circ \phi_1(a) = \phi_N \circ \cdots \circ \phi_1(b) \tag{2.1.2}$$

unless there exist $h_1, h_2 \in \langle S\rangle$, a complex linear polynomial $l(x) = ux + v$, $u \in \mathbb{C}^*$ and $v \in \mathbb{C}$, such that one of the following holds:

1. *$h_2 = P \circ P_d \circ l^{-1}$ and $h_1 = l \circ xQ(x^d) \circ l^{-1}$, for some polynomials P and Q, where $d \in \mathbb{N}_{\geqslant 2}$, $u = 1, v \in K$ and K contains a primitive d-th root of unity;*
2. *$h_2 = P \circ T_d \circ l^{-1}$ and $h_1 = l \circ \pm T_r \circ l^{-1}$, for some polynomials P, where $d|(r+1)$ or $d|(r-1)$ and one of the following holds:*
   (a) *$d = 2$ and $v \in K$;*
   (b) *$d > 2$ and there exists a primitive d-th root of unity ϵ such that the conic*

   $$V(Y^2 + (\epsilon + 1/\epsilon - 2)v \cdot (X+Y) - (\epsilon + 1/\epsilon)XY + X^2 + (2 - \epsilon - 1/\epsilon)v^2 + (\epsilon - 1/\epsilon)^2 u^2)$$

   inside $\mathbb{A}^2_K \subset \mathbb{P}^1_K \times \mathbb{P}^1_K$ has a K-point, where X and Y are variables; moreover, we can further refine the condition if r is even. In this case the conic having a K-point is equivalent to $u, v, \epsilon + 1/\epsilon \in K$.

Also h_1, h_2 in the statement, if they exist, can always be chosen as compositions of no more than N elements in S.

Remark 2.1.7. Moreover, if there exist h_1, h_2 and l as above such that one of (1) or (2) holds, then for any $j \in \mathbb{N}_{\geqslant 0}$, there exists an infinite set of K-points (a,b) in $\mathbb{P}^1_K \times \mathbb{P}^1_K$ such that

$$h_2 \circ h_1^j(a) = h_2 \circ h_1^j(b) \tag{2.1.3}$$

$$h_1^{\circ j}(a) \neq h_1^{\circ j}(b). \tag{2.1.4}$$

So Theorem 2.1.6 is an if and only if statement. We will also prove this remark in Section 3.

Remark 2.1.8. Notice that Theorem 2.1.6 is not necessarily stronger than Theorem 2.1.5 since there N can always be chosen as 2.

Remark 2.1.9. Given the number field K and the conic

$$C = V(Y^2 + (\epsilon + 1/\epsilon - 2)v \cdot (X + Y) + (\epsilon + 1/\epsilon)XY + X^2 + (2 - \epsilon - 1/\epsilon)v^2 + (\epsilon - 1/\epsilon)^2 u^2),$$

there are standard algorithms implemented in Sage to check if C has K-points. So the condition of case $(2)(b)$ in Theorem 2.1.6 can be checked with a finite number of computations.

We will also show that the number N is computable given the set S of polynomials. Therefore the two cases in Theorem 2.1.6 can be checked with a finite number of computations.

Remark 2.1.10. Notice that given a choice of h_2 and h_1 we can obtain a finite list of possible linear maps $l(x) = ux + v$ that can potentially make the conditions in (1) or (2) hold. For example, to make conditions in (1) holds, we check if

$$l^{-1} \circ h_1 \circ l = xQ(x^d).$$

In particular the vanishing of the constant term gives $h_1(v) - v = 0$ so that v can only take values from a finite set. Notice that $u = 1$. Thus, we have a finite list of possible $l(x) = ux + v$ and we can check if each linear map in the list satisfies the conditions one by one from here. Similar procedures work for checking conditions in (2).

2.2 Preliminary lemmas

In this section we provide some preliminary lemmas for the proof of Theorem 2.1.6. Before we start, we clarify some notations needed for the discussion. Let $f(X, Y)$ be a polynomial in variables X and Y. By $V(f)$ we mean an affine variety consisting of zeros of $f(X, Y)$ inside $\mathbb{C} \times \mathbb{C} \subset \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ and $V(f)(K)$ means the K-points of the variety $V(f)$ with a number field K.

Lemma 2.2.1. *Let $\pi(x) = x + 1/x$. If ϵ and ϵ' are roots of unity such that $V(Y - \epsilon X)$, $V(Y - \epsilon' X) \subseteq (\pi, \pi)^{-1}(C)$ for some irreducible curve $C \subseteq \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ then either $\epsilon = \epsilon'$ or $\epsilon = \epsilon'^{-1}$. Moreover, $V(Y - \epsilon X) \subseteq (\pi, \pi)^{-1}(C)$ if and only if $V(Y - \epsilon^{-1} X) \subseteq (\pi, \pi)^{-1}(C)$.*

Proof. Notice that

$$(\pi,\pi)(V(Y-\epsilon X)\cap(\mathbb{C}^*\times\mathbb{C}^*))=\{(x+1/x,\epsilon^{-1}/x+x/\epsilon^{-1}):x\in\mathbb{C}^*\}$$

is Zariski constructible by Chevalley's theorem [49, 1.8.4] as

$$V(Y-\epsilon X)\cap(\mathbb{C}^*\times\mathbb{C}^*)$$

is Zariski constructible in $\mathbb{C}^*\times\mathbb{C}^*$ and $(\pi,\pi):\mathbb{C}^*\times\mathbb{C}^*\to\mathbb{P}^1_{\mathbb{C}}\times\mathbb{P}^1_{\mathbb{C}}$ is a regular morphism of varieties. The same is true for the image of

$$V(Y-\epsilon' X)\cap(\mathbb{C}^*\times\mathbb{C}^*)$$

under (π,π). By the condition in the statement of the lemma, the Zariski closure of their images are both C as C is irreducible. Then there exist subsets

$$U\subseteq(\pi,\pi)(V(Y-\epsilon X)\cap(\mathbb{C}^*\times\mathbb{C}^*))$$

and

$$U'\subseteq(\pi,\pi)(V(Y-\epsilon' X)\cap(\mathbb{C}^*\times\mathbb{C}^*))$$

such that U and U' are open and dense in C since a constructible set of an irreducible curve is either finite or cofinite and in this case,

$$(\pi,\pi)(V(Y-\epsilon' X)\cap(\mathbb{C}^*\times\mathbb{C}^*))$$

is cofinite. Therefore $U\cap U'$ is an infinite set. Thus there exists infinitely many $x_0\in\mathbb{C}^*$ such that $(x_0+1/x_0,\epsilon^{-1}/x_0+x_0/\epsilon^{-1})\in U\cap U'$. Hence for each such x_0, there exists $y_0\in\mathbb{C}^*$ such that

$$(x_0+1/x_0,\epsilon^{-1}/x_0+x_0/\epsilon^{-1})=(y_0+1/y_0,\epsilon'^{-1}/y_0+y_0/\epsilon'^{-1}).$$

Therefore, $y_0=x_0$ or $y_0=1/x_0$ from $x_0+1/x_0=y_0+1/y_0$. If $y_0=x_0$, we have

$$\epsilon^{-1}/x_0+x_0/\epsilon^{-1}=\epsilon'^{-1}/x_0+x_0/\epsilon'^{-1}.$$

Multiplying $\epsilon\epsilon' x_0$ on both sides we get

$$\epsilon'+\epsilon^2\epsilon' x_0^2=\epsilon+\epsilon\epsilon'^2 x_0^2. \tag{2.2.1}$$

If $x_0=1/y_0$, then we have

$$\epsilon^{-1}/x_0+x_0/\epsilon^{-1}=\epsilon'^{-1}x_0+1/(x_0\epsilon'^{-1}).$$

Multiplying $\epsilon\epsilon' x_0$ on both sides we have

$$\epsilon' + \epsilon^2\epsilon' x_0^2 = \epsilon x_0^2 + \epsilon\epsilon'^2. \tag{2.2.2}$$

As the set of such pairs of (x_0, y_0) is infinite, there either exist infinitely many y_0 satisfying $x_0 = y_0$ or infinitely many y_0 satisfying $x_0 = 1/y_0$. Therefore either there are infinitely many x_0 satisfying Equation (2.2.1) or infinitely many x_0 satisfying Equation (2.2.2). But if Equation (2.2.1) has infinitely many solutions in x_0 then $\epsilon = \epsilon'$ and similarly Equation (2.2.2) having infinitely many solutions implies $\epsilon' = \epsilon^{-1}$. Thus ϵ' is either ϵ or ϵ^{-1}. The last statement is clear as the set $\{(x + 1/x, \epsilon^{-1}/x + x/\epsilon^{-1}) : x \in \mathbb{C}^*\}$ is the same as $\{(x + 1/x, \epsilon/x + x/\epsilon) : x \in \mathbb{C}^*\}$ under the map sending x to $1/x$. □

Lemma 2.2.2. *Let K be a number field and we view K as a subfield of $\mathbb{C}$ by fixing an embedding of K in $\mathbb{C}$. Let $\pi(x) = x + 1/x$, $l(x) = ax + b$ with $a \in \mathbb{C}^*$ and $b \in \mathbb{C}$, and $\epsilon \notin \{\pm 1\}$ be a root of unity. Let C be an irreducible curve in $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ such that $V(Y - \epsilon X) \subseteq (\pi, \pi)^{-1}(C)$. Then $(l, l)(C)$ is given by*

$$V(Y^2 + (\epsilon + 1/\epsilon - 2)b \cdot (X + Y) - (\epsilon + 1/\epsilon)XY + X^2 + (2 - 1/\epsilon - \epsilon)b^2 + (\epsilon - 1/\epsilon)^2 a^2),$$

and if it has infinitely many K-points then $\epsilon + 1/\epsilon \in K$, $a^2 \in K$ and $b \in K$.

Proof. From the condition that $V(Y - \epsilon X) \subset (\pi, \pi)^{-1}(C)$ we can explicitly deduce the expression of C since, from the proof of Lemma 2.2.1, we know that the Zariski closure of

$$(\pi, \pi)(V(Y - \epsilon X) \cap (\mathbb{C}^* \times \mathbb{C}^*))$$

is C. In other words, the irreducible polynomial in two variables vanishing on

$$\left\{ \left(\frac{1}{x} + x, \frac{1}{\epsilon x} + \epsilon x \right) : \forall x \in \mathbb{C}^* \right\}$$

is the defining equation of C. From Lemma 2.2.1 we know that the defining equation of the irreducible curve C is at least of degree 2 since both

$$(x + 1/x, \epsilon/x + x/\epsilon)$$

and

$$(x + 1/x, \epsilon x + 1/(\epsilon x))$$

are in C and also $\epsilon \notin \{\pm 1\}$. One can check that

$$C = V(Y^2 - (\epsilon + 1/\epsilon)XY + X^2 + (\epsilon - 1/\epsilon)^2). \tag{2.2.3}$$

Then $(l,l)(C)$ is given by

$$V(Y^2+(\epsilon+1/\epsilon-2)b(X+Y)-(\epsilon+1/\epsilon)XY+X^2+(2-\epsilon-1/\epsilon)b^2+(\epsilon-1/\epsilon)^2a^2). \quad (2.2.4)$$

If $(l,l)(C)$ has infinitely many K-points, we have the coefficients of the defining polynomial in Equation (2.2.4) are all in K. Therefore

$$a^2, b \text{ and } \epsilon+1/\epsilon \in K.$$

□

The following lemma is well known. In fact a similar statement is mentioned in the proof of [62, Proposition 2.28], but we are unaware of an explicit reference.

Lemma 2.2.3. *Let $C \subset \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ be an irreducible curve over $\mathbb{C}$ that is invariant under either (P_r, P_r) or $(-P_r, -P_r)$. Then either $C = V(X^mY^n - \nu)$ for $(m,n) \in \mathbb{Z}^2\backslash\{(0,0)\}$ with $\gcd(m,n) = 1$ and ν a root of unity or C is one of $\{0\} \times \mathbb{P}^1_{\mathbb{C}}$, $\mathbb{P}^1_{\mathbb{C}} \times \{0\}$, $\{\infty\} \times \mathbb{P}^1_{\mathbb{C}}$ or $\mathbb{P}^1_{\mathbb{C}} \times \{\infty\}$.*

Proof. We first look at the case that C is an invariant irreducible curve for (P_r, P_r). If $C \cap (\mathbb{C}^* \times \mathbb{C}^*) = \varnothing$, then we have C is one of $\{0\} \times \mathbb{P}^1_{\mathbb{C}}$, $\mathbb{P}^1_{\mathbb{C}} \times \{0\}$, $\{\infty\} \times \mathbb{P}^1_{\mathbb{C}}$ or $\mathbb{P}^1_{\mathbb{C}} \times \{\infty\}$. Otherwise, let $W = C \cap (\mathbb{C}^* \times \mathbb{C}^*)$, which is a curve inside $\mathbb{C}^* \times \mathbb{C}^*$ that is invariant under (P_r, P_r). We view $\mathbb{C}^* \times \mathbb{C}^*$ as an algebraic group with pointwise multiplication and C is a subvariety of $\mathbb{C}^* \times \mathbb{C}^*$ invariant under (P_r, P_r) (viewed as multiplication by r in the semi-abelian variety). Then by [51, Lemma 10], $C = \lambda H$ where $\lambda \in \mathbb{C}^* \times \mathbb{C}^*$ and H is a subgroups of $\mathbb{C}^* \times \mathbb{C}^*$. Thus $C = V(X^mY^n - \nu)$ for some $m, n \in \mathbb{Z}$, at least one of m and n is non-zero, and $\nu \in \mathbb{C}^*$. In fact $\gcd(m,n) = 1$ since otherwise $X^mY^n - \nu$ would be reducible. The fact that C is invariant under the map (P_r, P_r) implies that ν is a root of unity. The case for invariant curves under $(-P_r, -P_r)$ is similar and the only difference in the proof is that we view C as subvariety of the semi-abelian variety $\mathbb{C}^* \times \mathbb{C}^*$ invariant under $(-P_r, -P_r)$ which is a multiplication by r followed by a translation by $(-1,-1)$ in the semi-abelian variety $\mathbb{C}^* \times \mathbb{C}^*$ and then apply [51, Lemma 10]. □

2.3 Proof of Theorem 2.1.6

In this section we complete the proof of Theorem 2.1.6. We first construct $N \in \mathbb{N}^+$ and $Z \subset \mathbb{P}^1_K \times \mathbb{P}^1_K$.

Let $\Delta_K \subseteq (\mathbb{P}^1_K \times \mathbb{P}^1_K)$ be the diagonal subvariety. By [13, Lemma 5.7], all the possible irreducible curves over $\mathbb{C}$ different from Δ, having a Zariski dense set of K-points and living inside the preimages of Δ under elements of the form (f, f) with $f \in S$, are in a finite set Σ. Also, by the proof of [13, Theorem 1.6], there is a finite set $Z \subset \mathbb{P}^1_K \times \mathbb{P}^1_K$ with the property that if $C \in \Sigma$ and C' is an irreducible component of $(f, f)^{-1}(C)$ not in Σ then $C'(K) \subseteq Z$. Let

$$N = \#\Sigma. \tag{2.3.1}$$

Now we show that if there is a point (a, b) outside Z that makes the cancellation Equations (2.1.1) and (2.1.2) fail for some $(\phi_1, \ldots, \phi_k) \in S^k$, with $k > N$, then one of (1) or (2) in the statement of Theorem 2.1.6 must hold.

2.3.1 Producing h_1 and an irreducible invariant curve

To be precise, suppose that there exists $k \in \mathbb{N}$, $k > N$, $(a, b) \in (\mathbb{P}^1_K \times \mathbb{P}^1_K) \backslash Z$ and $\phi_j \in S$, $j = 1, 2, 3, \ldots, k$ such that

$$\phi_k \circ \cdots \circ \phi_1(a) = \phi_k \circ \cdots \circ \phi_1(b)$$

and

$$\phi_{k-1} \circ \cdots \circ \phi_1(a) \neq \phi_{k-1} \circ \cdots \circ \phi_1(b).$$

Claim 2.3.1. *There exists $k - N - 1 < m_1 < m_2 \leqslant k - 1$ and a $C' \in \Sigma$ which is an irreducible component of $(\phi_k \circ \cdots \circ \phi_{m_2+1}, \phi_k \circ \cdots \circ \phi_{m_2+1})^{-1}(\Delta)$ such that*

$$(\phi_{m_1} \circ \cdots \circ \phi_1(a), \phi_{m_1} \circ \cdots \circ \phi_1(b)) \in C' \tag{2.3.2}$$

and

$$(\phi_{m_2} \circ \cdots \circ \phi_{m_1+1}, \phi_{m_2} \circ \cdots \circ \phi_{m_1+1})(C') = C'. \tag{2.3.3}$$

Proof. Since $\phi_k \circ \cdots \circ \phi_1(a) = \phi_k \circ \cdots \circ \phi_1(b)$ we have

$$(\phi_k \circ \cdots \circ \phi_1(a), \phi_k \circ \cdots \circ \phi_1(b)) \in \Delta.$$

We take $C_0 = \Delta$. Then there exists a $C_1 \in \Sigma$ such that

$$(\phi_{k-1} \circ \cdots \circ \phi_1(a), \phi_{k-1} \circ \cdots \circ \phi_1(b)) \in C_1,$$

$$C_1 \subseteq (\phi_k, \phi_k)^{-1}(C_0).$$

In general, for $1 \leqslant j \leqslant k$ we can find $C_{k-j} \in \Sigma$ such that

$$(\phi_j \circ \cdots \circ \phi_1(a), \phi_j \circ \cdots \circ \phi_1(b)) \in C_{k-j},$$

$$C_{k-j} \subseteq (\phi_{j+1}, \phi_{j+1})^{-1}(C_{k-j-1})$$

where $1 \leqslant j < k$. Such curves $C_1, \ldots, C_k \in \Sigma$ exist as (a, b) is not in Z. However we know that $k > N$ and $N = \#\Sigma$. Thus there exist m_1 and m_2 with $k - N - 1 < m_1 < m_2 \leqslant k - 1$ such that $C_{k-m_1} = C_{k-m_2}$. Let $C' = C_{k-m_1} = C_{k-m_2}$. By the construction of C_i's, we have

$$(\phi_{m_1} \circ \cdots \circ \phi_{i_1}(a), \phi_{m_1} \circ \cdots \circ \phi_1(b)) \in C',$$

$$C' \subseteq (\phi_{m_2} \circ \cdots \circ \phi_{m_1+1}, \phi_{m_2} \circ \cdots \circ \phi_{m_1+1})^{-1}(C')$$

In particular, $(\phi_{m_2} \circ \cdots \circ \phi_{m_1+1}, \phi_{m_2} \circ \cdots \circ \phi_{m_1+1})(C') \subseteq C'$. Thus C' is an invariant curve under $(\phi_{i_{m_2}} \circ \cdots \circ \phi_{m_1+1}, \phi_{m_2} \circ \cdots \circ \phi_{m_1+1})$ as the map is a non-constant morphism and hence surjective [78, I §5, Theorem 1.12]; i.e., $(\phi_{m_2} \circ \cdots \circ \phi_{m_1+1}, \phi_{m_2} \circ \cdots \circ \phi_{m_1+1})(C') = C'$. □

We take

$$h_1 = \phi_{m_2} \circ \cdots \circ \phi_{m_1+1}, \tag{2.3.4}$$

$$h_2 = \phi_k \circ \cdots \circ \phi_{m_2+1}. \tag{2.3.5}$$

Then C' is an irreducible curve that is invariant under (h_1, h_1).

2.3.2 Reduction to cases (1) or (2) in Theorem 2.1.6

In this subsection, we continue using the notation from subsection 2.3.1. We have already seen that h_1, h_2 in Equation (2.3.4) can be written as compositions of no more than N elements in S. In this subsection, we will show that h_1 and h_2 satisfy either condition (1) or (2).

Notice that for every $c \in \mathbb{C}$ neither (∞, c) nor (c, ∞) is in C' since

$$\phi_k \circ \cdots \circ \phi_1(\infty) \neq \phi_k \circ \cdots \circ \phi_1(c).$$

Removing (∞, ∞) from C', we can view C' as an irreducible curve in $\mathbb{C} \times \mathbb{C}$ that is invariant under (h_1, h_1). (We adopt this view from now on and write a curve as an affine variety $V(f(X, Y))$, for some polynomial $f(X, Y)$, when there is no confusion.) Then by [66, Theorem 4.9], if h_1 is not linearly conjugate to P_r or $\pm T_r$ then C' is of the form $V(y - p(x))$ or $V(x - p(y))$ for some polynomial p commuting with h_1.

Case 1:

h_1 is not linearly conjugate to a cyclic polynomial or to a Chebyshev polynomial. Then, without loss of generality, we assume

$$C' = V(y - p(x)). \tag{2.3.6}$$

Since $C' \subseteq (h_2, h_2)^{-1}(\Delta)$, the polynomial p satisfies

$$h_2(x) = h_2(p(x)).$$

By a simple degree argument, this can only happen when $\deg(p) = 1$; i.e., $p(x) = sx + t$. Moreover, since $C' \neq \Delta$, we have $(s, t) \neq (1, 0)$. Then

$$h_2(x) = h_2(sx + t),$$

and since h_1 commutes with p, we also have

$$h_1(sx + t) = sh_1(x) + t.$$

If $s = 1$ and $t \neq 0$, then h_2 is a constant function, which is impossible. So, $s \neq 1$ and there exists $l(x) = x + t/(1 - s)$ such that

$$l^{-1} \circ h_1 \circ l(sx) = sl^{-1} \circ h_1 \circ l(x).$$

Then s must be a root of unity, which can be seen by looking at the leading term of the polynomials. Then $h_2 \circ l(sx) = h_2 \circ l(x)$ implies that $h_2 \circ l = P(x^d)$ for some polynomials P and integer $d > 1$ which is the order of s as a root of unity. Therefore

$$l^{-1} \circ h_1 \circ l(x) = xQ(x^d)$$

$$h_2 \circ l = P(x^d)$$

for some polynomials Q. Moreover since

$$C' = V(y - p(x)) = V(y - sx - t) \in \Sigma$$

contains a Zariski dense set of K-points, $s, t \in K$ and $l(x) = x + t/(1 - s)$. Thus we are in case (1) of Theorem 2.1.6.

Now we are left with the case that h_1 is linearly conjugate to P_r or $\pm T_r$.

Case 2:

h_1 is linearly related to P_r. Without loss of generality we may assume that $h_1 = P_r$ for some r. So C' is an irreducible curve invariant under (P_r, P_r). The irreducible curves over $\mathbb{C}$ invariant under power maps are given by $V(X^mY^n - \nu)$ for $(m,n) \in \mathbb{Z}^2\backslash\{(0,0)\}$, $\gcd(m,n) = 1$ and ν is a root of unity, or one of $\{0\} \times \mathbb{P}^1_{\mathbb{C}}$, $\mathbb{P}^1_{\mathbb{C}} \times \{0\}$, $\{\infty\} \times \mathbb{P}^1_C$ or $\mathbb{P}^1_C \times \{\infty\}$ by Lemma 2.2.3. Since we have $(h_2, h_2)(C') \subseteq \Delta$ by the construction of C', we have the first case holds with $m = -n \neq 0$ by a simple degree argument. Therefore the irreducible invariant curves are of the form $V(Y - \epsilon X)$ with some root of unity ϵ satisfying $(\epsilon x)^r = \epsilon x^r$. This implies that the order d of the ϵ satisfies $d|(r-1)$. The condition $(h_2, h_2)(C') \subseteq \Delta$ and $C' \neq \Delta$ gives that

$$h_2 = P \circ P_d \tag{2.3.7}$$

for some polynomial P and $d \geqslant 2$. In general, the conclusion is that there exists complex linear polynomials $l = ux + v$ such that

$$h_2 \circ l = P \circ P_d$$

$$l^{-1} \circ h_1 \circ l = P_r$$

for some polynomial P, $d|(r-1)$ and $d \geqslant 2$. Also

$$C' = (l,l)(V(Y - \epsilon X)) = V(Y - \epsilon X + (\epsilon - 1)v) \in \Sigma, \tag{2.3.8}$$

with some d-th root of unity $\epsilon \neq 1$, contains infinitely many K-points. This implies $V(Y - \epsilon X + (\epsilon - 1)v)$ is defined over K. Thus $\epsilon, v \in K$. Now $h_1(x) = u^{1-r}(x-v)^r + v$ and $h_2(x) = P \circ P_d \circ l^{-1}$. Let $l'(x) = x + v$. We have

$$l'^{-1} \circ h_1 \circ l'(x) = u^{1-r}x^r = xQ(x^d),$$

and $Q(x) = u^{1-r}x^{(r-1)/d}$.

$$h_2(x) \circ l' = P(u^{-d}(x-v)^d) \circ (x+v) = P(u^{-d}x^d) = P'(x^d),$$

for some polynomial $P'(x)$. Therefore we are in case (1) of Theorem 2.1.6.

Case 3 :

h_1 is linearly conjugate to $\pm T_r$. We may first assume without loss of generality that h_1 is either T_r or $-T_r$. Notice that $\pi : \mathbb{C}^* \to \mathbb{A}^1_{\mathbb{C}}$ given by $\pi(x) = 1/x + x$ is dominant

and $\pm T_r \circ \pi = \pi \circ (\pm P_r)$. Therefore the preimages under (π, π) of invariant subvarieties in $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ under $(\pm T_r, \pm T_r)$ are invariant subvarieties under $(\pm P_r, \pm P_r)$. Recall from Section 2.3.1 that C' is an irreducible curve invariant under (h_1, h_1) with Zariski dense set of K-points. Now, in this case, C' is an irreducible curve invariant under either (T_r, T_r) or $(-T_r, -T_r)$ over $\mathbb{C}$, and so

$$E = (\pi, \pi)^{-1}(C') \tag{2.3.9}$$

is an invariant subvariety under either (P_r, P_r) or $(-P_r, -P_r)$. Let $E_1, \ldots, E_t$ be the irreducible components of E. Since E is invariant under $(\pm P_r, \pm P_r)$, we have

$$\overline{(\pm P_r, \pm P_r)(E_i)} = E_{\tau(i)}$$

for some map $\tau : \{1, 2, \ldots, t\} \to \{1, 2, \ldots, t\}$. Then there exists $j \in \{1, 2, \ldots, t\}$ and $w \in \mathbb{N}^+$ such that $\tau^w(j) = j$ and therefore

$$(\pm P_r, \pm P_r)^{\circ w}(E_j) = E_j,$$

since $(\pm P_r, \pm P_r)^{\circ w}$ is not a constant morphism and therefore surjective. Now we have that E_j is an irreducible curve invariant under either $(P_r, P_r)^{\circ w}$ or $(-P_r, -P_r)^{\circ w}$.

In both cases, by Lemma 2.2.3, E_j is given by $V(X^m Y^n - \nu)$ for some coprime integers m, n, at least one of them is not zero, and a root of unity ν, since it cannot be one of the $\{0\} \times \mathbb{P}^1_{\mathbb{C}}$, $\mathbb{P}^1_{\mathbb{C}} \times \{0\}$, $\{\infty\} \times \mathbb{P}^1_{\mathbb{C}}$ or $\mathbb{P}^1_{\mathbb{C}} \times \{\infty\}$ as

$$(h_2, h_2) \circ (\pi, \pi)(E_j) \subseteq \Delta.$$

By the same reason, a degree argument implies that $m = -n \neq 0$. Therefore

$$E_j = V(Y - \epsilon X) \tag{2.3.10}$$

with a root of unity ϵ of order $d > 1$. The discussion from now on doesn't depend on the sign of $(\pm T_r, \pm T_r)$, so we only consider the case for $(h_1, h_1) = (T_r, T_r)$. Then $E_j = V(Y - \epsilon X) \subseteq E$ with the root of unity ϵ of order $d > 1$ and Lemma 2.2.1 implies that

$$V(Y - \epsilon X) \text{ and } V(Y - \epsilon^{-1} X)$$

are the only irreducible components of $E = (\pi, \pi)^{-1}(C')$. The fact that E is invariant under (P_r, P_r) implies

$$(P_r, P_r)(V(Y - \epsilon X)) \subset V(Y - \epsilon X) \cup V(Y - \epsilon^{-1} X). \tag{2.3.11}$$

In particular, Equation (2.3.11) implies $\epsilon^r \in \{\epsilon^{-1}, \epsilon\}$. Therefore

$$d \mid (r-1) \text{ or } d \mid (r+1).$$

On the other hand, $(h_2, h_2)(C') \subseteq \Delta$ implies

$$h_2(\epsilon^{\pm 1}/x + x/\epsilon^{\pm 1}) = h_2(1/x + x). \tag{2.3.12}$$

Notice that every polynomial can be written as a linear combination of Chebyshev polynomials; in particular,

$$h_2(x) = \sum_{i=0}^{v} a_i T_i(x)$$

for some constants $a_i \in \mathbb{C}$, with $v = \deg(h_2)$. For $\mu = \epsilon^{\pm 1}$ we have

$$h_2(\mu/x + x/\mu) = \sum_{i=0}^{v} a_i(\mu^i/x^i + x^i/\mu^i) = \sum_{i=0}^{v} a_i(1/x^i + x^i) \tag{2.3.13}$$

by Equation (2.3.12) for all $x \in \mathbb{C}^*$. So for each $i \in \{1, 2, \ldots, v\}$ such that $a_i \neq 0$, we have $\mu^i = 1$; i.e., $d|i$. In particular $d|v$. Therefore

$$h_2(x) = \sum_{j=0}^{v/d} a_{jd} T_j \circ T_d = P \circ T_d$$

for some polynomial P. Therefore, in general we have that there exists a complex linear polynomial $l(x) = ux + v$ such that

$$h_2 \circ l = P \circ T_d,$$

$$l^{-1} \circ h_1 \circ l = \pm T_r$$

where $d|(r+1)$ or $d|(r-1)$ and P is an arbitrary polynomial. Also Equations (2.3.9) and (2.3.10) implies that

$$V(Y - \epsilon X) \subset (\pi, \pi)^{-1}((l, l)^{-1}(C')) \tag{2.3.14}$$

with some root of unity $\epsilon \neq 1$ of order d, where $C' \in \Sigma$ has infinitely many K-points. If $d = 2$ then in this case from Equation (2.3.14)

$$C' = V(Y + X - 2v) = (l, l)(\overline{(\pi, \pi)(V(Y + X))}) \in \Sigma.$$

In particular, C' has infinitely many K-points is equivalent to $v \in K$. This is $2(a)$ of Theorem 2.1.6. If $d > 2$ then by Lemma 2.2.2 and Equation (2.3.14), we have that

$$C' = V(Y^2+(\epsilon+1/\epsilon-2)v(X+Y)+(\epsilon-1/\epsilon)XY+X^2+(2-\epsilon-1/\epsilon)v^2+(\epsilon-1/\epsilon)^2u^2). \quad (2.3.15)$$

has infinitely many K-points. Since this is a smooth conic and in particular a smooth genus zero curve, so it has infinitely many K-points if and only if there exists one. This is $2(b)$ of Theorem 2.1.6. If we further have that r is even, then

$$l \circ \pm T_r \circ l^{-1} \in K[x]$$

implies that $u^{r-1} \in K$. From Lemma 2.2.2 and recall $C' \in \Sigma$, we have

$$u^2, v, \epsilon + 1/\epsilon \in K.$$

So, $u \in K$. Notice that

$$(2u + v, u(\epsilon + 1/\epsilon) + v) \in K^2$$

is a point in C'. Therefore, C' has K-points if and only if $u, v, \epsilon + 1/\epsilon \in K$.

2.3.3 Sufficiency of (1), (2) in Theorem 2.1.6

For this we just need to verify the moreover statement of remark 2.1.7. We suppose there exist h_1, h_2 and l as in cases (1) or (2) in the statement of Theorem 2.1.6. The idea is that from previous sections we have already obtained the invariant curves under (h_1, h_1) and then the set of K-points on the invariant curves excluding the preimages of its intersection with Δ under some iterations of (h_1, h_1) will satisfy Equation (2.1.3) and (2.1.4) in Theorem 2.1.6. We will make these precise as follows.

If conditions in (1) of Theorem 2.1.6 hold, then take

$$C = V(Y - \epsilon X - v(1 - \epsilon))$$

for some primitive d-th root of unity $\epsilon \in K$. By the argument in the previous section or one can check directly that

$$(h_1, h_1)(C) \subseteq C.$$

Let j be a nonnegative integer and let

$$Z_1 = (h_1, h_1)^{-j}(C \cap \Delta) \cap C.$$

Notice that Z_1 is a finite set. So, we take $(a,b) \in C(K)\backslash Z_1$ and then

$$h_1^j(a) \neq h_1^j(b),$$
$$(h_1^j(a), h_1^j(b)) \in C$$

and also

$$h_2 \circ h_1^j(a) = h_2 \circ h_1^j(b)$$

by the construction of h_2.

For case (2) in Theorem 2.1.6, let ϵ be a primitive d-th root of unity and let C be

$$V(Y^2 + (\epsilon + 1/\epsilon - 2)v(X+Y) + (\epsilon + 1/\epsilon)XY + X^2 + (2 - \epsilon - 1/\epsilon)v^2 + (\epsilon - 1/\epsilon)^2 u^2)$$

if $2(b)$ of Theorem 2.1.6 holds; or

$$C = V(Y + X - 2v)$$

if $2(a)$ of Theorem 2.1.6 holds. Let j be a nonnegative integer and let

$$Z_1 = (h_1, h_1)^{-j}(C \cap \Delta) \cap C,$$

which is a finite set. Then for any K-points $(a,b) \in C(K)\backslash Z_1$ we have

$$h_1^j(a) \neq h_1^j(b).$$

Recall from the argument in the last subsection, C is invariant under (h_1, h_1) and

$$(h_2, h_2)(C) \subset \Delta.$$

Therefore

$$h_2 \circ h_1^j(a) = h_2 \circ h_1^j(b).$$

This concludes the proof.

2.4 Computing N in Theorem 2.1.6

It is natural to ask if the number N and the number of points in Z in Theorem 2.1.6 can be bounded. Bounding $\#Z$ leads us to look at the K-points of genus $\geqslant 2$ curves that appear in the preimages of $\Delta \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ under maps (f, f) with $f \in \langle S \rangle$. It is still an open problem to bound the number of K-points on curves uniformly in terms of the number field K and the genus of the curve [20]. Also, it is difficult to bound the number of K-points on a given curve in general. So it is not an easy task to bound $\#Z$. However, we can indeed get a computable bound on N:

Corollary 2.4.1. *Adopt the notation from Theorem 2.1.6. There is a computable number $N_0 \in \mathbb{N}$ depending on S such that we can take $N = N_0$ in the statement of Theorem 2.1.6.*

Proof. Recall that in the proof of Theorem 2.1.6, we take $N = \#\Sigma$ and Σ is the set of irreducible curves different from Δ, having a Zariski dense set of K-points and living inside the preimages of Δ under elements of the form (f, f) with $f \in S$. Let $\Sigma' \subseteq \Sigma$ be the subset of Σ consisting of curves of genus 0 and then we will show that we can in fact take N to be $N_0 = \#\Sigma'$. Suppose there exists h_1 and h_2 as in the case (1) or (2) of Theorem 2.1.6 and $h_2 = \phi_{k_1} \circ \cdots \circ \phi_1$, where $k_1 > N_0$. Then there exists a curve $C \in \Sigma'$ such that $(h_2, h_2)(C) = \Delta$ (C is a line in (1) and conic in (2) according to the proof of Theorem 2.1.6). We assume $(\phi_{k_1-1} \circ \cdots \circ \phi_1, \phi_{k_1-1} \circ \cdots \circ \phi_1)(C) \neq \Delta$, otherwise we just take $h_2 = \phi_{k_1-1} \circ \cdots \circ \phi_1$. Then there exists a sequence of curves $\{C_0, \ldots, C_{k_1-1}\} \subseteq \Sigma'$ such that $C_{i-1} \subseteq (\phi_i, \phi_i)^{-1}(C_i)$, $C_0 = C$ and $C_{k_1} = \Delta$, $0 \leqslant i \leqslant k_1$. The fact that all of these curves have to be genus 0 follows from the genus estimation lemma [13, Proposition 4.1] saying that for any irreducible curve C' over $\mathbb{C}$ and irreducible curve $C'' \subseteq (f, f)^{-1}(C')$, $f \in S$, genus of C'' is not less than genus of C'. So, if there is a curve C_j, $1 < j < k_1$, having genus greater than 0 in the middle of the sequence, then C is of genus greater than 0, which is a contradiction. Since $k_1 > N_0$, there exists $k_1 \geqslant m_2 > m_1 > k_1 - N_0 - 1$ such that $(\phi_{m_2} \circ \cdots \circ \phi_{m_1+1}, \phi_{m_2} \circ \cdots \circ \phi_{m_1+1})(C_{m_2}) = C_{m_2}$, $C_{m_2} = C_{m_1}$. Also, $(\phi_{k_1} \circ \cdots \circ \phi_{m_2+1}, \phi_{k_1} \circ \cdots \circ \phi_{m_2+1})(C_{m_2}) = \Delta$. Let

$$h_1' = \phi_{m_2} \circ \cdots \circ \phi_{m_1+1},$$

$$h_2' = \phi_{k_1} \circ \cdots \circ \phi_{m_2+1}.$$

Then repeating the argument in the proof of Theorem 2.1.6 after the Claim 2.3.1 we have h_1', h_2' also satisfy the condition in (1) or (2). Now suppose h_2 is written as a composition of no more than N_0 elements in S while h_1 is not. Then we replace h_2 with $h_2 \circ h_1$ and repeat the argument above. Therefore, we proved that we can take N to be N_0 and now we just need to bound N_0. Notice that we can obtain the set Σ' explicitly by the following procedures:

1. Let $L = \{\Delta\}$.
2. For every curve $C \in L$ and $f \in S$, factorize the defining equation of $(f, f)^{-1}(C)$ (polynomials in two variables) over the algebraic closure of the field of coefficients. We desingularize the curves defined by each factor and for each smooth curve obtained we compute its genus and if genus is equal to 0 then check if it contains infinitely many K-points. If yes, then record this curve (defined by an irreducible polynomial in two variables) in L if it is not already in L.

