

In Search of a Time-Independent Reset Hamiltonian

by

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A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Master of Mathematics
in
Computer Science

Waterloo, Ontario, Canada, 2024

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

In this work, we consider the problem of simulating time-independent Lindblad evolution in continuous time using time-independent Schrödinger evolution. The task is distinct from the usual computational problem of outputting the final state of a time evolution, as we seek to have our simulation be close to the original evolution at all times between the start and end. This goal is broken down into two parts, this work focusing only on the first: approximating an instantaneous reset of a qubit with time-independent Schrödinger evolution and combining multiple time-independent approximations into a single time-independent evolution that inherits characteristics of its constituents.

We propose and analyze a family of time-independent Hamiltonians and initial ancilla states that approximate in continuous time an instantaneous reset of a single qubit with four parameters: state error, time to perform the reset, time before the reset in which the initial state is unchanged, and time after the reset in which the state remains reset. The specific time-independent Hamiltonians are constructed using the idea of interpolating a unitary that exchanges the information between the input register and a register initialized to the $|0\rangle$ state, but only after a set number of applications and then leaves the input register untouched for a set number of additional applications.

Acknowledgements

I would like to thank my advisor, the University of Waterloo, and the University of California, San Diego for guiding me on my wonderful journey of learning that so far culminates in this thesis.

Dedication

This thesis is dedicated to my amazing family, especially my parents and little brother, for all their loving support through thick and thin.

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Chapter 1

Motivation

In quantum physics, the simplest mathematical model to describe continuous time evolution of quantum systems is that of Schrödinger’s equation, in which all state evolution within the Hilbert space is guided by the action of a fixed Hermitian operator, the Hamiltonian. However, the fixed Hamiltonian model is not the only model of time evolution. In the realm of quantum computation, letting the Hamiltonian depend on time allows a sequence of independent unitary gates to be applied to an input state in succession rather than as a single, net unitary. Additionally, in the realm of open systems, time evolution usually involves Markovian dissipation, which has elicited the Lindblad master equation as a description of those open systems.

1.1 Models of evolution

As a basis for this work, we consider the question of how well one model of evolution may simulate another. However, our notion of simulation is different from the usual computational sense, as we would like to have our simulating evolution match the original *at all times between the start and finish times*, not just momentarily at the start time and at the finish time. To make our discussion of simulating models of evolution precise, we first define the notion of a parameterized channel, which is just a function mapping some input to a set of channels.

Definition 1.1.1. *Let S be some set and $\mathcal{X}$ and $\mathcal{Y}$ be Hilbert spaces. A “parameterized channel” from $\mathcal{X}$ to $\mathcal{Y}$ over the parameter space S is a function from S to channels from $\mathcal{X}$ to $\mathcal{Y}$. Notationally, for the parameterized channel Φ over S from $\mathcal{X}$ to $\mathcal{Y}$, any $s \in S$,*

and any density operator ρ on $\mathcal{X}$, we write $\Phi[s]$ to denote the specific channel from $\mathcal{X}$ to $\mathcal{Y}$ corresponding to s , and we write $\Phi[s](\rho)$ to denote the density operator on $\mathcal{Y}$ that results from applying $\Phi[s]$ to ρ . Furthermore, we may optionally write $\Phi[s]_{\mathcal{X} \rightarrow \mathcal{Y}}$ to specify that the channel's source is the register labeled $\mathcal{X}$ and that its destination is the register labeled $\mathcal{Y}$, or just $\Phi[s]_{\mathcal{X}}$ when $\mathcal{X} = \mathcal{Y}$ to specify that the source and destination registers are both the same register.

The above definition allows us to describe an evolution as particular type of parameterized channel, mapping time, a real number, to a channel.

Definition 1.1.2. *Let $\mathcal{X}$ be a Hilbert space. An “evolution” or “process” on $\mathcal{X}$ is any parameterized channel Φ over $\mathbb{R}$ from $\mathcal{X}$.*

Using the above definition, we may think of a “model of evolution” as simply any set of evolutions.

Also, we utilize parameterized channels to define some common notation that appears frequently throughout the rest of this work.

Definition 1.1.3. *Let $\mathcal{X}$ be a Hilbert space. The unitary parameterized channel is a parameterized channel from $\mathcal{X}$ to $\mathcal{X}$ over the parameter space of unitaries acting on $\mathcal{X}$ that is defined as simply evolving the state by the parameter unitary. Notationally, we omit a symbol for the parameterized channel itself for brevity, and denote the channel that applies the unitary U simply as $[U]$. In other words, for ρ being a density operator and U being a unitary,*

$$[U](\rho) = U\rho U^\dagger. \quad (1.1)$$

Definition 1.1.4. *Let $\mathcal{X}$ and $\mathcal{Y}$ be Hilbert spaces. The ancilla parameterized channel, denoted Ξ , is a parameterized channel from $\mathcal{X}$ to $\mathcal{X} \otimes \mathcal{Y}$ over the parameter space of density operators on $\mathcal{Y}$ that is defined as adding an ancilla register initialized to the parameter density operator. In other words, for ρ being a density operator on $\mathcal{X}$ and σ being a density operator on $\mathcal{Y}$,*

$$\Xi[\sigma](\rho) = \rho \otimes \sigma.$$

As the input register is usually implied and the input register is reused as an output register, we may abbreviate $\Xi[\sigma]_{\mathcal{X} \rightarrow \mathcal{X}\mathcal{Y}}$ simply as $\Xi[\sigma]_{\rightarrow \mathcal{Y}}$.

Though they are not parameterized, we also use tr to denote the trace channel and Γ to denote the reset channel.

Definition 1.1.5. The trace channel, denoted tr , is the channel that maps a square linear operator to the sum of its eigenvalues. For a density operator ρ acting on $\mathcal{X} \otimes \mathcal{Y}$, we write the partial trace of ρ over $\mathcal{X}$ as $(\text{tr}_{\mathcal{X}} \otimes I_{\mathcal{Y}})(\rho)$.

Definition 1.1.6. Let $\mathcal{X}$ be a Hilbert space with basis $B = \{|0\rangle, \dots, |\dim(\mathcal{X}) - 1\rangle\}$. The reset channel, denoted Γ is defined as the channel that replaces its input with $|0\rangle\langle 0|_{\mathcal{X}}$. As such, we may express Γ as

$$\Gamma = \Xi[|0\rangle\langle 0|_{\mathcal{X}}] \circ \text{tr} \quad (1.2)$$

or, equivalently,

$$\Gamma = (I_{\mathcal{X}} \otimes \text{tr}_{\mathcal{Y}}) \circ [\text{SWAP}] \circ \Xi[|0\rangle\langle 0|_{\mathcal{Y}}], \quad (1.3)$$

for an extra register $\mathcal{Y} = \mathcal{X}$. The equivalency may be seen as

$$\Xi[|0\rangle\langle 0|_{\mathcal{X}}] \circ \text{tr} = (\text{tr}_{\mathcal{X}} \otimes I_{\mathcal{Y}}) \circ \Xi[|0\rangle\langle 0|_{\mathcal{Y}}] \quad (1.4)$$

$$= (I_{\mathcal{X}} \otimes \text{tr}_{\mathcal{Y}}) \circ [\text{SWAP}] \circ \Xi[|0\rangle\langle 0|_{\mathcal{Y}}]. \quad (1.5)$$

Returning to evolutions, the models mentioned above may now be formally expressed. We consider the model of time-independent Schrödinger evolutions for the Hilbert space $\mathcal{X}$ as solutions to the time-independent von Neumann equation, the density operator generalization of the Schrödinger equation, for the Hamiltonian H being a Hermitian operator on $\mathcal{X}$ and ρ being a density operator,

$$\dot{\rho} = -i[H, \rho], \quad (1.6)$$

in which the commutator $[A, B]$ is defined as $AB - BA$. Notably, each time-independent Schrödinger evolution Φ has the form

$$\Phi[t] = [e^{-iHt}] \quad (1.7)$$

for H being the Hermitian operator from above. Conversely, conjugation by any unitary operator U can be expressed as a time-independent Schrödinger evolution for some fixed time t by picking some $H = \frac{1}{t}(i \ln U)$, for which $\ln U$ is multi-valued. By the Stinespring dilation theorem, it follows that any arbitrary channel Ψ may be expressed as

$$\Psi = (I \otimes \text{tr}) \circ \Phi[t] \circ \Xi[\rho] \quad (1.8)$$

for some time-independent Schrödinger evolution Φ , some fixed time t , and some ancilla state ρ .

Similarly, we consider the model of time-dependent Schrödinger evolutions for the Hilbert space $\mathcal{X}$ as the solutions to the time-dependent von Neumann equation,

$$\dot{\rho} = -i[H(t), \rho], \quad (1.9)$$

for H now being a function from the real numbers to the Hermitian operators acting on $\mathcal{X}$. Clearly, the model of time-dependent Schrödinger evolutions subsumes the model of time-independent Schrödinger evolutions, as a time-dependent H can be defined to be constant in t .

Lastly, the model of time-independent Lindblad processes for the Hilbert space $\mathcal{X}$ is given by the solutions to the Lindblad master equation,

$$\dot{\rho} = -i[H, \rho] + \sum_{j=1}^m \left(L_j \rho L_j^\dagger - \frac{1}{2} L_j^\dagger L_j \rho - \frac{1}{2} \rho L_j^\dagger L_j \right), \quad (1.10)$$

for H being a Hermitian operator and $\{L_j : j \in [m]\}$ being a set of linear “jump” operators [9][10]. Again, the model of time-independent Lindblad processes clearly subsumes the model of time-independent Schrödinger evolutions, as the jump operators may all be set to 0. Though a time-dependent generalization of Lindblad processes exists, it is not relevant to this work.

We state that an evolution Φ is simulated for time t_f by another evolution Ψ on a larger Hilbert space with initial ancilla ρ if evolving by Ψ for any $t \in [0, t_f]$ with an ancilla initialized to ρ has the same effect on the main system register as evolving by Φ for the same amount of time. Note that this notion of simulation is distinct from the usual computational notion of simulation that is only concerned with the closeness of the final result, as we care about the intermediate times over the entire course of the evolution.

Definition 1.1.7. *An evolution Φ on Hilbert space $\mathcal{X}$ is said to be simulated by another evolution Ψ on Hilbert space $\mathcal{X} \otimes \mathcal{Y}$ with ancilla state ρ on $\mathcal{Y}$ for time $t_f \in \mathbb{R}$ with error $\epsilon \geq 0$ if, for any time $t \in [0, t_f]$,*

$$\|\Phi[t] - (I \otimes \text{tr}) \circ \Psi[t] \circ \Xi[\sigma]\|_\diamond \leq \epsilon, \quad (1.11)$$

in which $\|\cdot\|_\diamond$ denotes the completely bounded trace norm on quantum channels.

With this definition, we may also state that an entire model of evolution simulates a single target evolution Φ if the model contains any evolution that simulates Φ .

1.2 Simulation of Lindblad evolution

Based on their definitions, both the time-dependent Schrödinger model and the time-independent Lindblad model completely subsume the time-independent Schrödinger model, so it naturally follows that both models may simulate all time-independent Schrödinger evolutions with no error and infinite duration. On the other hand, time-independent Schrödinger evolutions must satisfy the Poincaré recurrence theorem, in that, for any error $\epsilon > 0$, there exists a time $t > 0$ such that evolution for time t returns the state to being within ϵ of the initial state. Time-dependent Schrödinger evolutions and time-independent Lindblad evolutions in general do not satisfy the Poincaré recurrence theorem, as the time-dependent Hamiltonian that flips a qubit just once before turning off and the Lindblad amplitude damping process are counterexamples [9][10]. Thus, the time-independent model cannot simulate all time-dependent or Lindblad evolutions with no error and infinite duration. In this work, we focus on whether or not time-independent Lindblad evolution may be simulated by time-independent Schrödinger evolution for arbitrarily small error and arbitrary duration.

The task of efficiently producing the final state of time-independent Schrödinger evolution using quantum algorithms, or, equivalently, the time-dependent Schrödinger model, has been well studied [7][8][1][2][3][4]. More recently, it has been shown that quantum algorithms can also be utilized to efficiently output the final state of an arbitrary time-independent Lindblad process for an arbitrary evolution time and arbitrarily small error [5][6].

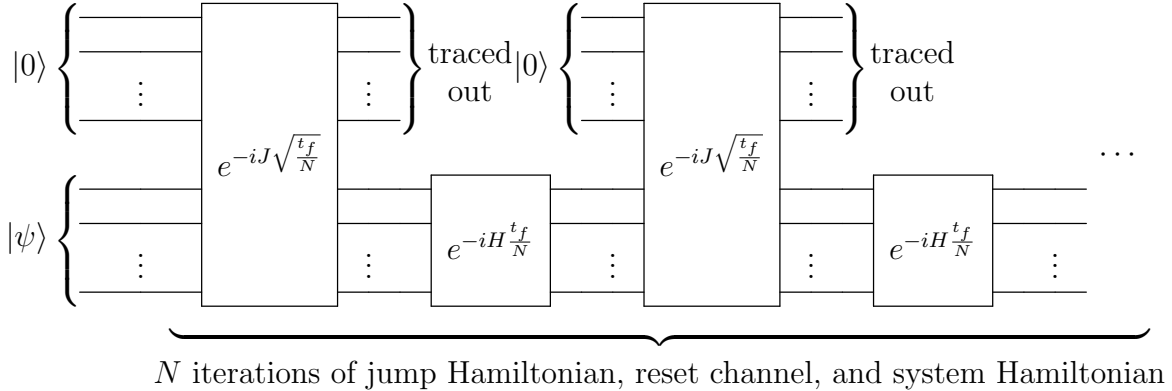


Figure 1.1: Circuit diagram simulating Lindblad evolution.

Though the task of outputting just the final state generally ignores intermediate times

and thus differs from the focus of this work, [6] does include a simple, though inefficient, method of simulating a time-independent Lindblad process via time-dependent Schrödinger evolution that does respect intermediate times. For a given time-independent Lindblad evolution Φ with Hamiltonian H and jump operators $L_1, \dots, L_m$ and an arbitrary time $t_f \in \mathbb{R}$, $\Phi[t_f]$ may be approximated by repeatedly interleaving the three channels

$$\left[e^{-iJ\sqrt{\frac{t_f}{N}}} \right], \quad \Gamma \otimes I, \quad I \otimes \left[e^{-iH\frac{t_f}{N}} \right], \quad (1.12)$$

as depicted in Figure 1.1, for which J is a Hermitian block matrix defined from the jump operators as

$$J = \begin{bmatrix} 0 & L_1^\dagger & \dots & L_m^\dagger \\ L_1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ L_m & 0 & \dots & 0 \end{bmatrix}. \quad (1.13)$$

Specifically, with

$$\Psi_N = \left(I \otimes \left[e^{-iH\frac{t_f}{N}} \right] \right) \circ (\Gamma \otimes I) \circ \left[e^{-iJ\sqrt{\frac{t_f}{N}}} \right], \quad (1.14)$$

it is the case that the channel

$$(\Psi_N)^N$$

converges to $\Phi[t_f]$ in the limit as $N \rightarrow \infty$. Moreover, fixing some $t < t_f$, a prefix of $k = \left\lceil \frac{tN}{t_f} \right\rceil$ steps of the above channel, given by the channel

$$(\Psi_N)^k,$$

approximates $\Phi[t]$ in the limit as $N \rightarrow \infty$ [6][10].

So far, our discussion of simulating time-independent Lindblad evolutions has been limited to discrete applications of unitary gates and quantum channels. However, in the world of continuous time evolution, the approach above suggests that we might simulate the entirety of Φ by applying Ψ_N instantaneously N equally spaced times in sequence, so long as N is sufficiently large. We can imagine this evolution as simply assigning the time $\frac{t_f k}{N}$ to the k th group of three operations in Figure 1.1, so that the state at time t can

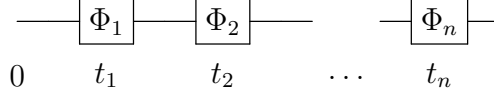


Figure 1.2: Illustration of Υ_s with s from (1.16) as a timed circuit of channels. The state at time t is given by applying all of the gates with times less than t starting from Φ_1 .

be determined by applying all of the operations whose times are less than t sequentially. Then, fixing some $\epsilon > 0$, as long as

$$\left\| \Phi \left[\frac{t_f k}{N} \right] - \Phi[t] \right\|_{\diamond} \leq \epsilon \quad (1.15)$$

for any $k \leq N$ and any $t \in \left(\frac{t_f k}{N}, \frac{t_f (k+1)}{N} \right)$, we may simulate Φ for time t_f with error ϵ by repeatedly instantaneously applying Ψ_N at N equally spaced points in time between 0 and t_f . To formalize this last notion of a timed circuit or an evolution that consists of a sequence of channels each occurring instantaneously at a specific time, we define the following parameterized channel:

Definition 1.2.1. *Let $\mathcal{X}$ be a Hilbert space and $C(\mathcal{X})$ be the set of channels acting on density operators on $\mathcal{X}$. Let the channel sequence*

$$s = ((\Phi_1, t_1), \dots, (\Phi_n, t_n)) \in (C(\mathcal{X}), \mathbb{R})^n \quad (1.16)$$

for some $n \in \mathbb{N}$ with $t_1 < \dots < t_n$ denote a sequence of pairs of channels and times at which the corresponding channel is instantly applied. The sequence of channels evolution for s , denoted Υ_s , is an evolution and parameterized channel from $\mathcal{X}$ to $\mathcal{X}$ over the parameter space $\mathbb{R}$. Then, for $t \in \mathbb{R}$, Υ_s is defined as follows:

$$\Upsilon_s[t] = \begin{cases} I & t < t_1 \\ \Phi_k \circ \dots \circ \Phi_1 & t_k \leq t \leq t_{k+1}, \quad k < n \\ \Phi_n \circ \dots \circ \Phi_1 & t \geq t_n \end{cases} \quad (1.17)$$

The above evolution is depicted in [Figure 1.2](#).

