Efficiency and Equilibria in Centralized and Decentralized Insurance Markets

by

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A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Doctor of Philosophy
in
Actuarial Science

Waterloo, Ontario, Canada, 2025

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Examining Committee Membership

The following served on the Examining Committee for this thesis. The decision of the Examining Committee is by majority vote.

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Author's Declaration

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of Contributions

I am the sole author of Chapters 1, 2, and 7.

Chapter 3 is joint work with Mario Ghossoub, and has been published in a peer-reviewed journal:

- [63] Ghossoub, M., and **Zhu, M.B.** (2024). Stackelberg Equilibria with Multiple Policyholders. *Insurance: Mathematics and Economics*, 116(1):189-201.

Chapter 4 is joint work with Tim Boonen and Mario Ghossoub, and has been published in a peer-reviewed journal:

- [113] **Zhu, M.B.**, Boonen, T.J., and Ghossoub, M. (2023). Equilibria and Efficiency in a Reinsurance Market. *Insurance: Mathematics and Economics*, 113(1):24-49.

Chapter 5 is joint work with Mario Ghossoub, and has been submitted to a peer-reviewed journal. The arXiv reference is below:

- [62] Ghossoub, M., and **Zhu, M.B.** Efficiency in Pure-Exchange Economies with Risk-Averse Monetary Utilities. https://arxiv.org/abs/2406.02712

Chapter 6 is joint work with Mario Ghossoub and Wing Fung (Alfred) Chong, and is forthcoming in a peer-reviewed journal:

- [64] Ghossoub, M., **Zhu, M.B.**, and Chong, W.F. (2025+). Pareto-Optimal Peer-to-Peer Risk Sharing with Robust Distortion Risk Measures. *ASTIN Bulletin*, forthcoming.

Abstract

This thesis studies Pareto efficiency and market equilibrium in the context of insurance markets. Given a specific model of insurance markets, it is of great practical interest in identifying those risk allocations that are deemed desirable to each agent in the market. Equally as important are market equilibria, the allocations and prices that result from agents' decisions given the structure of the market. The research presented in this thesis studies the relationship between these two concepts in both centralized and decentralized markets of insurance. Throughout, we provide various characterization results for Pareto-efficient contracts and market equilibria in a variety of settings. These results are illustrated with numerical examples, including an in-depth application to markets of flood risk insurance.

Chapters 3 and 4 focus on centralized markets of insurance and the interaction between policyholder and insurer. In a setting with a monopolistic insurer and multiple policyholders, we first obtain a characterization of Pareto efficiency in Chapter 3. We argue that the appropriate notion of equilibrium is the Stackelberg equilibrium (SE), and provide a characterization thereof. However, SE do not induce a welfare gain to the policyholders in this case, echoing the conclusions of recent work in the literature. In an effort to address this issue, we study a market with strategic price competition on the supply side of insurance in Chapter 4. We show that the Subgame Perfect Nash Equilibrium (SPNE) is the appropriate solution concept, and generalizes Stackelberg equilibria to the case of multiple insurers. We obtain a characterization of SPNE and show that the policyholder can realize a strict welfare gain, thereby illustrating the benefit to the policyholder of competition in the market.

As an alternative, decentralized markets of insurance allow for risk exchange between agents without the presence of centralized suppliers such as insurance companies. Chapter 5 situates this strand of the literature within the wider literature on Pareto-optimal allocations in pure-exchange economies. In that chapter, we study Pareto efficiency in a pure-exchange economy where agents' preferences are represented by risk-averse monetary utilities. These coincide with law-invariant monetary utilities, and they can be shown to correspond to the class of monotone, (quasi-)concave, Schur concave, and translation-invariant utility functionals. We show that Pareto optima exist and are comonotone, and we provide a novel characterization thereof in the case of law-invariant positively homogeneous monetary utilities. Our results show that the shape of the allocations depends primarily on each agent's assessment of the tail of the aggregate risk. This characterization provides an easily implementable algorithm that fully determines the shape of Pareto-optimal allocations. Chapter 6 extends these results to a setting of peer-to-peer

risk sharing, where each agent's preferences are given by robust distortion risk measures. Through a characterization of Pareto-efficient contracts, we perform an analysis of the differences between decentralized and centralized markets of insurance. We conclude with an in-depth discussion of the advantages and disadvantages of a decentralized insurance scheme in this setting.

Acknowledgements

I would like to thank my supervisor, Prof. Mario Ghossoub, for his invaluable support and guidance throughout my academic career. My journey from a naive undergraduate to a PhD would not have been possible without his encouragement and advice, in academia and beyond.

I greatly appreciate the efforts and insights of my committee members, Prof. Steven Vanduffel, Prof. Anqi Li, Prof. David Saunders, and Prof. Ruodu Wang.

I am grateful to my co-authors Tim Boonen and Alfred Chong for their help and insight on the research in this thesis. In particular, I would like to thank Tim for hosting me in Amsterdam, where Chapter 4 of this thesis was formulated.

I would also like to thank the M3 faculty for continuously supporting my academic endeavours. Special thanks to Prof. David Saunders, Prof. Ruodu Wang, Prof. David Landriault, Prof. Mary Hardy, and Prof. Lisa Gao. The M3 staff are the best university staff in the world, and are always available to lend a hand. Special thanks to Mary Lou Dufton, Greg Preston, and Jenny Tieu, who have all made my life immensely easier through their efforts.

My time at M3 was greatly enriched by my wonderful friends in the fourth floor hallway. You know who you are: Burnt Bro, Circle Moon, Cheechee, Pot Brother, Shadow Brother, Shi Sisters, and Sack.

I am eternally grateful for the love and encouragement from my immediate family, Xiao-Feng, Ying, and Jennifer.

My relationship with Meixi is Pareto efficient, a Nash equilibrium, and globally optimal. Thank you for your love.

Dedication

This thesis is dedicated to my parents.

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Chapter 1

Introduction

The two fundamental elements of an insurance market are the risky financial positions of interest and the preferences of the agents. A risk, in the present context, refers to the experience of a future uncertain quantifiable loss. Agents are the participants in an insurance market, who can be insurance or reinsurance companies, individual policyholders or groups thereof, etc. Given the differing attitudes of each of the agents towards risk, it may be mutually beneficial for these agents to engage in exchange. Insurance markets exist precisely to enable and facilitate this exchange between parties.

Given a model of an insurance market, the natural question is to identify an "optimal" allocation of risk among the agents. However, it is often unclear how optimality of an allocation should be defined, since an allocation favouring one agent may be detrimental to another in a scenario with competing interests. Instead, it is common in the economics literature to study *Pareto optimality* or *Pareto efficiency* of allocations – those allocations where no agent can improve without harming another. Here, the word "optimality" is somewhat of a misnomer, as the notion of Pareto optimality does not identify the best possible allocation by any specified criterion. It may be more appropriate to interpret Pareto optimality as a minimal requirement: those allocations that are not Pareto optimal are automatically disqualified from being considered a "good" outcome. Hence, identifying the set of Pareto-optimal allocations in a market is an essential first step when analyzing potential outcomes. In the remainder of this thesis, the terms *Pareto optimality*, *Pareto efficiency*, or simply *efficiency* all refer to the same concept: broadly speaking, an efficient allocation is one that "we would like to happen".

On the other hand, the risk allocations that emerge from the participation of agents within a market are also affected by the nature or structure of the market itself. After the

agents' preferences are specified, it is assumed that agents will act "rationally" in some sense, in accordance with these preferences. The resulting allocations are referred to as *equilibrium allocations* or *equilibria*, and their study in the economics literature is vast. The use of the term "equilibrium" suggests the notion of stability: when an equilibrium is reached, there is no incentive for any agent to deviate. Some classical examples of equilibria include Nash equilibria for competitive games, and Arrow-Debreu equilibria in general equilibrium theory. In contrast with efficient allocations, equilibrium allocations are those that "we would expect to happen".

In this thesis, we make the distinction between centralized and decentralized markets of insurance. A centralized market is one where the providers of insurance are a pre-specified set of agents, and the rest of the agents are buyers of insurance. The theory of efficient contracting in centralized markets of insurance has its roots in seminal work by Borch [23] and Arrow [5], who showed that stop-loss contracts are efficient in the two-agent model when both the policyholder's and insurer's preferences admit an Expected-Utility (EU) representation, the policyholder is risk-averse, and the insurer is risk neutral. It is shown in this case that premia are priced via the equivalence principle: that is, premia are determined by the expected indemnified loss (e.g., as in Arrow [5]), sometimes multiplied by a loading factor to represent variable costs to the reinsurer (e.g., as in Raviv [95]). The preferences of the risk-neutral insurer imply that in a PO allocation, the premium is a function of the indemnified risk, known as a premium principle. Under this premium principle, the efficient indemnification is obtained by solving a problem of demand for the insurer. This problem has been extensively studied in the two-agent case, and we refer to Gollier [66] and Schlesinger [98], for instance, for a review of the literature on optimal insurance with EU preferences. Of the numerous generalizations of the original EU model, we highlight the two that are most relevant to the present thesis: the use of distortion risk measures (DRMs), and the consideration of heterogeneous beliefs. The special case in which agents' preferences are given by distortion risk measures has seen recent interest, due to the connection between DRMs and Yaari's Dual Theory [112], a special case of the Rank-Dependent Utility model of decision-making under ambiguity (e.g., Quiggin [92, 93]). These risk measures satisfy certain desirable axioms (e.g., Wang et al. [108]), including translation invariance. Efficiency of insurance contracts has been of great interest in the literature, and closed-form solutions are given for Value-at-Risk (VaR) and Expected Shortfall (ES) (e.g., Cai et al. [26] and Chi and Tan [36]), and more general DRMs when indemnity functions are assumed to satisfy the so-called *no-sabotage condition* (e.g., Cui et al. [37] and Assa [8]). The latter paper introduces the Marginal Indemnification Function (MIF) approach, which has been fruitful in characterizing solutions for DRM preferences (e.g., Zhuang et al. [114]).

However, the focus on identifying Pareto-optimal allocations often neglects the impact of market forces in a given insurance market. Even in the two-agent model, the asymmetry of bargaining power between the insurer and the policyholder can invalidate certain Pareto-optimal allocations as likely outcomes of centralized markets of insurance. Furthermore, the study of Pareto efficiency takes prices (or premia) in the market as given, and it does not address the question of how premia are determined at a market equilibrium. A popular equilibrium concept that takes into account the aforementioned asymmetry between the insurer and the policyholder is the Stackelberg equilibrium (or Bowley optimum), which applies to a monopolistic insurance market where the insurer and the policyholder move sequentially in an economic game, with the insurer as the leader and the policyholder as the follower. Specifically, the insurer is able to set the prices of insurance before the policyholder has a chance to purchase insurance. The contracts that are expected to emerge in markets with this leader-follower structure are Stackelberg equilibria, which were first examined in an insurance context by Chan and Gerber [32] for agents with exponential utility functions. These results were further extended to the case of DRMs by Cheung et al. [35], and Li and Young [77] characterize Stackelberg equilibria when agents use a mean-variance functional. This setting has recently been examined in a dynamic context in continuous time, as in Chen and Shen [34] and Cao et al. [27].

As an alternative to centralized markets of insurance, decentralized insurance markets offer the agents the possibility of sharing risks directly among themselves, thereby avoiding interaction with traditional insurance providers. In a decentralized market, each agent can be interpreted as both a provider and buyer of insurance under the right circumstances. In practice, decentralized markets of insurance are often found in the form of risk-sharing pooling arrangements, in which every agent pays a contribution fee to a pool in return for partial coverage of their monetary risk. For more details on real-world examples of peer-to-peer risk sharing, we refer to Abdikerimova and Feng [2]. Decentralized insurance is also prevalent in markets for catastrophe risk: we refer to Bollman and Wang [15] for an overview, as well as Feng et al. [54] for the flood risk market with proportional insurance.

A primary concern of decentralized risk-sharing markets is the Pareto-efficiency of the allocations of the aggregate risk. The seminal work of Borch [24] and Wilson [110] showed that when agents have risk-averse expected-utility (EU) preferences, each individual's allocation of risk is a nondecreasing deterministic function of the aggregate risk in the market at a Pareto optimum. This, in turn, leads to a complete characterization of efficient allocations (through the so-called Borch rule) and shows that Pareto optima (PO) are in fact comonotone. The subsequent literature has then examined several extensions of the classical model beyond EU. Notably, the effect of ambiguity-sensitive agents on optimal allocations has attracted significant attention in the literature. For instance, Chateauneuf

et al. [33], Dana [40], De Castro and Chateauneuf [44], and Beissner and Werner [12] study economies with non-probabilistic uncertainty as in the Choquet-Expected Utility model of Schmeidler [101]. The more general class of variational preferences of Maccheroni et al. [84]) was examined by Dana and LeVan [42] and Ravanelli and Svindland [94], for instance. In the context of risk management, Pareto-optimal allocations have been studied when preferences satisfy the properties of translation invariance, risk aversion, convexity, coherence, or a combination thereof: we refer to Jouini et al. [69], Filipovic and Svindland [56], and Mao and Wang [85] and references therein for an overview of the topic. Of particular recent interest is the risk-sharing problem for agents with quantile-based risk measures that are not necessarily convex, as studied by Embrechts et al. [53, 52] and Liu [82]. In contrast with the literature on risk-averse preferences, optimal allocations can be counter-monotone rather than comonotone, as shown in Lauzier et al. [76] and Ghossoub et al. [60, 61].

While there is comparably less literature on the notion of equilibria in decentralized markets, research in this direction is closely linked with the classical literature on competitive equilibria. Each agent participates by buying or selling insurance in a risk-sharing market where the price of insurance is itself endogenously determined at an equilibrium. Competitive equilibria have been characterized in the case of EU (Aase [1]), the case of VaR and ES (Embrechts et al. [52, 53]), the case of convex DRMs (Boonen [16]), and the case of general DRMs for comonotonic indemnities (Boonen et al. [21]).

1.1 Structure of the Thesis

The research presented in this thesis explores the two notions of efficiency and equilibria within the context of insurance markets. In Chapter 2, economic models of these markets are formally introduced using modern tools of quantitative risk management. The notions of centralized and decentralized markets of insurance are defined and contrasted, and the mathematical foundations of these models, as well as important game-theoretic notions, are introduced. Chapters 3 and 4 focus on characterizing and comparing efficiency and equilibria in centralized markets of insurance. In Chapter 3, we apply the Stackelberg model to a market with one provider of insurance and n policyholders, an extension of the two-agent case previously considered in the literature. The main results of this chapter are the characterizations of both efficient contracts and Stackelberg equilibria under general assumptions on preferences, which also facilitate an analysis of the relationship between these notions. In particular, we find that Stackelberg equilibria have the undesirable property that they do not provide any welfare gain to the policyholders. To address

this issue, we introduce a market with multiple providers of insurance in Chapter 4. As an extension of the Stackelberg equilibrium, we introduce the notion of subgame-perfect Nash equilibrium and provide a characterization thereof.

Chapters 5 and 6 study Pareto efficiency in decentralized markets of insurance. In a departure from the setting of risk measures as is standard in the actuarial literature, Chapter 5 formulates the risk-sharing problem in terms of utility maximization. The main result provides a characterization of Pareto efficiency when preferences are given by positively homogeneous risk-averse utilities. Chapter 6 applies these results to the context of risk sharing where preferences are given by robust distortion risk measures. A comparison of efficient contracting between centralized and decentralized markets is explored through the real-world example of flood risk insurance. Concluding remarks and avenues for future research are discussed in Chapter 7.

Chapter 2

Models of Insurance Markets

2.1 Mathematical Background

Throughout, a risk in an insurance market is modelled as a random variable (i.e., a measurable function) on a measurable space (Ω, Σ). When X is a risk and $\omega \in \Omega$, we interpret $X(\omega)$ as the monetary loss associated with the risk X in the state of the world ω. Denote the set of bounded measurable functions on this space by $B(\Sigma)$, and its positive elements by $B_+(\Sigma)$. For all $X \in B(\Sigma)$, define the supnorm $||\cdot||_{\sup}$ by

$$||X||_{\sup} := \sup\{|X(\omega)| : \omega \in \Omega\} < \infty\,.$$

Then $B(\Sigma)$ is a Banach space (e.g., Dunford & Schwartz [51, IV.5.1]). In the presence of a probability measure $\mathbb{P}$, we also consider risks as elements of the standard L^p space on the probability space $(\Omega, \Sigma, \mathbb{P})$, for $1 \leq p \leq \infty$. These spaces, denoted $L^p(\Omega, \Sigma, \mathbb{P})$, are Banach spaces with respect to the p-norm, defined by

$$||X||_p := \begin{cases} \left(\int_\Omega |X|^p \, d\mathbb{P}\right)^{1/p}, & p \neq \infty\,, \\ \operatorname{ess\,sup} |X|\,, & p = \infty\,. \end{cases}$$

Of particular relevance are the cases of $p = 1$ and and $p = \infty$, which correspond to integrable random variables and essentially bounded random variables respectively. Here, with a slight abuse of notation, we consider both elements of $B(\Sigma)$ and elements of $L^p(\Omega, \Sigma, \mathbb{P})$ as "risks" and use the same notation accordingly. However, elements of $B(\Sigma)$ are measurable

functions, whereas elements of L^p are classes of measurable functions up to $\mathbb{P}$-almost-everywhere equivalence. This distinction will not be of particular relevance for this thesis, although we emphasize that expressions of the form $X(\omega)$ refer to X as a measurable function before any embedding into a L^p space. In the following, we use the notation $\mathcal{X}$ to be the space of "risks", where $\mathcal{X}$ is either $B(\Sigma)$ or $L^p(\Omega, \Sigma, \mathbb{P})$. As discussed above, $\mathcal{X}$ is a Banach space, which provides the mathematical foundation for the analysis of insurance markets. We use the notation $\mathcal{X}_+$ to refer to the (almost-surely) non-negative elements of $\mathcal{X}$.

2.2 Agents and Preferences

The participants within an insurance market are the *agents*. Each agent measures risk by a real-valued function on $\mathcal{X}$, often called a *risk measure* and denoted by ϱ. We use the convention that larger values of ϱ correspond to higher evaluations of risk, and hence an agent prefers smaller values of ϱ over larger ones. We say that an agent's preference is represented by ϱ if for all $X, Y \in \mathcal{X}$, the agent prefers X to Y if and only if

$$\varrho(X) < \varrho(Y)\,.$$

Some properties of risk measures are given below.

Definition 2.2.1. A risk measure $\varrho : \mathcal{X} \to \mathbb{R}$ is said to be:

- *Translation invariant* if $\varrho(X + c) = \varrho(X) + c$, for all $X \in \mathcal{X}$ and $c \in \mathbb{R}$.
- *Normalized* if $\varrho(c) = c$, for all $c \in \mathbb{R}$.
- *Monotone* if $\varrho(X) \leq \varrho(Y)$, for all $X, Y \in \mathcal{X}$ such that $X \leq Y$.
- *Positively homogeneous* if $\varrho(kX) = k\varrho(X)$, for all $X \in \mathcal{X}$ and $k \in \mathbb{R}^+$.
- *Subadditive* if $\varrho(X + Y) \leq \varrho(X) + \varrho(Y)$, for all $X, Y \in \mathcal{X}$.
- *Coherent* if it is translation invariant, monotone, positively homogeneous, and subadditive.

Of additional interest is how risk measures behave over risks that are comonotone: i.e., those risks are that exhibit the maximum positive dependence, as given in the following definition.

Definition 2.2.2. Two random variables $X, Y \in \mathcal{X}$ are said to be *comonotone* if for all $\omega_1, \omega_2 \in \Omega$,

$$[X(\omega_1) - X(\omega_2)]\,[Y(\omega_1) - Y(\omega_2)] \geq 0\,.$$

Similarly, a random vector $(X_1, \ldots, X_k) \in \mathcal{X}^k$ is said to be *comonotone* if

$$[X_i(\omega_1) - X_i(\omega_2)]\,[X_j(\omega_1) - X_j(\omega_2)] \geq 0$$

for all $\omega_1, \omega_2 \in \Omega$ and $i, j \in \{1, \ldots, k\}$.

Definition 2.2.3. A risk measure $\varrho : \mathcal{X} \to \mathbb{R}$ is said to be *comonotone additive* if $\varrho(X + Y) = \varrho(X) + \varrho(Y)$ for all $X, Y \in \mathcal{X}$ such that X and Y are comonotone.

Intuitively, comonotone additivity of a risk measure reflects the fact that there is no benefit to diversification that can be gained from combining risks that are maximally positively dependent. Many commonly used risk measures in practice are comonotone additive, including the Value-at-Risk (VaR) and the Expected Shortfall (ES).

Definition 2.2.4. The *Value-at-Risk (VaR)* at level $\alpha \in (0, 1)$ of a random variable $X \in \mathcal{X}$ under the probability measure $\mathbb{P}$ is

$$\mathrm{VaR}_\alpha^{\mathbb{P}}(X) := \inf\left\{t \in \mathbb{R} : \mathbb{P}(X > t) \leq \alpha\right\}\,.$$

Definition 2.2.5. The *Expected Shortfall (ES)* at level $\alpha \in (0, 1)$ of a random variable $X \in \mathcal{X}$ under the probability measure $\mathbb{P}$ is

$$\mathrm{ES}_\alpha^{\mathbb{P}}(Z) := \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}_u^{\mathbb{P}}(X)\,du\,.$$

More generally, a classical result of Schmeidler [100] shows that any monotone and comonotone additive risk measure has a representation as a Choquet integral. We refer to these risk measures as *Choquet risk measures*, and they are specified by the following definitions.

Definition 2.2.6. A set function $\upsilon : \Sigma \to \mathbb{R}$ is a *capacity* if:

- $\upsilon(\varnothing) = 0$ and $\upsilon(S) < \infty$; and,
- If $A, B \in \Sigma$ are such that $A \subseteq B$, then $\upsilon(A) \leq \upsilon(B)$.

A capacity υ is said to be *normalized* if $\upsilon(\Omega) = 1$. For $A \in \Sigma$, a capacity υ is said to be *continuous from below* at A if for any increasing sequence $(A_k)_{k=1}^{\infty}$ converging to A, we have $\lim_{k\to\infty} \upsilon(A_k) = \upsilon(A)$. We denote the set of all capacities by $\mathcal{C}$.

Definition 2.2.7. The *Choquet integral* of $X \in \mathcal{X}$ with respect to a capacity υ is defined as

$$\int X \, d\upsilon = \int_0^{\infty} \upsilon(X > t) \, dt + \int_{-\infty}^{0} [\upsilon(X > t) - \upsilon(\Omega)] \, dt \, .$$

Definition 2.2.8. A risk measure ϱ is a *Choquet risk measure* if there exists a normalized capacity υ such that for all $X \in \mathcal{X}$,

$$\varrho(X) = \int X \, d\upsilon \, .$$

Normalization of ν insures that ϱ is translation invariant. As an additional technical assumption, we require the capacity υ in this definition to be continuous from below at S.

For a detailed treatment of capacities and Choquet integration, we refer to Denneberg [47] and Marinacci and Montrucchio [86]. The following result characterizes Choquet risk measures in terms of its useful properties.

Proposition 2.2.9. *A risk measure ϱ is monotone, translation invariant, and comonotone additive if and only if it is a Choquet risk measure.*

Proof. See Föllmer and Schied [57, Theorem 4.88]. □

2.3 Efficiency of Allocations

The purpose of an insurance market is to ultimately determine a desirable allocation of risk to each agent through trade. Consider a market with n agents. For each $i \in \{1, \dots, n\}$, denote the risk measure of agent i by ρ_i and their initial endowment by $X_i \in \mathcal{X}$. The aggregate endowment, or the total amount of risk present in this market, is $S := \sum_{i=1}^{n} X_i$. The definition of an allocation is given by the following.

Definition 2.3.1. The set of *allocations* of S is

$$\mathcal{A} := \left\{ \{Y_i\}_{i=1}^{n} \in \mathcal{X}^n : \sum_{i=1}^{n} Y_i = S \right\} .$$

Implicit in this definition is the assumption that reallocating risk does not incur transaction costs. For an axiomatic study on the frictional costs associated with reallocation and the consequences thereof, we refer to Ghossoub et al. [59]. However, we assume throughout that there are no frictions in the transfer of risk, and so all possible outcomes of an insurance market can be represented as an element of $\mathcal{A}$. The context of the market model often dictates that only a subset of the allocations $\mathcal{A}$ are admissible as market outcomes. Let $\tilde{\mathcal{A}} \subseteq \mathcal{A}$ denote the set of admissible allocations. It is safe to assume that the allocation of initial endowments $(X_1, \dots, X_n)$ is always admissible, since this allocation corresponds to no market participation. Furthermore, we always allow each agent to decline to participate in the market, which motivates the following definition.

Definition 2.3.2. An allocation $(Y_1, \dots, Y_n) \in \mathcal{A}$ is said to be *individually rational (IR)* if it incentivizes each agent to participate in the market. That is,

$$\rho_i(Y_i) \leq \rho_i(X_i)\,, \quad \forall i \in \{1, \dots, n\}\,.$$

We denote the set of individually rational allocations by $\mathcal{IR}$.

In fact, if we allow each agent to decline to participate, then those allocations that are not individually rational should not be admissible as market outcomes. In other words, we assume that $\tilde{\mathcal{A}} \subseteq \mathcal{IR}$, which highlights a necessary distinction between $\tilde{\mathcal{A}}$ and $\mathcal{A}$. Those admissible allocations that are *efficient* are given by the following.

Definition 2.3.3. An allocation $(Y_1, \dots, Y_n) \in \tilde{\mathcal{A}}$ is said to be *Pareto efficient* or *Pareto optimal (PO)* if there does not exist another admissible allocation $(\tilde{Y}_1, \dots, \tilde{Y}_n) \in \tilde{\mathcal{A}}$ such that

$$\rho_i(\tilde{Y}_i) \leq \rho_i(Y_i)\,, \quad \forall i \in \{1, \dots, n\}\,,$$

with at least one strict inequality. We denote the set of Pareto-optimal allocations by $\mathcal{PO}$.

In other words, a Pareto-optimal allocation is one upon which no agent can improve their welfare without negatively impacting the welfare of another. A primary goal in any market of insurance is to determine which admissible allocations are Pareto optimal.

2.3.1 Centralized vs. Decentralized Markets

The definition of Pareto efficiency given in Definition 2.3.3 is valid under very general setups of market models. However, this definition depends heavily on the set of admissible

allocations $\tilde{\mathcal{A}}$ under consideration. Not only must a Pareto-optimal allocation be admissible, but the notion of Pareto optimality itself requires a comparison between admissible allocations. For different contexts of insurance markets, the set of admissible allocations can vary significantly. In the following, we distinguish between two main categories of insurance markets: centralized markets and decentralized markets.

A *centralized market* of insurance is one where the supply of insurance is centralized within a specified set of agents. The remaining agents are the buyers of insurance. Perhaps the most familiar example of a centralized insurance market is the two-agent model of traditional insurance with one insurance company and one prospective policyholder. Here, the supplier is the insurance company and the buyer is the individual who wishes to purchase insurance. Suppose that the initial endowment of the individual is a non-negative risk $X_1 \in \mathcal{X}$, and that the initial endowment of the insurance company is 0. In this context, the insurance company typically indemnifies a portion of the risk X_1 in exchange for a premium payment. Let π_1 denote the premium charged, and let $I_1 : \mathbb{R}_+ \to \mathbb{R}_+$ denote the indemnification function. The resulting risk exposure of the policyholder, after engaging in this insurance contract, is

$$X_1 - I_1(X_1) + \pi_1 \,.$$

The resulting risk exposure of the insurance company is

$$I_1(X_1) - \pi_1 \,.$$

It is customary for insurance companies to further specify the indemnity functions that are admissible. A common choice is the set of non-decreasing 1-Lipschitz indemnities:

$$\mathcal{I} := \{I : \mathbb{R}_+ \to \mathbb{R}_+ : I\,(0) = 0\,,\ 0 \leq I(x) - I(y) \leq x - y \text{ for all } x > y \geq 0\}\,.$$

These indemnity functions satisfy the so-called *no-sabotage condition* of Carlier and Dana [28, 29], which both disincentivizes under-reporting of risk and eliminates *ex post* moral hazard. Specifically, since I is increasing, underreporting the true risk amount would only result in a lower payout from the insurance contract. Furthermore, if $I(x) - I(y) > x - y$ for $x > y$, it follows that if a loss of y is experienced, the policyholder has an incentive to create more loss of the amount $x - y$, since the increase in payout exceeds the increase in loss suffered. This is precisely the moral hazard that is avoided by the indemnity functions in $\mathcal{I}$. This moral hazard also motivates the notion of a comonotone market, which we study in Chapter 5 in the context of pure-exchange economies.

Therefore the set of admissible allocations in this two-agent market is

$$\tilde{\mathcal{A}} = \{(X_1 - I_1(X_1) + \pi_1,\ I_1(X_1) - \pi_1)\ :\ I_1 \in \mathcal{I}\,, \pi_1 \in \mathbb{R}_+\} \cap \mathcal{IR}\,.$$

The study of Pareto optimality in this two-agent market is extensive in the literature.

More generally, we can consider centralized markets with m suppliers of insurance and n buyers of insurance. These are markets with $m + n$ agents, and insurance contracts are typically described in terms of premium payments and indemnification functions as above. For convenience, we refer to the suppliers of insurance as *insurers* and the buyers of insurance as *policyholders* in this thesis. Chapters 3 and 4 of this thesis are concerned with centralized markets of insurance.

On the other hand, a *decentralized market* of insurance is one where each agent is both a buyer and provider of insurance. Examples of these markets include risk-sharing pools, peer-to-peer insurance, and mutual aid. The admissible allocations in a decentralized market are those that result from a risk-sharing rule implemented by the administrators or managers of the pool. A common choice is the implementation of an aggregate risk-sharing rule, where the allocation of risk depends only on the aggregate risk S in the market. These allocations are of the form

$$(g_1(S), \dots, g_n(S))$$

where $\sum_{i=1}^n g_i = Id$, and arise naturally in the setting of risk sharing. More generally, when agents are risk averse, the classical comonotone improvement theorem yields the existence of a Pareto-optimal allocation of the above form, with each g_i being an increasing function. We will see in Chapter 5 that in this setting, there is no harm in restricting the set of admissible allocations to the following:

$$\tilde{\mathcal{A}} = \left\{ (g_1(S), \dots, g_n(S)) \, : \, \sum_{i=1}^n g_i = Id \, , \, g_i \text{ increasing} \right\} \cap \mathcal{IR} \, .$$

2.4 Market Equilibria

A market equilibrium is a concept that depends heavily on the structure of the market itself and the actions and strategies that each agent can take. Formally, an equilibrium is modelled within the context of game theory. When viewing the market as an economic game, an equilibrium is a set of strategies that is stable, in the sense that agents are not incentivized to change their strategies. It so follows that the study of equilibria requires a formalization of what a *strategy* is for each agent. However, in decentralized insurance markets, it can be unclear how the participating agents can strategically exert influence on the allocations of the market, which can be largely dependent on the third-party administrators of the risk-sharing pools. For this reason, we will not focus on the study of equilibria

in decentralized markets, although notions of equilibria are certainly still possible in such a setting. Instead, we primarily consider market equilibria in the context of centralized markets, where the actions and strategies of each agent are more intuitive and natural. Specifically, the strategy of a buyer of insurance is to decide which insurance contract to purchase. The strategy of the supplier of insurance is to decide how to price the available contracts, through a process we will refer to as *strategic pricing*.

2.4.1 Games and Strategies

The notion of a *game* is the central object of study in game theory. However, the structure of the game-theoretic models of insurance markets that we consider is fairly simple, and does not require the complete formal definition of these objects. In particular, we will not formalize the role of information in these games, nor do we consider games with probabilistic strategies. For a comprehensive overview of the extensive-form representation of games, we refer to Osborne and Rubinstein [90, Chapter 6] and Maschler et al. [87, Chapter 3], the latter of which formally introduces infinite games as is required in our setting. For our purposes, the following definition of a game will suffice.

Definition 2.4.1. A *game in extensive-form* is an ordered vector

$$(N, V, E, x_0, (V_i)_{i\in N}, u)\,,$$

where:

- N is a finite set of players.
- (V, E, x_0) is a directed and rooted tree. Here, V is the set of vertices, E is the set of edges, and $x_0 \in V$ is the root.
- $(V_i)_{i\in N}$ is a partition of the set of vertices V. The interpretation is that if a node is in V_i, then it is the i-th player's turn to move.
- u is a function associating every leaf of the tree with an outcome. For our purposes, outcomes are quantifiable in monetary terms, and can therefore be represented by real-valued vectors in $\mathbb{R}^{|N|}$. We will use the convention that agents prefer larger values of u to smaller ones.

Given a game, the goal is to understand which strategies are the "solutions" to the game. This first requires a definition of a strategy, which is given below.

Definition 2.4.2. A *strategy for player* i is a map from each node $x \in V_i$ to an edge leading out of x. In other words, player i's strategy completely specifies what move player i takes whenever it is their turn to move. We denote the set of all strategies for player i by S_i.

Definition 2.4.3. The set of all possible strategies is $S_1 \times \ldots \times S_{|N|}$. Elements of this set are called *strategies*.

A strategy $s = (s_1, \ldots, s_n) \in S_1 \times \ldots \times S_{|N|}$ therefore completely specifies the choices of each player in the game. In other words, each strategy corresponds to a unique path from the root node x_0 to a leaf node. We can therefore define the outcome of a strategy $u(s_1, \ldots, s_n)$ as the function u applied to the leaf node corresponding to the strategy $(s_1, \ldots, s_n)$. In the following, we use the notation $u_i(s_1, \ldots, s_n)$ to denote the i-th entry of the real-valued vector $u(s_1, \ldots, s_n)$.

When considering insurance markets, we are primarily concerned with identifying the strategies that are stable. The intuition is that when agents participate in a market, they may change their decisions over time when given feedback. This process can be described as a game played repeatedly until the agents eventually converge to a stable "equilibrium" strategy. Therefore, for our purposes, the solution concept for an insurance market is the Nash equilibrium: a strategy from which no agent has an incentive to deviate.

Definition 2.4.4. A strategy $s^* = (s_1^*, \ldots, s_n^*)$ is a *Nash equilibrium* if for each player $i \in N$ and each strategy $s_i \in S_i$, we have

$$u_i(s^*) \geq u_i(s_i, s_{-i}^*)\,,$$

where the notation (s_i, s_{-i}^*) refers to the vector s^* with its i-th element replaced by s_i.

2.4.2 The Stackelberg Game and Strategic Pricing

A centralized insurance market with only $m = 1$ provider of insurance is modelled by a Stackelberg game. In this model, the monopolistic insurer has the first-mover's advantage, which reflects the fact that the insurance company can set their pricing mechanism without the input of the policyholders. When this game is represented in extensive-form, the insurer is the player who moves at the root node by selecting their premium principle.

We assume that the insurer can charge a different premium to each policyholder, and that it always charges premia that are linear. Let $\mathcal{X}^*$ be the set of continuous linear

functionals from $\mathcal{X}$ to $\mathbb{R}$. We denote the set of all admissible premium principles charged to policyholder i by $\aleph_i \subseteq \mathcal{X}^*$, and we let $\Pi_i \in \aleph_i*$ refer to the specific premium principle charged by the insurer to policyholder i. The role of the insurer in the Stackelberg game is to select a premium principle for every policyholder in the market. Let $\aleph$ denote the Cartesian product of $\aleph_i$ for $i \in \{1, \ldots, n\}$. The insurer's strategy is then completely characterized by an element

$$(\Pi_1, \ldots, \Pi_n) \in \aleph\,.$$

The first level of the game tree consists of an edge from the root node x_0 to each possible selection of premium principles in $\aleph$. The next step is for each policyholder to select the indemnification function they wish to purchase. We assume that the set of admissible allocations is the set of non-decreasing 1-Lipschitz indemnities $\mathcal{I}$. Since the policyholder knows the premium principle being charged, their selection of indemnity depends on the premium principle. The policyholder's strategy is therefore determined by a function from the space of premium principles $\mathcal{X}^*$ to the set of admissible indemnities $\mathcal{I}$.

To simplify the structure of the game, note that in this model, after the premium principles have been specified, the choices of each policyholder are independent of those of the other policyholders. In other words, the decision of a policyholder is not affected by the premium principle charged to someone else, or by the indemnity selection of someone else. Hence, we can simplify the extensive-form representation of this game by considering the n policyholders as a collective. The second level of the game tree consists of an edge from each premium principle selection $\aleph$ to a vector of indemnities $\mathcal{I}^n$. We refer to elements $(I_1, \ldots, I_n) \in \mathcal{I}^n$ as *indemnity structures* or *indemnity selections*, with the interpretation that I_i refers to the indemnity selected by policyholder i. The actions of every agent in the market are completely represented by a pair $(\{I_i\}_{i=1}^n, \{\Pi_i\}_{i=1}^n) \in \mathcal{I}^n \times \aleph$. We refer to a pair of this form as a *market mechanism*.

This model is a tree of depth two, where the leaf nodes are completely determined by a market mechanism. The payoff function u of this game is

$$\begin{aligned} u : \mathcal{I}^n \times \aleph &\to \mathbb{R}^{n+1} \\ (\{I_i\}_{i=1}^n, \{\Pi_i\}_{i=1}^n) &\mapsto - \begin{bmatrix} \rho_1(X_1 - I_1(X_1) + \Pi_1(I_1(X_1))) \\ \vdots \\ \rho_n(X_n - I_n(X_n) + \Pi_n(I_n(X_n))) \\ \rho^{IN}\left(\sum_{i=1}^n \left(I_i^*\left(X_i\right) - \pi_i^*\right)\right) \end{bmatrix}. \end{aligned}$$

Here, the preference of each policyholder $i \in \{1, \ldots, n\}$ is represented by the risk measure ρ_i, and the preference of the insurer is represented by the risk measure ρ^{IN}. The minus

sign in this expression is due to our convention that agents prefer smaller values of risk measures to larger ones (but larger values of u to smaller ones).

Given the Stackelberg game, the goal of the insurer is therefore to select the premium principles that minimize its end-of-period risk exposure. This notion is captured in the definition of the Stackelberg equilibrium, which we provide below.

Definition 2.4.5. A market mechanism $(\{I_i^*\}_{i=1}^n, \{\Pi_i^*\}_{i=1}^n)$ is said to be a *Stackelberg Equilibrium* (SE) if it satisfies the following:

1. $I_i^* \in \arg\min_{I \in \mathcal{I}} \rho_i \left(X_i - I(X_i) + \Pi_i^*(I(X_i))\right)$, for all $i \in \{1, \ldots, n\}$;
2. $\rho^{IN}\left(\sum_{i=1}^n \left(I_i^*\left(X_i\right) - \Pi_i^*(I_i^*(X_i))\right)\right) \leq \rho^{IN}\left(\sum_{i=1}^n \left(\tilde{I}_i\left(X_i\right) - \tilde{\Pi}_i(\tilde{I}_i(X_i))\right)\right)$, for all $\tilde{\Pi} \in \aleph$, and all $\left\{\tilde{I}_i\right\}_{i=1}^n \in \mathcal{I}^n$ such that

$$\tilde{I}_i \in \arg\min_{I \in \mathcal{I}} \rho_i \left(X_i - I(X_i) + \tilde{\Pi}_i(I(X))\right), \quad \forall\, i \in \{1, \ldots, n\}.$$

However, implicit in this definition is the assumption that the policyholders act rationally. In fact, a Stackelberg equilibrium as defined by Definition 2.4.5 is not a strategy, since a strategy requires a specification of the policyholders' actions for every possible premium principle. A Stackelberg equilibrium only specifies the policyholders' strategies I^* given the premium principles Π^*. To fully describe this equilibrium in terms of strategies requires a refinement of the Nash equilibrium known as *subgame perfection*. This is the focus of the following section, which introduces the definition of a *subgame perfect Nash equilibrium (SPNE)*.

2.4.3 Subgame Perfection

The notion of subgame perfection was first developed by Selten [102], and we refer to Maschler et al. [87, Chapter 7] for an in-depth discussion. Roughly, an SPNE requires that each player's strategy be "rational" at all points of the game, including those states of the game that are not reached. In the context of a centralized insurance market, if certain "irrational" strategies were allowed, the policyholders could refuse to purchase insurance if premia are high, even when it would be beneficial to do so. This could coerce the insurers to lower their prices in order to convince the policyholders to purchase insurance. However, the insurers could elect to charge high premia regardless, if they deem that the irrational

strategy of the policyholders is not credible. This is precisely the situation that the concept of SPNE seeks to avoid, by requiring that a Nash equilibrium be induced at every possible game state. These game states are formally defined as *subgames*, and we refer to Maschler et al. [87, Definition 3.11] for a fully rigorous definition. The following provides a simplified description.

Definition 2.4.6. A *subgame* is a subtree of a game in extensive form, with the requirement that all players know with certainty whether or not the game has reached the subgame.

In this thesis, we will consider centralized market models with multiple providers of insurance, each of whom does not know the premium principles charged by the others until all premium principles are selected. After all premium principles have been specified, the buyers of insurance make their decisions. By Definition 2.4.6, a subtree starting from a node at which insurance is purchased represents a subgame. However, a subtree starting from a node at which only one insurer decides on a premium principle is not a subgame (unless they are the last to do so), since this insurer does not know of the decisions of the other insurers. An SPNE is defined by the following.

Definition 2.4.7. A strategy $s^* = (s_1^*, \ldots, s_n^*)$ is a *subgame perfect Nash equilibrium (SPNE)* if for every subgame, the restriction of the strategy s^* to the subgame is a Nash equilibrium.

Note that since the entire game is a subgame, SPNEs are NEs. However, the converse does not hold in general. For this reason, the concept of SPNE is often referred to as a refinement of the Nash equilibrium.

It is well-known that every SPNE can be characterized through the process of backward induction. Given a game tree of finite depth, this process iteratively determines an SPNE by solving for Nash equilibria at the final stages of the game. The payoff of the deepest subgames are then attached to the root of the subgame, reducing the size of the tree, upon which the process is repeated. For an in-depth discussion of this process, we refer to Maschler et al. [87, Remark 4.50] and Osborne and Rubinstein [90, Subsection 6.6.1]. Characterizing SPNEs in a centralized market with multiple suppliers of insurance is the focus of Chapter 4.

Chapter 3

Stackelberg Equilibria with Multiple Policyholders

3.1 Introduction

In this chapter, we examine Stackelberg equilibria in a centralized market with multiple policyholders. Closely related to both this chapter and the subsequent one is Boonen and Ghossoub [20], who were the first to examine the link between Stackelberg equilibria and Pareto-efficient contracting in the context of insurance. Their main result shows that the contracts induced by Stackelberg equilibria are Pareto optimal. However, the converse is not always true: instead, only those Pareto-optimal contracts that attribute no welfare gain to the policyholder can be obtained as Stackelberg equilibria. This is problematic, as it leaves the policyholder with no incentive to purchase insurance, which suggests that the first-mover advantage is too powerful a feature in these markets. The material in this chapter comprise an extension of this previous work from the two-agent case to the n-agent case. Specifically, we consider a market with multiple policyholders and a monopolistic central insurer, where each agent evaluates risk according to a Choquet risk measure. Our model is also similar to that of Boonen et al. [17], with the distinction that we only assume the insurer's risk measure to be coherent. As a first step, we provide a characterization of the set of Pareto-optimal contracts in this market using the marginal indemnity function (MIF) approach. We characterize Stackelberg equilibria in this market, thereby providing the first characterization of Stackelberg equilibria in the case of multiple policyholders. However, we show that even in this case, Stackelberg equilibria compose a subset of Pareto-optimal contracts that do not induce a welfare gain to the policyholders,

hence echoing the findings of Boonen and Ghossoub [20].

As an illustration of our results, we revisit the example of flood risk insurance examined by Boonen et al. [17]. Using historical data, they find that pooling flood risk into one entity generates welfare gains across the insurance market. Accordingly, they make the policy recommendation that the flood risk insurance market be centralized into a singular federal agency. In the present chapter, we expand on their numerical analysis in two ways. First, we relax the assumption that all policyholders use a coherent risk measure, and we show that the conclusions of Boonen et al. [17] are still valid in this case. Second, we show through the lens of Stackelberg equilibria that a monopolistic insurer has an incentive to drastically increase prices, and that this incentive is large when less correlated risks are aggregated. We therefore argue that additional regulation is necessary in such a market structure.

The remainder of this chapter is structured as follows. Section 3.2 introduces the insurance market model. We characterize Pareto optima in Section 3.3, and we characterize Stackelberg equilibria in Section 3.4. Section 3.5 provides an analysis of the efficiency of Stackelberg equilibria. We re-examine the flood risk insurance market in Section 3.6. Section 3.7 concludes. The proofs of our main results can be found at the end of this chapter in Section 3.8.

3.2 Problem Setup

For this chapter, we take the set of risks to be the vector space of all bounded measurable functions $B(\Sigma)$. We consider a centralized market with n agents in this market, each predisposed to an insurable loss. Let $X_i \in B^+(\Sigma)$ represent the insurable loss of agent $i \in \{1, \dots, n\}$, and let

$$M := \max_{i \in \{1,\dots,n\}} \|X_i\|_{sup} < \infty \, .$$

We assume that these agents all have access to a monopolistic insurance market, where insurance coverage is provided by a single centralized insurer (i.e., $m = 1$). We do not assume any specific dependence structure among the risks X_i.

3.2.1 Insurance Contracts

We assume that the collection of *ex ante* admissible indemnification functions in this market is the customary set of 1-Lipschitz indemnities $\mathcal{I}$, which prevents *ex post* moral hazard that could arise from misreporting of actual realized losses by policyholders.

An insurance contract consists of an indemnity function $I_i \in \mathcal{I}$ for each agent i, as well as a premium $\pi_i \in \mathbb{R}$ paid by each agent i.

Definition 3.2.1. An insurance contract is a pair $(\{I_i\}_{i=1}^n, \{\pi_i\}_{i=1}^n) \in \mathcal{I}^n \times \mathbb{R}^n$.

Given an insurance contract $(\{I_i\}_{i=1}^n, \{\pi_i\}_{i=1}^n) \in \mathcal{I}^n \times \mathbb{R}^n$, the resulting risk exposure of agent $i \in \{1, \dots, n\}$ is given by

$$X_i - I_i(X_i) + \pi_i \,.$$

The risk exposure of the insurer is then given by

$$\sum_{i=1}^{n} \left(I_i(X_i) - \pi_i\right) \,.$$

We assume that the preferences of each agent $i \in \{1, \dots, n\}$ and those of the insurer are represented by given risk measures ρ_i and ρ^{IN}, respectively. Additionally, we assume that $\rho^{IN}(0) = \rho_i(0) = 0$, for all $i \in \{1, \dots, n\}$.

3.2.2 Individual Rationality and Pareto Optimality

The goal of the insurance market is to ultimately achieve an individually rational and Pareto-optimal contract. These contracts are efficient, in the sense that each agent is incentivized to participate in the market, and no agent can improve their welfare without negatively impacting the welfare of another. The general definition of these notions are given in Definitions 2.3.2 and 2.3.3. In the present setting of a centralized market with a monopolistic insurer, these definitions are the following.

Definition 3.2.2. A contract $(\{I_i^*\}_{i=1}^n, \{\pi_i^*\}_{i=1}^n) \in \mathcal{I}^n \times \mathbb{R}^n$ is said to be:

- **Individually Rational** (IR) if it incentivizes the parties to participate in the market, that is,

$$\rho_i\left(X_i - I_i^*(X_i) + \pi_i^*\right) \leq \rho_i(X_i)\,, \quad \forall\, i \in \{1, \dots, n\}\,,$$
$$\rho^{IN}\left(\sum_{i=1}^{n} \left(I_i^*(X_i) - \pi_i^*\right)\right) \leq \rho^{IN}(0) = 0\,.$$

This definition can be equivalently stated in terms of welfare gain. The *welfare gain* for each agent is the improvement in welfare over the status quo. Hence, the welfare gain for policyholder i under this contract is

$$\rho_i\left(X_i\right) - \rho_i\left(X_i - I_i^*\left(X_i\right) + \pi_i^*\right),$$

and the welfare gain for the insurer is

$$-\rho^{IN}\left(\sum_{i=1}^{n}\left(I_i^*\left(X_i\right) - \pi_i^*\right)\right).$$

An IR contract is one that does not induce a negative welfare gain in any agent.

- **Pareto Optimal** (PO) if it is IR and there does not exist any other IR contract $\left(\{I_i\}_{i=1}^n, \{\pi_i\}_{i=1}^n\right)$ such that

$$\begin{aligned}\rho_i\left(X_i - I_i\left(X_i\right) + \pi_i\right) &\leq \rho_i\left(X_i - I_i^*\left(X_i\right) + \pi_i^*\right), \quad \forall\, i \in \{1,\ldots,n\},\\ \rho^{IN}\left(\sum_{i=1}^{n}\left(I_i\left(X_i\right) - \pi_i\right)\right) &\leq \rho^{IN}\left(\sum_{i=1}^{n}\left(I_i^*\left(X_i\right) - \pi_i^*\right)\right),\end{aligned}$$

with at least one strict inequality.

Let $\mathcal{IR}$ denote the set of all IR contracts, and let $\mathcal{PO} \subset \mathcal{IR}$ denote the set of all PO contracts. Then, in particular, $\mathcal{IR} \neq \varnothing$ since it contains the status-quo.

Remark 3.2.3. *Note that if ρ^{IN} is assumed to be monotone and translation-invariant, then for any $\left(\{I_i\}_{i=1}^n, \{\pi_i\}_{i=1}^n\right) \in \mathcal{IR}$,*

$$0 = \rho^{IN}(0) \leq \rho^{IN}\left(\sum_{i=1}^{n} I_i\left(X_i\right)\right) \leq \sum_{i=1}^{n}\pi_i. \tag{3.1}$$

The following result provides a useful characterization of PO contracts when the risk measures are translation invariant. A similar result can be found in Boonen et al. [17]. For completeness, we provide a proof in Section 3.8.

Proposition 3.2.4. *Suppose that ρ^{IN} and ρ_i are translation invariant, for all $i \in \{1,\ldots,n\}$. Then the following are equivalent:*

1. $(\{I_i^*\}_{i=1}^n, \{\pi_i^*\}_{i=1}^n) \in \mathcal{PO}$.