3. If there is no new curves added in L from step 2 then we stop and output L; otherwise, we repeat step 2.

Notice that $L \subseteq \Sigma' \cup \{\Delta\}$ and this process will stop eventually because we know $\Sigma' \subseteq \Sigma$ is a finite set. Also, we remark that the algorithm for desingularization, factorization, computing genus and testing if a genus 0 curve has infinitely many K-points are all known (See implementations on Maple and also [71, Section 4.1]). Therefore, we can compute $\#\Sigma' = \#L - 1$ and so is $N_0 = \#\Sigma'$. □

2.5 Proof of Theorem 2.1.1

In this section we give the proof of Theorem 2.1.1. Our first result shows that we can reduce to the case when our maps all have degree $\geqslant 2$.

Lemma 2.5.1. *Let $n \geqslant 2$ be a natural number and let $f = (f_1, \ldots, f_n) : (\mathbb{P}^1)^n \to (\mathbb{P}^1)^n$ be a split rational map defined over an algebraically closed characteristic zero field and suppose that there exists a positive integer $k \in \{1, \ldots, n-1\}$ such that $\deg(f_i) > 1$ when $1 \leqslant i \leqslant k$ and $\deg(f_i) = 1$ when $i > k$. If V is an irreducible subvariety of $(\mathbb{P}^1)^n$ that is invariant under f then there exist subvarieties $V_1 \subseteq (\mathbb{P}^1)^k$ and $V_2 \subseteq (\mathbb{P}^1)^{n-k}$ such that $V = V_1 \times V_2$ and such that V_1 is invariant under $g_1 = (f_1, \ldots, f_k)$ and V_2 is invariant under $g_2 = (f_{k+1}, \ldots, f_n)$.*

Proof. The proof is similar to [87, Proposition 3.14]. Let d denote the dimension of V and let $\pi_1 : (\mathbb{P}^1)^n \to (\mathbb{P}^1)^k$ and $\pi_2 : (\mathbb{P}^1)^n \to (\mathbb{P}^1)^{n-k}$ be respectively the projection onto the first k factors and the projection onto the last $n-k$ factors.

Let $V_i = \pi_i(V)$ and $d_i = \dim(V_i)$ for $i = 1, 2$. Since $V \subseteq V_1 \times V_2$ we have $d \leqslant d_1 + d_2$ and $d = d_1 + d_2$ if and only if $V = V_1 \times V_2$.

Thus we may assume without loss of generality that $d < d_1 + d_2$. We take α_1 to be the numerical class of the ample line bundle $\pi_1^* \mathcal{O}_{(\mathbb{P}^1)^k}(1, 1, \ldots, 1)$ and α_2 to be the numerical class of the ample line bundle $\pi_2^* \mathcal{O}_{(\mathbb{P}^1)^{n-k}}(1, 1, \ldots, 1)$. Furthermore, we let β_j denote the numerical class of the line bundle $p_j^* \mathcal{O}_{\mathbb{P}^1}(1)$ for each $j \in \{1, 2, \ldots, n\}$ and $p_j : (\mathbb{P}^1)^n \to \mathbb{P}^1$ the projection onto the j-th coordinate. Then

$$\alpha_1 = \beta_1 + \cdots + \beta_k, \tag{2.5.1}$$

$$\alpha_2 = \beta_{k+1} + \cdots + \beta_n. \tag{2.5.2}$$

Notice that for any $j \in \{0, 1, \ldots, d\}$, $\alpha_1^j \cdot \alpha_2^{d-j} \cdot V \geqslant 0$ and it is positive if $j = d_1$ or $d - j = d_2$. We denote $I_1 = \{1, 2, \ldots, k\}$ and $I_2 = \{k+1, \ldots, n\}$ from now on.

Since there are only finitely many collections of indices (allowing repetition) of size $d - d_2$ contained in I_1 and also only finitely many collections of indices (allowing repetition) of size d_2 which is inside I_2, there exists a $\{r_1, r_2, \ldots, r_{d-d_2}\} \subseteq I_1$, and a $\{e_1, e_2, \ldots, e_{d_2}\} \subseteq I_2$ such that

$$V \cdot \prod_{t=1}^{d-d_2} \beta_{r_t} \prod_{l=1}^{d_2} \beta_{e_l} > 0,$$

and $C = \prod_{t=1}^{d-d_2} c_{r_t}$, where $c_i = \deg(f_i)$ for $i \in I_1$, is the maximum in the set

$$\left\{ \prod_{i \in I'} c_i : V \cdot \prod_{i \in I'} \beta_i \prod_{v \in J'} \beta_v > 0, I' \subseteq I_1, J' \subseteq I_2, |I'| = d - d_2, |J'| = d_2 \right\}.$$

We let V' be an irreducible subvariety of $V \cap \bigcap_{t=1}^{d-d_2} \beta_{r_t}$ of dimension d_2 and here we abuse notation to view β_{r_t}'s as some suitable hypersurfaces in the numerical classes $p_{r_t}^* \mathcal{O}_{\mathbb{P}^1}(1)$'s. Then note that $\dim(\pi_1(V')) = d_1 - (d - d_2)$.

We have

$$V' \cdot (\alpha_1)^{d_1 - (d-d_2)} \cdot (\alpha_2)^{d-d_1} > 0.$$

This implies that there exists collections of indices

$$\{j_1 = r_1, j_2 = r_2, \ldots, j_{d-d_2} = r_{d-d_2}, j_{d-d_2+1}, \ldots, j_{d_1}\} \subseteq I_1$$

and

$$\{u_1, u_2, \ldots, u_{d-d_1}\} \subseteq I_2$$

such that

$$V \cdot \prod_{t=1}^{d_1} \beta_{j_t} \prod_{l=1}^{d-d_1} \beta_{u_l} > 0. \tag{2.5.3}$$

Notice that

$$\prod_{t=1}^{d_1} c_{j_t} > C \tag{2.5.4}$$

by construction and Equation (2.5.3) implies that

$$V \cdot \alpha_1^{d_1} \cdot \alpha_2^{d-d_1} > 0.$$

Let $u_1, u_2 \in \mathbb{R}$. Then, we have

$$\begin{aligned}(\deg(f|_V))(V \cdot (u_1\alpha_1 + u_2\alpha_2)^d) &= f_*(V) \cdot ((u_1\alpha_1 + u_2\alpha_2)^d) \\ &= V \cdot (u_1(g_1)^*(\alpha_1) + u_2(g_2)^*(\alpha_2))^d \\ &= V \cdot \left(u_1 \sum_{i=1}^{k} c_i\beta_i + u_2 \sum_{i=k+1}^{n} c_i\beta_i\right)^d.\end{aligned}$$

Now we compare the coefficients of the $u_1^{d_1}u_2^{d-d_1}$ and $u_1^{d-d_2}u_2^{d_2}$ terms and we obtain that for each positive integer m:

$$\deg(f^m|_V)\alpha_1^{d_1} \cdot \alpha_2^{d-d_1} \cdot V = \left(\sum_{i=1}^{k} c_i^m\beta_i\right)^{d_1} \cdot \left(\sum_{i=k+1}^{n} c_i^m\beta_i\right)^{d-d_1} \cdot V,$$

and

$$\deg(f^m|_V)\alpha_1^{d-d_2} \cdot \alpha_2^{d_2} \cdot V = \left(\sum_{i=1}^{k} c_i^m\beta_i\right)^{d-d_2} \cdot \left(\sum_{i=k+1}^{n} c_i^m\beta_i\right)^{d_2} \cdot V.$$

Therefore

$$\deg(f^m|_V) = \left(\sum_{i=1}^{k} c_i^m\beta_i\right)^{d_1} \cdot \alpha_2^{d-d_1} \cdot V / \left(\alpha_1^{d_1} \cdot \alpha_2^{d-d_1} \cdot V\right) \tag{2.5.5}$$

and

$$\deg(f^m|_V) = \left(\sum_{i=1}^{k} c_i^m\beta_i\right)^{d-d_2} \cdot \alpha_2^{d_2} \cdot V / \left(\alpha_1^{d-d_2} \cdot \alpha_2^{d_2} \cdot V\right), \tag{2.5.6}$$

since $c_i = 1$ when $i \in I_2$,

$$\alpha_1^{d_1} \cdot \alpha_2^{d-d_1} \cdot V > 0,$$

and

$$\alpha_1^{d-d_2} \cdot \alpha_2^{d_2} \cdot V > 0.$$

Now, by (2.5.4) and Equation (2.5.5), we have

$$\deg(f^m|_V) = aM^m + o(M^m)$$

for some positive integer a and M, such that $M > C \geqslant \prod_{t=1}^{d-d_2} c_{l_t}$ for all $\{l_1, \ldots, l_{d-d_2}\} \subseteq I_1$ and $\{v_1, \ldots, v_{d_2}\} \subseteq I_2$ with the property

$$\prod_{t=1}^{d-d_2} \beta_{l_t} \prod_{s=1}^{d_2} \beta_{v_s} \cdot V > 0.$$

On the other hand, Equation (2.5.6) gives that $\deg(f^m|_V) = \mathcal{O}(C^m)$. This is a contradiction and so the result follows.

□

Lemma 2.5.1 allows us to reduce to the case that all maps f_i in the statement of Theorem 2.1.1 have degree at least two when proving the result, and we now consider this restricted case. We begin with a result in which the maps are special.

Remark 2.5.2. We introduce the notation here to avoid confusion in the later discussion: Let $n, D \in \mathbb{N}$ and K be a field. For a quasi-projective variety $W \subseteq (\mathbb{P}^1_K)^n$ defined over K and a set L which is a union of the finite field extensions of K of degree not exceeding D, which is not necessarily a field or a ring, we let $W(L)$ denote the largest subset of points in $W(\overline{K})$ such that for every $i \in \{1, 2, \ldots, n\}$,

$$\pi_i(W(L)) \subseteq \bigcup_{K' : K' \subseteq L} \mathbb{P}^1_{K'},$$

where π_i is the projection to the i-th factor and K' ranges in the set of field extensions of K contained in L as a set. We call this set of points $W(L)$ the L-points of W.

More concretely, after fixing a coordinate of W defined over K, we have $W(L)$ is the set of points $(x_1, x_2, \ldots, x_n) \in W \subseteq (\mathbb{P}^1_{\overline{K}})^n$ such that $x_i \in L \cup \{\infty\}$, for each $i \in \{1, 2, \ldots, n\}$.

Moreover, we let L^* denote the set $L \backslash \{0\}$.

Proposition 2.5.3. *Let K be a number field, let n and D be positive integers that are at least 2, and let K' be the union of all the finite field extensions of K of degree not greater than D. Suppose that $f = (f_1, \ldots, f_n) : (\mathbb{P}^1_K)^n \to (\mathbb{P}^1_K)^n$ is a split rational map defined over K in which the f_i's all have the same degree $d \geqslant 2$ and each f_i is conjugate to either $x^{\pm d}$ or $\pm T_d$ for $i \in \{1, 2, \ldots, n\}$. If $V \subset (\mathbb{P}^1)^n$ is an irreducible f-invariant hypersurface defined over K, then there exists a non-negative integer s_0 such that*

$$(f^{-s-1}(V) \backslash f^{-s}(V))(K') = \varnothing$$

for all $s \geqslant s_0$.

Proof. We first enlarge our number field K to a larger number field so that each f_i can be conjugated to either $x^{\pm d}$ or to $\pm T_d$ by a map $v_i \in \mathrm{PGL}_2(K)$ and $\{i, e^{\pi i/4}\} \subseteq K$. Notice that it is enough to prove the statement in this larger field. We henceforth assume that K is a number field satisfying this condition. We can then perform a change of coordinates and may assume without the loss of the generality that f is of the form $f = (f_1(x_1), \ldots, f_n(x_n)) = (x_1^{\pm d}, x_2^{\pm d}, \ldots, x_k^{\pm d}, \pm T_d(x_{k+1}), \ldots, \pm T_d(x_n))$ for some $k \in \{0, \ldots, n\}$.

Let

$$\mu = (x_1, \ldots, x_k, (x_{k+1} + x_{k+1}^{-1})/2, \ldots, (x_n + x_n^{-1})/2) : (\mathbb{C}^*)^n \to (\mathbb{P}^1)^n,$$

and

$$G' = (x_1^{\pm d}, \ldots, x_k^{\pm d}, \pm x_{k+1}^d, \ldots, \pm x_n^d) : (\mathbb{C}^*)^n \to (\mathbb{C}^*)^n$$

be such that $\mu \circ G' = f \circ \mu$. Then take

$$\mu' = (x_1, \ldots, x_k, \xi_{k+1} x_{k+1}, \ldots, \xi_n x_n)$$

and

$$G = (x_1^{\pm d}, \ldots, x_k^{\pm d}, x_{k+1}^d, \ldots, x_n^d),$$

where $\xi_{k+1}, \ldots, \xi_n$ are some roots of unity such that $\mu' \circ G = G' \circ \mu'$. Enlarge K so that $\{\xi_{k+1}, \ldots, \xi_n\} \subseteq K$ and abuse notation from now on to let μ be $\mu \circ \mu'$. So we have

$$\mu \circ G = f \circ \mu.$$

Notice that further abusing notations by replacing G by G^2 and f by f^2, we may let

$$G = (x_1^d, \ldots, x_k^d, x_{k+1}^d, \ldots, x_n^d)$$

and

$$\mu \circ G = f \circ \mu.$$

Now we prove the statement by induction on the dimension n. If $n = 1$, then the statement is certainly true as V is just a finite set of points. Now, we suppose it is true when $n \leqslant n_0$ for some positive integer n_0 and we want to show it is true when $n = n_0 + 1$. From now on, let $n = n_0 + 1$. Let W be the preimage of V under μ, which is a subvariety in $(\mathbb{C}^*)^n$. Since $f(V) = V$, we have

$$G(W) \subseteq W.$$

Thus, there exists a subvariety $W' \subseteq W$ such that $G(W') = W'$, after replacing f and G by a suitable iterate, we have $W \subseteq G^{-1}(W')$. Therefore, by [51, Lemma 10], we have W' is

a finite union of translations of algebraic subgroups of $(\mathbb{C}^*)^n$. Notice that $(\xi_i x_i + \xi_i^{-1} x_i^{-1})/2$ maps $\{0, \infty\}$ to $\{\infty\}$, for each $i \in \{k+1, \ldots, n\}$. Thus, we have

$$V' = V \backslash \mu(W) = \left(\bigcup_{i=1}^{n} (V \cap H_i) \right) \cup \bigcup_{i=1}^{k} (V \cap P_i),$$

where H_i is the hypersurface in $(\mathbb{P}^1)^n$ defined by having the i-th coordinate equals to ∞ and P_i is the hypersurface with the i-th coordinate equals to 0. We write $V' = \bigcup_{j=1}^{k_0} V_j$ for some positive integer k_0, where V_j's are irreducible components of $V \cap H_i$ or $V \cap P_l$ for some $i \in \{1, 2, \ldots, n\}$ and $l \in \{1, 2, \ldots, k\}$. Also, $f(V') = V'$ because $f(V) = V$, $f(V') \subseteq V'$ and $f(\mu(W)) \subseteq \mu(W)$ as power maps take 0 or ∞ to themselves and Chebyshev polynomials take ∞ to itself.

After replacing f by some iteration we may assume each V_j is invariant under f. Notice that every irreducible component of $V \cap H_i$ or $V \cap P_l$ has dimension at least $n-2$, if not empty, for each $i \in \{1, 2, \ldots, n\}$ and $l \in \{1, 2, \ldots, k\}$. Thus, after some reorderings of the coordinates, each V_j projects onto either an irreducible hypersurface in $(\mathbb{P}^1)^{n-1}$ or $(\mathbb{P}^1)^{n-1}$ itself, denoted as V_j' in both case, and projects to either 0 or ∞ in the remaining $\mathbb{P}^1$ factor. If $V_j' = (\mathbb{P}^1)^{n-1}$, then its preimages under f restricted to $(\mathbb{P}^1)^{n-1}$ is itself and there is nothing to show, so from now on we assume V_j' is an irreducible hypersurface. Also, V_j' is invariant under f_j', the restriction of f to the corresponding $(\mathbb{P}^1)^{n-1}$ factors. Now the induction hypothesis tells us that there exists non-negative integer s_j' such that

$$((f_j')^{-s-1}(V_j') \backslash (f_j')^{-s}(V_j'))(K') = \varnothing$$

for every $s \geqslant s_j'$. But notice that this also implies

$$(f^{-s-1}(V_j) \backslash f^{-s}(V_j))(K') = \varnothing$$

for $s \geqslant s_j'$, since preimages of $\{\infty, 0\}$ under x^d and preimages of ∞ under $\pm T_d$ are just themselves. Take $s_0' = \max_{j=1}^{k_0} s_j'$.

Now we look at the preimages of V under μ. If this is empty, then V lives completely inside some H_i or P_j for some $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, k\}$ and, using arguments above, we go to the induction steps. From now on, we assume that $W = \mu^{-1}(V)$ is not empty. Notice that $\mu^{-1}((V \backslash V')(K')) \subseteq W(L)$, where L is the union of finite field extensions of degree $D \cdot 2^n$ over K. Recall that $W \subseteq G^{-1}(W')$ and $G(W') = W'$. It is enough to show that there exists a non-negative integer s_1 such that for any $x \in (L^*)^n$, if $G^s(x) \in W'(L)$ for some $s \geqslant 0$, then $G^{s_1}(x) \in W'(L)$. As, if this s_1 exists and there still exists a $y \in (\mathbb{P}^1_{K'})^n$ and a $s > s_1$ such that $f^s(y) \in \mu(W)(K')$ but $f^{s_1}(y) \notin \mu(W)$, then for any $x \in \mu^{-1}(y) \subset (L^*)^n$,

$$G^{s+1}(x) \in G(\mu^{-1}(f^s(x))) \subset G(W(L)) \subseteq W'(L).$$

This implies that $G^{s_1}(x) \in W'(L)$. But we have $G^{s_1}(x) \notin W$ from $f^{s_1}(y) \notin \mu(W)$. This is a contradiction.

Now, recall W' is a finite union of torsion translation of algebraic subgroups. It is enough to prove for each torsion translation of algebraic subgroup separately. So, without loss of generality, we assume

$$W' = V(x_1^{r_1} \dots x_n^{r_n} - \epsilon)$$

where $r_1, \dots, r_n$ are integers that are not all zero and ϵ is a torsion element such that $\epsilon^d = \epsilon$. Enlarge K, K' and L if necessary so that $\epsilon \in K$. Then L-points in preimages of $W'(L)$ under iterates of G live in the union of $V(x_1^{r_1} \dots x_n^{r_n} - \epsilon\lambda)$, where λ ranges over elements which is d^m-torsion for some non-negative integer m. Notice that if $(x_1, \dots, x_n) \in (L^*)^n$, then $x_i \in L_i$ where L_i is a finite field extension of K such that $[L_i : K] \leqslant D \cdot 2^n$ for each $i \in \{1, 2, \dots, n\}$. Therefore, $\lambda \in L' = L_1 L_2 \dots L_n$ where L' is a finite field extension of K such that $[L' : K] \leqslant (D \cdot 2^n)^n$. Therefore, λ is a root of unity such that

$$[\mathbb{Q}(\lambda) : \mathbb{Q}] \leqslant (D \cdot 2^n)^n \cdot [K : \mathbb{Q}].$$

We claim that

$$M = \{\lambda \in \overline{\mathbb{Q}} : \lambda^{d^m} = 1, \text{ for some } m \in \mathbb{N}^+, [\mathbb{Q}(\lambda) : \mathbb{Q}] \leqslant (D \cdot 2^n)^n \cdot [K : \mathbb{Q}]\}$$

is a finite set. This is because the set of roots of unity of bounded degree is finite, which is a special case of Northcott property, and M is a subset of it. Then we can take

$$s_1 = \max_{\lambda \in M} \operatorname{ord}(\lambda),$$

where $\operatorname{ord}(\lambda)$ is the minimum non-negative integer m such that $\lambda^{d^m} = 1$. Then for any $y \in (L^*)^n$ such that $G^s(y) \in W'(L)$ for some $s > s_1$, we have $y \in V(x_1^{r_1} \dots x_n^{r_n} - \epsilon\lambda)$ for some $\lambda \in M$ and thus $G^{s_1}(y) \in W'(L)$. This will conclude the proof in this case.

To conclude, we take $s_0 = \max(s_0', s_1)$. We have

$$(f^{-s-1}(V) \backslash f^{-s}(V))(K') = \varnothing$$

for any $s \geqslant s_0$. □

Proposition 2.5.4. *Let K be a number field, let n and D be positive integers that are at least 2, and let K' be the union of all the finite field extensions of K of degree not greater than D in some fixed algebraic closure of K. Suppose that $f = (f_1, \dots, f_n) : (\mathbb{P}^1_K)^n \to (\mathbb{P}^1_K)^n$ is a split rational map defined over K in which the f_i's all have the same degree $d \geqslant 2$ and*

each f_i is a Lattès map for $i \in \{1, 2, \ldots, n\}$. If $V \subset (\mathbb{P}^1)^n$ is an irreducible f-invariant hypersurface defined over K that projects dominantly onto any subset of $n-1$ coordinate axes, then exists a non-negative integer s_0 such that

$$(f^{-s-1}(V) \backslash f^{-s}(V))(K') = \varnothing$$

for all $s \geqslant s_0$.

Proof. In this case, $f = (f_1, f_2, \ldots, f_n)$ where each f_i is a Lattès map. We consider a diagram:

$$\begin{array}{ccc} E_1 \times E_2 \times \cdots \times E_n & \xrightarrow{G} & E_1 \times E_2 \times \cdots \times E_n \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ (\mathbb{P}^1)^n & \xrightarrow{\quad f \quad} & (\mathbb{P}^1)^n \end{array} \tag{2.5.7}$$

where $G = (g_1, g_2, \ldots, g_n)$ is also a split morphism on the abelian variety $E_1 \times E_2 \times \cdots \times E_n$ and $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$ is a projection map such that each g_i and π_i satisfies $f_i \circ \pi_i = \pi_i \circ g_i$ by the fact that f_i is a Lattès map. Let $W = \pi^{-1}(V)$, which is a subvariety in $E_1 \times E_2 \times \cdots \times E_n$. Since each π_i has degree bounded by 6 [82, Proposition 6.37] and we can enlarge K to a larger number field so that g_i's are defined over K, we replace K by this larger number field. It is enough to show that L-points of W are stabilized under preimages of G, for every finite field extension L of K of degree not greater than $(6^n \cdot D)^n$. We have $G(W) \subseteq W$ by construction and there exists a subvariety $W' \subseteq W$ such that, after replacing G and f by some suitable iterate, $G(W') = W'$ and $G(W) \subseteq W'$. We just need to show that there exists a non-negative integer s_0 such that for any $s \geqslant s_0$ and $[L : K] \leqslant (6^n \cdot D)^n$,

$$(G^{-s-1}(W') \backslash G^{-s}(W'))(L) = \varnothing. \tag{2.5.8}$$

This will imply that

$$(f^{-s-1}(V) \backslash f^{-s}(V))(K) = \varnothing, \tag{2.5.9}$$

since, if not, there exists $x \in (\mathbb{P}^1_K)^n$ such that $f^s(x) \in V(K)$ but $f^{s_0}(x) \notin V(K)$ for some $s > s_0$. Thus there exists a

$$y \in \pi^{-1}(x) \subset (E_1 \times E_2 \times \cdots \times E_n)(L),$$

where L is a finite field extension of K of degree not greater than $(6^n \cdot D)^n$, such that $G^{s+1}(y) \in W'(L)$, but $G^{s_0}(y) \notin W$ so is not in W'. This is a contradiction to (2.5.8).

Notice that it is enough to prove 2.5.8 for each irreducible component of W'. From now on we abuse notation and let W denote an arbitrary irreducible component of W'.

To prove the statement (2.5.8), we first consider $\tilde{W}$ which is a translation of W by some $\bar{K}$-point in $E_1 \times E_2 \times \cdots \times E_n$ such that $\tilde{W}$ is invariant under $\tilde{G}$, where $\tilde{G}$ is a group homomorphism of the abelian variety with the property that G is the composition of $\tilde{G}$ with a suitable translation.

Then, by [44, Theorem 3.1], $\tilde{W}$ contains a Zariski dense set of preperiodic points of $\tilde{G}$.

Note that the preperiodic points of $\tilde{G}$ are torsion points in

$$E_1 \times E_2 \times \cdots \times E_n$$

(see [44, Claim 3.2] and notice that $\tilde{G}$, G and f are polarizable). Therefore, we can apply the classical Manin-Mumford conjecture (proved in [72, 73]) to get that $\tilde{W}$ is a translation of some subabelian variety of $E_1 \times E_2 \times \cdots \times E_n$ and so is W. Thus G restricted to W is an étale morphism and therefore we apply [13, Theorem 2.3] to conclude that there exists a nonnegative integer s_0 such that for any $s \geqslant s_0$ we have

$$(G^{-s-1}(W) \backslash G^{-s}(W))(L) = \varnothing, \tag{2.5.10}$$

for any finite field extension L of K of degree not greater than $(6^n \cdot D)^n$. □

Lemma 2.5.5. *Let V be an irreducible hypersurface in $(\mathbb{P}^1)^n$ that projects dominantly onto every subset of $n-1$ coordinate axes. Let $f = (f_1, \ldots, f_n)$ be a split rational map with $\deg(f_i) > 1$ for all $1 \leqslant i \leqslant n$. If $f(V) = V$ then $\deg(f_1) = \deg(f_2) = \cdots = \deg(f_n)$.*

Proof. We consider for each $i \in \{1, 2, \ldots, n\}$ the following diagram:

$$\begin{array}{ccc} V & \xrightarrow{f|_V} & V \\ {\scriptstyle \iota}\downarrow & & \downarrow{\scriptstyle \iota} \\ (\mathbb{P}^1)^n & \xrightarrow{f} & (\mathbb{P}^1)^n \\ {\scriptstyle \pi_i}\downarrow & & \downarrow{\scriptstyle \pi_i} \\ (\mathbb{P}^1)^{n-1} & \xrightarrow{g_i} & (\mathbb{P}^1)^{n-1} \end{array} \tag{2.5.11}$$

where $\iota : V \to (\mathbb{P}^1)^n$ is the natural embedding, π_i is the projection to the $n-1$ coordinates excluding the i-th coordinate, and

$$g_i = (f_1, f_2, \ldots, f_{i-1}, f_{i+1}, \ldots, f_n),$$

i.e., we skip the i-th index and so $g_i : (\mathbb{P}^1)^{n-1} \to (\mathbb{P}^1)^{n-1}$. Since $\pi_i \circ \iota$ is a finite morphism by the assumption that V projects dominantly onto every subset of $n-1$ coordinate axes, we have

$$\deg(f|_V) = \deg(g_i) = \prod_{j \in I_i} \deg(f_j),$$

where $I_i = \{1, 2, \ldots, i-1, i+1, \ldots, n\}$. Since this argument and equation hold for each $i \in \{1, 2, \ldots, n\}$, we have

$$\prod_{j \in I_i} \deg(f_j) = \prod_{j \in I_k} \deg(f_j)$$

for any $i, k \in \{1, 2, \ldots, n\}$. This implies that $\deg(f_i) = \deg(f_k)$ for any pair of i, k in $\{1, 2, \ldots, n\}$. □

Lemma 2.5.6. *In the case where the degrees of $f_1, \ldots, f_n$ are all at least two, it is enough to prove Theorem 2.1.1 with the additional assumption that V is an irreducible hypersurface of dimension not less than 1 that projects dominantly onto every subset of $n-1$ coordinate axes in $(\mathbb{P}^1)^n$.*

Proof. The idea is the same as in [42, Proposition 2.1]. We assume throughout this proof that we have the hypotheses of Theorem 2.1.1 and that the maps $f_1, \ldots, f_n$ all have degree $\geqslant 2$.

We use induction on the dimension of V. If $\dim(V) = 0$, then V is a finite set of points and Theorem 2.1.1 is true since the preimages of K-points in V live inside a set of K-points of bounded height. Now suppose that the conclusion to Theorem 2.1.1 holds when $\dim(V) < D$, we prove the case that $\dim(V) = D$. Notice that it is enough to prove that the conclusion to Theorem 2.1.1 holds for each irreducible component of V separately after replacing f by a suitable iterate.

Thus we assume that V is irreducible. Then there exist D coordinate axes such that the projection π of V onto these axes is dominant.

Without loss of generality, we assume that they are the first D coordinates and for $j > D$ we let π_j denote the projection from V to the coordinate axes indexed by $\{1, \ldots, D, j\}$.

Therefore $\pi_j(V)$ is a hypersurface in $(\mathbb{P}^1)^{D+1}$ and also

$$H_j := \pi_j(V) \times (\mathbb{P}^1)^{n-D-1}$$

is a hypersurface in $(\mathbb{P}^1)^n$. Now we claim that V is a component of $\bigcap_{j=D+1}^n H_j$. Notice that $\dim\left(\bigcap_{j=D+1}^n H_j\right) \geqslant D$ and $V \subset \bigcap_{j=D+1}^n H_j$. So we just need to show that $\dim\left(\bigcap_{j=D+1}^n H_j\right) = D$.

Since $\pi_j(V)$ projects dominantly onto the first D coordinate axes, there exists a Zariski open subset $U \subset (\mathbb{P}^1)^D$ such that for each

$$\alpha = (\alpha_1, \ldots, \alpha_D) \in U$$

there exists a finite set $S_{\alpha,j}$ such that if $(\alpha_1, \ldots, \alpha_n) \in H_j$ and $(\alpha_1, \ldots, \alpha_D) \in \pi(V)$ then $\alpha_j \in S_{\alpha,j}$. This is saying for each $\alpha \in U$ there are only finitely many points in $\bigcap_{j=D+1}^n H_j$ such that the first D coordinates are equal to α. Therefore, $\dim\left(\bigcap_{j=D+1}^n H_j\right) = D$ and V is a component of $\bigcap_{j=D+1}^n H_j$. Notice that since $f(V) = V$, we also have $f(H_j) = H_j$.

We claim that if for each H_j, we have the statement of Theorem 2.1.1 holds, then certainly it holds for V. To prove the claim, we first notice that there exists a subvariety $H \subseteq \bigcap_{j=D+1}^n H_j$ containing V such that $f(H) = H$ and $f\left(\bigcap_{j=D+1}^n H_j\right) = H$ if we replace f by some iterate. We denote $V_1, \ldots, V_k$ as the irreducible components of H of dimension D and without loss of generality assume that $V = V_1$ and $f(V_i) = V_i$ for each $i \in \{1, \ldots, k\}$ by replacing f with suitable iterates. Assume that the conclusion to Theorem 2.1.1 holds for each H_j. Then there exist non-negative integers s_j such that each $x \in (\mathbb{P}^1_K)^n$ satisfies $f^s(x) \in H_j(K)$ for some non-negative integer s, and we have $f^{s_j}(x) \in H_j(K)$ for each $D+1 \leqslant j \leqslant n$.

Now for each $x \in (\mathbb{P}^1_K)^n$ such that $f^s(x) \in V(K)$ for some non-negative integer s, we have certainly $f^s(x) \in H_j(K)$ for each $D+1 \leqslant j \leqslant n$. Thus letting s_0' denote the quantity $\max_{D+1 \leqslant j \leqslant n}\{s_j\}$, we have $f^{s_0'}(x) \in \bigcap_{j=D+1}^n H_j(K)$ and $f^{s_0'+1}(x) \in H$. If $f^{s_0'+1}(x) \notin V$ then $f^{s_0'+1}(x) \in V_i$ for some $i \in \{2, \ldots, k\}$ such that $f^s(x) \in V \cap V_i$. Since $f(V) = V$ and $f(V_i) = V_i$, we have $f(V \cap V_i) \subseteq (V \cap V_i)$ and there exists $V' \subseteq V \cap V_i$ such that $f(V \cap V_i) = V'$ and $f(V') = V'$ after further replacing f by a suitable iterate. Notice that $\dim(V') < D$ and x is in the preimages of V' under f. By the induction hypothesis, there exists a positive integer s_1' such that $(f^{-s_1'-1}(V') \backslash f^{-s_1'}(V))(K) = \varnothing$. Therefore, taking $s_0 = \max\{s_1', s_0' + 1\}$, we have $f^{s_0}(x) \in V(K)$. We proved that it is enough to prove that the conclusion to Theorem 2.1.1 holds for each H_j.

Now for each H_j, it is equivalent to prove the statement of Theorem 2.1.1 for $\pi_j(V) \subset (\mathbb{P}^1)^{D+1}$ which is an irreducible hypersurface invariant under $f' = (f_1, \ldots, f_D)$ and projecting dominantly onto first D coordinate axes. We claim that, after reordering coordinate axes, we have $\pi_j(V) = W_j \times (\mathbb{P}^1)^{m_j}$ for some $0 \leqslant m_j \leqslant D$, where $W_j \subseteq (\mathbb{P}^1)^{D+1-m_j}$ projects dominantly onto any subset of $D - m_j$ coordinate axes. This can be proved by induction and the base case is that $D = 1$ and in this case $\pi_j(V)$ either projects dominantly onto any $\mathbb{P}^1$ factors or it is $W_j \times \mathbb{P}^1$, after reordering coordinate axes, where W_j is a finite set of points. Now, assuming the claim is true for $D = D_0$, $D_0 \geqslant 1$ is a positive integer, we prove

it for $D = D_0 + 1$. If there exists D subset of coordinate axes such that the projection, $\pi_1(\pi_j(V))$, onto those axes is not dominant and denote π_2 as the projection onto the other $\mathbb{P}^1$ factor, then $\dim(\pi_1(\pi_j(V))) = D - 1$ since we have

$$\dim(\pi_j(V)) \leqslant \dim(\pi_1(\pi_j(V))) + \dim(\pi_2(\pi_j(V))). \tag{2.5.12}$$

So $\dim(\pi_2(\pi_j(V))) = 1$, which implies $\pi_2(\pi_j(V)) = \mathbb{P}^1$. Then the equality actually holds in (2.5.12) and we have $\pi_j(V) = \pi_1(\pi_j(V)) \times \mathbb{P}^1$. Applying the induction hypothesis on $\pi_1(\pi_j(V))$ concludes the proof of the claim.

Recall, we have already shown that it is equivalent to prove the statement of Theorem 2.1.1 for $\pi_j(V)$. By the claim, proving for $\pi_j(V)$ is also equivalent to proving for W_j which is an irreducible hypersurface in $(\mathbb{P}^1)^{D+1-m_j}$ invariant under $f'' = (f_1, \ldots, f_{D+1-m_j})$, after reordering coordinate axes, and projects dominantly onto any subset of $D - m_j$ coordinate axes. □

Proposition 2.5.7. *Let K be a finitely generated field extension of $\mathbb{Q}$, let D be a positive integer and let L be the union of the finite extensions of K of degree less than or equal to D. Let $f : \mathbb{P}^1_K \to \mathbb{P}^1_K$ be a surjective morphism of degree greater than 1 defined over K. Then there exists a positive integer N such that if $a, b \in \mathbb{P}^1(L)$ satisfy $f^n(a) = f^n(b)$ for some $n \geqslant 0$, then $f^N(a) = f^N(b)$.*

Proof. [13, Theorem 3.1] proves this for L a number field. But the proof still works if we have L as above. The only changes to the proof that are needed are that instead of embedding L into $\mathbb{Q}_p$ for a suitable prime p directly such that, after embedding, $f \in \mathbb{Z}_p[[x]]$, we embed K into $\mathbb{Q}_p$ for a suitable prime p such that $f \in \mathbb{Z}_p[[x]]$. Then L is naturally embedded as a set in the union of the finite field extensions of $\mathbb{Q}_p$ of degree less or equal to D which is inside $\overline{\mathbb{Q}}_p$. Denote such an embedding map as ι. Since there are only finitely many finite extensions of $\mathbb{Q}_p$ of degree bounded by D^2, we can perform the argument separately for each finite extension K'_v of $\mathbb{Q}_p$ of degree bounded by D^2 to obtain a corresponding N and assume $\iota(a), \iota(b) \in K'_v$. In the end we simply take the maximum of these N's corresponding to each K'_v.

Notice that the key ingredient of the proof of [13, Theorem 3.1] is the p-adic uniformization which works over $\mathbb{C}_p$ and thus it can be applied here without changes. Also, in the proof [13, Theorem 3.1], N is only depending on f and the least common multiple of the orders of roots of unity in a finite extension of $\mathbb{Q}_p$ of degree bounded by $\deg(f)$ which is a finite set. And now, we only need to take it to be depending on f and a positive integer M—the least common multiple of orders of roots of unity in a finite extension of K'_v of degree bounded by $\deg(f)$. Notice that this is doable since the set of roots of unity in $\overline{\mathbb{Q}}_p$

such that their degrees over $\mathbb{Q}_p$ are all bounded by a constant is also finite [13, Proposition 3.6(3)] and the finite extension of K'_v is also a finite extension of $\mathbb{Q}_p$ of degree bounded by $\deg(f)D^2$. Thus, the proof follows just as in [13, Theorem 3.1]. □

Proof of Theorem 2.1.1 in the case when $\deg(f_1), \ldots, \deg(f_n) \geqslant 2$. Notice that by Lemma 2.5.6, it is enough to prove the statement with the assumption that V is an irreducible hypersurface projecting dominantly onto any subset of $(n-1)$ coordinate axes in $(\mathbb{P}^1)^n$. Since $f(V) = V$, we have that V contains a Zariski dense set of preperiodic points of f [31, Theorem 5.1]. Therefore, by [42, Theorem 2.2], we have if $n > 2$ then

1. either $f_1, \ldots, \ldots f_n$ are all Lattès maps;
2. or f_i's are all conjugated to $x^{\pm d_i}$ or $\pm T_{d_i}$, where T_{d_i}'s are Chebyshev polynomials of degree $d_i = \deg(f_i)$.