Equipped with the above definition and channel sequence

$$s = \left(\left(\Psi_N, \frac{t_f}{N} \right), \dots, (\Psi_N, t_f) \right), \quad (1.18)$$

constructed by assigning the times $\frac{t_f k}{N}$ for $k \leq N$ to each group of three operations in [Figure 1.1](#), we may formally state that as long as the condition from [\(1.15\)](#) holds,

$$\|\Phi[t] - \Upsilon_s[t]\|_{\diamond} \leq \epsilon \quad (1.19)$$

for all $t \in [0, t_f]$.

Even at this point, we are still outside the realm of time-independent Schrödinger evolution, as our simulation of Φ relies on instantaneous applications of Ψ_N . In particular, we still need to find some time-independent Hamiltonian and initial ancilla state whose evolution approximates the instantaneous applications of Ψ_N as defined in Υ_s . However, instead of tackling the problem of approximating all of the applications of Ψ_N in Υ_s directly, or even approximating a single instantaneous application of Ψ_N , this work concerns itself with approximating just the reset step Γ from Ψ_N . Whether or not such a time-independent Schrödinger simulation of just Γ can be composed with approximations of other steps into a single Hamiltonian and single ancilla that simulate Φ is left for discussion in [section 4.2](#).

1.3 The reset problem

As we are now solely interested in the reset step of Lindblad simulation, we will focus on finding a time-independent Hamiltonian and an initial ancilla state such that their evolution reasonably approximates $\Upsilon_{(\Gamma, t_f)}$ for an arbitrary time t_f and an arbitrary additional duration after t_f . Unfortunately, we cannot reuse our definition of simulation for evolutions from [Definition 1.1.7](#), as $\Upsilon_{(\Gamma, t_f)}$ has a discontinuity at time t_f when Γ is applied instantaneously while any time-independent Schrödinger evolution can be expressed as a continuous analytic function. Consequently, the requirement in [Definition 1.1.7](#) that the channel distance is no more than ϵ for all times cannot possibly be satisfied for small ϵ around t_f . As such, we define the property of a Hamiltonian and an initial ancilla state performing an (ϵ, δ) -approximation of a channel Φ with time t_b denoting the “before time” and t_a denoting the “after time” as evolution by the Hamiltonian and ancilla leaving the input state unchanged within error ϵ until time $t_b - \delta$, applying Φ to the input state in time 2δ , and finally returning to leaving the state unchanged within error ϵ for time $t_a - \delta$.

Definition 1.3.1. *Let $\mathcal{X}$ and $\mathcal{Y}$ be Hilbert spaces. A Hamiltonian H acting on Hilbert space $\mathcal{X} \otimes \mathcal{Y}$ and an initial state ρ on $\mathcal{Y}$ are said to perform an (ϵ, δ) -approximation of a channel Φ on $\mathcal{X}$ with time $t_b > 0$ before and time $t_a > 0$ after if the following two properties hold:*

1. For any $t \in [0, t_b - \delta]$,

$$\left\| (I_{\mathcal{X}} \otimes \text{tr}_{\mathcal{Y}}) \circ [e^{-iHt}] \circ \Xi[\rho]_{\rightarrow \mathcal{Y}} - I \right\|_{\diamond} \leq \epsilon. \quad (1.20)$$

2. For any $t \in [t_b + \delta, t_b + t_a]$,

$$\left\| (I_{\mathcal{X}} \otimes \text{tr}_{\mathcal{Y}}) \circ [e^{-iHt}] \circ \Xi[\rho]_{\rightarrow \mathcal{Y}} - \Phi \right\|_{\diamond} \leq \epsilon. \quad (1.21)$$

Furthermore, for $s = (\Phi, t_b)$, H and ρ are also said to perform an (ϵ, δ) -approximation of Υ_s with total duration $t_b + t_a$.

Chapter 2 proposes a family of Hamiltonians and ancillas that seem to satisfy the above (ϵ, δ) -approximation property for Γ with ϵ and δ being arbitrarily small, and Chapter 3 provides a rigorous analysis of those Hamiltonian and ancilla pairs.

Chapter 2

The Reset Hamiltonian

As motivated by the previous chapter, we would like to construct a time-independent Hamiltonian and initial state that perform an (ϵ, δ) -approximate reset for an arbitrary ϵ and δ . Unfortunately, there exists no such pair that performs a reset for $\epsilon = 0$ [11].

Theorem 2.0.1. *There exists no Hamiltonian and initial ancilla state that perform an (ϵ, δ) -approximation of Γ for $\epsilon = 0$ and $\delta < \min(t_b, t_a)$.*

Proof. Suppose that, for input register $\mathcal{S}$, such a Hamiltonian and initial state exist, denoted H and ρ , respectively, with ρ acting on ancilla register $\mathcal{E}$ and H acting on $\mathcal{S} \otimes \mathcal{E}$. Based on the first property of approximate reset, let the function $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined for any $t \in [0, t_b + t_a]$ as

$$f(t) = \|(I_{\mathcal{S}} \otimes \text{tr}_{\mathcal{E}}) \circ [e^{-iHt}] \circ \Xi[\rho]_{\rightarrow \mathcal{E}} - I\|_{\diamond}. \quad (2.1)$$

As f is a composition of analytic functions, f is also analytic. Furthermore, by definition of (ϵ, δ) -approximate reset, $f(t) = 0$ for $t \in [0, t_b - \delta]$. As $\delta < t_b$, f is constant on an interval of non-zero length. As f is analytic, f must be constant for all $t \in [0, t_b + t_a]$. It follows that

$$(I_{\mathcal{S}} \otimes \text{tr}_{\mathcal{E}}) \circ [e^{-iHt}] \circ \Xi[\rho]_{\rightarrow \mathcal{E}} = I \quad (2.2)$$

for all $t \in [0, t_b + t_a]$. However, using similar reasoning with the second property of approximate reset, we also obtain that

$$(I_{\mathcal{S}} \otimes \text{tr}_{\mathcal{E}}) \circ [e^{-iHt}] \circ \Xi[\rho]_{\rightarrow \mathcal{E}} = \Gamma \quad (2.3)$$

for all $t \in [0, t_b + t_a]$. Clearly, $\Gamma \neq I$, so contradiction proves that the initial supposition is false. $\square$

As no $\epsilon = 0$ reset can be performed by time-independent time evolution, this work proposes a construction of a Hamiltonian and initial state that perform an (ϵ, δ) -approximate reset for an arbitrarily small $\epsilon > 0$ and $\delta > 0$. Furthermore, for simplicity, we fix the time before and time after as $t_b = t_a = \frac{1}{4}$ from [Definition 1.3.1](#). In general, these parameters can be chosen arbitrarily with a few adjustments, but that case is left for the rigorous analysis in chapter 3.

2.1 Simple construction

Before tackling the more general construction that does nothing, performs the reset, and returns to doing nothing, we first consider the task of simulating the reset for just a single instant in time. In the Hilbert space of just two qubit registers, labeled $\mathcal{S}$ for the system qubit and $\mathcal{E}$ for the environment qubit, the most natural approach to simulating a reset for one instant is to initialize the ancilla to $|0\rangle$, evolve both registers according to some Hamiltonian that generates the SWAP operator, and then trace the environment out.

However, instead of considering SWAP, we choose a slight modification to SWAP that we will later generalize to perform resets that last more than just a single instant. Specifically, we define the SWAP-and- X unitary as follows:

Definition 2.1.1. *The unitary operator U_{sx} is defined as*

$$U_{\text{sx}} = (I \otimes X) \text{SWAP} . \quad (2.4)$$

Naturally, U_{sx} can be used in exactly the same way as SWAP to perform an instantaneous reset, as applying a Pauli X operator to a register right before tracing it out is equivalent to tracing out the register immediately. That said, unlike SWAP, U_{sx} has the special property that it is a cyclic permutation unitary, which is defined as follows:

Definition 2.1.2. *Let a cyclic permutation unitary for dimension $N \in \mathbb{N}$ be any unitary operator U that satisfies:*

1. *U permutes the computational basis states $\{|0\rangle, \dots, |N-1\rangle\}$.*
2. *For every $n \in \mathbb{Z}_N$, there exists an $m \in \mathbb{Z}_N$ such that $|n\rangle = U^m |0\rangle$.*

We may verify that $|00\rangle = U_{\text{sx}}^0 |00\rangle$, $|01\rangle = U_{\text{sx}}^1 |00\rangle$, $|10\rangle = U_{\text{sx}}^3 |00\rangle$, and $|11\rangle = U_{\text{sx}}^2 |00\rangle$, so U_{sx} is indeed a cyclic permutation operator.

$\mathcal{S}$	$\mathcal{E}$
0	0
0	1
1	1
1	0

Figure 2.1: Table form of U_{sx} . One application of U_{sx} advances each basis state to the basis state below it, wrapping from the bottom to the top.

One useful property of cyclic permutation unitaries is that such a unitary can be simply represented as an ordered table of computational basis states. The table is ordered such that one application of the unitary takes each basis state to the one below it, wrapping from the last basis state to the first. For instance, our U_{sx} unitary can be represented by the table in [Figure 2.1](#) acting on systems $\mathcal{S}$ and $\mathcal{E}$.

From the table it is simple to see that for either computational basis state with the environment initialized to $|0\rangle$, a single application of U_{sx} results in the system register ending in the state $|0\rangle$.

However, the primary useful property of cyclic permutation unitaries is that every such unitary, of any dimension, each has a simple, nicely structured Hamiltonian that generates evolution by that unitary. To obtain this result, we first note that any cyclic permutation unitary of dimension N can be expressed in terms of a *canonical* cyclic permutation unitary for dimension N , defined as follows:

Definition 2.1.3. *For any $N \in \mathbb{N}$, let the canonical cyclic permutation unitary for dimension N and computational basis $\{|j\rangle : j \in \mathbb{Z}_N\}$ be denoted by C_N and act on the basis states as follows:*

$$C_N |j\rangle = |(j+1) \bmod N\rangle \quad (2.5)$$

for any $j \in \mathbb{Z}_N$.

Subsequently, we have the following theorem.

Theorem 2.1.4. *An arbitrary cyclic permutation unitary C for dimension N can be decomposed as $C = PC_N P^\dagger$ for some permutation unitary P .*

Proof. Let C be a cyclic permutation unitary for dimension N and basis $\{|j\rangle : j \in \mathbb{Z}_N\}$. Then, by the definition of a cyclic permutation unitary, let $|\psi_j\rangle = C^j |0\rangle$ for each $j \in \mathbb{N}$.

As C is a permutation unitary, $\{|\psi_j\rangle : j \in \mathbb{Z}_N\} = \{|j\rangle : j \in \mathbb{Z}_N\}$. Let P be the permutation unitary such that $P|j\rangle = P|\psi_j\rangle$ for each $j \in \mathbb{N}$. It follows that $C_N = P^\dagger C P$, so $C = P C_N P^\dagger$. $\square$

By the above theorem, finding a Hamiltonian that generates powers of C at discrete time steps reduces to finding a Hamiltonian that generates C_N . Thus, we consider one Hamiltonian that generates C_N , based on the intuition that the eigenstates of C_N are the Fourier basis states.

Definition 2.1.5. Let ω_N be the N th primitive root of unity and F_N be the N -dimensional Fourier transform unitary that is defined as

$$F_N = \frac{1}{\sqrt{N}} \sum_{j \in \mathbb{Z}_N} \sum_{k \in \mathbb{Z}_N} \omega_N^{jk} |j\rangle \langle k|. \quad (2.6)$$

Let D_N be the N -dimensional Hermitian operator that is diagonal in the computational basis and is defined as

$$D_N = \sum_{j \in \mathbb{Z}_N} -2\pi j |j\rangle \langle j|. \quad (2.7)$$

Then, we define the canonical cyclic permutation Hamiltonian as $H_N = F_N^\dagger D_N F_N$. Notably, H_N has spectral norm $\|H_N\|_\infty = 2\pi(N-1)$, which is related to the maximum energy needed to evolve by the Hamiltonian.

By plugging in $t = \frac{1}{N}$, we may verify that time-independent Schrödinger evolution by H_N indeed generates C_N :

Theorem 2.1.6. For any $N \in \mathbb{N}$ and $t = \frac{1}{N}$,

$$e^{-iH_N t} = C_N. \quad (2.8)$$

Proof. Let $j \in \mathbb{Z}_N$

$$e^{-iH_N t} |j\rangle = F_N^\dagger e^{-iD_N \frac{1}{N}} F_N |j\rangle \quad (2.9)$$

$$= F_N^\dagger \left(\sum_{l \in \mathbb{Z}_N} e^{\frac{2\pi i l}{N}} |l\rangle \langle l| \right) \frac{1}{\sqrt{N}} \sum_{k \in \mathbb{Z}_N} \omega_N^{jk} |k\rangle \quad (2.10)$$

$$= F_N^\dagger \left(\sum_{l \in \mathbb{Z}_N} \omega_N^l |l\rangle \langle l| \right) \frac{1}{\sqrt{N}} \sum_{k \in \mathbb{Z}_N} \omega_N^{jk} |k\rangle \quad (2.11)$$

$$= F_N^\dagger \frac{1}{\sqrt{N}} \sum_{k \in \mathbb{Z}_N} \omega_N^{(j+1)k} |k\rangle \quad (2.12)$$

$$= |j+1\rangle \quad (2.13)$$

$$= C_N |j\rangle. \quad (2.14)$$

By linearity, $e^{-iH_N t} = C_N$. □

Corollary 2.1.7. *Let $C = PC_N P^\dagger$ be a cyclic permutation unitary. It is generated by the Hamiltonian $PH_N P^\dagger$.*

Returning to our 2-qubit construction, the unitary U_{sx} can be expressed using the dimension-4 canonical cyclic permutation operator as

$$U_{\text{sx}} = (I \otimes X) \text{SWAP} \quad (2.15)$$

$$= \text{CNOT}_{\mathcal{SE}} C_4 \text{CNOT}_{\mathcal{SE}}, \quad (2.16)$$

in which $\text{CNOT}_{\mathcal{SE}}$ performs a NOT operation on the $\mathcal{E}$ register, controlled on the $\mathcal{S}$ register.

Indeed, we have that

$$\text{CNOT}_{\mathcal{SE}} C_4 \text{CNOT}_{\mathcal{SE}} |00\rangle = \text{CNOT}_{\mathcal{SE}} C_4 |00\rangle \quad (2.17)$$

$$= \text{CNOT}_{\mathcal{SE}} |01\rangle \quad (2.18)$$

$$= |01\rangle, \quad (2.19)$$

$$\text{CNOT}_{\mathcal{SE}} C_4 \text{CNOT}_{\mathcal{SE}} |01\rangle = \text{CNOT}_{\mathcal{SE}} C_4 |01\rangle \quad (2.20)$$

$$= \text{CNOT}_{\mathcal{SE}} |10\rangle \quad (2.21)$$

$$= |11\rangle, \quad (2.22)$$

$$\text{CNOT}_{\mathcal{SE}} C_4 \text{CNOT}_{\mathcal{SE}} |11\rangle = \text{CNOT}_{\mathcal{SE}} C_4 |10\rangle \quad (2.23)$$

$$= \text{CNOT}_{\mathcal{SE}} |11\rangle \quad (2.24)$$

$$= |10\rangle, \quad (2.25)$$

$$\text{CNOT}_{\mathcal{SE}} C_4 \text{CNOT}_{\mathcal{SE}} |10\rangle = \text{CNOT}_{\mathcal{SE}} C_4 |11\rangle \quad (2.26)$$

$$= \text{CNOT}_{\mathcal{SE}} |00\rangle \quad (2.27)$$

$$= |00\rangle, \quad (2.28)$$

matching [Figure 2.1](#).

Using the cyclic permutation decomposition of U_{sx} , we find that U_{sx} is generated by the Hamiltonian

$$H_{\text{sx}} = \text{CNOT}_{\mathcal{SE}} H_4 \text{CNOT}_{\mathcal{SE}}. \quad (2.29)$$

Moreover, as $e^{-iH_4 t}$ has period 1, and the discrete reset occurs after one application of U_{sx} out of a cycle of 4 states, evolution by the H_{sx} Hamiltonian realizes a reset at time $t = \frac{1}{4}$.