2. *The contract* $(\{I_i^*\}_{i=1}^n, \{\pi_i^*\}_{i=1}^n)$ *solves*

$$\inf_{(\{I_i\}_{i=1}^n, \{\pi_i\}_{i=1}^n) \in \mathcal{IR}} \left\{ \rho^{IN}\left(\sum_{i=1}^n (I_i(X_i) - \pi_i)\right) + \sum_{i=1}^n \rho_i(X_i - I_i(X_i) + \pi_i) \right\}. \tag{3.2}$$

Remark 3.2.5. *By translation invariance, a contract* $(\{I_i^*\}_{i=1}^n, \{\pi_i^*\}_{i=1}^n)$ *solves* (3.2) *if and only if:*

1. *The indemnity functions* $\{I_i^*\}_{i=1}^n$ *solve*

$$\inf_{\{I_i\}_{i=1}^n \in \mathcal{I}^n} \left\{ \rho^{IN}\left(\sum_{i=1}^n I_i(X_i)\right) + \sum_{i=1}^n \rho_i(X_i - I_i(X_i)) \right\}. \tag{3.3}$$

2. $\{\pi_i^*\}_{i=1}^n$ *are chosen so that* $(\{I_i^*\}_{i=1}^n, \{\pi_i^*\}_{i=1}^n) \in \mathcal{IR}$.

We provide an implicit characterization of PO contracts in Section 3.3, under certain conditions.

3.2.3 A Sequential Insurance Market

In Section 3.4, we model the insurance market with a sequential structure in which the monopolistic insurer has the first-mover advantage. That is, the insurer determines pricing mechanisms first, and each policyholder selects their own optimal indemnity function afterwards. The notion of market equilibrium in this setting is that of a Stackelberg equilibrium, which was first introduced in a (re)insurance setting by Chan and Gerber [32].

Premium Principles

In our model, we assume that the insurer charges premia that are linear, similarly to the setting of Boonen and Ghossoub [20]. Let $ba(\Sigma)$ denote the linear space of all bounded finitely additive set functions on (S, Σ). When equipped with the variation norm $\|.\|_v$, $ba(\Sigma)$ is a Banach space, and $(ba(\Sigma), \|.\|_v)$ is isometrically isomorphic to the norm-dual of the Banach space $(B(\Sigma), \|.\|_{sup})$ (e.g., [51, IV.5.1]) via the duality

$$\langle \phi, \mu \rangle = \int \phi \, d\mu, \ \forall \mu \in ba(\Sigma), \ \forall \, \phi \in B(\Sigma).$$

Denote by $ca(\Sigma)$ the collection of all countably additive elements of $ba(\Sigma)$. Then $ca(\Sigma)$ is a $\|.\|_v$-closed (and hence complete) linear subspace of $ba(\Sigma)$, which includes the set of probability measures. Henceforth, a collection of (linear) premium principles will be identified with a subset of $ba(\Sigma)$ via the above duality, and it will be called weak*-compact if it is compact in the weak* topology inherited from $\sigma(ba(\Sigma), B(\Sigma))$.

For each $i \in \{1, \ldots, n\}$ and indemnity function $I \in \mathcal{I}$, the contract $I(X_i)$ is $\sigma(X_i)$-measurable. The premium principle for such an indemnity is a functional

$$\Pi_i : B^+(\sigma(X_i)) \to \mathbb{R}\,.$$

We impose the following additional technical assumption:

Assumption 3.2.6. The set of admissible premium principles $\aleph_i$ are those functionals Π_i such that

$$\Pi_i\left(I_i(X_i)\right) = \int I_i(X_i)\, d\mu_i\,,$$

for all $I_i \in \mathcal{I}$, where μ_i is a finite countably additive measure on $(S, \sigma(X_i))$.

Indeed, all linear premium principles that are commonly used in practice do indeed satisfy Assumption 3.2.6. This includes, for example, the expected value principle, given by $\Pi(I(X)) = (1+\theta)\,\mathbb{E}^{\mathbb{P}}[I(X)]$ for some probability measure $\mathbb{P}$ and some constant $\theta > 0$ (take, in this case, the measure $(1+\theta)\,\mathbb{P}$). Other examples include the economic premium principle introduced by Bühlmann [25] and the weighted capital allocation functionals of Furman and Zitikis [58]. These are of the form

$$\Pi\left(I(X)\right) = \frac{\mathbb{E}^{\mathbb{P}}[I(X)\, w(Z)]}{\mathbb{E}^{\mathbb{P}}[w(Z)]}\,,$$

where $Z := \sum_{i=1}^n X_i$ and w is a weighting function. Indeed, taking the measure $\mathbb{Q}$ corresponding to the Radon-Nikodym derivative

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{w(Z)}{\mathbb{E}^{\mathbb{P}}[w(Z)]}$$

gives the representation of Assumption 3.2.6 when $\mathbb{Q}$ is restricted to $\sigma(X_i)$. For instance, taking $w(Z) = e^{\alpha Z}$, for some $\alpha > 0$ gives the following modified Esscher premium principle:

$$\Pi_i\left(I_i(X_i)\right) = \mathbb{E}^{\mathbb{P}}\left[I_i(X_i)\, h_{\alpha,Z}\right]\,,$$

where $h_{\alpha,Z} = \frac{e^{\alpha Z}}{\mathbb{E}^{\mathbb{P}}[e^{\alpha Z}]}$.

The Stackelberg Equilibrium

When the insurer has the first-mover advantage, it is incentivized to choose the premium principles that minimize its end-of-period risk exposure, assuming that each policyholder behaves rationally (i.e., in accordance to their own preferences). This notion is captured in the definition of the Stackelberg equilibrium, as given in Definition 2.4.5.

Denote the set of all Stackelberg equilibria by $\mathcal{SE}$. We provide an implicit characterization of Stackelberg equilibria under certain conditions in Section 3.4.

3.2.4 Marginal Indemnification Functions

In this chapter, we make use of the Marginal Indemnification Function (MIF) approach in the proofs of our results. The MIF approach has been fruitful in the recent literature in addressing problems of optimal contracting (e.g., Assa [8] and Zhuang et al. [114]). This methodology focuses on first identifying the optimal derivative of the indemnity, which is then used to recover the indemnification function itself.

For our purposes, it will be useful to explicitly define the set of derivatives of elements of the set $\mathcal{I}$. To this end, let $\mathcal{B}[0, M]$ denote the Borel sigma-algebra on $[0, M]$, and let λ denote the Lebesgue measure. Then $L^\infty([0, M], \mathcal{B}([0, M]), \lambda)$ is the set of essentially bounded λ-measurable functions on $[0, M]$, which we denote by $L^\infty([0, M])$ for convenience.

Definition 3.2.7. We define the set of marginal indemnifications $\mathcal{MI}$ to be the set of all functions $h \in L^\infty([0, M])$ whose integral is a 1-Lipschitz indemnification function. That is, $h \in \mathcal{MI}$ if and only if the function

$$\begin{aligned} I_h : [0, M] &\to [0, M] \\ x &\mapsto \int_0^x h(t)\, dt \end{aligned}$$

is an element of $\mathcal{I}$.

The following proposition gives a useful characterization of the set $\mathcal{MI}$ as precisely the set of functions taking values in $[0, 1]$, almost everywhere.

Proposition 3.2.8. $\mathcal{MI} = \{h \in L^\infty([0, M]) : \|h\|_\infty \leq 1,\ h \geq 0\ \lambda\text{-}a.e.\}$

Proof. Each indemnification function $I \in \mathcal{I}$ is non-decreasing, 1-Lipschitz, and absolutely continuous, which implies that its derivative takes values in $[0, 1]$ almost everywhere. Conversely, the integral of an a.e. $[0, 1]$-valued function is a non-decreasing 1-Lipschitz function. □

As a consequence of Proposition 3.2.8, the following useful fact can be immediately verified:

Corollary 3.2.9. *The set $\mathcal{MI}$ is convex.*

However, $\mathcal{MI}$ is *not compact* in general, since the state space S may be infinite.

3.3 Pareto Optimality of Contracts

3.3.1 Characterization of Pareto Optima

In the following, we obtain an expression for PO contracts in the special case of Choquet risk measures. We assume the following.

Assumption 3.3.1. For each $i \in \{1, \dots, n\}$, there exists a normalized capacity ν_i such that for all $X \in B(\Sigma)$,

$$\rho_i(X) = \int X \, d\nu_i \, .$$

Choquet risk measures constitute a flexible class of comonotone additive and normalized risk measures, and they are therefore translation invariant. For more details on Choquet risk measures, we refer to Föllmer and Schied [57, Section 4.7] or Marinacci and Montrucchio [86, Proposition 4.11].

Assumption 3.3.2. There exists a convex and weak*-compact set $\mathcal{Q}$ of probability measures on (S, Σ) such that for all $X \in B(\Sigma)$,

$$\rho^{IN}(X) = \sup_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}^{\mathbb{Q}}[X] \, .$$

Remark 3.3.3. *Note that, in Assumption 3.3.2, the supremum is attained as a consequence of weak*-compactness of the set $\mathcal{Q}$. That is, for all $X \in B(\Sigma)$,*

$$\rho^{IN}(X) = \max_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}^{\mathbb{Q}}[X] \, .$$

Moreover, it is immediate to verify that under Assumption 3.3.2, the risk measure ρ^{IN} is coherent. However, the converse is not true in general: a coherent risk measure admits a representation as a maximum over a convex set of normalized, finitely additive measures (e.g., Delbaen [45, Theorem 2.2]). Under some additional conditions, it can be shown that a coherent measure admits the representation of Assumption 3.3.2 (e.g., Delbaen [45, Theorem 3.2]).

We wish to highlight that under our assumptions, only the risk measure used by the insurer is coherent. Introduced by Artzner et al. [6], coherent risk measures satisfy desirable properties and are widely used in practice. For these reasons, Assumption 3.3.2 is a fairly mild assumption. On the other hand, we allow for more flexibility in the specification of the policyholders' risk measures. Indeed, Choquet risk measures constitute a general class of translation-invariant risk measures that includes, for example, the class of distortion risk measures. We elaborate on the link between Choquet risk measures and coherent risk measures in the following remark.

Remark 3.3.4. *Suppose that ϱ is a Choquet risk measure of the form*

$$\varrho(X) = \int X \, d\upsilon \,, \ \forall \, X \in B(\Sigma) \,,$$

for some capacity υ. Then ϱ is coherent if and only if the capacity υ is submodular [1] *(e.g., Föllmer and Schied [57, Theorem 4.94]). In this case,*

$$\varrho(X) = \sup_{\mathbb{Q} \in \tilde{\mathcal{Q}}} \mathbb{E}^{\mathbb{Q}}[X] \,,$$

where

$$\tilde{\mathcal{Q}} := \{\mu \in \mathrm{ba}(\Sigma) : \mu(S) = 1, \, \mu(A) \leq \nu(A) \text{ for all } A \in \Sigma\} \ .$$

The set $\tilde{\mathcal{Q}}$ consists of countably additive measures if and only if ν is continuous (from below) at S (see Schmeidler [99, Theorem 3.2]). Hence, a Choquet risk measure satisfies Assumption 3.3.2 if its capacity is submodular and continuous at S.

The following theorem provides a characterization of Pareto optima under Assumptions 3.3.1 and 3.3.2. Its proof is given in Section 3.8.

Theorem 3.3.5. *Suppose that all agents use Choquet risk measures, and that the insurer's risk measure ρ satisfies Assumption 3.3.2. Then a necessary condition for the indemnities $\{I_i^*\}_{i=1}^n$ to be optimal for (3.3) is that for every solution $\mathbb{Q}^*$ to*

$$\sup_{\mathbb{Q} \in \mathcal{Q}} \sum_{i=1}^{n} \int_0^M \min \{\mathbb{Q}(X_i > t), \, \nu_i(X_i > t)\} \, dt \,, \tag{3.4}$$

we have, for each $i \in \{1, \ldots, n\}$ and for almost all $t \in (0, M)$,

$$(I_i^*)'(t) = \begin{cases} 1, & \text{if} \quad \mathbb{Q}^*(X_i > t) < \nu_i(X_i > t) \,, \\ h_i(t), & \text{if} \quad \mathbb{Q}^*(X_i > t) = \nu_i(X_i > t) \,, \\ 0, & \text{if} \quad \mathbb{Q}^*(X_i > t) > \nu_i(X_i > t) \,, \end{cases} \tag{3.5}$$

[1] A capacity $\upsilon : \Sigma \to \mathbb{R}$ is *submodular* if $\upsilon(A \cup B) + \upsilon(A \cap B) \leq \upsilon(A) + \upsilon(B)$ for all $A, B \in \Sigma$.

where each h_i is a $[0,1]$-valued measurable function.

Theorem 3.3.5 extends the characterization in Boonen et al. [17, Theorem 5.1] in several ways. Firstly, they assume a finite state space, whereas our result holds on a general state space S. We are able to obtain a more general result by applying certain properties of the set $\mathcal{MI}$, which was introduced in Subsection 3.2.4. Furthermore, under their setting, every policyholder uses a coherent risk measure, whereas we only impose this assumption on the monopolistic insurer. Hence, our setting is less restrictive from both a technical standpoint and an economic one.

As an immediate consequence of Theorem 3.3.5, we obtain the following two results.

Corollary 3.3.6 (Existence of Pareto Optima)**.** *Suppose that ρ^{IN} is a coherent risk measure and that, for each $i \in \{1,\dots,n\}$, ρ_i is a Choquet risk measure. Then $\mathcal{PO} \neq \varnothing$.*

Proof. This is a consequence of the proof of Theorem 3.3.5. □

Corollary 3.3.7 (Characterization of Pareto Optima)**.** *Suppose that all agents use Choquet risk measures, and that the insurer's risk measure ρ^{IN} satisfies Assumption 3.3.2. Then a necessary condition for $(\{I_i^*\}_{i=1}^n, \{\pi_i^*\}_{i=1}^n)$ to be Pareto optimal is that for every solution $\mathbb{Q}^*$ to*

$$\sup_{\mathbb{Q}\in\mathcal{Q}} \sum_{i=1}^{n} \int_0^M \min\left\{\mathbb{Q}(X_i > t),\, \nu_i(X_i > t)\right\} dt\,, \tag{3.6}$$

we have, for each $i \in \{1,\dots,n\}$ and for almost all $t \in (0,M)$,

$$(I_i^*)'(t) = \begin{cases} 1, & \text{if} \quad \mathbb{Q}^*(X_i > t) < \nu_i(X_i > t)\,, \\ h_i(t), & \text{if} \quad \mathbb{Q}^*(X_i > t) = \nu_i(X_i > t)\,, \\ 0, & \text{if} \quad \mathbb{Q}^*(X_i > t) > \nu_i(X_i > t)\,, \end{cases} \tag{3.7}$$

where each h_i is a $[0,1]$-valued measurable function and $\{\pi_i^*\}_{i=1}^n$ are chosen so that the contract $(\{I_i^*\}_{i=1}^n, \{\pi_i^*\}_{i=1}^n) \in \mathcal{IR}$.

Proof. This follows immediately from Proposition 3.2.4, Remark 3.2.5, and Theorem 3.3.5. □

3.3.2 Pareto-Optima in Terms of Worst-Case Measures

Similarly to Boonen et al. [17, Theorem 5.1], the result of Theorem 3.3.5 characterizes Pareto-optimal contracts in terms of the measure $\mathbb{Q}^*$ as defined through (3.4). This characterization is not explicit, since it is difficult in general to identify a closed-form solution for the measure $\mathbb{Q}^*$.

We now provide an interpretation of this measure $\mathbb{Q}^*$ as a "worst-case probability measure" on the level of the aggregate market. This interpretation is perhaps more familiar on the level of individual agents in the market. As discussed in Remark 3.3.3, it is well-known that a coherent risk measure ϱ can be represented as a supremum of expectations over a compact set of probability measures, and that this supremum is attained as a consequence of compactness. Hence, the measure that attains this expectation is interpreted as the worst-case measure.

In the setting of Theorem 3.3.5, for each $i \in \{1, \dots, n\}$ and each $\mathbb{Q} \in \mathcal{Q}$, define the set function $\aleph_i^{\mathbb{Q}}$ by

$$\aleph_i^{\mathbb{Q}}(A) = \min\{\mathbb{Q}(A), \nu_i(A)\}\,, \ \forall\, A \in \Sigma\,.$$

As we will see in Lemma 3.4.4, there exists a probability measure $\beth_i^{\mathbb{Q}}$ such that

$$\beth_i^{\mathbb{Q}}(X_i > t) = \aleph_i^{\mathbb{Q}}(X_i > t), \ \forall\, t \in \mathbb{R}\,.$$

Hence, the measure $\beth_i^{\mathbb{Q}}$ gives the more optimistic assessment between $\mathbb{Q}$ and ν_i for the likelihood of the tail loss event $X_i > t$. It then follows that

$$\begin{aligned}\int_0^M \min\{\mathbb{Q}(X_i > t), \nu_i(X_i > t)\}\, dt &= \int_0^M \aleph_i^{\mathbb{Q}}(X_i > t)\, dt \\ &= \int_0^M \beth_i^{\mathbb{Q}}(X_i > t)\, dt = \int_0^M X_i\, d\beth_i^{\mathbb{Q}}\,,\end{aligned}$$

which is an expectation with respect to the measure $\beth_i^{\mathbb{Q}}$. Summing over all agents yields

$$\sum_{i=1}^n \int_0^M \min\{\mathbb{Q}(X_i > t), \nu_i(X_i > t)\}\, dt = \sum_{i=1}^n \int_0^M X_i\, d\beth_i^{\mathbb{Q}}\,, \tag{3.8}$$

which can be interpreted as a measure of aggregate risk in a market where the preferences of each agent are given by $\beth_i^{\mathbb{Q}}$, and therefore induced by the measure $\mathbb{Q}$. Hence, each such $\mathbb{Q} \in \mathcal{Q}$ induces a market in which the agents have preferences determined by $\beth_i^{\mathbb{Q}}$ and the aggregate market risk is given by (3.8). Note that (3.8) is precisely the objective of the supremum in (3.4). Hence, the measure $\mathbb{Q}^*$ that attains this supremum (which was shown to exist in Theorem 3.3.5) is the measure in $\mathcal{Q}$ that induces the market with the largest aggregate risk.

3.4 A Characterization of Stackelberg Equilibria

We now provide a characterization of SE when, for each $i \in \{1, \cdots, n\}$, $\aleph_i$ is an expectation with respect to a probability measure, as outlined in Assumption 3.2.6. With a slight abuse of notation, we say that a measure μ is in $\aleph_i$ if its associated premium principle is an element of $\aleph_i$.

Remark 3.4.1. *For each $i \in \{1, \ldots, n\}$, the dual of $B(\sigma(X_i))$ is $ba(\sigma(X_i))$, the set of all finitely additive set functions on $(S, \sigma(X_i))$ (e.g., [51, Theorem IV.5.1]). Hence, each $\aleph_i$ is a subset of $ba(\sigma(X_i))$. In the following, we consider the relative weak*-topology on $\aleph_i$ inherited from $ba(\sigma(X_i))$.*

Remark 3.4.2. *The premium principles in Assumption 3.2.6 are represented by finite countably additive measures, but not necessarily probability measures. For example, the insurer is able to introduce a loading factor $\theta > 0$ if they so desire, by using the measure $\mu := (1 + \theta)\,\mathbb{P}$ for some probability measure $\mathbb{P}$.*

The following result gives a characterization of each policyholder's optimal indemnity, when presented with a fixed premium principle by the insurer. This provides the solution to the problem in point (1) of Definition 2.4.5.

Proposition 3.4.3. *For all $i \in \{1, \ldots, n\}$ a contract I_i^* solves*

$$\inf_{I \in \mathcal{I}} \rho_i \left(X_i - I\left(X_i\right) + \Pi_i \left(I(X_i) \right) \right)$$

if and only if

$$(I_i^*)'(t) = \begin{cases} 1, & \text{if} \quad \nu_i(X_i > t) > \mu_i(X_i > t)\,, \\ h_i(t), & \text{if} \quad \nu_i(X_i > t) = \mu_i(X_i > t)\,, \\ 0, & \text{if} \quad \nu_i(X_i > t) < \mu_i(X_i > t)\,, \end{cases} \tag{3.9}$$

where each h_i is a $[0,1]$-valued measurable function, and each μ_i is the measure identified with $\Pi_i \in \aleph_i$, as in Assumption 3.2.6.

Proof. By comonotone additivity, for any $I \in \mathcal{I}$,

$$\begin{aligned} \rho_i(X_i - I(X_i) + \Pi_i(I(X_i))) &= \rho_i(X_i) - \rho_i(I(X_i)) + \Pi_i\left(I(X_i)\right) \\ &= \rho_i(X_i) + \int_0^M \left(\mu_i(X_i > t) - \nu_i(X_i > t)\right) I'(t)\, dt\,, \end{aligned}$$

which immediately implies the result. □

A characterization of Stackelberg equilibria is provided by the following results. First, the following lemma shows that the insurer can mimic the preference of a policyholder by an appropriate choice of linear premium principle.

Lemma 3.4.4. *For each $i \in \{1, \dots, n\}$, there exists a probability measure τ_i on $(S, \sigma(X_i))$ such that*

$$\tau_i(X_i > t) = \nu_i(X_i > t), \ \forall\, t \in \mathbb{R}\,.$$

Proof. For each $i \in \{1, \dots, n\}$, the collection $\{\{X_i > t\}\}_{t\in\mathbb{R}}$ forms a chain in $\sigma(X_i)$. Therefore, by Marinacci and Montrucchio [86, Lemma 4.5], there exists $\tau_i \in ba(\sigma(X_i))$ such that $\tau_i(X_i > t) = \nu_i(X_i > t)$, for all $t \in \mathbb{R}$. Since ν_i is continuous from below, then so is τ_i and hence $\tau_i \in ca(\sigma(X_i))$. For $t < 0$, we have $\nu_i(X_i > t) = \nu_i(S) = \tau_i(S) = 1$, and so τ_i is a probability measure. □

Theorem 3.4.5 (Characterization of Stackelberg Equilibria)**.** *Suppose that the insurer's risk measure ρ^{IN} satisfies Assumption 3.3.2. Let $(\{I_i^*\}_{i=1}^n, \{\Pi_i^*\}_{i=1}^n)$ be a market mechanism, and let μ_i^* be the measure identified with Π_i^*, as in Assumption 3.2.6. Then a necessary condition for the market mechanism $(\{I_i^*\}_{i=1}^n, \{\Pi_i^*\}_{i=1}^n)$ to be an SE is that for every solution $\mathbb{Q}^*$ to*

$$\sup_{\mathbb{Q}\in\mathcal{Q}} \left\{ \sum_{i=1}^{n} \int_0^M \min\left\{\mathbb{Q}(X_i > t),\, \nu_i(X_i > t)\right\} dt \right\}, \tag{3.10}$$

we have, for each $i \in \{1, \dots, n\}$ and for almost all $t \in (0, M)$,

$$\begin{aligned}
&\mu_i^*(X_i > t) = \nu_i(X_i > t), \ \text{if} \quad \mathbb{Q}^*(X_i > t) < \nu_i(X_i > t)\,,\\
&\mu_i^*(X_i > t) \geq \nu_i(X_i > t), \ \text{if} \quad \mathbb{Q}^*(X_i > t) \geq \nu_i(X_i > t)\,,\\
&(I_i^*)'(t) = \begin{cases}
1 & \text{if} \quad \nu_i(X_i > t) = \mu_i^*(X_i > t) > \mathbb{Q}^*(X_i > t)\,,\\
h_i(t) & \text{if} \quad \nu_i(X_i > t) = \mu_i^*(X_i > t) = \mathbb{Q}^*(X_i > t)\,,\\
0 & \text{if} \quad \nu_i(X_i > t) < \mu_i^*(X_i > t) = \mathbb{Q}^*(X_i > t)\,,\\
0 & \text{if} \quad \nu_i(X_i > t) \leq \mu_i^*(X_i > t) < \mathbb{Q}^*(X_i > t)\,,
\end{cases}
\end{aligned} \tag{3.11}$$

where each h_i is a $[0,1]$-valued measurable function.

The proof of Theorem 3.4.5 can be found in Section 3.8. Similarly to the result of Theorem 3.3.5, the characterization given by Theorem 3.4.5 is an implicit one, since it is difficult to identify the solution to (3.10). However, we note that the optimal measures $\mathbb{Q}^*$ are the same for both Theorems 3.3.5 and 3.4.5, since (3.6) and (3.10) are identical. We elaborate on the relationship between Pareto optima and Stackelberg equilibria in Section 3.5. As a corollary, we have the following.

Corollary 3.4.6 (Existence of Stackelberg Equilibria)**.** *Suppose that ρ^{IN} is a coherent risk measure satisfying Assumption 3.3.2 and that, for each $i \in \{1, \dots, n\}$, ρ_i is a Choquet risk measure. Then under Assumption 3.2.6, $\mathcal{SE} \neq \varnothing$.*

Proof. This is a consequence of the proof of Theorem 3.4.5. □

3.5 Welfare Analysis of Equilibria

3.5.1 Pareto Efficiency of Equilibrium Contracts

We now examine the efficiency properties of SE. In particular, we are interested in the relationship between the contracts induced by an SE and Pareto optimality. We begin with the following proposition.

Proposition 3.5.1. *Suppose that ρ^{IN} satisfies Assumption 3.3.2 and that premium principles are of the form given in Assumption 3.2.6. Then every SE induces an individually rational contract. That is, if $(\{I_i^*\}_{i=1}^n, \{\Pi_i^*\}_{i=1}^n)$ is an SE, then $(\{I_i^*\}_{i=1}^n, \{\Pi_i^*(I_i^*)\}_{i=1}^n)$ is IR.*

Proof. By definition of SE, for each $i \in \{1, \dots, n\}$, we have that

$$I_i^* \in \arg\min_{I \in \mathcal{I}} \rho_i \left(X_i - I(X_i) + \Pi_i^* \left(I(X_i) \right) \right) .$$

Note that the indemnity that is identically zero is an admissible indemnity. Hence,

$$\rho_i \left(X_i - I_i^*(X_i) + \Pi_i^* \left(I_i^*(X_i) \right) \right) \leq \rho_i \left(X_i - 0 + \Pi_i^*(0) \right) = \rho_i(X_i) .$$

Furthermore, the proof of Theorem 3.4.5 shows that there is some measure $\mathbb{Q}^*$ such that

$$\begin{aligned} \rho^{IN} \left(\sum_{i=1}^n I_i^*(X_i) - \Pi_i^*(I_i^*(X_i)) \right) &= \sum_{i=1}^n \int_0^M \min \left\{ \mathbb{Q}^*(X_i > t) - \nu_i(X_i > t), 0 \right\} dt \\ &\leq 0 = \rho^{IN}(0) . \end{aligned}$$

Hence, all parties are at least as well off as the status quo, so $(\{I_i^*\}_{i=1}^n, \{\Pi_i^*(I_i^*)\}_{i=1}^n)$ is IR. □

In order to comment on the relationship between Stackelberg equilibria and Pareto optima, we first examine the situation under which the necessary conditions of Theorems 3.3.5 and 3.4.5 are also sufficient. Indeed, the condition is not sufficient in general, and a counterexample in a slightly different context is provided in Section 5.8. However, if the necessary condition characterizes a unique indemnity structure $\{I_i^*\}_{i=1}^n$, then the condition is also a sufficient one. This is made precise by the following.

Proposition 3.5.2. *Suppose that the maximizing measure $\mathbb{Q}^*$ to (3.4) is unique, and that for all $i \in \{1, \dots, n\}$, we have*

$$\mathbb{Q}^*(X_i > t) \neq \nu_i(X_i > t)$$

for almost all $t \in (0, M)$. Then the necessary conditions given in Theorems 3.3.5 and 3.4.5 are also sufficient.

Proof. We show the result for the condition in Theorem 3.3.5, as the case for Theorem 3.4.5 is similar. By Corollary 3.3.6, there exists a Pareto optimum. It then follows that if the indemnity structure $\{I_i^*\}_{i=1}^n$ given by Theorem 3.3.5 is unique and $\{\pi_i^*\}_{i=1}^n$ are chosen so that the contract $(\{I_i^*\}_{i=1}^n, \{\pi_i^*\}_{i=1}^n) \in \mathcal{IR}$, then the contract must also be Pareto optimal. By assumption, the maximizing measure $\mathbb{Q}^*$ is unique and $(I_i^*)'$ is uniquely determined almost everywhere, which completes the proof. □

Proposition 3.5.3. *Suppose that ρ^{IN} satisfies Assumption 3.3.2 and that premium principles are of the form given in Assumption 3.2.6. Suppose further that the conditions of Proposition 3.5.2 are satisfied. Then every SE induces a PO contract. That is, if $(\{I_i^*\}_{i=1}^n, \{\Pi_i^*\}_{i=1}^n)$ is an SE, then $(\{I_i^*\}_{i=1}^n, \{\Pi_i^*(I_i^*)\}_{i=1}^n)$ is PO.*

Proof. Suppose that $(\{I_i^*\}_{i=1}^n, \{\Pi_i^*\}_{i=1}^n)$ is an SE. We use the characterization provided by Theorem 3.4.5. Let $\mathbb{Q}^*$ be a solution to (3.10) such that for each $i \in \{1, \dots, n\}$, the marginal indemnity $(I_i^*)'$ is of the form given in (3.11). Then $\mathbb{Q}^*$ solves (3.6), and it is immediate to verify that $(I_i^*)'$ also satisfies (3.7). Hence, $\{I_i^*\}_{i=1}^n$ solves (3.3) by Theorem 3.3.5, and so $(\{I_i^*\}_{i=1}^n, \{\Pi_i^*(I_i^*)\}_{i=1}^n)$ is PO by Proposition 3.5.1 and Remark 3.2.5. □

3.5.2 Decentralization of Pareto Optima via Stackelberg Equilibria

We now provide a partial converse to Proposition 3.5.3. We show that every PO contract that leaves each policyholder indifferent with the *status quo* is associated with a Stackelberg equilibrium under some pricing rule.

Proposition 3.5.4. *Suppose that* $(\{I_i^*\}_{i=1}^n, \{\Pi_i^*\}_{i=1}^n)$ *is an SE. Then*

$$\rho_i\left(X_i - I_i^*(X_i) + \Pi_i^*\left(I_i^*(X_i)\right)\right) = \rho_i(X_i)\,,$$

for each $i \in \{1, \ldots, n\}$.

Proof. By comonotone additivity, it suffices to show that $\Pi_i^*(I_i^*(X_i)) - \rho_i(I_i^*(X_i)) = 0$. Let μ_i^* be the measure associated with Π_i^*. Then we have

$$\begin{aligned}\Pi_i^*\left(I_i^*(X_i)\right) - \rho_i\left(I_i^*(X_i)\right) &= \int_0^M \mu_i^*(X_i > t)\,(I_i^*)'(t)\,dt - \int_0^M \nu_i(X_i > t)\,(I_i^*)'(t)\,dt \\ &= \int_0^M \left(\mu_i^*(X_i > t) - \nu_i(X_i > t)\right)(I_i^*)'(t)\,dt\,.\end{aligned}$$

By Theorem 3.4.5, we see that for almost all $t \in (0, M)$, the marginal indemnification $(I_i^*)'(t) > 0$ only when $\nu_i(X_i > t) = \mu_i^*(X_i > t)$. Therefore the integrand vanishes almost everywhere, so $\Pi_i^*(I_i^*(X_i)) - \rho_i(I_i^*(X_i)) = 0$, as desired. □

Proposition 3.5.5. *Suppose that the conditions of Proposition 3.5.2 are satisfied, and that* (I_i^*, π_i^*) *is a PO contract that satisfies*

$$\pi_i^* = \rho_i\left(I_i^*(X_i)\right)\,, \quad \forall\, i \in \{1, \ldots, n\}\,.$$

Then there exists $\{\Pi_i^*\}_{i=1}^n \in \aleph$ *such that:*

1. $\Pi_i^*\left(I_i^*(X_i)\right) = \pi_i^*$*; and,*
2. *The market mechanism* $(\{I_i^*\}_{i=1}^n, \{\Pi_i^*\}_{i=1}^n)$ *is an SE.*

Proof. By Theorem 3.3.5, there exists a measure $\mathbb{Q}^*$ such that $\mathbb{Q}^*$ solves (3.6), and for almost all $t \in (0, M)$,

$$(I_i^*)'(t) = \begin{cases} 0, & \text{if} \quad \mathbb{Q}^*(X_i > t) > \nu_i(X_i > t)\,, \\ h_i(t), & \text{if} \quad \mathbb{Q}^*(X_i > t) = \nu_i(X_i > t)\,, \\ 1, & \text{if} \quad \mathbb{Q}^*(X_i > t) < \nu_i(X_i > t)\,. \end{cases}$$

By Lemma 3.4.4, for each $i \in \{1, \ldots, n\}$, there exists a probability measure μ_i^* such that $\mu_i^*(X_i > t) = \nu_i(X_i > t)$, for all $t \in \mathbb{R}$. Let the pricing mechanisms Π_i^* be such that

$$\Pi_i^*(Z) := \int Z\, d\mu_i^*\,,$$

for all $Z \in B(\Sigma)$. By construction, $\Pi_i^* \in \aleph_i$ for each $i \in \{1, \dots, n\}$. We have

$$(I_i^*)'(t) = \begin{cases} 0, & \text{if} \quad \mathbb{Q}^*(X_i > t) > \mu_i^*(X_i > t)\,, \\ h_i(t), & \text{if} \quad \mathbb{Q}^*(X_i > t) = \mu_i^*(X_i > t)\,, \\ 1, & \text{if} \quad \mathbb{Q}^*(X_i > t) < \mu_i^*(X_i > t)\,, \end{cases}$$

from which it follows directly that μ_i^* and $(I_i^*)'$ satisfy (3.11). Therefore $(\{I_i^*\}_{i=1}^n, \{\Pi_i^*\}_{i=1}^n)$ is an SE by Theorem 3.4.5, and $\Pi_i^*(I_i^*(X_i)) = \pi_i^* = \rho_i(I_i^*(X_i))$ by Proposition 3.5.4. □

As a consequence of Propositions 3.5.4 and 3.5.5, the contracts resulting from Stackelberg equilibria are precisely the set of PO contracts that leave each policyholder with no welfare gain. This echoes a result of Boonen and Ghossoub [20], who arrive at a similar conclusion in the two-agent setting. This implies that the policyholders have no incentive to purchase insurance, which is problematic in practice. Our results show that this undesirable situation happens even when multiple policyholders are introduced on the demand side, under the mild condition that the monopolistic insurer uses a coherent risk measure. The possible implications of this are explored in Section 3.6.

3.6 An Application to Flood Risk

We now illustrate Stackelberg equilibria in the market for flood risk insurance in the United States. Since 1968, flood risk insurance has been provided by the National Flood Insurance Program (NFIP), [2] which is managed by the Federal Emergency Management Agency (FEMA). The marketing, sale, and underwriting of flood insurance contracts is performed by private insurance companies, who may or may not act on behalf of the FEMA. In either case, the federal government ultimately bears the financial gains and losses. Historical claims data under the NFIP, dating back to 1970, are made publicly available.

A detailed analysis of this data is provided by Boonen et al. [17], who identify Pareto-optimal indemnities when every agent uses a coherent risk measure. Based on their results, they make the recommendation that the market for flood risk insurance be centralized into one entity (e.g., a federal agency). By aggregating the flood risks of different geographical regions, the single entity is able to offer insurance at a lower price than currently possible. This effect is most pronounced when risk is aggregated from regions that are geographically

[2] A comprehensive overview of the NFIP can be found on the website of the Congressional Research Service: https://sgp.fas.org/crs/homesec/R44593.pdf.

distant: in a concrete example, they show a large welfare gain in the market when the flood risks of California, New York, and Texas are combined. When more correlated risks from geographically closer states are combined, such as those of Alabama, Louisiana, and Mississippi, they find that a smaller but still significant welfare gain can be realized. They interpret this discrepancy as a consequence of the benefits of diversification, which are larger when the underlying risks are less correlated.

We now revisit this application in the context of a sequential market. Our goal is to complement the analysis of Boonen et al. [17] in two ways. Firstly, we relax their assumption of coherence for all agents by considering a setup where only the central insurer uses a coherent risk measure. This example is provided in Subsection 3.6.1, in which we verify that the conclusions of Boonen et al. [17] are still valid under this relaxation. That is, by consolidating all contracts within a central entity, a significant welfare gain can be realized. We also find that this gain is larger when aggregating risks that are less correlated. Furthermore, our results show that in this case, insurance coverage becomes available to states that would have otherwise been uninsured. Hence, this market structure has the additional benefit of reducing underinsurance of flood risk.

Next, by viewing this market through the lens of Stackelberg equilibria, we argue in Subsection 3.6.2 that the monopolistic insurer has an incentive to set high prices and erode the welfare gain of the policyholders. Consequently, we argue that third-party intervention is necessary to preserve the interests of the policyholders.

3.6.1 A Numerical Analysis

For each state, we have 621 months of claims data dating from August 1970 to June 2023. [3] Similar to Boonen et al. [17], we use the following five variables for our analysis:

- **dateOfLoss**: The date on which water first enters the insured property.
- **state**: The abbreviation of the state in which the insured property is located.
- **amountPaidOnBuildingClaim**: The dollar amount paid on the building claim.
- **amountPaidOnContentsClaim**: The dollar amount paid on the contents claim.
- **mountPaidOnIncreasedCostOfComplianceClaim**: The dollar amount paid on the increased cost of compliance (ICC) claim.

[3] This dataset is made publicly available by the Federal Emergency Management Agency on their website: https://www.fema.gov/openfema-data-page/fima-nfip-redacted-claims-v1.

In our numerical illustrations, we aggregate all flood insurance claims by the month of loss and geographical location (state), by summing the values of the claim dollar amounts given by the fields above. We assume that the distribution of a given month's flood risk adheres to the historical data: specifically, we assume a discrete state space with $S = \{\omega_1, \ldots, \omega_{635}\}$, and a probability measure $\mathbb{P}$ that gives equal weighting to each state in S.

We would like to mention that this setup has some limitations that may render it ineffective in the prediction of future flood events. For example, this setup does not account for the possible seasonality of flood claims, nor does it address possible trends in claim frequencies and severity over time. The accurate modelling of flood risk is a complicated problem and beyond the scope of this thesis. Instead, this numerical analysis can better be understood as quantifying the average welfare gain that would have been induced if such a risk-sharing scheme were in effect for the past ≈ 50 years. In other words, the analysis in this section is a backtest of the risk-sharing scheme given in Section 3.2.

The risk measures used by each policyholder are Choquet integrals in terms of the aforementioned probability measure $\mathbb{P}$. That is, for all $X \in B(\Sigma)$ and $i \in \{1, \ldots, n\}$,

$$\rho_i(X) = \int X \, dT_i \circ \mathbb{P} \,,$$

where each T_i is a distortion function. [4] We also assume throughout that the uncertainty set $\mathcal{Q}$ that generates the risk measure of the central insurer ρ^{IN} is given by the convex hull of the set $\{\mathbb{Q}_j\}_{j=1}^{635}$, where

$$\mathbb{Q}_j(\omega_k) = \begin{cases} 0.6 & \text{if } j = k \,, \\ 0.4/634 & \text{if } j \neq k \,. \end{cases}$$

The main computational hurdle is to solve Problem (3.6), which does not admit a closed-form representation. However, it may be verified that the objective of (3.6) is concave in the argument $\mathbb{Q}$. Therefore this problem can be efficiently computed through convex optimization methods, such as the "cvx" package in MATLAB.

California, New York, and Texas

As an example of three states that are geographically distant, we consider California (i=1), New York (i=2), and Texas (i=3). The correlation matrix for these three states, calculated

[4] A distortion function is a non-decreasing function $T : [0,1] \to [0,1]$ such that $T(0) = 0$ and $T(1) = 1$.

from historical data, is

$$\begin{bmatrix} 1 & -0.0086 & -0.0080 \\ -0.0086 & 1 & -0.0036 \\ -0.0080 & -0.0036 & 1 \end{bmatrix}.$$

We assume that the distortion functions T_1 and T_2 are inverse S-shaped functions, such as those used in Tversky and Kahneman [105]. That is, for $i = 1, 2$, the functional form of T_i is given by

$$T_i(t) = \frac{t^{\gamma_i}}{(t^{\gamma_i} + (1-t)^{\gamma_i})^{\frac{1}{\gamma_i}}},$$

where we take $\gamma_1 = 0.4$ and $\gamma_2 = 0.6$. Furthermore, we assume that the distortion function T_3 is given by a Proportional Hazard transform (e.g., Wang [106]):

$$T_3(t) = t^{\alpha_3},$$

with parameter $\alpha_3 = 0.3$.

Although inverse S-shaped distortion functions do not yield coherent risk measures, we may still apply the result of Theorem 3.3.5, since only the central insurer is required to use a coherent risk measure. Rieger et al. [96] show that inverse S-shaped distortion functions are descriptive of actual behaviour, and estimate a parameter of $\gamma = 0.5$ using a large international dataset. For this parameter, we choose a value slightly below and a value slightly above their estimated parameter (i.e., $\gamma_1 = 0.4$ and $\gamma_2 = 0.6$) as reasonable values for the agents in this market. On the other hand, the Proportional Hazard transform is one of the simplest concave distortion functions that yield a coherent risk measure, and it has seen extensive use in actuarial science. We choose $\alpha_3 = 0.3$, since it provides for a steep enough curvature to exhibit a good amount of concavity, but is not too extreme.

The expected welfare gain in this market for a future month is shown in Table 3.1a. When the central insurer aggregates the risk of California, New York, and Texas, the total welfare gain is a significant \$880,200 a month. This is substantial when compared to the case when no risk aggregation is possible. For example, if the central insurer can only indemnify California or Texas, no agreement can be made that benefits either counterparty. Hence, risk aggregation across geographical regions not only increases total welfare gain, but can also provide additional incentives for the insurer to cover previously underinsured areas. A Pareto-optimal indemnity structure when the central insurer insures all three states is given in Figure 3.1, in which we see that all states receive coverage.

Table 3.1: Aggregate Welfare Gain

States	Welfare Gain
{CA}	0
{NY}	0
{TX}	4.836×10^3
{CA,NY}	0
{CA,TX}	2.770×10^5
{NY,TX}	3.009×10^4
{CA,NY,TX}	8.802×10^5

(a) CA/NY/TX.

States	Welfare Gain
{AL}	0
{LA}	0
{MS}	1.063×10^3
{AL,LA}	0
{AL,MS}	1.052×10^4
{LA,MS}	1.063×10^3
{AL,LA,MS}	1.052×10^4

(b) AL/LA/MS.

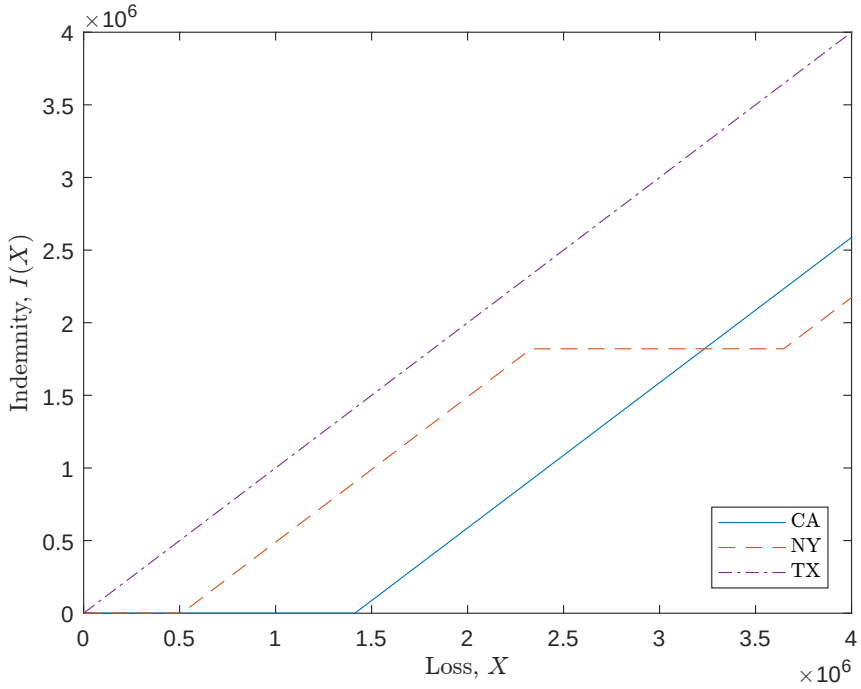


Figure 3.1: Pareto-Optimal Indemnity Structure, CA/NY/TX.

Alabama, Louisiana, and Mississippi

In contrast to the previous example, we now consider three states that are geographically close: Alabama (i=1), Louisiana (i=2), and Mississippi (i=3). Using monthly claims data from August 1970 to June 2023, we find that the correlation matrix for these three states is

$$\begin{bmatrix} 1 & 0.5960 & 0.5812 \\ 0.5960 & 1 & 0.9792 \\ 0.5812 & 0.9792 & 1 \end{bmatrix},$$

which indicates a significant positive correlation among the flood risk losses of this group. For comparability, we assume that the preferences of each state are given by the same forms of distortion functions T_i, for $i = \{1, 2, 3\}$, as in the previous example. In particular, the distortion functions of Alabama and Louisiana are non-concave.

The welfare gains from aggregating contracts from these states are shown in Table 3.1b. When the risks from all three states are aggregated, a welfare gain of \$10,520 per month can be achieved. However, this is significantly less than the benefits of aggregating less correlated risks, as we explored in the previous example. Hence, we find that the benefits of aggregation are most pronounced when contracts across geographically distant regions are combined, which is consistent with the results of Boonen et al. [17].

3.6.2 Policy Implications

The federal government of the United States undoubtedly exerts a significant influence on the industry of the nation. Boonen et al. [17] propose a new structure of flood insurance based on risk sharing, in which the federal government centralizes all contracts across the country. However, in the scenario that the federal government may set premia independently of the concerns of local policyholders and state-level insurance companies, it has a first-mover's advantage.

As argued in Section 3.4, we expect a Stackelberg equilibrium to arise in an insurance market of this form. The central insurer has an incentive to use a premium principle given by Theorem 3.4.5. The result of Proposition 3.5.4 shows that the use of this premium principle results in no welfare gain to any policyholder. This effectively means that each policyholder receives no benefit from purchasing insurance, since the central insurer ultimately keeps all the welfare gain for themselves. As a consequence, benefits to the

policyholder from this risk-sharing program are not possible. In particular, the aforementioned problem of underinsurance of flood risk is not addressed, since policyholders are not incentivized to purchase insurance.

This issue is particularly pronounced when risk is aggregated across distant geographical regions, since there is a higher welfare gain that the central insurer can try to capture. Table 3.2 shows the premia that California, New York, and Texas will pay on a monthly basis in a Stackelberg equilibrium. The final row shows the premia charged by the central insurer for the indemnity structures in Figure 3.1. We have seen in Subsection 3.6.1 that risk aggregation results in more indemnification, but this benefit is lost as the central insurer raises prices in response. We see that Texas pays over 20 times more premium when the central insurer includes risks from California and New York.

States	Premium Paid		
	CA	NY	TX
{CA}	0	0	0
{NY}	0	0	0
{TX}	0	0	3.590×10^5
{CA,NY}	0	0	0
{CA,TX}	8.678×10^5	0	5.525×10^6
{NY,TX}	0	7.793×10^7	9.776×10^5
{CA,NY,TX}	2.261×10^6	5.407×10^5	8.825×10^6

Table 3.2: Premium Paid for CA/NY/TX.

We therefore conclude that the first-mover's advantage poses a serious issue to policyholders in this insurance market. In order to realize the social benefits of risk aggregation, there needs to exist a mechanism that prevents the central insurer from exerting its influence to set prices independently of the concerns of its counterparties. For example, this could come in the form of regulation that ensures that premia are agreed upon between state-level and federal-level agencies. Alternatively, it may be possible to implement a procedure for splitting the aggregate welfare gain in a systematic and fair manner. A rigorous study of the precise mechanisms of such a split is beyond the scope of this chapter, and we suggest this as a direction for future study.

3.7 Conclusion

In this chapter, we examine an insurance market with one central monopolistic insurer and multiple policyholders, where all agents' preferences are given by Choquet risk measures. Without assuming any specific dependence structure among the risk positions of the policyholders in this market, we obtain a characterization of Pareto-optimal contracts in Theorem 3.3.5 and Corollary 3.3.7. This result extends that of Boonen et al. [17] in that we do not assume that the policyholders' risk measures are coherent.

We then introduce the first-mover's advantage to the central monopolistic insurer in this market, and we examine implications thereof. In Theorem 3.4.5, we provide a characterization of Stackelberg equilibria in this market, which is the first characterization of these objects in a market with multiple agents on the demand side. We show that these equilibria induce Pareto-optimal contracts, with the caveat that the monopolistic insurer has an incentive to raise prices to the detriment of the policyholders. Additionally, we show that even in this case, Stackelberg equilibria compose a subset of Pareto-optimal contracts that do not induce a welfare gain to the policyholders, hence echoing the findings of Boonen and Ghossoub [20]. Consequently, and in light of the findings of Chapter 4, a natural question to ask is whether introducing strategic competition between insurers on the supply side would remedy this shortcoming in a market with multiple policyholders on the demand side. We leave this as a possible direction for future research.

As an application of our main results, we revisit the example of the market for flood risk insurance in the USA, as examined by Boonen et al. [17]. Firstly, we perform an analysis without the assumption that each policyholder uses a coherent risk measure. We confirm that in this setting, there are still significant welfare gains that can be obtained through the consolidation of risk, and that the welfare gain is larger for risks that are relatively less correlated. This insurance scheme allows previously uninsured policyholders to access more affordable contracts in the market. However, we argue that realizing these benefits requires additional regulation to prevent the central insurer from drastically increasing the cost of insurance.

3.8 Proofs of Main Results

3.8.1 Proof of Proposition 3.2.4

We first show that (2) $\implies$ (1). Suppose for the sake of contradiction that $(\{I_i^*\}_{i=1}^n, \{\pi_i^*\}_{i=1}^n)$ is not PO. Then there exists a contract $(\{I_i\}_{i=1}^n, \{\pi_i\}_{i=1}^n) \in \mathcal{IR}$ such that

$$\begin{aligned}\rho_i\left(X_i - I_i\left(X_i\right) + \pi_i\right) &\leq \rho_i\left(X_i - I_i^*\left(X_i\right) + \pi_i^*\right), \quad \forall\, i \in \{1, \ldots, n\}, \\ \rho^{IN}\left(\sum_{i=1}^n \left(I_i\left(X_i\right) - \pi_i\right)\right) &\leq \rho^{IN}\left(\sum_{i=1}^n \left(I_i^*\left(X_i\right) - \pi_i^*\right)\right),\end{aligned}$$

with at least one strict inequality. Summing these inequalities yields

$$\begin{aligned}\rho^{IN}\left(\sum_{i=1}^n (I_i(X_i) - \pi_i)\right) + \sum_{i=1}^n \rho_i(X_i - I_i(X_i) + \pi_i) \\ < \rho^{IN}\left(\sum_{i=1}^n (I_i^*(X_i) - \pi_i^*)\right) + \sum_{i=1}^n \rho_i(X_i - I_i^*(X_i) + \pi_i^*),\end{aligned}$$

so $(\{I_i^*\}_{i=1}^n, \{\pi_i^*\}_{i=1}^n)$ does not solve (3.2).