Notice that by Lemma 2.5.5 in both cases $d_1 = d_2 = \cdots = d_n$. If $n = 2$, then by the proof of [42, Theorem 1.3] we have either f_1, f_2 are both Lattès maps or neither of them is. So overall we have three separate cases to show:

1. $n = 2$, f_1 and f_2 are not Lattès maps;
2. $n > 2$, f_i's are conjugated to either $x^{\pm d}$ or $\pm T_d$ for some $d > 1$;
3. f_i's are all Lattès maps.

Case (2) is implied by Proposition 2.5.3 and Case (3) is implied by Proposition 2.5.4. We are left to prove case (1): By [68, Corollary 4.5] and [68, Remark 4.3], we have that there exists rational functions U_1, U_2 and F defined over $\overline{K}$ such that

$$U_1 \circ F = f_1 \circ U_1,$$

$$U_2 \circ F = f_2 \circ U_2$$

and $W_1 = (U_1, U_2)^{-1}(V)$ is a subvariety such that $(F, F)(W_1) \subseteq W_1$ and it contains an irreducible component $V' \subseteq W$ such that $(F, F)(V') = V'$ and $(F, F)(W_1) \subseteq V'$ after replacing F, f_1 and f_2 by some suitable iterate. Also by replacing K with some larger number field, we assume U_1, U_2 and F are defined over K. Let K' be the union of the finite field extensions of K, whose degree is bounded by $\deg(U_1)^2 = \deg(U_2)^2$ of K. It is enough to show that there exists a positive integer s_0 such that

$$((F, F)^{-s-1}(V') \backslash (F, F)^{-s}(V'))(K') = \varnothing$$

for any integer $s \geqslant s_0$.

Now if F is Lattès map or is conjugate to either a power map or a Chebyshev polynomial or its negative, we obtain the result from Proposition 2.5.3 and Proposition 2.5.4 separately. So, we assume F is not a Lattès map nor conjugate to either a power map or a Chebyshev polynomial or its negative. Then by [68, Theorem 4.15], we have there exists rational functions U_3, U_4, F_1 and F_2 over $\mathbb{C}$ such that

$$U_3 \circ F_1 = F \circ U_3,$$

$$U_4 \circ F_2 = F \circ U_4$$

and $W = (U_3, U_4)^{-1}(W_1)$ is a subvariety such that $(F_1, F_2)(W) \subseteq W$ and it contains a subvariety W' such that $(F_1, F_2)(W') = W'$ after replacing F_1, F_2, f_1, f_2 and F by a suitable iterate. Furthermore, F_1 and F_2 are not generalized Lattès maps by the Theorem.

Now let $\tau_1 = U_1 \circ U_3$ and $\tau_2 = U_2 \circ U_4$ and replace K by a finitely generated field extension of K such that τ_1, τ_2, F_1 and F_2 are all defined over K. After replacing F and F_1, F_2 with a suitable iterate and abusing notation to let W be some irreducible component of W', we may assume $(F_1, F_2)(W) = W$. Let L be the union of the finite field extensions of K of degree bounded by $\deg(\tau_1)^2 = \deg(\tau_2)^2$. Similarly, it is enough to show that there exists a positive integer s_0 such that

$$((F_1, F_2)^{-s-1}(W) \backslash (F_1, F_2)^{-s}(W))(L) = \varnothing$$

for all integers $s \geqslant s_0$.

In this case, we use [68, Theorem 1.1] and we get that there exists rational functions X_1, X_2, Y_1, Y_2 and B such that there exists some positive integer d such that

$$F_1^d = X_1 \circ Y_1 \tag{2.5.13}$$

$$F_2^d = X_2 \circ Y_2 \tag{2.5.14}$$

$$B^d = Y_1 \circ X_1 = Y_2 \circ X_2, \tag{2.5.15}$$

$W \subseteq (Y_1, Y_2)^{-1}(\Delta)$ and also $(F_1^d, F_2^d)((Y_1, Y_2)^{-1}(\Delta)) = W$.

Enlarging K by adjoining the coefficients of X_1, Y_1, X_2 and Y_2 and abusing the notation to let K denote this larger field and L be the union of the finite field extensions of degree bounded by $\deg^2(\tau_1) = \deg^2(\tau_2)$ of K.

Now it is enough to show that there exists a non-negative integer s_0 such that for any $s \geqslant s_0$,

$$((F_1, F_2)^{(-s-1)d}(W) \backslash (F_1, F_2)^{-sd}(W))(L) = \varnothing. \tag{2.5.16}$$

We claim that to prove the above, it suffices to show that there exists s_0 such that for any $s \geqslant s_0$

$$((F_1, F_2)^{d(-s-1)}((Y_1, Y_2)^{-1}(\Delta))\backslash(F_1, F_2)^{-ds}((Y_1, Y_2)^{-1}(\Delta)))(L) = \varnothing. \tag{2.5.17}$$

To see this, suppose there exists s_0' such that for any $s \geqslant s_0'$ Equation (2.5.17) holds. If there exists a non-negative integer $s > s_0'$ and a $x \in (\mathbb{P}^1 \times \mathbb{P}^1)(L)$ such that $(F_1, F_2)^{sd}(x) \in W(L)$ but $(F_1, F_2)^{s_0' d}(x) \notin W$, then

$$(F_1, F_2)^{s_0' d}(x) \in (Y_1, Y_2)^{-1}(\Delta)\backslash W$$

by Equation (2.5.17). Thus $(F_1, F_2)^{(s_0'+1)d}(x) \in W$. Therefore, take $s_0 = s_0' + 1$, we have for any $s \geqslant s_0$

$$((F_1, F_2)^{(-s-1)d}(W)\backslash(F_1, F_2)^{-sd}(W))(L) = \varnothing.$$

Thus, it is enough to show that there exists s_0 such that for any $s \geqslant s_0$

$$((F_1, F_2)^{d(-s-1)}((Y_1, Y_2)^{-1}(\Delta))\backslash(F_1, F_2)^{-ds}((Y_1, Y_2)^{-1}(\Delta)))(L) = \varnothing \tag{2.5.18}$$

which is equivalent to, by Equation (2.5.13), (2.5.14) and (2.5.15),

$$((Y_1, Y_2)^{-1}((B^d, B^d)^{-s-1}(\Delta)\backslash(B^d, B^d)^{-s}(\Delta)))(L) = \varnothing. \tag{2.5.19}$$

Since, again, Y_1, Y_2 are defined over K, it is enough to show

$$((B^d, B^d)^{-s-1}(\Delta)\backslash(B^d, B^d)^{-s}(\Delta))(L) = \varnothing. \tag{2.5.20}$$

Notice that if there exists a non-negative integer s_0 such that for any non-negative integer s and any $x, y \in \mathbb{P}^1(L)$, we have

$$B^{ds}(x) = B^{ds}(y) \tag{2.5.21}$$

implies

$$B^{ds_0}(x) = B^{ds_0}(y), \tag{2.5.22}$$

then Equation (2.5.20) holds for $s \geqslant s_0$. While this is proved by Proposition 2.5.7. The result follows.

□

Proof of Theorem 2.1.1 in the general case. By replacing f by some suitable iterate, we have that each irreducible component of V is also an invariant subvariety of f. It is enough to prove the theorem for each irreducible component of V, so we assume that V is irreducible.

If there doesn't exist f_i such that $\deg(f_i) = 1$, for $i \in \{1, \ldots, n\}$, then the statement has been proved. Thus we may assume that at least one f_i is an automorphism. We reorder the coordinates so that there exists a positive integer $k \in \{2, \ldots, n-1\}$ such that $\deg(f_i) > 1$ when $i \leqslant k$ and $\deg(f_i) = 1$ when $i > k$. Now Lemma 2.5.1 implies that $V = V_1 \times V_2$, where $V_1 \subseteq (\mathbb{P}^1)^k$, $V_2 \subseteq (\mathbb{P}^1)^{n-k}$ such that $g_1(V_1) = V_1$ and $g_2(V_2) = V_2$, where $g_1 = (f_1, \ldots, f_k)$ and $g_2 = (f_{k+1}, \ldots, f_n)$. Notice that $g_2^{-1}(V_2) = V_2$ and there exists a non-negative integer s_0 such that $(g_1^{-s-1}(V_1) \backslash g_1^{-s}(V_1))(K) = \varnothing$ for all $s \geqslant s_0$, since we have established Theorem 2.1.1 in the case when the maps all have degree $\geqslant 2$. Thus, or any non-negative integer $s \geqslant s_0$,

$$(f^{-s-1}(V) \backslash f^{-s}(V))(K)$$

$$= (g_1^{-s-1}(V_1) \backslash g_1^{-s}(V_1))(K) \times (g_2^{-s-1}(V_2) \backslash g_2^{-s}(V_2))(K) = \varnothing.$$

The result follows.

□

Chapter 3

A finiteness result for common zeros of iterates of rational functions

3.1 Introduction

3.1.1 Historical background.

Bugeaud, Corvaja, and Zannier [18] provided an upper bound on the greatest common divisor (GCD) of two integer sequences using tools from Diophantine approximation. More precisely, they showed that for any multiplicatively independent integers $a, b \geqslant 2$ and for any $\epsilon > 0$, one has

$$\gcd(a^n - 1, b^n - 1) < \exp(\epsilon n)$$

for sufficiently large $n \geqslant 1$.

In 2004, Ailon and Rudnick [1] obtained the first instance of a GCD-type result over a function field of characteristic 0. In fact, they proved that for any nonconstant multiplicatively independent polynomials $a, b \in \mathbb{C}[x]$ there exists a polynomial $h \in \mathbb{C}[x]$ such that $\gcd(a^n - 1, b^n - 1)|h$ for all $n \geqslant 1$. Ostafe [65] pointed out that the result of Ailon-Rudnick holds generally for any pair $(m, n) \in \mathbb{N} \times \mathbb{N}$. Ostafe also provided an upper bound on the degree of $\gcd(a^m - 1, b^n - 1)$ in terms of degrees of a and b.

There are many generalizations and extensions of GCD-type results in various contexts, see for example [22], [39], [36], [47], [48], [53], [57], [58], [79], [80], [81] and [90].

Motivated by taking multiplicative powers of polynomials, one can study GCD-type results of iterates of rational functions. Let $f(x) \in \mathbb{C}(x)$ be a rational function with complex coefficients. Here and in what follows, f^n denotes the n-fold composition of f with itself.

Definition 3.1.1. Two rational functions f and g are said to be compositionally independent if there exist positive integers $i_1, ..., i_s, j_1, ..., j_s, l_1, ..., l_t, m_1, ..., m_t$ such that

$$f^{i_1} \circ g^{j_1} \circ \cdots \circ f^{i_s} \circ g^{j_s} = f^{l_1} \circ g^{m_1} \circ \cdots \circ f^{l_t} \circ g^{m_t}$$

then $s = t$ and $i_k = l_k, j_k = m_k$ for $1 \leqslant k \leqslant s$. In other words, the semigroup generated by f and g under composition is isomorphic to the free semigroup of two generators.

Hsia and Tucker [52] first addressed the question posed by Ostafe [65, §4.2] in dynamical setting. Under suitable compositional assumptions, they showed that for any polynomial maps f, g of degree $\geqslant 2$ with complex coefficients and $c(x) \in \mathbb{C}[x]$ there are only finitely many irreducible factors of

$$\gcd(f^m(x) - c(x), g^n(x) - c(x))$$

for some positive integers m and n (cf. [52, Theorem 1]). Interestingly, Theorem 2 in [52] can be viewed as a compositional analog result of Ailon-Rudnick for linear polynomials. However, for nonlinear polynomials, their result holds conditionally on the assumption that $c(x)$ is not a complex constant which is in ramified cycle of both f and g (cf. [52, Theorem 4]).

3.1.2 Statement of main results.

In this article, we answer the following question posed by Hsia and Tucker.

Question 3.1.2. [52, Question 18] Let f, g be two nonconstant, compositionally independent rational functions with complex coefficients. Let c be any rational function with complex coefficients. Is it true that there must be at most finitely many $\lambda \in \mathbb{C}$ such that $f^n(\lambda) = g^n(\lambda) = c(\lambda)$ for some positive integer n?

It turns out that Question 3.1.2 is not true in general. This can be seen from the counter-examples provided in Remark (3.3.5),(3.3.6), (3.3.7), (3.3.8) and (3.3.9).

However, after excluding these families of rational functions f and g, we obtain a finiteness result of common zeros of iterated rational functions in a similar spirit to Question 3.1.2. More precisely, we state our main results as follows.

Theorem 3.1.3. *Let $f(X)$, $g(X)$ be automorphisms on $\mathbb{P}^1$ defined over $\mathbb{C}$ and $c(X)$ be a rational function defined over $\mathbb{C}$. If the semigroup generated by $f(X)$ and $g(X)$ under compositions is free, then there are only finitely many $\lambda \in \mathbb{C}$ such that*

$$f^n(\lambda) = g^n(\lambda) = c(\lambda) \tag{3.1.1}$$

for some positive integer n unless $f(X)$, $g(X)$ are simultaneously conjugated by an automorphism on $\mathbb{P}^1(\mathbb{C})$ to one of the following:

1. *$\alpha X + \beta$, $X/(\gamma X + \delta)$, with some $\alpha, \delta, \gamma, \beta \in \mathbb{C}^*$ such that one of α/δ and $\alpha\delta$ is a root of unity;*
2. *$\alpha X + \beta$, $\delta X + \gamma$, with some $\alpha, \delta \in \mathbb{C}^*$ such that α and δ are not roots of unity, γ and β are not both 0, and either α/δ is a root of unity other than 1 or one of α^2/δ and δ^2/α is a root of unity.*

When at most one of rational functions f and g is of degree one, we can relax some conditions on $(m, n) \in \mathbb{N} \times \mathbb{N}$. Also, the finiteness result on the number of zeros of both equations $f^m(x) - c(x) = 0$ and $g^n(x) - c(x) = 0$ holds under natural assumptions. This is reminiscent of polynomial maps' counterpart (cf. [52, Theorem 1]).

Theorem 3.1.4. *Let f and g be two compositionally independent rational functions with complex coefficients, at least one of which has degree greater than one. Suppose that $c(x) \in \mathbb{C}(x)$ is not a compositional power of f or g. Then there are only finitely many $\lambda \in \mathbb{C}$ such that*

$$f^m(\lambda) = g^n(\lambda) = c(\lambda)$$

for some positive integers m and n.

Notice that the same statement allowing distinct n and m for the case that f and g are both automorphisms on $\mathbb{P}^1(\mathbb{C})$ doesn't hold. There is a counter-example in the beginning of Section 3 of [52].

3.1.3 Proof strategy.

Hsia and Tucker [52] combine various tools and techniques from Diophantine geometry, complex dynamics, and number theory. Our approach is akin to their method. There are technical difficulties to overcome for rational functions. We briefly explain our ideas of proving Theorem 3.1.3 and Theorem 3.1.4.

For automorphisms f and g defined over $\overline{\mathbb{Q}}$, we heavily hinge on tools from Diophantine geometry (e.g., Roth's theorem, Siegel's theorem, and S-unit theorem). After excluding suitable f, g (see §3.1.2), we show that f and g must share a common fixed point (see Lemma 3.3.1). Then we dynamically conjugate by moving such a fixed point to ∞, we may assume that $f(x) = \alpha x + \beta$ and $g(x) = \delta x + \gamma$ with $\alpha, \beta, \gamma, \delta \in \overline{\mathbb{Q}}^*$. We break our computation into several cases depending on whether α and δ is a root of unity to show that if there are infinitely many solutions to

$$f^n(x) = g^n(x) = c(x)$$

where $c(x)$ is any rational function with complex coefficients, then f and g are not compositionally independent. In other words, the semigroup generated by f and g under composition is not free (see Theorem 3.3.4). Using specialization argument, we obtain the desired result over $\mathbb{C}$.

When exactly one of rational functions $f, g \in \mathbb{C}(x)$ has degree greater than one, the proof follows verbatim in the polynomial maps setting as a consequence of Northcott's property explained in [52, Proposition 9].

In the case both rational functions $f, g \in \mathbb{C}(x)$ are of degree greater than 1 and $c(x) \in \mathbb{C}(x)$. Under suitable compositional assumptions, we show that the infinite sequence of solutions to

$$f^{m_i}(x_i) = g^{n_i}(x_i) = c(x_i)$$

as $i \to +\infty$ (so $m, n \to +\infty$) form a sequence of small dynamical height (Lemma 3.4.1). We, then, apply the arithmetic equidistribution theorem of small points (in reminiscence of [6], [5], [37], [38], and [70]) to deduce that dynamical height attached to f and g must be identical (Proposition 3.4.2) and hence f and g must share the same set of preperiodic points (i.e., points of finite forward orbit under iteration). Unlike polynomial maps setting, one cannot appeal to the classification of polynomials with the same Julia set of [3] and [77]. Alternatively, we apply the Tits alternative for rational functions of [12] to deduce that f and g are not compositionally independent.

3.2 Preliminaries

3.2.1 Absolute values over global fields.

Here and throughout this section, K denote a global field (unless otherwise stated) equipped with a set of inequivalent absolute values (places) satisfying the product formula. For example, K could be a number field or a function field of positive transcendence degree over

another field k. It was pointed out in [2] that a global field which has an archimedean place must be a number field.

If K is a number field, we let M_K to be the set of all absolute values of K which extends the absolute values of $\mathbb{Q}$. If K is a function field of finite transcendence degree $m \geqslant 1$ defined over a field k, then M_K is the set of all absolute values on K associated to the irreducible hypersurfaces of the projective space $\mathbb{P}^m_k$ (whose function field is $K' := k(t_1, ..., t_m)$ where $t_1, .., t_m$ are algebraically independent–over k–functions in K). Thus there exist $N_v \geqslant 1$ such that the product formula $\prod_{v \in M_K} |x|_v^{N_v} = 1$ holds for any nonzero $x \in K$. (see [16] and [56]).

Let K_v be the completion of K with respect to absolute value $|\cdot|_v$. Then $\mathbb{C}_v$ is the completion of the algebraic closure of K_v and it is algebraically closed and complete. Note that, by abuse notation, we still denote the absolute value on $\mathbb{C}_v$ which extends uniquely from $|\cdot|_v$ on K by $|\cdot|_v$.

Remark 3.2.1. When there is no confusion, we will simplify the notation to use $|\cdot|$ to denote the usual Archimedean norm on K.

3.2.2 Dynamical height and preperiodic points.

Let S be the finite set of Galois conjugates $\mathrm{Gal}(\overline{K}/K) \cdot x$ of x in $\mathbb{P}^1(\overline{K})$. Then we define the Weil height $h : \mathbb{P}^1(\overline{K}) \to [0, +\infty)$ by

$$h(x) := \frac{1}{|S|} \sum_{x \in S} \sum_{v \in M_K} N_v \log \max\{1, |x|_v\}.$$

In light of Baker-Rumely's construction [8, §7.9], our Weil height is defined as the average over Galois conjugate of points in $\overline{K}$. It was pointed out that this definition does not depend on the choice of embedding $\overline{K} \hookrightarrow \mathbb{C}_v$ as S is Galois stable subset of $\mathbb{P}^1(\overline{K})$.

Let f be a rational function defined over a global field K of degree $\geqslant 2$. Then the Call-Silverman canonical height $\hat{h}_f : \mathbb{P}^1(\overline{K}) \to [0, +\infty)$ is defined by

$$\hat{h}_f(x) = \lim_{n \to \infty} \frac{1}{(\deg f)^n} h(f^n(x)).$$

We record important properties of Weil height and dynamical canonical height as follows.

Proposition 3.2.2. *([82, §3], [19]) Let f, h and $\hat{h}_f$ be as above. Then*

(i) $h(f(x)) = (\deg f)h(x) + O(1)$.

(ii) $\hat{h}_f(x) = h(x) + O(1)$.

(iii) $\hat{h}_f(f(x)) = (\deg f)\hat{h}_f(x)$.

Over a number field, dynamical height is used to detect preperiodic points in the following sense.

Proposition 3.2.3. *([82, Theorem 3.22]) Let $f(x) \in K(x)$ be a rational function of degree $\geqslant 2$ defined over a number field K. Then $x \in \mathbb{P}^1(\overline{K})$ is f-preperiodic if and only if $\hat{h}_f(x) = 0$.*

The situation is different over function field. The dynamical height of a function defined on a function field (over an infinite field) fails to properly detect preperiodic points. As an interesting example shown by Benedetto [14], we consider $f(x) = x^2 \in K[x]$ defined over a function field $K = \overline{\mathbb{Q}}(T)$ of transcendence degree 1 with constant field $\overline{\mathbb{Q}}$. Notice that all points in $\mathbb{P}^1(\overline{\mathbb{Q}})$ have dynamical height zero but $0, \infty$ and roots of unity in $\overline{\mathbb{Q}}$ are the only f-preperiodic points. In fact, f is defined over the constant field of K.

Recall that a rational function $f(x) \in K(x)$ defined over a function field K is called **isotrivial** if there exists a finite extension K'/K and a Möbius transformation $\phi \in \mathrm{PGL}(2, K')$ such that $\phi^{-1} \circ f \circ \phi$ defined over the constant field of K'. More related results, see [4], [14], and [69].

After excluding maps with isotriviality, the dynamical height works as expected. The following result is due to Benedetto (for polynomial maps) and Baker (for rational functions) over $\mathbb{P}^1$.

Theorem 3.2.4. *([4], [14]) Let $f(x) \in K(x)$ be a rational function of degree $\geqslant 2$ defined over a function field K. Suppose that f is not isotrivial. Then $x \in \mathbb{P}^1(\overline{K})$ is f-preperiodic if and only if $\hat{h}_f(x) = 0$.*

3.2.3 Arakelov-Green's function.

The dynamical height associated to rational functions can be decomposed in terms of Arakelov-Green's function. We follow the construction in Baker-Rumely [8, §10.3]. Suppose that $f \in K(x)$ is a rational function of degree $\geqslant 2$ defined over a global field K. The v-adic Arakelov-Green's function of f is a function

$$G_{f,v} : \mathbb{P}^1_{\mathrm{Berk},v} \times \mathbb{P}^1_{\mathrm{Berk},v} \to \mathbb{R} \cup \{\infty\}$$

which is symmetric, finite and continuous off the diagonal in $\mathbb{P}^1(\mathbb{C}_v) \times \mathbb{P}^1(\mathbb{C}_v)$. Here $\mathbb{P}^1_{\mathrm{Berk},v}$ is the Berkovich projective line, we refer the reader to [15] and [8] for detailed exposition.

For each $y \in \mathbb{P}^1_{\mathrm{Berk},v}$, the function $G_{f,v}(x,y)$ is bounded differential variation on $\mathbb{P}^1_{\mathrm{Berk},v}$ and it satisfies the differential equation

$$\Delta_x G_{f,v}(x,y) = \mu_{f,v}(x) - \delta_y(x) \tag{3.2.1}$$

where $\mu_{f,v}$ is the equilibrium measure characterized by $f^*(\mu_{f,v}) = (\deg f)\mu_{f,v}$ and $G_{f,v}$ is normalized so that

$$\iint\limits_{\mathbb{P}^1_{\mathrm{Berk},v} \times \mathbb{P}^1_{\mathrm{Berk},v}} G_{f,v}(x,y) d\mu_{f,v}(x) d\mu_{f,v}(y) = 0.$$

Our Laplacian is the negative of the one introduced by Baker-Rumely [7, §3.5]. For any $x, y \in \mathbb{P}^1(\overline{K})$ such that $x \neq y$, we have

$$\hat{h}_f(x) + \hat{h}_f(y) = \sum_{v \in M_K} N_v G_{f,v}(x,y)$$

where the positive number N_v is the same normalization as appeared in the product formula.

3.2.4 Results from Diophantine geometry.

Here we collect some Diophantine geometry results that will be used in the proof.

Theorem 3.2.5. *[30, Theorem 6.1.3] Let K be a field of characteristic 0, let $n \geqslant 2$, let $a_1, \ldots, a_n \in K^*$ and let Γ be a subgroup of $(K^*)^n$ of finite rank r. Then the number of non-degenerate solutions of*

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = 1$$

in $(x_1, x_2, \ldots, x_n) \in \Gamma$ is finite.

Remark 3.2.6. We call a solution to the equation $a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = 1$ like above non-degenerate if

$$\sum_{i \in I} a_i x_i \neq 0$$

for any proper subset $I \subset \{1, 2, \ldots, n\}$.

3.2.5 The Ping-Pong Lemma

Lemma 3.2.7. *Let G be a group of maps under compositions. Let $S = \langle f_1, \ldots, f_r \rangle$, $r \geqslant 2$ and $f_i \in G$, for $i \in \{1, 2, \ldots, r\}$, be a semigroup of G. If there exists a collection of non-empty mutually non-intersecting sets $I_1, \ldots, I_r$ such that $f_i(\bigcup_{j=1}^{r} I_j) = I_i$ for all $1 \leqslant i \leqslant r$, then S is a free semigroup of rank r with free basis $f_1, \ldots, f_r$.*

Proof. This is basically the Ping-Pong Lemma stated in [55, The Ping-Pong Lemma]. The only difference is that it is stated for G being the group of affine linear maps on $\mathbb{R}$. However the proof doesn't rely on this assumption. □

3.2.6 Asymptotic notation

We clarify the asymptotic notation here:

Definition 3.2.8. Consider two functions $f(n)$ and $g(n)$ with $n \in \mathbb{N}$ and take values in $\mathbb{C}$. We define the asymptotic notation when n goes large as follows:

1. We write $f(n) \sim g(n)$ if there exists a non-zero constant $k \in \mathbb{C}$ such that
$$\lim_{n\to\infty} f(n)/g(n) = k;$$
2. We write $f(n) = o(g(n))$ if
$$\lim_{n\to\infty} f(n)/g(n) = 0;$$
3. We write $f(n) = O(g(n))$ if
$$\limsup_{n\to\infty} f(n)/g(n) < \infty.$$

Furthermore, for two functions $f(Z)$ and $g(Z)$ take values in $\mathbb{C}$ with $Z \in \mathbb{C}$. We define the asymptotic notation when Z is close to 0 similarly as above but taking limits with respect to $Z \to 0$.

3.3 Proof of Theorem 3.1.3

Lemma 3.3.1. *Let $f(X)$, $g(X)$ be automorphisms on $\mathbb{P}^1$ defined over $\mathbb{C}$ and $c(X)$ be a rational function defined over $\mathbb{C}$. If there are infinitely many $n \in \mathbb{N}$ such that*

$$c(X) = f^n(X) = g^n(X) \tag{3.3.1}$$

has solutions, then f and g share a common fixed point unless $f(X)$, $g(X)$ are simultaneously conjugated to $\alpha X + \beta$, $X/(\gamma X + \delta)$ respectively, with some $\alpha, \delta, \beta, \gamma \in \mathbb{C}^*$, such that at least one of the following conditions is satisfied:

1. *at least one of α, δ is a root of unity other than 1;*
2. *one of α/δ and $\alpha\delta$ is a root of unity.*

Remark 3.3.2. We record important notes in order:

(i) For the rest of the discussion, we will use the notation $c(X) = A(X)/B(X)$ for a pair of coprime $A(X), B(X) \in \mathbb{C}[X]$.

(ii) Recall that the field of Puiseux series with complex coefficients $\mathbb{C}$ is defined as the union

$$\mathcal{P}(Z) := \bigcup_{n \geqslant 1} \mathbb{C}(\!(Z^{1/n})\!) = \left\{ \sum_{m=m_0}^{\infty} c_m Z^{m/n} : c_m \in \mathbb{C}, m_0 \in \mathbb{Z}, n \geqslant 1 \right\}$$

of the fields of Laurent series in (indeterminate) $Z^{1/n}$. Then we define val_Z on $\mathcal{P}(Z)\backslash\{0\}$ by letting

$$\mathrm{val}_Z(F) = \min \left\{ \frac{m}{n} : c_m \neq 0 \right\}$$

for $F \in \mathcal{P}(Z)\backslash\{0\}$. For example, $\mathrm{val}_Z(Z^3 - 2Z^5) = 3$.

Proof. Suppose f and g don't share a fixed point, otherwise we have nothing to prove. After conjugation, we assume that

$$f(X) = \alpha X + \beta,$$

$$g(X) = \frac{X}{\gamma X + \delta}.$$

Notice that if γ or β is zero, then certainly $f(X)$ and $g(X)$ share a common fixed point. So we assume β ,γ are non-zero. We also have that α and δ are non-zero as otherwise it will result in a constant map. We break the discussion into two cases depending on α and δ.

Case I. Suppose α is not a root of unity and $\delta = 1$.

Lemma 3.3.3. *There are only finitely many $n \in \mathbb{N}$ such that*

$$c(X) = f^n(X) = g^n(X)$$

has solutions.

The same result for $\alpha = 1$ and δ is not a root of unity holds. We just need to further conjugate $c(X)$, $f(X)$ and $g(X)$ by $1/X$ to swap the role of α and δ. Then the same argument will give the result. This concludes these cases.

Case II. Now, suppose both α and δ are not roots of unity and neither α/δ nor $\alpha\delta$ is a root of unity. Then let

$$X_n^{\pm} = \left(-t_n \pm \sqrt{t_n^2 - 4\alpha^n\delta^n \frac{(1-\alpha^n)(1-\delta^n)}{(1-\alpha)(1-\delta)}\beta\gamma} \right) / \left(2\alpha^n \frac{1-\delta^n}{1-\delta}\gamma \right),$$

be the solution to the $f^n(X) = g^n(X)$, where

$$t_n = \frac{(1-\alpha^n)(1-\delta^n)}{(1-\alpha)(1-\delta)}\beta\gamma + \alpha^n\delta^n - 1.$$

We break the proof into steps:

- **Step I**: we will show that $c(X) = f^n(X) = g^n(X)$ has solution for infinitely many n will pose a restriction on α and δ: there exists a non-trivial polynomial in two variables $F(X, Y)$ such that $F(\alpha^n, \delta^n) = 0$ holds for infinitely many $n \in \mathbb{N}$.
- **Step II**: we use the S-unit Theorem to argue that this implies α and δ are multiplicatively dependent.
- **Step III**: we deduce that then $c(X) = f^n(X) = g^n(X)$ have solution for infinitely many n if $f(X)$ and $g(X)$ share a common fixed point.

Step I. Suppose there are infinitely many $n \in \mathbb{N}$ such that either

$$c(X_n^-) = f^n(X_n^-) \tag{3.3.2}$$

or

$$c(X_n^+) = f^n(X_n^+). \tag{3.3.3}$$

We claim that the following equation, which is obtained from above by replacing α^n with a variable Z and δ^n with a variable W,

$$c(X^{\pm}(Z,W)) = ZX^{\pm}(Z,W) + \frac{1-Z}{1-\alpha}\beta \tag{3.3.4}$$

doesn't hold constantly for every $Z, W \in \mathbb{C}$, where

$$X^{\pm}(Z,W) = \left(-t(Z,W) \pm \sqrt{t(Z,W)^2 - 4ZW\frac{(1-Z)(1-W)}{(1-\alpha)(1-\delta)}\beta\gamma}\right) / \left(2Z\frac{1-W}{1-\delta}\gamma\right), \tag{3.3.5}$$

$$t(Z,W) = \frac{(1-Z)(1-W)}{(1-\alpha)(1-\delta)}\beta\gamma + ZW - 1.$$

Therefore, we get the desired result of Step I as Equation (3.3.4) will give a non-trivial restriction on α^n and δ^n.

To prove the claim, taking a specialization at $W = 0$, we have

$$X^+(Z,W)\big|_{W=0} = \frac{-t(Z,0) + t(Z,0)}{2Z\gamma/(1-\delta)} = 0,$$

and Equation (3.3.4) implies

$$c(0) = \frac{1-Z}{1-\alpha}\beta$$

which only holds for finitely many values of Z. So,

$$c(X^+(Z,W)) = Z(X^+(Z,W)) + \frac{1-Z}{1-\alpha}\beta$$

doesn't hold constantly for every value of Z and W.

On the other hand, we have

$$X_0^-(Z) := X^-(Z,W)\big|_{W=0} = \frac{-2t(Z,0)}{2Z\gamma/(1-\delta)} = \frac{\beta}{1-\alpha} + \left(\frac{1-\delta}{\gamma} - \frac{\beta}{1-\alpha}\right)\frac{1}{Z}.$$

If

$$\frac{1-\delta}{\gamma} = \frac{\beta}{1-\alpha}$$

then $f(X)$ and $g(X)$ share a common fixed point and we are done. So, we assume

$$\frac{1-\delta}{\gamma} \neq \frac{\beta}{1-\alpha}.$$

Then Equation (3.3.4) implies

$$c(X_0^-(Z)) = A(X_0^-(Z))/B(X_0^-(Z)) = ZX_0^-(Z) + \frac{1-Z}{1-\alpha}\beta = \frac{1-\delta}{\gamma}. \tag{3.3.6}$$

This implies that there are only finitely many $Z \in \mathbb{C}$ satisfies the condition.

Thus

$$c(X^-(Z,W)) = f^n(X^-(Z,W))$$

doesn't hold constantly.

Step II. Now, either

$$c(X^+(Z,W)) = Z(X^+(Z,W)) + \frac{1-Z}{1-\alpha}\beta$$

or

$$c(X^-(Z,W)) = Z(X^-(Z,W)) + \frac{1-Z}{1-\alpha}\beta$$

is a non-trivial equation on variables Z and W with $(Z,W) = (\alpha^n, \delta^n)$ as solutions for infinitely many $n \in \mathbb{N}$. Thus, this gives a polynomial, by clearing the denominators and possible square roots on both sides of the equation,

$$\sum_{1 \leqslant i \leqslant k_1, 1 \leqslant j \leqslant k_2} c_{i,j}\alpha^{in}\delta^{jn} = 0 \tag{3.3.7}$$

for infinitely many $n \in \mathbb{N}$. Then, we want to apply the S-unit theorem (Theorem 3.2.5) to show that α and δ are multiplicatively dependent. Let $G \subset \mathbb{C}^*$ be the finitely generated subgroup generated by α and δ. Then Equation (3.3.7) can be rewritten as

$$\sum_{1 \leqslant i \leqslant m} c_i W_i = 0, \tag{3.3.8}$$

for some positive integer m, $c_i \in \mathbb{C}^*$ and $W_i \in G$ for each $i = 1, 2, \ldots, m$, which has solutions

$$(W_1, W_2, \ldots, W_m) = (\alpha^{s_1 n}\delta^{t_1 n}, \alpha^{s_2 n}\delta^{t_2 n}, \ldots, \alpha^{s_m n}\delta^{t_m n}) \in G^m,$$

for infinitely many $n \in \mathbb{N}$, where (s_i, t_i)'s are distinct pairs of integers. The S-unit theorem 3.2.5 says that there are only finitely many non-degenerate solutions to Equation (3.3.8). After an easy induction argument on the numbers of terms involved in Equation (3.3.7) we can get that there exists infinitely many $n \in \mathbb{N}$ such that

$$c_{i_1}\alpha^{s_{i_1} n}\delta^{t_{i_1} n} + c_{i_2}\alpha^{s_{i_2} n}\delta^{t_{i_2} n} = 0,$$

where i_1 and i_2 are distinct and in $\{1, 2, \ldots, m\}$. Since α and δ are not zero and not roots of unity, this is only possible if $s_{i_1} \neq s_{i_2}$ and $t_{i_1} \neq t_{i_2}$ and c_{i_1}/c_{i_2} is a root of unity. This implies that α and δ are multiplicative dependent. In another word, there exist non-zero integers k_1 and k_2 such that $\alpha^{k_1} = \delta^{k_2}$.

Step III. Let $k = k_1/k_2 \in \mathbb{Q}^*$ such that $\delta = \xi\alpha^k$, where ξ is a root of unity of order $k_2 \in \mathbb{N}$ and α^k is understood as a particular choice of k_2-th root of α^{k_1}. Then our assumption implies that there exists an $i \in \{1, 2, \ldots, k_2\}$ such that either the system of equations

$$A(X^+(Z, \xi^i Z^k)) = B(X^+(Z, \xi^i Z^k))\left(ZX^+(Z, \xi^i Z^k) + \frac{1-Z}{1-\alpha}\beta\right), \tag{3.3.9}$$

$$A(X^+(Z, \xi^i Z^k))\left(\frac{1-\xi^i Z^k}{1-\delta}\gamma X^+(Z, \xi^i Z^k) + \xi^i Z^k\right) = B(X^+(Z, \xi^i Z^k))X^+(Z, \xi^i Z^k) \tag{3.3.10}$$

or the system of equations

$$A(X^-(Z, \xi^i Z^k)) = B(X^-(Z, \xi^i Z^k))\left(ZX^-(Z, \xi^i Z^k) + \frac{1-Z}{1-\alpha}\beta\right), \tag{3.3.11}$$

$$A(X^-(Z, \xi^i Z^k))\left(\frac{1-\xi^i Z^k}{1-\delta}\gamma X^-(Z, \xi^i Z^k) + \xi^i Z^k\right) = B(X^-(Z, \xi^i Z^k))X^-(Z, \xi^i Z^k) \tag{3.3.12}$$

has infinitely many values of Z, given by $Z = \alpha^{nk_2+i}$ for infinitely many $n \in \mathbb{N}$, as solutions. This cannot happen unless the system of equations holds constantly for every possible value of Z as it only has one variable. From now on, we denote

$$X_i^{\pm}(Z) := X^{\pm}(Z, \xi^i Z^k).$$

Notice that

$$X_i^+(Z)X_i^-(Z) = \frac{\xi^i Z^k(1-Z)(1-\delta)\beta}{Z(1-\xi^i Z^k)(1-\alpha)\gamma}, \tag{3.3.13}$$

$$X_i^+(Z) + X_i^-(Z) = -2t(Z, \xi^i Z^k) \Big/ \left(2Z\frac{1-\xi^i Z^k}{1-\delta}\gamma\right), \tag{3.3.14}$$

for every $i \in \{1, 2, \ldots, k_2\}$.