Now that we have an explicit Hamiltonian H_{sx} , we may now consider how well it simulates a reset beyond the single moment when $t = \frac{1}{4}$. According to numerical simulation, as illustrated in [Figure 2.2](#), the 4-dimensional Hamiltonian does evolve a $|1\rangle$ state to an exact $|0\rangle$ state at time $t = \frac{1}{4}$ and stays somewhat close to $|0\rangle$ until time $t = \frac{1}{2}$. Furthermore, when starting in the state $|0\rangle$, evolution by the Hamiltonian maintains a state that is closer to $|0\rangle$ than $|1\rangle$ until time $t = \frac{1}{2}$. However, recalling the definition of (ϵ, δ) -approximate reset [Definition 1.3.1](#), we would like our reset Hamiltonian to approximate the identity on $\mathcal{S}$ before the reset, perform a reset in time δ , and persist the reset state. Unfortunately, our 4-dimensional Hamiltonian performs poorly according to our definition, as the evolution begins to reset the $|1\rangle$ state immediately, takes a long time to perform the reset, and does not maintain a steady $|0\rangle$ as the post-reset state. To resolve these issues, we consider a generalization of U_{sx} and H_{sx} for dimension larger than 4.

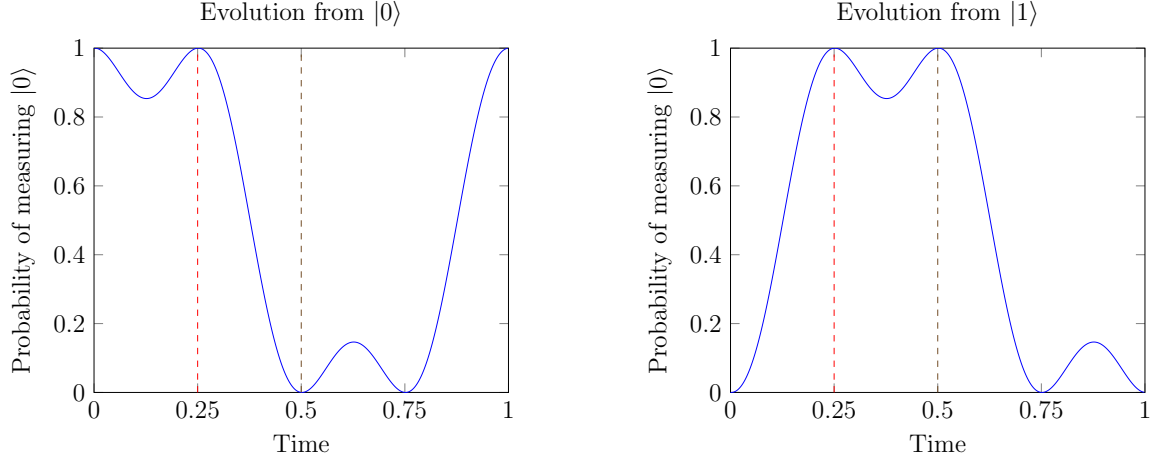


Figure 2.2: Probability of measuring $|0\rangle$ on $\mathcal{S}$ when evolving by H_{sx} with ancilla $|0\rangle$ on $\mathcal{E}$. t_b and $t_b + t_a$ are labeled in red and brown, respectively.

2.2 Clocked construction

Aside from the starting time and the single instant that H_{sx} exactly performs a reset, evolution by H_{sx} drifts away from the idealized reset behavior. The behavior of H_{sx} at these two times corresponds to how zero applications of U_{sx} followed by tracing out the environment performs the identity and how a single application of U_{sx} followed by tracing out the environment performs a reset. As such, in hopes of finding a Hamiltonian that drifts less from the initial state and reset state, we consider replacing U_{sx} in our construction process with a unitary that maintains the starting state and the reset state on the system register for more than just a single application each. It turns out that such unitaries do exist and may be obtained by generalizing U_{sx} to an $N = 2^n$ -dimensional unitary, denoted $U_{\text{CL},N}$, which shall be defined later, that maintains the starting state for $\frac{N}{4}$ applications and maintains the reset state for another $\frac{N}{4}$ applications.

Although such a unitary does not guarantee any behavior of a corresponding Hamiltonian except for the moments in time corresponding to discrete applications of the unitary, it turns out that the cyclic permutation Hamiltonian that is obtained from $U_{\text{CL},N}$, denoted $H_{\text{CL},N}$, which shall also be defined later, has most of the properties we desire.

In particular, according to numerical simulations of $H_{\text{CL},N}$ illustrated in Figure 2.3, we observe that for larger N , $H_{\text{CL},N}$ seems to converge pointwise to an idealized reset. Notably, the high amplitude troughs are pushed closer to time $t = 0$ for the $|0\rangle$ initial state and time $t = \frac{1}{4}$ for the $|1\rangle$ initial state, suggesting that $H_{\text{CL},N}$ performs an (ϵ, δ) -approximate

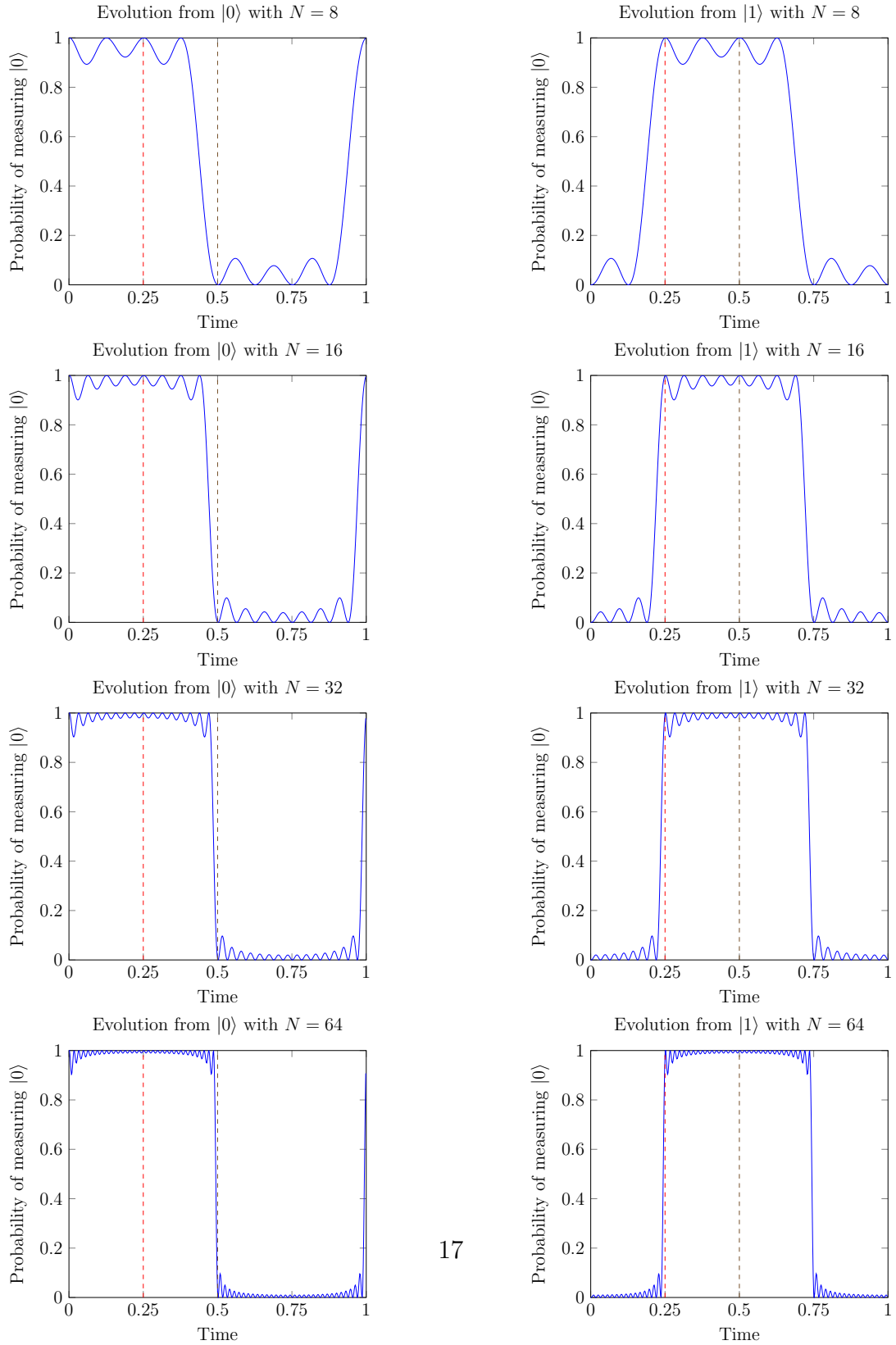


Figure 2.3: Numerical simulations approximating a reset with $t_b = t_a = \frac{1}{4}$ with t_b and $t_b + t_a$ marked in red and brown, respectively.

reset with smaller δ as N increases. Similarly, as N increases, the state remains closer to the initial state for a larger portion of time before $t = \frac{1}{4}$, and the state remains closer to $|0\rangle$ for a larger portion of the time after $t = \frac{1}{4}$. However, this last observation is not enough to suggest that ϵ decreases with larger N , as the depth of the first trough for the $|0\rangle$ case does not seem to change with larger N , though its width does. Fortunately, this issue may be resolved by shifting the entire curve leftward using an initial ancilla state $|j\rangle$ on $\mathcal{C}$ with $j > 0$. A complete, rigorous analysis of the reset property for $H_{\text{CL},N}$ is provided in Chapter 3.

To define the Hamiltonian $H_{\text{CL},N}$, we first generalize U_{SX} to the clocked-SWAP-and- X unitary $U_{\text{CL},N}$ for dimension $N = 2^n$ with $n \geq 2$. Conceptually, we partition the $(n-1)$ -qubit environment register $\mathcal{E}$ into a 1-qubit “eXchange register” labeled $\mathcal{X}$, and an $(n-2)$ -qubit “Clock” register labeled $\mathcal{C}$. Similar to $\mathcal{E}$ in the 4-dimensional case, $\mathcal{X}$ serves as the register into which the information of $\mathcal{S}$ is swapped into. On the other hand, the new register $\mathcal{C}$ simply maintains a clock or counter that maintains a memory of the number of unitary applications before the swap between $\mathcal{S}$ and $\mathcal{X}$ occurs. The above description can be captured by ensuring $U_{\text{CL},N}$ has two properties:

1. $U_{\text{CL},N}$ always increments the state of the $\frac{N}{4}$ -dimensional $\mathcal{C}$ register by 1, mapping $|0\rangle \rightarrow |1\rangle \rightarrow \dots \rightarrow |\frac{N}{4} - 1\rangle \rightarrow |0\rangle$.
2. $U_{\text{CL},N}$ applies the U_{SX} operator to the $\mathcal{S}$ and $\mathcal{X}$ registers when the clock overflows, or in other words, after exactly $\frac{N}{4}$ applications.

While the second property ensures that a reset is performed on $\mathcal{S}$ after exactly $\frac{N}{4}$ applications of $U_{\text{CL},N}$, the first property ensures that no change occurs to $\mathcal{S}$ after any of the first $\frac{N}{4} - 1$ applications of $U_{\text{CL},N}$ nor any of the $\frac{N}{4} - 1$ applications after the reset.

For instance, $U_{\text{CL},4}$ has a 2-qubit clock and corresponds to the table in [Figure 2.4](#). From the table we may observe that, indeed, for either state in which the environment is initialized to $|000\rangle$, the first three applications of $U_{\text{CL},4}$ leave the system state unchanged, whereas the fourth through seventh application leave the system qubit in the state $|0\rangle$.

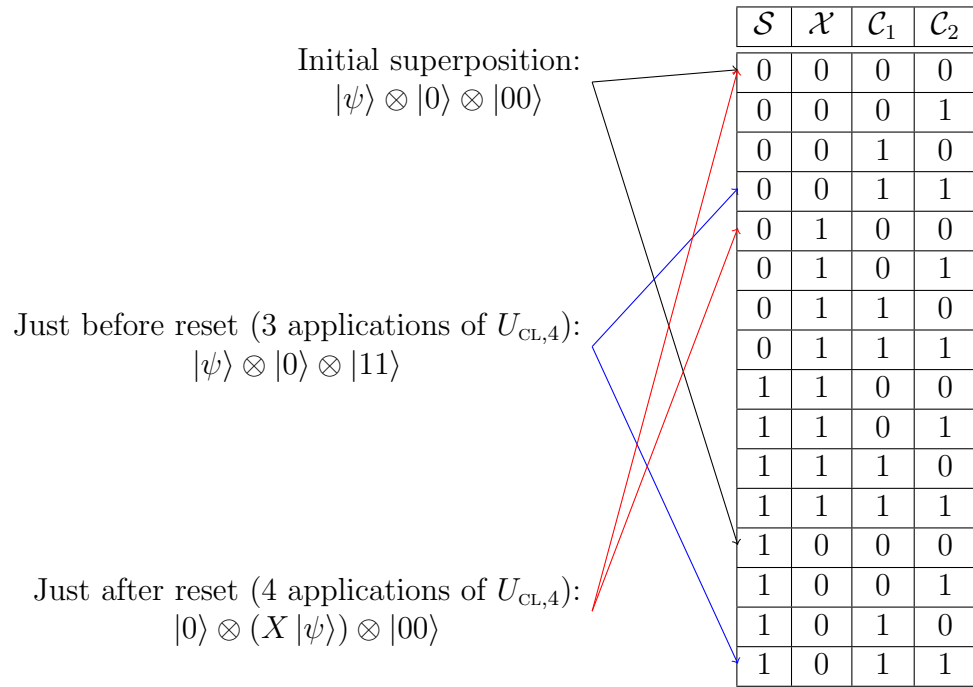


Figure 2.4: Depiction of an input qubit superposition $|\psi\rangle$ being reset after exactly 4 applications of $U_{\text{CL},4}$. Each application of $U_{\text{CL},4}$ advances the state down the table.

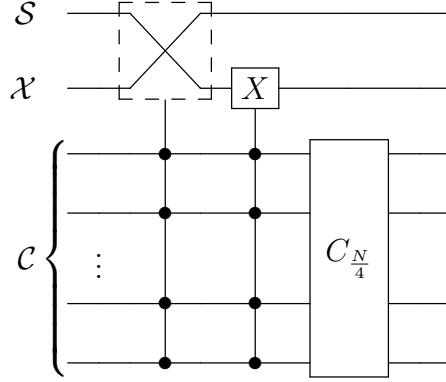


Figure 2.5: Circuit diagram of $U_{\text{CL},N}$ that resets register $\mathcal{S}$ after $\frac{N}{4}$ applications, though $U_{\text{CL},N}$ is never actually constructed from constituent gates.

The above properties give the intuition that we should define $U_{\text{CL},N}$ according to the circuit in Figure 2.5. However, we formally define $U_{\text{CL},N}$ in terms of its canonical cyclic permutation decomposition and then prove that it is equivalent to the circuit shown in the figure.

Definition 2.2.1. For $N = 2^n$ with $n \in \mathbb{N}$, the unitary operator $U_{\text{CL},N}$ is defined as

$$U_{\text{CL},N} = (\text{CNOT}_{\mathcal{S}\mathcal{X}} \otimes I_{\mathcal{C}}) C_N (\text{CNOT}_{\mathcal{S}\mathcal{X}} \otimes I_{\mathcal{C}}). \quad (2.30)$$

According to its definition, $U_{\text{CL},N}$ in its conjugated canonical cyclic permutation form is a natural generalization of that of U_{SX} from (2.16). However, we must still verify that the operator indeed satisfies the two desired properties described above.

Theorem 2.2.2. For $N = 2^n$ with $n \in \mathbb{N}$, $j \in \mathbb{Z}_{\frac{N}{4}}$, and $|\psi\rangle, |\phi\rangle$ as qubit pure states,

$$U_{\text{CL},N}^j |\psi\rangle |\phi\rangle |0\rangle = |\psi\rangle |\phi\rangle |j\rangle. \quad (2.31)$$

Proof.

$$U_{\text{CL},N}^j |\psi\rangle |\phi\rangle |0\rangle = \left((\text{CNOT}_{\mathcal{S}\mathcal{X}} \otimes I_{\mathcal{C}}) C_N (\text{CNOT}_{\mathcal{S}\mathcal{X}} \otimes I_{\mathcal{C}}) \right)^j |\psi\rangle |\phi\rangle |0\rangle \quad (2.32)$$

$$= (\text{CNOT}_{\mathcal{S}\mathcal{X}} \otimes I_{\mathcal{C}}) C_N^j (\text{CNOT}_{\mathcal{S}\mathcal{X}} \otimes I_{\mathcal{C}}) |\psi\rangle |\phi\rangle |0\rangle \quad (2.33)$$

$$= (\text{CNOT}_{\mathcal{S}\mathcal{X}} \otimes I_{\mathcal{C}}) C_N^j (\text{CNOT}_{\mathcal{S}\mathcal{X}} |\psi\rangle |\phi\rangle) |0\rangle. \quad (2.34)$$

Note that $\text{CNOT}_{\mathcal{SX}} |\psi\rangle |\phi\rangle$ may be expressed in the computational basis with amplitudes a_{00} , a_{01} , a_{10} , and a_{11} as

$$\text{CNOT}_{\mathcal{SX}} |\psi\rangle |\phi\rangle = \sum_{k=0}^1 \sum_{l=0}^1 a_{kl} |k\rangle |l\rangle. \quad (2.35)$$

It follows that

$$(\text{CNOT}_{\mathcal{SX}} \otimes I_{\mathcal{C}}) C_N^j (\text{CNOT}_{\mathcal{SX}} |\psi\rangle |\phi\rangle) |0\rangle = (\text{CNOT}_{\mathcal{SX}} \otimes I_{\mathcal{C}}) \left(\sum_{k=0}^1 \sum_{l=0}^1 a_{kl} C_N^j |k\rangle |l\rangle |0\rangle \right) \quad (2.36)$$

$$= (\text{CNOT}_{\mathcal{SX}} \otimes I_{\mathcal{C}}) \left(\sum_{k=0}^1 \sum_{l=0}^1 a_{kl} |k\rangle |l\rangle |j\rangle \right) \quad (2.37)$$

$$= (\text{CNOT}_{\mathcal{SX}} \otimes I_{\mathcal{C}}) (\text{CNOT}_{\mathcal{SX}} |\psi\rangle |\phi\rangle) |j\rangle \quad (2.38)$$

$$= |\psi\rangle |\phi\rangle |j\rangle. \quad (2.39)$$

□

Theorem 2.2.3. For $N = 2^n$ with $n \in \mathbb{N}$,

$$U_{\text{CL},N}^{\frac{N}{4}} = (U_{\text{SX}} \otimes I_{\mathcal{C}}). \quad (2.40)$$

Proof.

$$U_{\text{CL},N}^{\frac{N}{4}} = \left((\text{CNOT}_{\mathcal{SX}} \otimes I_{\mathcal{C}}) C_N (\text{CNOT}_{\mathcal{SX}} \otimes I_{\mathcal{C}}) \right)^{\frac{N}{4}} \quad (2.41)$$

$$= (\text{CNOT}_{\mathcal{SX}} \otimes I_{\mathcal{C}}) C_N^{\frac{N}{4}} (\text{CNOT}_{\mathcal{SX}} \otimes I_{\mathcal{C}}) \quad (2.42)$$

To rewrite $C_N^{\frac{N}{4}}$, we use the following lemma:

Lemma 2.2.4. For $N = 2^n$ with $n \in \mathbb{N}$ and $M = 2^m$ with $m < n$,

$$C_N^M = C_{\frac{N}{M}} \otimes I^{\otimes m}, \quad (2.43)$$

in which I denotes the qubit identity unitary.