To show that (1) $\implies$ (2), suppose for the sake of contradiction that $(\{I_i^*\}_{i=1}^n, \{\pi_i^*\}_{i=1}^n) \in \mathcal{IR}$ does not solve (3.2). Then there exists a contract $\left(\{\tilde{I}_i\}_{i=1}^n, \{\tilde{\pi}_i\}_{i=1}^n\right) \in \mathcal{IR}$ such that

$$\begin{aligned}\rho^{IN}\left(\sum_{i=1}^n (\tilde{I}_i(X_i) - \tilde{\pi}_i)\right) + \sum_{i=1}^n \rho_i(X_i - \tilde{I}_i(X_i) + \tilde{\pi}_i) \\ < \rho^{IN}\left(\sum_{i=1}^n (I_i^*(X_i) - \pi_i^*)\right) + \sum_{i=1}^n \rho_i(X_i - I_i^*(X_i) + \pi_i^*).\end{aligned}$$

For each $i \in \{1, \ldots, n\}$, define $\hat{\pi}_i$ by

$$\begin{aligned}\hat{\pi}_i &= \tilde{\pi}_i + \rho_i(X_i - I_i^*(X_i) + \pi_i^*) - \rho_i(X_i - \tilde{I}_i(X_i) + \tilde{\pi}_i) \\ &= \rho_i(X_i - I_i^*(X_i)) + \pi_i^* - \rho_i(X_i - \tilde{I}_i(X_i)).\end{aligned}$$

By translation invariance,

$$\rho_i(X_i - \tilde{I}_i(X_i) + \hat{\pi}_i) = \rho_i(X_i - I_i^*(X_i) + \pi_i^*) \leq \rho_i(X_i),$$

since $(\{I_i^*\}_{i=1}^n, \{\pi_i^*\}_{i=1}^n) \in \mathcal{IR}$. However,

$$\begin{aligned}\rho^{IN}\left(\sum_{i=1}^n (\tilde{I}_i(X_i) - \hat{\pi}_i)\right) &= \rho^{IN}\left(\sum_{i=1}^n (\tilde{I}_i(X_i) - \tilde{\pi}_i)\right) + \sum_{i=1}^n \rho_i(X_i - \tilde{I}_i(X_i) - \tilde{\pi}_i) \\ &\quad - \sum_{i=1}^n \rho_i(X_i - I_i^*(X_i) - \pi_i^*) \\ &< \rho^{IN}\left(\sum_{i=1}^n (I_i^*(X_i) - \pi_i^*)\right) \leq \rho^{IN}(0) = 0\,.\end{aligned}$$

Hence, the contract $\left(\{\tilde{I}_i\}_{i=1}^n, \{\hat{\pi}_i\}_{i=1}^n\right) \in \mathcal{IR}$ and strictly improves over $(\{I_i^*\}_{i=1}^n, \{\pi_i^*\}_{i=1}^n)$ for the insurer, implying that $(\{I_i^*\}_{i=1}^n, \{\pi_i^*\}_{i=1}^n)$ is not PO. □

3.8.2 Proof of Theorem 3.3.5

By Assumption 3.3.2, we may rewrite (3.3) as

$$\inf_{\{I_i\}_{i=1}^n \in \mathcal{I}^n} \sup_{\mathbb{Q}\in\mathcal{Q}} \left\{ \mathbb{E}^{\mathbb{Q}}\left[\sum_{i=1}^n I_i(X_i)\right] + \sum_{i=1}^n \rho_i(X_i - I_i(X_i)) \right\}. \tag{3.12}$$

By comonotonic additivity of ρ_i, we may remove the $\rho_i(X_i)$ term from the objective. Writing the objective in terms of the marginal indemnification yields the equivalent problem

$$\inf_{\{I_i\}_{i=1}^n \in \mathcal{I}^n} \sup_{\mathbb{Q}\in\mathcal{Q}} \left\{ \sum_{i=1}^n \int_0^M (\mathbb{Q}(X_i > t) - \nu_i(X_i > t)) I_i'(t)\, dt \right\}. \tag{3.13}$$

Recall that a function is 1-Lipschitz if and only if it is absolutely continuous (i.e., differentiable almost everywhere) and its derivative is essentially bounded by the Lipschitz constant. Therefore for each $I \in \mathcal{I}$, we may identify its derivative with an element of the set $\mathcal{MI}$. Hence, we may rewrite (3.13) as

$$\inf_{\{h_i\}_{i=1}^n \in \mathcal{MI}^n} \sup_{\mathbb{Q}\in\mathcal{Q}} \left\{ \sum_{i=1}^n \int_0^M (\mathbb{Q}(X_i > t) - \nu_i(X_i > t)) h_i(t)\, dt \right\}. \tag{3.14}$$

Note that the objective is linear and continuous in $\mathbb{Q}$, which is clear from (3.12). Since $\mathcal{MI}$ is the unit ball in $L^\infty((0, M))$, it is weak*-compact by the Banach-Alaoglu theorem. From

(3.14), it is evident that objective is linear and weak*-continuous in $\{h_i\}_{i=1}^n$. Furthermore, it is clear from (3.12) that the objective is linear and continuous in $\mathbb{Q}$, and $\mathcal{Q}$ is convex and weak*-compact. Compactness then implies that both the supremum and the infimum are attained, which implies the existence of Pareto optima. By Sion's Minimax Theorem (e.g., [10, Theorem 2.132]), the minimax equality holds for this problem. Exchanging the order of the infimum and supremum of (3.14) yields

$$\sup_{\mathbb{Q}\in\mathcal{Q}} \inf_{\{h_i\}_{i=1}^n \in \mathcal{MI}^n} \left\{ \sum_{i=1}^n \int_0^M (\mathbb{Q}(X_i > t) - \nu_i(X_i > t)) h_i(t)\, dt \right\} . \tag{3.15}$$

It follows from standard results on minimax problems (e.g., [10, Section 2.3]) that every solution $\{h_i^*\}_{i=1}^n$ to (3.14) must achieve the inner infimum of (3.15) when $\mathbb{Q}$ is replaced by any solution $\mathbb{Q}^*$ of the outer supremum. It is immediate that the inner infimum is achieved if and only if the marginal indemnification of the form given by (3.7).

Let $\{I_i^*\}_{i=1}^n$ be indemnities satisfying (3.7). The objective of (3.15) now simplifies to

$$\sum_{i=1}^n \int_0^M (\mathbb{Q}(X_i > t) - \nu_i(X_i > t))(I_i^*)'(t)\, dt = \sum_{i=1}^n \int_0^M \min\{\mathbb{Q}(X_i > t) - \nu_i(X_i > t),\, 0\}\, dt\,.$$

Adding $\sum_{i=1}^n \int_0^M \nu_i(X_i > t)\, dt$ to this expression yields

$$\sum_{i=1}^n \int_0^M \min\{\mathbb{Q}(X_i > t),\, \nu_i(X_i > t)\}\, dt\,,$$

which is precisely the objective of the problem (3.6). Hence, we have shown that (3.6) is equivalent to (3.3), which completes the proof. □

3.8.3 Proof of Theorem 3.4.5

For notational convenience, for each $i \in \{1, \dots, n\}$, define the set

$$\mathcal{A}_i(\Pi_i) := \arg\min_{I\in\mathcal{I}} \rho_i(X_i - I(X_i) + \Pi_i(I(X_i)))\,.$$

Define $\mathcal{A}(\Pi)$ to be the Cartesian product of $\mathcal{A}_i(\Pi_i)$, $i \in \{1, \dots, n\}$. By Proposition 3.4.3, the set $\mathcal{A}_i(\Pi_i)$ consists of precisely the indemnities satisfying (3.9). Furthermore, the set $\mathcal{A}(\Pi)$ consists of precisely the indemnities satisfying point (1) of Definition 2.4.5.

Point (2) of Definition 2.4.5 now simplifies to the following. A given market mechanism $(\{I_i^*\}_{i=1}^n, \{\Pi_i^*\}_{i=1}^n)$ is an SE if and only if it solves the problem

$$\inf_{\{\Pi_i\}_{i=1}^n \in \aleph} \inf_{\{I_i\}_{i=1}^n \in \mathcal{A}(\Pi)} \left\{ \rho^{IN} \left(\sum_{i=1}^n \left(I_i(X_i) - \Pi_i(I_i(X_i)) \right) \right) \right\} .$$

By translation invariance and Assumption 3.3.2, we may write the objective as

$$\begin{aligned} \rho^{IN} \left(\sum_{i=1}^n I_i(X_i) - \Pi_i(I_i(X_i)) \right) &= \sup_{\mathbb{Q} \in \mathcal{Q}} \left\{ \mathbb{E}^{\mathbb{Q}} \left(\sum_{i=1}^n I_i(X_i)) \right) - \sum_{i=1}^n \Pi_i(I_i(X_i)) \right\} \\ &= \sup_{\mathbb{Q} \in \mathcal{Q}} \left\{ \sum_{i=1}^n \left(\mathbb{E}^{\mathbb{Q}}(I_i(X_i)) - \Pi_i(I_i(X_i)) \right) \right\} \\ &= \sup_{\mathbb{Q} \in \mathcal{Q}} \left\{ \sum_{i=1}^n \int_0^M (\mathbb{Q}(X_i > t) - \mu_i(X_i > t)) I_i'(t)\, dt \right\} . \end{aligned} \tag{3.16}$$

Substituting this into the above and identifying each Π_i with its associated measure μ_i yields the equivalent problem

$$\inf_{\{\mu_i\}_{i=1}^n \in \aleph} \inf_{\{I_i\}_{i=1}^n \in \mathcal{A}(\Pi)} \sup_{\mathbb{Q} \in \mathcal{Q}} \left\{ \sum_{i=1}^n \int_0^M (\mathbb{Q}(X_i > t) - \mu_i(X_i > t)) I_i'(t)\, dt \right\} . \tag{3.17}$$

We now wish to check that the objective of (3.17) is linear and continuous in each argument, and that the domain of optimization is convex. Since every premium principle is a continuous linear functional, the objective of (3.17) is linear and continuous in each Π_i with respect to the topology induced by the weak* topology on $ba(\sigma(X_i))$ – this is clear from (3.16). Similarly, the objective is linear and continuous in $\mathbb{Q}$. Additionally, both $\aleph$ and $\mathcal{Q}$ are convex, and $\mathcal{Q}$ is weak* compact by assumption. Moreover, $\aleph$ is also weak* compact, since each $\aleph_i$ is homeomorphic to the unit ball.

However, the objective is not linear in I_i. Instead, we may identify each $I \in \mathcal{A}_i(\Pi_i)$ with its derivative. Define $\mathcal{MI}_i(\Pi_i)$ to be the set of all measurable functions g on $(0, M)$ such that almost everywhere,

$$g(t) = \begin{cases} 1, & \text{if} \quad \nu_i(X_i > t) > \mu_i(X_i > t)\,, \\ h(t), & \text{if} \quad \nu_i(X_i > t) = \mu_i(X_i > t)\,, \\ 0, & \text{if} \quad \nu_i(X_i > t) < \mu_i(X_i > t)\,, \end{cases}$$

where h is a $[0,1]$-valued measurable function. That is, $\mathcal{MI}_i(\Pi_i)$ is the set of derivatives of functions in $\mathcal{A}_i(\Pi_i)$. Note that $\mathcal{MI}_i(\Pi_i)$ is weak* closed, since it is the set of solutions to

$$\inf_{g\in\mathcal{MI}}\left\{\int_0^M (\mu_i(X_i>t)-\nu_i(X_i>t))g(t)\,dt\right\},$$

where the objective of the above problem is weak* continuous. Since $\mathcal{MI}_i(\Pi_i)$ is closed subset of the weak* compact set $\mathcal{MI}$, it is weak* compact as well. Hence, there exists a solution to (3.17), which implies the existence of Stackelberg equilibrium. Denote by $\mathcal{MI}(\Pi)$ the Cartesian product of each $\mathcal{MI}_i(\Pi_i)$, which can be immediately verified to be a convex set. Then (3.17) is equivalent to

$$\inf_{\{\mu_i\}_{i=1}^n\in\aleph}\ \inf_{\{g_i\}_{i=1}^n\in\mathcal{MI}(\Pi)}\ \sup_{\mathbb{Q}\in\mathcal{Q}}\left\{\sum_{i=1}^n\int_0^M (\mathbb{Q}(X_i>t)-\mu_i(X_i>t))g_i(t)\,dt\right\}, \tag{3.18}$$

from which it is clear that the objective is both linear and weak* continuous in g_i. By Sion's Minimax Theorem (e.g., [10, Theorem 2.132]), the minimax equality is achieved. Exchanging the order of the infimum and supremum yields

$$\sup_{\mathbb{Q}\in\mathcal{Q}}\ \inf_{\{\mu_i\}_{i=1}^n\in\aleph}\ \inf_{\{g_i\}_{i=1}^n\in\mathcal{MI}(\Pi)}\left\{\sum_{i=1}^n\int_0^M (\mathbb{Q}(X_i>t)-\mu_i(X_i>t))g_i(t)\,dt\right\}. \tag{3.19}$$

Similarly to the proof of Theorem 3.3.5, the necessary condition requires that the inner infimum of (3.19) is achieved for every solution $\mathbb{Q}^*$ of the outer supremum. We conclude the proof by solving the inner infimum of (3.19) for any fixed $\mathbb{Q}\in\mathcal{Q}$, by minimizing each integrand for each t. For each $i\in\{1,\dots,n\}$, we have

$$\begin{aligned}&(\mathbb{Q}(X_i>t)-\mu_i(X_i>t))g_i(t)\\ &\quad=\begin{cases}(\mathbb{Q}(X_i>t)-\mu_i(X_i>t)), & \text{if}\quad \nu_i(X_i>t)>\mu_i(X_i>t),\\ (\mathbb{Q}(X_i>t)-\mu_i(X_i>t))h_i(t), & \text{if}\quad \nu_i(X_i>t)=\mu_i(X_i>t),\\ 0, & \text{if}\quad \nu_i(X_i>t)<\mu_i(X_i>t),\end{cases}\end{aligned} \tag{3.20}$$

where h_i is some $[0,1]$-valued measurable function. We consider the following three cases:

- $\mathbb{Q}(X_i>t)<\nu_i(X_i>t)$: Then (3.20) is minimized when $\mu_i(X_i>t)=\nu_i(X_i>t)$ and $h_i(t)=1$. The minimum value is $\mathbb{Q}(X_i>t)-\nu_i(X_i>t)$.
- $\mathbb{Q}(X_i>t)>\nu_i(X_i>t)$: In this case, (3.20) is minimized when $\mu_i(X_i>t)\geq\nu_i(X_i>t)$, and $h_i(t)=0$ if $\mu_i(X_i>t)=\nu_i(X_i>t)$. The minimum value is 0.

- $\mathbb{Q}(X_i > t) = \nu_i(X_i > t)$: In this case, (3.20) also has a minimum value of 0, when $\mu_i(X_i > t) \geq \nu_i(X_i > t)$, with no further restriction on $h_i(t)$ in the case that $\mu_i(X_i > t) = \nu_i(X_i > t)$.

Hence, the infimum is attained if μ_i, g_i are chosen such that for almost every $t \in (0, M)$,

$$\begin{aligned}
&\mu_i(X_i > t) = \nu_i(X_i > t), \text{ if } \quad \mathbb{Q}(X_i > t) < \nu_i(X_i > t)\,, \\
&\mu_i(X_i > t) \geq \nu_i(X_i > t), \text{ if } \quad \mathbb{Q}(X_i > t) \geq \nu_i(X_i > t)\,, \\
&g_i(t) = \begin{cases} 1 & \text{if} \quad \nu_i(X_i > t) = \mu_i(X_i > t) > \mathbb{Q}(X_i > t)\,, \\ h_i(t) & \text{if} \quad \nu_i(X_i > t) = \mu_i(X_i > t) = \mathbb{Q}(X_i > t)\,, \\ 0 & \text{if} \quad \nu_i(X_i > t) < \mu_i(X_i > t) = \mathbb{Q}(X_i > t)\,, \\ 0 & \text{if} \quad \nu_i(X_i > t) \leq \mu_i(X_i > t) < \mathbb{Q}(X_i > t)\,. \end{cases}
\end{aligned}$$

By Lemma 3.4.4, there exists a measure μ_i that satisfies $\mu_i(X_i > t) = \nu_i(X_i > t)$ for all $t \in (0, M)$. Therefore, μ_i and g_i can indeed be chosen to satisfy the above, which implies that the inner infimum is attained. Substituting its minimum value into (3.19) yields

$$\sup_{\mathbb{Q} \in \mathcal{Q}} \left\{ \sum_{i=1}^{n} \int_0^M \min\left\{ \mathbb{Q}(X_i > t) - \nu_i(X_i > t),\, 0 \right\} dt \right\},$$

and adding $\sum_{i=1}^{n} \int_0^M \nu_i(X_i > t)\, dt$ to the objective function yields

$$\sup_{\mathbb{Q} \in \mathcal{Q}} \left\{ \sum_{i=1}^{n} \int_0^M \min\left\{ \mathbb{Q}(X_i > t),\, \nu_i(X_i > t) \right\} dt \right\}.$$

Hence, we have shown that a market mechanism $(\{I_i^*\}_{i=1}^n, \{\Pi_i^*\}_{i=1}^n)$ is an SE if and only if it is of the form given in (3.11), where $\mathbb{Q}^* \in \mathcal{Q}$ solves (3.10). □

Chapter 4

Equilibria with Supply Competition

4.1 Introduction

While Chapter 3 explores a market with multiple policyholders on the demand side, this chapter examines the opposite extreme by introducing multiple insurers on the supply side. Note that the Stackelberg model assumes monopolistic control of the insurance market by one entity, and it does not directly generalize to the case of multiple insurers. In the present chapter, we seek to address these issues by formulating a sequential model of an insurance market with multiple insurers. Similar to the Stackelberg model, the insurers collectively have the first-mover advantage, and they can choose their premium principles in anticipation of the policyholder's behaviour. However, the insurers are in strategic competition when it comes to setting the price of insurance. This model is, to the best of our knowledge, the first model that examines insurance contracting between a policyholder and multiple insurers from the perspective of market equilibrium. While Boonen and Ghossoub [18, 19] study optimal insurance between one policyholder and multiple insurers, the insurers' premium principles are fixed in their model. In contrast, the novelty of the model presented in this chapter is the strategic elements that emerge when insurers are allowed to select their own pricing functions. We assume that the preferences of each agent are given by DRMs under heterogeneous beliefs, which allows for a high level of flexibility. Furthermore, we allow the premium principles posted by each insurer to be fully general. In particular, in a departure from the efficiency literature, an insurer's premium principle is not required to be related to their true preferences, thereby reflecting the strategic nature of interaction on the supply side of the market. In our market model, we propose the notion of a Subgame Perfect Nash Equilibrium (SPNE),

a refinement of the Nash Equilibrium (NE), as the primary solution concept to illustrate strategic competition in this insurance market. We assume that each insurer acts in order to maximize their own welfare, which is in line with the literature on competitive equilibria (notably, this is in contrast to the cooperative model proposed by Asimit and Boonen [7]). We argue in this chapter that our model, as well as our suggested solution concept, act as a natural extension of the Stackelberg setting to multiple insurers. In particular, we show that if we consider a special case of our model in which there is only one insurer ($m = 1$), we can recover the Stackelberg model as a special case, in which all Stackelberg equilibria are SPNEs.

A standard result allows us to characterize SPNEs through the process of backward induction, which splits the problem into two steps: (i) the decision problem of the policyholder, and (ii) that of the insurers. The policyholder's decision problem reduces to a demand problem under fixed premium principles, and indemnities are fully characterized through the Marginal Indemnity Function (MIF) approach, as in Assa [8] or Zhuang et al. [114], for instance. To address the insurers' decision problem, we provide a set of sufficient conditions that explicitly characterize a class of SPNEs. In these equilibria, the policyholder is subject to prices corresponding to the second-lowest true preferences of the insurers as measured by distorted subjective survival probabilities, similarly to the premium principle suggested by Boonen et al. [22] to represent the maximum possible premium that preserves coalitional stability. Our results expand on this by showing that this premium principle plays a central role in determining stability of SPNEs as well.

We then use our characterization to examine efficiency properties of the resulting equilibrium contracts. Since SPNEs are market equilibria, it is not true in general that these coincide with PO allocations. However, we show that all allocations resulting from our class of SPNEs are indeed PO, thereby providing a link to the literature on efficient reinsurance contracting. We emphasize that PO in our setting results from the structure of the insurance market, and it is not assumed *a priori*. Furthermore, it is also important to examine allocations in terms of their relationship to the status quo. Market participants can only be incentivized to enter into contracts that induce a welfare gain, which is not true in the Stackelberg mode, as concluded by Boonen and Ghossoub [20] and Chapter 3 of this thesis. We find that within our class of SPNEs, each agent is able to realize a welfare gain under mild conditions, a consequence of both the competition in the insurance market and of the heterogeneity of beliefs. Moreover, one would expect that the welfare gain of the policyholder increases as the level of competition in the market increases. We formalize this notion by showing that the policyholder experiences an improvement in welfare if an additional insurer is added to the market. A consequence is that the presence of just two insurers is enough to realize a strict gain for the poliycholder compared to the Stackelberg

case. This point is illustrated by a numerical example.

The remainder of this chapter is organized as follows. Section 4.2 presents our sequential model for the insurance market, and it introduces SPNEs in this context. Section 4.3 provides a characterization of SPNEs and specific examples thereof. An analysis of SPNE contracts is given in Section 4.4, in which we examine efficiency of these contracts, as well as the relationship between SPNE contracts and the status quo. We also provide a numeral example, exhibiting an SPNE that achieves a welfare gain for all agents. Section 3.7 concludes. The proofs of most of this chapter's results are given in Section 4.6. Additional illustrative constructions of equilibria are presented in Section 4.7.

4.2 Problem Setup

For this chapter, we do not require that each risk be bounded, although we do assume that each agent's evaluation of the aggregate risk is finite. This allows for a slightly more general setup than that of Chapter 3. Specifically, let $\mathcal{X}$ denote the collection of all measurable real-valued functions on (Ω, Σ). We consider a centralized market with m suppliers of insurance, but only $n = 1$ policyholders. This policyholder is subject to a random insurable loss X, which is represented by a non-negative random variable $X \in \mathcal{X}$.

Throughout this chapter, the principal method for evaluating the risk of positions is the Choquet integral, and all risk measures are of this form. Given a capacity $\upsilon \in \mathcal{C}$, we use the notation ρ^{υ} to denote the Choquet integral with respect to υ.

We assume that these agents all have access to an insurance market with m suppliers of insurance. Given the presence of multiple suppliers, the primary purpose of this chapter is to study the effects of competition.

4.2.1 The Market Participants' Preferences

The policyholder is imbued with beliefs on the space (Ω, Σ), represented by a probability measure $\mathbb{P}$. We assume that the policyholder's preferences are represented by a distortion risk measure (DRM) consisting of the measure $\mathbb{P}$ distorted by a distortion function g. That is, for every risk $Y \in \mathcal{X}$, the risk measure of the policyholder is given by

$$\rho^{PH}(Y) := \rho^{g \circ \mathbb{P}}(Y) = \int Y \, dg \circ \mathbb{P} \, .$$

We assume that there are m insurers in this market, denoted by the set $\mathcal{M} := \{1, \dots, m\}$. Each insurer has heterogeneous beliefs on the states of nature, represented by probability measures $\mathbb{Q}_1, \dots, \mathbb{Q}_m$ for insurers $1, \dots, m$ respectively. We do not assume any relationship between $\mathbb{P}, \mathbb{Q}_1, \dots, \mathbb{Q}_m$, allowing for a flexible setup such as that in Boonen and Ghossoub [19]. Like the policyholder, each insurer evaluates risk according to a DRM. For each $i \in \mathcal{M}$, denote by g_i the distortion function of insurer i. Then the preference of insurer i is represented by the risk measure

$$\rho_i(Y) := \rho^{g_i \circ \mathbb{Q}_i}(Y) = \int Y \, dg_i \circ \mathbb{Q}_i \, .$$

To avoid pathological cases, we impose the restriction that all agents attribute finite risk to the initial position. Namely, for all $i \in \mathcal{M}$,

$$\rho_i(X) < \infty \text{ and } \rho^{PH}(X) < \infty \, , \tag{4.1}$$

which can be interpreted as an assumption of well-posedness of the problem. Under this restriction, all admissible premia are finite.

4.2.2 Insurance Contracts

We assume that the premium principles charged by each insurer follow a Choquet expectation. That is, the premium charged by insurer i to ensure a risk Y is

$$\pi_i(Y) := \int Y \, d\nu_i \, ,$$

where $\nu_i \in \mathcal{C}$. We also use the notation

$$\pi^{\nu_i}(Y) := \int Y \, d\nu_i \, . \tag{4.2}$$

In other words, the set of admissible premium principles $\aleph$ can be identified with $\mathcal{C}^m$. Throughout this chapter, we use the letter π to refer to premium principles and the letter ρ to refer to risk measures. Note here that we do not further restrict the set of possible premium principles in $\mathcal{C}$. In particular, we do not require $\nu \in \mathcal{C}$ to be normalized such that $\nu(S) = 1$. Hence, the insurer is free to introduce a loading factor if they so choose, to represent any frictional cost of insurance.

In this market, the policyholder may cede a portion of risk $I_i(X)$ to insurer i by paying the premium $\pi_i(I_i(X))$. Let the vector $\vec{I} := (I_1, \dots, I_m)$ denote the indemnity schedule

chosen by the policyholder. As before, we assume that the collection of admissible indemnification functions in this market is the set of 1-Lipschitz indemnities $\mathcal{I}$. However, we impose the additional assumption that the sum of indemnities must also be 1-Lipschitz.

Assumption 4.2.1. $\sum_{i\in\mathcal{M}} I_i \in \mathcal{I}$.

We refer to the class of indemnity schedules $\vec{I}$ satisfying Assumption 4.2.1 as

$$\vec{\mathcal{I}} \subset \mathcal{I}^n \,.$$

Restricting the set of admissible indemnities *a priori* to those in the above class rules out any potential moral hazard that might arise from the policyholder's misreporting of the true value of the loss in a given state of the world.

4.2.3 A Sequential Game Framework

We model this insurance market as a sequential game. First, all insurers simultaneously select pricing capacities $\nu_i \in \mathcal{C}$. The policyholder then views these pricing rules and selects an indemnity vector $(I_1, \ldots, I_m)$. The remaining random loss of the policyholder is $X - \sum_{i\in\mathcal{M}} I_i(X)$ and the policyholder pays the premium $\pi_i(I_i(X)) = \pi^{\nu_i}(I_i(X))$ to insurer i. Hence, the resulting risk exposure of the policyholder is $X - \sum_{i\in\mathcal{M}} I_i(X) + \sum_{i\in\mathcal{M}} \pi_i(I_i(X))$, and the policyholder evaluates this end-of-period risk exposure via

$$\rho^{PH}\left(X - \sum_{i\in\mathcal{M}} I_i(X) + \sum_{i\in\mathcal{M}} \pi_i(I_i(X))\right) = \rho^{PH}(X) - \rho^{PH}\left(\sum_{i\in\mathcal{M}} I_i(X)\right) + \sum_{i\in\mathcal{M}} \pi^{\nu_i}(I_i(X))\,, \tag{4.3}$$

where the simplification follows from comonotone additivity and translation invariance of the risk measure ρ^{PH}, and the fact that X and $\sum_{i\in\mathcal{M}} I_i(X)$ are comonotone by Assumption 4.2.1. Each insurer i will have assumed the risk $I_i(X)$ upon payment of the premium $\pi_i(I_i(X))$. Their resulting risk exposure is $I_i(X) - \pi_i(I_i(X))$, and they evaluate it via

$$\rho_i\left(I_i(X) - \pi_i(I_i(X))\right) = \rho_i(I_i(X)) - \pi^{\nu_i}(I_i(X))\,. \tag{4.4}$$

The goal of each agent is to minimize their risk exposure – as such, (4.3) and (4.4) represent the payoffs of the game to the policyholder and insurers respectively. We note that as a consequence of (4.1), the expressions (4.3) and (4.4) are always well-defined and finite.

Similar to the Stackelberg setting, insurers have the first-mover advantage, and their strategy is determined by their collective simultaneous choice of premium principles (which, by (4.2), depends only on pricing capacities). Therefore, the insurers' strategy can be identified by the choice of a vector of capacities $(\nu_1, \dots, \nu_n) \in \mathcal{C}^m$. Here, the strategy of insurer i is denoted by $\nu_i \in \mathcal{C}$. The policyholder's strategy is determined by its choice of indemnity schedule, after viewing the pricing rules selected by the insurers. This can be identified by a function

$$\begin{aligned} \mathfrak{I}: \quad & \mathcal{C}^m \to \vec{\mathcal{I}} \\ & (\nu_1, \dots, \nu_n) \mapsto \mathfrak{I}(\nu_1, \dots, \nu_n), \end{aligned} \tag{4.5}$$

which maps the observed pricing capacities to a feasible indemnity schedule. A strategy, in the formal game-theoretical sense, is therefore represented by the tuple $(\nu_1, \dots, \nu_n, \mathfrak{I}) \in \mathcal{C}^m \times (\vec{\mathcal{I}})^{\mathcal{C}^m}$.

4.2.4 Additional Notational Conventions

We use the following conventions in the remainder of this chapter. Recall that $\mathcal{M} = \{1, \dots, m\}$ denotes the set of insurers. For each insurer $i \in \mathcal{M}$, we define the capacity

$$\tau_i := g_i \circ \mathbb{Q}_i$$

to represent the true preferences of the insurer. For ease of notation, we use the index 0 to represent the policyholder. Hence, all agents in the insurance market are given by $\mathcal{M} \cup \{0\}$. Following this convention, we define

$$\rho_0 := \rho^{PH} \tag{4.6}$$

to represent the risk measure of the policyholder, and

$$I_0(X) := X - \sum_{i \in \mathcal{M}} I_i(X) \tag{4.7}$$

to represent the retained loss of the policyholder. Note that as a consequence of Assumption 4.2.1, we have $I_0 \in \mathcal{I}$. Additionally, we define

$$\tau_0 := g \circ \mathbb{P} \tag{4.8}$$

as the true preferences of the policyholder. It will also be convenient to define

$$\nu_0 := g \circ \mathbb{P} \tag{4.9}$$

as the "pricing rule" used by the policyholder in the insurance market. While (4.8) and (4.9) are different notational conventions for the same capacity, this will be convenient for the exposition of our main results. Note that we use the letter τ to represent true preferences of market agents, and we use the letter ν to represent pricing rules in the market. The interpretation for the policyholder is as follows: the policyholder uses their own true preferences as a premium principle when deciding which contracts to participate in (and which contracts to decline).

We use the unstyled capital letter I to refer to indemnity functions, i.e., elements of $\mathcal{I}$. Vectors of indemnities are notated with the vector symbol. For example, $\vec{I}$ is an element of $\vec{\mathcal{I}}$.

In the context of the economic game, the notation $\mathfrak{I}$ denotes a strategy, and it is therefore a map from $\mathcal{C}^m$ to $\vec{\mathcal{I}}$, as defined in (4.5) Hence, for any fixed insurance strategies $(\nu_1, \dots, \nu_n) \in \mathcal{C}^m$, the choice of indemnities $\mathfrak{I}(\nu_1, \dots, \nu_n)$ is a vector in $\vec{\mathcal{I}}$. For an insurer $i \in \mathcal{M}$, we use the notation $\mathfrak{I}_i(\nu_1, \dots, \nu_n)$ to refer to the i-th component of $\mathfrak{I}(\nu_1, \dots, \nu_n)$. That is, $\mathfrak{I}_i(\nu_1, \dots, \nu_n)$ is the indemnity ceded to the i-th policyholder under the strategy $\mathfrak{I}$, when the pricing capacities of the insurers are given by $(\nu_1, \dots, \nu_n)$. When pricing capacities $(\nu_1, \dots, \nu_n)$ are fixed in context and we are only concerned with the indemnities $\mathfrak{I}(\nu_1, \dots, \nu_n)$, we drop the capacities from the argument of the function and write $I_i = \mathfrak{I}_i(\nu_1, \dots, \nu_n)$.

4.2.5 Equilibria in Sequential Games

As discussed in Subsection 2.4.3, a common solution concept for sequential games is that of the *Subgame Perfect Nash Equilibrium* (SPNE). The definitions of equilibria concepts are reiterated below in the notation and context of this insurance market.

Definition 4.2.2. A strategy $(\nu_1^*, \dots, \nu_m^*, \mathfrak{I}^*)$ is a *Nash Equilibrium* (NE) if:

1. There does not exist a capacity $\tilde{\nu}$ and $i \in \mathcal{M}$ such that

$$\begin{aligned}\rho_i(\mathfrak{I}_i^*(\tilde{\nu}, \nu_{-i}^*)(X)) - \pi^{\tilde{\nu}}(\mathfrak{I}_i^*(\tilde{\nu}, \nu_{-i}^*)(X))) \\ < \rho_i(\mathfrak{I}_i^*(\nu_1^*, \dots, \nu_m^*)(X)) - \pi^{\nu_i^*}(\mathfrak{I}_i^*(\nu_1^*, \dots, \nu_m^*)(X))\,,\end{aligned}$$

   where $(\tilde{\nu}, \nu_{-i}^*)$ denotes the vector $(\nu_1^*, \dots, \nu_m^*)$ with the i-th component replaced by $\tilde{\nu}$.

2. There does not exist an indemnity selection $\tilde{\mathfrak{I}}$ such that

$$\rho^{PH}(X) - \sum_{i\in\mathcal{M}} \rho^{PH}\left(\tilde{\mathfrak{I}}_i(\nu_1^*,\dots,\nu_m^*)(X)\right) + \sum_{i\in\mathcal{M}} \pi^{\nu_i^*}(\tilde{\mathfrak{I}}_i(\nu_1^*,\dots,\nu_m^*)(X)) \\ < \rho^{PH}(X) - \rho^{PH}(\mathfrak{I}_i^*(\nu_1^*,\dots,\nu_n^*)(X)) + \sum_{i\in\mathcal{M}} \pi^{\nu_i^*}(\mathfrak{I}_i^*(\nu_1^*,\dots,\nu_m^*)(X))\,.$$

Note that in this model, the only (proper) subgames are when the policyholder decides how to purchase insurance after all premium principles are determined. The difference between SPNEs and NEs in this case is that in an SPNE, the policyholder must behave rationally given any possible choice of premium principles $(\nu_1,\dots,\nu_n) \in \mathcal{C}^m$. For the strategy to be an NE, it is only required that the policyholder behaves rationally for the *specific* choice of capacities $(\nu_1^*,\dots,\nu_m^*)$. This is made precise by the following.

Definition 4.2.3. In our setting, a strategy $(\nu_1^*,\dots,\nu_m^*,\mathfrak{I}^*)$ is a *Subgame Perfect Nash Equilibrium* (SPNE) if:

1. It is an NE.
2. For any choice of capacities $(\nu_1,\dots,\nu_m) \in \mathcal{C}^m$, there does not exist an indemnity selection $\tilde{\mathfrak{I}}$ such that

$$\rho^{PH}(X) - \sum_{i\in\mathcal{M}} \rho^{PH}\left(\tilde{\mathfrak{I}}_i(\nu_1,\dots,\nu_m)(X)\right) + \sum_{i\in\mathcal{M}} \pi^{\nu_i}(\tilde{\mathfrak{I}}_i(\nu_1,\dots,\nu_m)(X)) \\ < \rho^{PH}(X) - \sum_{i\in\mathcal{M}} \rho^{PH}(\mathfrak{I}_i^*(\nu_1,\dots,\nu_m)(X)) + \sum_{i\in\mathcal{M}} \pi^{\nu_i}(\mathfrak{I}_i^*(\nu_1,\dots,\nu_m)(X))\,.$$

While it is clear that every SPNE is an NE, the converse does not hold: in general, there exist NEs that are not SPNEs. This is true even in our setting, where the structure of the game is simple. We provide an example in Section 4.7.2, in which we also briefly comment on the advantages of SPNEs over NEs.

4.3 A Characterization of SPNEs

In this section, we provide sufficient conditions to characterize SPNEs in the sequential market outlined in Section 4.2. It is well known that all SPNEs can be found via the process backward induction (see, e.g., Osborne and Rubinstein [90, Proposition 99.2] for a proof). We restate this result in our setting in the following proposition.

Proposition 4.3.1. *A strategy $(\nu_1^*, \dots, \nu_m^*, \mathfrak{I}^*)$ is an SPNE if and only if it can be found through backward induction. That is, for any choice of pricing capacities $(\nu_1, \dots, \nu_m) \in \mathcal{C}^m$, the indemnity structure $\mathfrak{I}^*(\nu_1, \dots, \nu_m)$ solves*

$$\min_{I \in \vec{\mathcal{I}}} \left\{ \rho^{PH}(X) - \sum_{i \in \mathcal{M}} \rho^{PH}\left(I_i(X)\right) + \sum_{i \in \mathcal{M}} \pi^{\nu_i}\left(I_i(X)\right) \right\}, \tag{4.10}$$

and $(\nu_1^*, \dots, \nu_m^*)$ is an NE for the reduced game formed by fixing the policyholder's strategy $\mathfrak{I}^*$. That is, there does not exist $i \in \mathcal{M}$ and $\tilde{\nu} \in \mathcal{C}$ such that

$$\rho_i(\mathfrak{I}_i^*(\tilde{\nu}, \nu_{-i}^*)(X)) - \pi^{\tilde{\nu}}(\mathfrak{I}_i^*(\tilde{\nu}, \nu_{-i}^*)(X)) < \rho_i(\mathfrak{I}_i^*(\nu_1^*, \dots, \nu_m^*)(X)) - \pi^{\nu_i^*}(\mathfrak{I}_i^*(\nu_1^*, \dots, \nu_m^*)(X)). \tag{4.11}$$

We will see from the result of Proposition 4.3.3 that Problem (4.10) always has a solution. Therefore, by Proposition 4.3.1, to characterize an SPNE, we will:

1. Find Nash Equilibria in the subgames where the policyholder selects insurance. That is, for any pricing capacities $(\nu_1, \dots, \nu_m) \in \mathcal{C}^m$, we will determine the optimal strategy $\mathfrak{I}^*$ of the policyholder. This reduces to a demand problem for the policyholder, and we can characterize solutions via the MIF approach, as in Boonen and Ghossoub [19]. This is outlined in Subsection 4.3.1.

2. For the selected indemnity strategy $\mathfrak{I}^*$, we will find Nash Equilibria for the simultaneous selection of pricing capacities. We provide sufficient conditions in Subsection 4.3.2.

Remark 4.3.2. *In the special case when $m = 1$, we have only one insurer. Then Proposition 4.3.1 implies that a strategy $(\nu_1^*, \mathfrak{I}^*)$ is an SPNE if and only if (4.10) and (4.11) hold. The condition for the reduced game (4.10) simplifies to*

$$\mathfrak{I}^*(\nu_1^*) = \min_{I \in \vec{\mathcal{I}}} \left\{ \rho^{PH}(X) - \rho^{PH}\left(I(X)\right) + \pi^{\nu_1^*}\left(I(X)\right) \right\}.$$

Furthermore, the second condition (4.11) implies that

$$\rho_1(\mathfrak{I}^*(\nu_1^*)(X)) - \pi^{\nu_1^*}(\mathfrak{I}^*(\nu_1^*)(X)) = \min_{\tilde{\nu} \in \mathcal{C}} \left\{ \rho_1(\mathfrak{I}^*(\tilde{\nu})(X)) - \pi^{\tilde{\nu}}(\mathfrak{I}^*(\tilde{\nu})(X)) \right\}.$$

It follows from the definition of Stackelberg equilibria that every Stackelberg equilibrium is the result of an SPNE. Therefore, it is natural to interpret the SPNE in our model as a generalization of the Stackelberg setting to multiple insurers. We elaborate on this special case in Subsection 4.4.1.

4.3.1 Backward Induction Step One – An Optimal insurance Problem

In the first step of the backwards induction, we solve Problem (4.10) given any fixed strategy of the insurers $(\nu_1, \dots, \nu_m) \in \mathcal{C}^m$. A characterization is given in the following proposition, which can be found in Boonen and Ghossoub [19]. We provide a slightly condensed proof in Section 4.6.

Proposition 4.3.3. *Define the capacity*

$$\underline{\nu}(X > z) := \min_{j \in \mathcal{M}} \nu_j(X > z)\,,$$

and define the set

$$\mathcal{M}_z := \{i \in \mathcal{M} : \nu_i(X > z) = \underline{\nu}(X > z)\}\ . \tag{4.12}$$

The selection of indemnities $\vec{I}^* = (I_1^*, \dots, I_m^*)$ is optimal for Problem (4.10) if and only if for each $i \in \mathcal{M}$ and all $x \in \mathbb{R}_+$, there exists $[0,1]$-valued measurable functions h, h_i such that for almost all $z \in \mathbb{R}_+$,

$$\begin{aligned} I_i^*(x) &= \int_0^x \gamma_i^*(z)\, dz\,, \\ \gamma_i^*(z) &= h_i(z) \mathbb{1}_{\{i \in \mathcal{M}_z\}}\,, \\ \sum_{i \in \mathcal{M}} \gamma_i^*(z) &= \mathbb{1}_{\{g(\mathbb{P}(X>z)) > \underline{\nu}(X>z)\}} + h(z) \mathbb{1}_{\{g(\mathbb{P}(X>z)) = \underline{\nu}(X>z)\}}\,. \end{aligned}$$

We see that Proposition 4.3.3 gives a full characterization of optimal solutions to Problem (4.10). However, optimal indemnities are not unique in general: from the statement of Proposition 4.3.3, there is some flexibility in choosing the functions h_i when $i \in \mathcal{M}_z$. That is, the policyholder has many decisions that are optimal in the sense of Problem (4.10) when they are indifferent between their choice of insurer. We narrow the set of optimal solutions by imposing the following condition.

Definition 4.3.4. For each $z \in \mathbb{R}_+$, let $\mathcal{M}_z$ be as defined in (4.12). An optimal indemnity $\vec{I}^*$ *distributes generously* if for almost all $z \in \mathbb{R}_+$, we have:

1. If $g(\mathbb{P}(X > z)) = \underline{\nu}(X > z)$, and there exist $j \in \mathcal{M}_z$ such that $\tau_j(X > z) < g(\mathbb{P}(X > z))$, then
$$\sum_{i \in \mathcal{M}} \gamma_i^*(z) = 1\,.$$

2. If $i \in \mathcal{M}_z$ and either:

   - there exist $k \in \mathcal{M}_z \setminus \{i\}$ such that $\tau_k(X > z) < \tau_i(X > z)$; or,
   - $g(\mathbb{P}(X > z)) = \underline{\nu}(X > z)$ and $g(\mathbb{P}(X > z)) < \tau_i(X > z)$,

   then

$$\gamma_i^*(z) = 0\,.$$

Recall from (4.8) and (4.9) the conventions $\tau_0 = g \circ \mathbb{P}$ and $\nu_0 = g \circ \mathbb{P}$. Additionally, we may define γ_0^* such that for all $z \in \mathbb{R}_+$,

$$\gamma_0^*(z) := 1 - \sum_{i \in \mathcal{M}} \gamma_i^*(z)\,. \tag{4.13}$$

Under these conventions, Definition 4.3.4 can be stated more succinctly. An optimal indemnity $\vec{I}^*$ distributes generously if for almost all $z \in \mathbb{R}_+$ and all $i \in \mathcal{M} \cup \{0\}$ we have have $\gamma_i^*(z) = 0$ if both of the following conditions hold:

- $\nu_i(X > z) = \min_{j \in \mathcal{M} \cup \{0\}} \{\nu_j(X > z)\}$; and,
- there exist $k \in (\mathcal{M} \cup \{0\}) \setminus \{i\}$ such that $\nu_k(X > z) = \nu_i(X > z)$ but $\tau_k(X > z) < \tau_i(X > z)$.

These conditions state that when the policyholder is indifferent between insurers, it chooses the insurer that has the most to gain from the contract. This is interpreted as an act of good faith – it is in the policyholder's best interest to maintain good business relationships with its counterparties. For a fixed set of pricing capacities $(\nu_1, \ldots, \nu_m)$, we denote the set of generously distributed optimal indemnities in the sense of Definition 4.3.4 by $\mathcal{I}(\nu_1, \ldots, \nu_m)$. In the following, we assume that the policyholder always selects an optimal indemnity that distributes generously. We denote the set of such strategies by

$$\daleth := \left\{ \mathfrak{I} \in (\mathcal{C}^m)^{\vec{\mathcal{I}}} : \mathfrak{I}(\nu_1, \ldots, \nu_m) \in \mathcal{I}(\nu_1, \ldots, \nu_m),\ \forall\, (\nu_1, \ldots, \nu_m) \in \mathcal{C}^m \right\}. \tag{4.14}$$

In Section 4.3.2, we address the decision problem for the insurers, while fixing the policyholder's strategy $\mathfrak{I} \in \daleth$. In doing so, we show that generous distribution is sufficient to yield an explicit characterization of an SPNE.

Remark 4.3.5. *The notion of generous distribution outlined in Definition 4.3.4 is, to the best of our knowledge, a new concept. However, we note that a similar assumption has been applied in the Stackelberg setting by Cheung et al. [35]. They assume that the marginal indemnity function $h(z) \equiv 1$ in order to reduce the set of optimal policyholder strategies to a singleton. It can be verified that in the Stackelberg model, their assumption is stronger than ours, in the sense that $h(z) \equiv 1$ implies generous distribution, but that the converse does not hold.*

Remark 4.3.6. *Note that the set $\urcorner$ is in general a strict subset of the optimal solutions given by Proposition 4.3.3. Hence, maximizing coverage and generous distribution are not necessary conditions for the existence of SPNE, since there may exist SPNEs when the policyholder chooses a strategy in the solution set of Proposition 4.3.3 but not in $\urcorner$.*

4.3.2 Backward Induction Step Two – The insurers' Strategies

We now exhibit a characterization of SPNEs by completing the second step of the backward induction in Proposition 4.3.1. We begin by fixing an optimal strategy for the policyholder $\mathfrak{I}^* \in \urcorner$. Then, we characterize capacities $(\nu_1^*, \dots, \nu_m^*)$ such that there does not exist $\hat{\nu}$ that satisfies, for some $i \in \mathcal{M}$,

$$\rho_i(\mathfrak{I}_i^*(\hat{\nu}, \nu_{-i}^*)(X)) - \pi^{\hat{\nu}}(\mathfrak{I}_i^*(\hat{\nu}, \nu_{-i}^*)(X)) < \rho_i(\mathfrak{I}_i^*(\nu_1^*, \dots, \nu_m^*)(X)) - \pi^{\nu_i}(\mathfrak{I}_i^*(\nu_1^*, \dots, \nu_m^*)(X))\,.$$

We show that capacities satisfying certain conditions constitute an SPNE strategy. These conditions are closely related to the *second-lowest true preference* of all $m+1$ agents in the insurance market model. The second-lowest preferences are suggested by Boonen et al. [22] as an upper bound on premium principles that are coalitionally stable. In this chapter, SPNEs can be interpreted as an alternate notion of stability, in which the second-lowest preferences again play an important role.

Definition 4.3.7. Let $\bar{\tau}$ be constructed as the pointwise second-lowest function of the set of capacities $\{\tau_0, \dots, \tau_m\}$. Hence, for all $z \in \mathbb{R}_+$, there exist $i, j \in \mathcal{M} \cup \{0\}$ such that $i \neq j$, $\bar{\tau}(X > z) = \tau_j(X > z)$, $\tau_i(X > z) \leq \tau_j(X > z)$, and $\tau_k(X > z) \geq \tau_j(X > z)$ for all $k \neq i, j$. We refer to $\bar{\tau}$ as the *second-lowest true preference*.

Definition 4.3.8. For each $z \in \mathbb{R}_+$, let $\mathcal{M}_z$ be as defined in (4.12). Define the class of insurer strategies $\urcorner \subseteq \mathcal{C}^m$ as the set of all $(\nu_1, \dots, \nu_m)$ such that for almost all $z \in \mathbb{R}_+$:

(a) We have

$$\min_{j \in \mathcal{M}} \nu_j(X > z) = \bar{\tau}(X > z)\,. \tag{4.15}$$

(b) There exist $i, k \in \mathcal{M} \cup \{0\}$, $i \neq k$ such that

$$\min_{j \in \mathcal{M}} \nu_j(X > z) = \nu_i(X > z) = \nu_k(X > z)\,. \tag{4.16}$$

Recall from (4.9) that we use the convention $\nu_0 = g \circ \mathbb{P}$.

(c) We have

$$\mathcal{T}_z \neq \varnothing \implies \mathcal{T}_z \cap \mathcal{M}_z \neq \varnothing, \tag{4.17}$$

where

$$\mathcal{T}_z := \left\{ i \in \mathcal{M} : \tau_i(X > z) = \min_{j \in \mathcal{M} \cup \{0\}} \tau_j(X > z) \right\}.$$

That is, $\mathcal{T}_z$ are the indices in $\mathcal{M}$ that minimize $\tau_j(X > z)$ for $j \in \mathcal{M} \cup \{0\}$.

The conditions of Definition 4.3.8 can be summarized as follows. The first condition states that the lowest price for the loss layer $X > z$ is equal to the second-lowest true preference $\bar{\tau}$. The second condition guarantees that there are always at least two agents charging this price. Finally, the third condition states that out of the insurers who have the lowest true preference, at least one is posting the price based on $\bar{\tau}$.

We now provide the first main result of this chapter: if $\mathfrak{I}^* \in \beth$ and $(\nu_1^*, \dots, \nu_m^*) \in \gimel$, then the strategy $(\nu_1^*, \dots, \nu_m^*, \mathfrak{I}^*)$ is an SPNE. We begin with the following two results, whose proofs can be found in Section 4.6. Proposition 4.3.9 states that any such strategy $(\nu_1^*, \dots, \nu_m^*, \mathfrak{I}^*)$ achieves the same risk for each insurer. The following result, Proposition 4.3.10, completes the second step of backward induction by showing that every insurer strategy $(\nu_1^*, \dots, \nu_m^*) \in \gimel$ satisfies (4.11).