We also notice that, from Equation (3.3.5), if $k > 0$, we have

$$\begin{aligned} X_i^-(Z) &= \left(-\left(\frac{\beta\gamma}{(1-\alpha)(1-\delta)} - 1\right) + o(1) - \sqrt{\left(\frac{\beta\gamma}{(1-\alpha)(1-\delta)} - 1\right)^2 + o(1)}\right) \\ &\times \left(2\frac{\gamma}{1-\delta}Z + o(Z)\right)^{-1} \\ &= \left(-2\left(\frac{\beta\gamma}{(1-\alpha)(1-\delta)} - 1\right) + o(1)\right) \times \left(2\frac{\gamma}{1-\delta}Z + o(Z)\right)^{-1} \\ &= \frac{1-\delta}{2\gamma}Z^{-1}\left(-2\frac{\beta\gamma}{(1-\alpha)(1-\delta)} + 2 + o(1)\right), \end{aligned} \tag{3.3.15}$$

when Z is around 0, which in particular implies that $\mathrm{val}_Z(X_i^-(Z)) = -1$ as

$$\frac{\beta}{1-\alpha} \neq \frac{1-\delta}{\gamma},$$

since otherwise it implies $f(X)$ and $g(X)$ share a common fixed point. Then from Equation (3.3.13), we have

$$\mathrm{val}(X_i^+(Z)) = k. \tag{3.3.16}$$

Subcase I. We first suppose $k > 0$ and the system of Equation (3.3.9) and (3.3.10) have infinitely many zeros. That is saying they hold constantly for every Z. Then Equation (3.3.9) implies that

$$A(X_i^+(Z))/B(X_i^+(Z)) = \frac{\beta}{1-\alpha} - \frac{\beta}{1-\alpha}Z + o(Z), \tag{3.3.17}$$

when Z is around 0. However, if $k > 1$, we will have

$$A(X_i^+(Z))/B(X_i^+(Z)) = A(0)/B(0) + O(Z^k),$$

which contradicts Equation (3.3.17). If $0 < k < 1$, we can conjugate $f(X)$, $g(X)$ and $c(X)$ by $1/X$ to swap the role of α and δ and we get $k > 1$ again. The same argument shows that this case is also not possible. Thus, we are left with the case $k = 1$. However, our assumption implies that $k \neq 1$ since otherwise α/δ is a root of unity. So we are done in this case.

Subcase II. Now we suppose $k < 0$ and the system of Equation (3.3.9) and (3.3.10) have infinitely many zeros. Then Equation (3.3.5) implies that

$$\begin{aligned} X_i^+(Z) &= \left(-\left(-\frac{\beta\gamma}{(1-\alpha)(1-\delta)}\xi^i Z^k \right) + o(Z^k) + \sqrt{\left(-\frac{\beta\gamma}{(1-\alpha)(1-\delta)} \right)^2 \xi^{2i} Z^{2k} + o(Z^{2k})} \right) \\ &\times \left(-2\frac{\gamma}{1-\delta}\xi^i Z^{k+1} + o(Z^{k+1}) \right)^{-1} \\ &= 2\left(\frac{\beta\gamma}{(1-\alpha)(1-\delta)}\xi^i Z^k + o(Z^k) \right) \times \left(-2\frac{\gamma}{1-\delta}\xi^i Z^{k+1} + o(Z^{k+1}) \right)^{-1} \\ &= -\frac{1-\delta}{2\gamma\xi^i} Z^{-1-k} \left(2\frac{\beta\gamma}{(1-\alpha)(1-\delta)}\xi^i Z^k + o(Z^k) \right), \end{aligned} \tag{3.3.18}$$

when Z is around 0 and in particular we have that

$$\mathrm{val}_Z(X_i^+(Z)) = -1.$$

Now, if $-1 < k < 0$, we have Equation (3.3.10) implies that

$$\begin{aligned} -\deg(A) - 1 + k &= \mathrm{val}_Z\left(A(X_i^+(Z)) \left(\frac{1-\xi^i Z^k}{1-\delta}\gamma X_i^+(Z) + \xi^i Z^k \right) \right) \\ &= \mathrm{val}_Z(B(X_i^+(Z))X_i^+(Z)) = -\deg(B) - 1, \end{aligned} \tag{3.3.19}$$

which is impossible as k is not an integer. This again gives a contradiction. Now, similarly by the conjugation argument we also know that $k < -1$ is also impossible. Moreover, by the assumption we have $k \neq -1$ since otherwise $\alpha\delta$ is a root of unity. Thus, we conclude the proof for $k < 0$ and Equation (3.3.9) and (3.3.10) has infinitely many zeros.

Subcase III. We now suppose $k > 0$ and the system of Equation (3.3.11) and (3.3.12) have infinitely many zeros. We calculate similarly as in Equation (3.3.15), when Z is close to 0, but we assume first that $0 < k < 1$ to get the following:

$$t(Z, \xi^i Z^k) = \mathcal{K}_1 - 1 - \mathcal{K}_1 \xi^i Z^k + o(Z^k)$$

$$t(Z,\xi^i Z^k)^2 = (\mathcal{K}_1-1)^2 - 2\mathcal{K}_1(\mathcal{K}_1-1)\xi^i Z^k + o(Z^k)$$

$$\begin{aligned}\sqrt{t(Z,\xi^i Z^k)^2 - 4\mathcal{K}_1\xi^i Z^{k+1}(1-Z)(1-\xi^i Z^k)} &= \sqrt{(\mathcal{K}_1-1)^2 - 2\mathcal{K}_1(\mathcal{K}_1-1)\xi^i Z^k + o(Z^k)} \\ &= \mathcal{K}_1 - 1 - \mathcal{K}_1\xi^i Z^k + o(Z^k),\end{aligned}$$

where $\mathcal{K}_1 = \beta\gamma/((1-\alpha)(1-\delta))$. Therefore, Equation (3.3.5) gives us

$$\begin{aligned}X_i^-(Z) &= (-2(\mathcal{K}_1-1) + 2\mathcal{K}_1\xi^i Z^k + o(Z^k))(2\mathcal{K}_2 Z)^{-1}(1+\xi^i Z^k + o(\xi^i Z^k)) \\ &= -\frac{\mathcal{K}_1-1}{\mathcal{K}_2}Z^{-1} + \frac{\mathcal{K}_1}{\mathcal{K}_2}\xi^i Z^{k-1} - \frac{\mathcal{K}_1-1}{\mathcal{K}_2}\xi^i Z^{k-1} + o(Z^{k-1}) \\ &= -\frac{\mathcal{K}_1-1}{\mathcal{K}_2}Z^{-1} + \frac{1}{\mathcal{K}_2}\xi^i Z^{k-1} + o(Z^{k-1}),\end{aligned} \tag{3.3.20}$$

where $\mathcal{K}_2 = \gamma/(1-\delta)$. Notice that this is just a further expansion of the expression (3.3.15) when $0 < k < 1$. Since $\mathrm{val}_Z(X_i^-(Z)) = -1$ and

$$ZX_i^-(Z) + \frac{\beta}{1-\alpha} = \mathcal{K}_2^{-1} + o(1),$$

Equation (3.3.11) implies that $\deg(A) = \deg(B) = m$ for some positive integer m. Then the left hand side of Equation (3.3.11) is

$$\begin{aligned}&A\left(-\frac{\mathcal{K}_1-1}{\mathcal{K}_2}Z^{-1} + \mathcal{K}_2^{-1}\xi^i Z^{k-1} + o(Z^{k-1})\right) \\ &= a_m\left(-\frac{\mathcal{K}_1-1}{\mathcal{K}_2}Z^{-1} + \mathcal{K}_2^{-1}\xi^i Z^{k-1} + o(Z^{k-1})\right)^m + O(Z^{-m+1}) \\ &= a_m\left(-\frac{\mathcal{K}_1-1}{\mathcal{K}_2}\right)^m Z^{-m} + a_m m\left(-\frac{\mathcal{K}_1-1}{\mathcal{K}_2}\right)^{m-1}\mathcal{K}_2^{-1}\xi^i Z^{-m+k} + o(Z^{-m+k}),\end{aligned} \tag{3.3.21}$$

where a_m is a non-zero constant. The right hand side of Equation (3.3.11) is

$$
\begin{aligned}
&B\left(-\frac{\mathcal{K}_1-1}{\mathcal{K}_2}Z^{-1}+\mathcal{K}_2^{-1}\xi^i Z^{k-1}+o(Z^{k-1})\right)\left(-\frac{\mathcal{K}_1-1}{\mathcal{K}_2}+\mathcal{K}_2^{-1}\xi^i Z^k+o(Z^k)+\mathcal{K}_3-\mathcal{K}_3 Z\right)\\
&=b_m\left(-\frac{\mathcal{K}_1-1}{\mathcal{K}_2}Z^{-1}+\mathcal{K}_2^{-1}\xi^i Z^{k-1}+o(Z^{k-1})\right)^m\left(\mathcal{K}_3-\frac{\mathcal{K}_1-1}{\mathcal{K}_2}+\mathcal{K}_2^{-1}\xi^i Z^k+o(Z^k)\right)+O(Z^{-m+1})\\
&=b_m\left(\mathcal{K}_3-\frac{\mathcal{K}_1-1}{\mathcal{K}_2}\right)\left(-\frac{\mathcal{K}_1-1}{\mathcal{K}_2}\right)^m Z^{-m}+b_m\left(-\frac{\mathcal{K}_1-1}{\mathcal{K}_2}\right)^m\mathcal{K}_2^{-1}\xi^i Z^{-m+k}\\
&+b_m m\left(-\frac{\mathcal{K}_1-1}{\mathcal{K}_2}\right)^{m-1}\mathcal{K}_2^{-1}\left(\mathcal{K}_3-\frac{\mathcal{K}_1-1}{\mathcal{K}_2}\right)\xi^i Z^{-m+k}+o(Z^{-m+k})\\
&=b_m\left(\mathcal{K}_3-\frac{\mathcal{K}_1-1}{\mathcal{K}_2}\right)\left(-\frac{\mathcal{K}_1-1}{\mathcal{K}_2}\right)^m Z^{-m}+b_m(m+1)\left(-\frac{\mathcal{K}_1-1}{\mathcal{K}_2}\right)^m\mathcal{K}_2^{-1}\xi^i Z^{-m+k}\\
&+b_m m\mathcal{K}_3\left(-\frac{\mathcal{K}_1-1}{\mathcal{K}_2}\right)^{m-1}\mathcal{K}_2^{-1}\xi^i Z^{-m+k}+o(Z^{-m+k}),
\end{aligned}
\tag{3.3.22}
$$

where $\mathcal{K}_3=\beta/(1-\alpha)$ and b_m is a non-zero constant. Notice by definition that $\mathcal{K}_1=\mathcal{K}_2\mathcal{K}_3$, we obtain

$$-\frac{\mathcal{K}_1-1}{\mathcal{K}_2}=\mathcal{K}_2^{-1}-\mathcal{K}_3\neq 0,$$

and

$$\mathcal{K}_2^{-1}\neq 0.$$

Therefore, looking at the coefficients of Z^{-m} terms of Equation (3.3.11) gives us

$$a_m=\mathcal{K}_2^{-1}b_m.$$

Then, examining the coefficients of Z^{-m+k} terms of Equation (3.3.11) gives us

$$
\begin{aligned}
&\mathcal{K}_2^{-1}b_m m\left(-\frac{\mathcal{K}_1-1}{\mathcal{K}_2}\right)^{m-1}\mathcal{K}_2^{-1}\\
&=b_m(m+1)\left(-\frac{\mathcal{K}_1-1}{\mathcal{K}_2}\right)^m\mathcal{K}_2^{-1}+b_m m\mathcal{K}_3\left(-\frac{\mathcal{K}_1-1}{\mathcal{K}_2}\right)^{m-1}\mathcal{K}_2^{-1},
\end{aligned}
\tag{3.3.23}
$$

which is equivalent to

$$m(\mathcal{K}_2^{-1}-\mathcal{K}_3)b_m\left(\mathcal{K}_2^{-1}-\mathcal{K}_3\right)^{m-1}\mathcal{K}_2^{-1}=(m+1)b_m(\mathcal{K}_2^{-1}-\mathcal{K}_3)^m\mathcal{K}_2^{-1}.\tag{3.3.24}$$

This is a contradiction as $m\neq m+1$.

Similarly, if $k > 1$, we conjugate $f(X)$, $g(X)$ and $c(X)$ by $1/X$ to swap the role of α and δ and we get $0 < k < 1$ again. The same argument shows that this case is also not possible. Thus, we left with the case $k = 1$. However, our assumption implies that $k \neq 1$ since otherwise α/δ is a root of unity. So we are done in this case.

Subcase IV. We now suppose $k < 0$ and the system of Equation (3.3.11) and (3.3.12) have infinitely many zeros. We first suppose $k < -1$. To ease the notation, we denote

$$\mathcal{K} := \frac{\beta\gamma}{(1-\alpha)(1-\delta)}.$$

Notice that when $|Z| \to \infty$, we have

$$t(Z, \xi^i Z^k) = -\mathcal{K}Z + (\mathcal{K}-1) + (1+\mathcal{K})\xi^i Z^{k+1} - \xi^i Z^k \mathcal{K}, \tag{3.3.25}$$

$$t^2(Z, \xi^i Z^k) = \mathcal{K}^2 Z^2 - 2\mathcal{K}(\mathcal{K}-1)Z - 2\mathcal{K}(\mathcal{K}+1)\xi^i Z^{k+2} + o(Z^{k+2}), \tag{3.3.26}$$

$$4\xi^i Z^{k+1}(1-Z)(1-\xi^i Z^k)\mathcal{K} = -4\mathcal{K}\xi^i Z^{k+2} + o(Z^{k+2}). \tag{3.3.27}$$

Thus, by Equation (3.3.5), we have

$$\begin{aligned} X_i^-(Z) = &\Big(-t(Z, \xi^i Z^k) - \\ &\sqrt{\mathcal{K}^2 Z^2 - 2\mathcal{K}(\mathcal{K}-1)Z - 2\mathcal{K}(\mathcal{K}-1)\xi^i Z^{k+2} + o(Z^{k+2})}\Big) \\ &\times \left(-2\xi^i Z^{k+1}\frac{\gamma}{1-\delta} + 2Z\frac{\gamma}{1-\delta}\right)^{-1}, \end{aligned} \tag{3.3.28}$$

and, breaking the square root, we have that this equals to

$$\begin{aligned} &= \left(-t(Z, \xi^i Z^k) - \mathcal{K}Z + (\mathcal{K}-1) + (\mathcal{K}-1)\xi^i Z^{k+1} + o(Z^{k+1})\right) \\ &\times \left(-2\xi^i Z^{k+1}\frac{\gamma}{1-\delta} + 2Z\frac{\gamma}{1-\delta}\right)^{-1} \end{aligned} \tag{3.3.29}$$

$$= \frac{-2\xi^i Z^{k+1} + o(Z^{k+1})}{2\gamma/(1-\delta)} Z^{-1} \sum_{j=0}^{\infty} \xi^{ij} Z^{kj} \tag{3.3.30}$$

$$= -\frac{1-\delta}{\gamma}\xi^i Z^k + o(Z^k). \tag{3.3.31}$$

Now, Equation (3.3.11) implies that

$$A\left(-\frac{1-\delta}{\gamma}\xi^i Z^k + o(Z^k)\right) = B\left(-\frac{1-\delta}{\gamma}\xi^i Z^k + o(Z^k)\right)\left(-\frac{\beta}{1-\alpha}Z + o(Z)\right), \tag{3.3.32}$$

when $|Z| \to \infty$. Since $A(X)$ is a polynomial, the left hand side is bounded as $|Z|$ increasing. Therefore, to make the right hand side of Equation (3.3.32) bounded, we need

$$B(X) = X^m B_1(X)$$

for some positive integer m and polynomial $B_1(X)$ with non-trivial constant term b_m. Thus, we have

$$B\left(-\frac{1-\delta}{\gamma}\xi^i Z^k + o(Z^k)\right) = b_m\left(-\frac{1-\delta}{\gamma}\right)^m \xi^{im} Z^{mk} + o(Z^{mk}). \tag{3.3.33}$$

In particular, $B(0) = 0$.

However, since $k < -1$, we have $mk + 1 < 0$. Therefore, the right hand side of Equation (3.3.32) also goes to 0 when $|Z|$ approaches infinity. This implies that $A(0) = 0$, which contradicts that $A(X)$ and $B(X)$ don't share a common zero. Notice that when $-1 < k < 0$, we can swap the role of α and δ by conjugating $f(X)$, $g(X)$ and $c(X)$ by the map $1/X$ just like what we did above. Then we are back to $k < -1$ case. Also, by the assumption we have $k \neq -1$ as otherwise it will imply $\alpha\delta$ is a root of unity.

□

Proof of Lemma 3.3.3. Suppose there are infinitely many $n \in \mathbb{N}$ such that

$$c(X) = f^n(X) = g^n(X)$$

has solutions. We denote

$$X_n^{\pm} = \frac{-t_n \pm \sqrt{t_n^2 - 4\alpha^n n\gamma(1-\alpha^n)\beta/(1-\alpha)}}{2n\alpha^n\gamma},$$

$$t_n = \alpha^n + n\gamma\frac{1-\alpha^n}{1-\alpha}\beta - 1,$$

where $X_n^{\pm}$ satisfies $f^n(X_n^{\pm}) = g^n(X_n^{\pm})$.

We first suppose $|\alpha| > 1$ and there are infinitely many $n \in \mathbb{N}$ such that

$$c(X_n^+) = f^n(X_n^+) \tag{3.3.34}$$

holds. We denote $c(X) = A(X)/B(X)$ where $A(X)$ and $B(X)$ are two coprime polynomials. Since

$$\begin{aligned} t_n^2 - 4\alpha^n n\gamma\frac{1-\alpha^n}{1-\alpha}\beta = & \left(\frac{n\gamma\alpha^n\beta}{1-\alpha}\right)^2 + \frac{2n\gamma\beta\alpha^{2n}}{1-\alpha} + \alpha^{2n} - \frac{2n^2\gamma^2\alpha^n\beta^2}{(1-\alpha)^2} \\ - & \frac{2\gamma\beta}{1-\alpha}n\alpha^n - 2\alpha^n + \frac{\gamma^2\beta^2}{(1-\alpha)^2}n^2 - \frac{2n\gamma\beta}{1-\alpha} + o(n), \end{aligned} \tag{3.3.35}$$

we have

$$X_n^{\pm} = \left(\frac{\gamma\beta}{1-\alpha} n\alpha^n - \alpha^n - \frac{\gamma\beta}{1-\alpha} n + 1 \pm \left(-\frac{\gamma\beta}{1-\alpha} n\alpha^n - \alpha^n + \frac{n\gamma\beta}{1-\alpha} + 1 - 2\alpha^{-n} + o(\alpha^{-n}) \right) \right) \times (2n\alpha^n\gamma)^{-1}. \tag{3.3.36}$$

Therefore,

$$\lim_{n\to\infty} |X_n^+| = 0,$$
$$\lim_{n\to\infty} |\alpha^n X_n^+| = \infty.$$

Since we have

$$X_n^+ \sim 1/n$$

when n goes large, we have similarly

$$A(X_n^+)/B(X_n^+) \sim n^l,$$
$$\alpha^n X_n^+ + \frac{1-\alpha^n}{1-\alpha}\beta \sim \alpha^n$$

when n is large with some integer l, which contradicts Equation (3.3.34).

Suppose $|\alpha| > 1$ and there are infinitely many $n \in \mathbb{N}$ such that

$$c(X_n^-) = f^n(X_n^-) \tag{3.3.37}$$

holds. Notice that

$$X_n^- = \frac{\beta}{1-\alpha} - \frac{\beta}{1-\alpha}\alpha^{-n} + \gamma^{-1}n^{-1}\alpha^{-2n} + o(n^{-1}\alpha^{-2n}),$$

when n becomes large. Therefore,

$$f^n(X_n^-) \sim n^{-1}\alpha^{-n}$$
$$A(X_n^-)/B(X_n^-) \sim \alpha^{-nm}$$

for some integer m when n becomes large. This contradicts the assumption that Equation (3.3.37) has solutions for infinitely many $n \in \mathbb{N}$.

Now suppose $|\alpha| < 1$ and there are infinitely many $n \in \mathbb{N}$ such that Equation (3.3.37) holds. Now, similarly, since

$$t_n^2 - 4\alpha^n n\gamma\beta\frac{1-\alpha^n}{1-\alpha} = \frac{\gamma^2\beta^2}{(1-\alpha)^2}n^2 - 2\frac{\beta\gamma}{1-\alpha}n + 1 - 2\frac{\gamma^2\beta^2}{(1-\alpha)^2}n^2\alpha^n - 2\alpha^n + o(\alpha^n) \tag{3.3.38}$$

when n is large, we have

$$X_n^{\pm} = \left(\frac{\gamma\beta}{1-\alpha} n\alpha^n - \alpha^n - \frac{\beta\gamma}{1-\alpha} n + 1 \pm \left(\frac{\beta\gamma}{1-\alpha} n - 1 - \frac{\gamma\beta}{1-\alpha} n\alpha^n - \alpha^n + o(\alpha^n) \right) \right) \times (2n\alpha^n\gamma)^{-1}$$

So,

$$X_n^- = -\frac{\beta}{1-\alpha}\alpha^{-n} + \frac{1}{\gamma n \alpha^n} + o(n^{-1}\alpha^{-n}),$$

and we have

$$A(X_n^-)/B(X_n^-) \sim \alpha^{ln}$$

for some integer l when n is large. However,

$$\alpha^n X_n^- + \frac{1-\alpha^n}{1-\alpha}\beta \sim \frac{1}{n}$$

contradicting that Equation (3.3.37) has a solution for infinitely many $n \in \mathbb{N}$.

Suppose $|\alpha| < 1$ and there are infinitely many $n \in \mathbb{N}$ such that Equation (3.3.34) holds. Then $X_n^+ \sim \frac{1}{n}$ when n is large. Thus,

$$A(X_n^+)/B(X_n^+) \sim n^l$$

for some integer l and

$$\alpha^n X_n^+ + \frac{1-\alpha^n}{1-\alpha}\beta = \frac{\beta}{1-\alpha} - \frac{\beta}{1-\alpha}\alpha^n + o(\alpha^n)$$

when n is large. If $l \neq 0$, then these contradict the assumption that the Equation (3.3.34) has solutions for infinitely many $n \in \mathbb{N}$. If $l = 0$, then

$$A(X_n^+)/B(X_n^+) = A(0)/B(0) + O(1/n^m)$$

for some positive integer m. Still, these contradict the assumption that the Equation (3.3.34) has solution for infinitely many $n \in \mathbb{N}$ by comparing the rates they approach constants when n is large.

Now, suppose $|\alpha| = 1$ but not a root of unity and there are infinitely many $n \in \mathbb{N}$ such that Equation (3.3.37) holds. After eliminating denominators, combining terms and taking squares, we find that this is equivalent to the existence of a polynomial $F(Z, W)$

in two variables such that $F(n, \alpha^n) = 0$ for infinitely many $n \in \mathbb{N}$. By [11, Proposition 2.5.1.4], we have $\{a_n = F(n, \alpha^n)\}_{n\in\mathbb{N}}$ satisfies a linear recurrence. Then, by [11, Theorem 2.5.4.1], the set $\{n \in \mathbb{N} : a_n = 0\}$ is a finite union of arithmetic progressions and, since we assumed this set is infinite, we have that there exists s_1 and $s_2 \in \mathbb{N}$ such that $a_n = 0$ for every $n \in \{s_1m + s_2 : m \in \mathbb{N}\}$. This is equivalent to saying that Equation (3.3.37) holds for any $n \in \{s_1m + s_2 : m \in \mathbb{N}\}$. Then, we consider n within an infinite subsequence I of $\{s_1m + s_2 : m \in \mathbb{N}\}$ such that $|1 - \alpha^n|$ is bounded from below independently from n and not converging.

Notice that we can always find such a subsequence I. Indeed, since α^n, $n \in R = \{s_1m + s_2 : m \in \mathbb{N}\}$, is dense on the unit circle as α is not a root of unity, we can take a small $\epsilon > 0$ independent of n and then there are infinitely many $n \in R$ such that $\alpha^n \notin B(1, \epsilon)$. Denote the subset of R satisfies this condition as I'. Since, R is dense in the unit circle S^1, we have I' is also dense in $S^1\backslash B(1, \epsilon)$. Thus, we can take a subsequence $I \subseteq I'$, such that α^n won't converge to any point on the unit circle when n ranges in I. For example, we can achieve this by choosing two different points p_1 and p_2 on $S^1\backslash B(1, \epsilon)$ and for each $r \in \mathbb{N}$, we pick a $n_{1,r}$ and $n_{2,r} \in I'$ such that

$$|\alpha^{n_{1,r}} - p_1| < 1/r,$$

$$|\alpha^{n_{2,r}} - p_2| < 1/r,$$

and let $I = \bigcup_{r\in\mathbb{N}}\{n_{1,r}, n_{2,r}\} \cap \{n \in \mathbb{N} : \alpha^n \in S^1\backslash B(1, \epsilon)\}$.

Now, we expand $X_n^{\pm}$ when n is sufficiently large as follows:

$$X_n^{\pm} = \frac{-t_n \pm \sqrt{t_n^2 - 4\alpha^n n\gamma(1 - \alpha^n)\beta/(1 - \alpha)}}{2n\alpha^n\gamma}, \tag{3.3.39}$$

$$= \frac{-t_n \pm \sqrt{h_n + O(1)}}{2n\alpha^n\gamma},$$

where

$$h_n = \frac{n^2\gamma^2(1 - \alpha^n)^2\beta^2}{(1 - \alpha)^2} + 2\frac{(\alpha^n - 1)n\gamma(1 - \alpha^n)\beta}{(1 - \alpha)} - 4\frac{\alpha^n n\gamma(1 - \alpha^n)\beta}{(1 - \alpha)}.$$

Breaking the square root, we have

$$X_n^{\pm} = \frac{-t_n \pm (n\gamma(1 - \alpha^n)\beta/(1 - \alpha) - \alpha^n - 1 + o(1))}{2n\alpha^n\gamma}. \tag{3.3.40}$$

Thus, we have

$$X_n^+ = -\frac{1}{n\gamma} + o\left(\frac{1}{n}\right), \tag{3.3.41}$$

when $n \in I$ becomes large. Then Equation (3.3.34) implies

$$c\left(-\frac{1}{n\gamma} + o\left(\frac{1}{n}\right)\right) = \alpha^n\left(-\frac{1}{n\gamma} + o\left(\frac{1}{n}\right)\right) + \frac{1-\alpha^n}{1-\alpha}\beta. \tag{3.3.42}$$

However,

$$c\left(-\frac{1}{n\gamma} + o\left(\frac{1}{n}\right)\right) \sim (1/n)^l \tag{3.3.43}$$

for some integer l and

$$\alpha^n\left(-\frac{1}{n\gamma} + o\left(\frac{1}{n}\right)\right) + \frac{1-\alpha^n}{1-\alpha}\beta = (1-\alpha^n)\frac{\beta}{1-\alpha} + o(1) \tag{3.3.44}$$

when n is large. This gives a contradiction if $l \neq 0$.

Notice that if $l = 0$, then we still have

$$\lim_{n\to\infty} X_n^+ = 0,$$

by Equation (3.3.41). Then Equation (3.3.34) implies that

$$\lim_{n\to\infty} A(X_n^+) = \lim_{n\to\infty} B(X_n^+)\left(\alpha^n\left(-\frac{1}{n\gamma} + o\left(\frac{1}{n}\right)\right) + \frac{1-\alpha^n}{1-\alpha}\beta\right). \tag{3.3.45}$$

Now, if $\lim_{n\to\infty} B(X_n^+) = 0$, then this can only hold if $\lim_{n\to\infty} A(X_n^+) = 0$ which is impossible as $A(X)$ and $B(X)$ don't share common zeros. On the other hand, if $\lim_{n\to\infty} B(X_n^+) = b$, where b is a non-zero constant, then

$$\lim_{n\to\infty} A(X_n^+) = b \lim_{n\to\infty} \frac{1-\alpha^n}{1-\alpha}\beta, \tag{3.3.46}$$

which is not converging. This is a contradiction as $\lim_{n\to\infty} A(X_n^+) = A(0)$.

Now, suppose $|\alpha| = 1$ but not a root of unity and there are infinitely many $n \in \mathbb{N}$ such that Equation (3.3.34) holds. Similarly as above, there exists $s_3, s_4 \in \mathbb{N}$ such that Equation (3.3.34) holds for any $n \in \{s_3 m + s_4 | m \in \mathbb{N}\}$. We consider n within an infinite subsequence of $I \subseteq \{s_3 m + s_4 | m \in \mathbb{N}\}$ such that there exists a small $\epsilon > 0$ so that for all $n \in I$, we have $|1 - \alpha^n| > \epsilon$ and $\{\alpha^n : n \in I\}$ is dense in $S^1 \backslash B(1, \epsilon)$. This can be achieved in the same

way as in the previous case since α^n is dense on S^1, when n ranges in $\{s_3m + s_4 : m \in \mathbb{N}\}$. Then, again by Equation (3.3.40), we have

$$X_n^- = -\frac{\beta}{1-\alpha}\frac{1-\alpha^n}{\alpha^n} + \frac{1}{n\alpha^n\gamma} + o\left(\frac{1}{n}\right). \tag{3.3.47}$$

Therefore, Equation (3.3.37) implies that

$$c(X_n^-) = \frac{1}{n\gamma} + o\left(\frac{1}{n}\right). \tag{3.3.48}$$

However, for a non-zero rational function $c(X)$,

$$c\left(-\frac{\beta}{1-\alpha}\frac{1-\alpha^n}{\alpha^n} + \frac{1}{n\alpha^n\gamma} + o\left(\frac{1}{n}\right)\right) \tag{3.3.49}$$

cannot converge to 0 when n becomes large since otherwise, as $1/\alpha^n$ is dense on $S^1\backslash B(1,\epsilon)$, we would have

$$c\left(-\frac{\beta}{1-\alpha}u + \frac{\beta}{1-\alpha}\right) = 0,$$

for all $u \in S^1\backslash B(1,\epsilon)$ which implies that c is constantly zero. This gives a contradiction.

□

Now we can prove Theorem 3.1.3 and we restate it here for the convenience of readers:

Theorem 3.3.4. *Let $f(X)$, $g(X)$ be automorphisms on $\mathbb{P}^1$ defined over $\mathbb{C}$ and $c(X)$ be a rational function defined over $\mathbb{C}$. If the semigroup generated by $f(X)$ and $g(X)$ under compositions is free, then there are only finitely many $\lambda \in \mathbb{C}$ such that*

$$f^n(\lambda) = g^n(\lambda) = c(\lambda) \tag{3.3.50}$$

for some positive integer n unless $f(X)$, $g(X)$ are simultaneously conjugated by an automorphism on $\mathbb{P}^1(\mathbb{C})$ to one of the following:

1. *$\alpha X + \beta$, $X/(\gamma X + \delta)$, with some $\alpha, \delta, \gamma, \beta \in \mathbb{C}^*$ such that one of α/δ and $\alpha\delta$ is a root of unity;*
2. *$\alpha X + \beta$, $\delta X + \gamma$, with some $\alpha, \delta \in \mathbb{C}^*$ such that α and δ are not roots of unity, γ and β are not both 0, and either α/δ is a root of unity other than 1 or one of α^2/δ and δ^2/α is a root of unity.*

Before we start the proof, we gather here four counter-examples in Remark (3.3.5), (3.3.6), (3.3.7), (3.3.8) and (3.3.9) corresponding to the two exceptional cases mentioned in the statement of Theorem 3.3.4. Therefore, Theorem 3.3.4 is the best we can expect.

Remark 3.3.5. Suppose $f(X) = X + \beta$, $g(X) = X/(\gamma X + 1)$ and $c(X) = -\beta/(\gamma X)$, with some $\beta, \gamma \in \mathbb{C}^*$, we have that there are infinitely many $n \in \mathbb{N}$ such that Equation (3.3.50) has solutions. One of the solutions is given by

$$x_n = \left(-n\beta\gamma + \sqrt{n^2\beta^2\gamma^2 - 4\beta\gamma}\right)/(2\gamma).$$

However, $f(X)$ and $g(X)$ may still generate a free semigroup. For example, when $f(X) = X + 2$ and $g(X) = X/(2X + 1)$, we denote

$$V_1 = \{x \in \mathbb{R}^+ : x > 1\},$$

$$V_2 = \{x \in \mathbb{R}^+ : x < 1\}.$$

Then we have

$$f(V_1) \subseteq V_1,$$
$$f(V_2) \subseteq V_1,$$
$$g(V_2) \subseteq V_2,$$
$$g(V_1) \subseteq V_2.$$

Thus, the Ping-Pong Lemma 3.2.7 implies $\langle f, g\rangle$ is a free semigroup under composition.

Remark 3.3.6. Suppose $f(X) = \alpha X + \beta$ and $g(X) = X/(\gamma X + \alpha)$, where γ and β are non-zero and $\beta/(1-\alpha) \neq (1-\alpha)/\gamma$. Denote

$$K_1 = \frac{\beta}{1-\alpha} + \frac{1-\alpha}{\gamma},$$
$$K_2 = \frac{\beta}{1-\alpha} - \frac{1-\alpha}{\gamma},$$
$$K_3 = \frac{\beta}{\gamma}.$$

Let

$$c(X) = \frac{K_2X^2}{-2K_3 - 2X^2 + K_1X} + \frac{\beta}{1-\alpha} - \frac{\beta}{1-\alpha}\frac{K_2X}{-2K_3 - 2X^2 + K_1X}.$$

One can directly verify that

$$c(X) = f^n(X) = g^n(X)$$

has

$$X_n = \frac{-K_1 + K_2\alpha^{-n} - \sqrt{(-K_1 + K_2\alpha^{-n})^2 - 4}}{2}$$

as a solution for each $n \in \mathbb{N}$. Furthermore, suppose $\gamma = 1$, $\beta = 1$ and $\alpha = 2$ for example. Then, similarly as in Remark (3.3.5), one can use the Ping-Pong Lemma 3.2.7 to show that $\langle f(X), g(X)\rangle$ is a free semigroup.

Remark 3.3.7. Suppose $f(X) = \alpha X + \beta$ and $g(X) = \delta X + \gamma$, where α and δ are not roots of unity but $\xi = \delta/\alpha$ is a root of unity of order $l > 1$. For each $i \in \{1, 2, \ldots, l-1\}$, denote

$$K_{1,i} = \frac{1}{1-\xi^i}\left(\frac{\gamma}{1-\delta} - \frac{\beta}{1-\alpha}\right),$$

$$K_{2,i} = \frac{1}{1-\xi^i}\left(\frac{\xi^i\gamma}{1-\delta} - \frac{\beta}{1-\alpha}\right).$$

Let

$$c_i(X) = K_{1,i} + \frac{\beta}{1-\alpha} - \frac{K_{1,i}}{X + K_{2,i}}\left(K_{2,i} + \frac{\beta}{1-\alpha}\right).$$

We have that

$$X_{n,i} = K_{1,i}\alpha^{-nl-i} - K_{2,i}$$

is a solution to

$$f^{nl+i}(X) = g^{nl+i}(X) = c_i(X)$$

for every $n \in \mathbb{N}$ and $i \in \{1, 2, \ldots, l-1\}$. For an explicit example, if we take $f(X) = 2X + 1$ and $g(X) = -2X$ then $\langle f, g\rangle$ is a free semigroup by [55, Theorem 2] as $f \circ g \neq g \circ f$. So this provides a counter-example such that the finiteness of solutions to $f^n(X) = g^n(X) = c(X)$ doesn't hold even when f and g generate a free semigroup.

Remark 3.3.8. Suppose $f(X) = \alpha X + \beta$ and $g(X) = X/(\gamma X + \xi\alpha^{-1})$ where ξ is a root of unity of order m, $\alpha \in \mathbb{C}^*$ is not a root of unity, γ and β are non-zero in $\mathbb{C}$. We denote

$$K_1 := \frac{\beta}{1-\alpha},$$

$$K_2 := \frac{\gamma}{1-\xi\alpha^{-1}}.$$

We also define

$$c(X) = \frac{K_1K_2X - K_1}{K_1K_2 - K_2X} + K_1\left(1 - \frac{-K_1 + K_1K_2X}{K_2X(K_1 - X)}\right).$$

Then one can check directly that

$$\begin{aligned} X_n = &\big(-K_1K_2(1-\alpha^n)(1-\alpha^{-n}) \\ &-\sqrt{K_1^2K_2^2(1-\alpha^n)^2(1-\alpha^{-n})^2 - 4K_1K_2(1-\alpha^n)(1-\alpha^{-n})}\Big) \\ \times &\left(2\alpha^n\frac{1-\alpha^{-n}}{1-\xi\alpha^{-1}}\gamma\right)^{-1} \end{aligned} \tag{3.3.51}$$

when $n \in \{km : k \in \mathbb{N}\}$ is an infinite sequence satisfying

$$f^n(X_n) = g^n(X_n) = c(X_n). \tag{3.3.52}$$

Since the direct computation might be very complicated, we will demonstrate here on how to construct such a $c(X)$ to make this hold. Let $\delta = \xi\alpha^{-1}$. Then the sequence of equations:

$$f^n(X_n) = g^n(X_n), \tag{3.3.53}$$

when $n \in \{km : k \in \mathbb{N}\}$, will give a sequence of quadratic equations with variables X_n's:

$$\alpha^n(1-\alpha^{-n})K_2X_n^2 + K_1K_2(1-\alpha^n)(1-\alpha^{-n})X_n + K_1(1-\alpha^n)\alpha^{-n} = 0$$

and we will denote the two solutions of these quadratic equations as X_n^- and X_n^+. Then we have

$$X_n^-X_n^+ = \frac{\alpha^{-n}(1-\alpha^n)}{\alpha^n(1-\alpha^{-n})}\frac{K_1}{K_2} = -\alpha^{-n}\frac{K_1}{K_2}, \tag{3.3.54}$$

$$X_n^- + X_n^+ = \frac{-K_1K_2(1-\alpha^n)(1-\alpha^{-n})}{\alpha^nK_2(1-\alpha^{-n})} = K_1(1-\alpha^{-n}), \tag{3.3.55}$$

where $n \in \{km : k \in \mathbb{N}\}$. Combining these two equations, we get

$$K_1(1-\alpha^{-n})X_n^- - (X_n^-)^2 = -\alpha^{-n}\frac{K_1}{K_2}, \tag{3.3.56}$$

which implies

$$\alpha^n = \frac{-K_1 + K_1K_2X_n^-}{X_n^-K_2(K_1 - X_n^-)}. \tag{3.3.57}$$

Let $X_n = X_n^-$. Then using that

$$c(X_n) = f^n(X_n) = \alpha^nX_n + \frac{1-\alpha^n}{1-\alpha}\beta$$

and plugging in Equation (3.3.57) will give us the desired $c(X_n)$.