Proof. Note that

$$C_N^2 = C_{\frac{N}{2}} \otimes I, \quad (2.44)$$

proving the lemma for $m = 1$. Assume towards induction that for $m = k$,

$$C_N^M = C_{\frac{N}{M}} \otimes I^{\otimes m}. \quad (2.45)$$

Then, for $m = k + 1$,

$$C_N^M = (C_N^{2^k})^2 \quad (2.46)$$

$$= C_{\frac{N}{2^k}}^2 \otimes I^{\otimes k} \quad (2.47)$$

$$= (C_{\frac{N}{2^{k+1}}} \otimes I) \otimes I^{\otimes k} \quad (2.48)$$

$$= C_{\frac{N}{M}} \otimes I^{\otimes m}. \quad (2.49)$$

Thus, induction proves the lemma. $\square$

Using the lemma,

$$(\text{CNOT}_{\mathcal{SX}} \otimes I_{\mathcal{C}}) C_N^{\frac{N}{4}} (\text{CNOT}_{\mathcal{SX}} \otimes I_{\mathcal{C}}) = (\text{CNOT}_{\mathcal{SX}} \otimes I_{\mathcal{C}}) (C_4 \otimes I_{\mathcal{C}}) (\text{CNOT}_{\mathcal{SX}} \otimes I_{\mathcal{C}}) \quad (2.50)$$

$$= (U_{\text{sx}} \otimes I_{\mathcal{C}}). \quad (2.51)$$

$\square$

Corollary 2.2.5. *For $N = 2^n$ with $n \in \mathbb{N}$, $j \in \mathbb{Z}_{\frac{N}{4}}$, and $|\psi\rangle, |\phi\rangle$ as qubit pure states,*

$$U_{\text{CL},N}^{\frac{N}{4}} |\psi\rangle |\phi\rangle |j\rangle = |\phi\rangle (X |\psi\rangle) |j\rangle. \quad (2.52)$$

As $U_{\text{CL},N}$ is a cyclic permutation unitary, we may obtain a Hamiltonian that generates it from the cyclic permutation Hamiltonian using [Corollary 2.1.7](#):

Definition 2.2.6. *For $N = 2^n$ with $n \in \mathbb{N}$, the Hamiltonian $H_{\text{CL},N}$ is defined as*

$$H_{\text{CL},N} = (\text{CNOT}_{\mathcal{SX}} \otimes I_{\mathcal{C}}) H_N (\text{CNOT}_{\mathcal{SX}} \otimes I_{\mathcal{C}}). \quad (2.53)$$

As the Schatten norms are unitarily invariant, $H_{\text{CL},N}$ has spectral norm $\|H_{\text{CL},N}\|_{\infty} = \|H_N\|_{\infty} = 2\pi(N - 1)$.

Similar to evolution by H_{sx} , evolution by $H_{\text{CL},N}$ is periodic with period 1, and an exact reset is performed for an instant at time $t = \frac{1}{4}$. However, $H_{\text{CL},N}$ also exactly performs the identity for an instant at times $t = 0, \frac{1}{N}, \dots, \frac{1}{4} - \frac{1}{N}$, and exactly performs a reset for an instant at times $t = \frac{1}{4}, \frac{1}{4} + \frac{1}{N}, \dots, \frac{1}{2} - \frac{1}{N}$. In Chapter 3, we prove rigorously that $H_{\text{CL},N}$ may be used to satisfy the approximate reset property.

Chapter 3

Analysis

In this chapter, we prove that for any choice of $\epsilon, \delta > 0$, and $t_b, t_a > 0$, there exists an N such that the clocked Hamiltonian $H_{\text{CL},N}$ defined in [Definition 2.2.6](#) and an appropriate ancilla state $|\psi\rangle$ perform an (ϵ, δ) -approximate reset for those parameters, according to [Definition 1.3.1](#). That is, $H_{\text{CL},N}$ satisfies the properties in the following theorem:

Theorem 3.0.1. *Let $\epsilon, \delta > 0$ and $t_b, t_a > 0$. Let the timescale adjustment of the reset be denoted by*

$$\tau = \max(t_b, t_a) + \frac{\delta}{2}. \quad (3.1)$$

Let

$$N = 2^n = O(\epsilon^{-8} \delta^{-4} \tau^4) > \max\left(\frac{16\tau}{\delta}, 4\right) \quad (3.2)$$

with $n \in \mathbb{N}$. Let the approximate initial time offset be denoted by

$$j = \left\lceil \frac{N}{4\tau}(\tau - t_b) \right\rceil \in \mathbb{Z}_{\frac{N}{4}}. \quad (3.3)$$

Then, the Hamiltonian

$$H = \frac{1}{4\tau} H_{\text{CL},N} \quad (3.4)$$

acting on $\mathcal{S} \otimes \mathcal{X} \otimes \mathcal{C}$ and the ancilla state

$$\rho = |0\rangle\langle 0| \otimes |j\rangle\langle j| \quad (3.5)$$

on $\mathcal{X} \otimes \mathcal{C}$ satisfy the following two properties:

1. For any $t \in [0, t_b - \delta]$,

$$\|(I_S \otimes \text{tr}_{\mathcal{X}\mathcal{C}}) \circ [e^{-iHt}] \circ \Xi[\rho]_{\rightarrow \mathcal{X}\mathcal{C}} - I_S\|_{\diamond} \leq \epsilon. \quad (3.6)$$

2. For any $t \in [t_b + \delta, t_b + t_a]$,

$$\|(I_S \otimes \text{tr}_{\mathcal{X}\mathcal{C}}) \circ [e^{-iHt}] \circ \Xi[\rho]_{\rightarrow \mathcal{X}\mathcal{C}} - \Gamma_S\|_{\diamond} \leq \epsilon. \quad (3.7)$$

However, instead of proving the above theorem directly, we first prove a simpler version of the theorem in which ϵ and δ are still arbitrarily small, but t_b and t_a are fixed with $t_b = t_a = \frac{1}{4}$, and the behavior of the reset is allowed to be arbitrary on the intervals $[0, \delta]$ and $[t_b + t_a - \delta, t_b + t_a]$. Specifically, fixing t_b and t_a allows τ to be set to $\frac{1}{4}$, while allowing a small window of arbitrary behavior at either end of the time interval allows the state $|j\rangle$ to be replaced by the state $|0\rangle$.

Theorem 3.0.2. *Let $\epsilon, \delta > 0$. Let $N = 2^n = O(\epsilon^{-8}\delta^{-4})$ for $n \in \mathbb{N}$. Then, the Hamiltonian $H = H_{\text{CL},N}$ acting on $\mathcal{S} \otimes \mathcal{X} \otimes \mathcal{C}$ and the ancilla state $\rho = |0\rangle\langle 0|_{\mathcal{X}} \otimes |0\rangle\langle 0|_{\mathcal{C}}$ on $\mathcal{X} \otimes \mathcal{C}$ satisfy the following two properties:*

1. For any $t \in [\delta, \frac{1}{4} - \delta]$,

$$\|(I_S \otimes \text{tr}_{\mathcal{X}\mathcal{C}}) \circ [e^{-iHt}] \circ \Xi[\rho]_{\rightarrow \mathcal{X}\mathcal{C}} - I_S\|_{\diamond} \leq \epsilon. \quad (3.8)$$

2. For any $t \in [\frac{1}{4} + \delta, \frac{1}{2} - \delta]$,

$$\|(I_S \otimes \text{tr}_{\mathcal{X}\mathcal{C}}) \circ [e^{-iHt}] \circ \Xi[\rho]_{\rightarrow \mathcal{X}\mathcal{C}} - \Gamma_S\|_{\diamond} \leq \epsilon. \quad (3.9)$$

Proof. We first isolate a parameterized channel Ψ that acts as a subroutine for the overall simulated reset and ends up allowing us to bound both channel distances in [Theorem 3.0.2](#) by a single expression of Ψ .

Definition 3.0.3. *Let Ψ be a parameterized channel with three parameters, a Hamiltonian on $\mathbb{C}^N$, a density operator on $\mathbb{C}^{\frac{N}{2}}$, and a real number denoting time, for $n \in \mathbb{N}$ and $N = 2^n$ from $\mathcal{S} \otimes \mathcal{X}$ to $\mathcal{S} \otimes \mathcal{X}$ defined for H a Hamiltonian on n qubits, ρ a density operator on $n - 2$ qubits, and $t \in \mathbb{R}$ as*

$$\Psi[H, \rho, t] = (I_{\mathcal{S}\mathcal{X}} \otimes \text{tr}_{\mathcal{C}}) \circ [e^{-iHt}] \circ \Xi[\rho]_{\rightarrow \mathcal{C}}. \quad (3.10)$$

In particular, for σ an arbitrary density operator on $\mathcal{C}$ and $\rho = |0\rangle\langle 0|_{\mathcal{X}} \otimes \sigma_{\mathcal{C}}$, the proposed simulated reset may be expressed as

$$(I_{\mathcal{S}} \otimes \text{tr}_{\mathcal{X}\mathcal{C}}) \circ [e^{-iHt}] \circ \Xi[\rho]_{\rightarrow \mathcal{X}\mathcal{C}} = (I_{\mathcal{S}} \otimes \text{tr}_{\mathcal{X}}) \circ \Psi[H, \sigma, t] \circ \Xi[|0\rangle\langle 0|_{\mathcal{X}}]. \quad (3.11)$$

For ρ and t arbitrary, we may use [Theorem 2.2.3](#) to note that

$$\Psi[H_{\text{CL},N}, \rho, t] = (I_{\mathcal{S}\mathcal{X}} \otimes \text{tr}_{\mathcal{C}}) \circ [e^{-iH_{\text{CL},N}t}] \circ \Xi[\rho]_{\rightarrow \mathcal{C}} \quad (3.12)$$

$$= (I_{\mathcal{S}\mathcal{X}} \otimes \text{tr}_{\mathcal{C}}) \circ \left[U_{\text{CL},N}^{\frac{N}{4}} \right] \circ \left[e^{-iH_{\text{CL},N}(t-\frac{1}{4})} \right] \circ \Xi[\rho]_{\rightarrow \mathcal{C}} \quad (3.13)$$

$$= (I_{\mathcal{S}\mathcal{X}} \otimes \text{tr}_{\mathcal{C}}) \circ [U_{\text{SX}} \otimes I] \circ \left[e^{-iH_{\text{CL},N}(t-\frac{1}{4})} \right] \circ \Xi[\rho]_{\rightarrow \mathcal{C}} \quad (3.14)$$

$$= [U_{\text{SX}}] \circ (I_{\mathcal{S}\mathcal{X}} \otimes \text{tr}_{\mathcal{C}}) \circ \left[e^{-iH_{\text{CL},N}(t-\frac{1}{4})} \right] \circ \Xi[\rho]_{\rightarrow \mathcal{C}} \quad (3.15)$$

$$= [U_{\text{SX}}] \circ \Psi \left[H_{\text{CL},N}, \rho, t - \frac{1}{4} \right]. \quad (3.16)$$

Using the above fact, the channel distance from the second property [\(3.9\)](#) may be rewritten for H and ρ as in the theorem and $t \in [\frac{1}{4} + \delta, \frac{1}{2} - \delta]$, as

$$\left\| (I_{\mathcal{S}} \otimes \text{tr}_{\mathcal{X}\mathcal{C}}) \circ [e^{-iHt}] \circ \Xi[|0\rangle\langle 0|_{\mathcal{X}} \otimes \rho]_{\rightarrow \mathcal{X}\mathcal{C}} - \Gamma \right\|_{\diamond} \quad (3.17)$$

$$= \left\| (I_{\mathcal{S}} \otimes \text{tr}_{\mathcal{X}}) \circ \Psi[H, \rho, t] \circ \Xi[|0\rangle\langle 0|_{\mathcal{X}}] - \Gamma \right\|_{\diamond} \quad (3.18)$$

$$= \left\| (I_{\mathcal{S}} \otimes \text{tr}_{\mathcal{X}}) \circ [U_{\text{SX}}] \circ \left(\Psi \left[H, \rho, t - \frac{1}{4} \right] \circ \Xi[|0\rangle\langle 0|_{\mathcal{X}}] - \Xi[|0\rangle\langle 0|_{\mathcal{X}}] \right) \right\|_{\diamond} \quad (3.19)$$

$$\leq \left\| \Psi \left[H, \rho, t - \frac{1}{4} \right] \circ \Xi[|0\rangle\langle 0|_{\mathcal{X}}] - \Xi[|0\rangle\langle 0|_{\mathcal{X}}] \right\|_{\diamond}. \quad (3.20)$$

Similarly, the channel distance from the first property [\(3.8\)](#) may be rewritten for $t \in [\delta, \frac{1}{4} - \delta]$ as

$$\left\| (I_{\mathcal{S}} \otimes \text{tr}_{\mathcal{X}\mathcal{C}}) \circ [e^{-iHt}] \circ \Xi[|0\rangle\langle 0|_{\mathcal{X}} \otimes \rho]_{\rightarrow \mathcal{X}\mathcal{C}} - I \right\|_{\diamond} \quad (3.21)$$

$$= \left\| (I_{\mathcal{S}} \otimes \text{tr}_{\mathcal{X}}) \circ (\Psi[H, \rho, t] \circ \Xi[|0\rangle\langle 0|_{\mathcal{X}}] - \Xi[|0\rangle\langle 0|_{\mathcal{X}}]) \right\|_{\diamond} \quad (3.22)$$

$$\leq \left\| \Psi[H, \rho, t] \circ \Xi[|0\rangle\langle 0|_{\mathcal{X}}] - \Xi[|0\rangle\langle 0|_{\mathcal{X}}] \right\|_{\diamond}. \quad (3.23)$$

Consequently, to prove both of the properties from [Theorem 3.0.2](#), it is sufficient to prove that, for $t \in [\delta, \frac{1}{4} - \delta]$,

$$\left\| \Psi[H, \rho, t] \circ \Xi[|0\rangle\langle 0|_{\mathcal{X}}] - \Xi[|0\rangle\langle 0|_{\mathcal{X}}] \right\|_{\diamond} \leq \epsilon. \quad (3.24)$$

As such, we introduce the following lemma:

Lemma 3.0.4. *Let $N = 2^n$ with $n \in \mathbb{N}$, ρ be an arbitrary qubit state, and $\delta > 0$. Then, the following statement is asymptotically true for $t \in [\delta, \frac{1}{4} - \delta]$ and as N approaches ∞ and δ approaches 0:*

$$\|\Psi[H_{\text{CL},N}, |0\rangle\langle 0|_{\mathcal{C}}, t] \circ \Xi[|0\rangle\langle 0|_{\mathcal{X}}] - \Xi[|0\rangle\langle 0|_{\mathcal{X}}]\|_{\diamond} = O\left(N^{-\frac{1}{8}}\delta^{-\frac{1}{2}}\right). \quad (3.25)$$

The proof of the above lemma is quite involved, and it is provided in [section 3.1](#). Taking the lemma as true for the time being, it is simple to see that our choice of $N = 2^n = O(\epsilon^{-8}\delta^{-4})$ is indeed sufficient to obtain the bound from [\(3.24\)](#). $\square$

Contingent on [Lemma 3.0.4](#) being true, we may now use the simpler case of [Theorem 3.0.2](#) to prove the more general case of [Theorem 3.0.1](#). As the time intervals on which we wish to bound the channel distances by ϵ become $[0, t_b - \delta]$ and $[t_b + \delta, t_b + t_a]$ instead of $[\delta, \frac{1}{4} - \delta]$ and $[\frac{1}{4} + \delta, \frac{1}{2} - \delta]$, we prove that our modifications to the Hamiltonian and ancilla state simply apply an affine linear transformation to the time domain that map the second pair of intervals into the first pair of intervals.