Proposition 4.3.9. *Let $\mathfrak{I}^* \in \beth$ and $(\nu_1^*, \dots, \nu_m^*) \in \gimel$. Then for all $i \in \mathcal{M}$,*

$$\rho_i(\mathfrak{I}_i^*(\nu_1^*, \dots, \nu_m^*)(X) - \pi_i) = \int_{\mathcal{Z}_i} \tau_i(X > z) - \bar{\tau}(X > z)\, dz\,,$$

where

$$\mathcal{Z}_i := \{ z \in \mathbb{R}_+ : \tau_i(X > z) < \bar{\tau}(X > z) \}\,.$$

Proposition 4.3.10. *Let $\hat{\nu} \in \mathcal{C}$ be any pricing capacity, and let $\mathfrak{I}^* \in \beth$ and $(\nu_1^*, \dots, \nu_m^*) \in \gimel$. Then for each $i \in \mathcal{M}$, we have*

$$\rho_i(\mathfrak{I}_i^*(\hat{\nu}, \nu_{-i}^*)(X)) - \pi^{\hat{\nu}}(\mathfrak{I}_i^*(\hat{\nu}, \nu_{-i}^*)(X)) \geq \rho_i(\mathfrak{I}_i^*(\nu_1^*, \dots, \nu_m^*)(X)) - \pi^{\nu_i^*}(\mathfrak{I}_i^*(\nu_1^*, \dots, \nu_m^*)(X))\,.$$

That is, when the strategies $(\nu_1^*, \dots, \nu_m^*)$ are chosen by the insurers, no insurer has an incentive to deviate.

As a direct corollary of Propositions 4.3.1, 4.3.3, and 4.3.10, we obtain the following.

Theorem 4.3.11. *Let* $\mathfrak{I}^* \in \daleth$ *and* $(\nu_1^*, \dots, \nu_m^*) \in \beth$. *Then the strategy* $(\nu_1^*, \dots, \nu_m^*, \mathfrak{I}^*)$ *is an SPNE.*

We provide a few examples of such strategies in the following subsection. Note that while Theorem 4.3.11 provides sufficient conditions for a strategy to be an SPNE, these conditions are not necessary. An example of an SPNE *not* characterized by Theorem 4.3.11 is provided in Section 4.7.1.

Some Examples of SPNEs Characterized by Theorem 4.3.11

We now provide explicit constructions of insurer strategies $(\nu_1^*, \dots, \nu_m^*) \in \beth$. This results in an SPNE when combined with any policyholder strategy $\mathfrak{I}^* \in \daleth$, as a consequence of Theorem 4.3.11. First, recall that

$$\mathcal{T}_z := \left\{ i \in \mathcal{M} : \tau_i(X > z) = \min_{j \in \mathcal{M} \cup \{0\}} \tau_j(X > z) \right\}.$$

Definition 4.3.12. For each $i \in \mathcal{M}$, define ν_i^* by

$$\nu_i^*(X > z) := \begin{cases} \bar{\tau}(X > z), & i \in \mathcal{T}_z \\ \tau_i(X > z) & \text{otherwise} \end{cases}.$$

In the strategy, the insurer with the lowest true preference for every layer $X > z$ chooses the second-lowest true preference as their premium principle. Otherwise, the insurer quotes a price consistent with its underlying risk measure. The following result shows that this strategy is an SPNE, and its proof is given in Section 4.6.

Proposition 4.3.13. *The capacities* $(\nu_1^*, \dots, \nu_m^*)$ *given in Definition 4.3.12 are in* $\beth$.

Therefore by Theorem 4.3.11, for any $\mathfrak{I}^* \in \daleth$, the strategy $(\nu_1^*, \dots, \nu_m^*, \mathfrak{I}^*)$ is an SPNE. By increasing prices to match the true preference of their nearest competitors, insurers are able to profit if they are able to correctly identify the heterogeneous beliefs of the agents in the market. Note that in this case, each insurer quotes prices that are at least as high as their true preference. However, this is not necessary for all SPNEs, as shown by the following example.

Remark 4.3.14. *It follows easily from the definition of* $\beth$ *that* $(\bar{\tau}, \bar{\tau}, \dots, \bar{\tau}) \in \beth$.

Again by Theorem 4.3.11, for any $\mathfrak{I}^* \in \beth$, $(\bar{\tau}, \bar{\tau}, \ldots, \bar{\tau}, \mathfrak{I}^*)$ is an SPNE. This shows that there exists a pricing mechanism that every insurer can adopt for the market to achieve stability. That is, there is no incentive for any insurer to deviate from the pricing capacity $\bar{\tau}$, if every other insurer is also using the same premium principle.

However, this requires some insurers to quote prices that are below their true risk preference. The insurers do not experience a loss, since even when their price is too low, the policyholder cedes this portion of the risk to another insurer. This perhaps places too much hope in the policyholder's strategy, which is assumed to distribute generously. Nevertheless, the special form of this SPNE is worth mentioning, and further emphasizes the important role of the "second-lowest" pricing capacity $\bar{\tau}$.

This example is also an illustration of the wide range of pricing capacities that can yield an SPNE. On the levels $X > z$ that the insurer does not expect to receive any business, pricing can be fairly arbitrary. In this sense, $\beth$ is a non-trivial class that allows for many different selections of pricing capacities.

4.4 Welfare Analysis of SPNEs

It is clear that every strategy $(\nu_1, \ldots, \nu_m, \mathfrak{I})$ in this economic game determines a unique premium principle for each insurer and a unique indemnity structure demanded by the policyholder. This therefore determines the premia that the policyholder pays to each insurer, which determines the allocation of wealth (or risk) to each agent. In this section, we examine efficiency properties of the contracts induced by SPNEs. We consider both individual rationality (IR) and Pareto optimality of allocations.

Definition 4.4.1. An *allocation* is a pair $(\vec{I}, \vec{\pi}) \in \vec{\mathcal{I}} \times \mathbb{R}^n$, where $\vec{I}$ denotes the structure of the indemnities and π denotes the vector of premia paid by the policyholder. The resulting risk measures under this allocation are

$$\rho^{PH}\left(X - \sum_{i \in \mathcal{M}} I_i(X) + \pi_i\right) \quad \text{and} \quad \rho_i(I_i(X) - \pi_i),\ \forall i \in \mathcal{M}\,.$$

We say that an allocation $(\vec{I}, \vec{\pi})$ results from (or is induced by) a strategy $(\nu_1, \ldots, \nu_m, \mathfrak{I})$ if for all $i \in \mathcal{M}$, $I_i = \mathfrak{I}_i(\nu_1, \ldots, \nu_m)$ and $\pi_i = \pi_i^{\nu_i}(I_i(X)) = \displaystyle\int I_i(X)\, d\nu_i$.

4.4.1 Individual Rationality of SPNEs

The following restates the definition of individual rationality in the present setting.

Definition 4.4.2. An allocation $(\vec{I}, \vec{\pi}) \in \vec{\mathcal{I}} \times \mathbb{R}^n$ is IR if

$$\rho^{PH}\left(X - \sum_{i\in\mathcal{M}} I_i(X) + \sum_{i\in\mathcal{M}} \pi_i\right) \leq \rho^{PH}(X) \quad \text{and} \quad \rho_i(I_i(X) - \pi_i) \leq \rho_i(0) = 0, \ \forall i \in \mathcal{M}\,.$$

As before, we denote the set of all individually rational allocations by $\mathcal{IR}$.

That is, the allocation $(\vec{I}, \vec{\pi})$ is not worse than the status quo for any agent – hence, the policyholder and all insurers are willing to participate with these transactions. Note that if an allocation is IR, then each premium is non-negative, since by translation invariance the condition $\rho_i(I_i(X) - \pi_i) \leq \rho_i(0) = 0$ yields

$$\pi_i \geq \rho_i(I_i(X))\,,$$

and by monotonicity $\rho_i(I_i(X)) \geq 0$ since $I_i(X) \geq 0$ and τ_i is non-negative. We show that for every $\mathfrak{I}^* \in \daleth$ and $(\nu_1^*, \ldots, \nu_m^*) \in \beth$, the strategy $(\nu_1^*, \ldots, \nu_m^*, \mathfrak{I}^*)$ induces an IR allocation. The result follows readily from Proposition 4.3.9 and the following lemma.

Lemma 4.4.3. *Suppose $\mathfrak{I}^* \in \daleth$ and $(\nu_1^*, \ldots, \nu_m^*) \in \beth$. Then*

$$\begin{aligned}&\rho^{PH}\left(X - \sum_{i\in\mathcal{M}} \mathfrak{I}_i^*(\nu_1^*, \ldots, \nu_m^*)(X) + \sum_{i\in\mathcal{M}} \pi^{\nu_i^*}(\mathfrak{I}_i^*(\nu_1^*, \ldots, \nu_m^*)(X))\right)\\ &\quad = \int_{\mathcal{Z}_{PH}} g(\mathbb{P}(X > z))\, dz + \int_{\mathcal{Z}_{PH}^C} \bar{\tau}(X > z)\, dz\,,\end{aligned}$$

where

$$\mathcal{Z}_{PH} := \{z \in \mathbb{R}_+ : g(\mathbb{P}(X > z)) < \bar{\tau}(X > z)\}\,.$$

Theorem 4.4.4. *Suppose $\mathfrak{I}^* \in \daleth$ and $(\nu_1^*, \ldots, \nu_m^*) \in \beth$. Then the strategy $(\nu_1^*, \ldots, \nu_m^*, \mathfrak{I}^*)$ induces an IR allocation.*

Proof. By Lemma 4.4.3,

$$\begin{aligned}
&\rho^{PH}\left(X - \sum_{i\in\mathcal{M}} \mathfrak{I}_i^*(\nu_1^*,\dots,\nu_m^*)(X) + \sum_{i\in\mathcal{M}} \pi^{\nu_i}(\mathfrak{I}_i^*(\nu_1^*,\dots,\nu_m^*)(X))\right)\\
&= \int_{\mathcal{Z}_{PH}} g(\mathbb{P}(X>z))\,dz + \int_{\mathcal{Z}_{PH}^C} \bar{\tau}(X>z)\,dz\\
&\le \int_{\mathcal{Z}_{PH}} g(\mathbb{P}(X>z))\,dz + \int_{\mathcal{Z}_{PH}^C} g(\mathbb{P}(X>z))\,dz\\
&= \int_0^\infty g(\mathbb{P}(X>z))\,dz = \rho^{PH}(X)\,.
\end{aligned}$$

Also, by Proposition 4.3.9, we have for each $i \in \mathcal{M}$,

$$\rho_i(I_i^*(X) - \pi_i) = \int_{\mathcal{Z}_i} \tau_i(X>z) - \bar{\tau}(X>z)\,dz \le \int_{\mathcal{Z}_i} 0\,dz = 0\,.$$

□

Market Competition and the Policyholder's Welfare

Within the literature on Stackelberg equilibria (i.e., $m = 1$), solutions to the Stackelberg problem imply that the policyholder has no incentive to purchase insurance – that is, the policyholder is indifferent between insurance and the status quo. This argument is emphasized in Boonen and Ghossoub [20] as well as by Chapter 3 of this thesis, which conclude that the Stackelberg setting is problematic in problems of optimal insurance. It is also straightforward to confirm that the optimal indemnification proposed in Cheung et al. [35] does not provide a welfare gain for the policyholder. The following result shows that in our setting, the same conclusion applies: if there is only one insurer, then the policyholder does not improve upon the status quo.

Proposition 4.4.5. *Let $m = 1$, and suppose $\mathfrak{I}^* \in \beth$ and $\nu_1^* \in \daleth$. Then*

$$\rho^{PH}\left(X - I_1(X) + \pi^{\nu_1^*}(I_1(X))\right) = \rho^{PH}(X)\,.$$

However, as the number of insurers increases, the resulting risk measure for the policyholder decreases, with the decrease being strict under some mild conditions. This result is intuitive from an economic perspective: increased competition in the market lowers prices

and benefits the consumer. Hence, by extending the Stackelberg model to include multiple insurers, we address an apparent weakness of the Stackelberg setting by capturing value for the policyholder.

Fix a value of m, and suppose that the $(m+1)$-st insurer has preferences given by the capacity $g_{m+1} \circ \mathbb{Q}_{m+1} := \tau_{m+1}$. We wish to compare the SPNEs in the market with m insurers to that with $m+1$ insurers. Let $\daleth_m$ and $\daleth_{m+1}$ denote the sets of generously distributing optimal strategies for the market with m and $m+1$ insurers, respectively. Similarly, let $\gimel_m$ and $\gimel_{m+1}$ be the sets of insurer strategies satisfying the properties of Definition 4.3.8 for the market with m and $m+1$ insurers, respectively.

Proposition 4.4.6. *Let*

$$\begin{aligned} \mathfrak{I}^{*,m} &\in \daleth_m \,, \\ \mathfrak{I}^{*,m+1} &\in \daleth_{m+1} \,, \\ (\nu_1^{*,m}, \dots, \nu_m^{*,m}) &\in \gimel_m \,, \\ (\nu_1^{*,m+1}, \dots, \nu_{m+1}^{*,m+1}) &\in \gimel_{m+1} \,. \end{aligned}$$

We use the convention that $(I_1^{*,m}, \dots, I_m^{*,m}) := \mathfrak{I}^{*,m}(\nu_1^{*,m}, \dots, \nu_m^{*,m})$*, and similarly for* $m+1$*. Then*

$$\begin{aligned} &\rho^{PH}\left(X - \sum_{i=1}^{m+1} I_i^{*,m+1}(X) + \sum_{i=1}^{m+1} \pi^{\nu_i^{*,m+1}}(I_i^{*,m+1}(X))\right) \\ &\quad \le \rho^{PH}\left(X - \sum_{i=1}^{m} I_i^{*,m}(X) + \sum_{i=1}^{m} \pi^{\nu_i^{*,m}}(I_i^{*,m}(X))\right) . \end{aligned}$$

That is, when an insurer is added to the market, the welfare of the policyholder can only increase.

We can see from the proof of Proposition 4.4.6 that the above inequality can be strict when, for example, $\bar{\tau}_{m+1} < \bar{\tau}_m$. Hence, when an insurer joins the market by offering a cheaper or differentiated product, the policyholder can realize a strict welfare gain This point is explicitly demonstrated by a numerical example in Subsection 4.4.4.

4.4.2 Pareto Efficiency of SPNEs

We now show that allocations that result from the SPNE characterized in Section 4.3 are PO. In this section, we use the conventions $\rho_0 = \rho^{PH}$ and $I_0(X) = X - \sum_{i \in \mathcal{M}} I_i(X)$ as

defined in (4.6) and (4.7). As a direct consequence, $X = \sum_{i=0}^{n} I_i(X)$. Recall that by Assumption 4.2.1, $I_0 \in \mathcal{I}$.

The definition of Pareto optimality in this setting is given below.

Definition 4.4.7. An allocation $(\vec{I}, \vec{\pi}) \in \vec{\mathcal{I}} \times \mathbb{R}^n$ is PO if it is individually rational and there does not exist another allocation $(\tilde{I}, \tilde{\pi}) \in \mathcal{IR}$ such that

$$\rho^{PH}\left(X - \sum_{i\in\mathcal{M}} \tilde{I}_i(X) + \sum_{i\in\mathcal{M}} \tilde{\pi}_i\right) \leq \rho^{PH}\left(X - \sum_{i\in\mathcal{M}} I_i(X) + \sum_{i\in\mathcal{M}} \pi_i\right),$$
$$\rho_i(\tilde{I}_i(X) - \tilde{\pi}_i) \leq \rho_i(I_i(X) - \pi_i), \quad i \in \mathcal{M},$$

with at least one of these inequalities being strict.

It is well-known that when all risk measures are translation invariant, PO is related to minimizing the aggregate risk (see, e.g., Asimit and Boonen [7]), as shown by the following result. Note that this result is analogous to that of Remark 3.2.5, and its proof is similar.

Proposition 4.4.8. *An allocation $(\vec{I}, \vec{\pi})$ is PO if and only if*

1. *The indemnity structure $\vec{I}$ solves*

$$\inf_{\vec{I}\in\vec{\mathcal{I}}} \left\{ \sum_{i=0}^{n} \rho_i(I_i(X)) \right\}. \tag{4.18}$$

2. *$\vec{\pi}$ is chosen so that $(\vec{I}, \vec{\pi}) \in \mathcal{IR}$.*

In our case, we can obtain an explicit characterization of the indemnity structures that attain the minimum aggregate risk.

Proposition 4.4.9. *For each $z \in \mathbb{R}_+$, let*

$$\mathcal{L}_z := \{i \in \mathcal{M} \cup \{0\} : \tau_i(X > z) = \min_{j\in\mathcal{M}\cup\{0\}} \{\tau_j(X > z)\}\},$$

and $\mathcal{L}_z^C := (\mathcal{M} \cup \{0\}) \setminus \mathcal{L}_z$. For each $i \in \mathcal{M} \cup \{0\}$, let γ_i take values in $[0, 1]$ such that for almost every z,

$$\sum_{i\in\mathcal{L}_z} \gamma_i(z) = 1, \sum_{i\in\mathcal{L}_z^C} \gamma_i(z) = 0,$$

and define $I_i(z) := \int_0^z \gamma_i(z)\, dz$. Then $\vec{I}$ is a solution to (4.18). Furthermore, any indemnity structure that solves (4.18) is of this form.

We now provide the main result of this section: the SPNEs from Theorem 4.3.11 induce PO allocations. First, we show that for any $\mathfrak{I}^* \in \daleth$, the insurer strategy $(\bar{\tau}, \dots, \bar{\tau})$ given in Remark 4.3.14 induces a PO allocation. We then extend this result to all insurer strategies in $\gimel$ by applying Proposition 4.3.9 and Lemma 4.4.3.

Lemma 4.4.10. *The strategy $(\bar{\tau}, \bar{\tau}, \dots, \bar{\tau}, \mathfrak{I}^*)$ from Remark 4.3.14 induces an IR and PO allocation. That is, $(\vec{I}^*, \vec{\pi}^*)$ is IR and PO in the sense of Definitions 4.4.2 and 4.4.7, where $\vec{I}^*$ is an element of $\mathcal{I}(\bar{\tau}, \bar{\tau}, \dots, \bar{\tau})$ and $\pi_i^* = \rho^{\bar{\tau}}(I_i^*(X))$.*

Theorem 4.4.11. *Let $\mathfrak{I}^* \in \daleth$ and $(\nu_1^*, \dots, \nu_m^*) \in \gimel$. Then the strategy $(\nu_1^*, \dots, \nu_m^*, \mathfrak{I}^*)$ induces an IR and PO allocation.*

As a direct corollary of Theorem 4.4.11, we obtain the following.

Corollary 4.4.12. *The strategy $(\nu_1^*, \dots, \nu_m^*, \mathfrak{I}^*)$ with $\mathfrak{I}^* \in \daleth$ and $\nu_1^*, \dots, \nu_m^*$ as in Definition 4.3.12 induces an IR and PO allocation.*

4.4.3 Decentralization of Pareto-Efficient Allocations via SPNE

We now provide a partial converse to Theorem 4.4.11. That is, given a PO allocation $(\vec{I}^*, \vec{\pi}^*)$, we show that this can be generated by an SPNE as characterized in Theorem 4.3.11.

However, by Proposition 4.3.9 and Lemma 4.4.3, we see that every allocation induced by such an SPNE has the same resulting risk measure for each agent. Therefore, we must first impose the following assumption on the allocations.

Assumption 4.4.13. The allocation $(\vec{I}^*, \vec{\pi}^*)$ satisfies

$$\rho_i(I_i^*(X) - \pi_i^*) = \int_{\mathcal{Z}_i} \tau_i(X > z) - \bar{\tau}(X > z)\, dz\,,$$
$$\rho^{PH}\left(X - \sum_{i\in\mathcal{M}} I_i^*(X) + \sum_{i\in\mathcal{M}} \pi_i^*\right) = \int_{\mathcal{Z}_{PH}} g(\mathbb{P}(X > z))\, dz + \int_{\mathcal{Z}_{PH}^C} \bar{\tau}(X > z)\, dz\,.$$

We show that if $(\vec{I}^*, \vec{\pi}^*)$ is an allocation satisfying Assumption 4.4.13, then it is induced by the choice of insurer strategies $(\nu_1^*, \dots, \nu_m^*)$ given in Definition 4.3.12. This result follows from the following propositions.

Proposition 4.4.14. *Let $(\vec{I}^*, \vec{\pi}^*)$ be an allocation satisfying Assumption 4.4.13, and the pricing capacities $(\nu_1^*, \dots, \nu_m^*)$ be as in Definition 4.3.12. Then for each $i \in \mathcal{M}$,*

$$\pi_i^* = \int I_i^*(X)\, d\nu_i^* \,.$$

Proposition 4.4.15. *Let $(\vec{I}^*, \vec{\pi}^*)$ be an allocation satisfying Assumption 4.4.13, and the pricing capacities $(\nu_1^*, \dots, \nu_m^*)$ be the policyholder strategies from Definition 4.3.12. Then the indemnity profile $\vec{I}^*$ is optimal in the sense of Proposition 4.3.3. That is, the profile $\vec{I}^*$ solves*

$$\min_{I \in \vec{\mathcal{I}}} \left\{ \rho^{PH}\left(X - \sum_{i \in \mathcal{M}} I_i(X) + \sum_{i \in \mathcal{M}} \pi^{\nu_i^*}(I_i(X)) \right) \right\} .$$

Proposition 4.4.16. *Let $(\vec{I}^*, \vec{\pi}^*)$ be an allocation satisfying Assumption 4.4.13. Then for any choice of insurer strategies $(\nu_1^*, \dots, \nu_m^*) \in \beth$, the indemnities $\vec{I}^*$ distribute generously the sense of Definition 4.3.4.*

Theorem 4.4.17. *Suppose $(\vec{I}^*, \vec{\pi}^*)$ is an allocation satisfying Assumption 4.4.13. Then this allocation results from an SPNE with capacities $(\nu_1^*, \dots, \nu_m^*)$ as given in Definition 4.3.12. That is, there exists an $\mathfrak{I}^* \in \daleth$ such that $\vec{I}^* = \mathfrak{I}^*(\nu_1^*, \dots, \nu_m^*)$, and for all $i \in \mathcal{M}$,*

$$\pi_i^* = \int I_i^*(X)\, d\nu_i^* \,.$$

Proof. By Propositions 4.4.15 and 4.4.16, we see that $\vec{I}^*$ is optimal and distributes generously. Hence, there exists a strategy $\mathfrak{I}^* \in \daleth$ such that $\vec{I}^* = \mathfrak{I}^*(\nu_1^*, \dots, \nu_m^*)$. The rest follows directly from Proposition 4.4.14. □

4.4.4 A Numerical Example

We now illustrate our main results with a numerical example, where the policyholder is subject to a risk X distributed according to a censored exponential distribution. By applying Theorem 4.3.11, we explicitly characterize an SPNE in this market. We also show that in this SPNE, each agent realizes a strict welfare gain, which implies that each agent has an incentive to participate in this contract. This illustrates the result of Proposition 4.4.6, by showing that introducing competition on the supply side benefits the policyholder. Furthermore, by Theorems 4.4.4 and 4.4.11, the allocation resulting from the SPNE is IR and PO.

We assume that the policyholder's beliefs of the risk X are represented by the survival function

$$\mathbb{P}(X > z) = \begin{cases} \exp(-\beta_0 \cdot z)\,, & 0 \le z \le 5\,, \\ 0\,, & z \ge 5\,. \end{cases}$$

Here, we take $\beta_0 = 2.5$. We assume that there are two insurers in the market: i.e. $m = 2$. For $i = 1, 2$, let

$$\mathbb{Q}_i(X > z) = \begin{cases} \exp(-\beta_i \cdot z)\,, & 0 \le z \le 5\,, \\ 0\,, & z \ge 5\,, \end{cases}$$

with $\beta_1 = 2$ and $\beta_2 = 1.7$. These different parameters are due to the heterogeneity in beliefs among agents: while the insurers agree with the policyholder that X is distributed according to a censored exponential distribution and that this distribution is censored at 5, they differ in their belief of the parameter of the distribution.

We also assume that each agent measures risk according to the Expected Shortfall. That is, we have

$$g(t) = \min\left\{\frac{t}{\alpha_0}, 1\right\}, \qquad g_i(t) = \min\left\{\frac{t}{\alpha_i}, 1\right\}, i = 1, 2,$$

where $\alpha_0 = 1\%, \alpha_1 = 5\%$ and $\alpha_2 = 10\%$.

By the results of Section 4.3, we are primarily interested in the distorted subjective survival probabilities $\tau_i(X > z)$. These probabilities for τ_i (where $i = 0, 1, 2$) are displayed in Figure 4.1. The second-lowest capacity $\bar{\tau}$, as given in Definition 4.3.7, is displayed in Figure 4.2.

Let ν_1^*, ν_2^* be as in Definition 4.3.12. Then, the indemnity structure for an optimal policyholder strategy $\mathfrak{I}^* \in \daleth$ is shown in Figure 4.3. It can be seen that the indemnity function for insurer i is strictly increasing only if $\tau_i(X > z) = \min_{j=0,1,2} \tau_j(X > z)$.

Based on the insurers' strategies ν_1^{**} and ν_2^*, numerical calculation yields the following premia:

$$\pi_1^*(I_1^*(X)) = \pi^{\nu_1^*}(I_1^*(X)) = 1.4433 \quad \text{and} \quad \pi_2^*(I_2^*(X)) = 0.5450\,.$$

The risk measures for the agents resulting from these strategies are

$$\rho^{PH}\left(X - \sum_{i=1}^{2} I_i^*(X) + \sum_{i=1}^{2} \pi_i^*\right) = 2.001\,,$$
$$\rho_1(I_1^*(X) - \pi_1^*) = -0.0065\,,$$
$$\rho_2(I_2^*(X) - \pi_2^*) = -0.0725\,.$$

From the calculations, it is evident that $\rho^{PH}(X) \approx 2.2419$, indicating that the policyholder realizes a strict welfare gain from this allocation. Additionally, the initial risk of both insurers is 0, whereas the risk of both insurers are negative under the SPNE contracts. Therefore, both insurers also decrease their risk as a result of this allocation.

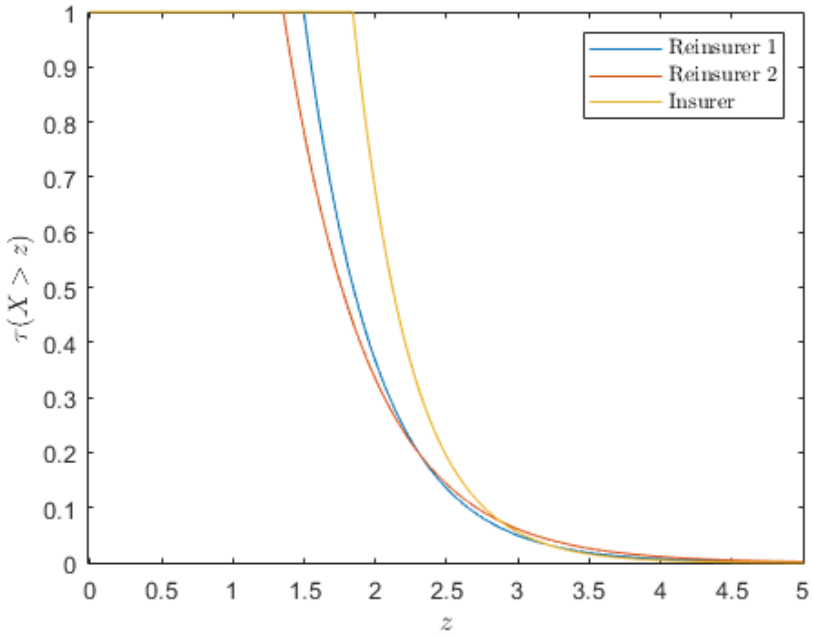


Figure 4.1: Distorted survival probabilities τ.

4.5 Conclusion

In this chapter, we provide a novel insurance market mechanism with multiple insurers, in which the insurers have the first-mover advantage. We assume distortion risk measure preferences for each agent and allow for heterogeneity in beliefs. Premium principles are nonlinear and are taken as Choquet expectations with respect to general capacities. Within this general setup, we argue that the notion of a Subgame Perfect Nash Equilibrium (SPNE) is the appropriate solution concept. Our main results characterize SPNEs, identify their properties, and provide a welfare analysis of resulting equilibrium allocations.

We first provide sufficient conditions which lead to an SPNE in Theorem 4.3.11. By applying backward induction, we isolate the decision problem of the policyholder and the

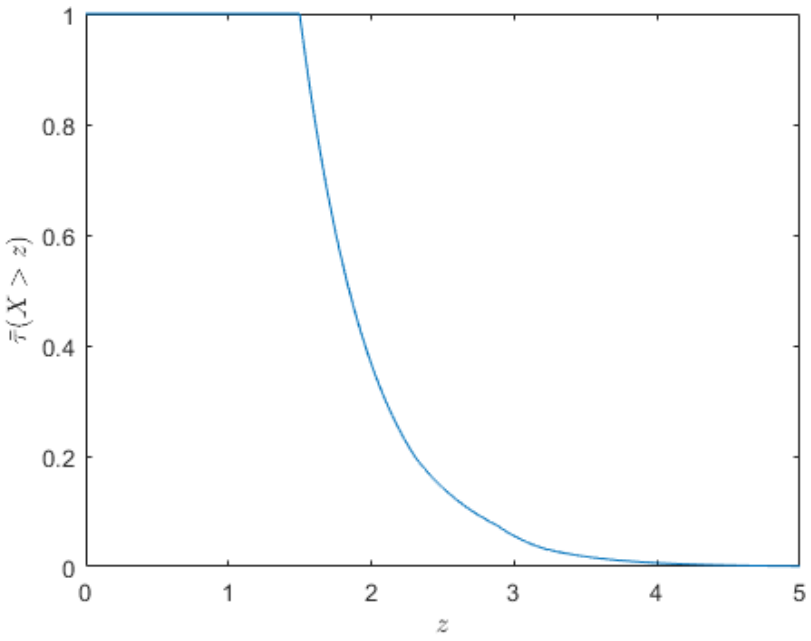


Figure 4.2: Second-lowest true preferences $\bar{\tau}$.

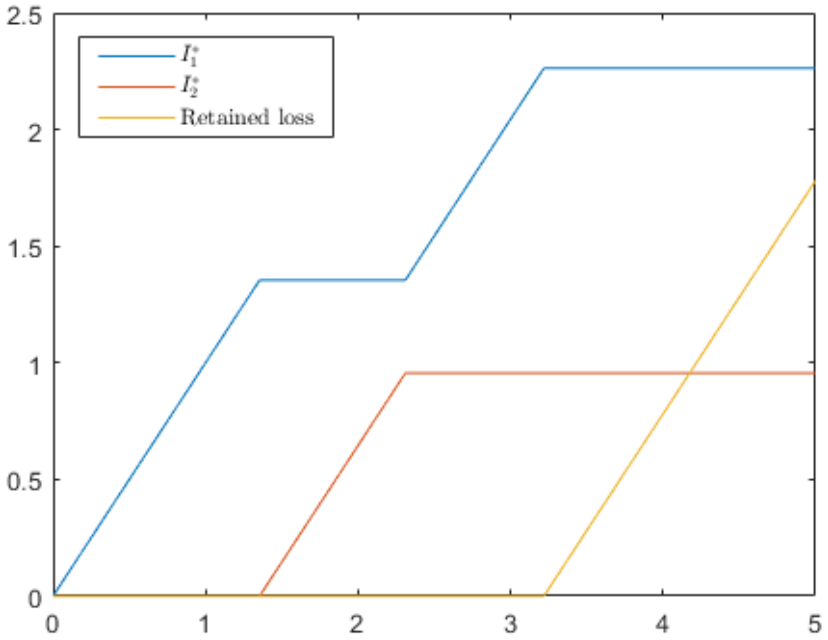


Figure 4.3: Optimal indemnity structure $\vec{I}^*$.

decision problem of the insurers. The former is addressed by applying the so-called marginal indemnification function approach to the case of heterogeneous beliefs, as done in Boonen and Ghossoub [19]. The latter is addressed by constructing strategies satisfying the conditions of Definition 4.3.8. In these equilibria, the policyholder faces prices induced by the second-lowest true preferences as measured by distorted subjective survival probabilities.

Additionally, we examine the Pareto efficiency properties of contracts resulting from SPNEs in Section 4.4. Since market equilibria are not efficient in general, we separately analyze the welfare of each agent. In Theorem 4.4.11, we demonstrate that such equilibria result in Pareto-efficient contracts in our market model. Conversely, we show in Theorem 4.4.17 that certain efficient allocations can be decentralized: that is, they can be induced by market forces. Since we identify market equilibria before considering efficiency, we do not need to assume that the agents in the market cooperate with each other, or that there exists a central planner influencing decisions.

Finally, our setting and results could be extended in several potential directions. For example, we consider a market with only one (representative) policyholder, whereas typically multiple policyholders, with heterogeneous risk preferences and risk exposures, participate in insurance markets. Moreover, insurers often operate with a background risk arising from other operational and financial decisions (e.g., Dana and Scarsini [39], Balbás et al. [9]). Another interesting extension would be to examine the effect of information asymmetry between the policyholder and the insurer. For instance, in markets with multiple policyholders with hidden types, an insurer can offer a menu of different premia to the policyholders. We refer to Liang et al. [78] for a recent discussion of this topic.

4.6 Proofs of Main Results

4.6.1 Proof of Proposition 4.3.1

Suppose first that the strategy $(\nu_1^*, \dots, \nu_m^*, \mathfrak{I}^*)$ was found through backward induction. Then it satisfies condition (2) of Definition 4.2.3. It remains to check that this is an NE. For the sake of contradiction, suppose $(\nu_1^*, \dots, \nu_m^*, \mathfrak{I}^*)$ is not an NE. By (4.10), there does not exist a strategy $\tilde{\mathfrak{I}}$ such that

$$\begin{aligned}\rho^{PH}(X) - \sum_{i\in\mathcal{M}} \rho^{PH}\left(\tilde{\mathfrak{I}}_i(\nu_1^*, \dots, \nu_m^*)(X)\right) + \sum_{i\in\mathcal{M}} \pi^{\nu_i^*}(\tilde{\mathfrak{I}}_i(\nu_1^*, \dots, \nu_m^*)(X)) \\ < \rho^{PH}(X) - \sum_{i\in\mathcal{M}} \rho^{PH}(\mathfrak{I}_i^*(\nu_1^*, \dots, \nu_m^*)(X)) + \sum_{i\in\mathcal{M}} \pi^{\nu_i}(\mathfrak{I}_i^*(\nu_1^*, \dots, \nu_m^*)(X))\,.\end{aligned}$$

Furthermore, by (4.11), there does not exist $i \in \mathcal{M}$ and $\tilde{\nu} \in \mathcal{C}$ such that

$$\rho_i(\mathfrak{I}_i^*(\tilde{\nu}, \nu_{-i}^*)(X)) - \pi^{\tilde{\nu}}(\mathfrak{I}_i^*(\tilde{\nu}, \nu_{-i}^*)(X)) < \rho_i(\mathfrak{I}_i^*(\nu_1^*, \ldots, \nu_m^*)(X)) - \pi^{\nu_i^*}(\mathfrak{I}_i^*(\nu_1^*, \ldots, \nu_m^*)(X)) \,.$$

Therefore, $(\nu_1^*, \ldots, \nu_m^*, \mathfrak{I}^*)$ is an NE.

Conversely, suppose that $(\nu_1^*, \ldots, \nu_m^*, \mathfrak{I}^*)$ is an SPNE. Then by condition (2) of Definition 4.2.3, for any choice of capacities $(\nu_1, \ldots, \nu_m) \in \mathcal{C}^m$, the indemnity structure $\mathfrak{I}^*(\nu_1, \ldots, \nu_m)$ solves (4.10). It remains to show that $(\nu_1^*, \ldots, \nu_m^*)$ is a Nash Equilibrium in the reduced game. Since the strategy $(\nu_1^*, \ldots, \nu_m^*, \mathfrak{I}^*)$ is an NE, there does not exist $i \in \mathcal{M}$ and $\tilde{\nu} \in \mathcal{C}$ such that

$$\rho_i(\mathfrak{I}_i^*(\tilde{\nu}, \nu_{-i}^*)(X)) - \pi^{\tilde{\nu}}(\mathfrak{I}_i^*(\tilde{\nu}, \nu_{-i}^*)(X)) < \rho_i(\mathfrak{I}_i^*(\nu_1^*, \ldots, \nu_m^*)(X)) - \pi^{\nu_i}(\mathfrak{I}_i^*(\nu_1^*, \ldots, \nu_m^*)(X)) \,.$$

Hence, $(\nu_1^*, \ldots, \nu_m^*)$ is an NE for the reduced game formed by fixing $\mathfrak{I}^*$, which completes the proof. □

4.6.2 Proof of Proposition 4.3.3

First we check feasibility. Suppose $(I_1^*, \ldots, I_m^*)$ is of the given form – then $I_i^* = \int_0^x \gamma_i^*(z)\, dz$ where $0 \le \gamma_i^* \le 1$. Then for all z, $(I_i^*)'(z) = \gamma_i^*(z) \ge 0$. Furthermore, $\sum_{i \in \mathcal{M}} (I_i^*)'(z) = \sum_{i \in \mathcal{M}} \gamma_i^*(z) \le 1$. Hence, $(I_1^*, \ldots, I_m^*)$ is a feasible choice of indemnities.

Let $\vec{I} = (I_1, \ldots, I_m)$ be any feasible indemnity, and write $I_i(x) = \displaystyle\int_0^x \gamma_i(z)\, dz$. Then,

$$\begin{aligned}
&\rho^{PH}\left(X - \sum_{i \in \mathcal{M}} I_i(X) + \sum_{i \in \mathcal{M}} \pi^{\nu_i}(I_i(X))\right) \\
&\quad = \rho^{PH}\left(X - \sum_{i \in \mathcal{M}} I_i(X)\right) + \sum_{i \in \mathcal{M}} \pi^{\nu_i}(I_i(X)) \\
&\quad = \rho^{PH}(X) - \rho^{PH}\left(\sum_{i \in \mathcal{M}} I_i(X)\right) + \sum_{i \in \mathcal{M}} \pi^{\nu_i}(I_i(X)) \\
&\quad = \rho^{PH}(X) - \sum_{i \in \mathcal{M}} \rho^{PH}(I_i(X)) + \sum_{i \in \mathcal{M}} \pi^{\nu_i}(I_i(X)) \\
&\quad = \rho^{PH}(X) + \sum_{i \in \mathcal{M}} \int_0^\infty (\nu_i(X > z) - g(\mathbb{P}(X > z)))\gamma_i(z)\, dz \\
&\quad = \rho^{PH}(X) + \int_0^\infty \sum_{i \in \mathcal{M}} (\nu_i(X > z) - g(\mathbb{P}(X > z)))\gamma_i(z)\, dz \,.
\end{aligned}$$

where the second equality follows from comonotonic additivity of ρ^{PH}. First we show that $(I_1^*, \ldots, I_m^*)$ solves Problem (4.10). We have

$$\begin{aligned}
&\rho^{PH}\left(X - \sum_{i\in\mathcal{M}} I_i^*(X) + \sum_{i\in\mathcal{M}} \pi^{\nu_i}(I_i^*(X))\right) \\
&\quad = \rho^{PH}(X) + \int_0^\infty \sum_{i\in\mathcal{M}} (\nu_i(X > z) - g(\mathbb{P}(X > z)))\gamma_i^*(z)\, dz \\
&\quad = \rho^{PH}(X) + \int_0^\infty \sum_{i\in\mathcal{M}_z} (\nu_i(X > z) - g(\mathbb{P}(X > z)))\gamma_i^*(z)\, dz \\
&\quad = \rho^{PH}(X) + \int_0^\infty \sum_{i\in\mathcal{M}_z} (\underline{\nu}(X > z) - g(\mathbb{P}(X > z)))\gamma_i^*(z)\, dz \\
&\quad = \rho^{PH}(X) + \int_0^\infty (\underline{\nu}(X > z) - g(\mathbb{P}(X > z))) \sum_{i\in\mathcal{M}_z} h_i(z)\, dz \\
&\quad = \rho^{PH}(X) + \int_{\mathcal{A}} (\underline{\nu}(X > z) - g(\mathbb{P}(X > z)))\, dz\,,
\end{aligned}$$

where we define

$$\mathcal{A} := \{z \in \mathbb{R}_+ : g(\mathbb{P}(X > z)) > \underline{\nu}(X > z)\}\,.$$

Then we have

$$\begin{aligned}
&\rho^{PH}(X) + \int_{\mathcal{A}} (\underline{\nu}(X > z) - g(\mathbb{P}(X > z)))\, dz \\
&\qquad \le \rho^{PH}(X) + \int_{\mathcal{A}} (\underline{\nu}(X > z) - g(\mathbb{P}(X > z))) \sum_{i\in\mathcal{M}} \gamma_i(z)\, dz \qquad (4.19) \\
&\qquad \le \rho^{PH}(X) + \int_{\mathcal{A}} (\underline{\nu}(X > z) - g(\mathbb{P}(X > z))) \sum_{i\in\mathcal{M}} \gamma_i(z)\, dz \qquad (4.20) \\
&\qquad\qquad + \int_{\mathcal{A}^c} (\underline{\nu}(X > z) - g(\mathbb{P}(X > z))) \sum_{i\in\mathcal{M}} \gamma_i(z)\, dz \\
&\qquad = \rho^{PH}(X) + \int_0^1 (\underline{\nu}(X > z) - g(\mathbb{P}(X > z))) \sum_{i\in\mathcal{M}} \gamma_i(z)\, dz \\
&\qquad \le \rho^{PH}(X) + \int_0^1 \sum_{i\in\mathcal{M}} (\nu_i(X > z) - g(\mathbb{P}(X > z)))\gamma_i(z)\, dz \qquad (4.21)
\end{aligned}$$

$$= \rho^{PH}\left(X - \sum_{i\in\mathcal{M}} \tilde{I}_i(X) + \sum_{i\in\mathcal{M}} \pi^{\nu_i}(\tilde{I}_i(X))\right),$$

and hence $(I_1^*, \ldots, I_m^*)$ is optimal. We now show the converse. Suppose that $(I_1, \ldots, I_m)$ is an allocation that does not satisfy the given form. Then there are three possibilities:

(i) There exists a set $\mathcal{A}_1$ of positive measure such that for each $z \in \mathcal{A}_1$,

$$\gamma_i(z) > 0 \text{ for at least one } i \notin \mathcal{M}_z\,.$$

(ii) There exists a set $\mathcal{A}_2$ of positive measure such that for each $z \in \mathcal{A}_2$,

$$\sum_{i\in\mathcal{M}} \gamma_i(z) < 1, \quad g(\mathbb{P}(X > t)) > \underline{\nu}(X > z)\,.$$

Note that $\mathcal{A}_2 \subseteq \mathcal{A}$.

(iii) There exists a set $\mathcal{A}_3$ of positive measure such that for each $z \in \mathcal{A}_3$,

$$\sum_{i\in\mathcal{M}} \gamma_i(z) > 0, \quad g(\mathbb{P}(X > t)) < \underline{\nu}(X > z)\,.$$

Note that $\mathcal{A}_3 \subseteq \mathcal{A}^C$.

Suppose that we are in the first case. Then for each $z \in \mathcal{A}_1$, we have

$$\sum_{i\in\mathcal{M}} \underline{\nu}(X > z)\gamma_i(z) < \sum_{i\in\mathcal{M}} \nu_i(X > z)\gamma_i(z)\,.$$

Hence,

$$\int_{\mathcal{A}_1} \sum_{i\in\mathcal{M}} (\underline{\nu}(X > z) - g(\mathbb{P}(X > z)))\gamma_i(z)\,dz < \int_{\mathcal{A}_1} \sum_{i\in\mathcal{M}} (\nu_i(X > z) - g(\mathbb{P}(X > z)))\gamma_i(z)\,dz\,,$$

where this inequality is strict. Also, since $\underline{\nu} \leq \nu_i$ for all i, we have

$$\int_{\mathcal{A}_1^C} \sum_{i\in\mathcal{M}} (\underline{\nu}(X > z) - g(\mathbb{P}(X > z)))\gamma_i(z)\,dz \leq \int_{\mathcal{A}_1^C} \sum_{i\in\mathcal{M}} (\nu_i(X > z) - g(\mathbb{P}(X > z)))\gamma_i(z)\,dz\,,$$

where $\mathcal{A}_1^C = \mathbb{R}_+ \setminus \mathcal{A}_1$. Adding these inequalities gives

$$\int_0^\infty \sum_{i\in\mathcal{M}} (\underline{\nu}(X > z) - g(\mathbb{P}(X > z)))\gamma_i(z)\,dz < \int_0^\infty \sum_{i\in\mathcal{M}} (\nu_i(X > z) - g(\mathbb{P}(X > z)))\gamma_i(z)\,dz\,,$$

and so inequality (4.21) is strict. Therefore, $(I_1, \ldots, I_m)$ is not optimal.

The remaining two cases are similar. Suppose that (ii) is true. Then for each $z \in \mathcal{A}_2$, we have

$$\underline{\nu}(X > z) - g(\mathbb{P}(X > z)) < (\underline{\nu}(X > z) - g(\mathbb{P}(X > z))) \sum_{i\in\mathcal{M}} \gamma_i(z)\,.$$

Hence,

$$\int_{\mathcal{A}_2} \underline{\nu}(X > z) - g(\mathbb{P}(X > z))\,dz < \int_{\mathcal{A}_2} (\underline{\nu}(X > z) - g(\mathbb{P}(X > z))) \sum_{i\in\mathcal{M}} \gamma_i(z)\,dz\,.$$

Also, since $g(\mathbb{P}(X > z)) > \underline{\nu}(X > z)$ on $\mathcal{A}$, and $\sum_{i\in\mathcal{M}} \gamma_i(z)\,dz \geq 0$, we have

$$\int_{\mathcal{A}\setminus\mathcal{A}_2} \underline{\nu}(X > z) - g(\mathbb{P}(X > z))\,dz \leq \int_{\mathcal{A}\setminus\mathcal{A}_2} (\underline{\nu}(X > z) - g(\mathbb{P}(X > z))) \sum_{i\in\mathcal{M}} \gamma_i(z)\,dz\,.$$

Adding these inequalities gives

$$\int_{\mathcal{A}} \underline{\nu}(X > z) - g(\mathbb{P}(X > z))\,dz < \int_{\mathcal{A}} (\underline{\nu}(X > z) - g(\mathbb{P}(X > z))) \sum_{i\in\mathcal{M}} \gamma_i(z)\,dz\,,$$

and so inequality (4.19) is strict. Hence, $(I_1, \ldots, I_m)$ is not optimal.

Finally, suppose that (iii) is true. Then for each $z \in \mathcal{A}_3$, we have

$$(\underline{\nu}(X > z) - g(\mathbb{P}(X > z))) \sum_{i\in\mathcal{M}} \gamma_i(z) > 0\,,$$

so

$$\int_{\mathcal{A}_3} (\underline{\nu}(X > z) - g(\mathbb{P}(X > z))) \sum_{i\in\mathcal{M}} \gamma_i(z)\,dz > 0\,.$$

Also, since $g(\mathbb{P}(X > z)) \leq \underline{\nu}(X > z)$ on $\mathcal{A}^C$ and $\sum_{i\in\mathcal{M}} \gamma_i(z) \geq 0$,

$$\int_{\mathcal{A}^C\setminus\mathcal{A}_3} (\underline{\nu}(X > z) - g(\mathbb{P}(X > z))) \sum_{i\in\mathcal{M}} \gamma_i(z)\,dz > 0\,.$$

Adding these inequalities gives

$$\int_{\mathcal{A}^C} (\underline{\nu}(X > z) - g(\mathbb{P}(X > z))) \sum_{i\in\mathcal{M}} \gamma_i(z)\,dz > 0\,,$$

and therefore inequality (4.20) is strict. Thus, $(I_1, \ldots, I_m)$ is not optimal. □

4.6.3 Proof of Proposition 4.3.9

Let $i \in \mathcal{M}$ and $\mathfrak{I}^* \in \beth$. First, we have

$$\begin{aligned}\rho_i(I_i^*(X) - \pi_i) &= \int_0^\infty \tau_i(X > z)\gamma_i^*(z)\,dz - \int_0^\infty \nu_i^*(X > z)\gamma_i^*(z)\,dz \\ &= \int_0^\infty (\tau_i(X > z) - \nu_i^*(X > z))\gamma_i^*(z)\,dz\,.\end{aligned}$$

Let

$$\mathcal{A}_1 = \{z : \gamma_i^*(z) = 1\} \quad \text{and} \quad \mathcal{A}_2 = \{z : \gamma_i^*(z) \in (0,1)\}\,,$$

so that

$$\begin{aligned}&\int_0^\infty (\tau_i(X > z) - \nu_i^*(X > z))\gamma_i^*(z)\,dz \\ &\qquad = \int_{\mathcal{A}_1} \tau_i(X > z) - \nu_i^*(X > z)\,dz + \int_{\mathcal{A}_2} (\tau_i(X > z) - \nu_i^*(X > z))\gamma_i^*(z)\,dz\,.\end{aligned}$$

We first show that for almost all $z \in \mathcal{A}_2$, $\tau_i(X > z) = \nu_i^*(X > z)$, so the second integral vanishes. We then identify the first integral with $\int_0^\infty (\tau_i(X > z) - \nu_i^*(X > z))\gamma_i^*(z)\,dz$ as desired. Note that on $\mathcal{A}_1 \cup \mathcal{A}_2$, we have $g(\mathbb{P}(X > z)) \geq \underline{\nu}(X > z)$.

Suppose that $z \in \mathcal{A}_2$, and z satisfies (4.15) and (4.17). Since $\sum_{j=0}^n \gamma_j^*(z) = 1$, there must exist $k \in (\mathcal{M} \cup \{0\}) \setminus \{i\}$ for which $\gamma_k^*(z) \in (0,1)$. This implies that $\nu_i^*(X > z) = \nu_k^*(X > z) = \underline{\nu}(X > z) = \bar{\tau}(X > z)$ by (4.15), where we use the convention that $\nu_0^* = g \circ \mathbb{P}$. Since the indemnity distributes generously in the sense of Definition 4.3.4, there does not exist any $j \in \mathcal{M} \cup \{0\}$ such that $\nu_j^*(X > z) = \min\{g(\mathbb{P}(X > z), \underline{\nu}(X > z)\} = \underline{\nu}(X > z)$ and $\tau_j(X > z) < \tau_i(X > z)$. Therefore, it must be true that $\tau_k(X > z) \geq \tau_i(X > z)$. A symmetric argument by switching the indices i, k implies $\tau_k(X > z) \leq \tau_i(X > z)$. Hence, $\tau_k(X > z) = \tau_i(X > z)$ for any k with $\gamma_k^*(z) \in (0,1)$.