Furthermore, notice that when $\alpha = 2$, $\beta = \gamma = \xi = 1$, we can use the Ping-Pong lemma, Lemma 3.2.7, similarly as in Remark 3.3.5 by choosing

$$V_1 = \{x \in \mathbb{R}^+ : x > 1\},$$

$$V_2 = \{x \in \mathbb{R}^+ : x < 1\},$$

to argue that $\langle f(X), g(X)\rangle$ is a free semigroup under the compositions.

Remark 3.3.9. Let α be any non-zero complex number that is not a root of unity. Let μ be a root of unity of order $m \in \mathbb{N}$. Let $\delta = \mu\alpha^2$ and $\beta = \alpha - 1$. Then we have $X_{mn} = \alpha^{-mn}$ satisfies

$$f^{mn}(X_{mn}) = g^{mn}(X_{mn}) = 1/X_{mn} \tag{3.3.58}$$

for all $n \in \mathbb{N}$, where $f(x) = \alpha x + \beta$ and $g(x) = \delta x$.

Furthermore, when $\alpha = 2$, $\delta = 4$ and $\beta = 1$, let V_1 denote the set of odd natural numbers and V_2 denote the set of even natural numbers. We have

$$f(V_1 \cup V_2) \subseteq V_1,$$

$$g(V_1 \cup V_2) \subseteq V_2.$$

Thus, Lemma 3.2.7 tells us that $\langle f, g\rangle$ is a free semigroup under composition.

Proof of Theorem 3.3.4. After conjugation, we assume that

$$f(X) = \alpha X + \beta,$$

$$g(X) = \frac{X}{\gamma X + \delta}.$$

We always assume that α and δ are non-zero as otherwise it will result in a constant map.

Suppose that at least one of α and δ is a root of unity other than 1. Assume without loss of generality it is δ. Then there exists a positive integer m such that $\delta^m = 1$ and

$$g^m(X) = \frac{X}{\delta^m} = X.$$

Therefore, $\langle g^m(X), f(X)\rangle$ is not a free semigroup under composition and there is nothing to show.

Now, we assume that neither α nor δ is a root of unity other than 1 and α/δ, $\alpha\delta$ are not roots of unity. By Lemma 3.3.1, we can assume that f and g share at least one common fixed point.

If f and g share two fixed points, then $\beta = \gamma = 0$ and they are both scaling maps and certainly not generating a free semigroup under composition.

Suppose f and g share only one common fixed point. Now we first assume that $c(X)$, $f(X)$ and $g(X)$ are all defined over $\overline{\mathbb{Q}}^*$. Then after conjugating $c(X)$, $f(X)$ and $g(X)$ by a common automorphism, we assume that $f(X) = \alpha X + \beta$ and $g(X) = \delta X + \gamma$, where $\alpha, \delta \in \overline{\mathbb{Q}}^*$, $\beta, \gamma \in \overline{\mathbb{Q}}$ and β, γ are not both 0. Now our discussion is divided into several cases:

Case I. Suppose at least one of δ and α is a root of unity other than 1. Without loss of generality we assume it is δ. Then there exists m such that $g^m(X) = X$ and we have $g^m \circ f = f \circ g^m$. Thus, $\langle f, g \rangle$ is not a free semigroup. Same argument works if α is a root of unity other than 1 and δ is not a root of unity.

Case II. Suppose $\delta = 1$ and α is not a root of unity.

Lemma 3.3.10. *There are only finitely many $n \in \mathbb{N}$ such that $c(X) = f^n(X) = g^n(X)$ has solution.*

This concludes this case.

Case III. Suppose both α and δ are not roots of unity and α/δ, α^2/δ and δ^2/α are not roots of unity. Without loss of generality, we assume that $\beta \neq 0$ since γ and β are not both 0 and we can always relabel f and g if necessary. We have

$$X_n = \left(\frac{1-\delta^n}{1-\delta}\gamma - \frac{1-\alpha^n}{1-\alpha}\beta \right) / (\alpha^n - \delta^n)$$

is the solution to

$$f^n(X) = g^n(X).$$

If there are infinitely many $n \in \mathbb{N}$ such that the Equation (3.3.50) has solutions, then

$$c(X_n) = f^n(X_n)$$

for infinitely many $n \in \mathbb{N}$. It means that

$$A(X_n) = \alpha^n B(X_n) X_n + \frac{1-\alpha^n}{1-\alpha}\beta B(X_n)$$

holds for infinitely many $n \in \mathbb{N}$. This is equivalent to say

$$F(Z,W) = A(X(Z,W)) - ZB(X(Z,W))X(Z,W) - \frac{1-Z}{1-\alpha}\beta B(X(Z,W)) = 0 \quad (3.3.59)$$

holds for infinitely many $\{(\alpha^n, \delta^n) : n \in \mathbb{N}\}$, where

$$X(Z,W) = \left(\frac{1-W}{1-\delta}\gamma - \frac{1-Z}{1-\alpha}\beta\right) \Big/ (Z-W).$$

We first claim that the rational function $F(Z,W)$ is not constantly zero. To prove this, we suppose on the contrary that Equation (3.3.59) holds constantly for any choice of values of Z and W. We denote $m_1 = \deg(A)$ and $m_2 = \deg(B)$.

Then choose a specialization $Z = W + 1$, we have

$$X(Z, Z-1) = \left(\frac{\beta}{1-\alpha} - \frac{\gamma}{1-\delta}\right) Z + \frac{2\gamma}{1-\delta} - \frac{\beta}{1-\alpha}$$

in this case. Then Equation (3.3.59) holds constantly implies that $m_1 = m_2 + 2$ unless

$$\frac{\beta}{1-\alpha} = \frac{\gamma}{1-\delta}. \quad (3.3.60)$$

While Equation (3.3.60) implies $f(X)$ and $g(X)$ share two fixed points and this case has already been handled. So, we assume Equation (3.3.60) doesn't hold and thus $m_1 = m_2 + 2$.

On the other hand, we can choose another specialization $Z = 0$. Then

$$X(0,W) = \frac{\gamma}{1-\delta} + \left(\frac{\beta}{1-\alpha} - \frac{\gamma}{1-\delta}\right) W^{-1},$$

and Equation (3.3.59) implies that

$$A(X(0,W)) = \frac{\beta}{1-\alpha} B(X(0,W)).$$

This holds constantly for all W will require that $m_1 = m_2$, which gives a contradiction. Therefore, Equation (3.3.59) doesn't hold constantly for every values of Z and W.

We then write

$$F(Z,W) = \sum_{0 \leqslant i \leqslant k_1, 0 \leqslant j \leqslant k_2} c_{i,j} Z^i W^j,$$

for some positive integers k_1 and k_2 after clearing the denominators. Then we use the same argument as in Lemma 3.3.1 by applying S-unit Theorem to conclude that α and δ are multiplicatively dependent. Thus, there exists $k \in \mathbb{Q}^*$ such that $\xi\alpha^k = \delta$, for some root of unity ξ of order $k_3 \in \mathbb{N}$. Note that, we can further relabel f and g if necessary (we will not assume $\beta \neq 0$ from now on which is an assumption obtained by a specific choice of labeling) so that $k \geqslant 1$ or $k \leqslant -1$.

Therefore, we have

$$X_n = \left(\frac{1-\xi^n\alpha^{nk}}{1-\xi\alpha^k}\gamma - \frac{1-\alpha^n}{1-\alpha}\beta\right) \Big/ (\alpha^n - \xi^n\alpha^{nk}).$$

Then following the same idea as how we argue in the proof of Lemma 3.3.1 (Case **II**, Step **III**), it is enough to show the claim that

$$A(X_i(Z)) = \left(ZX_i(Z) + \frac{1-Z}{1-\alpha}\beta\right) B(X_i(Z)) \tag{3.3.61}$$

doesn't hold constantly for any $i \in \{1, 2, \ldots, k_3\}$ unless f and g share two common fixed points, where

$$X_i(Z) = \left(\frac{1-\xi^i Z^k}{1-\xi\alpha^k}\gamma - \frac{1-Z}{1-\alpha}\beta\right) \Big/ (Z - \xi^i Z^k).$$

This is because, similarly as in Lemma 3.3.1, Equation (3.3.61) is an equation on one variable and having infinitely many roots implies it is constantly zero.

To prove the claim, we can assume that $k \neq 0$ since otherwise δ is a root of unity and we are back to the case that we have discussed. We first suppose $k > 0$ and suppose the contrary that there exists a

$$i \in \{1, 2, \ldots, k_3\}$$

such that Equation (3.3.61) holds constantly. Notice that in the beginning of Case **III** we assumed that α/δ, α^2/δ and δ^2/α are not roots of unity, which is equivalent to $k \neq 1, 2$ as we assumed $k \geqslant 1$. Also, if f and g only share one common fixed point, then

$$\frac{\gamma}{1-\delta} \neq \frac{\beta}{1-\alpha}$$

and

$$\begin{aligned} \mathrm{val}_Z(X_i(Z)) &= -\min\{1, k\} = -1, \\ \mathrm{val}_Z(A(X_i(Z))) &= -m_1 \min\{1, k\} = -m_1, \end{aligned}$$

$$\mathrm{val}_Z\left(\left(ZX_i(Z)+\frac{1-Z}{1-\alpha}\beta\right)B(X_i(Z))\right)\geqslant -m_2\min\{1,k\}=-m_2.$$

Suppose, Equation (3.3.61) holds constantly for every values of Z, then

$$m_2\geqslant m_1.$$

Denote $k=l_1/l_2$ where l_1, l_2 are coprime natural numbers and $l_1>l_2$ and denote l_3 as the order of ξ^{-i}. Consider the change of variable by introducing T such that $T^{l_2}=Z$. Then

$$X_i(T)=\left(\frac{1-\xi^iT^{l_1}}{1-\delta}\gamma-\frac{1-T^{l_2}}{1-\alpha}\beta\right)/(T^{l_2}-\xi^iT^{l_1}),$$

and

$$A(X_i(T))=\left(T^{l_2}X_i(T)+\frac{1-T^{l_2}}{1-\alpha}\beta\right)B(X_i(T)). \tag{3.3.62}$$

Since $k\neq 1,2$, we have $l_2\neq l_1-l_2$. For a $(l_1-l_2)l_3$-th root of unity ξ_0 such that $\xi_0^{l_1-l_2}=\xi^{-i}$, we have that for any $\mu\in\mathbb{C}^*$ such that $\mu^{l_1-l_2}=1$ the root of unity $\mu\xi_0$ is a zero of $T^{l_2}-\xi^iT^{l_1}$. On the other hand, for any $\mu\in\mathbb{C}$ such that $\mu^{l_1-l_2}=1$,

$$\frac{1-\xi^i(\mu\xi_0)^{l_1}}{1-\delta}\gamma-\frac{1-(\mu\xi_0)^{l_2}}{1-\alpha}\beta=\left(\frac{\gamma}{1-\delta}-\frac{\delta}{1-\alpha}\right)(1-\mu^{l_2}\xi_0^{l_2}). \tag{3.3.63}$$

Since $\gamma/(1-\delta)\neq\beta/(1-\alpha)$, if $l_1-l_2>1$, then there exists a $\mu_0\in\mathbb{C}$ such that $\mu_0^{l_1-l_2}=1$ and $1-\mu_0^{l_2}\xi_0^{l_2}\neq 0$, since otherwise $l_2=m(l_1-l_2)$ for some $m\in\mathbb{N}^*$ and so $l_1=(m+1)(l_1-l_2)$, contradicting that l_1 and l_2 are coprime. Note that this implies $T_0:=\mu_0\xi_0$ is a pole of $X_i(T)$.

But then, $\mathrm{val}_{T-T_0}(X_i(T))\leqslant -1$ and Equation (3.3.62) implies that

$$\begin{aligned}\mathrm{val}_{T-T_0}(A(X_i(T)))&=m_1\,\mathrm{val}_{T-T_0}(X_i(T)) \qquad (3.3.64)\\ =(1+m_2)\,\mathrm{val}_{T-T_0}(X_i(T))&=\mathrm{val}_{T-T_0}\left(\left(T^{l_2}X_i(T)+\frac{1-T^{l_2}}{1-\alpha}\beta\right)B(X_i(T))\right),\end{aligned}$$

which further implies that $m_1=m_2+1$. This contradicts the inequality $m_2\geqslant m_1$ we obtained above.

Now, Suppose $l_1=l_2+1$. If $\xi^{l_2i}\neq 1$, then $T_1:=\xi^{-i}$ is a pole of

$$X_i(T)=\left(\frac{\gamma}{1-\delta}(1-\xi^iT^{l_2+1})-\frac{\delta}{1-\alpha}(1-T^{l_2})\right)\left(T^{l_2}(1-\xi^iT)\right).$$

Then the valuation with respect to $T - T_1$ again gives the contradiction to $m_1 \leqslant m_2$.

Now, we are only left to show the case when $l_1 = l_2 + 1$ and also $\xi^{il_2} = 1$. Note that we only need to obtain a contradiction when $l_2 > 1$, since we assumed $k \neq 2$. In this case, we can rewrite $X_i(T)$ as follows

$$\begin{aligned} X_i(T) &= \left(\frac{\gamma}{1-\delta}(1-(\xi^i T)^{l_2+1}) - \frac{\beta}{1-\alpha}(1-(\xi^i T)^{l_2})\right)\left(T^{l_2}(1-\xi^i T)\right) \\ &= \left(\left(1+\xi^i T+\cdots+(\xi^i T)^{l_2-1}\right)\left(\frac{\gamma}{1-\delta}-\frac{\beta}{1-\alpha}\right)+\frac{\gamma}{1-\delta}T^{l_2}\right)/T^{l_2}. \end{aligned} \tag{3.3.65}$$

When T is close to 0, we have

$$X_i(T) = L(T^{-l_2} + \xi^i T^{1-l_2}) + o(T^{1-l_2}), \tag{3.3.66}$$

where we denote $L \coloneqq \gamma/(1-\delta) - \beta/(1-\alpha)$. Then, we similarly have

$$T^{l_2} X_i(T) + \frac{1-T^{l_2}}{1-\alpha}\beta = \frac{\gamma}{1-\delta} + L\xi^i T + o(T). \tag{3.3.67}$$

Now, by looking at the val_T, if $\gamma = 0$, then Equation (3.3.62) implies that

$$-\deg(A)l_2 = -\deg(B)l_2 + 1,$$

which is impossible since $l_2 > 1$. If $\gamma \neq 0$, then by looking at the val_T of Equation (3.3.62), we have $\deg(A) = \deg(B)$. Denote $m \coloneqq \deg(A)$, a_m and b_m the leading coefficients of A and B. Then

$$A(X_i(T)) = a_m L^m T^{-ml_2} + m a_m L^m \xi^i T^{1-ml_2} + o(T^{1-ml_2}), \tag{3.3.68}$$

$$B(X_i(T)) = b_m L^m T^{-ml_2} + m b_m L^m \xi^i T^{1-ml_2} + o(T^{1-ml_2}). \tag{3.3.69}$$

Then Equation (3.3.62) implies that

$$\begin{aligned} &a_m L^m T^{-ml_2} + m a_m L^m \xi^i T^{1-ml_2} + o(T^{1-ml_2}) \\ =& \left(\frac{\gamma}{1-\delta} + L\xi^i T + o(T)\right)\left(b_m L^m T^{-ml_2} + m b_m L^m \xi^i T^{1-ml_2} + o(T^{1-ml_2})\right). \end{aligned} \tag{3.3.70}$$

Comparing the coefficients of the first two leading terms, we have

$$a_m = \frac{\gamma}{1-\delta} b_m, \tag{3.3.71}$$

$$\frac{\gamma}{1-\delta} m b_m L^m + b_m L^{m+1} = m a_m L^m. \tag{3.3.72}$$

Since $b_m \neq 0$, these together imply that $L = 0$, which implies that f and g share two common fixed points. Hence $\langle f, g\rangle$ is not a free semigroup under the compositions.

Thus, the claim is proved for $k > 0$.

Now suppose $k < 0$. Then, it means that there exists a pair of positive integers (l_1, l_2) such that $\alpha^{l_1}\delta^{l_2} = 1$. Thus we can check directly

$$f^{l_1}(X) = \alpha^{l_1} X + \beta', \tag{3.3.73}$$

$$g^{l_2}(X) = \delta^{l_2} X + \gamma', \tag{3.3.74}$$

$$f^{l_1} \circ g^{l_2} = X + c_1, \tag{3.3.75}$$

$$g^{l_2} \circ f^{l_1} = X + c_2, \tag{3.3.76}$$

for some constants c_1, c_2, β' and γ'. Then obviously $f^{l_1} \circ g^{l_2}$ commutes with $g^{l_2} \circ f^{l_1}$, which implies that $f(X)$ and $g(X)$ don't generate a free semigroup under composition. Thus, it contradicts the assumption.

Case IV. Lastly, suppose $\alpha = \delta$, then $f^n(X) = g^n(X)$ only has a solution if $f = g$ and certainly $\langle f, g\rangle$ is not free in this case.

Now, consider the general case that $f(X)$, $g(X)$ and $c(X)$ are all defined over $\mathbb{C}$. We use a similar specialization argument as in the proof of [52, Theorem 2]. We assume α, δ and α/δ are not roots of unity, $\gamma/(1-\delta) - \beta/(1-\alpha) \neq 0$ and β, γ are not both zero, since otherwise the argument above directly shows that $\langle f, g\rangle$ is not free. We construct a ring extension R over $\mathbb{Z}$ generated by coefficients of $f(X)$, $g(X)$ and $c(X)$. Denote $K = \mathrm{Frac}(R)$. We do induction on the transcendence degree, N, of K. The base case is that $N = 0$, which means everything is algebraic and then we showed from the discussion above that Equation (3.3.50) can only have solution for finitely many $n \in \mathbb{N}$.

Now, assume that if the transcendence degree is $N-1$, then the above assumptions on γ , β, α, δ, α/δ and $\gamma/(1-\delta) - \beta/(1-\alpha)$ implies that the equation only admits solutions for finitely many $n \in \mathbb{N}$. Take L as a subfield of K of transcendence degree $N-1$. We will use the notation $\hat{h}_{\overline{L},f}$ to denote the canonical height function associated to the function f defined over $\overline{L}$. Applying [19, Theorem 4.1], we have that for all specializations $s : R \to \overline{L}$ of sufficiently large height, we have $\hat{h}_{\overline{L},x^2}(\alpha_s)$, $\hat{h}_{\overline{L},x^2}(\delta_s)$ and $\hat{h}_{\overline{L},x^2}(\alpha_s/\delta_s)$ are all greater than 0 and if β, γ and $\gamma/(1-\delta) - \beta/(1-\alpha)$ are not roots of unity or zero, we also have

$\hat{h}_{\overline{L},x^2}(\beta_s)$,$\hat{h}_{\overline{L},x^2}(\gamma_s)$ and $\hat{h}_{\overline{L},x^2}(\gamma_s/(1-\delta_s)-\beta_s/(1-\alpha_s))$ are non-zero. This, together with the assumption that

$$\gamma/(1-\delta)-\beta/(1-\alpha)\neq 0,$$

β, γ are not both zero and α, δ, α/δ are not roots of unity, implies, in particular, that none of α_s, δ_s and α_s/δ_s is a root of unity or zero, $\gamma_s/(1-\delta_s)-\beta_s/(1-\alpha_s)$ are nonzero and β_s and γ_s are not both zero. Then the induction hypothesis shows that

$$f_s^n(X)=g_s^n(X)=c_s(X)$$

has solution for only finitely many $n\in\mathbb{N}$ and thus

$$f^n(X)=g^n(X)=c(X)$$

has solution for only finitely many $n\in\mathbb{N}$.

□

Proof of Lemma 3.3.10. We have that

$$f^n(X)=\alpha^nX+\frac{1-\alpha^n}{1-\alpha}\beta$$

$$g^n(X)=X+n\gamma.$$

If $\gamma=0$, then $\langle f,g\rangle$ is obviously not free, so we assume that $\gamma\neq 0$. Then

$$X_n=\frac{\beta}{1-\alpha}-\frac{n\gamma}{1-\alpha^n} \tag{3.3.77}$$

is the solution to

$$f^n(X_n)=g^n(X_n).$$

Recall the notation that $c(X)=A(X)/B(X)$, for some coprime polynomials $A(X)$ and $B(X)$. Now suppose the equation

$$A(X_n)=B(X_n)\left(\alpha^nX_n+\frac{1-\alpha^n}{1-\alpha}\beta\right) \tag{3.3.78}$$

is satisfied by infinitely many $n\in\mathbb{N}$. If $|\alpha|>1$, then

$$A(X_n)/B(X_n)\sim(n/\alpha^n)^l$$

for some integer l when n is large. However,

$$\left(\alpha^n X_n + \frac{1-\alpha^n}{1-\alpha}\beta\right) \sim \gamma n$$

when n is large, which contradicts the assumption that Equation (3.3.78) holds for infinitely many $n \in \mathbb{N}$.

On the other hand, suppose $|\alpha| < 1$. Now, take a set of places S such that for any $v \notin S$, we have $A(X)$ and $B(X)$ have coefficients with norm 1 with respect to v and also

$$|1-\alpha|_v = |\alpha|_v = |\beta|_v = |\gamma|_v = 1,$$

if $\beta \neq 0$. Then, by the proof of [52, Proposition 15, **Case II**], when n is large enough, there is a v outside of S such that

$$|n/(1-\alpha^n)|_v > 1.$$

Thus, Equation (3.3.78) implies that

$$\deg(A) = \deg(B) + 1 \tag{3.3.79}$$

since $|X_n|_v > 1$ with the choice of n and v such that $|n/(1-\alpha^n)|_v > 1$. Now, since

$$\lim_{n\to\infty} |\alpha^n X_n| = 0,$$

we have

$$\lim_{n\to\infty} \left(A(X_n) - \frac{\beta}{1-\alpha}B(X_n)\right) = 0$$

by Equation (3.3.78) and the fact

$$\lim_{n\to\infty} B(X_n)\alpha^n X_n = 0.$$

Notice that if $\beta = 0$, we would have $A(X) \equiv 0$, which contradicts Equation (3.3.79). Now, if $\beta \neq 0$, this together with

$$\lim_{n\to\infty} |X_n| = \infty,$$

imply that

$$\deg(A) = \deg(B)$$

which contradicts Equation (3.3.79).

Now, suppose $|\alpha| = 1$. By the same argument from the last case, we have $\deg(A) = \deg(B) + 1$. Let

$$F(n, \alpha^n) = \left(A(X_n) - B(X_n) \left(\alpha^n X_n + \frac{1-\alpha^n}{1-\alpha} \beta \right) \right) (1-\alpha^n)^{\deg(A)}$$

be a polynomial in n and α^n. Then by [11, Proposition 2.5.1.4] we have $\{F(n, \alpha^n)\}_{n \in \mathbb{N}}$ satisfies a linear recurrence. Thus, by [11, Theorem 2.5.4.1], the set $\{n \in \mathbb{N} : F(n, \alpha^n) = 0\}$ is a finite union of arithmetic progressions. Since we assumed that $F(n, \alpha^n) = 0$ holds for infinitely many n, there exists $s_1, s_2 \in \mathbb{N}$ such that $F(n, \alpha^n) = 0$ for any $n \in \{s_1 m + s_2 : m \in \mathbb{N}\}$. Now, we consider n within an infinite subsequence I of $\{s_1 m + s_2 : m \in \mathbb{N}\}$ such that $|1 - \alpha^n| > \epsilon$ for some $\epsilon > 0$ independent from n and α^n is not converging. The existence of such an infinite subsequence I is argued in the last part of the proof of Lemma 3.3.3. Thus

$$\lim_{n \to \infty,\ n \in I} F(n, \alpha^n) = 0,$$

where n ranges in the chosen subsequence I. Notice that by Equation (3.3.77), we have

$$X_n = -\gamma n/(1-\alpha^n) + o(n/(1-\alpha^n))$$

for $n \in I$ and n becomes large. By looking at the leading terms of $F(n, \alpha^n)$ as n grows, we obtain that

$$\lim_{n \to \infty, n \in I} (a - b\alpha^n) \left(-\frac{\gamma}{1-\alpha^n} \right)^{\deg(A)} n^{\deg(A)} + o(n^{\deg(A)}) = 0, \tag{3.3.80}$$

where a and b are the leading coefficients of $A(X)$ and $B(X)$ respectively, which implies that

$$\lim_{n \to \infty,\ n \in I} \alpha^n = a/b,$$

since $|1/(1-\alpha^n)|$ is bounded from both below and above. This gives a contradiction as α^n doesn't converge. □

3.4 Proof of Theorem 3.1.4

The following lemma asserts that an infinite sequence $\{x_n\}$ of solutions to the equation $f^n(x_n) = c(x_n)$ for all $n \in \mathbb{N}$ is a sequence of small dynamical height.

Lemma 3.4.1. *Let K be a global field. Let $\{x_n\}$ be a sequence in $\mathbb{P}^1(\overline{K})$ so that $f^n(x_n) = c(x_n)$ for all $n \in \mathbb{N}$ where $f, c \in K(x)$ and $\deg(f) > 1$. Then*

$$\lim_{n\to\infty} \hat{h}_f(x_n) = 0.$$

Proof. For each $x \in \mathbb{P}^1(\overline{K})$, we have by Theorem 3.2.2 that

$$\hat{h}_f(c(x)) = (\deg c)\hat{h}_f(x) + O(1)$$

where the implied constants do not depend on x (but coefficients of f). For each $n \in \mathbb{N}$, the sequence $\{x_n\}_{n\in\mathbb{N}}$ satisfies $f^n(x_n) = c(x_n)$ and the functorial property of canonical height yields

$$\hat{h}_f(c(x_n)) = \hat{h}_f(f^n(x_n)) = (\deg f)^n \hat{h}_f(x_n).$$

This implies

$$(\deg f)^n \hat{h}_f(x_n) = (\deg c)\hat{h}_f(x_n) + O(1)$$

where the implied constant does not depend on x_n and so does not depend on n. Thus the quantity $((\deg f)^n - \deg c)\hat{h}_f(x_n)$ is bounded by a constant independent of n. As the term $((\deg f)^n - \deg c)$ becomes arbitrary large as n gets large because $\deg f > 1$, it follows that $\hat{h}_f(x_n)$ must become arbitrary small and approaches 0 as $n \to +\infty$. □

The following result is well known in arithmetic dynamics and it asserts that if two rational functions share the same (non-repeating) sequence of points of small height, their dynamical height must be identical. The strategy of the proof is similar to that of Baker-DeMarco [6, Theorem 1.2] but we include the proof here for completeness and for the convenience of the reader. For different proof (in the language of metrics on line bundle of $\mathbb{P}^1$) and ingredient we refer the reader to [70, Corollary 14].

Proposition 3.4.2. *Let K be a global field and let $f, g \in K(x)$ be rational functions of degree greater than 1. Suppose that there exists an infinite distinct sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{P}^1(\overline{K})$ such that*

$$\lim_{n\to\infty} \hat{h}_f(x_n) = \lim_{n\to\infty} \hat{h}_g(x_n) = 0.$$

Then $\hat{h}_f = \hat{h}_g$.

Proof. Suppose that $\{x_n\}_{n\geqslant 1}$ is a non-repeating sequence in $\mathbb{P}^1(\overline{K})$ such that $\hat{h}_f(x_n) \to 0$ and $\hat{h}_g(x_n) \to 0$ as $n \to +\infty$.

Case I. Suppose that $f, g \in \overline{\mathbb{Q}}(x)$. Suppose that L is a number field for which both f and g are defined. Thus $\{x_n\}$ must be a sequence of algebraic points in $\mathbb{P}^1(\overline{L})$ for all $n \in \mathbb{N}$. Since $\{x_n\}$ is a sequence of point of small dynamical height with respect to both f and g, so the sequence of probability measures supported equally on the sets $\mathrm{Gal}(\overline{L}/L) \cdot x_n$ of conjugates of x_n converges weakly to the canonical measures $\mu_{f,v}$ and $\mu_{g,v}$ at all places v of L in the Berkovich projective line $\mathbb{P}^1_{\mathrm{Berk},v}$ by the arithmetic equidistribution theorem of Baker-Rumely [7], Favre-Rivera-Letelier [34], and Chambert-Loir [21]. Thus $\mu_{f,v} = \mu_{g,v}$. From the definition of the Laplacian (3.2.1), for all $v \in M_L$ and all $x, y \in \mathbb{P}^1_{\mathrm{Berk},v}$, we have that the v-adic Arakelov Green's functions of f and g must differ by a constant $C_v(y) \in \mathbb{R}$ ([8, Proposition 5.28])

$$G_{\mu_f,v}(x, y) = G_{\mu_g,v}(x, y) + C_v(y). \tag{3.4.1}$$

Using the fact that the Arakelov-Green's function is symmetric, fixing x and viewing as a function of y, we can deduce that $C_v(y) = C_v \in \mathbb{R}$ is independent of y. Thus Equation (3.4.1) becomes

$$G_{\mu_f,v}(x, y) = G_{\mu_g,v}(x, y) + C_v. \tag{3.4.2}$$

Summing equation (3.4.2) over all places $v \in M_L$ with appropriate local degree, it follows that

$$\hat{h}_f(x) + \hat{h}_f(y) = \hat{h}_g(x) + \hat{h}_g(y) + C$$

where $C = \sum_{v \in M_L} N_v C_v$ and is independent of y. Setting $y = x_n$ and letting $n \to \infty$, we have

$$\hat{h}_f(x) = \hat{h}_g(x) + C.$$

Taking x to be f-preperiodic, we obtain $C = -\hat{h}_g(x) \leqslant 0$. Similarly, taking x to be g-preperiodic, it yields $C = \hat{h}_f(x) \geqslant 0$. Hence $C = 0$ and the desired result follows immediately.

Case II. Let L be a finitely generated field extension of $\overline{\mathbb{Q}}$ which is generated by all coefficients of f and g. Suppose that neither f nor g is isotrivial (i.e., they are not conjugate to a rational function defined over $\overline{\mathbb{Q}}$). Thus the sequence $\{x_n\}_{n \in \mathbb{N}}$ is defined over $\mathbb{P}^1(\overline{L})$. Then the proof follows along similar lines to that of Case I. The last step to conclude that $C = 0$, we instead use the Theorem 3.2.4 for function field.

Case III. Suppose that at least one of f and g is isotrivial. Without loss of generality, assume that f is isotrivial (i.e., it is conjugate to a rational function defined over $\overline{\mathbb{Q}}$). Then we follow the argument used in Baker-DeMarco [6, Theorem 2] to deduce that g must also be isotrivial. Hence both f and g are conjugate to a rational function defined over $\overline{\mathbb{Q}}$ and we are back to Case I. Therefore, the desired result is established. □

Recall that

$$\mathrm{Prep}(f) := \{x \in \mathbb{C} \mid f^n(x) = f^m(x), \text{ for } m, n \in \mathbb{N}\}$$

the collection of all preperiodic points of f. It is infinite and countable [9, §6.2].

Proposition 3.4.3. *Let $f, g \in \mathbb{C}(x)$ be two rational functions of degree greater than 1. Suppose that the semigroup generated by f and g under composition is free. Then there are finitely many $\lambda \in \mathbb{C}$ such that there are $m, n \in \mathbb{N}$ with the following properties:*

(i) $f^m(x) \neq c(x)$

(ii) $g^n(x) \neq c(x)$

(iii) $f^m(\lambda) = g^n(\lambda) = c(\lambda)$.

Proof. Let K be a field generated by all coefficients of f, g, and c over $\mathbb{Q}$. Thus K is either a number field or a function field of finite transcendence degree with constant field $K \cap \overline{\mathbb{Q}}$.

Suppose, on the contrary, that there is an infinite distinct sequence $\{\lambda_i\}_{i\in\mathbb{N}}$ such that for every i, there exist m_i and n_i so that $f^{m_i}(x) \neq c(x), g^{n_i}(x) \neq c(x)$, and λ_i is solution to both $f^{m_i}(x) = c(x)$ and $g^{n_i}(x) = c(x)$. Note that $m_i, n_i \to +\infty$ as $i \to +\infty$. It follows from Lemma 3.4.1 that

$$\lim_{i\to\infty} \hat{h}_f(\lambda_i) = \lim_{i\to\infty} \hat{h}_g(\lambda_i) = 0.$$

It then follows immediately from Proposition 3.4.2 that $\hat{h}_f = \hat{h}_g$.

Case I. Suppose that K is a number field. Proposition 3.2.3 asserts that the canonical heights $\hat{h}_f$ and $\hat{h}_g$ vanish precisely on the set of f-preperiodic points and g-preperiodic points, respectively. Then it follows that $\mathrm{Prep}(f) = \mathrm{Prep}(g)$ (i.e., f and g share the same set of preperiodic points). Applying the result of Bell-Haung-Peng-Tucker [12, Corollary 4.12] to deduce that the semigroup generated by f and g under composition is not free. Hence it contradicts the assumption.

Case II. Suppose that K is a function field of finite transcendence degree and neither f nor g is isotrivial. Thus f and g must share the same set of preperiodic points by Theorem 3.2.4. Again, applying the result of [12], we have that $\langle f, g\rangle$ is not free.

Case III. Assume that at least one of f and g is isotrivial. Without loss of generality, assume that f is isotrivial. We claim that g must also be isotrivial. This follows immediately from a weak Northcott result of Baker [4, Theorem 1.6]. Thus both f and g are conjugate to a rational function defined over $\overline{\mathbb{Q}}$. We are back to Case I. □

Next, we treat the case when one of rational functions f and g is of degree 1. Here we briefly discuss the proof idea due to Hsia-Tucker [52] and it goes through for rational functions f and g without any changes.

Proposition 3.4.4. *Let $f, g \in \mathbb{C}(x)$ be two rational functions such that $\deg(f) > 1$ and $\deg(g) = 1$. Suppose that the semigroup generated by f and g under composition is free. Then there are finitely many $\lambda \in \mathbb{C}$ such that there are $m, n \in \mathbb{N}$ with the following properties:*

(i) $f^m(x) \neq c(x)$

(ii) $g^n(x) \neq c(x)$

(iii) $f^m(\lambda) = g^n(\lambda) = c(\lambda)$.

Proof. Again, K is a field generated by all coefficients of f, g and c over $\mathbb{Q}$. Thus K is either a number field or a function field of finite transcendence degree.
Case I. Suppose that K is a number field. We follow closely the argument of Hsia-Tucker [52, Proposition 9]. For any infinite distinct sequence of $\{\lambda_i\}$ and $\{m_i\}$ such that $f^{m_i}(\lambda_i) = c(\lambda_i)$ for every $i \in \mathbb{N}$, we have that $\{\lambda_i\}$ is a sequence of small height with respect to f by Lemma 3.4.1. Also, the condition (iii) yields that any solution $\lambda \in \mathbb{C}$ to the equation $g^n(\lambda) = c(\lambda)$ is of bounded degree (i.e., degree at most $\deg(c)$ since $\deg(g) = 1$) over K. Hence the Northcott's property implies that there are only finitely many $\lambda \in \mathbb{C}$ satisfying (i), (ii), and (iii).
Case II. Suppose that K is a function field of finite transcendence degree. We refer the reader to Hsia-Tucker [52, Proposition 9] as the proof is verbatim to the polynomial maps setting. □

We are now ready to prove the main result of this section.

Theorem 3.4.5. *Let $f(x)$ and $g(x)$ be rational functions in $\mathbb{C}(x)$, at least one of which has degree greater than one. Suppose that semigroup generated by f and g under composition is free and $c(x)$ is not a compositional power of f or g. Then there are only finitely many $\lambda \in \mathbb{C}$ such that*

$$f^m(\lambda) = g^n(\lambda) = c(\lambda).$$

for some $m, n \in \mathbb{N}$.

Proof. The proof divides into two cases:
Case I. If $\deg(f), \deg(g) > 1$, then the desired result follows immediately from Proposition 3.4.3.
Case II. If one of rational functions f and g is of degree one (without loss of generality, assume that $\deg(f) > 1$ and $\deg(g) = 1$), then the desired conclusion follows from Proposition 3.4.4. □

Chapter 4

Polynomial endomorphisms of the affine plane with many periodic curves

4.1 Introduction

The Dynamical Manin-Mumford Conjecture is the dynamical generalization of the well known Manin-Mumford Conjecture, which conjectures that a subvariety that contains many periodic points of an endomorphism must itself be special with respect to this endomorphism. After a series of studies and refinements by [91], [92] and [43], the precise statement of the conjecture is the following:

Conjecture 4.1.1. *Let $f : X \to X$ be a polarized endomorphism of a smooth projective variety over a field of characteristic zero, and $Z \subset X$ be a subvariety containing a Zariski dense set of preperiodic points. Then either Z is preperiodic or Z is special, in the sense that it is contained in some subvariety Y that is both f^n-and ψ-invariant, for some $n \geqslant 1$, where ψ is another polarized endomorphism commuting with f^n on Y, and Z is preperiodic under ψ.*

The conjecture remains widely open, and only a few cases are known. For example, the conjecture is proved for split morphisms on $(\mathbb{P}^1)^n$ by [41], [42] and for polynomial endomorphisms on $\mathbb{A}^2$ extendable to endomorphisms on $\mathbb{P}^2$ with some constraint on the ramification at the line at infinity by [28].