Proof of Theorem 3.0.1. Recall from the statement of the theorem our definition of $\tau = \max(t_b, t_a) + \frac{\delta}{2}$, which represents the overall timescale adjustment needed going from of the simple reset from [Theorem 3.0.2](#) with fixed times before and after the reset to the simulated reset with arbitrary times before and after the reset. As the more general theorem requires that the Hamiltonian $\frac{1}{4\tau}H_{\text{CL},N}$ transitions from the initial state to the reset state in approximately time δ , it must be the case that the Hamiltonian $H_{\text{CL},N}$ on its own transitions from the initial state to the reset state in approximately time $\frac{\delta}{4\tau}$. As such, let $\tilde{\delta} = \frac{\delta}{16\tau}$ denote a timescale-adjusted δ . Then, our choice of $N = O(\epsilon^{-8}\delta^{-4}\tau^4) = O(\epsilon^{-8}\tilde{\delta}^{-4})$ in the statement of the current theorem is consistent with the choice of N in [Theorem 3.0.2](#), but with $\tilde{\delta}$ substituted in place of δ . However, while the time before the reset and the time after the reset in [Theorem 3.0.2](#) are identically $\frac{1}{4}$, in the current theorem, t_b and t_a may differ. As such, we denote the approximate initial time offset as $j = \lceil \frac{N}{4\tau}(\tau - t_b) \rceil$, which we use to initialize the ancilla state. With this choice of variables, the proposed simulated reset may be expressed in terms of Ψ as

$$(I_{\mathcal{S}} \otimes \text{tr}_{\mathcal{X}\mathcal{C}}) \circ [e^{-iHt}] \circ \Xi[\rho]_{\rightarrow \mathcal{X}\mathcal{C}} \quad (3.26)$$

$$= (I_{\mathcal{S}} \otimes \text{tr}_{\mathcal{X}\mathcal{C}}) \circ \left[e^{-iH_{\text{CL},N} \frac{t}{4\tau}} \right] \circ \Xi[|0\rangle\langle 0|_{\mathcal{X}} \otimes |j\rangle\langle j|_{\mathcal{C}}] \quad (3.27)$$

$$= (I_{\mathcal{S}} \otimes \text{tr}_{\mathcal{X}\mathcal{C}}) \circ \left[e^{-iH_{\text{CL},N} \left(\frac{t}{4\tau} + \frac{j}{N} \right)} \right] \circ \Xi[|0\rangle\langle 0|_{\mathcal{X}} \otimes |0\rangle\langle 0|_{\mathcal{C}}] \quad (3.28)$$

$$= (I_{\mathcal{S}} \otimes \text{tr}_{\mathcal{X}}) \circ \Psi \left[H_{\text{CL},N}, |0\rangle\langle 0|_{\mathcal{C}}, \frac{t}{4\tau} + \frac{j}{N} \right] \circ \Xi[|0\rangle\langle 0|_{\mathcal{X}}]. \quad (3.29)$$

As in the proof of [Theorem 3.0.2](#), we note that the channel distance from the first property (3.6) for $t \in [0, t_b - \delta]$ is bounded by

$$\|(I_{\mathcal{S}} \otimes \text{tr}_{\mathcal{X}\mathcal{C}}) \circ [e^{-iHt}] \circ \Xi[\rho]_{\rightarrow \mathcal{X}\mathcal{C}} - I\|_{\diamond} \quad (3.30)$$

$$= \left\| (I_{\mathcal{S}} \otimes \text{tr}_{\mathcal{X}}) \circ \Psi \left[H_{\text{CL},N}, |0\rangle \langle 0|_{\mathcal{C}}, \frac{t}{4\tau} + \frac{j}{N} \right] \circ \Xi[|0\rangle \langle 0|_{\mathcal{X}}] - I \right\|_{\diamond} \quad (3.31)$$

$$\leq \left\| \Psi \left[H_{\text{CL},N}, |0\rangle \langle 0|_{\mathcal{C}}, \frac{t}{4\tau} + \frac{j}{N} \right] \circ \Xi[|0\rangle \langle 0|_{\mathcal{X}}] - \Xi[|0\rangle \langle 0|_{\mathcal{X}}] \right\|_{\diamond}. \quad (3.32)$$

Recalling [Lemma 3.0.4](#), to prove that (3.32) is bounded above by ϵ , it is sufficient to show that for any $t \in [0, t_b - \delta]$, $\frac{t}{4\tau} + \frac{j}{N} \in [\tilde{\delta}, \frac{1}{4} - \tilde{\delta}]$. In fact, as $\frac{t}{4\tau} + \frac{j}{N}$ is linear in t , we need only verify that the images of the endpoints of $[0, t_b - \delta]$ fall within $[\tilde{\delta}, \frac{1}{4} - \tilde{\delta}]$. As such, we indeed find that

$$\frac{0}{4\tau} + \frac{j}{N} = \frac{1}{N} \left[\frac{N}{4\tau}(\tau - t_b) \right] \quad (3.33)$$

$$\geq \frac{1}{4\tau}(\tau - t_b) \quad (3.34)$$

$$\geq \frac{1}{4\tau} \left(t_b + \frac{\delta}{2} - t_b \right) \quad (3.35)$$

$$= \frac{\delta}{8\tau} \quad (3.36)$$

$$\geq \frac{\delta}{16\tau} \quad (3.37)$$

$$= \tilde{\delta}, \quad (3.38)$$

and

$$\frac{t_b - \delta}{4\tau} + \frac{j}{N} = \frac{t_b - \delta}{4\tau} + \frac{1}{N} \left\lceil \frac{N}{4\tau} (\tau - t_b) \right\rceil \quad (3.39)$$

$$\leq \frac{t_b - \delta}{4\tau} + \frac{1}{4\tau} (\tau - t_b) + \frac{1}{N} \quad (3.40)$$

$$= \frac{1}{4} - \frac{\delta}{\tau} + \frac{1}{N} \quad (3.41)$$

$$\leq \frac{1}{4} - \frac{\delta}{\tau} + \frac{\delta}{16\tau} \quad (3.42)$$

$$\leq \frac{1}{4} - \frac{\delta}{16\tau} \quad (3.43)$$

$$= \frac{1}{4} - \tilde{\delta}. \quad (3.44)$$

Thus, the first property is proven.

Similarly, the channel distance from the second property (3.7) for $t \in [t_b + \delta, t_b + t_a]$ is bounded by

$$\|(I_{\mathcal{S}} \otimes \text{tr}_{\mathcal{X}\mathcal{C}}) \circ [e^{-iHt}] \circ \Xi[\rho]_{\rightarrow \mathcal{X}\mathcal{C}} - \Gamma\|_{\diamond} \quad (3.45)$$

$$= \|(I_{\mathcal{S}} \otimes \text{tr}_{\mathcal{X}}) \circ \Psi \left[H_{\text{CL},N}, |0\rangle \langle 0|_{\mathcal{C}}, \frac{t}{4\tau} + \frac{j}{N} \right] \circ \Xi[|0\rangle \langle 0|_{\mathcal{X}}] - \Gamma\|_{\diamond} \quad (3.46)$$

$$\leq \left\| \Psi \left[H_{\text{CL},N}, |0\rangle \langle 0|_{\mathcal{C}}, \frac{t}{4\tau} + \frac{j}{N} - \frac{1}{4} \right] \circ \Xi[|0\rangle \langle 0|_{\mathcal{X}}] - \Xi[|0\rangle \langle 0|_{\mathcal{X}}] \right\|_{\diamond}. \quad (3.47)$$

As with the first property, to prove that (3.47) is bounded above by ϵ , it is sufficient to show that for any $t \in [t_b + \delta, t_b + t_a]$, $\frac{t}{4\tau} + \frac{j}{N} - \frac{1}{4} \in [\tilde{\delta}, \frac{1}{4} - \tilde{\delta}]$. Again, $\frac{t}{4\tau} + \frac{j}{N} - \frac{1}{4}$ is linear in t , so we need only verify that the images of the endpoints of $[t_b + \delta, t_b + t_a]$ fall within $[\tilde{\delta}, \frac{1}{4} - \tilde{\delta}]$, and it is indeed the case that

$$\frac{t_b + \delta}{4\tau} + \frac{j}{N} - \frac{1}{4} = \frac{t_b + \delta}{4\tau} + \frac{1}{N} \left\lceil \frac{N}{4\tau} (\tau - t_b) \right\rceil - \frac{1}{4} \quad (3.48)$$

$$\geq \frac{t_b - \delta}{4\tau} + \frac{1}{4\tau} (\tau - t_b) - \frac{1}{4} \quad (3.49)$$

$$= \frac{\delta}{4\tau} \quad (3.50)$$

$$\geq \frac{\delta}{16\tau} \quad (3.51)$$

$$= \tilde{\delta}, \quad (3.52)$$

and

$$\frac{t_b + t_a}{4\tau} + \frac{j}{N} - \frac{1}{4} = \frac{t_b + t_a}{4\tau} + \frac{1}{N} \left\lceil \frac{N}{4\tau} (\tau - t_b) \right\rceil - \frac{1}{4} \quad (3.53)$$

$$\leq \frac{t_b + t_a}{4\tau} + \frac{1}{4\tau} (\tau - t_b) + \frac{1}{N} - \frac{1}{4} \quad (3.54)$$

$$= \frac{t_a}{4\tau} + \frac{1}{N} \quad (3.55)$$

$$\leq \frac{\tau - \frac{\delta}{2}}{4\tau} + \frac{\delta}{16\tau} \quad (3.56)$$

$$= \frac{1}{4} - \frac{\delta}{8\tau} + \frac{\delta}{16\tau} \quad (3.57)$$

$$= \frac{1}{4} - \frac{\delta}{16\tau} \quad (3.58)$$

$$= \frac{1}{4} - \tilde{\delta}. \quad (3.59)$$

Thus, the second property is proven, completing the proof of [Theorem 3.0.1](#). $\square$

3.1 Proof of main lemma

Proof of [Lemma 3.0.4](#). To prove the lemma, we first relate the channel distance between the two channels from the lemma, given by

$$\|\Psi[H_{\text{CL},N}, |0\rangle\langle 0|_{\mathcal{C}}, t] \circ \Xi[|0\rangle\langle 0|_{\mathcal{X}}] - \Xi[|0\rangle\langle 0|_{\mathcal{X}}]\|_{\diamond}, \quad (3.60)$$

to the trace distance between the outputs of the two channels after they have both been applied to the same arbitrary state ρ on $\mathcal{S}$, given by

$$\left\| \left(\Psi[H_{\text{CL},N}, |0\rangle\langle 0|_{\mathcal{C}}, t] \circ \Xi[|0\rangle\langle 0|_{\mathcal{X}}] - \Xi[|0\rangle\langle 0|_{\mathcal{X}}] \right) (\rho) \right\|_1. \quad (3.61)$$

As $\Xi[|0\rangle\langle 0|_{\mathcal{X}}]$ is an isometric channel, we use a generalization of [[12](#), Theorem 3.56]¹. The original theorem bounds the distance from a channel to the identity channel based on the trace distance between an arbitrary state and the output of the channel after being applied to that state, and is stated as follows:

¹Corrections to the theorem are contained in that text's errata, found at <https://johnwatrous.com/wp-content/uploads/2023/08/TQI.errata.pdf>.

Theorem 3.1.1. *Let $\epsilon > 0$ and Φ be a channel from $\mathcal{X}$ to $\mathcal{X}$ such that, for any ρ being a density operator on $\mathcal{X}$,*

$$\|(\Phi - I)(\rho)\|_1 \leq \epsilon. \quad (3.62)$$

Then, it follows that

$$\|\Phi - I\|_\diamond \leq 2\sqrt{\epsilon}. \quad (3.63)$$

However, for this proof, we require a generalization of the above theorem that replaces the identity channel with an arbitrary isometric channel subject to a similar trace distance closeness constraint. While the case of an arbitrary *unitary* channel follows trivially from the original theorem due to the invertibility of unitaries, isometries are not generally invertible. Fortunately, the proof technique for the original theorem can be directly generalized to isometric channels, yielding the following theorem:

Theorem 3.1.2. *Let $\epsilon > 0$, Ψ be an isometric channel from $\mathcal{X}$ to $\mathcal{Y}$, and Φ be another channel from $\mathcal{X}$ to $\mathcal{Y}$ such that, for any ρ being a density operator on $\mathcal{X}$,*

$$\|(\Psi - \Phi)(\rho)\|_1 \leq \epsilon. \quad (3.64)$$

Then, it follows that

$$\|\Psi - \Phi\|_\diamond \leq 2\sqrt{\epsilon}. \quad (3.65)$$

Although the proof of the above theorem is almost identical to that of [12, Theorem 3.56], it is reproduced in [section 3.2](#) for completeness. In summary, we bound the channel distance by a fidelity between the actions of Ψ and Φ on a pure state, which has a nice form due to Ψ being an isometric channel. Using above theorem, the proof of [Lemma 3.0.4](#) reduces to asymptotically bounding (3.61) by

$$\left\| \left(\Psi [H_{\text{CL},N}, |0\rangle \langle 0|_{\mathcal{C}}, t] \circ \Xi[|0\rangle \langle 0|_{\mathcal{X}}] - \Xi[|0\rangle \langle 0|_{\mathcal{X}}] \right) (\rho) \right\|_1 = O\left(N^{-\frac{1}{4}}\delta^{-1}\right) \quad (3.66)$$

as N approaches ∞ and δ approaches 0. Using the Fuchs-van de Graaf inequalities, we relate the above trace distance to a fidelity

$$\left\| \left(\Psi [H_{\text{CL},N}, |0\rangle \langle 0|_{\mathcal{C}}, t] \circ \Xi[|0\rangle \langle 0|_{\mathcal{X}}] - \Xi[|0\rangle \langle 0|_{\mathcal{X}}] \right) (\rho) \right\|_1 \quad (3.67)$$

$$= \|\Psi [H_{\text{CL},N}, |0\rangle \langle 0|_{\mathcal{C}}, t] (\rho \otimes |0\rangle \langle 0|_{\mathcal{X}}) - \rho \otimes |0\rangle \langle 0|_{\mathcal{X}}\|_1 \quad (3.68)$$

$$\leq 2\sqrt{1 - F\left(\Psi [H_{\text{CL},N}, |0\rangle \langle 0|_{\mathcal{C}}, t] (\rho \otimes |0\rangle \langle 0|_{\mathcal{X}}), \rho \otimes |0\rangle \langle 0|_{\mathcal{X}}\right)}. \quad (3.69)$$

For now, we make the simplifying assumption that ρ is a pure state, as we consider the non-pure case later. As such, let $\rho = |\psi\rangle\langle\psi|$ for some pure state $|\psi\rangle = \psi_0|0\rangle + \psi_1|1\rangle$. Since the input state is pure, the fidelity may be expressed as

$$F\left(\Psi[H_{\text{CL},N}, |0\rangle\langle 0|_C, t](|\psi\rangle\langle\psi| \otimes |0\rangle\langle 0|_{\mathcal{X}}), |\psi\rangle\langle\psi| \otimes |0\rangle\langle 0|_{\mathcal{X}}\right) \quad (3.70)$$

$$= \langle\psi| \langle 0| \left(\Psi[H_{\text{CL},N}, |0\rangle\langle 0|_C, t](|\psi\rangle\langle\psi| \otimes |0\rangle\langle 0|_{\mathcal{X}}) \right) |\psi\rangle |0\rangle \quad (3.71)$$

$$= \langle\psi| \langle 0| \left(\left((I_{\mathcal{S}} \otimes I_{\mathcal{X}} \otimes \text{tr}_C) \circ [e^{-iH_{\text{CL},N}t}] \circ \Xi[|0\rangle\langle 0|_C] \right) (|\psi\rangle\langle\psi| \otimes |0\rangle\langle 0|_{\mathcal{X}}) \right) |\psi\rangle |0\rangle \quad (3.72)$$

$$= \sum_{x \in \mathbb{Z}_{\frac{N}{4}}} \langle\psi| \langle 0| \langle x| e^{-iH_{\text{CL},N}t} (|\psi\rangle\langle\psi| \otimes |0\rangle\langle 0|_{\mathcal{X}} \otimes |0\rangle\langle 0|_C) e^{iH_{\text{CL},N}t} |\psi\rangle |0\rangle |x\rangle \quad (3.73)$$

$$= \sum_{x \in \mathbb{Z}_{\frac{N}{4}}} \left| \langle\psi| \langle 0| \langle x| e^{-iH_{\text{CL},N}t} |\psi\rangle |0\rangle |0\rangle \right|^2. \quad (3.74)$$

For every $x \in \mathbb{Z}_{\frac{N}{4}}$, the inner product from (3.74) can be expanded by decomposing $|\psi\rangle$ in the computational basis, yielding

$$\langle\psi| \langle 0| \langle x| e^{-iH_{\text{CL},N}t} |\psi\rangle |0\rangle |0\rangle \quad (3.75)$$

$$= \left(\psi_0 \langle 0| + \psi_1 \langle 1| \right) \langle 0| \langle x| e^{-iH_{\text{CL},N}t} \left(\psi_0 |0\rangle + \psi_1 |1\rangle \right) |0\rangle |0\rangle \quad (3.76)$$