It remains to show that $\bar{\tau}(X > z) = \tau_i(X > z)$, which combined with the above, imply that $\tau_i(X > z) = \nu_i^*(X > z)$. Since $\tau_i(X > z) = \tau_k(X > z)$, it suffices to show that $\tau_i(X > z) = \tau_k(X > z) = \min_{j \in \mathcal{M}} \tau_j(X > z)$. To this end, note that $\gamma_i^*(z) > 0$ implies $g(\mathbb{P}(X > z)) \geq \bar{\tau}(X > z)$. Hence, we must have $g(\mathbb{P}(X > z)) \geq \min_{j \in \mathcal{M}} \tau_j(X > z)$, which implies that $\mathcal{T}_z \neq \varnothing$. Therefore (4.17) implies that there exists a $k' \in \mathcal{T}_z \cap \mathcal{M}_z$: that is, $\tau_{k'}(X > z) = \min_{j \in \mathcal{M}} \tau_j(X > z)$ and $\nu_{k'}^*(X > z) = \bar{\tau}(X > z) = \nu_i^*(X > z)$.

Assume for the sake of contradiction that $\tau_i(X > z) > \min_{j \in \mathcal{M}} \tau_j(X > z)$. Then since $\tau_{k'} = \min_{j \in \mathcal{M}} \tau_j(X > z) < \tau_i(X > z)$ and $\nu_{k'}^*(X > z) = \bar{\tau}(X > z)$, generous distribution

implies that $\gamma_i^*(z) = 0$, a contradiction. Hence, $\tau_i(X > z) = \min_{j \in \mathcal{M}} \tau_j(X > z)$, and so $\bar{\tau}(X > z) = \tau_i(X > z) = \tau_k(X > z)$ as desired. Since $\bar{\tau}(X > z) = \nu_i^*(X > z)$ by the above, we have $\tau_i(X > z) = \nu_i^*(X > z)$ for almost all $z \in \mathcal{A}_2$, which implies that the integral $\int_{\mathcal{A}_2} (\tau_i(X > z) - \nu_i^*(X > z))\gamma_i^*(z)\, dz$ vanishes.

Consider now the first integral $\int_{\mathcal{A}_1} \tau_i(X > z) - \nu_i^*(X > z)\, dz$. Let $z \in \mathcal{A}_1$, and suppose z satisfies (4.15) and (4.17). Since $\gamma_i^*(z) = 1$, we must have $i \in \mathcal{M}_z$, so $\nu_i^*(X > z) = \underline{\nu}(X > z) = \bar{\tau}(X > z)$ by (4.15). Hence, we have

$$\int_{\mathcal{A}_1} \tau_i(X > z) - \nu_i^*(X > z)\, dz = \int_{\mathcal{A}_1} \tau_i(X > z) - \bar{\tau}(X > z)\, dz\,.$$

Note that on $\mathcal{A}_1$, we must have $\tau_i(X > z) \leq \nu_i^*(X > z) = \bar{\tau}(X > z)$. Suppose for the sake of contradiction that $\tau_i(X > z) > \nu_i^*(X > z)$. Since $\gamma_i^*(z) = 1$, we have $g(\mathbb{P}(X > z)) \geq \bar{\tau}(X > z)$, so there must exist some insurer $k \in \mathcal{M}$ such that $\tau_k(X > z) = \min_{j \in \mathcal{M} \cup \{0\}} \tau_j(X > z)$. Then $k \in \mathcal{T}_z$, so there exist $k' \in \mathcal{T}_z \cap \mathcal{M}_z$ by (4.17). Therefore $\tau_{k'}(X > z) \leq \bar{\tau}(X > z) < \tau_i(X > z)$, and $\nu_{k'}^*(X > z) = \bar{\tau}(X > z) = \nu_i^*(X > z)$. However, since the indemnity distributes generously, this implies $\gamma_i^*(z) = 0$, which contradicts $z \in \mathcal{A}_1$.

Therefore $\tau_i(X > z) \leq \nu_i^*(X > z)$, so the integrand is non-zero only when $\tau_i(X > z) < \nu_i^*(X > z) = \bar{\tau}(X > z)$. Let $\mathcal{A}_3 := \{z \in \mathbb{R}_+ : \tau_i(X > z) < \nu_i^*(X > z)\}$. Then the above simplifies to

$$\begin{aligned}
&\int_{\mathcal{A}_1} \tau_i(X > z) - \bar{\tau}(X > z)\, dz \\
&\quad = \int_{\mathcal{A}_1 \cap \mathcal{A}_3} \tau_i(X > z) - \bar{\tau}(X > z)\, dz + \int_{\mathcal{A}_1 \setminus \mathcal{A}_3} \tau_i(X > z) - \bar{\tau}(X > z)\, dz \\
&\quad = \int_{\mathcal{A}_1 \cap \mathcal{A}_3} \tau_i(X > z) - \bar{\tau}(X > z)\, dz + \int_{\mathcal{A}_1 \setminus \mathcal{A}_3} 0\, dz \\
&\quad = \int_{\mathcal{A}_1 \cap \mathcal{A}_3} \tau_i(X > z) - \bar{\tau}(X > z)\, dz\,.
\end{aligned}$$

For $z \in \mathcal{A}_1 \cap \mathcal{A}_3$, we have $\tau_i(X > z) < \nu_i^*(X > z) = \bar{\tau}(X > z)$, so $\mathcal{A}_1 \cap \mathcal{A}_3 \subseteq \mathcal{Z}_i$. We now show that $\mathcal{Z}_i \subseteq \mathcal{A}_1$. Let $z \in \mathcal{Z}_i$, and suppose for the sake of contradiction that $\gamma_i^*(z) < 1$. Then since $\sum_{j=0}^n \gamma_j^*(z) = 1$, we must have $\gamma_k^*(z) > 0$ for some $k \in (\mathcal{M} \cup \{0\}) \setminus \{i\}$, which implies $\nu_k^*(X > z) = \bar{\tau}(X > z)$. However, since $z \in \mathcal{Z}_i$, we have $\tau_i(X > z) < \bar{\tau}(X > z)$, which implies that $\mathcal{T}_z = \{i\}$. Then by (4.17), $i \in \mathcal{M}_z$, so $\nu_i^*(X > z) = \bar{\tau}(X > z) = \nu_k^*(X >$

z). Since the indemnity distributes generously and $\tau_i(X > z) < \tau_k(X > z)$, $\gamma_k^*(z) = 0$, which is a contradiction – hence, $\mathcal{Z}_i \subseteq \mathcal{A}_1$. Putting the above together, we have

$$\begin{aligned}
&\int_{\mathcal{A}_1} \tau_i(X > z) - \bar{\tau}(X > z)\, dz \\
&\quad = \int_{\mathcal{A}_1 \cap \mathcal{A}_3} \tau_i(X > z) - \bar{\tau}(X > z)\, dz \\
&\quad = \int_{\mathcal{A}_1 \cap \mathcal{A}_3} \tau_i(X > z) - \bar{\tau}(X > z)\, dz + \int_{\mathcal{Z}_i \cap (\mathcal{A}_1 \setminus \mathcal{A}_3)} 0\, dz \\
&\quad = \int_{\mathcal{A}_1 \cap \mathcal{A}_3} \tau_i(X > z) - \bar{\tau}(X > z)\, dz + \int_{\mathcal{Z}_i \cap (\mathcal{A}_1 \setminus \mathcal{A}_3)} \tau_i(X > z) - \bar{\tau}(X > z)\, dz \\
&\quad = \int_{\mathcal{Z}_i} \tau_i(X > z) - \bar{\tau}(X > z)\, dz\,,
\end{aligned}$$

which completes the proof. □

4.6.4 Proof of Proposition 4.3.10

Let $i \in \mathcal{M}$. For notational convenience, we define $\hat{\gamma}_i$ such that for each $x \in \mathbb{R}_+$, we have $\mathfrak{I}_i^*(\hat{\nu}, \nu_{-i}^*)(x) = \int_0^x \hat{\gamma}_i(z)\, dz$. We now fix the pricing capacities $(\hat{\nu}, \nu_{-i}^*)$, and use our previous notation as introduced in Proposition 4.3.3. For example, $\mathcal{M}_z$ is the set of indices that minimize over the set $\{\hat{\nu}(X > z), \nu_j^*(X > z)\}$, for $j \in \mathcal{M} \setminus \{i\}$. Then we can write

$$\rho_i(\mathfrak{I}_i^*(\hat{\nu}, \nu_{-i}^*)(X)) - \pi^{\hat{\nu}}(\mathfrak{I}_i^*(\hat{\nu}, \nu_{-i}^*)(X)) = \int_0^\infty (\tau_i(X > z) - \hat{\nu}(X > z))\hat{\gamma}_i(z)\, dz.$$

On the other hand, by Proposition 4.3.9, we have that

$$\rho_i(\mathfrak{I}_i^*(\nu_1^*, \ldots, \nu_m^*)(X)) - \pi^{\nu_i}(\mathfrak{I}_i^*(\nu_1^*, \ldots, \nu_m^*)(X)) = \int_{\mathcal{Z}_i} \tau_i(X > z) - \bar{\tau}(X > z)\, dz\,,$$

where

$$\mathcal{Z}_i := \{z \in \mathbb{R}_+ : \tau_i(X > z) < \bar{\tau}(X > z)\}\,.$$

We first show that for all $z \in \mathcal{Z}_i$, $(\tau_i(X > z) - \hat{\nu}(X > z))\hat{\gamma}_i(z) \geq \tau_i(X > z) - \bar{\tau}(X > z)$. Then for all $z \in \mathcal{Z}_i^C$, we show that $(\tau_i(X > z) - \hat{\nu}(X > z))\hat{\gamma}_i(z) \geq 0$. Combining these equations yields the desired relation.

1. First, suppose that $z \in \mathcal{Z}_i$. Then $\tau_i(X > z) < \bar{\tau}(X > z)$, and we can distinguish three cases.

   (a) $\hat{\nu}(X > z) > \bar{\tau}(X > z)$: In this case, by (4.16), there exist some $k \neq i$, $k \in \mathcal{M} \cup \{0\}$ such that $\nu_k^*(X > z) = \bar{\tau}(X > z)$, where we use the convention $\nu_0^*(X > z) := g(\mathbb{P}(X > z))$. If there exists any such $k \neq 0$, then $i \notin \mathcal{M}_z$, so $\hat{\gamma}_i(z) = 0$, so $(\tau_i(X > z) - \hat{\nu}(X > z))\hat{\gamma}_i(z) = 0$. On the other hand, if the only $k \neq i$ such that $\nu_k^*(X > z) = \bar{\tau}(X > z)$ is $k = 0$, then $\nu_0^*(X > z) < \nu_j^*(X > z)$ for all $j \neq i$. This, along with the assumption $\hat{\nu}(X > z) > \bar{\tau}(X > z) = \nu_0^*(X > z)$, implies that $\hat{\gamma}_i(z) = 0$, so again $(\tau_i(X > z) - \hat{\nu}(X > z))\hat{\gamma}_i(z) = 0$. Note that $0 \geq \tau_i(X > z) - \bar{\tau}(X > z)$ on $\mathcal{Z}_i$.

   (b) $\hat{\nu}(X > z) < \bar{\tau}(X > z)$: Recall that we have $\bar{\tau}(X > z) = \min_{j \in \mathcal{M}} \nu_j^*(X > z)$ by (4.15). Then by assumption, $\hat{\nu}(X > z) < \min_{j \neq i} \nu_j^*(X > z)$ and $\hat{\nu}(X > z) < g(\mathbb{P}(X > z))$, since $g(\mathbb{P}(X > z)) \geq \bar{\tau}(X > z)$ whenever $\tau_i(X > z) < \bar{\tau}(X > z)$. This implies that $\mathcal{M}_z = \{i\}$ and $\sum_{j=1}^n \hat{\gamma}_j(z) = 1$. This implies $\hat{\gamma}_j(z) = 0$ for $j \neq i$, so it must be true that $\hat{\gamma}_i(z) = 1$. Hence,

   $$(\tau_i(X > z) - \hat{\nu}(X > z))\hat{\gamma}_i(z) = \tau_i(X > z) - \hat{\nu}(X > z) \geq \tau_i(X > z) - \bar{\tau}(X > z)\,.$$

   (c) $\hat{\nu}(X > z) = \bar{\tau}(X > z)$: In this case, note that $\tau_i(X > z) - \bar{\tau}(X > z) < 0$, so $\tau_i(X > z) - \hat{\nu}(X > z) < 0$. Since $\hat{\gamma}_i(z) \leq 1$,

   $$(\tau_i(X > z) - \hat{\nu}(X > z))\hat{\gamma}_i(z) \geq (\tau_i(X > z) - \hat{\nu}(X > z)) \cdot 1 = \tau_i(X > z) - \bar{\tau}(X > z)\,.$$

2. Now, suppose that $z \in \mathcal{Z}_i^C$. We distinguish two cases.

   (a) $\hat{\nu}(X > z) > \bar{\tau}(X > z)$: By the same logic as Case (1-a), we have $(\tau_i(X > z) - \hat{\nu}(X > z))\hat{\gamma}_i(z) = 0$.

   (b) $\hat{\nu}(X > z) \leq \bar{\tau}(X > z)$: In this case, since $z \in \mathcal{Z}_i^C$, $\bar{\tau}(X > z) \leq \tau_i(X > z)$. Therefore,

   $$(\tau_i(X > z) - \hat{\nu}(X > z))\ \hat{\gamma}_i(z) \geq 0\,.$$

Hence, we have

$$\begin{aligned}
&\rho_i\left(\mathfrak{I}_i^*(\hat{\nu},\nu_{-i}^*)(X)\right)-\pi^{\hat{\nu}}\left(\mathfrak{I}_i^*(\hat{\nu},\nu_{-i}^*)(X)\right)\\
&=\int_0^\infty(\tau_i(X>z)-\hat{\nu}(X>z))\hat{\gamma}_i(z)\,dz\\
&=\int_{\mathcal{Z}_i}(\tau_i(X>z)-\hat{\nu}(X>z))\hat{\gamma}_i(z)\,dz+\int_{\mathcal{Z}_i^C}(\tau_i(X>z)-\hat{\nu}(X>z))\hat{\gamma}_i(z)\,dz\\
&\geq\int_{\mathcal{Z}_i}\tau_i(X>z)-\bar{\tau}(X>z)\,dz+\int_{\mathcal{Z}_i^C}0\,dz\\
&=\int_{\mathcal{Z}_i}\tau_i(X>z)-\bar{\tau}(X>z)\,dz\\
&=\rho_i(\mathfrak{I}_i^*(\nu_1^*,\ldots,\nu_m^*)(X))-\pi^{\nu_i^*}(\mathfrak{I}_i^*(\nu_1^*,\ldots,\nu_m^*)(X))\,,
\end{aligned}$$

as desired. □

4.6.5 Proof of Proposition 4.3.13

We check properties (4.15), (4.16), and (4.17) respectively. First, let $z\in\mathbb{R}_+$. Then by definition of $\bar{\tau}$, there exist $i,j\in\mathcal{M}\cup\{0\}$, $i\neq j$ such that

$$\tau_i(X>z)\leq\tau_j(X>z)=\bar{\tau}(X>z)\leq\tau_k(X>z)\,,$$

for all $k\in\mathcal{M}$, $k\neq i,j$. This implies that either $i\in\tilde{\mathcal{M}}_z$, or $i=0$. In the first case, we have $\underline{\nu}(X>z)\leq\nu_i^*(X>z)=\bar{\tau}(X>z)$. In the second case, $j\in\tilde{\mathcal{M}}_z$, so $\underline{\nu}(X>z)=\tau_j^*(X>z)=\bar{\tau}(X>z)$. Therefore $\underline{\nu}(X>z)\leq\bar{\tau}(X>z)$.

On the other hand, we have $\bar{\tau}(X>z)=\tau_j(X>z)$, and so for $k\neq i,j$, $k\in\mathcal{M}$, $\bar{\tau}(X>z)\leq\tau_k(X>z)$. Then we have $\bar{\tau}(X>z)=\nu_j^*(X>z)$ and $\bar{\tau}(X>z)\leq\nu_k^*(X>z)$ by definition of ν_j^*,ν_k^*. If $i\neq0$, then $\nu_i^*(X>z)=\bar{\tau}(X>z)$ by the above. This gives $\bar{\tau}(X>z)\leq\nu_l^*(X>z)$ for all $l\in\mathcal{M}$. That is, $\bar{\tau}(X>z)\leq\underline{\nu}(X>z)$. Therefore $\underline{\nu}(X>z)=\bar{\tau}(X>z)$ as desired, so (4.15) holds.

Next, we check (4.16). Let $z\in\mathbb{R}_+$ be fixed. By definition of $\bar{\tau}$, there exist $i,j\in\mathcal{M}$, $i\neq j$ such that $\bar{\tau}(X>z)=\tau_j(X>z)$, $\tau_i(X>z)\leq\tau_j(X>z)$, and $\tau_k(X>z)\geq\tau_j(X>z)$ for all $k\neq i,j$. Then by construction of $(\nu_1^*,\ldots,\nu_m^*)$, $\nu_j^*(X>z)=\bar{\tau}(X>z)=\tau_j(X>z)$, and $\bar{\tau}\leq\nu_k^*(X>z)$ for all $k\neq i,j$. Assume for the sake of contradiction that $\nu_i^*(X>z)<\bar{\tau}(X>z)$. Then by definition of ν_i^*, we must have $\nu_i^*(X>z)=\tau_i(X>z)$, so

$\tau_i(X > z) \leq \tau_k(X > z)$ for all $k \neq i$ as above. But then $i \in \tilde{\mathcal{M}}_z$, so $\nu_i^*(X > z) = \bar{\tau}(X > z)$, a contradiction. Therefore $\nu_i^*(X > z) = \bar{\tau}(X > z) = \nu_j^*(X > z)$ as desired, so (4.16) holds.

Finally, we check (4.17). For each $z \in \mathbb{R}_+$, if $\mathcal{T}_z \neq \varnothing$, take $i \in \mathcal{T}_z$. Then we have $\nu_i^*(X > z) = \bar{\tau}(X > z) = \min_{j \in \mathcal{M}} \nu_j^*(X > z)$, so $i \in \mathcal{M}_z$. Hence, $\mathcal{T}_z \cap \mathcal{M}_z \neq \varnothing$, so (4.17) holds. □

4.6.6 Proof of Lemma 4.4.3

Note that for $z \in \mathcal{Z}_{PH}$, we have $\sum_{i \in \mathcal{M}} \gamma_i^*(z) = 0$, and so $\gamma_i^*(z) = 0$ for all $i \in \mathcal{M}$. Then we have

$$\rho^{PH}\left(X - \sum_{i \in \mathcal{M}} \mathfrak{I}_i^*(\nu_1^*, \ldots, \nu_m^*)(X) + \sum_{i \in \mathcal{M}} \pi^{\nu_i^*}(\mathfrak{I}_i^*(\nu_1^*, \ldots, \nu_m^*)(X))\right)$$
$$= \int_0^\infty g(\mathbb{P}(X > z))\left(1 - \sum_{i \in \mathcal{M}} \gamma_i^*(z)\right) dz + \sum_{i \in \mathcal{M}} \int_0^\infty \nu_i^*(X > z)\gamma_i^*(z)\, dz\,.$$

Define the sets

$$\mathcal{Z}_{PH}^{=} := \{z \in \mathbb{R}_+ : g(\mathbb{P}(X > z)) = \bar{\tau}(X > z)\}\,,$$
$$\mathcal{Z}_{PH}^{+} := \{z \in \mathbb{R}_+ : g(\mathbb{P}(X > z)) > \bar{\tau}(X > z)\}\,.$$

Since $\vec{I}^*$ is optimal, the first term simplifies to

$$\int_0^\infty g(\mathbb{P}(X > z))\left(1 - \sum_{i \in \mathcal{M}} \gamma_i^*(z)\right) dz = \int_{\mathcal{Z}_{PH}} g(\mathbb{P}(X > z))\, dz + \int_{\mathcal{Z}_{PH}^{=}} g(\mathbb{P}(X > z))\gamma_0^*(z)\, dz$$
$$= \int_{\mathcal{Z}_{PH}} g(\mathbb{P}(X > z))\, dz + \int_{\mathcal{Z}_{PH}^{=}} \bar{\tau}(X > z)\gamma_0^*(z)\, dz\,.$$

For second term, since $\gamma_i^*(z) > 0$ implies $\nu_i^*(X > z) = \bar{\tau}(X > z)$, we have

$$\sum_{i \in \mathcal{M}} \int_0^\infty \nu_i^*(X > z)\gamma_i^*(z)\, dz = \sum_{i \in \mathcal{M}} \int_{\mathcal{Z}_{PH}^C} \nu_i^*(X > z)\gamma_i^*(z)\, dz$$
$$= \int_{\mathcal{Z}_{PH}^C} \sum_{i \in \mathcal{M}} \nu_i^*(X > z)\gamma_i^*(z)\, dz = \int_{\mathcal{Z}_{PH}^C} \bar{\tau}(X > z) \sum_{i \in \mathcal{M}} \gamma_i^*(z)\, dz$$
$$= \int_{\mathcal{Z}_{PH}^{=}} \bar{\tau}(X > z) \sum_{i \in \mathcal{M}} \gamma_i^*(z)\, dz + \int_{\mathcal{Z}_{PH}^{-}} \bar{\tau}(X > z)\, dz\,.$$

Adding this to the above, we have

$$\begin{aligned}
&\int_0^\infty g(\mathbb{P}(X > z)) \left(1 - \sum_{i \in \mathcal{M}} \gamma_i^*(z)\right) dz + \sum_{i \in \mathcal{M}} \int_0^\infty \nu_i^*(X > z)\gamma_i^*(z)\, dz \\
&\quad = \int_{\mathcal{Z}_{PH}} g(\mathbb{P}(X > z))\, dz + \int_{\mathcal{Z}_{PH}^=} \bar{\tau}(X > z)\gamma_0^*(z)\, dz \\
&\qquad + \int_{\mathcal{Z}_{PH}^=} \bar{\tau}(X > z) \sum_{i \in \mathcal{M}} \gamma_i^*(z)\, dz + \int_{\mathcal{Z}_{PH}^-} \bar{\tau}(X > z)\, dz \\
&\quad = \int_{\mathcal{Z}_{PH}} g(\mathbb{P}(X > z))\, dz + \int_{\mathcal{Z}_{PH}^=} \bar{\tau}(X > z)\, dz + \int_{\mathcal{Z}_{PH}^-} \bar{\tau}(X > z)\, dz \\
&\quad = \int_{\mathcal{Z}_{PH}} g(\mathbb{P}(X > z))\, dz + \int_{\mathcal{Z}_{PH}^C} \bar{\tau}(X > z)\, dz\,.
\end{aligned}$$

□

4.6.7 Proof of Proposition 4.4.5

Note that when $m = 1$, if $g(\mathbb{P}(X > z)) \geq \bar{\tau}(X > z)$, then it must be true that $g(\mathbb{P}(X > z)) = \bar{\tau}(X > z)$. Therefore by Lemma 4.4.3, we have

$$\begin{aligned}
\rho^{PH}\left(X - I_1(X) + \pi^{\nu_1^*}(I_1(X))\right) &= \int_{\mathcal{Z}_{PH}} g(\mathbb{P}(X > z))\, dz + \int_{\mathcal{Z}_{PH}^C} \bar{\tau}(X > z)\, dz \\
&= \int_{\mathcal{Z}_{PH}} g(\mathbb{P}(X > z))\, dz + \int_{\mathcal{Z}_{PH}^C} g(\mathbb{P}(X > z))\, dz \\
&= \int_0^\infty g(\mathbb{P}(X > z))\, dz = \rho^{PH}(X)\,.
\end{aligned}$$

□

4.6.8 Proof of Proposition 4.4.6

Let $\bar{\tau}_n$ be the second-lowest function of the set of capacities $\{\tau_0, \ldots, \tau_m\}$, and let $\bar{\tau}_{m+1}$ be the second-lowest function of $\{\tau_0, \ldots, \tau_{m+1}\}$. Define

$$\begin{aligned}
\mathcal{Z}_{PH}^n &:= \{z \in \mathbb{R}_+ : g(\mathbb{P}(X > z)) < \bar{\tau}_n(X > z)\}\,, \\
\mathcal{Z}_{PH}^{m+1} &:= \{z \in \mathbb{R}_+ : g(\mathbb{P}(X > z)) < \bar{\tau}_{m+1}(X > z)\}\,.
\end{aligned}$$

Then $\bar{\tau}_{m+1} \leq \bar{\tau}_n$, and so $\mathcal{Z}_{PH}^{m+1} \subseteq \mathcal{Z}_{PH}^n$. By Lemma 4.4.3, we have

$$\begin{aligned}
&\rho^{PH}\left(X - \sum_{i=1}^n I_i^{*,n}(X) + \sum_{i=1}^n \pi^{\nu_i^{*,n}}(I_i^{*,n}(X))\right) \\
&\quad = \int_{\mathcal{Z}_{PH}^n} g(\mathbb{P}(X > z))\, dz + \int_{(\mathcal{Z}_{PH}^n)^C} \bar{\tau}_n(X > z)\, dz \\
&\quad = \int_{\mathcal{Z}_{PH}^{m+1}} g(\mathbb{P}(X > z))\, dz + \int_{\mathcal{Z}_{PH}^n \setminus \mathcal{Z}_{PH}^{m+1}} g(\mathbb{P}(X > z))\, dz + \int_{\left(\mathcal{Z}_{PH}^n\right)^C} \bar{\tau}_n(X > z)\, dz \\
&\quad \geq \int_{\mathcal{Z}_{PH}^{m+1}} g(\mathbb{P}(X > z))\, dz + \int_{\mathcal{Z}_{PH}^n \setminus \mathcal{Z}_{PH}^{m+1}} \bar{\tau}_{m+1}(X > z)\, dz + \int_{\left(\mathcal{Z}_{PH}^n\right)^C} \bar{\tau}_{m+1}(X > z)\, dz \\
&\quad = \int_{\mathcal{Z}_{PH}^{m+1}} g(\mathbb{P}(X > z))\, dz + \int_{\left(\mathcal{Z}_{PH}^{m+1}\right)^C} \bar{\tau}_{m+1}(X > z)\, dz \\
&\quad = \rho^{PH}\left(X - \sum_{i=1}^{m+1} I_i^{*,m+1}(X) + \sum_{i=1}^{m+1} \pi^{\nu_i^{*,m+1}}(I_i^{*,m+1}(X))\right).
\end{aligned}$$

□

4.6.9 Proof of Proposition 4.4.8

Let $(\vec{I}, \vec{\pi})$ be an allocation, and suppose for the sake of contradiction that $\sum_{i=0}^n \rho_i(I_i(X))$ does not solve (4.18). Then there exist $\tilde{\vec{I}} \in \vec{\mathcal{I}}$ such that $\sum_{i=0}^n \rho_i(\tilde{I}_i(X)) < \sum_{i=0}^n \rho_i(I_i(X))$. For $i \in \mathcal{M}$, define $\tilde{\pi}_i$ by

$$\tilde{\pi}_i := \rho_i(\tilde{I}_i(X)) - \rho_i(I_i(X)) + \pi_i\,.$$

Then we have

$$\rho_i(I_i(X)) - \pi_i = \rho_i(\tilde{I}_i(X)) - \rho_i(\tilde{I}_i(X)) + \rho_i(I_i(X)) - \pi_i = \rho_i(\tilde{I}_i(X)) - \tilde{\pi}_i\,.$$

On the other hand, we have

$$
\begin{aligned}
&\rho^{PH}\left(X-\sum_{i\in\mathcal{M}}I_i(X)+\sum_{i\in\mathcal{M}}\pi_i\right)\\
&=\rho^{PH}\left(X-\sum_{i\in\mathcal{M}}I_i(X)\right)+\sum_{i\in\mathcal{M}}\pi_i\\
&=\rho^{PH}\left(X-\sum_{i\in\mathcal{M}}I_i(X)\right)+\sum_{i\in\mathcal{M}}\tilde{\pi}_i-\sum_{i\in\mathcal{M}}\rho_i(\tilde{I}_i(X))+\sum_{i\in\mathcal{M}}\rho_i(I_i(X))\\
&=\sum_{i=0}^{n}\rho_i(I_i(X))+\sum_{i\in\mathcal{M}}\tilde{\pi}_i-\sum_{i\in\mathcal{M}}\rho_i(\tilde{I}_i(X))\\
&>\sum_{i=0}^{n}\rho_i(\tilde{I}_i(X))+\sum_{i\in\mathcal{M}}\tilde{\pi}_i-\sum_{i\in\mathcal{M}}\rho_i(\tilde{I}_i(X))\\
&=\rho^{PH}\left(X-\sum_{i\in\mathcal{M}}\tilde{I}_i(X)\right)+\sum_{i\in\mathcal{M}}\rho_i(\tilde{I}_i(X))+\sum_{i\in\mathcal{M}}\tilde{\pi}_i-\sum_{i\in\mathcal{M}}\rho_i(\tilde{I}_i(X))\\
&=\rho^{PH}\left(X-\sum_{i\in\mathcal{M}}\tilde{I}_i(X)\right)+\sum_{i\in\mathcal{M}}\tilde{\pi}_i\\
&=\rho^{PH}\left(X-\sum_{i\in\mathcal{M}}\tilde{I}_i(X)+\sum_{i\in\mathcal{M}}\tilde{\pi}_i\right),
\end{aligned}
$$

implying that the allocation $(\tilde{I},\tilde{\pi})$ improves over $(\vec{I},\vec{\pi})$, with a strict improvement for the policyholder. Hence, $(\vec{I},\vec{\pi})$ is not PO.

Conversely, if $(\vec{I},\vec{\pi})$ is not PO, then there exist an allocation $\left(\tilde{I}_i,\tilde{\pi}\right)$ such that

$$
\begin{aligned}
\rho^{PH}\left(X-\sum_{i\in\mathcal{M}}\tilde{I}_i(X)+\sum_{i\in\mathcal{M}}\tilde{\pi}_i\right)&\leq\rho^{PH}\left(X-\sum_{i\in\mathcal{M}}I_i(X)+\sum_{i\in\mathcal{M}}\pi_i\right)\\
\rho_i(\tilde{I}_i(X)-\tilde{\pi}_i)&\leq\rho_i(I_i(X)-\pi_i),\ \forall i\in\mathcal{M},
\end{aligned}
$$

with at least one strict inequality. Summing these inequalities gives

$$\rho^{PH}\left(X-\sum_{i\in\mathcal{M}}\tilde{I}_i(X)+\sum_{i\in\mathcal{M}}\tilde{\pi}_i\right)+\sum_{i\in\mathcal{M}}^{n}\rho_i(\tilde{I}_i(X)-\tilde{\pi}_i)$$
$$<\rho^{PH}\left(X-\sum_{i\in\mathcal{M}}I_i(X)+\sum_{i\in\mathcal{M}}\pi_i\right)+\sum_{i\in\mathcal{M}}^{n}\rho_i(I_i(X)-\pi_i)\,,$$

which, by translation invariance, yields

$$\rho^{PH}\left(X-\sum_{i\in\mathcal{M}}\tilde{I}_i(X)\right)+\sum_{i\in\mathcal{M}}^{n}\rho_i(\tilde{I}_i(X))<\rho^{PH}\left(X-\sum_{i\in\mathcal{M}}I_i(X)\right)+\sum_{i\in\mathcal{M}}\rho_i(I_i(X)),$$

and thus

$$\sum_{i=0}^{n}\rho_i(\tilde{I}_i(X))<\sum_{i=0}^{n}\rho_i(I_i(X))\,.$$

Therefore $\vec{I}$ does not solve (4.18). □

4.6.10 Proof of Proposition 4.4.9

First we check feasibility. We have

$$\sum_{i=0}^{n}I_i(z)=\sum_{i=0}^{n}\int_0^z\gamma_i(z)\,dz=\int_0^z\sum_{i=0}^{n}\gamma_i(z)\,dz=\int_0^z 1\,dz=z\,.$$

Therefore $\sum_{i=0}^{n}X_i=\sum_{i=0}^{n}I_i(X)=X$. Also, since the derivative of I_i is non-negative, I_i is increasing, and so $I_i(z)$ is comonotonic with X Hence, $\vec{I}\in\vec{\mathcal{I}}$, so it is feasible.

Next, we show that $\vec{I}$ solves (4.18). Let $\vec{\tilde{I}}$ be any other profile of indemnities in $\vec{\mathcal{I}}$. We have

$$\sum_{i=0}^{n}\rho_i(\tilde{I}_i(z))=\sum_{i=0}^{n}\int_0^\infty\tau_i(X>z)\tilde{I}_i'(z)\,dz=\int_0^\infty\sum_{i=0}^{n}\tau_i(X>z)\tilde{I}_i'(z)\,dz$$
$$\geq\int_0^\infty\sum_{i=0}^{n}\min_{j\in\mathcal{M}\cup\{0\}}\{\tau_j(X>z)\}\tilde{I}_i'(z)\,dz \tag{4.22}$$

$$
\begin{aligned}
&= \int_0^\infty \min_{j\in\mathcal{M}\cup\{0\}}\{\tau_j(X>z)\}\sum_{i=0}^n \tilde{I}_i'(z)\,dz \\
&= \int_0^\infty \min_{j\in\mathcal{M}\cup\{0\}}\{\tau_j(X>z)\}\,dz \qquad (4.23)\\
&= \int_0^\infty \left[\min_{j\in\mathcal{M}\cup\{0\}}\{\tau_j(X>z)\}\cdot 1 + \sum_{i\in\mathcal{L}_z^C}\tau_i(X>z)\cdot 0\right] dz \\
&= \int_0^\infty \left[\min_{j\in\mathcal{M}\cup\{0\}}\{\tau_j(X>z)\}\cdot \sum_{i\in\mathcal{L}_z}\gamma_i(z) + \sum_{i\in\mathcal{L}_z^C}\tau_i(X>z)\cdot\gamma_i(z)\right] dz \\
&= \int_0^\infty \left[\sum_{i\in\mathcal{L}_z}\min_{j\in\mathcal{M}\cup\{0\}}\{\tau_j(X>z)\}\cdot\gamma_i(z) + \sum_{i\in\mathcal{L}_z^C}\tau_i(X>z)\cdot\gamma_i(z)\right] dz \\
&= \int_0^\infty \sum_{i\in\mathcal{L}_z}\tau_i(X>z)\cdot\gamma_i(z) + \sum_{i\in\mathcal{L}_z^C}\tau_i(X>z)\cdot\gamma_i(z)\,dz \\
&= \int_0^\infty \sum_{i=0}^n \tau_i(X>z)\gamma_i(z)\,dz = \sum_{i=0}^n\int_0^\infty \tau_i(X>z)\gamma_i(z)\,dz = \sum_{i=0}^n \rho_i(I_i(z))\,.
\end{aligned}
$$

Therefore $\vec{I}$ solves (4.18).

We now show the converse. Suppose that $\vec{\tilde{I}}$ is a feasible indemnity structure not of the specified form: that is, $\sum_{i\in\mathcal{L}_z^C}\tilde{I}_i'(z) > 0$ on a set $\mathcal{A}$ of positive measure. Then for every z in $\mathcal{A}$, we have

$$
\begin{aligned}
\sum_{i=0}^n \tau_i(X>z)\tilde{I}_i'(z) &= \sum_{i\in\mathcal{L}_z}\tau_i(X>z)\tilde{I}_i'(z) + \sum_{i\in\mathcal{L}_z^C}\tau_i(X>z)\tilde{I}_i'(z) \\
&= \sum_{i\in\mathcal{L}_z}\min_{j\in\mathcal{M}\cup\{0\}}\{\tau_j(X>z)\}\tilde{I}_i'(z) + \sum_{i\in\mathcal{L}_z^C}\tau_i(X>z)\tilde{I}_i'(z) \\
&> \sum_{i\in\mathcal{L}_z}\min_{j\in\mathcal{M}\cup\{0\}}\{\tau_j(X>z)\}\tilde{I}_i'(z) + \sum_{i\in\mathcal{L}_z^C}\min_{j\in\mathcal{M}\cup\{0\}}\{\tau_j(X>z)\}\tilde{I}_i'(z) \\
&= \sum_{i=0}^n \min_{j\in\mathcal{M}\cup\{0\}}\{\tau_j(X>z)\}\tilde{I}_i'(z)\,,
\end{aligned}
$$

where the strict inequality follows because $\tilde{I}_i'(z)$ are not all zero. Taking the integral over

the set $\mathcal{A}$ gives

$$\int_{\mathcal{A}} \sum_{i=0}^{n} \tau_i(X > z)\tilde{I}_i'(z)\,dz > \int_{\mathcal{A}} \sum_{i=0}^{n} \min_{j\in\mathcal{M}\cup\{0\}}\{\tau_j(X > z)\}\tilde{I}_i'(z)\,dz\,,$$

where the inequality is strict since $\mathcal{A}$ has positive measure. Therefore in this case, the inequality (4.22) is strict, so $\vec{\tilde{I}}$ does not solve (4.18). □

4.6.11 Proof of Lemma 4.4.10

Since $\mathfrak{I}^* \in \beth$ and $(\nu_1^*, \dots, \nu_m^*) \in \gimel$, we know that $(\vec{I}^*, \vec{\pi}^*)$ is IR by Theorem 4.4.4.

For PO, by Proposition 4.4.8, it suffices to show that the indemnities $\vec{I}^*$ solve (4.18). We have seen in (4.23) that the value of (4.18) is

$$\int_0^\infty \min_{j\in\mathcal{M}\cup\{0\}}\{\tau_j(X > z)\}\,dz\,.$$

Therefore it suffices to show that

$$\sum_{i=0}^{n} \rho_i(I_i^*(X)) = \int_0^\infty \min_{j\in\mathcal{M}\cup\{0\}}\{\tau_j(X > z)\}\,dz\,.$$

First recall that $I_0^*(x) = \int_0^x \gamma_0^*(z)\,dz$ as defined in (4.13). Then we can write

$$\sum_{i=0}^{n} \rho_i(I_i^*(X)) = \sum_{i=0}^{n} \int_0^\infty \tau_i(X > z)\gamma_i^*(z)\,dz = \int_0^\infty \sum_{i=0}^{n} \tau_i(X > z)\gamma_i^*(z)\,dz\,.$$

We show that for all $z \in \mathbb{R}_+$, we have $\sum_{i=0}^n \tau_i(X > z)\gamma_i^*(z) = \min_{j\in\mathcal{M}\cup\{0\}}\{\tau_j(X > z)\}$. It suffices to show that $\gamma_i^*(z) > 0$ implies that $\tau_i(X > z) = \min_{j\in\mathcal{M}\cup\{0\}} \tau_j(X > z)$. We consider two cases:

1. $\underline{i = 0}$: Then $\gamma_0^*(z) > 0$ implies that $g(\mathbb{P}(X > z)) \le \bar{\tau}(X > z)$. If this inequality is strict, then we automatically have $g(\mathbb{P}(X > z)) = \min_{j\in\mathcal{M}\cup\{0\}} \tau_j(X > z)$. Otherwise, if $g(\mathbb{P}(X > z)) = \bar{\tau}(X > z)$, then since every insurer is quoting the same price $\bar{\tau}(X > z)$, generous distribution implies that there does not exist $k \in \mathcal{M}$ such that $\tau_k(X > z) < g(\mathbb{P}(X > z))$. Therefore $g(\mathbb{P}(X > z)) = \min_{j\in\mathcal{M}\cup\{0\}} \tau_j(X > z)$ as desired.

2. $\underline{i \in \mathcal{M}}$: In this case, since every insurer is using the strategy $\bar{\tau}$, there does not exist $k \in \mathcal{M}$ such that $\tau_k(X > z) < \tau_i(X > z)$, so $\tau_i(X > z) = \min_{j \in \mathcal{M}} \tau_j(X > z)$.

   It remains to show that $g(\mathbb{P}(X > z)) \geq \tau_i(X > z)$. To this end, note that $\gamma_i^*(z) > 0$ implies $g(\mathbb{P}(X > z)) \geq \bar{\tau}(X > z)$. Then there exist $k' \in \mathcal{M}$ such that $\tau_{k'}(X > z) \leq \bar{\tau}(X > z) \leq g(\mathbb{P}(X > z))$, which implies that $\tau_i(X > z) \leq g(\mathbb{P}(X > z))$. Therefore $\tau_i(X > z) = \min_{j \in \mathcal{M} \cup \{0\}} \tau_j(X > z)$ as desired.

To conclude the proof, we see that since $\tau_i(X > z) = \min_{j \in \mathcal{M} \cup \{0\}} \{\tau_j(X > z)\}$ whenever $\gamma_i^*(z) > 0$, we have

$$\begin{aligned}\sum_{i=0}^{n} \tau_i(X > z)\gamma_i^*(z) &= \sum_{i \in \mathcal{M}, \gamma_i^*(z) > 0} \tau_i(X > z)\gamma_i^*(z) = \sum_{i \in \mathcal{M}, \gamma_i^*(z) > 0} \min_{j \in \mathcal{M} \cup \{0\}} \{\tau_j(X > z)\}\gamma_i^*(z) \\ &= \min_{j \in \mathcal{M} \cup \{0\}} \{\tau_j(X > z)\} \sum_{i \in \mathcal{M}, \gamma_i^*(z) > 0} \gamma_i^*(z) = \min_{j \in \mathcal{M} \cup \{0\}} \{\tau_j(X > z)\}\,.\end{aligned}$$

Therefore,

$$\sum_{i=0}^{n} \rho_i(I_i^*(X)) = \int_0^\infty \min_{j \in \mathcal{M} \cup \{0\}} \{\tau_j(X > z)\}\, dz\,.$$

□

4.6.12 Proof of Theorem 4.4.11

By Theorem 4.4.4, we know that $(\nu_1^*, \ldots, \nu_m^*, \mathfrak{I}^*)$ induces an IR allocation. It remains to show PO.

First, let $(\bar{I}_1^*, \ldots, \bar{I}_n^*) := \mathfrak{I}^*(\bar{\tau}, \bar{\tau}, \ldots, \bar{\tau})$. Then since $\mathfrak{I}^* \in \beth$ and $(\bar{\tau}, \bar{\tau}, \ldots, \bar{\tau}) \in \daleth$, we have

$$\rho_i(\bar{I}_i^*(X) - \pi^{\bar{\tau}}(\bar{I}_i^*(X))) = \int_{\mathcal{Z}_i} \tau_i(X > z) - \bar{\tau}(X > z)\, dz,$$

$$\rho^{PH}\left(X - \sum_{i \in \mathcal{M}} \bar{I}_i^*(X) + \sum_{i \in \mathcal{M}} \pi^{\bar{\tau}}(\bar{I}_i^*(X))\right) = \int_{\mathcal{Z}_{PH}} g(\mathbb{P}(X > z))\, dz + \int_{\mathcal{Z}_{PH}^C} \bar{\tau}(X > z)\, dz\,,$$

by Proposition 4.3.9 and Lemma 4.4.3. Summing these inequalities and applying Lemma 4.4.10, we have

$$\int_{\mathcal{Z}_i} \tau_i(X > z) - \bar{\tau}(X > z)\, dz + \int_{\mathcal{Z}_{PH}} g(\mathbb{P}(X > z))\, dz + \int_{\mathcal{Z}_{PH}^C} \bar{\tau}(X > z)\, dz$$

$$
\begin{aligned}
&= \rho^{PH}\left(X - \sum_{i\in\mathcal{M}} \bar{I}_i^*(X) + \sum_{i\in\mathcal{M}} \pi^{\bar{\tau}}(\bar{I}_i^*(X))\right) + \sum_{i\in\mathcal{M}} \rho_i(\bar{I}_i^*(X) - \pi^{\bar{\tau}}(\bar{I}_i^*(X))) \\
&= \rho^{PH}\left(X - \sum_{i\in\mathcal{M}} \bar{I}_i^*(X)\right) + \sum_{i\in\mathcal{M}} \pi^{\bar{\tau}}(\bar{I}_i^*(X)) + \sum_{i\in\mathcal{M}} \rho_i(\bar{I}_i^*(X)) - \sum_{i\in\mathcal{M}} \pi^{\bar{\tau}}(\bar{I}_i^*(X)) \\
&= \rho^{PH}\left(X - \sum_{i\in\mathcal{M}} \bar{I}_i^*(X)\right) + \sum_{i\in\mathcal{M}} \rho_i(\bar{I}_i^*(X)) = \sum_{i=0}^{n} \rho_i(\bar{I}_i^*(X)) \\
&= \int_0^\infty \min_{j\in\mathcal{M}\cup\{0\}} \{\tau_j(X > z)\}\, dz\,.
\end{aligned}
$$

Now suppose that $\mathfrak{I}^* \in \beth$ and $(\nu_1^*, \ldots, \nu_m^*) \in \daleth$. As before, we use the notation $(I_1^*, \ldots, I_m^*) = \mathfrak{I}^*(\nu_1^*, \ldots, \nu_m^*)$. By Proposition 4.3.9 and Lemma 4.4.3, we have

$$
\begin{aligned}
\sum_{i=0}^{n} \rho_i(I_i^*(X)) &= \rho^{PH}\left(X - \sum_{i\in\mathcal{M}} I_i^*(X) + \sum_{i\in\mathcal{M}} \pi^{\nu_i^*}(I_i^*(X))\right) + \sum_{i\in\mathcal{M}} \rho_i\left(I_i^*(X) - \pi^{\nu_i^*}(I_i^*(X))\right) \\
&= \int_{\mathcal{Z}_i} \tau_i(X > z) - \bar{\tau}(X > z)\, dz + \int_{\mathcal{Z}_{PH}} g(\mathbb{P}(X > z))\, dz + \int_{\mathcal{Z}_{PH}^C} \bar{\tau}(X > z)\, dz\,,
\end{aligned}
$$

so the resulting allocation solves (4.18). Therefore by Proposition 4.4.8, the strategy $(\nu_1^*, \ldots, \nu_m^*, \mathfrak{I}^*)$ induces a PO allocation, as desired. □

4.6.13 Proof of Proposition 4.4.14

Recall that $\mathcal{Z}_i = \{z \in \mathbb{R}_+ : \tau_i(X > z) < \bar{\tau}(X > z)\}$. Note that this condition is equivalent to $\tau_i(X > z) < \tau_j(X > z)$ for $j \in (\mathcal{M}\cup\{0\})\setminus\{i\}$ – that is, $\tau_i(X > z)$ attains the minimum $\min_j \tau_j(X > z)$ uniquely. Then for all $z \in \mathcal{Z}_i$, we have $\gamma_i^*(z) = 1$ by Proposition 4.4.9. Now define

$$
\bar{\mathcal{Z}}_i := \{z \in \mathbb{R}_+ : \tau_i(X > z) = \min_j \tau_j(X > z)\} \setminus \mathcal{Z}_i\,.
$$

That is, $\bar{\mathcal{Z}}_i$ is the set over which $\tau_i(X > z)$ attains the minimum, but not uniquely. Note that on this set, $\tau_i(X > z) = \bar{\tau}_i(X > z)$. By Proposition 4.4.9, we have that $\gamma_i^*(z) = 0$ on $(\mathcal{Z}_i \cup \bar{\mathcal{Z}}_i)^C$. By Assumption 4.4.13, we have

$$
\int_{\mathcal{Z}_i} \tau_i(X > z) - \bar{\tau}(X > z)\, dz = \rho_i(I_i^*(X) - \pi_i^*)
$$

$$
\begin{aligned}
&= \int_0^\infty \tau_i(X > z)\gamma_i^*(z)\,dz - \pi_i^* \\
&= \int_{\mathcal{Z}_i} \tau_i(X > z)\cdot 1\,dz + \int_{\bar{\mathcal{Z}}_i} \tau_i(X > z)\gamma_i^*(z)\,dz \\
&\qquad + \int_{(\mathcal{Z}_i \cup \bar{\mathcal{Z}}_i)^C} \tau_i(X > z)\cdot 0\,dz - \pi_i^* \\
&= \int_{\mathcal{Z}_i} \tau_i(X > z)\,dz + \int_{\bar{\mathcal{Z}}_i} \tau_i(X > z)\gamma_i^*(z)\,dz - \pi_i^*\,.
\end{aligned}
$$

Subtracting $\int_{\mathcal{Z}_i} \tau_i(X > z)\,dz$ from both sides and rearranging yields

$$
\begin{aligned}
\pi_i^* &= \int_{\bar{\mathcal{Z}}_i} \tau_i(X > z)\gamma_i^*(z)\,dz + \int_{\mathcal{Z}_i} \bar{\tau}(X > z)\,dz \\
&= \int_{\bar{\mathcal{Z}}_i} \bar{\tau}(X > z)\gamma_i^*(z)\,dz + \int_{\mathcal{Z}_i} \bar{\tau}(X > z)\,dz \\
&= \int_{\bar{\mathcal{Z}}_i} \bar{\tau}(X > z)\gamma_i^*(z)\,dz + \int_{\mathcal{Z}_i} \bar{\tau}(X > z)\cdot 1\,dz + \int_{(\mathcal{Z}_i \cup \bar{\mathcal{Z}}_i)^C} \tau_i(X > z)\cdot 0\,dz \\
&= \int_0^\infty \nu_i^*(X > z)\gamma_i^*(z)\,dz = \int I_i^*(X)\,d\nu_i^*\,,
\end{aligned}
$$

as desired. □

4.6.14 Proof of Proposition 4.4.15

By Proposition 4.4.14 and Lemma 4.4.3, we have

$$
\begin{aligned}
\rho^{PH}\left(X - \sum_{i\in\mathcal{M}} I_i^*(X) + \sum_{i\in\mathcal{M}} \pi^{\nu_i^*}(I_i^*(X))\right) &= \rho^{PH}\left(X - \sum_{i\in\mathcal{M}} I_i^*(X) + \sum_{i\in\mathcal{M}} \pi_i^*\right) \\
&= \int_{\mathcal{Z}_{PH}} g(\mathbb{P}(X > z))\,dz + \int_{\mathcal{Z}_{PH}^C} \bar{\tau}(X > z)\,dz \\
&= \min_{I\in\vec{\mathcal{I}}} \left\{\rho^{PH}\left(X - \sum_{i\in\mathcal{M}} I_i(X) + \sum_{i\in\mathcal{M}} \pi^{\nu_i^*}(I_i(X))\right)\right\}\,.
\end{aligned}
$$

□

4.6.15 Proof of Proposition 4.4.16

Suppose for the sake of contradiction that $\vec{I}^*$ does not distribute generously. Then there exists a $k \in \mathcal{M} \cup \{0\}$ and a set of positive measure $\mathcal{A}$ such that for $z \in \mathcal{Z}$, we have

$$\gamma_k^*(z) > 0\,,$$

and there exists a $k' \in (\mathcal{M} \cup \{0\}) \setminus \{k\}$ such that

$$\tau_{k'}(X > z) < \tau_k(X > z)\,.$$

Then $\sum_{i=0}^n \gamma_i^*(z)\tau_i(X > z) > \sum_{i=0}^n \gamma_i^*(z) \min_{j\in\mathcal{M}\cup\{0\}} \tau_j(X > z) = \min_{j\in\mathcal{M}\cup\{0\}} \tau_j(X > z)$, so

$$\begin{aligned}
\sum_{i=0}^n \rho_i(I_i^*(X)) &= \sum_{i=0}^n \int_0^\infty \tau_i(X > z)\gamma_i^*(z)\,dz = \int_0^\infty \sum_{i=0}^n \tau_i(X > z)\gamma_i^*(z)\,dz \\
&= \int_{\mathcal{A}} \sum_{i=0}^n \tau_i(X > z)\gamma_i^*(z)\,dz + \int_{\mathcal{A}^C} \sum_{i=0}^n \tau_i(X > z)\gamma_i^*(z)\,dz \\
&> \int_{\mathcal{A}} \min_{j\in\mathcal{M}\cup\{0\}} \tau_j(X > z)\,dz + \int_{\mathcal{A}^C} \sum_{i=0}^n \tau_i(X > z)\gamma_i^*(z)\,dz \\
&\geq \int_{\mathcal{A}} \min_{j\in\mathcal{M}\cup\{0\}} \tau_j(X > z)\,dz + \int_{\mathcal{A}^C} \sum_{i=0}^n \min_{j\in\mathcal{M}\cup\{0\}} \tau_j(X > z)\gamma_i^*(z)\,dz \\
&\geq \int_0^\infty \min_{j\in\mathcal{M}\cup\{0\}} \tau_j(X > z)\,dz\,.
\end{aligned}$$

Hence, $\vec{I}^*$ does not solve (4.18), so $\vec{I}^*$ is not PO by Proposition 4.4.8, which contradicts our initial assumption. □

4.7 Additional Examples of Equilibria

4.7.1 An SPNE Not Characterized by Theorem 4.3.11

As an example of an SPNE that is not characterized by Theorem 4.3.11, consider the following example for $m = 2$. We construct an SPNE such that the insurers' strategy does not satisfy (4.15), and therefore not in $\beth$. Suppose for all $z \in \mathbb{R}_+$, we have

$$\tau_1(X > z) < \tau_2(X > z) < g(\mathbb{P}(X > z))\,.$$

Let $\mathfrak{I}^* \in \urcorner$, and define ν_1, ν_2 by

$$\nu_1 = \nu_2 = \tau_1 \,.$$

Note that by construction, since $\bar{\tau} = \tau_2 > \tau_1 = \underline{\nu}$, this does not satisfy (4.15). We claim that $(\nu_1, \nu_2, \mathfrak{I}^*)$ is an SPNE. Let $(I_1^*, I_2^*) = \mathfrak{I}^*(\nu_1, \nu_2)$. It is easy to verify that since the policyholder distributes generously, $I_1^*(X) = X$ and $I_2^*(X) = 0$. For insurer 1, we have $\rho_1(I_1^*(X) - \pi^{\tau_1}(I_1^*(X))) = 0$. It is impossible for insurer 1 to profit, as quoting a higher price than their true preferences τ_1 will result in being undercut by insurer 2. On the other hand, we have $\rho_2(I_2^*(X) - \pi^{\tau_1}(I_2^*(X))) = 0$ as well. It is impossible for insurer 2 to profit, as quoting a higher price than their true preferences τ_2 will result in being undercut by insurer 1.