4.1.1 Relative Dynamical Manin-Mumford Conjecture

Recently, inspired by Gao and Habegger's work on Relative Manin-Mumford Conjecture for families of abelian varieties [35, Theorem 1.1 and 1.3], DeMarco and Mavraki proposed the generalized Dynamical Manin-Mumford Conjecture for families of dynamical systems in [27].

We follow [27] to introduce the necessary notations. An algebraic family of endomorphisms of $\mathbb{P}^n$ of degree d is a morphism

$$\Phi : S \times \mathbb{P}^n \to S \times \mathbb{P}^n$$

given by $\Phi(s, z) = (s, f_s(z))$ where f_s is an endomorphism of $\mathbb{P}^n$ of degree d. Let $\mathcal{X} \subseteq S \times \mathbb{P}^n$ denote a closed irreducible subvariety which is flat over S. We let $\mathbf{X}$ denote the generic fiber of $\mathcal{X}$ and let $\mathbf{\Phi} : \mathbf{P}^n \to \mathbf{P}^n$ be the map induced by Φ, viewed as an endomorphism over the function field $\mathbb{C}(S)$.

We say $\mathcal{X}$ is Φ-special if there exists a subvariety $\mathbf{Z} \subseteq \mathbf{P}^n$ over the algebraic closure $\mathbb{C}(S)$ containing the generic fiber $\mathbf{X}$, a polarizable endomorphism $\mathbf{\Psi} : \mathbf{Z} \to \mathbf{Z}$, and a positive integer n such that the following hold:

- $\mathbf{\Phi}^n(\mathbf{Z}) = \mathbf{Z}$;
- $\mathbf{\Phi}^n \circ \mathbf{\Psi} = \mathbf{\Psi} \circ \mathbf{\Phi}^n$ on $\mathbf{Z}$; and
- $\mathbf{X}$ is preperiodic under $\mathbf{\Psi}$.

We denote $r_{\Phi,\mathcal{X}}$ the relative special dimension of $\mathcal{X}$ over S. This is given by

$$r_{\Phi,\mathcal{X}} \coloneqq \min\{\dim_S \mathcal{Y} : \mathcal{X} \subseteq \mathcal{Y} \text{ and } \mathcal{Y} \text{ is } \Phi\text{-special}\},$$

where $\dim_S \mathcal{Y} = \dim \mathcal{Y} - \dim S$ is the dimension of a generic fiber of $\mathcal{Y}$ over S.

With the notations from above, DeMarco and Mavraki proposed the following relative version of Dynamical Manin-Mumford Conjecture:

Conjecture 4.1.2. *Let $\Phi : S \times \mathbb{P}^N \to S \times \mathbb{P}^N$ be an algebraic family of morphisms of degree > 1, and let $\mathcal{X} \subseteq S \times \mathbb{P}^N$ be a complex, irreducible subvariety which is flat over S. The following are equivalent:*

- *$\mathcal{X}$ contains a Zariski dense set of Φ-preperiodic points.*

- $\hat{T}_\Phi^{r_{\Phi,\mathcal{X}}} \wedge [X] \neq 0$ *for the relative special dimension* $r_{\Phi,\mathcal{X}}$.

Here $\hat{T}_\Phi$ is the canonical Green current associated to Φ on $S \times \mathbb{P}^N$.

This is a very ambitious conjecture and remains open in most cases. When $\dim S = 0$, the Conjecture 4.1.2 reduces to the Dynamical Manin-Mumford Conjecture. When $N = 1$, the Conjecture is already known. Moreover, several important problems can be viewed as special cases of Conjecture 4.1.2, including the Dynamical André-Oort Conjecture proposed by [5] and [38], and questions on the uniform bounds for common preperiodic points between two rational functions on $\mathbb{P}^1$, see for example [26]. See [27, Section 3] for detailed discussion.

The recent work of Mavraki and Schmidt [61] establish a special case of a weaker version of the conjecture if one replaces $\mathbb{P}^N$ with $(\mathbb{P}^1)^N$ (see also the discussion in [27, Conjecture 1.2]).

4.1.2 Main results

In this chapter, we study **regular polynomial endomorphisms** of $\mathbb{P}^2$ of degree > 1, that is, endomorphisms of $\mathbb{P}^2$ extending polynomial endomorphisms on $\mathbb{A}^2$. We focus on the *constant family* case, where

$$\Phi(s, p) = (s, F(p))$$

for a fixed regular polynomial endomorphism F. Within this setting, we consider a family $\mathcal{X}$ that contains a Zariski dense set of periodic curves. Our first main result shows that such a family must be invariant under F, and moreover, that the degree of a generic member of the family remains stable under the action of F.

Theorem 4.1.3. *Let $Z \subseteq \mathcal{M}_d$ be a subvariety of the space of effective divisors of degree d in $\mathbb{P}^2$. Let F be a regular polynomial endomorphism on $\mathbb{P}^2$ of degree > 1. Suppose that Z contains a Zariski dense set of divisors corresponding to periodic curves under F. Then, after replacing F with some iterate, for a generic curve C such that $[C] \in Z$, we have $\deg(C) = \deg(F(C))$. Moreover, for any curve C' such that $[C'] \in Z$, we have $[F(C')] \in Z$.*

Remark 4.1.4. Suppose Z is a subvariety of the Chow variety of pure dimension D and degree d cycles inside $\mathbb{P}^K$, for some $K \geqslant 2$. Throughout the chapter, for a subvariety $C \subseteq \mathbb{P}^K$ of dimension D, we let $[C]$ denote the algebraic cycle corresponding to C, and we write $[C] \in Z$ if there exists a $z \in Z$ whose geometric support as a variety in $\mathbb{P}^K$ is equal to the union of the irreducible components of C.

A key ingredient in the proof of Theorem 4.1.3 builds on ideas from [87, Sections 5–6]. These techniques play a central role in handling invariant curves through an unramified point on the line at infinity and form the foundation of our approach.

Theorem 4.1.3 implies a special case of Dynamical Manin-Mumford Conjecture under a stronger assumption (see Remark 4.5.2), and also proves a weaker form of the Conjecture 4.1.2:

Corollary 4.1.5. *Let S be a smooth and irreducible quasi-projective variety defined over $\mathbb{C}$. Let $\Phi : S \times \mathbb{P}^2 \to S \times \mathbb{P}^2$ be a constant family of endomorphisms such that for every point $(s, p) \in S \times \mathbb{P}^2$*

$$\Phi(s, p) = (s, F(p)),$$

where F is a regular polynomial endomorphism of degree > 1 that does not depend on s. Let $\mathcal{X} \subseteq S \times \mathbb{P}^2$ be an irreducible hypersurface that is flat over S and projects dominantly to S. Suppose there exists a Zariski dense set of $s \in S$ such that X_s, the fiber of $\mathcal{X}$ at s, is a periodic curve under F. Then for all $N \in \mathbb{Z}^+$, we have

$$\hat{T}_{\Phi^{\times N}}^{\wedge r_{\Phi^{\times N}, \mathcal{X}^N}} \wedge [\mathcal{X}^N] \neq 0,$$

and $r_{\Phi^{\times N}, \mathcal{X}^N} \leqslant \min\{2N, N + \dim S\}$.

Remark 4.1.6. Throughout this paper, we will let $\mathcal{X}^N$ and $\Phi^{\times N}$ denote the N-th fiber product of $\mathcal{X}$ and Φ over S.

As another consequence of Theorem 4.1.3, we combine it with the classification of endomorphisms on $\mathbb{P}^2$ preserving an algebraic k-webs obtained in [33] to describe all regular polynomial endomorphisms that admit infinitely many periodic curves of bounded degrees.

Theorem 4.1.7. *Let D be any positive integer. Let F be a regular polynomial endomorphism on $\mathbb{P}^2$ of degree > 1 and suppose that it admits infinitely many periodic curves of degree bounded by D. Then, after replacing F with some iterate, it is of one of the following forms:*

- *F is a polynomial skew product;*
- *F is homogeneous;*
- *there exists a generic finite rational map $\mu : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2$ and a split polynomial endomorphism (f, g) on $\mathbb{P}^1 \times \mathbb{P}^1$ such that*

$$\mu \circ (f, g) = F \circ \mu.$$

A more detailed description of the third case is given in the last section. This is an extension of the work in [87, Section 6] where Xie proved that for a regular polynomial endomorphism whose restriction on the line at infinity is not a polynomial, the existence of infinitely many periodic curves forces the map to be homogeneous [87, Theorem 6.2].

4.1.3 Some conditional results on degree stabilization

Theorem 4.1.3 can also be interpreted as a Dynamical Manin-Mumford type statement on the moduli space of divisors: instead of looking at subvarieties with a dense set of preperiodic points in $\mathbb{P}^2$, we look at the subvarieties in the moduli space $\mathcal{M}_d$ containing a dense set of periodic curves under F. The theorem asserts that such a subvariety must be special in the sense that a generic divisor it contains has stabilized degree under the action of F, and the subvariety itself is invariant under the induced map.

We expect the degree stabilization should hold in a more general setting, and we propose the following question which was raised in a discussion with Junyi Xie:

Question 4.1.8. Let F be an endomorphism of degree > 1 on $\mathbb{P}^N$ and $Ch_{D,d}(\mathbb{P}^N)$ be the Chow variety consisting of algebraic cycles of $\mathbb{P}^N$ of dimension D and degree d. Suppose $Z \subseteq Ch_{D,d}(\mathbb{P}^N)$ is a subvariety containing a Zariski dense set of algebraic cycles corresponding to irreducible periodic subvarieties of $\mathbb{P}^N$ of degree d under F. Then is it true that, after replacing F with some iterate, for a generic cycle $V \in Z$, we have $\deg(V) = \deg(F(V))$?

Example 4.1.9. Note that even if a curve $C \subseteq \mathbb{P}^2$ is periodic under an endomorphism F, the curves in the periodic cycle can have various degrees. For example, under the regular polynomial endomorphism $F(x, y) = (x^2, y^2 - x)$, we have the following periodic cycle of curves

$$V(y) \xrightarrow{F} V(y^2 - x) \xrightarrow{F} V(y)$$

With a mild assumption on F, we will demonstrate that the Dynamical Manin-Mumford Conjecture implies a positive answer to the Question 4.1.8, and a broader version of Theorem 4.1.3 should hold:

Definition 4.1.10. Let S be a smooth, irreducible, quasi-projective variety over $\mathbb{C}$ of positive dimension, and let $M \geqslant 2$ be an integer. Let

$$\Phi : S \times \mathbb{P}^M \to S \times \mathbb{P}^M$$

be a family of surjective endomorphisms, and let $\mathcal{X} \subseteq S \times \mathbb{P}^M$ be an irreducible subvariety of dimension greater than $\dim S$, which projects dominantly onto S and is flat over S.

We say that $(\Phi, \mathcal{X})$ is of **general type** if for any subvariety $\mathbf{Y} \subseteq \mathbf{P}^M$ containing $\mathbf{X}$ and invariant under $\mathbf{\Phi}^n$ for some $n > 0$, the only polarizable endomorphisms on $\mathbf{P}^M$ that commute with $\mathbf{\Phi}^n$ on $\mathbf{Y}$ are iterates of Φ itself.

In the special case where $\Phi(s, p) = (s, F(p))$ for a fixed surjective endomorphism F, we say that the pair $(F, \mathcal{X})$ is of general type, meaning the same as $(\Phi, \mathcal{X})$ being of general type.

Theorem 4.1.11. *Let F be a surjective endomorphism on $\mathbb{P}^K$ of degree > 1. Let $Z \subseteq Ch_{D,d}(\mathbb{P}^K)$ be a subvariety in the Chow variety consisting of algebraic cycles of $\mathbb{P}^K$ of dimension $D \leqslant K - 1$ and degree d. Suppose Z contains a Zariski dense set of algebraic cycles corresponding to irreducible periodic subvarieties of $\mathbb{P}^K$ under F of degree d. Assuming (F, Z) is of general type and Conjecture 4.1.1 is true, then for every subvariety $C \subseteq \mathbb{P}^K$ and a generic subvariety $C' \subseteq \mathbb{P}^K$ such that $[C], [C'] \in Z$, we have that $[F(C)] \in Z$ and $\deg(C') = \deg(F(C'))$, after replacing F with some iterate.*

Furthermore, we show that Conjecture 4.1.2 implies an even stronger stabilization property for the degrees of images in families of endomorphisms:

Proposition 4.1.12. *Let S be a smooth and irreducible quasi-projective variety defined over $\mathbb{C}$ of positive dimension and $M \geqslant 2$ be a positive integer. Let $\Phi : S \times \mathbb{P}^M \to S \times \mathbb{P}^M$ be a family of endomorphisms such that for every point $(s, p) \in S \times \mathbb{P}^M$*

$$\Phi(s, p) = (s, F_s(p))$$

for an endomorphism F_s of degree > 1. Let $\mathcal{X} \subseteq S \times \mathbb{P}^M$ be an irreducible subvariety of dimension $> \dim S$ that projects dominantly to S and is flat over S. Assume $(\Phi, \mathcal{X})$ is of general type.

Suppose that for a Zariski dense subset of $s \in S$, $\mathcal{X}_s$ is preperiodic under Φ_s. If Conjecture 4.1.2 holds, then there exists positive integers n, m and an infinite set of positive integers I such that for any pair of integers $k_1, k_2 \in I$ we have that,

$$\deg(F_s^{m+nk_1}(\mathcal{X}_s)) = \deg(F_s^{m+nk_2}(\mathcal{X}_s)),$$

for a generic point $s \in S$.

We also establish a degree stabilization result for families $\Phi : \mathbb{A}^1 \times \mathbb{P}^2 \to \mathbb{A}^1 \times \mathbb{P}^2$ of regular polynomial endomorphisms, under suitable assumptions on the behavior of $\mathcal{X}$ at infinity (Theorem 4.6.3 and 4.6.5). Our argument combines analysis of branches along H_∞ with results from [32] and, conditionally, on [25, Conjecture 6.1] and the classification of semiconjugate pairs in [68].

4.2 Outline of the chapter

Section 4.3 establishes several technical lemmas that will be essential for the subsequent arguments.

In Section 4.4, we prove Theorem 4.1.3. A key technique used in the proof of Theorem 4.1.3 is inspired by [87, Sections 5 and 6]. It establishes that, for a regular polynomial endomorphism, the invariant algebraic curve passing through a non-superattracting fixed point of $F|_{H_\infty}$ is unique whenever it exists, where $H_\infty = \mathbb{P}^2 \backslash \mathbb{A}^2$. Moreover, the intersection must be transverse. A slight adaptation of this result (Lemma 4.3.4), combined with suitable blow-up procedures along H_∞, allows us to obtain a reparametrization of a family of divisors containing infinitely many periodic curves. This forms a crucial step in the proof of Theorem 4.1.3.

We also explain at the end of this section how the Dynamical Manin–Mumford Conjecture suggests that the same expectation should hold for endomorphisms on $\mathbb{P}^K$ in general.

In Section 4.5, we show that our main result, Theorem 4.1.3, implies a weaker version of the *relative Dynamical Manin–Mumford Conjecture* (Conjecture 4.1.2) proposed by DeMarco and Mavraki.

Section 4.6 contains conditional results. We show that the relative Dynamical Manin–Mumford Conjecture predicts a degree stabilization phenomenon for families of endomorphisms and subvarieties containing a Zariski dense set of periodic subvarieties. We verify this expectation for families $\Phi : \mathbb{A}^1 \times \mathbb{P}^2 \to \mathbb{A}^1 \times \mathbb{P}^2$ of regular polynomial endomorphisms, under unramification assumptions on the behavior of $\mathcal{X}$ at infinity. Our argument relies on identifying when a marked point $a(t) \in \mathbf{X} \cap H_\infty$ becomes superattracting for infinitely many parameters $t_0 \in \mathbb{A}^1$. In the polynomial case, this follows from [32] (see Theorem 4.6.3); in the general case, the proof depends conditionally on [25, Conjecture 6.1] and the argument relies on the classification of semiconjugate pairs in [68] (see Theorem 4.6.5).

Finally, Section 4.7 introduces the notion of k-webs following [33]. Using Theorem 4.1.3, we show that a regular polynomial endomorphism admitting infinitely many periodic curves of bounded degree must preserve either an algebraic k-web or a pencil of curves. Combining these with the classification results of [33] and [23] yields a complete description of such endomorphisms, proving Theorem 4.1.7.

4.3 Preliminaries

In this section, we collect some technical lemmas that will be useful in later proofs.

We begin by clarifying the notion of a resolution tree:

Definition 4.3.1. By a **resolution tree of depth N at p with respect to a curve** C, where $p \in \mathbb{P}^2$, $N \in \mathbb{Z}^+$, and $C \subset \mathbb{P}^2$ passes through p, we mean the tree obtained by iteratively blowing up points where the strict transform of C intersects the exceptional divisors. The root of the tree is p (level 0), and the process is repeated until the tree has depth N.

Lemma 4.3.2. *Let D be a positive integer. If C_1 and C_2 are two affine curves in $\mathbb{A}^2$ passing through $(0,0)$ of degree $\leqslant D$ that have the same resolution tree of depth D^2+1 at the point $(0,0)$, then $C_1 = C_2$.*

Proof. Let π_i, $i \in \{1, 2, \ldots, D^2+1\}$ denotes the (sequence of) blow-up maps at $(0,0)$ which gives us the resolution tree of depth i at $(0,0)$ with respect to both C_1 and C_2, as they share the same resolution tree.

We show inductively that for any positive integer $M \leqslant D^2+1$, we have

$$(C_1, C_2)_{(0,0)} \geqslant M + (\tilde{C}_1^{(M)}, \tilde{C}_2^{(M)})_{p_M}$$

where $\tilde{C}_i^{(M)}$ is the strict transformation of C_i under π_M, $i \in \{1,2\}$ and p_M is a point in the intersection of $\tilde{C}_1^{(M)}$, $\tilde{C}_2^{(M)}$ and an exceptional divisor introduced when we construct the M-th level of the resolution tree. The existence of such a p_M is guaranteed by our assumption that C_1 and C_2 share the resolution tree up to depth D^2+1.

Note that π_1 is the first blow up map at $(0,0)$, which introduces an exceptional divisor E_1 in the preimages of $(0,0)$ under π_1. Then we have

$$\pi_1^* C_i = \tilde{C}_i^{(1)} + m_i E_1$$

where $m_i > 0$ is the multiplicity of E_1 in the total transformation of C_i under π_1, for $i \in \{1,2\}$. Then, since $\tilde{C}_i^{(1)} \cdot E_1 = m_i$ for $i \in \{1,2\}$, we have

$$\begin{aligned}(C_1 \cdot C_2)_{(0,0)} &= (\tilde{C}_1^{(1)} \cdot \tilde{C}_2^{(1)})_{E_1} + m_2 \tilde{C}_1^{(1)} \cdot E_1 + m_1 \tilde{C}_2^{(1)} \cdot E_1 + m_1 m_2 E_1 \cdot E_1 \\ &= (\tilde{C}_1^{(1)} \cdot \tilde{C}_2^{(1)})_{E_1} + m_1 m_2 \geqslant 1 + (\tilde{C}_1^{(1)} \cdot \tilde{C}_2^{(1)})_{E_1}, \end{aligned} \tag{4.3.1}$$

where

$$(\tilde{C}_1^{(1)} \cdot \tilde{C}_2^{(1)})_{E_1} \coloneqq \sum_{p \in \tilde{C}_1^{(1)} \cap \tilde{C}_2^{(1)} \cap E_1} (\tilde{C}_1^{(1)} \cdot \tilde{C}_2^{(1)})_p.$$

This verifies the base case.

Now, we assume that the statement holds for a positive integer $M < D^2 + 1$ and we show that it also holds for $M + 1$. We do the further expansion of

$$(\tilde{C}_1^{(M)}, \tilde{C}_2^{(M)})_{p_M}$$

by taking a blow-up π_{p_M} at p_M. Then, we have

$$\begin{aligned}
&(\tilde{C}_1^{(M)}, \tilde{C}_2^{(M)})_{p_M} \\
&= (\tilde{C}_1^{(M),p_M} \cdot \tilde{C}_2^{(M),p_M})_{E_{p_M}} + m_1^{p_M} \tilde{C}_2^{(M),p_M} \cdot E_{p_M} \\
&+ m_2^{p_M} \tilde{C}_1^{(M),p_M} \cdot E_{p_M} + m_1^{p_M} m_2^{p_M} E_{p_M} \cdot E_{p_M} \\
&= (\tilde{C}_1^{(M),p_M} \cdot \tilde{C}_2^{(M),p_M})_{E_{p_M}} + m_1^{p_M} m_2^{p_M}, \qquad (4.3.2)
\end{aligned}$$

where

$$\pi_{p_M}^* \tilde{C}_i^{(M)} = \tilde{C}_i^{(M),p_M} + m_i^{p_M} E_{p_M},$$

$m_i^{p_M} > 0$, $i \in \{1, 2\}$ and E_{p_M} is the exceptional divisor introduced by π_{p_M}.

Notice that the assumption that C_1 and C_2 share the resolution tree at $(0, 0)$ up to depth $D^2 + 1$ implies that there exists a

$$p_{M+1} \in \tilde{C}_1^{(M),p_M} \cap \tilde{C}_2^{(M),p_M} \cap E_{p_M}.$$

Also, there exists a sequence of blow-up π'_{M+1} such that

$$\pi_{M+1} = \pi'_{M+1} \circ \pi_{p_M} \circ \pi_M$$

and $\tilde{C}_i^{(M+1)}$ is the strict transformation of $\tilde{C}_i^{(M),p_M}$ under π'_{M+1}, where $i \in \{1, 2\}$.

Then

$$\begin{aligned}
(\tilde{C}_1^{(M),p_M} \cdot \tilde{C}_2^{(M),p_M})_{E_{p_M}} &\geqslant (\tilde{C}_1^{(M),p_M} \cdot \tilde{C}_2^{(M),p_M})_{p_{M+1}} \\
= ((\pi'_{M+1})^* \tilde{C}_1^{(M),p_M} \cdot (\pi'_{M+1})^* \tilde{C}_2^{(M),p_M})_{p_{M+1}} &\geqslant (\tilde{C}_1^{(M+1)} \cdot \tilde{C}_2^{(M+1)})_{p_{M+1}}. \qquad (4.3.3)
\end{aligned}$$

Now the induction hypothesis

$$(C_1, C_2)_{(0,0)} \geqslant M + (\tilde{C}_1^{(M)}, \tilde{C}_2^{(M)})_{p_M}$$

together with (4.3.2) and (4.3.3) imply that

$$(C_1, C_2)_{(0,0)} \geqslant M + 1 + (\tilde{C}_1^{(M+1)} \cdot \tilde{C}_2^{(M+1)})_{p_{M+1}}.$$

Then the induction concludes the statement. Hence, plugging in $M = D^2 + 1$, we have

$$(C_1, C_2)_{(0,0)} \geqslant D^2 + 1.$$

Since, $\deg(C_1) = \deg(C_2) = D$, by Bézout's intersection theorem we conclude that $C_1 = C_2$.

□

Lemma 4.3.3. *Let $\mathcal{C}$ be an infinite set of curves in $\mathbb{P}^2$ of uniformly bounded degree $\leqslant D$ passing through a single point $p \in \mathbb{P}^2$. There exists a blow up π of length bounded by D^2+1 at the point p and an exceptional divisor $E \subseteq \pi^{-1}(p)$ such that $\bigcup_{C\in\mathcal{C}} \pi^{\#}C \cap E$ contains infinitely many distinct points in E.*

Proof. Suppose, for the sake of contradiction, that for any blow up π at p of length not greater than D^2+1 and any exceptional divisor E in $\pi^{-1}(p)$ we have $|\bigcup_{C\in\mathcal{C}} \pi^{\#}C \cap E| < \infty$. We build a resolution tree of depth $D^2 + 1$ at p by recursively blowing up all the points in the intersections of the strict transformation of $C \in \mathcal{C}$ with exceptional divisors. By our assumption, there are only finitely many exceptional divisors in this resolution tree and each exceptional divisor only intersects with $\mathcal{C}$ at a finite set of points. Therefore, by the Lemma 4.3.2 and the pigeonhole principle, there are only finitely many distinct curves in $\mathcal{C}$, which is a contradiction. Thus, there must exist a divisor $E \subseteq \pi'^{-1}(p)$, with a blow up π' at p of length not greater than $D^2 + 1$ (pick a branch of the resolution tree), such that $\bigcup_{C\in\mathcal{C}} \pi'^{\#}C \cap E$ is an infinite set. □

The following lemma is an adaptation of [87, Lemma 5.11] for our purposes. The proof follows the original almost verbatim; we will explain here the necessary changes.

Lemma 4.3.4. *Let F be a regular polynomial endomorphism on $\mathbb{A}^2$ of degree > 1. Let X be a compactification of $\mathbb{A}^2$ which is either $\mathbb{P}^2$ or constructed by a sequence of blow-up of some points on $H_\infty = \mathbb{P}^2\backslash\mathbb{A}^2$. Let E be an irreducible component of $X\backslash\mathbb{A}^2$ such that F, after being replaced with some iterate, extends to a finite morphism, f, on E. Let $o \in E$ be a non-superattracting fixed point of f. Then there exists a unique irreducible formal curve $\hat{C}$ at o which is F-invariant and not contained in E. Moreover, it intersects E transversely at o.*

Proof. In [87, Lemma 5.11], the author proves this lemma with $X = \mathbb{P}^2$ and $E = H_\infty$. But the approach only requires the data locally around o and so can be adapted to this setting.

Let $L = \mathbb{C}((t))$ with t-adic norm. Let U be the affinoid subdomain of X_L^{an}, which, in the local affine chart around $o = (0,0)$, is given by

$$U := \{(x,y) : |x| \leqslant |t|, |y| \leqslant |t|\}.$$

Since F induces a finite morphism on E, we have that F induces a well-defined endomorphism

$$g|_U : U \to U$$

which fixes the hyperplane $Y = E_L^{an} \cap U$, where E within the affine chart is given by $V(x)$ by a coordinate choice. Now, the rest of the proof follows exactly the same line as in [87, Lemma 5.11] with $g|_U$, U and Y as above. □

Remark 4.3.5. The key ingredient of the proof of this lemma, as explained in [87, Proof of Lemma 5.11], is [88, Theorem 8.3], which is the main result of the Appendix A in the paper. In this particular case of Lemma 4.3.4, it shows that $g|_U$ is semiconjugated to $g|_Y$.

Notice that Lemma 4.3.4 has a similar flavor to [28, Theorem 2.1], which shows that there exists a unique smooth analytic curve through o which is transverse to E and invariant under F.

4.4 Proof of Theorem 4.1.3

In this section, we prove Theorem 4.1.3. We begin with a series of lemmas that form the intermediate steps toward the proof. Throughout this section, let F be a regular polynomial endomorphism of $\mathbb{A}^2$ of degree > 1.

Lemma 4.4.1. *Suppose $F : \mathbb{P}^2 \to \mathbb{P}^2$ admits an infinite set of periodic curves $\mathcal{C}$. Then one of the following holds:*

(1) There exists a point $p \in H_\infty$ such that infinitely many curves in $\mathcal{C}$ pass through p.

(2) There exists an integer $e \in \{1,2\}$ and an infinite subset $\mathcal{C}' \subseteq \mathcal{C}$ consisting of periodic curves of degree e, together with a finite morphism

$$\sigma : \mathbb{P}^1 \longrightarrow Z,$$

where Z is the Zariski closure of $\mathcal{C}'$ in the space of effective divisors of degree e, $\mathcal{M}_e$, such that:

- *for a generic* $C \in Z$*, we have* $\deg(F(C)) = \deg(C)$*; and*
- *for any curve* $C' \subseteq \mathbb{P}^2$ *with* $[C'] \in Z$*, we have* $[F(C')] \in Z$*.*

Proof. Suppose that for every point $p \in H_\infty$, only finitely many periodic curves of F pass through p. We will show that (2) must then occur.

By [87, Lemma 6.6 and Remark 6.7], every periodic curve of F, except the line at infinity $H_\infty = \mathbb{P}^2 \backslash \mathbb{A}^2$, has at most two branches at infinity. Since $F|_{H_\infty}$ has only finitely many superattracting periodic cycles, and each periodic point in H_∞ lies on finitely many periodic curves by assumption, there must exist infinitely many periodic curves

$$\mathcal{C}' \subseteq \mathcal{C}$$

such that each $C \in \mathcal{C}'$ meets H_∞ only at non-superattracting periodic points of $F|_{H_\infty}$.

For such curves, Lemma 4.3.4 implies that each intersection $C \cap H_\infty$ is transverse. Consequently, the intersection multiplicity $C \cdot H_\infty$ equals the number of branches of C at infinity, which is at most two. Hence, $\deg(C) \leqslant 2$ for infinitely many periodic curves $C \in \mathcal{C}$. Moreover, the same reasoning applies to all curves lying in the periodic cycles of these curves, showing that $\deg(C) = \deg(F(C)) \in \{1, 2\}$ for all such C.

We now regard these periodic curves as points in the projective space $\mathcal{M}_e$ of effective divisors of degree e on $\mathbb{P}^2$, where $e \in \{1, 2\}$. By enlarging $\mathcal{C}'$ if necessary, we may assume that it is closed under the action of F, i.e., it contains all curves in the periodic cycles of its members.

Consider the incidence variety

$$\Gamma = \overline{\{(C, x) : C \in \mathcal{C}',\, x \in C \cap H_\infty\}} \subseteq \mathcal{M}_e \times H_\infty,$$

and let π_i denote the projection onto the i-th factor for $i \in \{1, 2\}$. Since non-superattracting periodic points are dense in $H_\infty \cong \mathbb{P}^1$, π_2 is surjective; by Lemma 4.3.4, π_2 is in fact bijective. Also, $\pi_1 : \Gamma \to \mathcal{M}_e$ is a morphism of degree e. We may therefore define

$$\sigma := \pi_1 \circ \pi_2^{-1} : \mathbb{P}^1 \cong H_\infty \longrightarrow \mathcal{M}_e,$$

which is a finite morphism with image $Z = \overline{\pi_1(\Gamma)}$. Note that even without enlarging $\mathcal{C}'$, the same Z is obtained after taking closure, since Z is irreducible, parametrized by $\mathbb{P}^1$, and contains all members of $\mathcal{C}'$ and their periodic cycles.

Finally, let Z_t be the irreducible curve in $\mathbb{P}^2_{\mathbb{C}(t)}$ corresponding to Z, where $\mathbb{C}(t)$ is the function field of H_∞. For any $t_0 \in \mathbb{C}$ such that $Z_{t_0} \in \mathcal{C}'$, we have $F(Z_{t_0}) = Z_{f(t_0)}$, where

$f = F|_{H_\infty}$. As the set of such t_0 (corresponding to non-superattracting periodic points of f) is dense in $\mathbb{P}^1$, it follows that

$$F(Z_t) = Z_{f(t)} \quad \text{for all } t.$$

Consequently, for every curve C' with $[C'] \in Z$, we have $[F(C')] \in Z$, completing the proof.

□

Now, by Lemma 4.4.1, one of the following two situations occurs. Either we can construct a one-dimensional family of curves, parametrized by $\mathbb{P}^1$, that contains infinitely many periodic curves and whose generic member has stabilized degree; or there exists a point $p \in H_\infty$ through which infinitely many periodic curves pass.

It remains to handle the second case.

Proposition 4.4.2. *Let $\mathcal{C}$ be an infinite set of periodic curves under F in $\mathbb{P}^2$ of degree D passing through a point $p \in H_\infty$. Then there exists an infinite subset of curves $\mathcal{C}' \subseteq \mathcal{C}$ such that we have a parametrization $\tau : \mathbb{P}^1 \to Z$, where Z is the Zariski closure of $\mathcal{C}'$ in the space of effective divisors of degree $\leqslant D$ in $\mathbb{P}^2$. Moreover, after replacing F with a suitable iterate, for any curve C such that $[C] \in Z$, we have that $[F(C)] \in Z$ and, in particular, for a generic curve C' such that $[C'] \in Z$, we have* $\deg(F(C')) = \deg(C')$.

Proof. We first replace F with some iterate and so p is fixed under F. Moreover, by Lemma 4.3.4, it is a super-attracting fixed point under $F|_{H_\infty}$. By Lemma 4.3.3, there exists a blow up π_0 of length at most $D^2 + 1$ at p such that $\mathcal{C}$ intersects a divisor $E_0 \subseteq \pi_0^{-1}(p)$ at infinitely many points. This in particular implies that $F|_{E_0}$, the induced map of F on E after replacing F with some iterate, is a finite morphism as curves in $\mathcal{C}$ are periodic and so are their intersections with E_0. Then there exists an infinite subset $\mathcal{C}' \subseteq \mathcal{C}$ such that for every curve C in $\mathcal{C}'$, the intersection $C \cap E_0$ contains some non-superattracting periodic points of $F|_{E_0}$. This is because $F|_{E_0}$ is a finite endomorphism on E_0 and thus has only finitely many super-attracting periodic points.

Now, if there exists another point $p_1 \in \pi_0^{-1}(p) \cup H_\infty$ such that infinitely many curves from $\{\pi_0^{\#} C : C \in \mathcal{C}'\}$ pass through p_1. Then we again use Lemma 4.3.3 to obtain a blow up π_1 at p_1, so that there exists a $E_1 \subseteq \pi_1^{-1}(p_1)$ whose intersection with strict transformations of curves in $\mathcal{C}'$ at infinitely many distinct points. Similarly, shrink $\mathcal{C}'$ and abuse notation to also let $\mathcal{C}'$ denote the infinite subset of $\mathcal{C}'$ only containing those curves with strict transformations intersecting E_1 at some non-superattracting periodic points of $F|_{E_1}$, after replacing F with some iterate.

Now, we repeat this process if we can still find a point in $(\pi_1 \circ \pi_0)^{-1}(H_\infty)$ such that infinitely many strict transformations of curves in $\mathcal{C}'$ pass through it. Then, after the process stop, we obtain a sequence of blow-ups $\pi_0, \cdots, \pi_r$ and exceptional divisors $E_0, \ldots, E_r$, such that there exists an infinite set of periodic curves $\mathcal{C}'$ whose strict transformations only intersect E_i, $i \in \{1, 2, \ldots, r\}$, at non-superattracting periodic points of $F|_{E_i}$, where $r \in \mathbb{Z}^+$.

This process must terminate after at most D iterations, and $r + 1 \leqslant D$, as, denoting $\pi = \pi_r \circ \cdots \circ \pi_0$,

$$D \geqslant \deg(C) \geqslant (\pi^{\#}C, E_0 + \cdots + E_r) \geqslant r + 1,$$

for any $C \in \mathcal{C}'$. If the process did not stop, we would obtain $r + 1 = D + 1$, contradicting $\deg(C) \leqslant D$.

Hence, after finitely many steps, we obtain a blow-up π above H_∞ such that through any point of $\pi^{-1}(H_\infty)$ only finitely many strict transforms $\pi^{\#}C$ (with $C \in \mathcal{C}'$) pass. Excluding from $\mathcal{C}'$ those finitely many curves whose strict transforms meet $E \subseteq \pi^{-1}(H_\infty)$ at some superattracting periodic points, we may assume that for all $C \in \mathcal{C}'$ and all components $E \subseteq \pi^* H_\infty$, the intersection $\pi^{\#}C \cap E$ consists only of non-superattracting periodic points of $F|_E$. We further choose an infinite subset of $\mathcal{C}'$ so that the strict transformation of every curve in it intersects every exceptional divisor (including H_∞) at the same number of points. We abuse notation to still call it $\mathcal{C}'$.

Now, notice that

$$\deg(C) = (\pi^{\#}C, \pi^* H_\infty),$$

and by Lemma 4.3.4 we have every intersection between $\pi^{\#}C$ and every $E \subseteq \pi^* H_\infty$ is transverse (if non-empty) and so is the intersection between $\pi^{\#}(F(C))$ and E, since the intersection of $\pi^{\#}C$ and E's are non-superattracting periodic points. Notice that there is a one-to-one correspondence between them as they live in the same periodic cycles under $F|_E$. Thus,

$$(\pi^{\#}F(C), E) = (\pi^{\#}C, E),$$

for every exceptional divisor $E \subseteq \pi^* H_\infty$. Hence, we have

$$\deg(F(C)) = (\pi^{\#}F(C), \pi^* H_\infty) = (\pi^{\#}C, \pi^* H_\infty) = \deg(C).$$

From now on, fix $E := E_0$.

From now on, fix $E := E_0$ and by our construction the strict transformation of every curve in $\mathcal{C}'$ intersects E at $D' \leqslant D$ points. Define

$$\mathcal{C}_0 = \{F^n(C) : C \in \mathcal{C}',\ n \geqslant 0\}$$

and let Z' be the closure of $\mathcal{C}_0$ in $\mathcal{M}_D$.

Consider

$$W = \{(p, C) \in E \times Z' : p \in \pi^{\#}C \cap E\} \subseteq E \times Z'.$$

By Lemma 4.3.4, through each non-superattracting periodic point of $F|_E$ there passes a unique periodic curve (other than E itself) intersecting E transversely. Thus, the projection $\tau_1 : W \to E$ is generically one-to-one, hence birational, and therefore an isomorphism since $E \cong \mathbb{P}^1$. Composing with the projection $\tau_2 : W \to Z'$, we obtain

$$\tau = \tau_2 \circ \tau_1^{-1} : E \to Z',$$

a morphism of degree D', giving the desired parametrization.