Then, we define $a_{b_1 b_2}(x)$ for $b_1, b_2 \in \{0, 1\}$ such that the inner product above can be expressed as a sum of four terms,

$$\langle\psi| \langle 0| \langle x| e^{-iH_{\text{CL},N}t} |\psi\rangle |0\rangle |0\rangle = a_{00}(x) + a_{01}(x) + a_{10}(x) + a_{11}(x), \quad (3.77)$$

with

$$a_{b_1 b_2}(x) = \psi_{b_1} \psi_{b_2}^* \langle b_2| \langle 0| \langle x| e^{-iH_{\text{CL},N}t} |b_1\rangle |0\rangle |0\rangle. \quad (3.78)$$

Recall that $H_{\text{CL},N}$ as defined in [Definition 2.2.6](#) has a canonical cyclic permutation decomposition given by

$$H_{\text{CL},N} = (\text{CNOT}_{12} \otimes I) H_N (\text{CNOT}_{12} \otimes I). \quad (3.79)$$

Starting with $a_{11}(x)$, the term may be rewritten as

$$a_{11}(x) = |\psi_1|^2 \langle 1| \langle 0| \langle x| e^{-iH_{\text{CL},N}t} |1\rangle |0\rangle |0\rangle \quad (3.80)$$

$$= |\psi_1|^2 \langle 1| \langle 0| \langle x| (\text{CNOT}_{12} \otimes I) e^{-iH_N t} (\text{CNOT}_{12} \otimes I) |1\rangle |0\rangle |0\rangle \quad (3.81)$$

$$= |\psi_1|^2 \langle 1| \langle 1| \langle x| e^{-iH_N t} |1\rangle |1\rangle |0\rangle \quad (3.82)$$

It turns out that the $a_{11}(x)$ term may be transformed and combined with the $a_{00}(x)$ term by observing another property of the cyclic permutation Hamiltonian. As C_N is simply a permutation of the basis states and exponentiation by H_N generates C_N , each basis state can be expressed as a time-independent Schrödinger evolution by H_N from the state $|0\rangle$, which is captured in the following lemma:

Lemma 3.1.3. *For any $N \in \mathbb{N}$, $j, k \in \mathbb{Z}_N$, and $t \in \mathbb{R}$,*

$$\langle j | e^{-iH_N t} | k \rangle = \langle 0 | e^{-iH_N(t + \frac{k}{N} - \frac{j}{N})} | 0 \rangle. \quad (3.83)$$

Proof. As $e^{-iH_N} = C_N$ is the cyclic permutation Hamiltonian and $C_N^l |0\rangle = |l\rangle$, $\langle j|$ and $|k\rangle$ can each be rewritten as

$$\langle j| = \langle 0| C_N^{-j}, \quad |k\rangle = C_N^k |0\rangle \quad (3.84)$$

$$= \langle 0| e^{iH_N \frac{j}{N}}, \quad = e^{-iH_N \frac{k}{N}} |0\rangle. \quad (3.85)$$

Thus,

$$\langle j | e^{-iH_N t} | k \rangle = \langle 0 | e^{-iH_N(t + \frac{k}{N} - \frac{j}{N})} | 0 \rangle. \quad (3.86)$$

□

Equipped with this new lemma, $a_{11}(x)$ may be expressed as

$$a_{11}(x) = |\psi_1|^2 \langle 1| \langle 1| \langle x| e^{-iH_N t} |1\rangle |1\rangle |0\rangle \quad (3.87)$$

$$= |\psi_1|^2 \langle 0| \langle 0| \langle x| e^{-iH_N t} |0\rangle |0\rangle |0\rangle. \quad (3.88)$$

Using the same steps, the four terms may each be expressed as

$$a_{00}(x) = |\psi_0|^2 \langle 0| \langle 0| \langle x| e^{-iH_N t} |0\rangle |0\rangle |0\rangle, \quad (3.89)$$

$$a_{01}(x) = \psi_0 \psi_1^* \langle 1| \langle 1| \langle x| e^{-iH_N t} |0\rangle |0\rangle |0\rangle, \quad (3.90)$$

$$a_{10}(x) = \psi_1 \psi_0^* \langle 0| \langle 1| \langle x| e^{-iH_N t} |0\rangle |0\rangle |0\rangle, \quad (3.91)$$

$$a_{11}(x) = |\psi_1|^2 \langle 0| \langle 0| \langle x| e^{-iH_N t} |0\rangle |0\rangle |0\rangle. \quad (3.92)$$

As mentioned, we may now combine the first and last terms, since

$$a_{00}(x) + a_{11}(x) = (|\psi_0|^2 + |\psi_1|^2) \langle 0| \langle 0| \langle x| e^{-iH_N t} |0\rangle |0\rangle |0\rangle \quad (3.93)$$

$$= \langle 0| \langle 0| \langle x| e^{-iH_N t} |0\rangle |0\rangle |0\rangle. \quad (3.94)$$

Subsequently, our expression for fidelity from (3.74) can be lower-bounded as

$$\sum_{x \in \mathbb{Z}_{\frac{N}{4}}} \left| \langle \psi | \langle 0 | \langle x | e^{-iH_{\text{cl}, N} t} | \psi \rangle | 0 \rangle | 0 \rangle \right|^2 \quad (3.95)$$

$$= \sum_{x \in \mathbb{Z}_{\frac{N}{4}}} |a_{00}(x) + a_{01}(x) + a_{10}(x) + a_{11}(x)|^2 \quad (3.96)$$

$$\geq \sum_{x \in \mathbb{Z}_{\frac{N}{4}}} \left(\left| \langle 0 | \langle 0 | \langle x | e^{-iH_N t} | 0 \rangle | 0 \rangle | 0 \rangle \right| - |a_{01}(x)| - |a_{10}(x)| \right)^2 \quad (3.97)$$

Furthermore, we simplify our notation for $a_{01}(x)$ and $a_{10}(x)$ by defining a function α_N that outputs the amplitude still in the state $|0\rangle$ after evolving from the state $|0\rangle$ by H_N for some time t .

Definition 3.1.4. For any $N \in \mathbb{N}$, let $\alpha_N : \mathbb{R} \rightarrow \mathbb{C}$ be defined for $t \in \mathbb{R}$ as

$$\alpha_N(t) = \langle 0 | e^{-iH_N t} | 0 \rangle. \quad (3.98)$$

We may then express the absolute values of a_{01} and a_{10} in terms of α_N as

$$|a_{01}(x)| = |\psi_0 \psi_1^*| \left| \langle 1 | \langle 1 | \langle x | e^{-iH_N t} | 0 \rangle | 0 \rangle | 0 \rangle \right| \quad (3.99)$$

$$\leq \frac{1}{2} \left| \langle 0 | \langle 0 | \langle 0 | e^{-iH_N (t - \frac{x}{N} - \frac{3}{4})} | 0 \rangle | 0 \rangle | 0 \rangle \right| \quad (3.100)$$

$$= \frac{1}{2} \left| \alpha_N \left(t - \frac{x}{N} - \frac{3}{4} \right) \right| \quad (3.101)$$

and

$$|a_{10}(x)| = |\psi_1 \psi_0^*| \left| \langle 0 | \langle 1 | \langle x | e^{-iH_N t} | 0 \rangle | 0 \rangle | 0 \rangle \right| \quad (3.102)$$

$$\leq \frac{1}{2} \left| \langle 0 | \langle 0 | \langle 0 | e^{-iH_N (t - \frac{x}{N} - \frac{1}{4})} | 0 \rangle | 0 \rangle | 0 \rangle \right| \quad (3.103)$$

$$= \frac{1}{2} \left| \alpha_N \left(t - \frac{x}{N} - \frac{1}{4} \right) \right|. \quad (3.104)$$

Since everything is expressed in terms of the absolute value of α_N , we define another function β_N and prove that the new function is an upper bound of α_N .

Definition 3.1.5. For any $N \in \mathbb{N}$, let $\beta_N : \mathbb{R} \rightarrow \mathbb{R}^+$, be a periodic function with period 1 that is defined for $t \in (0, 1)$ as follows:

$$\beta_N(t) = \begin{cases} \frac{1}{2Nt}, & t \in (0, \frac{1}{2}) \\ \frac{1}{2N(1-t)}, & t \in [\frac{1}{2}, 1) \end{cases}. \quad (3.105)$$

Lemma 3.1.6. For any $N \in \mathbb{N}$ and $t \in \mathbb{R}$,

$$|\alpha_N(t)| \leq \beta_N(t). \quad (3.106)$$

A plot comparing $|\alpha_N|$ and β_N is shown in [Figure 3.1](#).

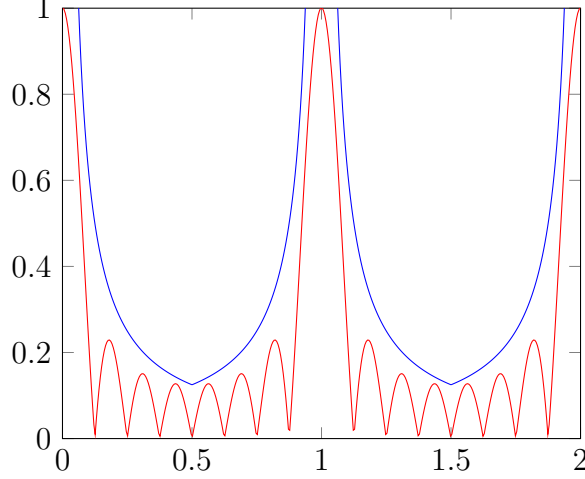


Figure 3.1: Plot of $|\alpha_N(t)|$ (red) and $\beta_N(t)$ (blue) for $N = 8$ and $t \in [0, 2]$.

Proof. Using the definition of H_N , we have that

$$|\alpha_N(t)| = |\langle 0 | e^{-iH_N t} | 0 \rangle| \quad (3.107)$$

$$= |\langle 0 | F_N^\dagger e^{-iD_N t} F_N | 0 \rangle| \quad (3.108)$$

$$= \left| \left(\frac{1}{\sqrt{N}} \sum_{l \in \mathbb{Z}_N} \langle l | \right) \left(\sum_{k \in \mathbb{Z}_N} e^{2\pi i k t} |k\rangle \langle k| \right) \left(\frac{1}{\sqrt{N}} \sum_{j \in \mathbb{Z}_N} |j\rangle \right) \right| \quad (3.109)$$

$$= \frac{1}{N} \left| \sum_{k \in \mathbb{Z}_N} e^{2\pi i k t} \right| \quad (3.110)$$

$$= \frac{|1 - e^{2\pi i N t}|}{N |1 - e^{2\pi i t}|}, \quad (3.111)$$

which is periodic with period 1. To bound the above expression, we use the trigonometric law of cosines as depicted in [Figure 3.2](#) and the half angle identity for sine to attain

$$|1 - e^{i\theta}| = \sqrt{2 - 2\cos\theta} = 2 \left| \sin\left(\frac{\theta}{2}\right) \right|. \quad (3.112)$$

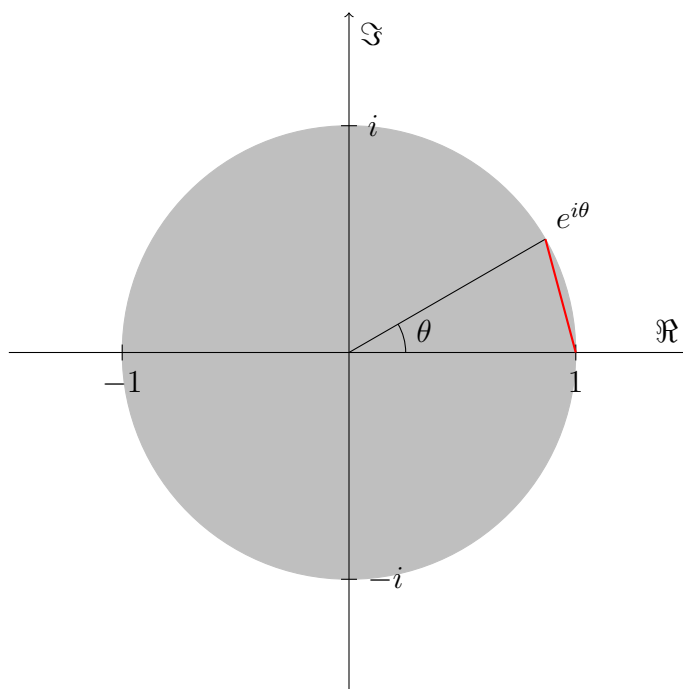


Figure 3.2: Geometric depiction of $|1 - e^{i\theta}|$ being given as the length of the red chord of the unit circle.

As the sine function is concave on $[0, \frac{\pi}{2}]$,

$$\frac{2\theta}{\pi} \leq \sin(\theta). \quad (3.113)$$

Thus, for $\theta \in [-\pi, \pi]$,

$$\frac{2|\theta|}{\pi} \leq 2 \left| \sin \left(\frac{\theta}{2} \right) \right| \leq 2. \quad (3.114)$$

Consequently, we may bound (3.111) for $t \in [-\frac{1}{2}, \frac{1}{2}]$ as

$$\frac{|1 - e^{2\pi i N t}|}{N |1 - e^{2\pi i t}|} = \frac{2 \left| \sin \left(\frac{2\pi N t}{2} \right) \right|}{2N \left| \sin \left(\frac{2\pi t}{2} \right) \right|} \quad (3.115)$$

$$\leq \frac{2}{N \left(\frac{2|2\pi t|}{\pi} \right)} \quad (3.116)$$

$$= \frac{1}{2N|t|} \quad (3.117)$$

$$= \beta_N(t), \quad (3.118)$$

completing the proof. $\square$

It turns out that we may use the fact that β_N is decreasing on $(0, \frac{1}{2}]$ and increasing on $[\frac{1}{2}, 1]$ to bound the expressions from (3.101) and (3.104) purely in terms on N and δ .

Lemma 3.1.7. *For any $t \in [\delta, \frac{1}{4} - \delta]$, $x \in [0, \frac{N}{4}]$, and $y \in [\frac{1}{4}, \frac{3}{4}]$, it is the case that*

$$\left| \alpha_N \left(t - \frac{x}{N} - y \right) \right| \leq \frac{1}{2N\delta}. \quad (3.119)$$

Proof. By choice of t , x , and y , we have that $t - \frac{x}{N} - y \in [\delta - 1, -\delta]$, which is equivalent to $[\delta, 1 - \delta]$ under periodicity 1. Using Lemma 3.1.6,

$$\left| \alpha_N \left(t - \frac{x}{N} - y \right) \right| \leq \beta_N \left(t - \frac{x}{N} - y \right) \quad (3.120)$$

$$\leq \max_{\tilde{t} \in [\delta, 1 - \delta]} \beta_N(\tilde{t}). \quad (3.121)$$

As β_N is decreasing on $(0, \frac{1}{2}]$ and increasing on $[\frac{1}{2}, 1)$,

$$\max_{\tilde{t} \in [\delta, 1 - \delta]} \beta_N(\tilde{t}) = \beta_N(\delta) = \beta_N(1 - \delta) = \frac{1}{2N\delta}. \quad (3.122)$$

$\square$

Using [Lemma 3.1.7](#), the fidelity bound from [\(3.97\)](#) may be further rewritten as

$$\sum_{x \in \mathbb{Z}_{\frac{N}{4}}} \left(\left| \langle 0 | \langle 0 | \langle x | e^{-iH_N t} | 0 \rangle | 0 \rangle | 0 \rangle \right| - \frac{1}{2} \left| \alpha_N \left(t - \frac{x}{N} - \frac{3}{4} \right) \right| - \frac{1}{2} \left| \alpha_N \left(t - \frac{x}{N} - \frac{1}{4} \right) \right| \right)^2 \quad (3.123)$$

$$\geq \sum_{x \in \mathbb{Z}_{\frac{N}{4}}} \left(\left| \langle 0 | \langle 0 | \langle x | e^{-iH_N t} | 0 \rangle | 0 \rangle | 0 \rangle \right| - \frac{1}{2N\delta} \right)^2 \quad (3.124)$$

$$\geq \sum_{x \in \mathbb{Z}_{\frac{N}{4}}} \left| \langle 0 | \langle 0 | \langle x | e^{-iH_N t} | 0 \rangle | 0 \rangle | 0 \rangle \right|^2 - \frac{1}{N\delta} \sum_{x \in \mathbb{Z}_{\frac{N}{4}}} \left| \langle 0 | \langle 0 | \langle x | e^{-iH_N t} | 0 \rangle | 0 \rangle | 0 \rangle \right|. \quad (3.125)$$

We bound the above expression by individually lower-bounding the first and second terms. The second term from [\(3.125\)](#) may be lower-bounded using Cauchy-Schwarz as

$$- \frac{1}{N\delta} \sum_{x \in \mathbb{Z}_{\frac{N}{4}}} \left| \langle 0 | \langle 0 | \langle x | e^{-iH_N t} | 0 \rangle | 0 \rangle | 0 \rangle \right| \quad (3.126)$$

$$\geq - \frac{1}{N\delta} \left(\sqrt{\sum_{x \in \mathbb{Z}_{\frac{N}{4}}} 1} \right) \left(\sqrt{\sum_{x \in \mathbb{Z}_{\frac{N}{4}}} \left| \langle 0 | \langle 0 | \langle x | e^{-iH_N t} | 0 \rangle | 0 \rangle | 0 \rangle \right|^2} \right) \quad (3.127)$$

$$= - \frac{1}{2\sqrt{N}\delta} \sqrt{\sum_{x \in \mathbb{Z}_{\frac{N}{4}}} \left| \langle 0 | \langle 0 | \langle x | e^{-iH_N t} | 0 \rangle | 0 \rangle | 0 \rangle \right|^2}. \quad (3.128)$$