However, by Remark 4.3.14, we see that $\nu_1^* = \nu_2^* = \bar{\tau} = \tau_2$ is also an SPNE, which charges a strictly higher price for insurance. From the perspective of the insurers (in particular insurer 1), this SPNE would be preferable.

4.7.2 A Remark on NE vs. SPNE

Throughout this thesis, we focus on finding SPNEs, as opposed to the perhaps more familiar notion of NE. It is clear from Definition 4.2.3 that all SPNEs are NEs. However, we show in this section that the converse statement is not true, even within our model where the game structure is not overly complicated.

Suppose that the strategy $(\nu_1^*, \ldots, \nu_m^*, \mathfrak{I}^*)$ is an SPNE, as given in Theorem 4.3.11. We construct a strategy as follows. Define by $\hat{\nu} \in \mathcal{C}$ a capacity such that:

1. $\hat{\nu} \neq \nu_i^*$ for all $i \in \mathcal{M}$; and,
2. $\hat{\nu}(X > z) > g(\mathbb{P}(X > z))$ on a set of positive measure.

Now define by $\hat{\mathfrak{I}}$ a strategy in $(\vec{\mathcal{I}})^{\mathcal{C}^m}$ such that

$$\begin{aligned} \hat{\mathfrak{I}} : \mathcal{C}^m &\to \vec{\mathcal{I}} \\ (\hat{\nu}, \hat{\nu}, \ldots, \hat{\nu}) &\mapsto 0 \,, \\ (\nu_1, \ldots, \nu_m) &\mapsto \mathfrak{I}^*(\nu_1, \ldots, \nu_m), \quad \text{otherwise.} \end{aligned}$$

Under the strategy $\hat{\mathfrak{I}}$, the policyholder refuses to do business with any insurer in the scenario that all insurers quote the price $\hat{\nu}$. This is admittedly strange behaviour, which

is indeed not optimal for the policyholder. Since $\hat{\nu}(X > z) > g(\mathbb{P}(X > z))$ on a set of positive measure, the policyholder can benefit from insurance that is priced via $\hat{\nu}$. Hence, 0 does not solve

$$\min_{I \in \vec{\mathcal{I}}} \left\{ \rho^{PH} \left(X - \sum_{i \in \mathcal{M}} I_i(X) + \sum_{i \in \mathcal{M}} \pi^{\hat{\nu}}(I_i(X)) \right) \right\},$$

so $(\nu_1^*, \dots, \nu_m^*, \hat{\mathfrak{I}})$ is not an SPNE.

However, we can see that $(\nu_1^*, \dots, \nu_m^*, \hat{\mathfrak{I}})$ is an NE. Let $\tau \in \mathcal{C}$ be any non-negative capacity, and $i \in \mathcal{M}$. By Proposition 4.3.10, we have

$$\begin{aligned}
\rho_i(\hat{\mathfrak{I}}_i(\tau, \nu_{-i}^*)(X)) - \rho^{\tau}(\hat{\mathfrak{I}}_i(\tau, \nu_{-i}^*)(X)) &= \rho_i(\mathfrak{I}_i^*(\tau, \nu_{-i})(X)) - \rho^{\tau}(\mathfrak{I}_i^*(\tau, \nu_{-i})(X)) \\
&\geq \rho_i(\mathfrak{I}_i^*(\nu_1^*, \dots, \nu_m^*)(X)) - \pi^{\nu_i^*}(\mathfrak{I}_i^*(\nu_1^*, \dots, \nu_m^*)(X)) \\
&= \rho_i(\hat{\mathfrak{I}}_i(\nu_1^*, \dots, \nu_m^*)(X)) - \pi^{\nu_i^*}(\hat{\mathfrak{I}}_i(\nu_1^*, \dots, \nu_m^*)(X)) .
\end{aligned}$$

Also, by construction, $\hat{\mathfrak{I}}(\nu_1^*, \dots, \nu_m^*) \in \mathcal{I}(\nu_1^*, \dots, \nu_m^*)$, so it solves

$$\min_{I \in \vec{\mathcal{I}}} \left\{ \rho^{PH} \left(X - \sum_{i \in \mathcal{M}} I_i(X) + \sum_{i \in \mathcal{M}} \pi^{\nu_i^*}(I_i(X)) \right) \right\} .$$

Hence, $(\nu_1^*, \dots, \nu_m^*, \hat{\mathfrak{I}})$ is an NE.

Although this is an NE, we see that the strategy of the policyholder is not a reasonable one. A possible interpretation is that the policyholder's sub-optimal behaviour when the pricing capacity is $\hat{\nu}$ is a threat to the insurers, to dissuade them from actually choosing such a pricing rule. The notion of SPNE can be seen as a refinement that eliminates this possibility. That is, we assume that the policyholder is unable or unwilling to issue such threats, or that the insurers correctly identify that the policyholder will never follow through with such threats.

Chapter 5

Efficiency with Risk-Averse Monetary Utilities

5.1 Introduction

In both this chapter and the next, we examine Pareto efficiency within decentralized insurance markets. In a departure from the risk allocation problem formulation in the insurance literature, this chapter considers the setting of utility maximization and the optimal allocation problem thereof. This setting may be more familiar to mathematical economists, and its existing literature is vast, as outlined below.

In a context of choice under uncertainty, risk aversion is defined as consistency with second-order stochastic dominance (SSD), as in Rothschild and Stiglitz [97], for instance. Preferences that are SSD preserving are known as strongly risk averse preferences, and their study in the literature has been extensive. The study of optimal allocations in pure-exchange economies with risk-averse agents has its roots in the seminal work of Borch [24] and Wilson [110] in the framework of Expected Utility Theory (EUT), who show that each individual's optimal allocation must be a non-decreasing deterministic function of the aggregate endowment in the market. In particular, optimal allocations are *comonotone*. This important property of allocations has been shown to extend to more general models of preferences. A landmark result in this direction is the *comonotone improvement theorem* of Landsberger and Meilijson [75], who show that for any given allocation of an aggregate risk between two agents, a Pareto-improving comonotone feasible allocation is always possible. The only requirement is that the agents' preferences must exhibit strong risk aversion, or more generally, be *Schur concave*, i.e., consistent with the concave stochastic order. While

the original result was established for a discrete state space, the comonotone improvement result was extended by Dana [43] to atomless probability spaces, by Ludkovski and Rüschendorf [83] to more than two agents and unbounded random variables, and by Carlier et al. [31] to vectors of random variables. Recently, Denuit et al. [48] provide a proof of the comonotone improvement result in the general case, using a constructive algorithmic approach. Comonotonicity of allocations is important enough to warrant the study of the so-called *comonotone market*, an incomplete market in which only comonotone allocations are available (see, e.g., Boonen et al. [21]).

The comonotone improvement theorem laid the groundwork for an extensive literature focusing on the characterization of Pareto-optimal allocations under different models of agents' preferences. The original result of Borch [24] includes an explicit formula for optimal allocations as a function of each agent's decreasing marginal utility of wealth (the so-called Borch rule). Allocations in markets with Knightian uncertainty, as modeled through either the Choquet Expected Utility model of Schmeidler [101] or the Maxmin Expected Utility model of Gilboa and Schmeidler [65], are studied by Chateauneuf et al. [33], [40], Tsanakas and Christofides [104], De Castro and Chateauneuf [44], and Beissner and Werner [12], for instance. Through the relationship of these models with the classical EUT framework, it is possible to characterize the shape of Pareto-optimal allocations. However, for more general models of preferences such as the popular class of monetary utilities (i.e., monotone, concave, and translation-invariant utility functionals), explicit characterizations of Pareto-optimal allocations are more difficult to obtain. Instead, the focus has been to show the existence of Pareto-optimal allocations, within the set of comonotone allocations. A seminal result in this direction is due to Jouini et al. [69], who prove existence of comonotone allocations between two agents whose preferences are represented by law-invariant monetary utilities. In particular, given translation invariance, identifying Pareto optima reduces to solving the *sup-convolution* of preferences, as observed by Barrieu and el Karoui [11], for instance. Filipović and Svindland [56] extend this result to markets with more than two agents, as well as to more general spaces of random variables that allow for unbounded endowments. Further results concerning existence of Pareto optima include Acciaio [3] for non-monotone preferences, Mastrogiacomo and Rosazza Gianin [88] for quasi-concave utilities, and Ravanelli and Svindland [94] for a class of law-invariant variational preferences. Recently, it has been shown that for SSD-preserving and translation invariant preferences, the assumption of concavity can be relaxed (e.g., Mao and Wang [85] and Liebrich [79]). Pareto-optimal allocations have also be proven to exist in more complicated market models, including markets with a portfolio of multiple assets (e.g., Kiesel and Rüschendorf [71, 72]) and markets with predetermined sets of admissible endowments (e.g., Liebrich and Svindland [80]). However, results concerning the shape

of Pareto-optimal allocations are more rare in this strand of the literature, and require more stringent assumptions on agents' preferences. Notably, Embrechts et al. [53] and Liu et al. [81] provide closed-form characterizations of Pareto-optimal allocations when preferences are given by quantile-based risk measures such as the expected shortfall, and Liu [82] extends these results to distortion risk measures.

In this chapter, our main result (Theorem 5.4.11) provides a characterization of Pareto optima for law-invariant and positively homogeneous monetary utilities (hence SSD preserving). Our characterization relies on the dual representation of these functionals in the spirit of Kusuoka [74], who shows that these functionals can be expressed as an infimum of expectations over a certain set of probability measures. This representation is generalized to concave Schur-concave functionals by Dana [38], who also provides a representation in terms dual utilities. The key to our result is expressing the sup-convolution problem in terms of the dual representation of preferences. The supremum from the sup-convolution and the infimum from the dual representation can be exchanged, and the resulting expression simplifies to yield our characterization. Notably, this transforms the domain of the optimization problem from the space of feasible allocations to the space of distortion functions. Once an appropriate problem is solved, the structure of the (comonotone) Pareto-optimal allocations is determined explicitly.

In the context of risk measures, the dual concept to a positively homogeneous monetary utility is the notion of a coherent risk measure introduced by Artzner et al. [6]. Indeed, these notions are equivalent up to a change in sign: if U is a positively homogeneous monetary utility, then $\rho := -U$ is a coherent risk measure. Our result therefore also provides a characterization of Pareto optima in the case of a risk-sharing problem among multiple agents who evaluate risk via law-invariant coherent risk measures. As a special case, we show that when each risk measure is comonotone-additive, we recover the closed-form characterization obtained by Liu [82]. In a numerical illustration, we examine Pareto-optimal risk-sharing allocations in a market where a regulatory entity imposes capital requirements to all agents. In such a market, we apply an algorithm based on our main result to characterize the unique comonotone Pareto-optimal allocation.

The remainder of this chapter is structured as follows. Section 5.2 introduces the optimal allocation problem. Section 5.3 provides some background on the comonotone improvement result for Schur-concave functionals, and motivates the notion of a comonotone market. Our main characterization result for law-invariant and positively homogeneous monetary utilities is provided in Section 5.4. Section 5.5 provides an application to risk-sharing markets with coherent risk measures, and includes some numerical illustrations. Section 5.6 concludes. The proofs of our main results, as well as some related analysis, can be found in Section 5.7.

5.2 Problem Setup

In this chapter, we take $\mathcal{X} = L^\infty(\Omega, \Sigma, \mathbb{P})$, the set of essentially bounded random variables on the atomless probability space $(\Omega, \Sigma, \mathbb{P})$. There are $n \in \mathbb{N}$ agents wishing to reallocate their initial endowments among themselves without central authority involvement. For each $i \in \mathcal{N} := \{1, \dots, n\}$, let $X_i \in \mathcal{X}$ denote the initial endowment of the i-th agent. We consider a one-period economy, where all financial gains and losses are realized at the end of the period. As before, we denote the aggregate endowment by $S := \sum_{i=1}^n X_i$. Since S is a sum of essentially bounded random variables, it is also essentially bounded by some constant $M < \infty$.

At the end of the period, the aggregate endowment S is redistributed among the agents in the market. For each $i \in \mathcal{N}$, we denote the end-of-period, post-transfer payout of agent i by Y_i. Therefore, the *ex ante* admissible set of decision variables, henceforth referred to as the set of allocations, is given by

$$\mathcal{A} = \left\{ \{Y_i\}_{i=1}^n \in \mathcal{X}^n : \sum_{i=1}^n Y_i = S \right\}.$$

We assume that each agent $i \in \mathcal{N}$ has a preference $\succ_i$ on $\mathcal{X}$ that admits a representation by a utility functional $U_i : \mathcal{X} \to \mathbb{R}$. We recall below some standard properties of such functionals, which are analogous to those properties of risk measures given in Definition 2.2.1.

Definition 5.2.1. A utility functional $U : \mathcal{X} \to \mathbb{R}$ is said to be:

- *Monotone* if $U(Z_1) \leq U(Z_2)$, for all $Z_1, Z_2 \in \mathcal{X}$ such that $Z_1 \leq Z_2$.
- *Translation invariant* if $U(Z + c) = U(Z) + c$, for all $Z \in \mathcal{X}$ and $c \in \mathbb{R}$.
- *Concave* if for all $Z_1, Z_2 \in \mathcal{X}$ and $t \in [0, 1]$,

$$t\, U(Z_1) + (1 - t)\, U(Z_2) \leq U(t\, Z_1 + (1 - t)\, Z_2).$$

- *Positively Homogeneous* if $U(t\, Z) = t\, U(Z)$, for all $Z \in \mathcal{X}$ and $t \geq 0$.
- *Law-invariant* if for all $Z_1, Z_2 \in \mathcal{X}$ with the same distribution under $\mathbb{P}$, we have $U(Z_1) = U(Z_2)$.

The definitions of individual rationality and Pareto optimality are reiterated in this context. Recall the convention that agents prefer higher values of utility functions to lower ones. Furthermore, we also introduce the notion of weak Pareto optimality in this setting.

Definition 5.2.2. An allocation $\{Y_i^*\}_{i=1}^n \in \mathcal{A}$ is said to be:

- **Individually Rational** (IR) if it incentivizes the parties to participate in the market, that is,
$$U_i\left(Y_i^*\right) \geq U_i\left(X_i\right), \quad \forall\, i \in \mathcal{N}.$$
- **Pareto Optimal** (PO) if it is IR and there does not exist any other allocation $\{Y_i\}_{i=1}^n$ such that
$$U_i\left(Y_i\right) \geq U_i\left(Y_i^*\right), \quad \forall\, i \in \mathcal{N},$$
with at least one strict inequality.
- **Weakly Pareto Optimal** if it is IR and there does not exist any other allocation $\{Y_i\}_{i=1}^n$ such that
$$U_i\left(Y_i\right) > U_i\left(Y_i^*\right), \quad \forall\, i \in \mathcal{N}.$$

It follows immediately that if an allocation is Pareto optimal, then it is weakly Pareto optimal. The converse holds under fairly mild conditions on the monotonicity of the preferences. For example, Xia [111] shows that weak Pareto optimality and Pareto optimality are equivalent in the expected-utility setting when utility functions are strictly increasing. For our purposes, the following condition will suffice. A similar condition can be found in Ravanelli and Svindland [94], where it is used for the same purpose.

Assumption 5.2.3. For all $i \in \mathcal{N}$ and $Z \in \mathcal{X}$, the function
$$\begin{aligned} \mathbb{R} &\to \mathbb{R} \\ c &\mapsto U_i(Z + c) \end{aligned}$$
is continuous, strictly increasing, and satisfies $\lim\limits_{c\to\infty} U_i(Z + c) = \infty$.

Lemma 5.2.4. *Under Assumption 5.2.3, an allocation $\{Y_i^*\}_{i=1}^n \in \mathcal{A}$ is PO if and only if it is weakly Pareto-optimal.*

Proof. Clearly, Pareto optimality implies weak Pareto optimality. For the reverse direction, suppose that $\{Y_i^*\}_{i=1}^n \in \mathcal{A}$ is not PO. Then there exists an IR allocation $\{Y_i\}_{i=1}^n$ such that
$$U_i\left(Y_i\right) \geq U_i\left(Y_i^*\right), \quad \forall\, i \in \mathcal{N},$$

with at least one strict inequality. Without loss of generality, suppose that the inequality is strict for U_1. Then by Assumption 5.2.3, there exists $\varepsilon > 0$ such that

$$U_1\left(Y_1 - \varepsilon\right) > U_1\left(Y_1^*\right) .$$

Let $\tilde{Y}_1 := Y_1 - \varepsilon$ and $\tilde{Y}_i := Y_i + \left(\frac{1}{n-1}\right)\varepsilon$, for $i = 2, \ldots, n$. Then $\sum_{i=1}^n \tilde{Y}_i = \sum_{i=1}^n Y_i = S$, and hence $\left\{\tilde{Y}_i\right\}_{i=1}^n \in \mathcal{A}$. Furthermore, for $i = 2, \ldots, n$, we have

$$U_i\left(\tilde{Y}_i\right) > U_i\left(Y_i\right) \geq U_i\left(Y_i^*\right) ,$$

implying that $\{Y_i^*\}_{i=1}^n \in \mathcal{A}$ is not weakly Pareto optimal either. □

Let $\mathcal{IR}$ denote the set of all IR allocations. Then, in particular, $\mathcal{IR} \neq \varnothing$ since it contains the status-quo, i.e., the no-risk-sharing allocation $Y_i = X_i$ under which each agent retains their initial endowment.

Let $\mathcal{PO} \subseteq \mathcal{IR}$ denote the set of all PO allocations. The following provides a useful characterization of the set $\mathcal{PO}$ under the assumption that all U_i are concave. Define the set Λ as follows:

$$\Lambda := \left\{(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}_+^n\right\} \setminus \{0\} .$$

For a given vector $\lambda \in \Lambda$, let $\mathcal{S}_\lambda$ be the set of all maximizers for the following sum-maximization problem:

$$\sup_{\{Y_i\}_{i=1}^n \in \mathcal{IR}} \sum_{i=1}^n \lambda_i \, U_i(Y_i) . \tag{5.1}$$

By a classical result, the set $\mathcal{PO}$ coincides with the solutions to problem (5.1) for some choice of $\lambda \in \Lambda$. Elements of Λ in this context are often referred to as Negishi weights in the literature. Hence, every Pareto-optimal allocation is associated with a vector of Negishi weights by this characterization.

The argument of the proof follows the framework of Dana and Jeanblanc [41, Proposition 6.3.3], who show this result without the consideration of individual rationality. The included proof in Section 5.7 shows that the individual rationality constraint is compatible with the classical methodology.

Proposition 5.2.5. *Suppose that Assumption 5.2.3 holds and that U_i is monotone and concave for all $i \in \mathcal{N}$. Then $\mathcal{PO} = \bigcup_{\lambda \in \Lambda} \mathcal{S}_\lambda$.*

5.3 Comonotone Allocations

While more explicit characterizations of solutions to (5.1) are difficult to determine in practice, it is well known that Pareto-optimal allocations are comonotone when each agent's preference preserves the concave order (i.e., preferences are Schur concave). This classical result was first obtained by Landsberger and Meilijson [75] in the two-agent case over a discrete state space, and later extended to continuous random variables by Dana and Meilijson [43] and Filipović and Svindland [56].

The comonotone improvement theorem motivates the notion of a comonotone market, a special incomplete market where the only admissible allocations are those that are comonotone with the aggregate endowment. This market was previously studied by Boonen et al. [21], who provide characterizations of competitive equilibria when each agent's preference admits a representation in terms of the Rank-Dependent Expected-Utility (RDU) model of Quiggin [92]. In this section, we examine the relationship between Pareto optima in a comonotone market and those in the general case. We find that when preferences are SSD preserving, each Pareto-optimal allocation in the comonotone market is also Pareto optimal in the original unconstrained market, therefore providing justification for the existence of the comonotone market itself.

5.3.1 Schur Concave and SSD-Preserving Maps

First, we recall some standard definitions for the concave and increasing concave orders. For an in-depth overview of the mathematical properties of these orders, we refer to Shaked and Shanthikumar [103, Section 3.A].

Definition 5.3.1 (Concave Order)**.** For all random variables $Z_1, Z_2 \in \mathcal{X}$, we say that Z_2 dominates Z_1 in the concave order, and we write

$$Z_1 \preccurlyeq_{CCV} Z_2\,,$$

if and only if for every concave function ϕ,

$$\mathbb{E}[\phi(Z_1)] \leq \mathbb{E}[\phi(Z_2)]\,,$$

when the above expectations are defined. If, in addition, the above inequality is strict for every strictly concave function ϕ, then Z_2 is said to strictly dominate Z_1 in the concave order, denoted by $Z_1 \prec_{CCV} Z_2$.

Definition 5.3.2 (Schur Concavity)**.** A functional $U : \mathcal{X} \to \mathbb{R}$ is Schur concave if for all $Z_1, Z_2 \in \mathcal{X}$ such that $Z_1 \preccurlyeq_{CCV} Z_2$, we have

$$U(Z_1) \leq U(Z_2)\,.$$

Similarly, U is strictly Schur concave if for all $Z_1, Z_2 \in \mathcal{X}$ such that $Z_1 \prec_{CCV} Z_2$, we have

$$U(Z_1) < U(Z_2)\,.$$

Definition 5.3.3 (Increasing Concave Order)**.** For all random variables Z_1, Z_2 on a probability space $(\Omega, \Sigma, \mathbb{P})$, we say that Z_2 dominates Z_1 in the increasing concave order, and we write

$$Z_1 \preccurlyeq_{ICV} Z_2\,,$$

if and only if for every increasing concave function ϕ,

$$\mathbb{E}[\phi(Z_1)] \leq \mathbb{E}[\phi(Z_2)]\,,$$

when the above expectations are defined. If, in addition, the above inequality is strict for every increasing strictly concave function ϕ, then Z_2 is said to strictly dominate Z_1 in the concave order, denoted by $Z_1 \prec_{ICV} Z_2$.

In economic theory, the increasing concave order is a classical notion of risk aversion, and it is commonly referred to as second-order stochastic dominance. A preference functional that is consistent with respect to this order is said to be strongly risk averse. The literature on the behaviour-theoretic foundations of strong risk aversion is vast (e.g., Rothschild and Stiglitz [97] and Quiggin [93]). We refer to these strongly risk averse maps as SSD-preserving maps, as is common in the literature. For more on preferences modelled directly by SSD-preserving maps, see Dana and Meilijson [43].

Definition 5.3.4 (SSD Consistency)**.** A functional $U : \mathcal{X} \to \mathbb{R}$ is SSD consistent (or SSD preserving) if for all $Z_1, Z_2 \in \mathcal{X}$ such that $Z_1 \preccurlyeq_{ICV} Z_2$, we have

$$U(Z_1) \leq U(Z_2)\,.$$

Similarly, U is strictly SSD preserving if for all $Z_1, Z_2 \in \mathcal{X}$ such that $Z_1 \prec_{ICV} Z_2$, we have

$$U(Z_1) < U(Z_2)\,.$$

For more on the relationship between the concave and increasing concave orders, we refer to Shaked and Shanthikumar [103] and Dana [38]. In particular, the following two results will be relevant for the present chapter. The first result provides an equivalence between Schur concavity and SSD consistency for monotone functionals. The next result shows that given concavity and an additional regularity assumption, Schur concavity and SSD consistency are equivalent to law invariance. Together, these results characterize monotone, concave, and law-invariant functionals as a class of strongly risk averse preferences.

Proposition 5.3.5. *A functional $U : \mathcal{X} \to \mathbb{R}$ is SSD preserving if and only if it is monotone and Schur concave.*

Proof. See Dana [38, Proposition 2.1]. □

Proposition 5.3.6. *Suppose that a functional $U : \mathcal{X} \to \mathbb{R}$ is concave and upper semicontinuous with respect to the norm topology on L^∞. Then Schur concavity of U is equivalent to law invariance of U.*

Proof. The result follows directly from the following three equivalences. By Grechuk and Zabarankin [67, Corollary 3.3], for a concave Schur-concave functional, norm upper semicontinuity is equivalent to $\sigma(L^\infty, L^1)$ upper semicontinuity. By Dana [38, Theorem 4.1], for a concave and $\sigma(L^\infty, L^1)$ upper semicontinuous functional, Schur concavity and law invariance are equivalent. Finally, by Jouini et al. [68, Theorem 2.2], for a concave law-invariant functional, $\sigma(L^\infty, L^1)$ upper semicontinuity and norm upper semicontinuity are equivalent. □

Corollary 5.3.7. *Suppose that a functional $U : \mathcal{X} \to \mathbb{R}$ is monotone, concave, and upper semicontinuous with respect to the norm topology on L^∞. Then U is SSD preserving if and only if it is law-invariant.*

Proof. Direct corollary of Propositions 5.3.5 and 5.3.6. □

5.3.2 Pareto Optimality in Comonotone Markets

Denote the set of all comonotone allocations by

$$\mathcal{A}_C := \left\{ \{Y_i\}_{i=1}^n \in \mathcal{A} : \{Y_i\}_{i=1}^n \text{ is comonotone} \right\}.$$

The notion of Pareto optimality in this comonotone market is defined below.

Definition 5.3.8. An allocation $\{Y_i^*\}_{i=1}^n \in \mathcal{A}_C$ is *Comonotone Pareto Optimal* (CPO) if it is IR and there does not exist any other allocation $\{Y_i\}_{i=1}^n \in \mathcal{A}_C$ such that

$$U_i(Y_i) \geq U_i(Y_i^*), \quad \forall\, i \in \mathcal{N},$$

with at least one strict inequality.

That is, CPO allocations are those that are not dominated by any other *comonotone* allocation, and are therefore Pareto optimal in the (restricted) comonotone market. However, we will show that given SSD consistency, CPO allocations are also Pareto optimal in the original market. Hence, there is no ambiguity in referring to these allocations as "comonotone Pareto optimal".

Let $\mathcal{CPO}$ denote the set of all CPO allocations. Analogous to the result of Proposition 5.2.5, we can obtain the following characterization for the set $\mathcal{CPO}$. For a given $\lambda \in \Lambda$, let $\mathcal{CS}_\lambda$ be the set of all maximizers of the following sum-maximization problem:

$$\sup_{\{Y_i\}_{i=1}^n \in \mathcal{IR} \cap \mathcal{A}_C} \sum_{i=1}^n \lambda_i\, U_i(Y_i).$$

Corollary 5.3.9. *Suppose Assumption 5.2.3 holds If U_i is monotone and concave for all $i \in \mathcal{N}$, then $\mathcal{CPO} = \bigcup_{\lambda \in \Lambda} \mathcal{CS}_\lambda$.*

Proof. The proof is identical to that of Proposition 5.2.5, with the set $\mathcal{IR}$ replaced by $\mathcal{IR} \cap \mathcal{A}_C$. Note that convex combinations of comonotone allocations are also comonotone. □

Proposition 5.3.10. *For each $\{Y_i\}_{i=1}^n \in \mathcal{A}$ there exists a comonotone allocation $\left\{\tilde{Y}_i\right\}_{i=1}^n \in \mathcal{A}_C$ such that*

$$Y_i \preccurlyeq_{CCV} \tilde{Y}_i, \quad \forall i \in \mathcal{N}.$$

If, in addition, $\{Y_i\}_{i=1}^n \notin \mathcal{A}_C$, then the comonotone allocation $\left\{\tilde{Y}_i\right\}_{i=1}^n \in \mathcal{A}_C$ can be taken such that

$$Y_j \prec_{CCV} \tilde{Y}_j,$$

for some $j \in \mathcal{N}$.

Proof. See, e.g., Carlier et al. [31, Theorem 3.1] or Denuit et al. [48, Theorem 3.1]. □

Corollary 5.3.11. *Suppose that U_i is Schur concave, for each $i \in \mathcal{N}$. Then for each $\{Y_i\}_{i=1}^n \in \mathcal{A}$ there exists a comonotone allocation $\left\{\tilde{Y}_i\right\}_{i=1}^n \in \mathcal{A}_C$ such that*

$$U_i(Y_i) \leq U_i(\tilde{Y}_i), \quad \forall i \in \mathcal{N}\,.$$

If, in addition, U_i is strictly Schur concave for each $i \in \mathcal{N}$ and $\{Y_i\}_{i=1}^n \notin \mathcal{A}_C$, then there exists a comonotone allocation $\left\{\tilde{Y}_i\right\}_{i=1}^n \in \mathcal{A}_C$ such that

$$U_j(Y_j) < U_j(\tilde{Y}_j)\,,$$

for some $j \in \mathcal{N}$.

Proof. Direct consequence of Proposition 5.3.10 and Definition 5.3.1. □

The following results illustrate the relationship between PO allocations and CPO allocations, given Schur concavity of preferences.

Theorem 5.3.12. *Suppose that each U_i is concave and SSD preserving, and that Assumption 5.2.3 holds. Then $\mathcal{PO} \neq \varnothing$ if and only if $\mathcal{CPO} \neq \varnothing$. Furthermore, when Pareto optima and comonotone Pareto optima exist, we have for all $\lambda \in \Lambda$,*

$$\sup_{\{Y_i\}_{i=1}^n \in \mathcal{IR}} \left\{ \sum_{i=1}^n \lambda_i\, U_i(Y_i) \right\} = \sup_{\{Y_i\}_{i=1}^n \in \mathcal{IR} \cap \mathcal{A}_C} \left\{ \sum_{i=1}^n \lambda_i\, U_i(Y_i) \right\}. \tag{5.2}$$

We obtain the following result as a corollary, which describes the relationship between PO and CPO in this case.

Corollary 5.3.13. *Suppose that each U_i is concave and SSD preserving, and Assumption 5.2.3 holds. Then $\mathcal{PO} \cap \mathcal{A}_C = \mathcal{CPO}$. If, in addition, each U_i is strictly SSD preserving, then $\mathcal{PO} = \mathcal{CPO}$.*

Proof. Suppose that $\{Y_i^*\}_{i=1}^n \in \mathcal{PO} \cap \mathcal{A}_C$. Since this allocation is Pareto optimal, it is not Pareto-dominated by any other allocation. That is, there exists no other allocation $\{\tilde{Y}_i\}_{i=1}^n \in \mathcal{IR}$ such that $U_i(\tilde{Y}_i) \geq U_i(Y_i^*)$ for all $i \in \mathcal{N}$, with at least one strict inequality. Hence, there exists no such allocation in $\mathcal{IR} \cap \mathcal{A}_C$ either. Since $\{Y_i^*\}_{i=1}^n$ is a comonotone allocation, this implies that $\{Y_i^*\}_{i=1}^n \in \mathcal{CPO}$, and so $\mathcal{PO} \cap \mathcal{A}_C \subseteq \mathcal{CPO}$.

For the reverse inclusion, suppose that $\{Y_i^*\}_{i=1}^n \in \mathcal{CPO}$. Then by Theorem 5.3.12, there exists $\lambda \in \Lambda$ for which the allocation $\{Y_i^*\}_{i=1}^n$ achieves both suprema in (5.2). Hence,

$\{Y_i^*\}_{i=1}^n \in \mathcal{S}_\lambda \subseteq \mathcal{PO}$ by Proposition 5.2.5. Now suppose that each U_i is strictly Schur concave and that $\{Y_i\}_{i=1}^n \in \mathcal{A} \setminus \mathcal{A}_C$. Then by Corollary 5.3.11, there exists a comonotone allocation $\{\tilde{Y}_i\}_{i=1}^n$ such that

$$U_i(Y_i) \leq U_i(\tilde{Y}_i) \quad \forall i \in \mathcal{N}\,,$$

with at least one strict inequality. Therefore, $\{Y_i\}_{i=1}^n \notin \mathcal{PO}$, which implies $\mathcal{PO} = \mathcal{PO} \cap \mathcal{A}_C = \mathcal{CPO}$ by the above. □

Define the *utility possibility frontier* to be the set of all n-dimensional real vectors that represent the utility achievable by each agent in a Pareto-optimal allocation. That is,

$$\mathcal{UPF} := \{(u_1, \ldots, u_n) \in \mathbb{R}^n : u_i = U_i(Y_i^*),\ \forall i \in \mathcal{N},\ \text{for some } \{Y_i^*\}_{i=1}^n \in \mathcal{PO}\}\ .$$

A similar concept can be defined for the comonotone market. Let the *comonotone utility possibility frontier* be the utility achievable by each agent in a comonotone Pareto-optimal allocation:

$$\mathcal{CUPF} := \{(u_1, \ldots, u_n) \in \mathbb{R}^n : u_i = U_i(Y_i^*),\ \forall i \in \mathcal{N},\ \text{for some } \{Y_i^*\}_{i=1}^n \in \mathcal{CPO}\}\,.$$

A consequence of Theorem 5.3.12 is that for concave Schur-concave utilities, both of these possibility frontiers coincide, as shown by the following.

Corollary 5.3.14. *Suppose that each U_i is concave and SSD preserving, and that Assumption 5.2.3 holds. Then $\mathcal{UPF} = \mathcal{CUPF}$.*

Proof. By Corollary 5.3.13, we have $\mathcal{CPO} \subseteq \mathcal{PO}$, and so $\mathcal{CUPF} \subseteq \mathcal{UPF}$. For the reverse inclusion, suppose that $\{Y_i^*\}_{i=1}^n \in \mathcal{PO}$. Then by Corollary 5.3.11, there exists a comonotone allocation $\left\{\tilde{Y}_i\right\}_{i=1}^n \in \mathcal{A}_C$ such that

$$U_i\left(\tilde{Y}_i\right) \geq U_i(Y_i^*) \quad \forall i \in \{1, \ldots, n\}\,.$$

However, since $\{Y_i^*\}_{i=1}^n$ is Pareto optimal, equality must hold for each i, implying that $\mathcal{UPF} \subseteq \mathcal{CUPF}$. □

These results provide additional justification for the existence of a comonotone market whenever preferences are SSD preserving. In general, the set of all Pareto-optimal allocations is difficult to characterize. However, whenever Pareto optima exist, comonotone Pareto optima must also exist by Theorem 5.3.12. Furthermore, Corollary 5.3.13 implies

that these comonotone Pareto optima are indeed Pareto optimal in the unconstrained market. Hence, imposing the restriction that all allocations must be comonotone does not adversely affect the total welfare gain that is possible, as highlighted by Corollary 5.3.14.

Finally, it is possible to show that $\mathcal{PO} \neq \varnothing$ (equivalently, $\mathcal{CPO} \neq \varnothing$) under some additional conditions. Notably, by Filipovic and Svindland [56, Theorem 2.5], comonotone Pareto optima exist when preferences are concave, Schur concave, translation invariant, and $\sigma(L^\infty, L^1)$ upper semicontinuous.

5.4 Characterization of Pareto Optima for Positively Homogeneous Risk-Averse Monetary Utilities

In this section, we provide the main result of this chapter: a characterization of all (comonotone) Pareto-optimal allocations when each agent's preference is represented by a positively homogeneous risk-averse monetary utility.

5.4.1 Monetary Utilities

First, we recall the definition of a monetary utility.

Definition 5.4.1 (Monetary Utility)**.** A functional $U : \mathcal{X} \to \mathbb{R}$ is a *monetary utility* if it is monotone, concave, and translation invariant.

Remark 5.4.2. *It can be shown that under translation invariance, quasi-concavity and concavity are equivalent for monotone functionals: see Delbaen [46, Proposition 5]. Hence, concavity in Definition 5.4.1 can be replaced by quasi-concavity.*

We refer to Delbaen [46] for an extensive overview of monetary utilities. The key property of monetary utilities is translation invariance, which allows for a simplified characterization of $\mathcal{PO}$ and $\mathcal{CPO}$ in terms of solutions of a maximization problem. We note that in contrast to the result of Proposition 5.2.5, the following characterization does not require the functionals to be concave. Let $\mathbf{1} \in \mathbb{R}^n$ denote the vector with 1 in every entry. When all functionals U_i are translation invariant, we obtain the following result.

Proposition 5.4.3. *If U_i is translation invariant for all $i \in \mathcal{N}$, then $\mathcal{PO} = \mathcal{S}_{\mathbf{1}}$.*

This is a well-known result in the literature, and implies that when preferences are translation invariant, the Negishi weights may be taken to be equal. When each of these weights is 1, finding the Pareto-optimal allocations is equivalent to solving the problem

$$\sup_{\{Y_i\}_{i=1}^n \in \mathcal{IR}} \sum_{i=1}^n U_i(Y_i)\,,$$

which is also known as the sup-convolution of the preferences U_i, for $i \in \mathcal{N}$. A proof of this result can be found in Embrechts et al. [53, Proposition 1] in the context of risk sharing, without the consideration of individual rationality. For completeness, we provide a proof of Proposition 5.4.3 in Section 5.7. This characterization also applies to comonotone Pareto optima, as per the following result.

Corollary 5.4.4. *If U_i is translation invariant for all $i \in \mathcal{N}$, then $\mathcal{CPO} = \mathcal{CS}_{\mathbf{1}}$.*

Proof. The proof is identical to that of Proposition 5.4.3, with the set $\mathcal{IR}$ replaced by $\mathcal{IR} \cap \mathcal{A}_C$. □

The property of translation invariance has some convenient implications on the regularity of preferences. Firstly, it is immediate that under translation invariance, Assumption 5.2.3 is automatically satisfied. Furthermore, monetary utilities are norm continuous, which implies the following result.

Corollary 5.4.5. *A monetary utility U is SSD preserving if and only if it is law-invariant.*

Proof. Monetary utilities are 1-Lipschitz continuous with respect to the supremum norm on L^∞ (e.g., Föllmer and Schied [57, Lemma 4.3]). The result then follows from Corollary 5.3.7. □

Corollary 5.4.5 implies that when each U_i is a law-invariant monetary utility, the results of Subsections 5.3.1 and 5.3.2 apply to describe Pareto optimality through comonotone allocations.

Remark 5.4.6. *In the event that each U_i is a law-invariant monetary utility, Propositions 5.2.5 and 5.4.3 imply that*

$$\mathcal{PO} = \bigcup_{\lambda\in\Lambda} \mathcal{S}_\lambda = \mathcal{S}_{\mathbf{1}} \quad \text{and} \quad \mathcal{CPO} = \bigcup_{\lambda\in\Lambda} \mathcal{CS}_\lambda = \mathcal{CS}_{\mathbf{1}}\,.$$

Indeed, it is possible to show that for all $\lambda \in \Lambda$, translation invariance implies that $\mathcal{S}_\lambda \subseteq \mathcal{S}_{\mathbf{1}}$.

5.4.2 Law-Invariant Positively Homogeneous Monetary Utilities

In the following, we obtain a more explicit characterization of the set $\mathcal{CPO}$, under the additional assumption that each utility functional is positively homogeneous. Positively homogeneous monetary utilities are well-studied in the insurance and risk management literature, where they are commonly known as coherent risk measures, as introduced by Artzner et al. [6]. We will use the fact that these functionals admit a representation in terms of Choquet integrals, as given by the following.

Lemma 5.4.7. *Let $U : \mathcal{X} \to \mathbb{R}$ be a law-invariant and positively homogeneous monetary utility. Then there exists a $\sigma(L^1, L^\infty)$ closed, convex, law-invariant set $\mathcal{H} \subseteq L^1_+$ such that*

$$\mathbb{E}[H] = 1 \text{ for all } H \in \mathcal{H}\,,$$

and

$$U(Z) = \inf_{H \in \mathcal{H}} \mathbb{E}[HZ] = \inf_{T \in \{\phi_H : H \in \mathcal{H}\}} \int Z \, dT \circ \mathbb{P}\,,$$

where for every $H \in L^1_+$, the function $\phi_H : [0,1] \to [0,1]$ is defined by

$$\phi_H(x) := \int_0^x F^{-1}_{H,\mathbb{P}}(t)\, dt, \ \ \forall x \in [0,1]\,.$$

Proof. See Dana [38, Corollary 4.3]. □

Remark 5.4.8. *By Bellini et al. [13, Proposition 5.1], the set H in the representation of Lemma 5.4.7 can be taken to be in L^∞_+ instead of in L^1_+. However, the statement of Lemma 5.4.7 will suffice for our purposes.*

In a comonotone market, every admissible post-transfer payout Y_i is comonotone with the aggregate endowment S. Denote by $\mathcal{X}^\uparrow$ the set of all random variables in $\mathcal{X}$ that are comonotone with S. Under this setting, we may restrict the domain of each utility functional to $\mathcal{X}^\uparrow$, which admits the following representation. The proof of this result is provided in Section 5.7.

Lemma 5.4.9. *Let $U : \mathcal{X} \to \mathbb{R}$ be a law-invariant and positively homogeneous monetary utility. Then for all $Z \in \mathcal{X}^\uparrow$, we have*

$$U(Z) = \inf_{T \in \mathcal{T}} \int Z \, dT \circ \mathbb{P}\,,$$

where $\mathcal{T}$ is a convex set of convex distortion functions that is sequentially closed under pointwise convergence.

To facilitate our characterization result, we first show that all comonotone allocations are translations of a suitable non-negative function of the aggregate endowment. First, define the set $\mathcal{G}$ by the following:

$$\mathcal{G} := \left\{ \{g_i\}_{i=1}^n \,\middle|\, g_i : \mathbb{R}_+ \to \mathbb{R}_+ \text{ non-decreasing, and } \sum_{i=1}^n g_i(\cdot) = \mathrm{Id} \right\}.$$

Let $\underline{s} := \operatorname{ess\,inf} S$. Since $S \in L^\infty$, we have $\underline{s} > -\infty$. The following lemma shows that every comonotone allocation can be represented in terms of functions in $\mathcal{G}$.

Lemma 5.4.10. *Let $\{Y_i\}_{i=1}^n \in \mathcal{A}_C$. Then there exist functions $\{g_i\}_{i=1}^n \in \mathcal{G}$ and constants $\{c_i\}_{i=1}^n \in \mathbb{R}^n$ such that*

$$Y_i = g_i(S - \underline{s}) + c_i \quad \forall i \in \mathcal{N},$$

and $\sum_{i=1}^n c_i = \underline{s}$.

Proof. Let $\{Y_i\}_{i=1}^n$ be a comonotone allocation. By a standard result (see, e.g., Denneberg [47, Proposition 4.5]), there exist increasing 1-Lipschitz functions $f_i : \mathbb{R} \mapsto \mathbb{R}$ such that $f_i(S) = Y_i$ and $\sum_{i=1}^n f_i(x) = x$ for all $x \in \mathbb{R}$. Let $c_i := f_i(\underline{s})$, and define the function g_i by

$$\begin{aligned} g_i : \mathbb{R}_+ &\to \mathbb{R}_+ \\ x &\mapsto f_i(\underline{s} + x) - f_i(\underline{s}). \end{aligned}$$

Then g_i is non-decreasing, and

$$\sum_{i=1}^n g_i(x) = \sum_{i=1}^n f_i(\underline{s} + x) - \sum_{i=1}^n f_i(\underline{s}) = \underline{s} + x - \underline{s} = x,$$

for all $x \geq 0$, which implies that $\{g_i\}_{i=1}^n \in \mathcal{G}$. It follows from the construction of g_i that

$$f_i(x) = g_i(x - \underline{s}) + c_i,$$

from which we conclude that $Y_i = f_i(S) = g_i(S - \underline{s}) + c_i$ for all $i \in \mathcal{N}$. □

Our main result is given below. The interpretation of this result is discussed in Subsection 5.4.3.

Theorem 5.4.11. *Suppose that for each $i \in \mathcal{N}$, the utility functional U_i is a law-invariant and positively homogeneous monetary utility. Let $\mathcal{T}_i$ denote the representing convex set of convex distortions for U_i that is sequentially closed under pointwise convergence, in the sense of Lemma 5.4.9. Then the following hold:*

(i) There exists a solution to the problem

$$\min_{\{T_i\}_{i=1}^n \in \prod_{i=1}^n \mathcal{T}_i} \int_0^\infty \max_{i\in\mathcal{N}} \{T_i(\mathbb{P}(S > \underline{s} + x))\}\, dx\,. \tag{5.3}$$

(ii) A necessary condition for an allocation $\{Y_i^*\}_{i=1}^n$ to be comonotone Pareto optimal is that

$$Y_i^* = g_i^*(S - \underline{s}) + c_i^*\,,$$

where $\{c_i^*\}_{i=1}^n \in \mathbb{R}^n$ is chosen such that $\sum_{i=1}^n c_i^* = \underline{s}$ and $\{g_i^*(S-\underline{s}) + c_i^*\}_{i=1}^n \in \mathcal{IR}$, and $\{g_i^*\}_{i=1}^n \in \mathcal{G}$ can be written in terms of the integrals of suitable functions h_i. Specifically, for each $i \in \mathcal{N}$, we can write $g_i^*(x) = \int_0^x h_i(z)\,dz$, where each $h_i : \mathbb{R}_+ \to [0,1]$ is a function such that for each $(T_1^*, \dots, T_n^*)$ that solves (5.3) and for almost every $x \in \mathbb{R}_+$, we have

$$\sum_{i\in L_{x,T_1^*,\dots,T_n^*}} h_i(x) = 1 \text{ and } \sum_{i\in\mathcal{N}\setminus L_{x,T_1^*,\dots,T_n^*}} h_i(x) = 0\,,$$

where

$$L_{x,T_1^*,\dots,T_n^*} := \left\{ i \in \mathcal{N} : T_i^*(\mathbb{P}(S > \underline{s} + x)) = \max_{j\in\mathcal{N}} \left\{ T_j^*(\mathbb{P}(S > \underline{s} + x)) \right\} \right\}.$$

Remark 5.4.12. *In the statements of Lemma 5.4.10 and Theorem 5.4.11, we may replace $\underline{s}$ with any lower bound of S. For example, if the aggregate endowment S is non-negative, we may take $\underline{s} = 0$ for simplicity.*

Corollary 5.4.13. *Suppose that for each $i \in \mathcal{N}$, the utility functional U_i is a strictly SSD-preserving, law-invariant, and positively homogeneous monetary utility. A necessary condition for an allocation $\{Y_i^*\}_{i=1}^n$ to be Pareto optimal is that*

$$Y_i^* = g_i^*(S - \underline{s}) + c_i^*\,,$$

where $\sum_{i=1}^n c_i^* = \underline{s}$ and g_i^* are of the form given in Theorem 5.4.11. That is, $g_i^*(x) = \int_0^x h_i(z)\,dz$, where $h_i : \mathbb{R}_+ \to [0,1]$ such that for every $(T_1^*, \dots, T_n^*)$ that solves (5.3) and for almost every $x \in \mathbb{R}_+$,

$$\sum_{i\in L_{x,T_1^*,\dots,T_n^*}} h_i(x) = 1 \text{ and } \sum_{i\in\mathcal{N}\setminus L_{x,T_1^*,\dots,T_n^*}} h_i(x) = 0\,,$$

where

$$L_{x,T_1^*,\dots,T_n^*} := \left\{ i \in \mathcal{N} : T_i^*(\mathbb{P}(S > \underline{s} + x)) = \max_{j\in\mathcal{N}} \left\{ T_j^*(\mathbb{P}(S > \underline{s} + x)) \right\} \right\}.$$

Proof. By Corollary 5.3.13, the sets $\mathcal{CPO}$ and $\mathcal{PO}$ coincide. It then follows that the characterization in Theorem 5.4.11 applies to all Pareto-optimal allocations. □

5.4.3 An Algorithmic Approach to Finding Pareto Optima

The main advantage of the characterization in Theorem 5.4.11 is that it describes an algorithm that can be implemented to find the shape of comonotone Pareto optima. First, problem (5.3) must be solved. This problem can be interpreted in the following manner:

- Define the function $\psi : \prod_{i=1}^{n} \mathcal{T}_i \times \mathbb{R}_+ \to \mathbb{R}$ by

$$\psi\left((T_1, \ldots, T_n), x\right) := \max_{i \in \mathcal{N}} \left\{T_i(\mathbb{P}(S > \underline{s} + x))\right\} .$$

  That is, for a given vector of distortion functions $(T_1, \ldots, T_n)$ and a positive real number x, the function ψ provides the most optimistic assessment of the likelihood of the tail event $\mathbb{P}(S > \underline{s} + x)$ among all agents.