We now show that for every C with $[C] \in Z'$, we have $[F(C)] \in Z'$, after replacing F with some iterate. View Z' as a curve

$$Z'_t \subseteq \mathbb{P}^2_{\mathbb{C}(E)} \cong \mathbb{P}^2_{\mathbb{C}(t)},$$

where $\mathbb{C}(E) \cong \mathbb{C}(t)$ is the function field of $E \cong \mathbb{P}^1$. The map F induces a morphism

$$\tilde{F} : \mathbb{P}^2_{\mathbb{C}(t)} \to \mathbb{P}^2_{\mathbb{C}(t)},$$

and since $\deg(F(C')) = \deg(C')$ for infinitely many $C' \in Z'$, we have

$$\deg(\tilde{F}(Z'_t)) = \deg(Z'_t).$$

For any $C_1 = Z'_{t_1} \in \mathcal{C}'$, where t_1 is a non-superattracting periodic point of $F|_E$, Lemma 4.3.4 and the construction of Z' ensures that

$$F(C_1) = Z'_{f(t_1)},$$

where $f = F|_E$. As such points t_1 form a Zariski dense subset of $E \cong \mathbb{P}^1$, we obtain

$$\tilde{F}(Z'_t) = Z'_{f(t)}.$$

Thus, for every C with $[C] \in Z'$, we have $[F(C)] \in Z'$.

Finally, since $Z \subseteq Z'$ and Z' is irreducible, we have $Z = Z'$. Therefore, for any C with $[C] \in Z$, we have $[F(C)] \in Z$, and for a generic C,

$$\deg(F(C)) = \deg(C).$$

□

We now summarize the results obtained for the case where F admits an infinite set of periodic curves.

Proposition 4.4.3. *Let F be a regular polynomial endomorphism of degree $d > 1$ admitting an infinite set $\mathcal{C}$ of periodic curves of degree at most D. Then there exists a parametrization*

$$\tau : \mathbb{P}^1 \longrightarrow Z,$$

where $Z \subseteq \mathcal{M}_D$ is the Zariski closure, in the space of effective divisors of degree at most D, of the effective divisors corresponding to an infinite subset of $\mathcal{C}$. Moreover, after replacing F by a suitable iterate, for any curve C and for a generic curve C' with $[C], [C'] \in Z$, we have

$$[F(C)] \in Z \quad \textit{and} \quad \deg(F(C')) = \deg(C').$$

Proof. By Lemma 4.4.1, either there exists an infinite subset of $\mathcal{C}$ consisting of curves all passing through a single point $p \in H_\infty$, or the statement already holds for an infinite subset of $\mathcal{C}$, in which case we are done.

The first case is covered by Proposition 4.4.2. □

We are now ready to prove Theorem 4.1.3.

Proof. Without loss of generality, we may assume that Z is irreducible. We first show that for any curve C with $[C] \in Z$, one has $[F(C)] \in Z$.

The pushforward map

$$F_* : \mathcal{M}_d \longrightarrow \mathcal{M}_{\deg(F)d}$$

is well defined. Define

$$\psi : \mathcal{M}_d \longrightarrow \mathcal{M}_{\deg(F)d}, \quad \psi(A) = \deg(F)\, A,$$

which is an isomorphism onto its image. Let $\mathcal{C}$ denote the Zariski dense set of periodic curves of F whose corresponding effective divisors lie in Z. By Proposition 4.4.3, after replacing F by an iterate, there exists an infinite subset $\mathcal{C}' \subseteq \mathcal{C}$ whose Zariski closure Z' satisfies $[F(C')] \in Z'$ for all $C' \in \mathcal{C}'$.

If $\mathcal{C}'$ is not Zariski dense in Z, then $\mathcal{C} \backslash \mathcal{C}'$ must still be infinite. We may then apply Proposition 4.4.3 again to $\mathcal{C} \backslash \mathcal{C}'$ and enlarge $\mathcal{C}'$ and Z', where Z' becomes a union of projective lines contained in the Zariski closure of $\mathcal{C}'$. By Zorn's Lemma, there exists a maximal

subset $\mathcal{C}'' \subseteq \mathcal{C}$ whose Zariski closure satisfies this invariance property. Our argument implies that $|\mathcal{C} \backslash \mathcal{C}''| < \infty$, since otherwise $\mathcal{C}''$ could be enlarged, contradicting maximality. We rename $\mathcal{C}''$ as $\mathcal{C}'$, noting that it remains Zariski dense in Z.

Define

$$G := \psi^{-1} \circ F_*.$$

This map is well defined on the set of effective divisors corresponding to $\mathcal{C}'$, a Zariski dense subset of Z, and its image also lies in Z. Since $\psi(\mathcal{M}_d)$ is closed in $\mathcal{M}_{\deg(F)d}$ and contains $\deg(F)\mathcal{C}'$, the composition $\psi^{-1} \circ F_*$ extends to an endomorphism on a closed subset of $\mathcal{M}_d$ containing $\mathcal{C}'$. Hence G extends to an endomorphism of Z, and we have $G(Z) = Z$. Consequently, for any curve C with $[C] \in Z$, one has $[F(C)] \in Z$.

Let $\tilde{Z}$ denote the curve in $\mathbb{P}^2_{\mathbb{C}(Z)}$ whose fibers over points of Z correspond to the curves in Z, where $\mathbb{C}(Z)$ is the function field of Z. Let $\tilde{F} : \mathbb{P}^2_{\mathbb{C}(Z)} \to \mathbb{P}^2_{\mathbb{C}(Z)}$ be the induced map of F. The discussion above shows that $\tilde{F}(\tilde{Z})$ is a curve in $\mathbb{P}^2_{\mathbb{C}(Z)}$ containing a Zariski dense set of fibers corresponding to curves in $\mathcal{C}'$. Hence

$$\deg(\tilde{F}(\tilde{Z})) = \deg(\tilde{Z}).$$

Therefore, for a generic curve C with $[C] \in Z$, we have

$$\deg(F(C)) = \deg(C),$$

completing the proof. □

In fact, a more general version of Theorem 4.1.3 is predicted by the Dynamical Manin–Mumford Conjecture:

Theorem 4.4.4 (Theorem 4.1.11)**.** *Let F be a surjective endomorphism of $\mathbb{P}^K$ of degree > 1. Let*

$$Z \subseteq Ch_{D,d}(\mathbb{P}^K)$$

be a subvariety of the Chow variety parameterizing algebraic cycles of $\mathbb{P}^K$ of dimension $D \leqslant K - 1$ and degree d. Suppose Z contains a Zariski dense set of algebraic cycles corresponding to irreducible periodic subvarieties of $\mathbb{P}^K$ under F of degree d.

Assuming (F, Z) is of general type and Conjecture 4.1.1, then for every subvariety $C \subseteq \mathbb{P}^K$ and a generic subvariety $C' \subseteq \mathbb{P}^K$ with $[C], [C'] \in Z$, we have

$$[F(C)] \in Z \quad \text{and} \quad \deg(F(C')) = \deg(C'),$$

after possibly replacing F by some iterate.

Proof. Without loss of generality, we may assume Z is irreducible, since the argument can be applied to each irreducible component separately. Set $N \coloneqq \dim(Z) + 1$. Inspired by the fiber products constructions used in [27], we consider the subvariety

$$Y \coloneqq \overline{\bigcup_{z \in Z} \{(x_1, \ldots, x_N) : x_i \in C \subseteq \mathbb{P}^K,\, [C] = z,\, i = 1, \ldots, N\}} \subseteq (\mathbb{P}^K)^N.$$

Since Z contains a Zariski dense set of periodic subvarieties under F, the set of preperiodic points of $F^{\times N}$ in Y is Zariski dense. By Conjecture 4.1.1, there exists a subvariety W invariant under $(F^{\times N})^l$ for some $l > 0$, containing Y and a polarizable endomorphism G on W such that

$$G \circ (F^{\times N})^l = (F^{\times N})^l \circ G \quad \text{on } W.$$

Moreover, Y is preperiodic under G.

Since Y contains every C^N whose class $[C]$ lies in Z, and since G becomes a split morphism after replacing it by a suitable iterate, we have

$$G_i \circ F^l = F^l \circ G_i \quad \text{on } W_i,$$

for each $i \in \{1, 2, \ldots, N\}$, where W_i denotes the projection of W onto the i-th factor of $(\mathbb{P}^M)^N$, and W_i contains C for every $[C] \in Z$.

By our assumption that (F, Z) is of general type, it follows that each G_i is an iterate of F, and therefore G itself is an iterate of $F^{\times N}$. Hence, Y is preperiodic under $F^{\times N}$.

Then, there exist $m, n \in \mathbb{Z}^+$ such that

$$(F^{\times N})^m(Y) \subseteq (F^{\times N})^n(Y).$$

Assuming $n > m$, it follows that

$$(F^{\times N})^{m+(n-m)k}(Y) \subseteq (F^{\times N})^m(Y) \quad \text{for all } k \in \mathbb{Z}^+.$$

Let $Z_0 \subseteq Z$ denote the set of cycles corresponding to the Zariski dense set of periodic subvarieties in Z. Then there exists some $j \in \{1, \ldots, n-m\}$ such that

$$\bigcup_{z \in Z_0'} \{(x_1, \ldots, x_N) \in C^N : [C] = z\} \subseteq (F^{\times N})^{m+j}(Y),$$

where $Z_0' \subseteq Z_0$ is a Zariski dense subset of Z. Note that

$$\overline{\bigcup_{z \in Z_0'} \{(x_1, \ldots, x_N) \in C^N : [C] = z\}} = Y,$$

hence

$$Y = (F^{\times N})^{m+j}(Y),$$

so Y is periodic under $F^{\times N}$.

Replacing F by F^{m+j}, we have

$$Y = \bigcup_{z \in Z} \{(x_1, \ldots, x_N) \in C^N : [C] = z\}.$$

Indeed, this follows because the union above equals

$$\pi(\mathcal{Z}),$$

where $\mathcal{Z}$ is the closed subvariety

$$\mathcal{Z} \coloneqq \{(z, x_1, \ldots, x_N) : z \in Z,\ (x_1, \ldots, x_N) \in C^N,\ [C] = z\} \subseteq Z \times (\mathbb{P}^K)^N,$$

and $\pi : Z \times (\mathbb{P}^K)^N \to (\mathbb{P}^K)^N$ is the natural projection, which is a closed map.

Now let $C \subseteq \mathbb{P}^K$ be any subvariety with $[C] \in Z$. We claim that

$$(F(C))^N \subseteq \bigcup_{z \in Z} \{(x_1, \ldots, x_N) \in C'^N : [C'] = z\}$$

implies $[F(C)] \in Z$.

We prove this claim by induction on $N \geqslant 1$. For the base case $N = 1$, we have $\dim(Z) = 0$, and Y is a finite union of subvarieties of $\mathbb{P}^K$ parameterized by Z. In this case, $F(C) \subseteq Y$ clearly implies $[F(C)] \in Z$.

Assume the statement holds for $N \leqslant M$ for some $M \geqslant 1$, and consider $N = M + 1$. There exists a point $p \in F(C)$ such that the subvariety

$$Z_1 \coloneqq \{z \in Z : p \in \operatorname{Supp}(z)\}$$

has $\dim(Z_1) < N-1$. Otherwise, if every C' with $[C'] \in Z$ contains $F(C)$, then $\dim(Z) = 0$, contradicting our assumption that Z parametrizes a Zariski dense family of irreducible subvarieties.

Hence,

$$\{p\} \times (F(C))^{N-1} \subseteq \bigcup_{z \in Z_1} \{(p, x_2, \ldots, x_N) : (x_2, \ldots, x_N) \in C'^{N-1}, [C'] = z\},$$

which implies

$$(F(C))^{N-1} \subseteq \bigcup_{z \in Z_1} \{(x_2, \ldots, x_N) : (x_2, \ldots, x_N) \in C'^{N-1}, [C'] = z\}.$$

By the induction hypothesis, we conclude that $[F(C)] \in Z$, proving the claim.

Since $(F(C))^N \subseteq Y$, the claim yields $[F(C)] \in Z$. Let $\mathbb{C}(Z)$ be the function field of Z, and let

$$\tilde{Z} \subseteq \mathbb{P}^K_{\mathbb{C}(Z)}$$

denote the subvariety corresponding to the generic fiber, so that for any $z \in Z$, we have $[\tilde{Z}_z] = z$. Denote by

$$\tilde{F} : \mathbb{P}^K_{\mathbb{C}(Z)} \to \mathbb{P}^K_{\mathbb{C}(Z)}$$

the map induced by F under base change. Then

$$[\tilde{F}(\tilde{Z}_z)] = [F(\tilde{Z}_z)] \in Z \quad \text{for all } z \in Z.$$

In particular, this implies $\deg(\tilde{F}(\tilde{Z})) = \deg(\tilde{Z})$, and hence for a generic $C' \subseteq \mathbb{P}^K$ with $[C'] \in Z$, we have

$$\deg(F(C')) = \deg(C').$$

□

4.5 Implications toward the Conjecture 4.1.2

In this section, we explore how our main result relates to the conjectures proposed in [27]. When we restrict to the case where the family

$$\Phi : S \times \mathbb{P}^2 \longrightarrow S \times \mathbb{P}^2$$

is constant with respect to the parameter space S, we show that our main theorem implies a weaker form of Conjecture 4.1.2, stated below as Corollary 4.1.5.

For the convenience of the reader, we restate Corollary 4.1.5 here.

Corollary 4.5.1 (Corollary 4.1.5)**.** *Let S be a smooth and irreducible quasi-projective variety defined over $\mathbb{C}$. Let $\Phi : S \times \mathbb{P}^2 \to S \times \mathbb{P}^2$ be a constant family of endomorphisms such that for every point $(s, p) \in S \times \mathbb{P}^2$*

$$\Phi(s, p) = (s, F(p))$$

for a regular polynomial endomorphism F of degree > 1 that does not depend on s. Let $\mathcal{X} \subseteq S \times \mathbb{P}^2$ be an irreducible hypersurface that is flat over S and projects dominantly to S. Suppose there exists a Zariski dense set of $s \in S$ such that $\mathcal{X}_s$, the fiber of $\mathcal{X}$ at s, is a periodic curve under F. Then for all $N \in \mathbb{Z}^+$, we have

$$\hat{T}_{\Phi^{\times N}}^{\wedge r_{\Phi^{\times N},\mathcal{X}^N}} \wedge [\mathcal{X}^N] \neq 0,$$

and $r_{\Phi^{\times N},\mathcal{X}^N} \leqslant \min\{2N, N + \dim S\}$.

Proof. There exists an irreducible subvariety

$$Z \subseteq \mathcal{M}_d,$$

where $\mathcal{M}_d$ denotes the space of effective divisors of degree at most $d > 0$ in $\mathbb{P}^2$, such that the induced map by the family $\mathcal{X}$

$$\sigma : S \to Z$$

is dominant. When no confusion arises, we will also regard $\mathcal{M}_d$ as the space of (possibly reducible) curves in $\mathbb{P}^2$, and similarly for $Z \subseteq \mathcal{M}_d$. By assumption, there exists a Zariski dense set of $z \in Z$ such that z is a periodic curve under F. Then Theorem 4.1.3 implies that, after replacing F by some iterate, $F(z) \in Z$ for all $z \in Z$.

Consider the subvariety

$$Y \coloneqq \overline{\{(p_1, \dots, p_N) \in (\mathbb{P}^2)^N : p_1, \dots, p_N \in \mathcal{X}_s \text{ for some } s \in S\}},$$

which is the image of the projection of the fiber product $\mathcal{X}^N$ onto $(\mathbb{P}^2)^N$. Then

$$\dim(Y) = r \leqslant \min\{2N, \dim(\mathcal{X}^N)\} \leqslant \min\{2N, N + \dim S\}.$$

We claim that Y is invariant under $F^{\times N}$. Indeed, for any $(p_1, \dots, p_N) \in (\mathcal{X}_s)^N \subseteq (\mathbb{P}^2)^N$, we have

$$(F(p_1), \dots, F(p_N)) \in F(\mathcal{X}_s)^N = F(\sigma(s))^N = z^N$$

for some $z \in Z$. Since $\sigma : S \to Z$ is dominant and Y is Zariski closed, we have $z^N \subseteq Y$, hence $F^{\times N}(Y) \subseteq Y$.

Applying the projection formula for

$$\tau_N : S \times (\mathbb{P}^2)^N \to (\mathbb{P}^2)^N,$$

we obtain

$$\hat{T}_{\Phi^{\times N}}^{\wedge r} \wedge [\mathcal{X}^N] = \tau_N^*(\hat{T}_{F^{\times N}}^{\wedge r}) \wedge [\mathcal{X}^N] = \hat{T}_{F^{\times N}}^{\wedge r} \wedge (\tau_N)_*[\mathcal{X}^N] \geqslant \hat{T}_{F^{\times N}}^{\wedge r} \wedge [Y].$$

Let $T = \left(\sum_{i=1}^{N} \pi_i^* \omega\right)^{\wedge r}$ be an r-current on $(\mathbb{P}^2)^N$, where π_i denotes projection onto the i-th factor and ω is the Fubini–Study form on $\mathbb{P}^2$. Then

$$\int_{(\mathbb{P}^2)^N} T \wedge [Y] > 0.$$

Let $d = \deg(F^{\times N})$. Observe that

$$\begin{aligned}\int_{(\mathbb{P}^2)^N} d^{-rn}(F^{\times N})^{n*} T \wedge [Y] &= \int_{(\mathbb{P}^2)^N} d^{-rn} T \wedge (F^{\times N})_*^n [Y] \\ &= \int_{(\mathbb{P}^2)^N} d^{-rn} T \wedge d^{rn} [Y] \\ &= \int_{(\mathbb{P}^2)^N} T \wedge [Y].\end{aligned}$$

Moreover, by the local uniform convergence of the potentials of $(F^{\times N})^{n*} T$ to those of $\hat{T}_{F^{\times N}}^{\wedge r}$ (see [24, Chapter III, Corollary 3.6]), we have

$$d^{-rn}(F^{\times N})^{n*} T \wedge [Y] \longrightarrow \hat{T}_{F^{\times N}}^{\wedge r} \wedge [Y] \quad \text{as } n \to \infty.$$

Hence,

$$\int_{(\mathbb{P}^2)^N} \hat{T}_{F^{\times N}}^{\wedge r} \wedge [Y] > 0,$$

and therefore,

$$\hat{T}_{F^{\times N}}^{\wedge r} \wedge [Y] > 0.$$

Finally, by definition, we have $r_{\Phi^{\times N}, \mathcal{X}^N} \leqslant r$ since $S \times Y$ is $\Phi^{\times N}$-special. Consequently,

$$\hat{T}_{\Phi^{\times N}}^{\wedge r_{\Phi^{\times N}, \mathcal{X}^N}} \wedge [\mathcal{X}^N] \neq 0.$$

□

Remark 4.5.2. The proof above in fact shows that we have established a version of the Dynamical Manin–Mumford Conjecture for $F^{\times N}$ and $Y \subseteq (\mathbb{P}^2)^N$. However, our result holds under the stronger assumption that Y arises from a family of curves containing a Zariski dense set of *periodic curves*, rather than merely requiring that Y contains a Zariski dense set of *preperiodic points*.

Corollary 4.5.3. *Under the same assumptions of Corollary 4.1.5, for any positive integer $N > 0$, we have $\mathcal{X}^N$ has codimension $\leqslant \dim S$ in a $\Phi^{\times N}$-special subvariety in $S \times (\mathbb{P}^2)^N$.*

Proof. This follows directly from Corollary 4.1.5, in the same way that [27, Conjecture 1.1] implies [27, Conjecture 1.2]. Since we have

$$r_{\Phi^{\times N},\mathcal{X}^N} \leqslant \min\{2N,\ N + \dim S\} \quad \text{and} \quad \dim \mathcal{X}^N = N + \dim S,$$

it follows that there exists a $\Phi^{\times N}$-special subvariety

$$Y \subseteq S \times (\mathbb{P}^2)^N$$

of dimension at most

$$\min\{2N + \dim S,\ N + 2\dim S\}$$

containing $\mathcal{X}^N$. Hence, $\mathcal{X}^N$ has codimension at most $\dim S$ within Y.

□

Corollary 4.1.5 establishes a special case of the following conjecture.

Conjecture 4.5.4. *Let S be a smooth and irreducible quasi-projective variety defined over $\mathbb{C}$, and let K be a positive integer. Let $\Phi : S \times \mathbb{P}^K \to S \times \mathbb{P}^K$ be a family of endomorphisms such that for every point $(s, p) \in S \times \mathbb{P}^K$,*

$$\Phi(s, p) = (s, F_s(p)),$$

where F_s is an endomorphism of $\mathbb{P}^K$ of degree greater than 1. Let $\mathcal{X} \subseteq S \times \mathbb{P}^K$ be an irreducible subvariety that projects dominantly onto S, is flat over S, and such that $\mathcal{X}_s$ is a preperiodic subvariety under F_s for a Zariski dense set of $s \in S$. Then for any positive integer N, we have

$$\hat{T}_{\Phi^{\times N}}^{r_{\Phi^{\times N},\mathcal{X}^N}} \wedge [\mathcal{X}^N] \neq 0.$$

We note explicitly that this conjecture is a direct consequence of Conjecture 4.1.2 by embedding $(\mathbb{P}^K)^N$ into some projective space $\mathbb{P}^M$ via the Segre embedding and viewing $\mathcal{X}^N$ as a subvariety of $\mathbb{P}^M$. This makes clear how our result connects to Conjecture 4.1.2 proposed in [27].

Lemma 4.5.5. *Conjecture 4.1.2 implies Conjecture 4.5.4*

Proof. Notice that we can view $\mathcal{X}^N$ as a subvariety of $S\times\mathbb{P}^M$ for some positive integer M via the Segre embedding. Under this embedding, $\Phi^{\times N}$ extends naturally to an endomorphism on $S \times \mathbb{P}^M$, since each F_s is polarizable for $s \in S$. Let

$$\iota : S \times (\mathbb{P}^K)^N \longrightarrow S \times \mathbb{P}^M$$

denote this embedding map, and let

$$\Psi : S \times \mathbb{P}^M \longrightarrow S \times \mathbb{P}^M$$

be the extended endomorphism induced by $\Phi^{\times N}$. Set $Y = \iota(\mathcal{X}^N)$. Then

$$\hat{T}_{\Psi}^{r_{\Psi,Y}} \wedge [Y] \neq 0 \quad \Longrightarrow \quad \hat{T}_{\Phi^{\times N}}^{r_{\Psi,Y}} \wedge [\mathcal{X}^N] \neq 0.$$

Observe that $S \times \iota((\mathbb{P}^K)^N) \subseteq S \times \mathbb{P}^M$ is clearly Ψ-special and contains Y, so that $r_{\Psi,Y} = r_{\Phi^{\times N},\mathcal{X}^N}$. Moreover, the assumption that for a Zariski dense set of $s \in S$, the fiber $\mathcal{X}_s$ is preperiodic under F_s implies that $\mathrm{Prep}(\Phi) \cap \mathcal{X}_s$ is Zariski dense in $\mathcal{X}_s$ for such s. Consequently, for those s, the fiber $(\mathcal{X}^N)_s$ contains a Zariski dense subset of $\mathrm{Prep}(\Phi_s^{\times N})$, and hence $\mathrm{Prep}(\Phi^{\times N}) \cap \mathcal{X}^N$ is Zariski dense in $\mathcal{X}^N$. Therefore, $\iota(\mathrm{Prep}(\Phi^{\times N}) \cap \mathcal{X}^N)$ is Zariski dense in Y, and Conjecture 4.1.2 directly implies the desired statement. □

4.6 Degree stabilization for families of endomorphisms

In this section, we collect some conditional results on degree stabilization allowing for a family of varying endomorphisms. We first show that Conjecture 4.1.2 implies the degree stabilization phenomenon in this setting of a family of endomorphisms.

Proposition 4.6.1. *Let S be a smooth and irreducible quasi-projective variety defined over $\mathbb{C}$ of positive dimension and let M be a positive integer. Let $\Phi : S \times \mathbb{P}^M \to S \times \mathbb{P}^M$ be a family of endomorphisms such that for every point $(s, p) \in S \times \mathbb{P}^M$*

$$\Phi(s, p) = (s, F_s(p))$$

for an endomorphism F_s of degree > 1. Let $\mathcal{X} \subseteq S \times \mathbb{P}^M$ be an irreducible subvariety of dimension $> \dim S$ that projects dominantly to S and is flat over S.

If there exist a positive integer N and a proper irreducible $\Phi^{\times N}$-preperiodic subvariety $\mathcal{Y} \subseteq S \times (\mathbb{P}^M)^N$ containing $(\Phi^{\times N})^l(\mathcal{X}^N)$ with some non-negative integer l as a codimension $\leqslant \min\{N-1, \dim S\}$ subvariety, then there exists positive integers n, m and an infinite set of positive integers I such that for any pair of integers $k_1, k_2 \in I$ we have that for a generic point $s \in S$,

$$\deg(F_s^{m+nk_1}(\mathcal{X}_s)) = \deg(F_s^{m+nk_2}(\mathcal{X}_s)).$$

Proof. By our assumption, we know that there exists positive integers n, m such that $(\Phi^{\times N})^m(\mathcal{Y}) = (\Phi^{\times N})^{m+nk}(\mathcal{Y})$ for all $k \in \mathbb{N}$.

We prove it inductively on N. The base case is $N = 1$. In this case we have $\Phi^l(\mathcal{X})$ itself is Φ-special and, by our assumption on Φ, this implies $\mathcal{X}$ is preperiodic under Φ. Therefore, for every $s \in S$, we have

$$F_s^m(\mathcal{X}_s) = F_s^{m+nk}(\mathcal{X}_s)$$

and in particular

$$\deg(F_s^m(\mathcal{X}_s)) = \deg(F_s^{m+nk}(\mathcal{X}_s)),$$

for all positive integer k.

Now, suppose $N \geqslant 2$ and assume that we have the statement proved for any $N' \leqslant N-1$. If the projection of $\mathcal{Y}$ to $S \times (\mathbb{P}^M)^{N-1}$, denote as $\pi(\mathcal{Y})$, contains $\mathcal{X}^{N-1}$ as a codimension $\leqslant \min\{N-2, \dim S\}$ subvariety, then we apply the induction hypothesis to conclude the proof.

Otherwise, we will have

$$\dim(\mathcal{X}^{N-1}) + \min\{N-1, \dim S\} \geqslant \dim(\pi(\mathcal{Y})) > \dim(\mathcal{X}^{N-1}) + \min\{N-2, \dim S\},$$

since $\dim(\mathcal{Y}) \leqslant \dim(\mathcal{X}^N) + \min\{N-1, \dim S\}$. By [50, Chapter III, Theorem 9.9 and Corollary 9.10], there exists a non-empty Zariski open subset U in $\pi(\mathcal{Y})$ such that $\deg(\mathcal{Y}_p)$ is constant with p varies in U, where $\mathcal{Y}_p$ is the fiber of $\mathcal{Y}$ over p.

Suppose there exists positive integers k_0, k_1 such that for every $k > k_0$, we have that $(\Phi^{\times(N-1)})^{m+nk_1k}(\mathcal{X}^{N-1}) \subseteq U^c$. Then there exists a closed subset

$$V \coloneqq \overline{\bigcup_{k \geqslant k_0} (\Phi^{\times(N-1)})^{m+nk_1k}(\mathcal{X}^{N-1})} \subseteq U^c$$

which is $\Phi^{\times(N-1)}$ preperiodic and contains $(\Phi^{\times(N-1)})^{l'}(\mathcal{X}^{N-1})$ for some positive integer l'. Then the induction hypothesis will conclude the proof as $\dim(V) < \dim(\pi(\mathcal{Y}))$.

Suppose such a pair of integers does not exist. This in particular implies that there is an infinite set of positive integers I such that for any $j \in I$, $(\Phi^{\times(N-1)})^{m+nj}(\mathcal{X}^{N-1})$ is not contained in U^c. Then for any pair of integers $k_3, k_4 \in I$, we have

$$(\Phi^{\times(N-1)})^{m+nk_i}(\mathcal{X}^{N-1})$$

is not contained in U^c, for $i \in \{3,4\}$. Since $U \cap (\Phi^{\times(N-1)})^{m+nk_i}(\mathcal{X}^{N-1})$ is a non-empty open subset in $(\Phi^{\times(N-1)})^{m+nk_i}(\mathcal{X}^{N-1})$ for $i \in \{3,4\}$, this implies that for a generic $s_0 \in S$, there exists

$$p_{k_i} \in U \cap (\Phi^{\times(N-1)})^{m+nk_i}(\mathcal{X}^{N-1})$$

such that $\pi_0(p_{k_i}) = s_0 \in S$ and

$$F_{s_0}^{m+nk_i}(\mathcal{X}_{s_0}) \subseteq \mathcal{Y}_{p_{k_i}},$$

with $i \in \{3,4\}$. Therefore, since $\dim(F_{s_0}^{m+nk_i}(\mathcal{X}_{s_0})) = \dim(\mathcal{Y}_{p_{k_i}})$, for $i \in \{3,4\}$, $\deg(\mathcal{Y}_{p_{k_3}}) = \deg(\mathcal{Y}_{p_{k_4}})$, we have

$$\deg(\mathcal{Y}_{p_{k_3}}) \geqslant \deg(F_{s_0}^{m+nk_3}(\mathcal{X}_{s_0})),$$
$$\deg(\mathcal{Y}_{p_{k_4}}) \geqslant \deg(F_{s_0}^{m+nk_4}(\mathcal{X}_{s_0})).$$

This implies that for all pairs of $k_3, k_4 \in I$, we have

$$\deg(F_{s_0}^{m+nk_3}(\mathcal{X}_{s_0})), \deg(F_{s_0}^{m+nk_4}(\mathcal{X}_{s_0})) \leqslant R$$

for a generic $s_0 \in S$ and a positive integer R only depending on U and $\mathcal{Y}$. Then, by the pigeonhole principle, there exists an infinite subset $I' \subseteq I$ such that for all pairs of $k_3, k_4 \in I'$, we have

$$\deg(F_{s_0}^{m+nk_3}(\mathcal{X}_{s_0})) = \deg(F_{s_0}^{m+nk_4}(\mathcal{X}_{s_0})) \leqslant R.$$

□

Proposition 4.6.2 (Proposition 4.1.12)**.** *Under the same assumptions as Proposition 4.6.1, suppose that for a Zariski dense subset of $s \in S$, the fiber $\mathcal{X}_s$ is preperiodic under Φ_s. Assume $(\Phi, \mathcal{X})$ is of general type. If Conjecture 4.1.2 holds, then there exist positive integers n, m and an infinite set of positive integers I such that for any pair of integers $k_1, k_2 \in I$, we have, for a generic point $s \in S$,*

$$\deg(F_s^{m+nk_1}(\mathcal{X}_s)) = \deg(F_s^{m+nk_2}(\mathcal{X}_s)).$$

Proof. Note

$$\hat{T}_{\Phi^{\times N}}^{r_{\Phi^{\times N},\mathcal{X}^N}} \wedge [\mathcal{X}^N] \neq 0,$$

implies $\dim(\mathcal{X}^N) \geqslant r_{\Phi^{\times N},\mathcal{X}^N}$, which implies the existence of a $\Phi^{\times N}$-special subvariety of dimension at most $\dim(\mathcal{X}^N) + \dim S$. Since we have already shown that Conjecture 4.1.2

implies Conjecture 4.5.4, our assumption implies that we can find a $N > \dim S$ such that there exists a proper $\Phi^{\times N}$-special subvariety $\mathcal{Y} \subseteq S \times (\mathbb{P}^M)^N$ of dimension at most

$$\dim(\mathcal{X}^N) + \dim S = N + 2\dim S < \dim(S \times (\mathbb{P}^M)^N),$$

containing $\mathcal{X}^N$. Since $\mathcal{X}^N$ is irreducible, there exists an irreducible component of $\mathcal{Y}$ containing $\mathcal{X}^N$; by abuse of notation, we denote this component again by $\mathcal{Y}$.

Note that $\mathcal{Y}$ being $\Phi^{\times N}$-special means that there exists a subvariety $\mathbf{Z} \subseteq (\mathbf{P}^M)^N$ containing $\mathbf{Y}$ which is invariant under a polarizable endomorphism $\mathbf{\Psi}$ on $(\mathbf{P}^M)^N$ and also under $(\mathbf{\Phi}^{\times N})^n$ for some $n > 0$, such that

$$\mathbf{\Psi} \circ (\mathbf{\Phi}^{\times N})^n = (\mathbf{\Phi}^{\times N})^n \circ \mathbf{\Psi} \quad \text{on } \mathbf{Z}.$$

Moreover, $\mathbf{Y}$ is preperiodic under $\mathbf{\Psi}$.

Since $\mathbf{\Psi}$ is polarizable, by replacing it with a suitable iterate, we may assume that $\mathbf{\Psi}$ is a split morphism. Because $\mathcal{Y}$ contains $\mathcal{X}^N$ and $\mathbf{Z}$ contains $\mathbf{Y}$, for each $i \in \{1, 2, \ldots, N\}$ our assumptions imply that

$$\mathbf{\Psi}_i \circ \mathbf{\Phi}^n = \mathbf{\Phi}^n \circ \mathbf{\Psi}_i \quad \text{on } \mathbf{Z}_i,$$

where $\mathbf{Z}_i$ is the projection of $\mathbf{Z}$ to the i-th factor of $(\mathbf{P}^M)^N$ and contains $\mathbf{X}$, and $\mathbf{\Psi}_i$ denotes the corresponding projection of $\mathbf{\Psi}$ to that factor.

By the assumption that $(\Phi, \mathcal{X})$ is of general type, it follows that $\mathbf{\Psi}$ must be an iterate of $\mathbf{\Phi}^{\times N}$. Consequently, $\mathbf{Y}$ is preperiodic under $\mathbf{\Phi}^{\times N}$, and hence $\mathcal{Y}$ is preperiodic under $\Phi^{\times N}$.

Then Proposition 4.6.1 applies, yielding the existence of positive integers n, m and an infinite set of positive integers I such that for any pair $k_1, k_2 \in I$, we have, for a generic point $s \in S$,

$$\deg(F_s^{m+nk_1}(\mathcal{X}_s)) = \deg(F_s^{m+nk_2}(\mathcal{X}_s)).$$

□

4.6.1 One-dimensional families of polynomial endomorphisms on $\mathbb{P}^2$

In this subsection, we verify the degree stabilization for one-dimensional families of regular polynomial endomorphisms under some ramification restriction on the line at infinity.

Suppose $S = \mathbb{A}^1$, and we view

$$\Phi : \mathbb{A}^1 \times \mathbb{P}^2 \longrightarrow \mathbb{A}^1 \times \mathbb{P}^2$$

as a family $F_t : \mathbb{P}^2_{\overline{\mathbb{C}(t)}} \to \mathbb{P}^2_{\overline{\mathbb{C}(t)}}$ parametrized by $t \in \mathbb{A}^1$. Then, the technique used in Section 3 can be applied to prove degree stabilization when, for each $a_i(t) \in \mathcal{X}_t \cap H_\infty$, the pair $(F_t|_{H_\infty}, a_i(t))$ does not share its forward orbit with a marked critical point.

Theorem 4.6.3. *Let F_t be a family of regular polynomial endomorphisms of $\mathbb{P}^2$ of degree > 1, parametrized by $t \in \mathbb{A}^1$, and let $\mathcal{X}_t \subset \mathbb{P}^2$ be a family of curves. Suppose there are infinitely many $t_0 \in \mathbb{A}^1_{\mathbb{C}}$ such that the fiber $\mathcal{X}_{t_0}$ is periodic under F_{t_0}, and set $f_t := F_t|_{H_\infty}$ satisfying:*

1. *f_t is a polynomial of degree > 1 over $\overline{\mathbb{C}(t)}$;*
2. *For any $a_i(t) \in \mathcal{X}_t \cap H_\infty$, there do not exist $n \in \mathbb{N}$ and a polynomial h_t sharing the same Julia set with f_t such that*

$$f_t^n(a_i(t)) = h_t(c(t)),$$

where $c(t)$ is a marked critical point of h_t, i.e., $(h_t)'(c(t)) \equiv 0$.

Then, for a generic $t_0 \in \mathbb{A}^1_{\mathbb{C}}$, we have

$$\deg(\mathcal{X}_{t_0}) = \deg(F_{t_0}^k(\mathcal{X}_{t_0}))$$

for all positive integers k.

Proof. We first show that under the above assumptions, for any $a(t) \in \mathcal{X}_t \cap H_\infty$, there are only finitely many $t_1 \in \mathbb{A}^1$ such that $a(t_1)$ lies in a superattracting periodic cycle of f_{t_1}.

Suppose, for contradiction, that there are infinitely many $t_1 \in \mathbb{A}^1$ and some $a(t) \in \mathcal{X}_t \cap H_\infty$ such that $a(t_1)$ is superattracting periodic under f_{t_1}. Since f_t has bounded degree, it has finitely many marked critical points. By the pigeonhole principle, there exists a marked critical point $c(t)$ such that

$$\{t_1 \in \mathbb{A}^1 : c(t_1) \in \mathrm{Per}(f_{t_1}),\, a(t_1) = f_{t_1}^{n_{t_1}}(c(t_1)),\, n_{t_1} \in \mathbb{N}\}$$

is infinite. Note that if $c(t)$ is passive, then there exists $m, n \in \mathbb{N}$ such that $f_t^n(c(t))$ is periodic of order m. Since there are infinitely many $t_1 \in \mathbb{A}^1$ such that $a(t_1)$ lives in the same periodic cycle of $f_{t_1}^n(c(t_1))$. By the pigeonhole principle, there exits a $n_1 \in \mathbb{N}$ such

that $f_{t_1}^{n_1}(c(t_1)) = a(t_1)$ for infinitely many $t_1 \in \mathbb{A}^1$. Hence, $f_t^{n_1}(c(t)) = a(t)$ and, also, $f_t^{n_1+1}(c(t)) = f_t(a(t))$. This is the second exceptional case in the condition and, hence, we rule out this case and assume $c(t)$ is active from now on.