Note that $e^{-iH_N t} | 0 \rangle | 0 \rangle | 0 \rangle$ is a unit vector, so

$$\sum_{a, b \in \{0, 1\}} \sum_{x \in \mathbb{Z}_{\frac{N}{4}}} \left| \langle a | \langle b | \langle x | e^{-iH_N t} | 0 \rangle | 0 \rangle | 0 \rangle \right|^2 = 1. \quad (3.129)$$

Thus,

$$- \frac{1}{2\sqrt{N}\delta} \sqrt{\sum_{x \in \mathbb{Z}_{\frac{N}{4}}} \left| \langle 0 | \langle 0 | \langle x | e^{-iH_N t} | 0 \rangle | 0 \rangle | 0 \rangle \right|^2} \geq - \frac{1}{2\sqrt{N}\delta}. \quad (3.130)$$

Reusing the fact that $e^{-iH_N t} |0\rangle |0\rangle |0\rangle$ is a unit vector, [Lemma 3.1.3](#), [Definition 3.1.4](#), and [Lemma 3.1.7](#) we may lower-bound the first term from (3.125) as

$$\sum_{x \in \mathbb{Z}_{\frac{N}{4}}} \left| \langle 0 | \langle 0 | \langle x | e^{-iH_N t} |0\rangle |0\rangle |0\rangle \right|^2 \quad (3.131)$$

$$= 1 - \sum_{\substack{a,b \in \{0,1\} \\ (a,b) \neq (0,0)}} \sum_{x \in \mathbb{Z}_{\frac{N}{4}}} \left| \langle a | \langle b | \langle x | e^{-iH_N t} |0\rangle |0\rangle |0\rangle \right|^2 \quad (3.132)$$

$$= 1 - \sum_{y \in \{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}} \sum_{x \in \mathbb{Z}_{\frac{N}{4}}} \left| \langle 0 | \langle 0 | \langle 0 | e^{-iH_N(t - \frac{x}{N} - y)} |0\rangle |0\rangle |0\rangle \right|^2 \quad (3.133)$$

$$= 1 - \sum_{y \in \{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}} \sum_{x \in \mathbb{Z}_{\frac{N}{4}}} \left| \alpha_N \left(t - \frac{x}{N} - y \right) \right|^2 \quad (3.134)$$

$$\geq 1 - \sum_{y \in \{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}} \sum_{x \in \mathbb{Z}_{\frac{N}{4}}} \left(\frac{1}{2N\delta} \right)^2 \quad (3.135)$$

$$= 1 - \frac{3}{16N\delta^2}. \quad (3.136)$$

Combining all of the terms, we find that the fidelity is lower bounded by

$$F\left(\Psi[H_{\text{CL},N}, |0\rangle \langle 0|_{\mathcal{C}}, t](|\psi\rangle \langle \psi| \otimes |0\rangle \langle 0|_{\mathcal{X}}), |\psi\rangle \langle \psi| \otimes |0\rangle \langle 0|_{\mathcal{X}}\right) \quad (3.137)$$

$$\geq 1 - \frac{3}{16N\delta^2} - \frac{1}{N\delta} \frac{\sqrt{N}}{2} \quad (3.138)$$

$$= 1 - \left(\frac{3}{16N\delta^2} + \frac{1}{2\sqrt{N}\delta} \right). \quad (3.139)$$

Returning to the original trace distance from (3.61), we have that, for ρ a pure state,

$$\left\| \left(\Psi[H_{\text{CL},N}, |0\rangle \langle 0|_{\mathcal{C}}, t] \circ \Xi[|0\rangle \langle 0|_{\mathcal{X}}] - \Xi[|0\rangle \langle 0|_{\mathcal{X}}] \right)(\rho) \right\|_1 \leq 2\sqrt{\frac{3}{16N\delta^2} + \frac{1}{2\sqrt{N}\delta}} \quad (3.140)$$

$$\leq 2\sqrt{\frac{3}{16\sqrt{N}\delta^2} + \frac{1}{2\sqrt{N}\delta^2}} \quad (3.141)$$

$$= O\left(N^{-\frac{1}{4}}\delta^{-1}\right). \quad (3.142)$$

Now, instead of a pure state, let ρ be an arbitrary qubit mixed state with spectral decomposition

$$\rho = \sum_{j=1}^2 p_j |\psi_j\rangle \langle \psi_j|. \quad (3.143)$$

Then, we may simply apply the linearity of quantum channels and the triangle inequality over a convex combination to attain

$$\left\| \left(\Psi [H_{\text{CL},N}, |0\rangle \langle 0|_{\mathcal{C}}, t] \circ \Xi[|0\rangle \langle 0|_{\mathcal{X}}] - \Xi[|0\rangle \langle 0|_{\mathcal{X}}] \right) (\rho) \right\|_1 \quad (3.144)$$

$$= \left\| \left(\Psi [H_{\text{CL},N}, |0\rangle \langle 0|_{\mathcal{C}}, t] \circ \Xi[|0\rangle \langle 0|_{\mathcal{X}}] - \Xi[|0\rangle \langle 0|_{\mathcal{X}}] \right) \left(\sum_{j=1}^2 p_j |\psi_j\rangle \langle \psi_j| \right) \right\|_1 \quad (3.145)$$

$$= \left\| \sum_{j=1}^2 p_j \left(\Psi [H_{\text{CL},N}, |0\rangle \langle 0|_{\mathcal{C}}, t] \circ \Xi[|0\rangle \langle 0|_{\mathcal{X}}] - \Xi[|0\rangle \langle 0|_{\mathcal{X}}] \right) (|\psi_j\rangle \langle \psi_j|) \right\|_1 \quad (3.146)$$

$$\leq \sum_{j=1}^2 p_j \left\| \left(\Psi [H_{\text{CL},N}, |0\rangle \langle 0|_{\mathcal{C}}, t] \circ \Xi[|0\rangle \langle 0|_{\mathcal{X}}] - \Xi[|0\rangle \langle 0|_{\mathcal{X}}] \right) (|\psi_j\rangle \langle \psi_j|) \right\|_1 \quad (3.147)$$

$$= O\left(N^{-\frac{1}{4}}\delta^{-1}\right), \quad (3.148)$$

which completes the proof of [Lemma 3.0.4](#). $\square$

3.2 Proof of generalized channel distance theorem

Proof of [Theorem 3.1.2](#). According to [\[12, Theorem 3.51\]](#), as Φ and Ψ are channels, for the identity channel I from $\mathcal{X}$ to $\mathcal{X}$, it is the case that

$$\|\Psi - \Phi\|_{\diamond} = \max_{|\psi\rangle \in \mathcal{X} \otimes \mathcal{X}} \|(\Psi \otimes I - \Phi \otimes I)(|\psi\rangle \langle \psi|)\|_1. \quad (3.149)$$

For an arbitrary $|\psi\rangle \in \mathcal{X} \otimes \mathcal{X}$, we may relate the trace distance above to a fidelity using the Fuchs-van de Graaf inequalities as

$$\|(\Psi \otimes I - \Phi \otimes I)(|\psi\rangle \langle \psi|)\|_1 \leq 2\sqrt{1 - F((\Psi \otimes I)(|\psi\rangle \langle \psi|), (\Phi \otimes I)(|\psi\rangle \langle \psi|))}. \quad (3.150)$$

Towards bounding the above expression, we let $\rho = |\psi\rangle\langle\psi|$ and ψ have Schmidt decomposition given by

$$|\psi\rangle = \sum_{j=1}^s \sqrt{p_j} |\phi_{A,j}\rangle |\phi_{B,j}\rangle, \quad (3.151)$$

for Schmidt rank s and probability vector $(p_1, \dots, p_s)$. Because Ψ is an isometric channel, $\Psi \otimes I$ is also isometric, and

$$(\Psi \otimes I)(\rho) = |\chi\rangle\langle\chi| \quad (3.152)$$

for some pure state $|\chi\rangle \in \mathcal{Y} \otimes \mathcal{X}$. In particular, the squared fidelity from (3.150) may be rewritten as

$$F((\Psi \otimes I)(\rho), (\Phi \otimes I)(\rho))^2 = F(|\chi\rangle\langle\chi|, (\Phi \otimes I)(\rho))^2 \quad (3.153)$$

$$= \langle\chi| (\Phi \otimes I)(\rho) |\chi\rangle \quad (3.154)$$

$$= \text{tr}(|\chi\rangle\langle\chi| (\Phi \otimes I)(\rho)) \quad (3.155)$$

$$= \text{tr}((\Psi \otimes I)(\rho)^\dagger (\Phi \otimes I)(\rho)). \quad (3.156)$$

At this point, we may expand $(\Psi \otimes I)(\rho)^\dagger$ and $(\Phi \otimes I)(\rho)$ using the Schmidt decomposition of $|\psi\rangle$, attaining

$$(\Psi \otimes I)(\rho)^\dagger = (\Psi \otimes I) \left(\sum_{j=1}^s \sum_{k=1}^s \sqrt{p_j p_k} |\phi_{A,j}\rangle \langle\phi_{A,k}| \otimes |\phi_{B,j}\rangle \langle\phi_{B,k}| \right)^\dagger \quad (3.157)$$

$$= \sum_{j=1}^s \sum_{k=1}^s \sqrt{p_j p_k} \left(\Psi(|\phi_{A,j}\rangle \langle\phi_{A,k}|)^\dagger \right) \otimes |\phi_{B,k}\rangle \langle\phi_{B,j}|, \quad (3.158)$$

$$(\Phi \otimes I)(\rho) = (\Phi \otimes I) \left(\sum_{l=1}^s \sum_{m=1}^s \sqrt{p_l p_m} |\phi_{A,l}\rangle \langle\phi_{A,m}| \otimes |\phi_{B,l}\rangle \langle\phi_{B,m}| \right) \quad (3.159)$$

$$= \sum_{l=1}^s \sum_{m=1}^s \sqrt{p_l p_m} \left(\Phi(|\phi_{A,l}\rangle \langle\phi_{A,m}|) \right) \otimes |\phi_{B,l}\rangle \langle\phi_{B,m}|. \quad (3.160)$$

The above two expressions may be substituted into (3.156) to get that

$$\text{tr} \left((\Psi \otimes I)(\rho)^\dagger (\Phi \otimes I)(\rho) \right) \quad (3.161)$$

$$= \text{tr} \left(\sum_{j=1}^s \sum_{k=1}^s \sum_{m=1}^s \sqrt{p_j^2 p_k p_m} \left(\Psi(|\phi_{A,j}\rangle \langle \phi_{A,k}|)^\dagger \Phi(|\phi_{A,j}\rangle \langle \phi_{A,m}|) \right) \otimes |\phi_{B,k}\rangle \langle \phi_{B,m}| \right) \quad (3.162)$$

$$= \text{tr} \left(\sum_{j=1}^s \sum_{k=1}^s p_j p_k \left(\Psi(|\phi_{A,j}\rangle \langle \phi_{A,k}|)^\dagger \Phi(|\phi_{A,j}\rangle \langle \phi_{A,k}|) \right) \right) \quad (3.163)$$

$$= \sum_{j=1}^s \sum_{k=1}^s p_j p_k \text{tr} \left(\Psi(|\phi_{A,j}\rangle \langle \phi_{A,k}|)^\dagger \Phi(|\phi_{A,j}\rangle \langle \phi_{A,k}|) \right) \quad (3.164)$$

$$= \sum_{j=1}^s \sum_{k=1}^s p_j p_k \langle \Psi(|\phi_{A,j}\rangle \langle \phi_{A,k}|), \Phi(|\phi_{A,j}\rangle \langle \phi_{A,k}|) \rangle. \quad (3.165)$$

The entire right-hand side of (3.150) may now be rewritten as

$$2\sqrt{1 - F((\Psi \otimes I)(|\psi\rangle \langle \psi|), (\Phi \otimes I)(|\psi\rangle \langle \psi|))^2} \quad (3.166)$$

$$= 2\sqrt{1 - \sum_{j=1}^s \sum_{k=1}^s p_j p_k \langle \Psi(|\phi_{A,j}\rangle \langle \phi_{A,k}|), \Phi(|\phi_{A,j}\rangle \langle \phi_{A,k}|) \rangle} \quad (3.167)$$

$$= 2\sqrt{\sum_{j=1}^s \sum_{k=1}^s p_j p_k \langle \Psi(|\phi_{A,j}\rangle \langle \phi_{A,k}|), (\Psi - \Phi)(|\phi_{A,j}\rangle \langle \phi_{A,k}|) \rangle}. \quad (3.168)$$

When considering the summands under the radical, there are two cases: that $j = k$ and that $j \neq k$. First, consider the case that $j = k$. Then, we may let $\sigma = |\phi_{A,j}\rangle \langle \phi_{A,k}| = |\phi_{A,j}\rangle \langle \phi_{A,j}|$ be the density operator of a pure state. It follows by Hölder's inequality for Schatten norms that

$$\langle \Psi(\sigma), (\Psi - \Phi)(\sigma) \rangle \leq \|\Psi(\sigma)\|_\infty \|(\Psi - \Phi)(\sigma)\|_1 \quad (3.169)$$

$$\leq \epsilon. \quad (3.170)$$

Second, consider the case that $j \neq k$. Then, let $\sigma = |\phi_{A,j}\rangle \langle \phi_{A,k}|$ be the outer product of orthogonal states, which is not the density operator of a quantum state. It follows by

Hölder's inequality for Schatten norms that

$$\langle \Psi(\sigma), (\Psi - \Phi)(\sigma) \rangle \leq \|\Psi(\sigma)\|_1 \|(\Psi - \Phi)(\sigma)\|_\infty \quad (3.171)$$

$$\leq \|\sigma\|_1 \|(\Psi - \Phi)(\sigma)\|_\infty \quad (3.172)$$

$$= \|(\Psi - \Phi)(\sigma)\|_\infty. \quad (3.173)$$

As Ψ and Φ are trace-preserving, $(\Psi - \Phi)(\sigma)$ is traceless. It turns out that the tracelessness of $(\Psi - \Phi)(\sigma)$ may be used to bound the spectral norm from (3.173) by a trace norm.

Lemma 3.2.1. *Let ρ be a traceless operator. Then,*

$$\|\rho\|_\infty \leq \frac{1}{2} \|\rho\|_1. \quad (3.174)$$

Proof. Let $|\lambda_1| \geq |\lambda_2| \geq \dots \geq |\lambda_r|$ for r being the rank of ρ denote the absolute values of the eigenvalues of ρ . As ρ is traceless,

$$\lambda_1 = \lambda_1 - \text{tr}(\rho) \quad (3.175)$$

$$= - \sum_{j=2}^r \lambda_j. \quad (3.176)$$

Consequently,

$$\|\rho\|_\infty = |\lambda_1| \quad (3.177)$$

$$= \frac{1}{2} |\lambda_1| + \frac{1}{2} \left| \sum_{j=2}^r \lambda_j \right| \quad (3.178)$$

$$\leq \frac{1}{2} \sum_{j=1}^r |\lambda_j| \quad (3.179)$$

$$= \frac{1}{2} \|\rho\|_1, \quad (3.180)$$

completing the proof. $\square$

Applying the lemma,

$$\|(\Psi - \Phi)(\sigma)\|_\infty \leq \frac{1}{2} \|(\Psi - \Phi)(\sigma)\|_1. \quad (3.181)$$

Unfortunately, we cannot apply the premise from the theorem immediately, as σ is not a density operator. However, it happens that σ may be expressed as a linear combination of four density operators.

Lemma 3.2.2. *For arbitrary orthogonal pure states $|\psi\rangle$ and $|\phi\rangle$,*

$$|\psi\rangle\langle\phi| = \frac{1}{2} \sum_{j=1}^4 i^j |\xi_j\rangle\langle\xi_j| \quad (3.182)$$

for $|\xi_j\rangle$ being a pure state for each $j \in \{1, 2, 3, 4\}$.