- Define the function $\Psi : \prod_{i=1}^{n} \mathcal{T}_i \to \mathbb{R}$ by

$$\Psi\left(T_1, \ldots, T_n\right) := \int_0^\infty \psi((T_1, \ldots, T_n), x)\, dx\,.$$

  The function Ψ provides an aggregate measure of the most optimistic assessment over all possible tail event likelihoods.

- Problem (5.3) minimizes the function Ψ over all possible choices of distortion functions. In other words, a solution to (5.3) represents the worst case scenario of the most optimistic assessment of tail event likelihood.

- The shape of a Pareto-optimal comonotone allocation can now be explicitly determined through the following steps:

  – Determine all possible solutions to (5.3).

  – For each solution $(T_1^*, \ldots, T_n^*)$ to (5.3) and each $x \in \mathbb{R}_+$, define the set

$$L_{x,T_1^*,\ldots,T_n^*} := \left\{ i \in \mathcal{N} : T_i^*(\mathbb{P}(S > \underline{s} + x)) = \max_{j \in \mathcal{N}} \left\{T_j^*(\mathbb{P}(S > \underline{s} + x))\right\} \right\} .$$

    The set $L_{x,T_1^*,\ldots,T_n^*}$ then represents the set of agents who have the most optimistic assessment of the likelihood of the tail event $\mathbb{P}(S > \underline{s} + x)$, when the preferences of each agent $i \in \mathcal{N}$ is fixed as the distortion risk measure with respect to T_i^*.

– For each $i \in \mathcal{N}$, choose a function $h_i : \mathbb{R}_+ \to [0,1]$ such that for every solution $(T_1^*, \dots, T_n^*)$ to (5.3) and almost every $x \in \mathbb{R}_+$,

$$\sum_{i \in L_{x,T_1^*,\dots,T_n^*}} h_i(x) = 1 \quad \text{and} \quad \sum_{i \in \mathcal{N} \setminus L_{x,T_1^*,\dots,T_n^*}} h_i(x) = 0\,. \tag{5.4}$$

The functions h_i represent the post-exchange marginal endowment of each agent i.

– Let $g_i^*(x) := \displaystyle\int_0^x h_i(z)\,dz$, and take

$$Y_i^* := g_i^*(S - \underline{s}) + c_i^*\,,$$

where $\{c_i^*\}_{i=1}^n \in \mathbb{R}^n$ is chosen such that $\sum_{i \in \mathcal{N}} c_i^* = \underline{s}$ and $\{Y_i^*\}_{i=1}^n \in \mathcal{IR}$.

– Reiterate this process for every choice of h_i that satisfies (5.4). The allocation that gives the highest value of aggregate welfare

$$\sum_{i=1}^n U_i(Y_i^*)$$

must be an element of $\mathcal{CS}_\mathbf{1}$, and hence comonotone Pareto optimal.

In the above algorithm, the only step that does not contain an explicit formula is the minimization of (5.3) itself. Indeed, a closed-form expression for this problem is difficult to obtain. However, the advantage of (5.3) is that the domain of the minimization is a product of sets of distortion functions. In many practical scenarios, this is a more tractable domain for numerical optimization, unlike the domain of all comonotone allocations, as is the case with the original sup-convolution problem $\mathcal{CS}_\mathbf{1}$. This advantage is illustrated through a numerical simulation in Section 5.5.

Finally, note that the form imposed by Theorem 5.4.11 only provides a *necessary* condition for Pareto optimality. This condition is *not sufficient* in general, and we provide a counterexample in Section 5.8. Therefore, identifying the Pareto-optimal allocations among those suggested by Theorem 5.4.11 requires evaluating the utility of each candidate allocation, as described in the final step of the above algorithm. Nonetheless, in practice, the condition imposed on the marginal endowments (5.4) often specifies a narrow range of candidate allocations, from which the Pareto-optimal allocations can be easily identified. Our numerical example in Section 5.5 provides one such example, where both the solution to (5.3) and the allocation satisfying (5.4) are unique.

5.4.4 Concave Dual Utilities

In the special case where each set $\mathcal{T}_i$ is a singleton, the solution to (5.3) is immediate. In fact, these are precisely the dual utility functionals introduced by Yaari [112]. We show below that if U_i satisfies the conditions of Theorem 5.4.11 as well as the additional condition of *comonotone additivity*, then $\mathcal{T}_i$ can be taken to be a singleton. In this case, it is possible to obtain a representation of $\mathcal{CPO}$ in closed form.

Definition 5.4.14. A functional U is a *dual utility functional* if there exists a distortion function T such that for all random variables $Z \in \mathcal{X}$,

$$U(Z) = \int Z \, dT \circ \mathbb{P}\,.$$

Proposition 5.4.15. *(Yaari [112, Theorem 2]) Yaari's dual utility functional* $Z \mapsto \int Z \, dT \circ \mathbb{P}$ *is concave and Schur concave if and only if the distortion function* T *is convex.*

It is well known that concave dual utilities can be characterized as the set of functionals that satisfy some standard properties, including comonotone additivity. In particular, for monetary utilities, comonotone additivity implies positive homogeneity (e.g., [57, Lemma 4.83]). The following result shows that given the additional property of comonotone additivity, the setting of Theorem 5.4.11 reduces precisely to the case of dual utilities.

Proposition 5.4.16. *(Kusuoka [74], Jouini et al. [69, Theorem 2.3]) A functional* U *is a concave dual utility functional if and only if it is a law-invariant comonotone-additive monetary utility.*

The following result assumes that for each $i \in \mathcal{N}$, the utility functional U_i is a dual utility functional with respect to a convex distortion. That is, there exists a convex distortion function T_i such that

$$U_i(Z) = \int Z \, dT_i \circ \mathbb{P},$$

for all random variables $Z \in \mathcal{X}$. In this case, we can fully characterize the set of solutions $\mathcal{CS}_\mathbf{1}$, and hence the set of all comonotone Pareto-optimal allocations $\mathcal{CPO}$.

Corollary 5.4.17. *Suppose that each* U_i *is a concave dual utility functional. For each* $x \in \mathbb{R}_+$, *let*

$$L_x := \left\{ i \in \mathcal{N} : T_i(\mathbb{P}(S > \underline{s} + x)) = \max_{j \in \mathcal{N}} \{T_j(\mathbb{P}(S > \underline{s} + x))\} \right\},$$

and let $L_x^C := \mathcal{N} \setminus L_x$. *For each* $i \in \mathcal{N}$, *let* $h_i : \mathbb{R}_+ \to [0,1]$ *be a function such that for almost every* x,

$$\sum_{i \in L_x} h_i(x) = 1 \text{ and } \sum_{i \in L_x^C} h_i(x) = 0\,,$$

and let $g_i^*(x) := \int_0^x h_i(z)\,dz$. *Let* $\{c_i^*\}_{i=1}^n \in \mathbb{R}^n$ *be chosen such that* $\sum_{i=1}^n c_i^* = \underline{s}$ *and* $\{g_i^*(S-\underline{s}) + c_i^*\}_{i=1}^n \in \mathcal{IR}$. *Then*

$$\sup_{\{Y_i\}_{i=1}^n \in \mathcal{IR} \cap \mathcal{A}_C} \left\{ \sum_{i=1}^n U_i\left(Y_i\right) \right\} = \sum_{i=1}^n U_i\left(g_i^*(S-\underline{s}) + c_i^*\right) = \underline{s} + \int_0^\infty \max_{i \in \mathcal{N}} \left\{ T_i(\mathbb{P}(S > \underline{s} + x)) \right\} dx\,,$$

and therefore $\{g_i^*(S-\underline{s}) + c_i^*\}_{i=1}^n$ *is comonotone Pareto optimal. Furthermore, all solutions are of this form. That is, if* $\{Y_i^*\}_{i=1}^n \in \mathcal{CS}_{\mathbf{1}}$, *then*

$$Y_i^* = g_i^*(S-\underline{s}) + c_i^*\,,$$

where $\{c_i^*\}_{i=1}^n \in \mathbb{R}^n$ *and* $\{g_i^*\}_{i=1}^n \in \mathcal{G}$ *are of the form given above.*

Proof. Note that each U_i has the representation

$$U_i(Z) = \inf_{\tilde{T} \in \mathcal{T}_i} \int Z\, d\tilde{T} \circ \mathbb{P}\,,$$

where for each i, the set $\mathcal{T}_i$ is the singleton set $\{T_i\}$. The result then follows from the proof of Theorem 5.4.11. □

5.4.5 Competitive Equilibria

We now consider equilibria in a competitive market where each agent's preference is represented by a law-invariant positively homogeneous monetary utility. Let $\mathcal{P}$ denote the set of all probability measures on (Ω, Σ) that are absolutely continuous with respect to $\mathbb{P}$. We assume that the price of a random wealth $Z \in \mathcal{X}$ is given by the expectation of this random variable with respect to an element of $\mathcal{P}$, which we call the pricing measure. For a fixed pricing measure $\mathbb{Q} \in \mathcal{P}$, the demand problem faced by each agent $i \in \mathcal{N}$ is therefore

$$\max_{Y_i \in \mathcal{X}} U_i(Y_i) \quad s.t. \quad \mathbb{E}^{\mathbb{Q}}[Y_i] \leq \mathbb{E}^{\mathbb{Q}}[X_i]\,. \tag{5.5}$$

That is, each agent maximizes their utility given the budgetary constraint that the price of their random payoff does not exceed the value of their initial endowment. The definition of a classical Arrow-Debreu equilibrium in this setting is the following.

Definition 5.4.18. A pair $((Y_1^*, \dots, Y_n^*), \mathbb{Q}^*) \in \mathcal{X}^n \times \mathcal{P}$ is *an equilibrium* if

(i) For each agent $i \in \mathcal{N}$, Y_i^* solves the demand problem (5.5) under the pricing measure $\mathbb{Q}^*$.

(ii) $\sum_{i=1}^n Y_i^* = \sum_{i=1}^n X_i = S$. This condition is often referred to as the market-clearing condition.

The following result establishes the existence of equilibria in our setting, as well as the first and second welfare theorems.

Proposition 5.4.19. *Suppose that for each $i \in \mathcal{N}$, the utility functional U_i is a law-invariant and positively homogeneous monetary utility. Then there exists an equilibrium $((Y_1^*, \dots, Y_n^*), \mathbb{Q}^*) \in \mathcal{X}^n \times \mathcal{P}$. Furthermore,*

(i) If $((Y_1^*, \dots, Y_n^*), \mathbb{Q}^*)$ is an equilibrium, then the allocation $(Y_1^*, \dots, Y_n^*)$ is Pareto optimal.

(ii) If $(Y_1^*, \dots, Y_n^*)$ is a Pareto-optimal allocation, then there exists a pricing measure $\mathbb{Q}^* \in \mathcal{P}$ such that the pair

$$\left(\left(Y_1^* + \mathbb{E}^{\mathbb{Q}^*}[X_1 - Y_1^*], \dots, Y_n^* + \mathbb{E}^{\mathbb{Q}^*}[X_n - Y_n^*]\right), \mathbb{Q}^*\right)$$

is an equilibrium.

Note that when preferences are translation invariant, Pareto-optimal allocations are only unique up to translation. Specifically, if $(Y_1, \dots, Y_n^*)$ is a Pareto-optimal allocation and $\{b_i\}_{i=1}^n \in \mathbb{R}^n$ satisfies $\sum_{i=1}^n b_i = 0$, then the allocation $(Y_1^* + b_1, \dots, Y_n^* + b_n)$ is also Pareto optimal. However, for a given pricing measure $\mathbb{Q}^*$, only one of these translations can possibly lead to an equilibrium allocation under the pricing measure $\mathbb{Q}^*$. This allocation is precisely what is given in part (ii) of Proposition 5.4.19, as shown by the following result.

Lemma 5.4.20. *Let $(Y_1^*, \dots, Y_n^*)$ be a Pareto-optimal allocation, and suppose that $\mathbb{Q}^*$ is a pricing measure such that $((Y_1^* + b_1, \dots, Y_n^* + b_n), \mathbb{Q}^*)$ is an equilibrium, where $\{b_i\}_{i=1}^n \in \mathbb{R}^n$ satisfies $\sum_{i=1}^n b_i = 0$. Then $b_i = \mathbb{E}^{\mathbb{Q}^*}[X_i - Y_i^*]$, for all $i \in \mathcal{N}$.*

However, a more explicit characterization of the equilibrium pricing measure is difficult to obtain in general. In the special case of Yaari utilities, the pricing measure can be identified in terms of distortion functions, as shown in Section 3.1 of Boonen et al. [21], for instance. Their characterization result is given below.

Proposition 5.4.21. *Suppose that for each $i \in \mathcal{N}$, the utility functional U_i is a dual utility with respect to a convex distortion function T_i. Let $(Y_1^*, \dots, Y_n^*) \in \mathcal{CPO}$. If $((Y_1^* + \mathbb{E}^{\mathbb{Q}^*}[X_1 - Y_1^*], \dots, Y_n^* + \mathbb{E}^{\mathbb{Q}^*}[X_n - Y_n^*]), \mathbb{Q}^*)$ is an equilibrium, then the pricing measure $\mathbb{Q}^*$ must satisfy*

$$T_{(n-1)}(\mathbb{P}(S > \underline{s} + x)) \le \mathbb{Q}^*(S > \underline{s} + x) \le T_{(n)}(\mathbb{P}(S > \underline{s} + x)), \ \forall\, x \in \mathbb{R}_+\,,$$

where for each $t \in [0,1]$, $T_{(k)}(t)$ denotes the k-th smallest value of $T_i(t)$ over all $i \in \mathcal{N}$.

For completeness, we provide a self-contained proof of Proposition 5.4.21 in Section 5.9. However, this result does not easily extend to the more general setting of law-invariant positively homogeneous monetary utilities, and we leave the problem of further characterization of equilibrium pricing measures for future research.

5.5 Pareto Optima in Risk-Sharing Markets

In this section, we provide an application to risk-sharing markets. Specifically, we consider a decentralized insurance market with n agents, each measuring their risk exposure via a coherent risk measure. Hence, coherent risk measures and positively homogeneous monetary utilities are equivalent up to a change in sign. Our previous results therefore apply to the setting of risk sharing considered in this section.

For each $i \in \mathcal{N}$, let $X_i \in \mathcal{X}$ denote the initial risk exposure of the i-th agent, which is assumed to be non-negative. In this section, we again use the convention that positive values of X_i denote positive values of liabilities or risk exposures (i.e., negative values of endowments). The aggregate risk in the market is $S := \sum_{i=1}^n X_i$. All agents are assumed to participate in a risk-sharing pool, which allocates the aggregate risk S among the n agents. The risk distributed to agent i is denoted by Y_i.

5.5.1 Law-Invariant Coherent Risk Measures

We assume that each agent $i \in \mathcal{N}$ uses a law-invariant coherent risk measure ρ_i. By Theorem 5.4.11, we obtain a full characterization of all comonotone Pareto optima. This result is restated below in the context of risk sharing. Note that since we assume that the aggregate risk is non-negative, we may take the lower bound $\underline{s} = 0$ to simplify the expressions.

Corollary 5.5.1. *Suppose that for each $i \in \mathcal{N}$, the risk measure ρ_i is law-invariant and coherent. Let $\mathcal{T}_i$ denote the representing convex set of concave distortions for ρ_i that is sequentially closed under pointwise convergence, as per Lemma 5.4.9. Then the following hold:*

(i) There exists a solution to the problem

$$\max_{\{T_i\}_{i=1}^n \in \prod_{i=1}^n \mathcal{T}_i} \int_0^\infty \min_{i \in \mathcal{N}} \{T_i(\mathbb{P}(S > x))\} \, dx \, . \tag{5.6}$$

(ii) A necessary condition for an allocation $\{Y_i^*\}_{i=1}^n$ to be comonotone Pareto optimal is that

$$Y_i^* = g_i^*(S) + c_i^* \, ,$$

where $\{c_i^*\}_{i=1}^n \in \mathbb{R}^n$ is chosen such that $\sum_{i=1}^n c_i^* = 0$ and $\{g_i^*(S) + c_i^*\}_{i=1}^n \in \mathcal{IR}$, and $\{g_i^*\}_{i=1}^n \in \mathcal{G}$ can be written in terms of the integrals of suitable functions h_i. Specifically, for each $i \in \mathcal{N}$, we can write $g_i^*(x) := \int_0^x h_i(z) \, dz$, where each $h_i : \mathbb{R}_+ \to [0,1]$ is a function such that for each $(T_1^*, \dots, T_n^*)$ that solves (5.6) and for almost every $x \in \mathbb{R}_+$,

$$\sum_{i \in L_{x, T_1^*, \dots, T_n^*}} h_i(x) = 1 \quad \text{and} \quad \sum_{i \in \mathcal{N} \setminus L_{x, T_1^*, \dots, T_n^*}} h_i(x) = 0 \, ,$$

where

$$L_{x, T_1^*, \dots, T_n^*} := \left\{ i \in \mathcal{N} : T_i^*(\mathbb{P}(S > x)) = \min_{j \in \mathcal{N}} \{T_j^*(\mathbb{P}(S > x))\} \right\} .$$

The case of dual utilities, as examined in Subsection 5.4.4, is analogous to the case where every agent in a risk-sharing market uses a coherent distortion risk measure. These risk measures are of the form

$$\begin{aligned} \rho_i : \mathcal{X} &\to \mathbb{R} \\ Z &\mapsto \int Z \, dT_i \circ \mathbb{P} \, , \end{aligned}$$

where T_i is a concave distortion function. The result of Corollary 5.4.17 applies in this case, and it is restated below in the setting of risk sharing. This characterization is known in the literature on distortion risk measures. See Liu [82, Theorem 3.3] and Boonen et al. [21, Proposition 1], for example.

Corollary 5.5.2. *Suppose that each agent uses a coherent distortion risk measure. For each $x \in \mathbb{R}_+$, let*

$$L_x := \left\{ i \in \mathcal{N} : T_i(\mathbb{P}(S > x)) = \min_{j \in \mathcal{N}} \{T_j(\mathbb{P}(S > x))\} \right\},$$

and let $L_x^C := \mathcal{N} \setminus L_x$. For each $i \in \mathcal{N}$, let $h_i : \mathbb{R}_+ \to [0,1]$ be a function such that for almost every x,

$$\sum_{i \in L_x} h_i(x) = 1 \text{ and } \sum_{i \in L_x^C} h_i(x) = 0\,,$$

and let $g_i^*(x) := \int_0^x h_i(z)\,dz$. Let $\{c_i^*\}_{i=1}^n \in \mathbb{R}^n$ be chosen such that $\sum_{i=1}^n c_i^* = 0$ and $\{g_i^*(S) + c_i^*\}_{i=1}^n \in \mathcal{IR}$. Then

$$\inf_{\{Y_i\}_{i=1}^n \in \mathcal{IR} \cap \mathcal{A}_C} \left\{ \sum_{i=1}^n \rho_i\left(Y_i\right) \right\} = \sum_{i=1}^n \rho_i\left(g_i^*(S) + c_i^*\right) = \int_0^\infty \min_{i \in \mathcal{N}} \left\{ T_i(\mathbb{P}(S > x)) \right\} dx\,,$$

and therefore $\{g_i^*(S) + c_i^*\}_{i=1}^n$ is comonotone Pareto optimal. Furthermore, all comonotone Pareto-optimal allocations are of this form. That is, if $\{Y_i^*\}_{i=1}^n$ is comonotone Pareto optimal, then

$$Y_i^* = g_i^*(S) + c_i^*\,,$$

where $\{c_i^*\}_{i=1}^n \in \mathbb{R}^n$ and $\{g_i^*\}_{i=1}^n \in \mathcal{G}$ are of the form given above.

We note that in an insurance context, the structure of the allocations $\{Y_i^*\}_{i=1}^n$ admit the following interpretation. For each $i \in \mathcal{N}$, we have $Y_i^* = g_i^*(S) + c_i^*$. Here, the function g_i^* can be seen as the retained risk allocated to agent i, as a function of the value of the loss S. Since each g_i^* is increasing and 1-Lipschitz, these retention functions satisfy the so-called *no-sabotage condition* of Carlier and Dana [28, 29], which guarantees that no agent has an incentive to misreport their actual realized loss. The constants c_i^* can be interpreted as the fixed premia that each agent i pays to participate in the risk-sharing scheme. In the following, we will refer to the functions g_i^* as retention functions.

5.5.2 An Example – A Risk Sharing Problem

As an application of the explicit characterization of comonotone Pareto optima, we consider a risk-sharing market from the perspective of the risk-bearing agents. As is common in risk management, risk measures are used to determine the amount of capital that each

agent must hold in reserve to offset future liabilities. That is, $\rho_i(Z)$ represents the amount of capital that agent i must reserve when faced with the risk Z. It is in the agents' best interest to reach a risk-sharing arrangement that allows them to minimize the amount of capital reserve required. Each agent's capital reserve must meet two targets. First, the agent must adhere to international capital requirement reporting standards, which typically prescribe the same risk measure to be used for all participating agents in the market. Second, each agent also uses an internal capital model for risk management at the institutional level. These internal models can vary based on the differences among agents in management procedures, accounting practices, etc.

We assume that every risk measure in this scenario is a law-invariant coherent risk measure, and that admissible allocations are constrained to be comonotone. The results of Section 5.3 imply that this restriction does not negatively affect the total welfare gain that is possible from risk sharing. Furthermore, as argued by Embrechts et al. [53], comonotonicity is an important property in the context of risk-sharing arrangements since it eliminates the possibility for moral hazard among collaborative agents.

We first examine the Pareto-optimal allocations that arise from regulation. A prominent set of standards on capital reserves reporting are those that regulate the international banking sector, as set by the Basel Committee on Banking Supervision. Also known as Basel IV, these standards suggest that institutions report their Expected Shortfall calculated with the parameter $\alpha = 2.5\%$.

It is well known that $\mathrm{ES}_\alpha^{\mathbb{P}}$ is a coherent distortion risk measure, with the distortion function $T(t) = \min\{t/\alpha, 1\}$ (e.g., Denuit et al. [49, Section 2.6.3.2]). However, if each agent uses only the standardized capital requirement, then the following result implies that while comonotone allocations are Pareto optimal, it is not possible to find a Pareto improvement beyond any comonotone allocation.

Proposition 5.5.3. *If each agent uses the same coherent distortion risk measure, then every individually rational comonotone allocation is Pareto optimal.*

Proof. Let $\{Y_i^*\}_{i=1}^n$ be any individually rational comonotone allocation. Then by Lemma 5.4.10, $Y_i^* = g_i^*(S) + c_i^*$ where $\{g_i^*\}_{i=1}^n \in \mathcal{G}$ and $\{c_i^*\}_{i=1}^n \in \mathbb{R}^n$. Since $\{g_i^*\}_{i=1}^n \in \mathcal{G}$, each g_i^* is 1-Lipschitz and therefore absolutely continuous. Hence, for each i, there exists $h_i : \mathbb{R}_+ \to [0,1]$ such that $g_i^*(x) = \int_0^x h_i(z)\,dz$, a.e. Since $\sum_{i=1}^n g_i^*(x) = x$, we also have $\sum_{i=1}^n h_i(x) = 1$, a.e. Additionally, since each agent uses the same coherent distortion risk measure, it follows that $L_x = \mathcal{N}$, for all $x \in \mathbb{R}_+$. Thus,

$$\sum_{i \in L_x} h_i(x) = \sum_{i=1}^n h_i(x) = 1 \quad \text{and} \quad \sum_{i \in L_x^C} h_i(x) = 0\,,$$

satisfying the conditions of Corollary 5.5.2. It then follows that $\{Y_i^*\}_{i=1}^n$ is CPO and PO. □

The result of Proposition 5.5.3 is not surprising, since distortion risk measures are comonotone-additive maps. We may interpret the objective in problem (5.6) as a measure of aggregate post-transfer risk in the market. If $\rho = \rho_1 = \ldots = \rho_n$, it is straightforward to verify that

$$\inf_{\{Y_i\}_{i=1}^n \in \mathcal{IR} \cap \mathcal{A}_C} \left\{ \sum_{i=1}^n \rho_i \left(Y_i \right) \right\} = \rho(S) \, .$$

A Numerical Illustration

Suppose that there are $n = 3$ agents in the market, each using a different coherent distortion risk measure for internal capital management. We assume that the aggregate risk S follows a Gamma distribution with shape parameter $k = 2$ and scale parameter $\theta = 10$. The mean of this distribution is 20, and the variance of this distribution is 200. Its probability density function is shown in Figure 5.1.

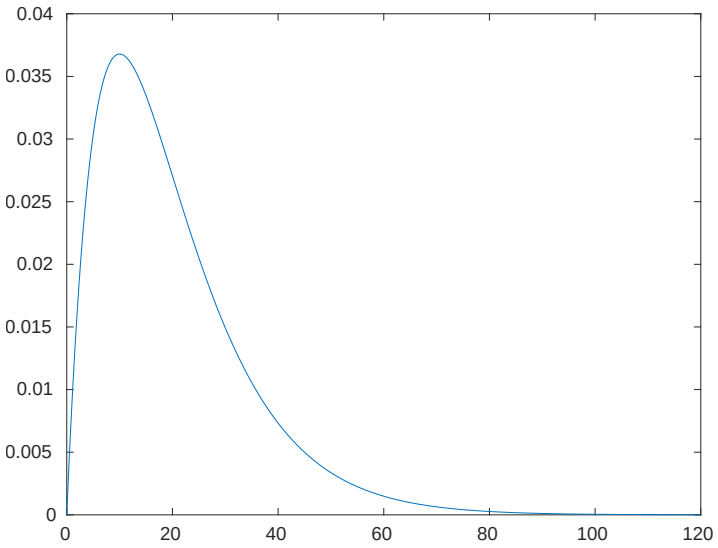


Figure 5.1: Probability Density Function of S.

We assume that all agents in this market are subject to the Basel IV regulatory standard of $\mathrm{ES}^{\mathbb{P}}_{2.5\%}$. This risk measure can be represented as a distortion risk measure, by using the

distortion function

$$\widehat{T}(t) := \min\{t/0.025, 1\}\,.$$

However, each agent also possesses their individual capital requirements for internal usage. Agent 1 has decided that they would like to take a more conservative approach and reserve capital equal to the Expected Shortfall at the 1% level. This is represented by the distortion function

$$\widehat{T}_1(t) := \min\{t/0.01, 1\}\,.$$

Since this agent must meet both the regulatory requirement and the internal capital requirement, their risk measure is therefore given by

$$\begin{aligned}\rho_1 : \mathcal{X} &\to \mathbb{R}\\ Z &\mapsto \max\left\{\int Z\, d\widehat{T}\circ\mathbb{P}, \int Z\, d\widehat{T}_1\circ\mathbb{P}\right\}\,.\end{aligned}$$

To apply the result of Corollary 5.5.1, we need to express the risk measure ρ_1 in terms of a closed convex set of distortion functions. This is given in the following lemma, which provides the desired expression for each agent in this market. We use the notation for Agent 1 in the statement and proof of this result for convenience, but the result applies to the other two agents as well.

Lemma 5.5.4. *Suppose that* $\rho_1(Z) = \max\left\{\int Z\, d\widehat{T}\circ\mathbb{P}, \int Z\, d\widehat{T}_1\circ\mathbb{P}\right\}$*, for each* $Z \in \mathcal{X}$*, and let* $\mathcal{T}_1 := \left\{\lambda\widehat{T} + (1-\lambda)\widehat{T}_1 \,:\, \lambda\in[0,1]\right\}$ *be the convex hull of the set* $\left\{\widehat{T}, \widehat{T}_1\right\}$*. Then for all* $Z \in \mathcal{X}$*, we have*

$$\rho_1(Z) = \sup_{T\in\mathcal{T}_1}\int Z\, dT\circ\mathbb{P}\,,$$

and $\mathcal{T}_1$ *is convex and sequentially closed under pointwise convergence.*

Proof. The equality $\rho_1(Z) = \sup_{T\in\mathcal{T}_1}\int Z\, dT\circ\mathbb{P}$ follows from linearity of the Choquet integral in T. Furthermore, $\mathcal{T}_1$ is convex by definition. It remains to show that $\mathcal{T}_1$ is sequentially closed under pointwise convergence. To this end, let $\{\lambda_k\widehat{T} + (1-\lambda_k)\widehat{T}_1\}_{k=1}^{\infty}$ be a sequence in $\mathcal{T}_1$ that converges to a distortion function $\tilde{T}$ pointwise. Since $\lambda_k \in [0,1]$ for all k, the sequence $\{\lambda_k\}_{k=1}^{\infty}$ admits a converging subsequence $\{\lambda_{k_l}\}_{l=1}^{\infty}$ with limit $\lambda \in [0,1]$. It then follows that for each $t\in[0,1]$, we have

$$\lim_{l\to\infty}\left\{\lambda_{k_l}\widehat{T}(t) + (1-\lambda_{k_l})\widehat{T}_1(t)\right\} = \lambda\widehat{T}(t) + (1-\lambda)\widehat{T}_1(t) = \tilde{T}(t)\,.$$

Hence, $\tilde{T} = \lambda\widehat{T} + (1-\lambda)\widehat{T}_1 \in \mathcal{T}_1$. □

We construct the risk measures for Agents 2 and 3 in a similar manner. Note that in the particular case of Agent 1, the Expected Shortfall at a level of 1% is always greater than that at a level of 2.5%, and so the agent's risk measure can be instead just represented as $\mathrm{ES}^{\mathbb{P}}_{1\%}$. However, this will not be the case for the other agents.

We assume that Agent 2 also wishes to meet the requirement given by the distortion function

$$\widehat{T}_2(t) := \min\{(t/0.05)^{0.3}, 1\}\,.$$

This is similar to the Expected Shortfall at a level of 5%, but assigns more weight to the extreme tail risk. Note that $\widehat{T}_2$ is concave. Let $\mathcal{T}_2$ be the convex hull of the set $\left\{\widehat{T}, \widehat{T}_2\right\}$. By Lemma 5.5.4, the risk measure of Agent 2 is represented by the set $\mathcal{T}_2$, in the sense that

$$\rho_2(Z) = \sup_{T\in\mathcal{T}_2} \int Z\, dT \circ \mathbb{P},\ \forall Z \in \mathcal{X}\,.$$

Finally, Agent 3 chooses to implement a distortion function given by Wang's transform (see Wang [107]) as follows:

$$\widehat{T}_3(t) := \Phi(\Phi^{-1}(t) + 2.8)\,,$$

where Φ denotes the distribution function of the standard normal distribution. Similarly, we let $\mathcal{T}_3$ denote the convex hull of the set $\left\{\widehat{T}, \widehat{T}_3\right\}$. A comparison of all distortion functions used in this scenario is illustrated in Figure 5.2.

We can now apply Corollary 5.5.1 to solve for comonotone Pareto optima in this risk-sharing market. First, it is determined through numerical optimization that the vector

$$(T_1^*,\, T_2^*,\, T_3^*) := (\widehat{T}_1,\, \widehat{T}_2,\, 0.2269\,\widehat{T} + 0.7731\,\widehat{T}_3)$$

is the unique solution to (5.6). This yields the retention functions shown in Figure 5.4. Furthermore, the optimal retention functions are unique in this case. Indeed, the marginal retention for each layer of the risk $S > x$ depends only on the agents with the lowest value of $T_i^*(\mathbb{P}(S > x))$, that is, the most optimistic likelihood assessment of the risk layer $S > x$. The agents' assessments of the likelihoods of tail events are shown in Figure 5.3.

Now, recall from Corollary 5.5.1 that for each $x \in \mathbb{R}_+$, the set L_{x,T_1^*,T_2^*,T_3^*} denotes the set of agents with the most optimistic likelihood assessment of the tail event $S > x$, when the preference of agent i is represented by a distortion risk measure with respect to T_i^*. It can be numerically verified that the agents with the most optimistic view towards the

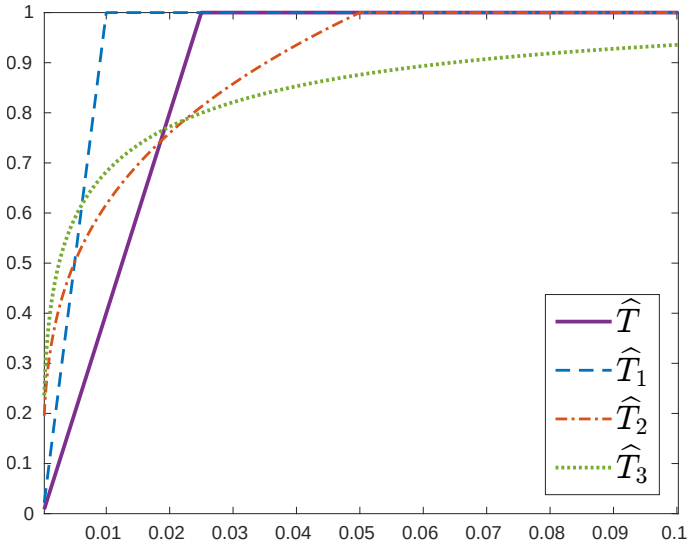


Figure 5.2: Probability Distortion Functions.

likelihood of such tail events are as follows:

$$L_{x,T_1^*,T_2^*,T_3^*} = \begin{cases} \{3\}, & x \in (0, 53.302) \cup (68.164, 74.287) \\ \{2\}, & x \in (53.302, 68.164) \\ \{1\}, & x > 74.287 \end{cases}.$$

Figure 5.4 shows how the retention function for agent i increases with a slope of 1 whenever $L_{x,T_1^*,T_2^*,T_3^*} = \{i\}$, that is, agent i absorbs that tranche of the aggregate risk Since L_{x,T_1^*,T_2^*,T_3^*} is a singleton for a.e. x, it follows that each marginal retention is a.e. unique, which implies that retention functions are unique. That is, the retention structure shown in Figure 5.4 is the only possible structure that is comonotone Pareto optimal. Note that the retention is only increasing for an agent when that agent is most optimistic about the likelihood of the tail risk.

5.6 Conclusion

In this chapter, we provide a characterization of Pareto-optimal allocations in a pure-exchange economy, in which agents have preferences represented by law-invariant positively homogeneous monetary utilities. Since these preferences are consistent with respect to the

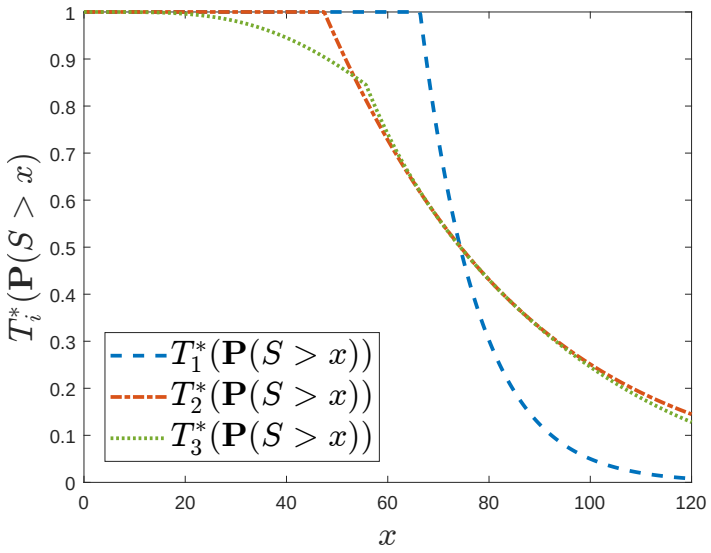


Figure 5.3: Assessments of Tail Risks $T_i^*(\mathbb{P}(S > x))$.

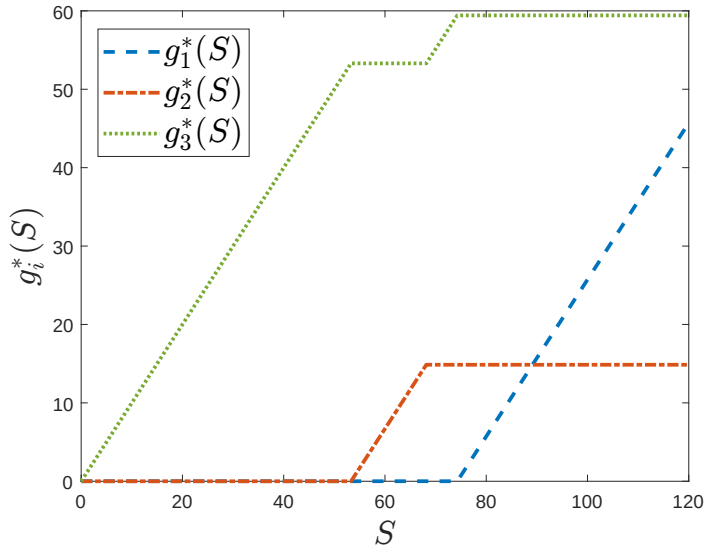


Figure 5.4: Pareto-Optimal Retention Structure.

concave order, the classical comonotone improvement result applies. Identifying Pareto-optimal allocations then reduces to solving a sup-convolution problem over the set of comonotone allocations. By combining this result with duality representations of these preferences, we obtain a new characterization of Pareto-optimal allocations. The main advantage of our result is that it provides explicit formulas for the shape of the comonotone optimal allocations themselves, as well as a clear and easily implementable algorithm for finding these optima.

Our results may also be interpreted in the context of risk sharing, where these preferences are more commonly known as law-invariant coherent risk measures. This is a wide class of risk measures that encompasses many popular functionals in the insurance and risk management literature, including the expected shortfall. As a special case of our characterization result, we recover an explicit characterization of Pareto-optimal allocations when agents use law-invariant coherent risk measures. We apply this characterization to a problem of risk sharing in a numerical illustration.

5.7 Proofs of Main Results

5.7.1 Proof of Proposition 5.2.5

Suppose that $\{Y_i^*\}_{i=1}^n \in \mathcal{A}$ is not PO, and hence not weakly PO under our assumptions. Then by Lemma 5.2.4, there exists another IR allocation $\{Y_i\}_{i=1}^n$ such that

$$U_i(Y_i) > U_i(Y_i^*), \ \forall\, i \in \mathcal{N}\,.$$

Therefore, since $\lambda \geq 0$ and not all entries of λ are zero,

$$\sum_{i=1}^n \lambda_i\, U_i(Y_i) > \sum_{i=1}^n \lambda_i\, U_i(Y_i^*)\,.$$

Conversely, suppose $\{Y_i^*\}_{i=1}^n$ is PO. Define subsets of $\mathbb{R}^n$ by the following:

$$\begin{aligned}
\mathcal{U} &:= co\left\{\{U_i(Y_i)\}_{i=1}^n : \{Y_i\}_{i=1}^n \in \mathcal{IR}\right\},\\
\mathcal{U}^- &:= \{r \in \mathbb{R}^n : r \leq u \text{ for some } u \in \mathcal{U}\}\,,\\
\mathcal{V} &:= \{r \in \mathbb{R}^n : r_i \geq U_i(Y_i^*)\} \setminus \{U_i(Y_i^*)\}_{i=1}^n\,,
\end{aligned}$$

where *co* denotes the convex hull. Then by construction, both $\mathcal{U}$ and $\mathcal{V}$ are convex sets, and it is easy to verify that $\mathcal{U}^-$ is convex.

We claim that $\mathcal{U}^- \cap \mathcal{V} = \varnothing$. Suppose for the sake of contradiction that $(r_1, \ldots, r_n) \in \mathcal{U}^- \cap \mathcal{V}$. Then $(r_1, \ldots, r_n)$ is dominated by some convex combination of elements in $\mathcal{U}$. That is, for each $i \in \mathcal{N}$, we have

$$r_i \leq \sum_{k=1}^{m} t_k \, U_i \left(Y_i^{(k)} \right) ,$$

where $\sum_{k=1}^{m} t_k = 1$ and $\left\{ Y_i^{(k)} \right\}_{i=1}^{n} \in \mathcal{IR}$ for all $k \in 1, \ldots, m$. Recall that for each $i \in \mathcal{N}$, the initial endowment of agent i is denoted by X_i. By concavity of each U_i, we have

$$U_i(X_i) = \sum_{k=1}^{m} t_k \, U_i(X_i) \leq \sum_{k=1}^{m} t_k \, U_i \left(Y_i^{(k)} \right) \leq U_i \left(\sum_{k=1}^{m} t_k \, Y_i^{(k)} \right) . \tag{5.7}$$

Furthermore,

$$U_i(Y_i^*) \leq r_i \leq \sum_{k=1}^{m} t_k \, U_i \left(Y_i^{(k)} \right) \leq U_i \left(\sum_{k=1}^{m} t_k \, Y_i^{(k)} \right) , \tag{5.8}$$

where the inequality is strict for some $i \in \mathcal{N}$. On the other hand,

$$\sum_{i=1}^{n} \sum_{k=1}^{m} t_k \, Y_i^{(k)} = \sum_{k=1}^{m} t_k \sum_{i=1}^{n} Y_i^{(k)} = \sum_{k=1}^{m} t_k \, S = S \, ,$$

implying that $\left\{ \sum_{k=1}^{m} t_k \, Y_i^{(k)} \right\}_{i=1}^{n} \in \mathcal{A}$, and hence $\left\{ \sum_{k=1}^{m} t_k \, Y_i^{(k)} \right\}_{i=1}^{n} \in \mathcal{IR}$ by (5.7). Therefore $\left\{ \sum_{k=1}^{m} t_k \, Y_i^{(k)} \right\}_{i=1}^{n}$ is an IR allocation that improves upon $\{Y_i^*\}_{i=1}^{n}$, which contradicts the assumption that $\{Y_i^*\}_{i=1}^{n} \in \mathcal{PO}$. We conclude that $\mathcal{U}^- \cap \mathcal{V} = \varnothing$.

The hyperplane separation theorem [4, Theorem 5.61] then implies that there exists $\lambda \in \mathbb{R}^n \setminus \{0\}$ such that

$$\lambda \cdot u \leq \lambda \cdot v, \ \forall \, (u, v) \in \mathcal{U}^- \times \mathcal{V} \, .$$

Since $\{U_i(Y_i^*)\}_{i=1}^{n}$ is a limit point of $\mathcal{V}$, the above implies

$$\lambda \cdot u \leq \lambda \cdot \{U_i(Y_i^*)\}_{i=1}^{n}, \ \forall \, u \in \mathcal{U}^- \, . \tag{5.9}$$

We now show that $\lambda \geq 0$. To this end, for $i \in \mathcal{N}$, let $e_i \in \mathbb{R}^n$ be the vector with 1 in its i-th coordinate, and 0 elsewhere. Then since $\{U_i(Y_i^*)\}_{i=1}^{n} \in \mathcal{U}$, we have $\{U_i(Y_i^*)\}_{i=1}^{n} - e_j \in \mathcal{U}^-$ for any $j \in \mathcal{N}$. Substituting this into (5.9) gives

$$\lambda \cdot \{U_i(Y_i^*)\}_{i=1}^{n} - \lambda_j \leq \lambda \cdot \{U_i(Y_i^*)\}_{i=1}^{n} \ \text{ and } \ \lambda_j \geq 0.$$

To complete the proof, note that (5.9) implies that

$$\sum_{i=1}^{n} \lambda_i U(Y_i^*) \geq \lambda \cdot u, \ \forall u \in \mathcal{U}.$$

However, (5.7) and (5.8) imply that for any $(u_1, \ldots, u_n) \in \mathcal{U}$, it is possible to find an allocation $\{Y_i\}_{i=1}^n \in \mathcal{IR}$ such that

$$U_i(Y_i) \geq u_i, \ \forall i \in \mathcal{N}.$$

Hence, we have

$$\sum_{i=1}^{n} \lambda_i U(Y_i^*) \geq \sup_{\{Y_i\}_{i=1}^n \in \mathcal{IR}} \sum_{i=1}^{n} \lambda_i U(Y_i),$$

and since $\{Y_i^*\}_{i=1}^n$ is feasible for this problem, it must be a solution. □

5.7.2 Proof of Theorem 5.3.12

By Proposition 5.2.5 and Corollary 5.3.9, we have $\mathcal{PO} = \cup_{\lambda \in \Lambda} \mathcal{S}_\lambda$ and $\mathcal{CPO} = \cup_{\lambda \in \Lambda} \mathcal{CS}_\lambda$. For a given $\lambda \in \Lambda$, we will first show that if $\mathcal{S}_\lambda$ is non-empty, then $\mathcal{CS}_\lambda$ is also non-empty and (5.2) holds. To conclude the proof, we show that $\mathcal{CS}_\lambda \neq \varnothing$ implies $\mathcal{S}_\lambda \neq \varnothing$.

Suppose that $\mathcal{S}_\lambda$ is non-empty, and let $\{Y_i^*\}_{i=1}^n \in \mathcal{S}_\lambda$. Then it is immediate that

$$\sum_{i=1}^{n} \lambda_i U_i\left(Y_i^*\right) = \sup_{\{Y_i\}_{i=1}^n \in \mathcal{IR}} \left\{ \sum_{i=1}^{n} \lambda_i U_i(Y_i) \right\} \geq \sup_{\{Y_i\}_{i=1}^n \in \mathcal{IR} \cap \mathcal{A}_C} \left\{ \sum_{i=1}^{n} \lambda_i U_i(Y_i) \right\}.$$

By Proposition 5.3.10, there exists a comonotone allocation $\left\{\tilde{Y}_i\right\}_{i=1}^n \in \mathcal{A}_C$ such that

$$Y_i^* \preccurlyeq_{CCV} \tilde{Y}_i, \ \forall i \in \{1, \ldots, n\}.$$

Since each U_i is Schur concave, this implies that

$$U_i\left(\tilde{Y}_i\right) \geq U_i(Y_i^*), \ \forall i \in \{1, \ldots, n\}.$$

However, since $\{Y_i^*\}_{i=1}^n$ is Pareto optimal, equality must hold for each i. That is,

$$U_i\left(\tilde{Y}_i\right) = U_i(Y_i^*), \ \forall i \in \{1, \ldots, n\},$$

implying that $\left\{\tilde{Y}_i\right\}_{i=1}^n \in \mathcal{IR} \cap \mathcal{A}_C$. Hence,

$$\sup_{\{Y_i\}_{i=1}^n \in \mathcal{IR} \cap \mathcal{A}_C} \left\{ \sum_{i=1}^n \lambda_i \, U_i(Y_i) \right\} \geq \sum_{i=1}^n \lambda_i \, U_i \left(\tilde{Y}_i \right) = \sup_{\{Y_i\}_{i=1}^n \in \mathcal{IR}} \left\{ \sum_{i=1}^n \lambda_i \, U_i(Y_i) \right\},$$

so (5.2) holds.

It remains to show that if $\mathcal{CS}_\lambda$ is non-empty, then so is $\mathcal{S}_\lambda$. Let $\{Y_i^*\}_{i=1}^n \in \mathcal{CS}_\lambda$. It is immediate that

$$\sup_{\{Y_i\}_{i=1}^n \in \mathcal{IR}} \left\{ \sum_{i=1}^n \lambda_i \, U_i(Y_i) \right\} \geq \sup_{\{Y_i\}_{i=1}^n \in \mathcal{IR} \cap \mathcal{A}_C} \left\{ \sum_{i=1}^n \lambda_i \, U_i(Y_i) \right\}.$$

Suppose for the sake of contradiction that this inequality is strict. That is, there exists an allocation $\{\hat{Y}_i\}_{i=1}^n \in \mathcal{IR}$ such that

$$\sum_{i=1}^n \lambda_i \, U_i \left(\hat{Y}_i \right) > \sup_{\{Y_i\}_{i=1}^n \in \mathcal{IR} \cap \mathcal{A}_C} \left\{ \sum_{i=1}^n \lambda_i \, U_i(Y_i) \right\} = \sum_{i=1}^n \lambda_i \, U_i(Y_i^*).$$

Then by Proposition 5.3.10, there exists a comonotone allocation $\left\{\tilde{Y}_i\right\}_{i=1}^n \in \mathcal{A}_C$ such that

$$U_i \left(\tilde{Y}_i \right) \geq U_i \left(\hat{Y}_i \right), \ \forall i \in \{1, \ldots, n\},$$

which implies that $\left\{\tilde{Y}_i\right\}_{i=1}^n$ is individually rational as well. Therefore

$$\begin{aligned} \sup_{\{Y_i\}_{i=1}^n \in \mathcal{IR} \cap \mathcal{A}_C} \left\{ \sum_{i=1}^n \lambda_i \, U_i \left(Y_i \right) \right\} &\geq \sum_{i=1}^n \lambda_i \, U_i \left(\tilde{Y}_i \right) \geq \sum_{i=1}^n \lambda_i \, U_i \left(\hat{Y}_i \right) > \sum_{i=1}^n \lambda_i \, U_i \left(Y_i^* \right) \\ &= \sup_{\{Y_i\}_{i=1}^n \in \mathcal{IR} \cap \mathcal{A}_C} \left\{ \sum_{i=1}^n \lambda_i \, U_i \left(Y_i \right) \right\}, \end{aligned}$$

a contradiction. Hence, (5.2) holds, and $\{Y_i^*\}_{i=1}^n \in \mathcal{S}_\lambda$, implying that $\mathcal{S}_\lambda \neq \varnothing$, as desired. □

5.7.3 Proof of Proposition 5.4.3

If $\{Y_i^*\}_{i=1}^n \notin \mathcal{PO}$, then there exists $\left\{\tilde{Y}_i\right\}_{i=1}^n \in \mathcal{IR}$ such that

$$U_i \left(\tilde{Y}_i \right) \geq U_i \left(Y_i^* \right),$$

with at least one strict inequality, which implies that

$$\sum_{i=1}^{n} U_i\left(\tilde{Y}_i\right) > \sum_{i=1}^{n} U_i\left(Y_i^*\right) .$$

Therefore, $\{Y_i^*\}_{i=1}^n \notin \mathcal{S}_{\mathbf{1}}$, and hence $\mathcal{S}_1 \subseteq \mathcal{PO}$.