Case 1: f_t is conjugate to a power map or Chebyshev polynomial. Then there exists $l_t \in \overline{\mathbb{C}(t)}[x]$ such that

$$l_t \circ S_d \circ l_t^{-1} = f_t,$$

where S_d is either a power map or Chebyshev polynomial of degree $d = \deg(f_t)$. The marked critical points are

$$\{l_t(c_i) : c_i \text{ is a critical point of } S_d\}.$$

Thus, for infinitely many t_1,

$$a(t_1) = f_{t_1}^{n_{t_1}}(c(t_1)) = l_{t_1} \circ S_d^{n_{t_1}}(c),$$

where c is a critical point of S_d. Since all critical points of S_d are preperiodic, this implies

$$a(t_0) = l_{t_0} \circ S_d^n(c)$$

for infinitely many $t_0 \in \mathbb{A}^1$, contradicting assumption (2).

Case 2: f_t is non-special. Then there exists a primitive polynomial g_t sharing the same Julia set as f_t over $\overline{\mathbb{C}(t)}$ such that

$$f_t = \sigma \circ g_t^k$$

for some $k \in \mathbb{N}$ and $\sigma \in \mathrm{Aut}(\mathbb{P}^1)$ preserving $J(f_t)$ [77]. Since f_t and g_t share the same preperiodic points, there exist infinitely many t_1 such that $a(t_1)$ and $c(t_1)$ are preperiodic under g_{t_1}. By [32, Theorem D], there exists $\sigma_1 \in \mathrm{Aut}(\mathbb{P}^1)$ preserving $J(g_t)$ and integers $m, n > 0$ such that

$$\sigma_1 \circ g_t^m(c(t)) = g_t^n(a(t)).$$

Applying $\sigma' \circ g_t^{k'}$ on both sides of the equation above with sufficiently large k' and some $\sigma' \in \mathrm{Aut}(\mathbb{P}^1)$ preserving $J(g_t)$ yields

$$\sigma_2 \circ g_t^{l_1}(c(t)) = f_t^{l_2}(a(t)),$$

with $l_1, l_2 > k$ and $\sigma_2 \in \mathrm{Aut}(\mathbb{P}^1)$ preserving $J(g_t)$. Since

$$g_t^{l_1} = g_t^{l_1-k} \circ \sigma^{-1} \circ f_t,$$

$c(t)$ is a marked critical point of $g_t^{l_1}$. Let $h_t = \sigma_2 \circ g_t^{l_1}$, then $c(t)$ is also a marked critical point of h_t, contradicting assumption (2).

Hence, each $a(t) \in \mathcal{X}_t \cap H_\infty$ lies in a superattracting periodic orbit for only finitely many t_1. Since there are infinitely many $t_2 \in \mathbb{A}^1$ such that $\mathcal{X}_{t_2}$ is periodic under F_{t_2}, every $a(t_2) \in \mathcal{X}_{t_2} \cap H_\infty$ is periodic under f_{t_2}.

By selecting an infinite subset of such t_2, we may assume each $a(t_3) \in \mathcal{X}_{t_3} \cap H_\infty$ is non-superattracting periodic. Then, by Lemma 4.3.4,

$$\deg(\mathcal{X}_{t_3}) = |\mathcal{X}_{t_3} \cap H_\infty| = |F_{t_3}^k(\mathcal{X}_{t_3}) \cap H_\infty| = \deg(F_{t_3}^k(\mathcal{X}_{t_3}))$$

for all $k \in \mathbb{Z}^+$. Since $F_t^k(\mathcal{X}_t)$ and $\mathcal{X}_t$ both form an algebraic family of curves in $\mathbb{P}^2$, it follows that for a generic $t_0 \in \mathbb{A}^1$,

$$\deg(F_{t_0}^k(\mathcal{X}_{t_0})) = \deg(\mathcal{X}_{t_0}).$$

□

The following conjecture is a special case of the conjecture stated explicitly in [25, Conjecture 6.1]:

Conjecture 4.6.4. *Let $f_t(x)$ be a non-isotrivial algebraic family of rational functions of degree > 1 defined over $\mathbb{C}$ parametrized by t varies in $\mathbb{A}^1_{\mathbb{C}}$ and $a(t)$, $b(t)$ are two marked points defined over $\overline{\mathbb{C}(t)}$. Suppose there are infinitely many $t_0 \in \mathbb{A}^1$ such that $a(t_0), b(t_0) \in \mathrm{Prep}(f_{t_0})$. Then*

$$\overline{\mathrm{Orb}_{(f_t,f_t)}((a(t),b(t)))} \subseteq (\mathbb{P}^1 \times \mathbb{P}^1)(\overline{\mathbb{C}(t)})$$

is proper Zariski closed.

Assuming this Conjecture, we can strengthen Theorem 4.6.3:

Theorem 4.6.5. *Assume Conjecture 4.6.4. Let F_t be a family of regular polynomial endomorphisms on $\mathbb{P}^2$ defined over $\mathbb{C}$ of degree greater than 1, parametrized by $t \in \mathbb{A}^1$, and let $\mathcal{X}_t$ be a family of curves in $\mathbb{P}^2$ defined over $\mathbb{C}$. Suppose there are infinitely many $t_0 \in \mathbb{P}^1$ such that $\mathcal{X}_{t_0}$ is periodic under F_{t_0}. Denote by $f_t := F_t|_{H_\infty}$ the restriction of F_t to the line at infinity. Then, for a generic set of parameters $t_0 \in \mathbb{A}^1_{\mathbb{C}}$, we have*

$$\deg(\mathcal{X}_{t_0}) = \deg(F_{t_0}^k(\mathcal{X}_{t_0}))$$

for all positive integers k, unless there exists a point $a_i(t) \in \mathcal{X}_t \cap H_\infty$, a pair of integers $n, m \geqslant 0$, a rational map $X_t : \mathbb{P}^1 \to \mathbb{P}^1$, and two rational functions $g_t, h_t \in \overline{\mathbb{C}(t)}[x]$ such that

1. *$f_t \circ X_t = X_t \circ g_t$;*

2. *g_t commutes with h_t and they share a common iterate;*

3. *$f_t^n(a_i(t)) = X_t \circ h_t^m(c(t))$, where $c(t)$ is a marked critical point of h_t^m.*

Proof. Suppose that for every $a(t) \in \mathcal{X}_t \cap H_\infty$, there exist only finitely many parameters $t_1 \in \mathbb{A}^1$ such that $a(t_1)$ lies in a superattracting periodic cycle of f_{t_1}. Then, by the same argument as in the proof of Theorem 4.6.3, we conclude that

$$\deg(F_{t_0}^k(\mathcal{X}_{t_0})) = \deg(\mathcal{X}_{t_0})$$

for a generic set of $t_0 \in \mathbb{A}^1$ and all $k > 0$.

Now suppose that f_t lies entirely in the Lattès locus. Since all periodic points of a Lattès map are repelling, it follows that for any $t_0 \in \mathbb{A}^1$ such that $\mathcal{X}_{t_0}$ is periodic under F_{t_0}, every $a(t_0) \in \mathcal{X}_{t_0} \cap H_\infty$ is a repelling periodic point. Hence this case is complete.

Assume next that f_t is not contained in the Lattès locus. Suppose there exist infinitely many parameters $t_1 \in \mathbb{A}^1$ and some $a(t) \in \mathcal{X}_t \cap H_\infty$ such that $a(t_1)$ is a superattracting periodic point of f_{t_1}. By the pigeonhole principle, there exists a marked critical point $c(t)$ of f_t such that

$$\{\, t_1 \in \mathbb{A}^1 : c(t_1) \in \mathrm{Per}(f_{t_1}),\ a(t_1) = f_{t_1}^{n_{t_1}}(c(t_1)),\ n_{t_1} \in \mathbb{N} \,\}$$

is an infinite set. Then, Conjecture 4.6.4 implies that there exists an invariant curve $V \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ under (f_t, f_t) such that $\mathrm{Orb}_{(f_t,f_t)}(a(t), c(t)) \subseteq V$.

If V is vertical or horizontal, then one of $a(t)$ or $c(t)$ is passive. Suppose $c(t)$ is passive. Then there exists $m \in \mathbb{N}$ such that $f_t^m(c(t)) = c(t)$. Since $a(t_1)$ and $c(t_1)$ lie in the same periodic cycle for infinitely many t_1, we must also have $f_t^m(a(t)) = a(t)$ and $f_t^n(a(t)) = c(t)$ for some $n \in \mathbb{N}$. Taking X_t to be the identity map and $g_t = h_t = f_t$, the statement follows. The same reasoning applies when $a(t)$ is passive.

Hence, we may assume both $a(t)$ and $c(t)$ are active, so V projects dominantly onto both coordinates. If f_t is conjugate to a power map or a Chebyshev map, then, as in the proof of Theorem 4.6.3, we again have $a(t) = f_t^m(c(t))$ for some $m \geqslant 0$, and the theorem holds.

Now assume that f_t is non-special. By [68, Theorem 4.15], there exist rational maps $X_t, g_t \in \overline{\mathbb{C}(t)}(x)$, where g_t is not a generalized Lattès map (in the sense of [67]), such that

$$f_t \circ X_t = X_t \circ g_t,$$

and an invariant curve W under (g_t, g_t) satisfying

$$V = (X_t, X_t)(W),$$

after replacing f_t and g_t by suitable iterates if necessary. Then, by [68, Theorem 1.2], there exist rational maps U_1, U_2, V_1, V_2 commuting with g_t such that

$$U_1 \circ V_1 = V_1 \circ U_1 = g_t^l, \quad U_2 \circ V_2 = V_2 \circ U_2 = g_t^l$$

for some positive integer l, and W is parametrized by $u \mapsto (U_1(u), U_2(u))$. Consequently, there exists $u(t) \in \mathbb{P}^1_{\overline{\mathbb{C}(t)}}$ such that

$$a(t) = X_t \circ U_1(u(t)), \quad c(t) = X_t \circ U_2(u(t)).$$

For any $n > 0$, we have

$$f_t^n(c(t)) = f_t^n \circ X_t \circ U_2(u(t)) = X_t \circ U_2 \circ g_t^n(u(t)).$$

Since $(f_t^n)'|_{c(t)} \equiv 0$, by the chain rule $u(t)$ is a marked critical point of $X_t \circ U_2 \circ g_t^n$. If $u(t)$ were preperiodic under g_t, then $c(t)$ would be preperiodic under f_t, contradicting the assumption that $c(t)$ is active. Hence $u(t) \notin \mathrm{Prep}(g_t)$. Because the set of critical points of $X_t \circ U_2$ is finite, there exists $s \in \mathbb{N}$ such that $u(t)$ is a marked critical point of g_t^s. That is, choosing s so that $g_t^s(u(t))$ avoids $\mathrm{Crit}(X_t \circ U_2)$ and $(X_t \circ U_2 \circ g_t^s)'|_{u(t)} = 0$, we obtain

$$f_t^s(a(t)) = X_t \circ U_1 \circ g_t^s(u(t)),$$

with $u(t)$ a marked critical point of g_t^s.

Finally, by [75], unless g_t is conjugate to a power map, a Chebyshev polynomial, or a Lattès map (in particular unless g_t is generalized Lattès), every rational function commuting with g_t shares a common iterate with g_t. Thus, U_1 shares a common iterate with g_t. Enlarging s if necessary so that $U_1^{m-1} = g_t^s$ for some $m > 0$, we conclude that $u(t)$ is a marked critical point of U_1^m and

$$f_t^s(a(t)) = X_t \circ U_1^m(u(t)).$$

This completes the proof. □

Remark 4.6.6. There is in fact additional information about X_t in Theorem 4.6.5. After introducing an appropriate orbifold structure on $\mathbb{P}^1$, the map X_t becomes a Galois covering of $\mathbb{P}^1$. We omit a detailed discussion of this aspect here and refer the reader to [68] for a comprehensive treatment of orbifold structures and the associated covering map notation.

Notice that if we restrict to a constant family of regular polynomial endomorphisms, then Theorem 4.1.3 establishes this degree stabilization statement under the assumption that $\mathcal{X}_s$ is a periodic curve under F for a dense subset of $s \in S$.

The following example shows that, in general, a family of curves can easily have degrees blow up under iteration by a family of regular polynomial endomorphisms.

Example 4.6.7. Let $L_s = V(y - sx)$ be a family of lines parametrized by $s \in \mathbb{C}$ and let $\Phi : \mathbb{A}^1 \times \mathbb{P}^2 \to \mathbb{A}^1 \times \mathbb{P}^2$ be an endomorphism given by

$$\Phi(s, p) = (s, F_s(p)),$$

for $s \in \mathbb{A}^1$ and $p \in \mathbb{P}^2$, where for a fixed $\mathbb{A}^2$ chart we have

$$F_s(x, y) = (x^2, y^2 + sx).$$

Then we obviously have that

$$F_s^n(L_s) \cap \mathbb{A}^2 = \{(u^{2^n}, s^{2^n}u^{2^n} + 2^{n-1}s^{2^n-1}u^{2^n-1} + o(u^{2^n-1})) : u \in \mathbb{C}\}.$$

For any $s \in \mathbb{C}^*$, suppose there exists a polynomial $P_n(x, y) \in \mathbb{C}[x, y]$ such that

$$V(P_n(x, y)) = \overline{F_s^n(L_s) \cap \mathbb{A}^2},$$

for a $n \in \mathbb{N}$. Then, after changing the coordinate by letting $z = x$ and $w = y - s^{2^n}x$, we have

$$\deg(P) \geqslant 2^n,$$

since $\deg_u(z) = 2^n$, $\deg_u(w) = 2^n - 1$ and

$$P_n(z(u), w(u)) = 0$$

for all $u \in \mathbb{C}$. Therefore, for any $s \in \mathbb{C}^*$, we have

$$\deg(F_s^n(L_s)) \geqslant 2^n > 1 = \deg(L_s)$$

for any positive integer n.

4.7 Endomorphisms with infinitely many periodic curves of bounded degree

In this section, let $F(x, y)$ denote a regular polynomial endomorphism of degree greater than 1. We apply Theorem 4.1.3 to classify regular polynomial endomorphisms that admit infinitely many periodic curves of bounded degree.

We adopt the notion of *k-webs* from [33], and for the reader's convenience we recall their definition following the same reference.

A *k-web* $\mathcal{W}$ on a smooth complex surface X is determined by a section

$$\omega \in H^0\big(X, \operatorname{Sym}^k \Omega^1_X \otimes \mathcal{N}_{\mathcal{W}}\big)$$

for some line bundle $\mathcal{N}_{\mathcal{W}}$ on X, satisfying the following conditions:

1. the zero set of ω has codimension 2 or is empty;
2. for any point p outside a proper hypersurface of X, after trivializing the line bundle $\mathcal{N}_{\mathcal{W}}$, the k-symmetric 1-form $\omega(p) \in \operatorname{Sym}^k \Omega^1_{X,p}$ can be written as the product of k pairwise distinct linear forms.

Two sections ω and ω' determine the same k-web if and only if they differ by multiplication by a global nowhere-vanishing holomorphic function.

Intuitively, a k-web $\mathcal{W}$ on X can be viewed as a multi-foliation consisting of k distinct local leaves through a generic point of X. At a generic point, $\mathcal{W}$ is the superposition of k pairwise transverse foliations $\mathcal{F}_1, \ldots, \mathcal{F}_k$, and we may write

$$\mathcal{W} = \mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_k.$$

An endomorphism F of X is said to *preserve* a k-web $\mathcal{W}$ if it maps each local leaf of $\mathcal{W}$ to a leaf of $\mathcal{W}$.

We now show that if F admits infinitely many periodic curves, then either there exists a preserved k-web with $k > 1$, or F preserves a pencil of curves.

Proposition 4.7.1. *Suppose that F admits an infinite set of irreducible periodic curves of degree bounded by some $D \in \mathbb{Z}^+$. Then either F preserves a k-web with $k > 1$, or F preserves a pencil of curves.*

Proof. Without loss of generality, we may assume that F admits an infinite set of irreducible periodic curves all of degree exactly D. By Proposition 4.4.3, there exists an irreducible subvariety $Z \subseteq \mathcal{M}_D$ parameterized by a morphism $\tau : \mathbb{P}^1 \to Z$, where Z is the closure in $\mathcal{M}_D$ of an infinite subset of irreducible periodic curves of degree D. Moreover, for any curve C with $[C] \in Z$, we have $[F(C)] \in Z$. For convenience, we also view $\mathcal{M}_D$ as the space of (possibly reducible) curves of degree at most D.

Our goal is to construct either a k-web or a pencil preserved by F.

We first show that the set

$$\mathcal{S} := \{p \in \mathbb{P}^2 : p \in C_1 \cap C_2,\ C_1 \neq C_2,\ C_1, C_2 \in Z\}$$

is either Zariski dense in $\mathbb{P}^2$ or finite.

Consider the space

$$W = \overline{\{(C_1, C_2, x) : C_1, C_2 \in Z,\ C_1 \neq C_2,\ x \in C_1 \cap C_2\}} \subseteq Z \times Z \times \mathbb{P}^2.$$

For any point $x \in \mathbb{P}^2$, let $Z_x := \{C \in Z : x \in C\}$. Let $\pi_3 : W \to \mathbb{P}^2$ be the projection onto the third factor. We have that $\pi_3^{-1}(x)$ contains $(Z_x \times Z_x)\backslash\Delta$ which is of dimension $2\dim(Z_x)$. Hence $\dim(\pi_3^{-1}(x))$ is either 0 or 2. As any two distinct curves in Z intersect in finitely many points bounded in terms of D, we have $\dim(W) = 2\dim(Z) = 2$, since $\dim(Z) = 1$. Hence,

$$\dim(\overline{\mathcal{S}}) = \dim(\pi_3(W)) \in \{0, 2\},$$

and thus $\mathcal{S}$ is either Zariski dense in $\mathbb{P}^2$ or finite.

Let us first suppose that $\mathcal{S}$ is finite. Define

$$\mathcal{O} := \bigcap_{C \in Z} C.$$

Since $\mathcal{O} \subseteq \mathcal{S}$, it is finite as well. Choose two sufficiently generic curves $C_1, C_2 \in Z$ such that

$$C_1 \cap C_2 = \mathcal{O}.$$

Then, for any sufficiently generic $C_3 \in Z$ (avoiding the points in $\mathcal{S}\backslash\mathcal{O}$), Bézout's Theorem gives

$$\sum_{p \in \mathcal{O}} (C_3, C_i)_p = D^2,$$

where $i \in \{1, 2\}$ and D is the common degree of the generic curves in Z. Moreover, for every $p \in \mathcal{O}$ we have

$$(C_1, C_3)_p = (C_2, C_3)_p,$$

assuming that C_1, C_2, and C_3 are chosen generically in Z.

We now show that each such C_3 can be written in the form

$$\lambda_0 C_1 + \mu_0 C_2$$

for some $[\lambda_0 : \mu_0] \in \mathbb{P}^1_{\mathbb{C}}$, so that C_3 lies in the pencil of curves generated by C_1 and C_2. Suppose, on the contrary, that C_3 is not of this form. Then, for a generic choice of $[\lambda_0 : \mu_0] \in \mathbb{P}^1_{\mathbb{C}}$, there exists a point $q \notin \mathcal{O}$ such that we have

$$q \in C_3 \cap (\lambda_0 C_1 + \mu_0 C_2).$$

In that case,

$$(C_3, \lambda_0 C_1 + \mu_0 C_2) = \sum_{p \in \mathcal{O}} (C_3, \lambda_0 C_1 + \mu_0 C_2)_p + (C_3, \lambda_0 C_1 + \mu_0 C_2)_q > D^2,$$

since for each $p \in \mathcal{O}$,

$$(C_3, \lambda_0 C_1 + \mu_0 C_2)_p \geqslant \min\{(C_3, C_1)_p, (C_3, C_2)_p\}.$$

This contradicts Bézout's theorem, which forces

$$C_3 = \lambda_0 C_1 + \mu_0 C_2.$$

Hence, Z is a pencil of curves in $\mathbb{P}^2$ preserved by F.

We now assume $\mathcal{S}$ is Zariski dense and construct a k-web, with $k \geqslant 2$, using the family of curves in Z. For each $s \in \mathbb{P}^1$, let $C_s = V(P_s(x, y, z))$ denote the corresponding curve, where $P_s(x, y, z)$ is a homogeneous polynomial varying algebraically with s. Define a differential form

$$\omega([x : y : z]) := \left(\prod_{s : P_s(x,y,z) = 0} dP_s(x, y, z) \right)^{1/e},$$

where $e = \deg(\tau)$. Then ω is a global section of $H^0(\mathbb{P}^2, \operatorname{Sym}^k \Omega_{\mathbb{P}^2})$, where k is the number of curves in Z passing through a general point of $\mathbb{P}^2$ (so that $ke = |\{s \in \mathbb{P}^1 : P_s(x, y, z) = 0\}|$ for general $[x : y : z]$).

We will construct from ω a section in

$$H^0(\mathbb{P}^2, \operatorname{Sym}^k \Omega_{\mathbb{P}^2} \otimes \mathcal{NW}),$$

for some line bundle $\mathcal{NW}$, defining a k-web preserved by F.

We first claim that the zero locus of ω has codimension at least 2, unless empty. Suppose instead that $\omega(p) = 0$ for infinitely many points $p \in \mathbb{P}^2$. If this set is Zariski dense, then $\omega \equiv 0$, which implies that $dP_s(x, y, z) = 0$ for all $[x : y : z] \in V(P_s)$ for infinitely many s. Hence, each such C_s would consist entirely of singular points, forcing it to be reducible. This contradicts the fact that Z is generically irreducible, being the closure of infinitely many irreducible periodic curves.

Next, suppose that the zero locus of ω contains finitely many irreducible curves $C_i = V(Q_i(x, y, z))$ for $1 \leqslant i \leqslant \ell$. Then for each i, there exists $m_i \geqslant 1$ such that $Q_i^{-m_i}\omega$ vanishes only at finitely many points of C_i. Define

$$\omega' = \omega \prod_{i=1}^{\ell} Q_i^{-m_i},$$

which can be viewed as a section of

$$H^0\left(\mathbb{P}^2,\, \mathrm{Sym}^k \Omega_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}\left(\sum_{i=1}^{\ell} m_i C_i\right)\right).$$

Since ω encodes the tangent directions of the family of curves Z preserved by F, both ω and ω' are preserved by F.

It remains to show that ω' can be locally written as a product of k distinct linear factors outside a proper Zariski closed subset of $\mathbb{P}^2$. Suppose the contrary: there exists a Zariski dense set of points p such that at p there are distinct curves $C_s, C_t \in Z$ tangent to each other at p.

Let $a_i(s) \in \mathbb{C}[s]$ denote the coefficients of $P_s(x, y, z)$ for $i = 1, \ldots, h$, for some $h \in \mathbb{N}^+$, and define

$$H(s, t) = \gcd\big\{a_i(s)a_j(t) - a_j(s)a_i(t) : 1 \leqslant i \neq j \leqslant h\big\}.$$

Write

$$\nabla P_s(x, y, z) = (\partial_x P_s, \partial_y P_s, \partial_z P_s), \quad (A_{s,t}, B_{s,t}, C_{s,t}) = \nabla P_s \times \nabla P_t,$$

so $A_{s,t}, B_{s,t}, C_{s,t} \in \mathbb{C}[s, t][x, y, z]$. Consider the generalized resultant

$$R([x : y : z]) = \mathrm{Res}_{s,t}\big((P_s - P_t)/H(s, t), P_s, A_{s,t}, B_{s,t}, C_{s,t}\big),$$

whose zero locus consists of points where two distinct curves in Z are tangent to each other. Our assumption implies that $R(p) = 0$ for a Zariski dense set of p, hence $R \equiv 0$. Therefore, at every $p \in \mathbb{P}^2$, there exist distinct $C_s, C_t \in Z$ tangent to each other at p.

In particular, this holds for points $p \in H_\infty$. If the intersections of the curves in $Z\backslash\{H_\infty\}$ with H_∞ form a finite set, we reach a contradiction, since periodic curves are dense in Z but would then meet H_∞ only finitely often, whereas $R \equiv 0$ forces every point of H_∞ to lie on at least two such curves. On the other hand, if these intersections are infinite, then they occur at infinitely many periodic points of $F|_{H_\infty}$. By Lemma 4.4.1, Z can be parameterized by a finite morphism $\tau' : H_\infty \to Z$, implying that there is a non-superattracting periodic point of $F|_{H_\infty}$ such that there is a unique curve in $Z\backslash\{H_\infty\}$ passing through it which is a periodic curve, and the intersection is transverse (Lemma 4.3.4). This again contradicts $R \equiv 0$, which would require tangency at this point. □

Now we have established that any polynomial endomorphism admitting infinitely many periodic curves of bounded degree must preserve either a k-web with $k > 1$, or a pencil of curves. Consequently, the classification of endomorphisms of $\mathbb{P}^2$ that preserve k-webs, as given in [33], yields a corresponding classification of polynomial endomorphisms that possess infinitely many periodic curves of bounded degree.

We begin by recalling the classification results from [33].

Theorem 4.7.2. *[33, Theorem C (2)] Suppose $\phi : \mathbb{P}^2 \to \mathbb{P}^2$ is a regular polynomial endomorphism extended from $\mathbb{A}^2$ that preserves a k-web $\mathcal{W}$. Then there exists a projective toric surface X, and a finite group G of automorphisms of X, such that the quotient space X/G is isomorphic to $\mathbb{P}^2$. The image of $(\mathbb{C}^*)^2$ in $\mathbb{P}^2$ is a Zariski open subset U which is totally invariant by ϕ. Moreover, one can find an integer $d \geqslant 2$, and a collection of complex numbers $\lambda_1, \ldots, \lambda_l$ such that the triple $(U, \phi^N, \mathcal{W})$ is the push forward under the natural quotient map of*

$$\left((\mathbb{C}^*)^2, (x^d, y^d), \left(\lambda_1 \frac{dx}{x} + \frac{dy}{y}\right) \boxtimes \cdots \boxtimes \left(\lambda_l \frac{dx}{x} + \frac{dy}{y}\right)\right),$$

with $N \in \{1, 2, 3\}$.

Theorem 4.7.3. *[33, Theorem E] Let $\phi : \mathbb{P}^2 \to \mathbb{P}^2$ be a holomorphic map of degree at least 2 preserving a 2-web $\mathcal{W}$. If ϕ does not preserve any other web then the pair $(\mathcal{W}, \phi)$ is one of the following:*

1. *$\mathcal{W}$ is the union of two pencil of lines, and in suitable homogeneous coordinates $\phi(x, y) = (P(x), Q(y))$ or $(Q(y), P(x))$ where P and Q are polynomials of the same degree which are not conjugated to monomial maps nor to Chebyshev maps;*
2. *$\mathcal{W}$ is the algebraic web dual to a smooth conic and ϕ is an Ueda map whose associated map on $\mathbb{P}^1$ is not a finite quotient of an affine map;*

3. *$\mathcal{W}$ is the union of the pencil of lines $\{x/y = cst\}$ and the pencil of curves $\{x^p y^q = cst\}$, with $p, q \in \mathbb{N}^*$, $\gcd\{p, q\} = 1$ and $\phi(x, y) = (x^d/R(x, y), y^d/R(x, y))$ where $R(x, y) = \prod_{i=1}^{l}(1+c_i x^p y^q)$, $l(p+q) \leqslant d$, $c_i \in \mathbb{C}^*$ and the rational map $\theta \to \theta^d/(\prod_{i=1}^{l}(1+c_i\theta))^{p+q}$ is conjugated neither to a monomial nor to a Chebyshev map;*

4. *$\mathcal{W}$ is the union of the pencil of lines $x/y = cst$ and the pencil of curves $xy = cst$, and $\phi(x, y) = (y^d/R(x, y), x^d/R(x, y))$ where $R(x, y) = \prod_{i=1}^{l}(1 + c_i xy)$, $2l \leqslant d$, $c_i \in \mathbb{C}^*$ and the rational map $\theta \to \theta^d/(\prod_{i=1}^{l}(1 + c_i\theta))^2$ is conjugated neither to a monomial nor to a Chebyshev map;*

5. *$\mathcal{W}$ is the quotient of the foliation $[dx]$ on $\mathbb{P}^1 \times \mathbb{P}^1$ by the action of $(\mathbb{Z}/2)^2$ generated by $(x, y) \to (-x, -y)$ and $(x, y) \to (y, x)$; and ϕ is the quotient of a polynomial endomorphism of $\mathbb{P}^1 \times \mathbb{P}^1$ of the form $(f(x), f(y))$ where f is an odd or even polynomial which is conjugated neither to a monomial nor to a Chebyshev map;*

6. *$\mathcal{W}$ is the quotient of the foliation $[dx]$ on $\mathbb{P}^1 \times \mathbb{P}^1$ by the action of $(\mathbb{Z}/2)^3$ generated by $(x, y) \to (y, x)$, $(x, y) \to (x^{-1}, y^{-1})$, and $(x, y) \to (-x, -y)$; and ϕ is the quotient of an endomorphism of $\mathbb{P}^1 \times \mathbb{P}^1$ of the form $(f(x), f(y))$ where f is a rational map that commutes with the group generated by $x \to -x$ and $x \to x^{-1}$ which is not a finite quotient of an affine map.*

Together, these two theorems provide a complete classification of regular polynomial endomorphisms that preserve a k-web with $k \geqslant 2$. The following theorem from [23] gives the classification of endomorphisms of $\mathbb{P}^2$ that preserve a pencil of curves.

Theorem 4.7.4. *[23] There are two types of irreducible pencils $\mathcal{P}$ invariant under an endomorphism F of $\mathbb{P}^2$.*

1. *$\mathcal{P}$ is an elementary pencil, that is, the pencil of lines through a point. Under a suitable homogeneous coordinates on $\mathbb{P}^2$, F takes the form $F[x : y : z] = [P(x, y) : Q(x, y) : R(x, y, z)]$.*

2. *$\mathcal{P}$ is a binomial pencil. Under a suitable homogeneous coordinates on $\mathbb{P}^2$, F takes the form $F[x : y : z] = [x^d : y^d : R]$ or $F[x : y : z] = [y^d : x^d : R]$ for $R(x, y, z) = z^{d-kl}\prod_{i=1}^{l}(z^k + c_i x^h y^{k-h})$, where $k \geqslant 2$, $0 \leqslant l \leqslant d/k$, $c_i \in \mathbb{C}^*$, $0 < h < k$ and $\gcd(h, k) = 1$.*

Since we have shown in Proposition 4.7.1 that a regular polynomial endomorphism admitting infinitely many periodic curves of bounded degrees must preserve either a k-web with $k \geqslant 2$ or a pencil of curves, Theorems 4.7.2, 4.7.3, and 4.7.4 together provide a complete classification of the possible polynomial endomorphisms.

It then follows directly that Theorem 4.1.7 gives a coarse summary of all possible situations.

Proof of Theorem 4.1.7. By Proposition 4.7.1, the map F must belong to one of the classes described in Theorems 4.7.3, 4.7.2, or 4.7.4.

Suppose first that F preserves a pencil of curves, so that F falls into one of the cases of Theorem 4.7.4. If it is in the first case, then there exists a pencil of lines passing through a single common point $p \in \mathbb{P}^2$ that are preserved by F. If p is not contained in the totally invariant line at infinity, then after a change of coordinates, we assume $p = (0, 0)$ and F preserves a pencil of lines through $(0, 0)$, which implies that F is homogeneous. If $p \in H_\infty$, after a suitable change of coordinates, we have the pencil of lines contains $\{V(x - k) : k \in \mathbb{C}\}$. This implies that it is a regular polynomial skew product as it sends vertical lines to vertical lines. If F falls in the second case of Theorem 4.7.4, it is again clear that F becomes a regular polynomial skew product after replacing F with an iterate.

Next, if F belongs to the class of Theorem 4.7.2, then there exists a generically finite rational map

$$\mu : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^2,$$

which restricts to a morphism on $(\mathbb{C}^*)^2$ arising from a finite quotient, such that

$$\mu \circ (x^d, y^d) = F \circ \mu.$$

Finally, if F is in one of the classes of Theorem 4.7.3, then in subcases (3) and (4), F becomes a regular polynomial skew product after replacing it with an iterate and choosing a suitable affine coordinate on some $\mathbb{A}^2 \subseteq \mathbb{P}^2$. In all other cases, after possibly taking a further iterate, there exists a finite morphism

$$\mu : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^2,$$

induced by a finite quotient, satisfying

$$\mu \circ (f, g) = F \circ \mu,$$

where (f, g) is a split endomorphism of $\mathbb{P}^1 \times \mathbb{P}^1$. In cases (1) and (5), both f and g are polynomials. In cases (2) and (6), we have

$$\mu \circ (f, f) = F \circ \mu$$

for some rational function f, and we now show that f must in fact be a polynomial after being replaced with a proper iterate.

Since F is a regular polynomial endomorphism, the line at infinity H_∞ is totally invariant under F. Let

$$E := \mu^{-1}(H_\infty) \subseteq \mathbb{P}^1 \times \mathbb{P}^1.$$

since H_∞ is totally invariant under F. This gives that E is totally invariant under (f, f). After replacing (f, f) and F with some iterate, we may assume that there exists an irreducible component of E that is totally invariant under (f, f). We abuse notation to still call it E.

If E is a vertical or horizontal line, then we conclude directly that f is a polynomial after being replaced with some iterate. Suppose now that E projects dominantly to both factors. Let p be a fixed point of f and let $L_p := V(x - p)$. Then

$$(f, f)|_{L_p}^{-1}(L_p \cap E) = L_p \cap (f, f)^{-1}(E) = L_p \cap E.$$

Therefore $L_p \cap E$ is a finite exceptional set of $(f, f)|_{L_p} = f$. Thus, f is a polynomial after an iteration.

□

Here we provide an example of a regular polynomial endomorphism of $\mathbb{P}^2$ admitting infinitely many periodic curves, corresponding to the third case of Theorem 4.1.7.

Example 4.7.5. Consider the regular polynomial endomorphism on $\mathbb{A}^2$

$$A_2(x, y) = (x^2 - y^2 - 2x,\ 2xy + 2y),$$

which arises as the *folding map* associated to the Lie algebra $\mathcal{A}_2$ (see [83] for details). It admits non-trivial commuting pairs; indeed, it commutes with

$$A_3(x, y) = (x^3 - 3xy^2 - 3x^2 - 3y^2 + 3,\ 3x^2y - y^3),$$

the folding map constructed from the Lie algebra $\mathcal{A}_3$.

Under the change of coordinates

$$z = x + iy, \qquad w = x - iy,$$

the map becomes

$$A_2(z, w) = (z^2 - 2w,\ w^2 - 2z),$$

and we will work with this form for convenience.

One can verify that for any $k \in \mathbb{C}^*$ that is periodic under $x \mapsto x^2$, the curve

$$V(y - kx - k^{-1} + k^2)$$

is periodic under A_2.

Moreover, define the generically finite rational map

$$\Phi(x, y) = (x + y + x^{-1}y^{-1},\ x^{-1} + y^{-1} + xy)$$

from $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^2$. Then the following commutative diagram holds:

$$\begin{array}{ccc} \mathbb{P}^1 \times \mathbb{P}^1 & \xrightarrow{(x^2,\, y^2)} & \mathbb{P}^1 \times \mathbb{P}^1 \\ \Phi\downarrow & & \downarrow\Phi \\ \mathbb{P}^2 & \xrightarrow{\quad A_2 \quad} & \mathbb{P}^2. \end{array}$$

For a generic point in $\mathbb{A}^2$, there are three curves in the family

$$\mathcal{C} = \{\, V(y - kx - k^{-1} + k^2) \mid k \in \mathbb{C}^* \,\}$$

passing through it. Notice that $\mathcal{C}$ is preserved by A_2, since

$$A_2\big(V(y - kx - k^{-1} + k^2)\big) = V(y - k^2x - k^{-2} + k^4).$$

Thus, $\mathcal{C}$ forms a 3-web preserved by A_2, represented by the global differential

$$\omega = \prod_{k:\, y - kx - k^{-1} + k^2 = 0} (-k\,\mathrm{d}x + \mathrm{d}y + (k^2 - k^{-1})\,\mathrm{d}z).$$

Substituting the symmetric functions of the roots in terms of x and y, this can be expressed as

$$\begin{aligned} \omega = {} & (\mathrm{d}y - y\,\mathrm{d}z)^3 + x\,(\mathrm{d}y - y\,\mathrm{d}z)^2(-\mathrm{d}x + x\,\mathrm{d}z) \\ & + y\,(\mathrm{d}y - y\,\mathrm{d}z)(-\mathrm{d}x + x\,\mathrm{d}z)^2 + (-\mathrm{d}x + x\,\mathrm{d}z)^3. \end{aligned}$$

Now, viewing Φ as a map from $(\mathbb{C}^*)^2$ to $\mathbb{P}^2$, we have

$$\Phi^{-1}\big(V(y - kx - k^{-1} + k^2)\big) = V(y - k^{-1}x) \cup V(x - k) \cup V(y - k),$$

and therefore

$$\Phi^*\omega = \mathcal{F}_1 \boxtimes \mathcal{F}_2 \boxtimes \mathcal{F}_3,$$

where $\mathcal{F}_1$, $\mathcal{F}_2$, and $\mathcal{F}_3$ are the foliations on $(\mathbb{C}^*)^2$ corresponding respectively to the families $V(y - k^{-1}x)$, $V(x - k)$, and $V(y - k)$ as k ranges over $\mathbb{C}^*$. This is precisely the phenomenon described in Theorem 4.7.2.

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