Proof. Let $|\xi_j\rangle$ for $j \in \{1, 2, 3, 4\}$ be a pure state defined as

$$|\xi_j\rangle = \frac{|\psi\rangle + i^j |\phi\rangle}{\sqrt{2}}. \quad (3.183)$$

Then,

$$\frac{1}{2} \sum_{j=1}^4 i^j |\xi_j\rangle\langle\xi_j| = \frac{1}{4} \sum_{j=1}^4 i^j (|\psi\rangle\langle\psi| + i^j |\phi\rangle\langle\psi| + (-i)^j |\psi\rangle\langle\phi| + |\phi\rangle\langle\phi|) \quad (3.184)$$

$$= \frac{1}{4} \left(\sum_{j=1}^4 i^j |\psi\rangle\langle\psi| + \sum_{j=1}^4 (-1)^j |\phi\rangle\langle\psi| + \sum_{j=1}^4 1^j |\psi\rangle\langle\phi| + \sum_{j=1}^4 i^j |\phi\rangle\langle\phi| \right) \quad (3.185)$$

$$= \frac{1}{4} \left(|\phi\rangle\langle\psi| + \sum_{j=1}^4 1^j |\psi\rangle\langle\phi| \right) \quad (3.186)$$

$$= |\psi\rangle\langle\phi|, \quad (3.187)$$

as desired. $\square$

By the lemma,

$$\sigma = \frac{1}{2} \sum_{l=1}^4 i^l \sigma_l \quad (3.188)$$

with σ_l being some density operator for each $l \in \{1, 2, 3, 4\}$. It follows that

$$\|(\Psi - \Phi)(\sigma)\|_\infty \leq \frac{1}{2} \|(\Psi - \Phi)(\sigma)\|_1 \quad (3.189)$$

$$= \frac{1}{2} \left\| (\Psi - \Phi) \left(\frac{1}{2} \sum_{l=1}^4 i^l \sigma_l \right) \right\|_1 \quad (3.190)$$

$$\leq \frac{1}{4} \sum_{l=1}^4 \|i^l (\Psi - \Phi)(\sigma_l)\|_1 \quad (3.191)$$

$$\leq \epsilon. \quad (3.192)$$

Thus, in both cases,

$$\langle \Psi(|\phi_{A,j}\rangle \langle \phi_{A,k}|), (\Psi - \Phi)(|\phi_{A,j}\rangle \langle \phi_{A,k}|) \rangle \leq \epsilon. \quad (3.193)$$

Ultimately, we may bound (3.168) by

$$2 \sqrt{\sum_{j=1}^s \sum_{k=1}^s p_j p_k \langle \Psi(|\phi_{A,j}\rangle \langle \phi_{A,k}|), (\Psi - \Phi)(|\phi_{A,j}\rangle \langle \phi_{A,k}|) \rangle} \leq 2\sqrt{\epsilon}, \quad (3.194)$$

completing the proof of [Theorem 3.1.2](#). □

Chapter 4

Conclusion

In the previous chapter, we rigorously proved that for any arbitrarily small choice of ϵ and δ , there exists a corresponding time-independent Hamiltonian H and ancilla state ρ such that H and ρ perform an (ϵ, δ) -approximation of an idealized reset. It turns out that our choice of Hamiltonian and ancilla incur certain costs in performing the approximate reset.

4.1 Cost model

In particular, we consider as costs the system size of the ancilla and the largest energy required for the Hamiltonian, given by the spectral norm of the Hamiltonian. By the main reset theorem, [Theorem 3.0.1](#), the Hamiltonian given by our construction that satisfies our choices of ϵ , δ , t_b , and t_a is given by

$$\frac{1}{4\tau} H_{\text{CL},N} \tag{4.1}$$

for, asymptotically, a timescale constant

$$\tau = O(\max(t_b, t_a) + \delta) \tag{4.2}$$

and system dimension

$$N = O(\epsilon^{-8} \delta^{-4} \tau^4). \tag{4.3}$$

By our definition of $H_{\text{CL},N}$ from [Definition 2.2.6](#), $H_{\text{CL},N}$ is N -dimensional and has spectral norm $\|H_{\text{CL},N}\|_\infty = 2\pi(N-1)$. As such, the asymptotic cost of performing an approximate reset by our construction that satisfies our parameters is a maximum energy that is

$$O(\epsilon^{-8}\delta^{-4}\tau^3) \tag{4.4}$$

and an ancilla with

$$O(\log(\tau) - \log(\epsilon) - \log(\delta)) \tag{4.5}$$

qubit registers. Intuitively, it makes sense that the required energy asymptotically tends to infinity as δ approaches 0, as the Hamiltonian must act more and more quickly on the interval to traverse between the initial state and the reset state in time 2δ as δ approaches 0. The intuition for the required energy asymptotically tending to infinity when ϵ approaches 0 is a little less clear, but recall the analyticity argument in the proof that no $\epsilon = 0$ reset approximation exists for [Theorem 2.0.1](#). In particular, as ϵ approaches 0, the distance from the approximation to the identity channel approaches a constant 0 on the interval $t \in [0, t_b - \delta]$ while approaching a constant 1 on the interval $t \in [t_b + \delta, t_b + t_a]$, which requires a polynomial with degree approaching infinity for approximation, though work on a thorough analysis of this statement is ongoing.

It is still unclear whether there is an asymptotic lower bound for either the energy or dimension in terms of the parameters. As such, a natural avenue of further investigation is to determine whether better asymptotic bounds on energy and system size are possible for the same set of parameters and what trade-offs may exist between the two costs.

In a related direction, we may also consider the relationship between ϵ and δ . In our construction, for N being fixed, ϵ and δ are polynomially related. In other words, to get within ϵ of an idealized reset, our construction requires an amount of time that is polynomial in ϵ . It turns out that if the constraints of our reset approximation problem are relaxed by fixing $t_b = 0$ instead of allowing the parameter to be arbitrary, it is possible to achieve an exponentially fast approximate reset by using a different construction that approximates the channel distance as function of time with a Gaussian centered at time 0 [\[11\]](#).

4.2 Connecting reset to Lindblad simulation

Recall that the timed sequence of channels that was proposed to simulate arbitrary Lindblad evolution, Υ_s with s from [\(1.18\)](#), consisted of repeatedly interleaving a Γ step along

with two other time-independent Hamiltonian evolution channels at equally spaced times. Of that sequence, we constructed a Hamiltonian and ancilla pair that approximates just the Γ . Supposing that we find Hamiltonian and ancilla pairs that individually approximate each of the two Hamiltonian evolution channels, we still need to merge those Hamiltonians and ancillas with our reset Hamiltonian and ancilla to get just one Hamiltonian and ancilla pair that simulates the entire sequence of channel applications.

Towards that goal, let us consider the problem of merging just two pairs of Hamiltonian and ancilla state. Formally, suppose that we have our first Hamiltonian H_1 on $\mathcal{X} \otimes \mathcal{Y}$ and its corresponding initial ancilla state ρ_1 on $\mathcal{Y}$ that perform an (ϵ, δ) -approximation of Υ_{s_1} for $s_1 = (\Phi_1, t_1)$ and a second Hamiltonian H_2 on $\mathcal{X} \otimes \mathcal{Z}$ and its corresponding ancilla state ρ_2 on $\mathcal{Z}$ that perform an (ϵ, δ) -approximation of Υ_{s_2} for $s_2 = (\Phi_2, t_1 + t_2)$. Our hope is that it is possible to construct H_3 on $\mathcal{X} \otimes \mathcal{Y} \otimes \mathcal{Z}$ and ρ_3 on $\mathcal{Y} \otimes \mathcal{Z}$ simply from H_1 , H_2 , ρ_1 , and ρ_2 such that H_3 and ρ_3 perform an (ϵ, δ) -approximation of $\Upsilon_{s_1 s_2}$. Indeed, we conjecture that it may be possible to simply construct a Hamiltonian and ancilla with the desired properties as

$$H_3 = H_1 \otimes I_{\mathcal{Z}} + H_2 \otimes I_{\mathcal{Y}} \quad (4.6)$$

and

$$\rho_3 = \rho_1 \otimes \rho_2 \quad (4.7)$$

as long as δ is sufficiently small compared to t_1 and t_2 , and our pairs H_1 and ρ_1 and H_2 and ρ_2 both satisfy an additional property termed *dormancy*, which shall be defined shortly.

In particular, the two concerns that arise when simply adding H_1 and H_2 as above is that evolution by H_2 around t_1 might interfere with the approximation of Υ_{s_1} and that evolution by H_1 around t_2 might interfere with the approximation of Υ_{s_2} . It turns out that the definition of (ϵ, δ) approximation from [Definition 1.3.1](#) protects us from the first concern, as its first property requires that H_2 and ρ_2 leave the system state completely unchanged up to error ϵ before time t_2 regardless of what the system state is. It might appear as though the second property of [Definition 1.3.1](#) protects us from the second concern, as it requires that H_1 and ρ_1 keep the state unchanged after the channel is applied at t_1 .

However, in the specific case of the channel being approximated being a replacement channel, the second property does not distinguish between whether the Hamiltonian and ancilla are continuously applying the identity channel after t_1 or whether the Hamiltonian and ancilla are continuously reapplying the replacement channel after t_1 , as illustrated in [Figure 4.1](#). For instance, if a Hamiltonian and ancilla pair (H_1, ρ_1) is expected to perform

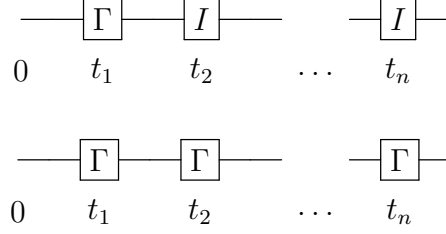


Figure 4.1: The two behaviors that [Definition 1.3.1](#) cannot distinguish between, as both take an arbitrary input state to $|0\rangle$ at time t_1 and maintain that state.

a reset just at t_1 , but it performs the reset at t_1 and also at a second time $t_2 > t_1$, the pair cannot be expected to compose well with another pair (H_2, ρ_2) that performs an operation between t_1 and t_2 , as the action of (H_2, ρ_2) will be reset by (H_1, ρ_1) .

In fact, a Hamiltonian and ancilla pair can be constructed from our reset approximation that indeed reapplies the reset channel multiple times. Specifically, consider extending the exchange register $\mathcal{X}$ to m qubits with the respective ancilla qubits initialized to $|0\rangle$, and generalizing the unitary behavior of $U_{\text{CL},N}$ that applies U_{SX} to the combined $\mathcal{S} \otimes \mathcal{X}$ register when the clock overflows to a new repeated-reset unitary $U_{\text{REP},n,m}$ that applies an X on $\mathcal{S}$ followed by a cyclic shift of the $(m+1)$ -qubit combined $\mathcal{S} \otimes \mathcal{X}$ register when the clock of n qubits overflows, as shown in [Figure 4.2](#). Applying $U_{\text{REP},n,m}$ results in a reset being performed the first m times that the clock register overflows. Furthermore, as $U_{\text{REP},n,m}$ is a cyclic permutation unitary like $U_{\text{CL},N}$, the Hamiltonian that generates it will perform exact resets at the times corresponding to its discrete applications.

As such, we define the dormancy property as an extension of (ϵ, δ) -approximation such that the Hamiltonian truly leaves the target register unchanged, up to error ϵ , even after applying an arbitrary unitary U after the channel is applied around time t_b .

Definition 4.2.1. *Let H on $\mathcal{X} \otimes \mathcal{Y}$ and ρ on $\mathcal{Y}$ perform an (ϵ, δ) -approximation of Φ with time t_b before the application of Φ and time t_a after the application of Φ . The approximation is said to be dormant if, for any unitary U on $\mathcal{X}$ and $t_1, t_2 \in [t_b + \delta, t_b + t_a]$ with $t_1 < t_2$,*

$$\|(I \otimes \text{tr}) \circ [e^{-iH(t_2-t_1)}] \circ [U \otimes I] \circ [e^{-iHt_1}] \circ \Xi[\rho] - [U] \circ \Phi\|_{\diamond} \leq \epsilon. \quad (4.8)$$

We conjecture that our construction approximating Γ from [Theorem 3.0.1](#) satisfies dormancy and that any approximation of an isometric channel also satisfies dormancy. If the two above conjectures turn out to be true, it seems plausible that Υ_s might have an (ϵ, δ) -approximation constructed by a simple sum of Hamiltonians and tensor product of

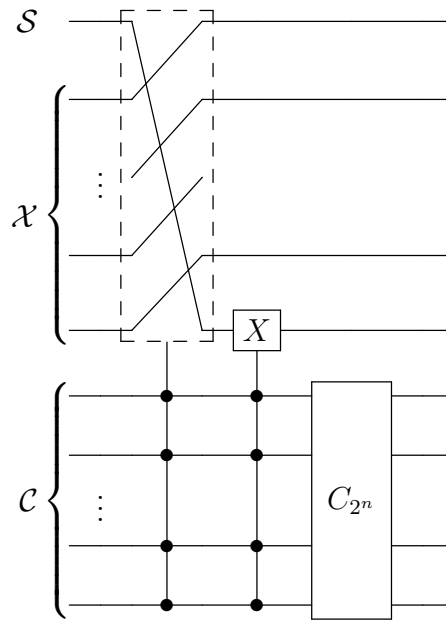


Figure 4.2: Circuit diagram of $U_{\text{REP},n,m}$ that resets register $\mathcal{S}$ every 2^n applications with up to m resets for register $\mathcal{X}$ consisting of m qubits.

ancilla states from approximations of the constituent channels from (1.12), which would be a time-independent Schrödinger simulation of the original time-independent Lindblad evolution.

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APPENDICES

Appendix A

Python Code for Numerical Plots

Below is the Python code used with Python 3.11.1, numpy 1.26.1, scipy 1.11.3, and matplotlib 3.8.0 to generate the numerical plots in [Figure 2.3](#) and [Figure 2.2](#):

```
import sys
import argparse

import numpy as np
from scipy import linalg
import matplotlib.pyplot as plt
import matplotlib.animation as animation

# matrix representation of cnot gate
CNOT = np.asmatrix(
    np.array(
        [
            [1, 0, 0, 0],
            [0, 1, 0, 0],
            [0, 0, 0, 1],
            [0, 0, 1, 0],
        ]
    )
)

def get_canon_unitary(dim):
    # get canonical cyclic permutation unitary
    base = np.asmatrix(np.zeros((dim, dim)))
    for j in range(dim):
```

```

        base[(j + 1) % dim, j] = 1
    return base

def get_canon_hamiltonian(dim):
    # get canonical cyclic permutation hamiltonian
    dft = np.asmatrix(linalg.dft(dim, scale="sqrt{n}"))
    base = np.asmatrix(np.diag(np.arange(0, dim) * 2 * np.pi))
    return dft.H @ base @ dft

def get_reset_conjugator(dim):
    # get permutation operator used in canonical decomposition of reset unitary
    return np.asmatrix(np.kron(CNOT, np.eye(dim // 4)))

def get_reset_unitary(dim):
    # get reset unitary
    conjugator = get_reset_conjugator(dim)
    return conjugator @ get_canon_unitary(dim) @ conjugator

def get_reset_hamiltonian(dim):
    # get reset hamiltonian
    conjugator = get_reset_conjugator(dim)
    return conjugator @ get_canon_hamiltonian(dim) @ conjugator

if __name__ == "__main__":
    # set up args
    parser = argparse.ArgumentParser(
        description="Run numerical simulation of reset Hamiltonian."
    )
    subparsers = parser.add_subparsers(required=True)

    parser_test = subparsers.add_parser("test", help="run sanity checks")
    parser_test.set_defaults(test=True)

    parser_sim = subparsers.add_parser("sim", help="run simulation")
    parser_sim.add_argument(
        "dim",
        metavar="n",
        type=int,
        help="dimension of environment. must be power of 2 and at least 4.",
    )

```

```

parser_sim.add_argument(
    "steps",
    metavar="t",
    type=int,
    help="number of steps to subdivide the simulation time interval.",
)
parser_sim.add_argument(
    "-s",
    "--state",
    metavar="s",
    type=int,
    help="start state of system. must be 0 or 1.",
)
parser_sim.add_argument(
    "-p",
    "--plot",
    action="store_true",
    help="whether simulation data should be plotted.",
)
parser_sim.set_defaults(test=False, state=0, plot=False)

# get args
args = parser.parse_args()

if args.test:
    # sanity check reset hamiltonian
    dim = 32
    print("performing sanity check...")

    out = "FAIL"
    if np.allclose(linalg.expm(-1j * get_reset_hamiltonian(dim)), np.eye(dim)):
        out = "PASS"
    print("exp(-ih) close to identity: {}".format(out))
    out = "FAIL"
    if np.allclose(
        linalg.expm(-1j * get_reset_hamiltonian(dim) / dim), get_reset_unitary(dim)
    ):
        out = "PASS"
    print("exp(-ih/d) close to reset unitary: {}".format(out))
else:
    # check args
    dim = args.dim
    steps = args.steps
    start_state = args.state

```

```

if (dim < 4) or (dim & (dim - 1) != 0):
    print("dimension must be power of 2 and at least 4")
    sys.exit(1)

if (start_state != 0) and (start_state != 1):
    print("start state must be 0 or 1")
    sys.exit(1)

# set up state vector
vec_table = []
vec_table.append(np.asmatrix(np.zeros((dim, 1), dtype=np.cfloat)))
if start_state == 0:
    vec_table[0][0] = 1
else:
    vec_table[0][dim // 2] = 1

# get hamiltonian
h = get_reset_hamiltonian(dim)

# run simulation
for j in range(steps - 1):
    print("iter {} of {}".format(j + 1, steps))
    vec_table.append(linalg.expm(h * (-1j) * j / steps) @ vec_table[0])

# get probability of observing classical state 0
probs = np.array(
    [
        sum([np.square(np.abs(x[y])).item() for y in range(dim // 2)])
        for x in vec_table
    ]
)

if args.plot:
    # output plot of probabilities over time to file
    fig, ax = plt.subplots()
    ax.set_title(
        "Probability of measuring ket 0 for reset Hamiltonian with N = {},\n"
        "starting with basis state {}".format(dim, start_state)
    )
    ax.set_xlabel("Time")
    ax.set_ylabel("Probability".format(dim // 2))
    ax.plot(np.arange(steps) / steps, probs)
    fig.savefig("plot_prob_from_{}_with_dim_{}.png".format(start_state, dim))
else:

```

```
# output probabilities to file
with open(
    "data_prob_from_{}_with_dim_{}.txt".format(start_state, dim), "w"
) as f:
    f.write("dat\n")
    f.write("\n".join((str(x) for x in probs)))
```