To show the reverse inclusion, assume, by way of contradiction, that there exists $\{Y_i^*\}_{i=1}^n \in \mathcal{PO}$ such that $\{Y_i^*\}_{i=1}^n \notin \mathcal{S}_{\mathbf{1}}$. Then, there exists $\left\{\tilde{Y}_i\right\}_{i=1}^n \in \mathcal{IR}$ such that

$$\sum_{i=1}^{n} U_i\left(\tilde{Y}_i\right) > \sum_{i=1}^{n} U_i\left(Y_i^*\right) . \tag{5.10}$$

Define $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3 \subseteq \mathcal{N}$ such that,

$$U_i\left(\tilde{Y}_i\right) < U_i\left(Y_i^*\right), \quad \forall i \in \mathcal{N}_1 ,$$

$$U_i\left(\tilde{Y}_i\right) = U_i\left(Y_i^*\right), \quad \forall i \in \mathcal{N}_2 ,$$

$$U_i\left(\tilde{Y}_i\right) > U_i\left(Y_i^*\right), \quad \forall i \in \mathcal{N}_3 .$$

Note that $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3$ is a partition of $\mathcal{N}$. Moreover, by (5.10), $\mathcal{N}_3 \neq \varnothing$. By assumption, $\{Y_i^*\}_{i=1}^n \in \mathcal{PO}$, which implies that $\mathcal{N}_1 \neq \varnothing$. Define, for $i \in \mathcal{N}_1$,

$$\varepsilon_i := U_i\left(Y_i^*\right) - U_i\left(\tilde{Y}_i\right) > 0 .$$

Then, by (5.10), there exist $\{\varepsilon_i\}_{i\in\mathcal{N}_3}$ such that, (i) $\varepsilon_i \geq 0$, for $i \in \mathcal{N}_3$; (ii) $U_i\left(\tilde{Y}_i - \varepsilon_i\right) \geq U_i\left(Y_i^*\right)$, for $i \in \mathcal{N}_3$, with at least one strict inequality; and (iii) $\sum_{i\in\mathcal{N}_3} \varepsilon_i = \sum_{i\in\mathcal{N}_1} \varepsilon_i$. Let

$$\hat{Y}_i := \begin{cases} \tilde{Y}_i + \varepsilon_i , & \forall i \in \mathcal{N}_1 \\ \tilde{Y}_i & \forall i \in \mathcal{N}_2 \\ \tilde{Y}_i - \varepsilon_i , & \forall i \in \mathcal{N}_3 \end{cases} .$$

Note that $\left\{\hat{Y}_i\right\}_{i=1}^n \in \mathcal{A}$, since

$$\sum_{i=1}^{n} \hat{Y}_i = \sum_{i=1}^{n} \tilde{Y}_i - \sum_{i\in\mathcal{N}_1} \varepsilon_i + \sum_{i\in\mathcal{N}_3} \varepsilon_i = \sum_{i=1}^{n} \tilde{Y}_i = \sum_{i=1}^{n} X_i .$$

Moreover, $\left\{\hat{Y}_i\right\}_{i=1}^n \in \mathcal{IR}$, since

$$\begin{aligned} U_i\left(\hat{Y}_i\right) &= U_i\left(\tilde{Y}_i\right) + \varepsilon_i = U_i\left(\tilde{Y}_i\right) + \left(U_i\left(\tilde{Y}_i\right) - U_i\left(Y_i^*\right)\right) \\ &= U_i\left(Y_i^*\right) \geq U_i\left(X_i\right), \quad \forall i \in \mathcal{N}_1, \\ U_i\left(\hat{Y}_i\right) &= U_i\left(\tilde{Y}_i\right) = U_i\left(Y_i^*\right) \geq U_i\left(X_i\right), \quad \forall i \in \mathcal{N}_2, \\ U_i\left(\hat{Y}_i\right) &= U_i\left(\tilde{Y}_i - \varepsilon_i\right) \geq U_i\left(Y_i^*\right) \geq U_i\left(X_i\right), \quad \forall i \in \mathcal{N}_3, \end{aligned} \tag{5.11}$$

in which (5.11) has at least one strict inequality. Hence, $\{Y_i^*\}_{i=1}^n \notin \mathcal{PO}$, a contradiction. □

5.7.4 Proof of Lemma 5.4.9

By Lemma 5.4.7, there exists a convex law-invariant set $\mathcal{H} \subseteq L_+^\infty$ such that

$$\mathcal{U}(Z) = \inf_{T \in \{\phi_H : H \in \mathcal{H}\}} \int Z \, dT \circ \mathbb{P}.$$

We claim that defining $\mathcal{T}$ to be the pointwise closure of the set

$$\{\phi_H : H \in \mathcal{H} \cap \mathcal{X}^\uparrow\}$$

yields the desired convex set of distortions. It is immediate that

$$\inf_{T \in \{\phi_H : H \in \mathcal{H}\}} \int Z \, dT \circ \mathbb{P} \leq \inf_{T \in \{\phi_H : H \in \mathcal{H} \cap \mathcal{X}^\uparrow\}} \int Z \, dT \circ \mathbb{P}.$$

Suppose for the sake of contradiction that there exists $\tilde{H} \in \mathcal{H}$ such that

$$\int Z \, d\phi_{\tilde{H}} \circ \mathbb{P} < \inf_{T \in \{\phi_H : H \in \mathcal{H} \cap \mathcal{X}^\uparrow\}} \int Z \, dT \circ \mathbb{P}.$$

Then since the probability space is non-atomic and $\mathcal{H}$ is a law-invariant set, there exists $H^\uparrow \in \mathcal{H} \cap \mathcal{X}^\uparrow$ such that $H^\uparrow$ and $\tilde{H}$ have the same distribution (see Dana [38, Lemma 4.2]). However, by the Hardy-Littlewood inequality (e.g., Föllmer and Schied [57, Theorem A.28]), we have

$$\int Z \, d\phi_{H^\uparrow} \circ \mathbb{P} = \int_0^1 F_Z^{-1}(t) F_{H^\uparrow}^{-1}(1-t) \, dt \leq \int_0^1 F_Z^{-1}(t) F_{\tilde{H}}^{-1}(1-t) \, dt = \int Z \, d\phi_{\tilde{H}} \circ \mathbb{P},$$

a contradiction. Hence,

$$\mathcal{U}(Z) = \inf_{T \in \{\phi_H : H \in \mathcal{H}\}} \int Z \, dT \circ \mathbb{P} = \inf_{T \in \{\phi_H : H \in \mathcal{H} \cap \mathcal{X}^\uparrow\}} \int Z \, dT \circ \mathbb{P} \, .$$

Next, we show that $\{\phi_H : H \in \mathcal{H} \cap \mathcal{X}^\uparrow\}$ is convex. To this end, let $\lambda \in (0,1)$ and $H_1, H_2 \in \mathcal{H} \cap \mathcal{X}^\uparrow$. For every $x \in [0,1]$, we have

$$\begin{aligned}
\lambda \phi_{H_1}(x) + (1-\lambda)\, \phi_{H_2}(x) &= \lambda \int_0^x F_{H_1}^{-1}(t)\, dt + (1-\lambda) \int_0^x F_{H_2}^{-1}(t)\, dt \\
&= \int_0^x \lambda\, F_{H_1}^{-1}(t) + (1-\lambda)\, F_{H_2}^{-1}(t)\, dt = \int_0^x F^{-1}_{\lambda\, H_1 + (1-\lambda)\, H_2}(t)\, dt \\
&= \phi_{\lambda\, H_1 + (1-\lambda)\, H_2}(x),
\end{aligned}$$

where the third equality holds since H_1 and H_2 are comonotone. Since $\mathcal{H} \cap \mathcal{X}^\uparrow$ is convex, we have $\lambda\, Y_1 + (1-\lambda)\, Y_2 \in \mathcal{H} \cap \mathcal{X}^\uparrow$, and so $\{\phi_H : H \in \mathcal{H} \cap \mathcal{X}^\uparrow\}$ is convex as well.

Finally, we show that the Choquet integral $\int Z \, dT \circ \mathbb{P}$ is sequentially continuous in T with respect to pointwise convergence. Let $\{T^{(k)}\}_{k=1}^\infty$ be a sequence of distortion functions such that for all $t \in [0,1]$, we have $\lim_{k\to\infty} T^{(k)}(t) = T(t)$, for some function T. It is immediate that if $\{T^{(k)}\}_{k=1}^\infty$ converges pointwise to T and each $T^{(k)}$ is a convex distortion function, then T is also a convex distortion function. Let $Z \in \mathcal{X}$, and let $K := ||Z||_\infty < \infty$. Then we have

$$\begin{aligned}
\lim_{k\to\infty} \int Z \, dT^{(k)} \circ \mathbb{P} &= \lim_{k\to\infty} \int_0^\infty T^{(k)}(\mathbb{P}(Z > x))\, dx + \lim_{k\to\infty} \int_{-\infty}^0 [T^{(k)}(\mathbb{P}(Z > x)) - 1]\, dx \\
&= \lim_{k\to\infty} \int_0^K T^{(k)}(\mathbb{P}(Z > x))\, dx + \lim_{k\to\infty} \int_{-K}^0 [T^{(k)}(\mathbb{P}(Z > x)) - 1]\, dx \\
&= \int_0^K \lim_{k\to\infty} T^{(k)}(\mathbb{P}(Z > x))\, dx + \int_{-K}^0 \Big[\lim_{k\to\infty} T^{(k)}(\mathbb{P}(Z > x)) - 1\Big]\, dx \\
&= \int_0^\infty T(\mathbb{P}(Z > x))\, dx + \int_{-\infty}^0 [T(\mathbb{P}(Z > x)) - 1]\, dx = \int Z \, dT \circ \mathbb{P} \, ,
\end{aligned}$$

where we may apply the dominated convergence theorem to exchange the order of the limit and the integral, since for any $x \in \mathbb{R}$,

$$|T(\mathbb{P}(Z > x))| \le 1 \quad \text{and} \quad |T(\mathbb{P}(Z > x)) - 1| \le 1 \, .$$

Hence, the Choquet integral is sequentially continuous in T. Therefore

$$\mathcal{U}(Z) = \inf_{T \in \{\phi_H : H \in \mathcal{H} \cap \mathcal{X}^\uparrow\}} \int Z \, dT \circ \mathbb{P} = \inf_{T \in \mathcal{T}} \int Z \, dT \circ \mathbb{P} \, ,$$

where $\mathcal{T}$ has the desired properties. □

5.7.5 Proof of Theorem 5.4.11

(i) Since the objective function of (5.3) is non-negative, we have

$$-\infty < V := \inf_{\{T_i\}_{i=1}^n \in \prod_{i=1}^n \mathcal{T}_i} \int_0^\infty \max_{i\in\mathcal{N}} \{T_i(\mathbb{P}(S > \underline{s} + x))\}\, dx\,.$$

Let $\{(T_1^{(k)}, \ldots, T_n^{(k)})\}_{k=1}^\infty$ be a sequence such that

$$\int_0^\infty \max_{i\in\mathcal{N}} \{T_i^{(k)}(\mathbb{P}(S > \underline{s} + x))\}\, dx \le V + \frac{1}{k}\,.$$

Then since each distortion is a monotone function on the bounded interval $[0,1]$, by Helly's compactness theorem (e.g., Doob [50, pp. 165-166]), there exists a subsequence $\left\{\left(T_1^{(k_j)}, \ldots, T_n^{(k_j)}\right)\right\}_{j=1}^\infty$ for which $\left\{T_1^{(k_j)}\right\}_{j=1}^\infty$ converges pointwise to a limit T_1^*. Since $\mathcal{T}_1$ is closed under pointwise convergence by Lemma 5.4.9, we have $T_1^* \in \mathcal{T}_1$. Applying Helly's compactness theorem again to this subsequence gives another sequence $\left\{(T_1^{(k_l)}, \ldots, T_n^{(k_l)})\right\}_{l=1}^\infty$ such that $\left\{T_1^{(k_l)}\right\}_{l=1}^\infty$ converges pointwise to $T_1^* \in \mathcal{T}_1$ and $\left\{T_2^{(k_l)}\right\}_{l=1}^\infty$ converges pointwise to $T_2^* \in \mathcal{T}_2$. Hence, iterating this process n times yields a subsequence $\left\{\left(T_1^{(k_m)}, \ldots, T_n^{(k_m)}\right)\right\}_{m=1}^\infty$ that converges pointwise to a limit

$$(T_1^*, \ldots, T_n^*) \in \prod_{i=1}^n \mathcal{T}_i\,.$$

Therefore,

$$\begin{aligned} V &\le \int_0^\infty \max_{i\in\mathcal{N}} \{T_i^*(\mathbb{P}(S > \underline{s} + x))\}\, dx = \int_0^\infty \max_{i\in\mathcal{N}} \left\{\lim_{m\to\infty} T_i^{(k_m)}(\mathbb{P}(S > \underline{s} + x))\right\} dx \\ &= \lim_{m\to\infty} \int_0^\infty \max_{i\in\mathcal{N}} \left\{T_i^{(k_m)}(\mathbb{P}(S > \underline{s} + x))\right\} dx \le V + \lim_{m\to\infty} \frac{1}{k_m} = V, \end{aligned}$$

where we can exchange the limit and the integral by dominated convergence. Hence, $(T_1^*, \ldots, T_n^*)$ is a solution to (5.3).

(ii) By Corollary 5.4.4, it suffices to characterize $\mathcal{CS}_\mathbf{1}$, i.e., solutions to the problem

$$\sup_{\{Y_i\}_{i=1}^n \in \mathcal{IR}\cap\mathcal{A}_C} \left\{\sum_{i=1}^n U_i\left(Y_i\right)\right\}.$$

By Lemma 5.4.10, this allocation can be written in terms of functions $\{g_i\}_{i=1}^n \in \mathcal{G}$ and constants $\{c_i\}_{i=1}^n \in \mathbb{R}^n$ where $\sum_{i=1}^n c_i = \underline{s}$. Conversely, if $\{g_i\}_{i=1}^n \in \mathcal{G}$ and $\{c_i\}_{i=1}^n \in \mathbb{R}^n$ with $\sum_{i=1}^n c_i = \underline{s}$, then $\{g_i(S-\underline{s})+c_i\}_{i=1}^n \in \mathcal{A}_C$. Therefore

$$\begin{aligned}\sup_{\{Y_i\}_{i=1}^n \in \mathcal{IR}\cap\mathcal{A}_C} \left\{\sum_{i=1}^n U_i\left(Y_i\right)\right\} &= \sup_{\substack{\left(\{g_i\}_{i=1}^n,\{c_i\}_{i=1}^n\right)\in\mathcal{IR}\cap(\mathcal{G}\times\mathbb{R}^n) \\ \sum_{i=1}^n c_i=\underline{s}}} \left\{\sum_{i=1}^n U_i\left(g_i(S-\underline{s})+c_i\right)\right\} \\ &= \sup_{\left(\{g_i\}_{i=1}^n,\{c_i\}_{i=1}^n\right)\in\mathcal{IR}\cap(\mathcal{G}\times\mathbb{R}^n)} \left\{\sum_{i=1}^n U_i\left(g_i(S-\underline{s})\right)\right\}+\underline{s}\,,\end{aligned}$$

where we write $(\{g_i\}_{i=1}^n, \{c_i\}_{i=1}^n) \in \mathcal{IR}$ when the allocation $\{g_i(S-\underline{s})+c_i\}_{i=1}^n \in \mathcal{IR}$. This problem is solved by $(\{g_i^*\}_{i=1}^n, \{c_i^*\}_{i=1}^n) \in \mathcal{G}\times\mathbb{R}^n$ if and only if $\{g_i^*\}_{i=1}^n$ solves

$$\sup_{\{g_i\}_{i=1}^n\in\mathcal{G}} \left\{\sum_{i=1}^n U_i(g_i(S-\underline{s}))\right\}, \tag{5.12}$$

and the constants c_i^* are chosen such that $\{g_i^*(S-\underline{s})+c_i^*\}_{i=1}^n \in \mathcal{IR}$ and $\sum_{i=1}^n c_i^* = \underline{s}$. We will show that the form given in the statement of the theorem is a necessary condition for $\{g_i^*\}_{i=1}^n$ to be a solution to (5.12).

By Lemma 5.4.7, we may rewrite (5.12) as follows:

$$\sup_{\{g_i\}_{i=1}^n\in\mathcal{G}} \left\{\sum_{i=1}^n U_i\left(g_i(S-\underline{s})\right)\right\} = \sup_{\{g_i\}_{i=1}^n\in\mathcal{G}} \left\{\sum_{i=1}^n \inf_{T_i\in\mathcal{T}_i} \int g_i(S-\underline{s})\,dT_i\circ\mathbb{P}\right\}.$$

For each $i \in \mathcal{N}$, let

$$A_i := \left\{\int g_i(S-\underline{s})\,dT_i\circ\mathbb{P} : T_i\in\mathcal{T}_i\right\}.$$

Then we have

$$\sup_{\{g_i\}_{i=1}^n \in \mathcal{G}} \left\{ \sum_{i=1}^n \inf_{T_i \in \mathcal{T}_i} \int g_i(S - \underline{s})\, dT_i \circ \mathbb{P} \right\} \tag{5.13}$$

$$= \sup_{\{g_i\}_{i=1}^n \in \mathcal{G}} \left\{ \sum_{i=1}^n \inf A_i \right\}$$

$$= \sup_{\{g_i\}_{i=1}^n \in \mathcal{G}} \left\{ \inf \left(\sum_{i=1}^n A_i \right) \right\}$$

$$= \sup_{\{g_i\}_{i=1}^n \in \mathcal{G}} \inf_{\{T_i\}_{i=1}^n \in \prod_{i=1}^n \mathcal{T}_i} \sum_{i=1}^n \int g_i(S - \underline{s})\, dT_i \circ \mathbb{P}\,, \tag{5.14}$$

since the infimum commutes with the Minkowski sum $\sum_{i=1}^n A_i$.

Note that the range of $S - \underline{s}$ is contained within the interval $[0, 2M]$, where M is the essential supremum of S. Let $C([0, 2M])$ denote the set of continuous functions on $[0, 2M]$, which is a Banach space under the supremum norm. Let $\mathcal{D} := \mathbb{R}^{[0,1]}$ denote the space of functions from $[0, 1] \to \mathbb{R}$, which is a topological vector space with the topology of pointwise convergence. Consider the objective function of (5.14) as a function from $C([0, 2M])^n \times \mathcal{D}^n$ to $\mathbb{R}$ with the product topology of the spaces $C([0, 2M])$ and $\mathcal{D}$.

Since the Choquet integral is comonotone additive, this objective function is linear in both $\{g_i\}_{i=1}^n$ and $\{T_i\}_{i=1}^n$. Both $\mathcal{G}$ and $\prod_{i=1}^n \mathcal{T}_i$ are convex, with the latter due to Lemma 5.4.9. Furthermore, since $\mathcal{G}$ is a closed subset of a Cartesian product of 1-Lipschitz functions on the interval $[0, 2M]$, it is also compact by the Arzela-Ascoli Theorem (see e.g., Dunford and Schwartz [51, IV.6.7]).

We now verify some continuity properties of the objective function. Firstly, for each $i \in \mathcal{N}$, let $\left\{ g_i^{(k)} \right\}_{k=1}^{\infty}$ be a sequence that converges to g_i uniformly (i.e., with respect to the supremum norm on $C([0, 2M])$). Then $g_i^{(k)}(S - \underline{s}) \to g_i(S - \underline{s})$ uniformly on L^∞. The objective function is therefore continuous in $\{g_i\}_{i=1}^n$, since the Choquet integral is continuous with respect to the L^∞ norm.[1] Furthermore, by the proof of Lemma 5.4.9, the objective is sequentially continuous in each T_i under pointwise convergence.

Therefore, by Sion's minimax theorem (e.g., Komiya [73]), the minimax equality holds for this problem. Exchanging the order of the supremum and infimum yields

[1] In fact, it is Lipschitz continuous: see Marinacci and Montrucchio [86, Proposition 4.11].

the minimax problem

$$\inf_{\{T_i\}_{i=1}^n \in \prod_{i=1}^n \mathcal{T}_i} \max_{\{g_i\}_{i=1}^n \in \mathcal{G}} \sum_{i=1}^n \int g_i(S - \underline{s})\, dT_i \circ \mathbb{P}\,, \tag{5.15}$$

where the inner supremum is attained due to compactness.

Recall that a solution to (5.12) exists (by Filipovic and Svindland [56, Theorem 2.5]). Let $\{g_i^*\}_{i=1}^n$ be a solution to (5.12). Since the minimax equality holds, it follows from standard results on minimax problems (e.g., Barbu and Precupanu [10, Section 2.3]) that for every vector of distortions $\{T_i^*\}_{i=1}^n$ solving (5.15), the pair $(\{T_i^*\}_{i=1}^n, \{g_i^*\}_{i=1}^n)$ is a saddle point of (5.15) as a minimax problem. Hence, it must be true that

$$\begin{aligned}\sum_{i=1}^n \int g_i^*(S - \underline{s})\, dT_i^* \circ \mathbb{P} &= \inf_{\{T_i\}_{i=1}^n \in \prod_{i=1}^n \mathcal{T}_i} \max_{\{g_i\}_{i=1}^n \in \mathcal{G}} \sum_{i=1}^n \int g_i(S - \underline{s})\, dT_i \circ \mathbb{P} \\ &= \max_{\{g_i\}_{i=1}^n \in \mathcal{G}} \sum_{i=1}^n \int g_i(S - \underline{s})\, dT_i^* \circ \mathbb{P}\,.\end{aligned}$$

To complete the proof, we will now show that for any fixed vector of convex distortion functions $(T_1, \dots, T_n)$, we have

$$\max_{\{g_i\}_{i=1}^n \in \mathcal{G}} \sum_{i=1}^n \int g_i(S - \underline{s})\, dT_i \circ \mathbb{P} = \int_0^\infty \max_{i \in \mathcal{N}} \{T_i(\mathbb{P}(S > \underline{s} + x))\}\, dx\,,$$

and hence problems (5.15) and (5.3) are equivalent. Furthermore, this maximum is attained at $\{g_i^*\} \in \mathcal{G}$ if and only if $g_i^*(x) = \int_0^x h_i(z)\, dz$, and

$$\sum_{i \in L_x} h_i(x) = 1 \text{ and } \sum_{i \in \mathcal{N} \setminus L_x} h_i(x) = 0\,,$$

where

$$L_x := \left\{ i \in \mathcal{N} : T_i(\mathbb{P}(S > \underline{s} + x)) = \max_{j \in \mathcal{N}} \{T_j(\mathbb{P}(S > \underline{s} + x))\} \right\}.$$

This is the characterization of optimal allocations for Yaari utilities, and a proof can be found in Liu [82], for instance. We provide the full argument below for completeness.

Since each g_i is 1-Lipschitz and non-negative, by a standard result (e.g., Zhuang et al. [114, Lemma 2.1]), we may rewrite the Choquet integral as

$$\max_{\{g_i\}_{i=1}^n \in \mathcal{G}} \sum_{i=1}^n \int_0^\infty T_i(\mathbb{P}(S > \underline{s} + x))\, g_i'(x)\, dx\,. \tag{5.16}$$

Here, $g_i'(x)$ is understood as a function for which $g_i(x) = \int_0^x g_i'(z)\, dz$, since each g_i is absolutely continuous. Let $\{g_i^*\}_{i=1}^n \in \mathcal{G}$ satisfy the form given in the statement of the theorem. We first check that $\{g_i^*\}_{i=1}^n \in \mathcal{G}$. By construction, each g_i^* is increasing. Furthermore, for all $x \in \mathbb{R}_+$, we have

$$\sum_{i=1}^n g_i^*(x) = \sum_{i=1}^n \int_0^x h_i(z)\, dz = \int_0^x \sum_{i=1}^n h_i(z)\, dz = \int_0^x 1\, dz = x\,,$$

and so $\{g_i^*\}_{i=1}^n \in \mathcal{G}$.

Now suppose that $\{\tilde{g}_i\}_{i=1}^n \in \mathcal{G}$. For ease of notation, let $L_x^C := \mathcal{N} \setminus L_x$. Then we have

$$\begin{aligned}
&\sum_{i=1}^n U_i(\tilde{g}_i(S - \underline{s}) \\
&\quad = \sum_{i=1}^n \int_0^\infty T_i(\mathbb{P}(S > \underline{s} + x))\tilde{g}_i'(x)\, dx \\
&\quad = \int_0^\infty \sum_{i=1}^n T_i(\mathbb{P}(S > \underline{s} + x))\tilde{g}_i'(x)\, dx \\
&\quad \le \int_0^\infty \sum_{i=1}^n \max_{i \in \mathcal{N}} \{T_i(\mathbb{P}(S > \underline{s} + x))\}\tilde{g}_i'(x)\, dx \qquad (5.17) \\
&\quad = \int_0^\infty \max_{i \in \mathcal{N}} \{T_i(\mathbb{P}(S > \underline{s} + x))\} \sum_{i=1}^n \tilde{g}_i'(x)\, dx \\
&\quad = \int_0^\infty \max_{i \in \mathcal{N}} \{T_i(\mathbb{P}(S > \underline{s} + x))\}\, dx \\
&\quad = \int_0^\infty \max_{i \in \mathcal{N}} \{T_i(\mathbb{P}(S > \underline{s} + x))\} + \sum_{i \in L_x^C} \left\{T_i(\mathbb{P}(S > \underline{s} + x)) \cdot 0\right\} dx \\
&\quad = \int_0^\infty \max_{i \in \mathcal{N}} \{T_i(\mathbb{P}(S > \underline{s} + x))\} \cdot \sum_{i \in L_x} h_i(x) + \sum_{i \in L_x^C} \left\{T_i(\mathbb{P}(S > \underline{s} + x)) \cdot h_i(x)\right\} dx
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^\infty \sum_{i\in L_x} \left\{ \max_{i\in\mathcal{N}} \{T_i(\mathbb{P}(S > \underline{s}+x))\} \cdot h_i(x) \right\} + \sum_{i\in L_x^C} \left\{ T_i(\mathbb{P}(S > \underline{s}+x)) \cdot h_i(x) \right\} dx \\
&= \int_0^\infty \sum_{i\in L_x} \left\{ T_i(\mathbb{P}(S > \underline{s}+x)) \cdot h_i(x) \right\} + \sum_{i\in L_x^C} \left\{ T_i(\mathbb{P}(S > \underline{s}+x)) \cdot h_i(x) \right\} dx \\
&= \int_0^\infty \sum_{i=1}^n T_i(\mathbb{P}(S > \underline{s}+x)) h_i(x)\, dx \\
&= \sum_{i=1}^n \int_0^\infty T_i(\mathbb{P}(S > \underline{s}+x)) h_i(x)\, dx \\
&= \sum_{i=1}^n U_i(g_i^*(S-\underline{s}))\,,
\end{aligned}
$$

implying that $\{g_i^*\}_{i=1}^n$ solves (5.16). The above also shows that the optimal value of (5.16) is

$$
\int_0^\infty \max_{i\in\mathcal{N}} \{T_i(\mathbb{P}(S > \underline{s}+x))\}\, dx\,.
$$

Conversely, suppose that $\tilde{g}_i$ are not of the specified form. That is , we have $\sum_{i\in L_x^C} \tilde{g}_i'(x) > 0$ on a set $\mathbf{A}$ of positive measure. Then for every x in $\mathbf{A}$,

$$
\begin{aligned}
&\sum_{i=1}^n T_i(\mathbb{P}(S > \underline{s}+x)) \tilde{g}_i'(x) \\
&\qquad = \sum_{i\in L_x} T_i(\mathbb{P}(S > \underline{s}+x)) \tilde{g}_i'(x) + \sum_{i\in L_x^C} T_i(\mathbb{P}(S > \underline{s}+x)) \tilde{g}_i'(x) \\
&\qquad = \max_{i\in\mathcal{N}} \{T_i(\mathbb{P}(S > \underline{s}+x))\} \tilde{g}_i'(x) + \sum_{i\in L_x^C} T_i(\mathbb{P}(S > \underline{s}+x)) \tilde{g}_i'(x) \\
&\qquad < \max_{i\in\mathcal{N}} \{T_i(\mathbb{P}(S > \underline{s}+x))\} \tilde{g}_i'(x) + \sum_{i\in L_x^C} \max_{i\in\mathcal{N}} \{T_i(\mathbb{P}(S > \underline{s}+x))\} \tilde{g}_i'(x) \\
&\qquad = \max_{i\in\mathcal{N}} \{T_i(\mathbb{P}(S > \underline{s}+x))\} = \sum_{i=1}^n \max_{i\in\mathcal{N}} \{T_i(\mathbb{P}(S > \underline{s}+x))\} \tilde{g}_i'(x)\,,
\end{aligned}
$$

where the strict inequality follows because L_x^C is non-empty and $\tilde{g}_i'(x)$ are not all zero for $i \in L_x^C$. Taking the integral over the set $\mathbf{A}$ of positive measure gives

$$
\int_{\mathbf{A}} \sum_{i=1}^n T_i(\mathbb{P}(S > \underline{s}+x)) \tilde{g}_i'(x)\, dx < \int_{\mathbf{A}} \sum_{i=1}^n \max_{i\in\mathcal{N}} \{T_i(\mathbb{P}(S > \underline{s}+x))\} \tilde{g}_i'(x)\, dx\,.
$$

Therefore the inequality (5.17) is strict in this case, implying that $\tilde{g}_i(x)$ does not solve (5.16). □

Remark 5.7.1. *In our application of Sion's minimax theorem in the proof of Theorem 5.4.11 above, we show that the objective function (5.14) is sequentially continuous on each $\mathcal{T}_i$. However, standard statements of Sion's minimax theorem in the literature (e.g., Komiya [73], Barbu and Precupanu [10, Theorem 2.132]) require that the objective function be lower semicontinuous with respect to the topological vector space over which the infimum is taken. Nonetheless, a careful examination of the proof of Sion's minimax theorem shows that sequential lower semicontinuity is sufficient for the result. See, in particular, Komiya [73, Lemma 1].*

This slight generalization is particularly relevant in our case, since our objective function (5.14) is not, in general, continuous on each $\mathcal{T}_i$ with respect to the topology of pointwise convergence. This is due to the fact that the dominated convergence theorem does not generalize from sequences of functions to nets of functions. Since the topology of pointwise convergence is not metrizable, it follows that sequential continuity does not necessarily imply continuity.

5.8 Counterexample to Sufficiency of Theorem 5.4.11 (ii)

Suppose that there are $n = 2$ agents in the market, and that the aggregate endowment S is a continuous random variable. Let T_1 and T_2 be two convex distortion functions such that $T_1 < T_2$ with positive measure, and $T_1 > T_2$ also with positive measure. Suppose further that $\int S\, dT_1 \circ \mathbb{P} \neq \int S\, dT_2 \circ \mathbb{P}$, and without loss of generality, let $\int S\, dT_1 \circ \mathbb{P} > \int S\, dT_2 \circ \mathbb{P}$.

Let Agent 1's preferences be represented by a Yaari dual utility with respect to the convex distortion function T_1. This preference can be represented by a the singleton set

$$\mathcal{T}_1 := \{T_1\}\,,$$

in the sense of Lemma 5.4.9. That is, for each $Z \in \mathcal{X}$,

$$U_1(Z) = \inf_{T \in \mathcal{T}_1} \int Z\, dT \circ \mathbb{P} = \int Z\, dT_1 \circ \mathbb{P}\,.$$

Let $\mathcal{T}_2$ be the closed convex hull of T_1 and T_2, and suppose that

$$U_2(Z) = \inf_{T \in \mathcal{T}_2} \int Z\, dT \circ \mathbb{P}\,.$$

Since $T_1 \in \mathcal{T}_2$, Agent 2's utility functional is always dominated by Agent 1's utility functional. This implies that it is comonotone Pareto optimal for Agent 1 to assume all of the variability in the market. Indeed, for any $(g_1, g_2) \in \mathcal{G}$, we have

$$U_1(g_1(S)) + U_2(g_2(S)) \leq U_1(g_1(S)) + U_1(g_2(S)) = U_1(g_1(S) + g_2(S)) = U_1(S)\,.$$

However, applying the result of Theorem 5.4.11 does not easily identify the comonotone Pareto-optimal allocations in this scenario. Firstly, since $T_2 > T_1$ on a set of positive measure and S is a continuous random variable, it follows that the set

$$\left\{x \in \mathbb{R}_+ : T_2(\mathbb{P}(S > \underline{s} + x)) > T_1(\mathbb{P}(S > \underline{s} + x))\right\}$$

also has positive measure. This implies that for each $\lambda \in (0, 1]$, we have

$$\begin{aligned}
&\int_0^\infty \max\left\{T_1(\mathbb{P}(S > \underline{s} + x)), \lambda T_2(\mathbb{P}(S > \underline{s} + x)) + (1-\lambda)T_1(\mathbb{P}(S > \underline{s} + x))\right\} dx \\
&\quad = \int_0^\infty T_1(\mathbb{P}(S > \underline{s} + x))\, dx + \lambda \int_0^\infty \max\left\{T_2(\mathbb{P}(S > \underline{s} + x)) - T_1(\mathbb{P}(S > \underline{s} + x)), 0\right\} dx \\
&\quad > \int_0^\infty T_1(\mathbb{P}(S > \underline{s} + x))\, dx\,.
\end{aligned}$$

Hence, the unique optimizer to (5.3) in this market is the pair $\{T_1, T_1\}$. When applying part (ii) of Theorem 5.4.11, we see that $L_{x,T_1,T_1} = \{1, 2\}$ for all $x \in \mathbb{R}_+$. Therefore, the necessary condition for comonotone Pareto optimality provided by Theorem 5.4.11 allows for any comonotone allocation in $\mathcal{G}$.

Nonetheless, not all comonotone allocations are Pareto optimal. Consider the two allocations $(S - c^*, c^*)$ and $(\tilde{c}, S - \tilde{c})$, where c^* and $\tilde{c}$ are chosen in $\mathbb{R}$ such that both allocations are individually rational. Then we have

$$U_1(\tilde{c}) + U_2(S - \tilde{c}) = U_2(S) = \int S\, dT_2 \circ \mathbb{P} < \int S\, dT_1 \circ \mathbb{P} = U_1(S) = U_1(S - c^*) + U_2(c^*)\,,$$

implying that the allocation $(\tilde{c}, S - \tilde{c})$ is comonotone but not comonotone Pareto optimal. Hence, the necessary condition provided by Theorem 5.4.11 is not sufficient in this case.

5.9 Proofs for Section 5.4.5

5.9.1 Proof of Proposition 5.4.19

The result follows from Filipovic and Kupper [55] (Theorems 3.1 and 3.2), provided that the preferences are $\sigma(L^\infty, L^1)$ upper semicontinuous and that the value of the sup-convolution

problem $\mathcal{S}_1$ is finite. Since monetary utilities are norm-continuous, law invariance implies $\sigma(L^\infty, L^1)$ upper semicontinuity by Theorem 2.2 of Jouini et al. [68]. Moreover, Theorems 5.3.12 and 5.4.11 imply that the value of $\mathcal{S}_1$ is the value of (5.3), which is finite. □

5.9.2 Proof of Lemma 5.4.20

This follows from the fact that the inequality in the demand problem (5.5) must hold with equality at optimum. Indeed, if we have $\mathbb{E}^{\mathbb{Q}^*}[Y_i^* + b_i] < \mathbb{E}^{\mathbb{Q}^*}[X_i]$, then

$$U_i\left(Y_i^* + b_i + \mathbb{E}^{\mathbb{Q}^*}[X_i] - \mathbb{E}^{\mathbb{Q}^*}[Y_i^* + b_i]\right) = U_i(Y_i^* + b_i) + \mathbb{E}^{\mathbb{Q}^*}[X_i] - \mathbb{E}^{\mathbb{Q}^*}[Y_i^* + b_i] > U_i(Y_i^* + b_i)\,,$$

and so $Y_i^* + b_i$ does not solve (5.5). Therefore we must have $\mathbb{E}^{\mathbb{Q}^*}[Y_i^* + b_i] = \mathbb{E}^{\mathbb{Q}^*}[X_i]$, which implies that $b_i = \mathbb{E}^{\mathbb{Q}^*}[X_i - Y_i^*]$. □

5.9.3 Proof of Proposition 5.4.21

Note that since $(Y_1^*, \ldots, Y_n^*) \in \mathcal{CPO}$, $(Y_1^* + \mathbb{E}^{\mathbb{Q}^*}[X_1 - Y_1^*], \ldots, Y_n^* + \mathbb{E}^{\mathbb{Q}^*}[X_n - Y_n^*])$ is an allocation, the market-clearing condition is satisfied. Suppose that, for each $i \in \mathcal{N}$, $Y_i^* + \mathbb{E}^{\mathbb{Q}^*}[X_i - Y_i^*]$ solves the demand problem (5.5) under the pricing measure $\mathbb{Q}^*$. Recall from Lemma 5.4.10 that every comonotone allocation $(Y_1, \ldots, Y_n)$ has the form

$$Y_i = g_i(S - \underline{s}) + c_i\,,$$

where $\{g_i\}_{i=1}^n \in \mathcal{G}$ and $\{c_i\}_{i=1}^n \in \mathbb{R}^n$ satisfies $\sum_{i=1}^n c_i = \underline{s}$. In particular, we can write $Y_i^* + \mathbb{E}^{\mathbb{Q}^*}[X_i - Y_i^*] = g_i^*(S - \underline{s}) + c_i^*$, where $\{g_i^*\}_{i=1}^n \in \mathcal{G}$ and $\{c_i^*\}_{i=1}^n \in \mathbb{R}^n$. Then for each $i \in \mathcal{N}$, since $Y_i^* + \mathbb{E}^{\mathbb{Q}^*}[X_i - Y_i^*]$ solves the demand problem (5.5), it follows that g_i^* and c_i^* must be a solution to

$$\begin{aligned} \max_{g_i \in \mathcal{I},\, c_i \in \mathbb{R}} \quad & U_i(g_i(S - \underline{s}) + c_i) \\ s.t. \quad & \mathbb{E}^{\mathbb{Q}^*}[g_i(S - \underline{s}) + c_i] = \mathbb{E}^{\mathbb{Q}^*}[X_i]\,, \end{aligned}$$

where $\mathcal{I}$ is the set of non-decreasing 1-Lipschitz functions on $\mathbb{R}_+$. By applying translation invariance of U_i, we see that g_i^* must be a solution to

$$\max_{g_i \in \mathcal{I}} \left\{U_i(g_i(S - \underline{s})) + \mathbb{E}^{\mathbb{Q}^*}[X_i] - \mathbb{E}^{\mathbb{Q}^*}[g_i(S - \underline{s})]\right\}\,.$$

Since $\mathbb{E}^{\mathbb{Q}^*}[X_i]$ is a constant, if g_i^* solves the above, it must also be a solution to

$$\max_{g_i \in \mathcal{I}} \left\{U_i(g_i(S - \underline{s})) - \mathbb{E}^{\mathbb{Q}^*}[g_i(S - \underline{s})]\right\}\,. \tag{5.18}$$

We now provide an explicit solution to (5.18). By rewriting the Choquet integral according to Zhuang et al. [114, Lemma 2.1], the objective of (5.18) becomes

$$\int_0^\infty T_i(\mathbb{P}(S > \underline{s} + x))\, g_i'(x)\, dx - \int_0^\infty \mathbb{Q}^*(S > \underline{s} + x)\, g_i'(x)\, dx$$
$$= \int_0^\infty \big(T_i(\mathbb{P}(S > \underline{s} + x)) - \mathbb{Q}^*(S > \underline{s} + x)\big)\, g_i'(x)\, dx\,.$$

It follows that g_i^* solves (5.18) if and only if $g_i^*(x) = \int_0^x h_i(z)\, dz$, where h_i satisfies

$$h_i(x) = \begin{cases} 1\,, & \text{if } T_i(\mathbb{P}(S > \underline{s} + x)) > \mathbb{Q}^*(S > \underline{s} + x)\,, \\ 0\,, & \text{if } T_i(\mathbb{P}(S > \underline{s} + x)) < \mathbb{Q}^*(S > \underline{s} + x)\,, \end{cases} \tag{5.19}$$

for almost every $x \in \mathbb{R}_+$.

We conclude the proof by showing that h_i satisfies (5.19) for all $i \in \mathcal{N}$ if and only if

$$T_{(n-1)}(\mathbb{P}(S > \underline{s} + x)) \le \mathbb{Q}^*(S > \underline{s} + x) \le T_{(n)}(\mathbb{P}(S > \underline{s} + x))\,, \tag{5.20}$$

for almost every $x \in \mathbb{R}_+$. First, suppose that (5.20) holds. Recall from our characterization of $\mathcal{CPO}$ in Corollary 5.4.17 that h_i satisfies

$$\sum_{i \in L_x} h_i(x) = 1 \text{ and } \sum_{i \in L_x^C} h_i(x) = 0\,.$$

If $T_i(\mathbb{P}(S > \underline{s} + x)) > \mathbb{Q}^*(S > \underline{s} + x)$, then (5.20) implies that $L_x = \{i\}$, and hence $h_i(x) = 1$. Similarly, if $T_i(\mathbb{P}(S > \underline{s} + x)) < \mathbb{Q}^*(S > \underline{s} + x)$, then (5.20) implies that $i \notin L_x$, and hence $h_i(x) = 0$.

For the converse, suppose that (5.20) does not hold. If $\mathbb{Q}^*(S > \underline{s} + x) < T_{(n-1)}(\mathbb{P}(S > \underline{s} + x)$ and h_i satisfies (5.19) for all $i \in \mathcal{N}$, then there exists $j, k \in \mathcal{N}$ such that $j \neq k$ and $h_j(x) = h_k(x) = 1$. Therefore $\sum_{i=1}^n h_i(x) \ge 2 \neq 1$, contradicting the results of Corollary 5.4.17. Hence, h_i does not satisfy (5.19) for all $i \in \mathcal{N}$. Similarly, if $\mathbb{Q}^*(S > \underline{s} + x) > T_{(n)}(\mathbb{P}(S > \underline{s} + x)$, then (5.19) would imply that $\sum_{i=1}^n h_i(x) = 0 \neq 1$, a contradiction. Therefore h_i satisfies (5.19) for all $i \in \mathcal{N}$ if and only if (5.20) holds, thereby completing the proof. □

Chapter 6

Efficiency with Robust Distortion Risk Measures

6.1 Introduction

In this chapter, we study Pareto optimality in a decentralized peer-to-peer risk-sharing market where agents' preferences are represented by robust distortion risk measures that are not necessarily convex. In our setting, robustness is with respect to the distortion function used, not with respect to the probability measure on the space. This is in contrast to Bernard et al. [14], but it is in line with Wang and Xu [109], who argue that these risk measures can be motivated by the ambiguity present in each agent's preference. This type of robustness also emerges naturally in the context of coherent risk measures (e.g., Dana [38]). Moreover, we allow for heterogeneity both in the baseline probability measures used by the agents and in the sets of distortion functions used by the agents. The set of feasible allocations is assumed to consist of comonotone allocations, that is, allocations that are 1-Lipschitz functions of the aggregate risk in the market. This is related to the so-called *no-sabotage* condition of Carlier and Dana [28, 29] typically assumed in centralized insurance markets, which guarantees that no agent has an incentive to misreport their actual realized loss *ex post*.

Our main result (Theorem 6.3.3) provides a characterization of Pareto-optimal allocations of the aggregate risk in the market. In particular, we show that the shape of the allocations depends primarily on each agent's assessment of the tail of the aggregate risk. We quantify the latter via an index of probabilistic risk aversion. In a special case of our result where the set of distortion functions for each agent is in fact a singleton (Corollary

6.3.4), we recover the explicit form of optimal allocations under distortion risk measures previously attained by Liu [82]. We then provide several illustrations of our results using concrete examples of popular families of distortion functions.

As an application of our results, we revisit the market for flood risk insurance in the United States through the lens of peer-to-peer insurance. We present the decentralized risk sharing arrangement as an alternative to the current centralized market structure, and we characterize the optimal allocations in a numerical study with historical flood data. Specifically, we revisit the numerical study of Boonen et al. [17], which examines a centralized market with the federal government acting as the sole provider of insurance. However, as argued in Chapter 3, this centralized structure provides an incentive for the central authority to charge extremely high prices at the expense of the policyholders. We therefore present the decentralized risk-sharing scheme as an alternative, and we characterize the optimal allocations thereof using historical data. We find that while the decentralized risk-sharing market avoids Stackelberg equilibria, the trade-off is that the average welfare gain is lower than that of the centralized market.

The remainder of this chapter is structured as follows. Section 6.2 formulates the optimal allocation problem for decentralized markets. Section 6.3 provides the characterization of Pareto optima in this setting, as well as some general results for common families of distortion risk measures. We apply our results to the market for flood risk insurance in the United States in Section 6.4, in which we also discuss the advantages and disadvantages of a decentralized market scheme. Section 6.5 concludes.

6.2 Problem Setup

In this chapter, we take $\mathcal{X} = L^\infty(\Omega, \Sigma, \mathbb{P})$, the set of essentially bounded random variables on the atomless probability space $(\Omega, \Sigma, \mathbb{P})$. As in Chapter 5, there are $n \in \mathbb{N}$ agents wishing to reallocate their initial endowments among themselves without central authority involvement. For each $i \in \mathcal{N} := \{1, \dots, n\}$, let $X_i \in \mathcal{X}$ denote the initial endowment of the i-th agent to be contributed to the risk-sharing pool. We consider a one-period, risk-sharing economy, where all risks are realized at the end of the period. The risk-sharing mechanism in this market is as follows.

At the end of the period, the aggregate loss $S := \sum_{i=1}^n X_i$ is redistributed to the participants of the pool. If $S \geq 0$, the pool is subject to a residual aggregate loss, whereas if $S \leq 0$, the pool has a monetary surplus to be shared among the agents at the end of the period. In this chapter, we decompose the redistribution to each agent into two parts.

Firstly, let $Y_i \in \mathcal{X}_+$ denote the component of the aggregate loss S that is borne by the i-th agent, for each $i \in \mathcal{N}$. This risky component is then adjusted by a real number $\pi_i \in \mathbb{R}$. A positive value of π_i represents a premium paid by agent i for participation in the pool, which reflects the scenario where this agent is being insured by the pool. In other words, the allocation of the component Y_i to agent i is enough of an improvement to justify the collection of a premium payment as compensation. On the other hand, a negative value of π_i represents a premium paid to the agent i for participation in the pool. Under these conventions, for each $i \in \mathcal{N}$, the end-of-period, post-transfer risk exposure of agent i is given by $Y_i + \pi_i$. Note that $S = \sum_{i=1}^n X_i = Y_i + \pi_i$.

Remark 6.2.1. *Note that this formulation slightly generalizes the usual formulation in the literature by decomposing the allocation into a risky part and a cash transfer. This can be compared to the usual formulation by defining $Z_i := Y_i + \pi_i$ to be the resulting allocation to agent i. Note that for each $i \in \mathcal{N}$, the allocation Z_i is almost surely bounded by below.*

We assume that the market is a comonotone risk-sharing market, as previously studied by Boonen et al. [21]. This is a market in which the admissible contracts are only those that are comonotone with the aggregate initial risk. Under this restriction, the contracts all satisfy the aforementioned *no-sabotage* condition, which guarantees that no agent has an incentive to misreport their actual realized loss. This is one possible justification for the existence of such a market, since preventing *ex-post* moral hazard is in the best interest of every agent. An alternative justification arises from the situation where each agent's preference is monotone with respect to the convex order, as explored in Chapter 5.

The set of *ex ante* admissible decision variables is therefore given by:

$$\tilde{\mathcal{A}} := \left\{ (\{Y_i\}_{i=1}^n, \{\pi_i\}_{i=1}^n) \in \mathcal{X}_+^n \times \mathbb{R}^n : \sum_{i=1}^n (Y_i + \pi_i) = \sum_{i=1}^n X_i,\ \{Y_i\}_{i=1}^n \text{ is comonotone} \right\}.$$

Remark 6.2.2. *Note that $\{Y_i + \pi_i\}_{i=1}^n$ is comonotone if and only if $\{Y_i\}_{i=1}^n$ is comonotone. Recall that $\sum_{i=1}^n (Y_i + \pi_i) = \sum_{i=1}^n X_i = S$ is the aggregate risk present in the insurance market. By a standard result (e.g., Denneberg [47, Proposition 4.5]), if $\{Y_i + \pi_i\}_{i=1}^n$ is comonotone, then there exist increasing 1-Lipschitz functions f_i such that $Y_i + \pi_i = f_i(S)$.*

The preferences of each agent i, for $i \in \mathcal{N}$, are represented by given risk measures ρ_i. Consequently, each agent i measures their pre-transfer risk exposure by $\rho_i(X_i)$, and their post-transfer risk exposure by $\rho_i(Y_i + \pi_i)$. We will assume throughout that all risk measures are monotone.

To characterize Pareto optimality in this setting, we apply the methods of Chapter 5. In the case where every agent uses a translation invariant risk measure, we again have a characterization of PO in terms of solutions of a minimization problem. Let $\mathcal{S}$ denote the set of all solutions to the problem

$$\inf_{(\{Y_i\}_{i=1}^n, \{\pi_i\}_{i=1}^n) \in \mathcal{IR}} \sum_{i=1}^n \rho_i(Y_i + \pi_i)\,. \tag{6.1}$$

Then by Proposition 5.4.3, $\mathcal{PO} = \mathcal{S}$.

Moreover, all comonotone allocations are translations of a suitable non-decreasing function of the aggregate endowment. Recall the set of vectors of functions $\mathcal{G}$:

$$\mathcal{G} := \left\{ \{g_i\}_{i=1}^n \,\middle|\, g_i : \mathbb{R}_+ \to \mathbb{R}_+ \text{ non-decreasing, and } \sum_{i=1}^n g_i(\cdot) = \mathrm{Id} \right\}.$$

By Lemma 5.4.10, there exist functions $\{g_i\}_{i=1}^n \in \mathcal{G}$ and constants $\{c_i\}_{i=1}^n \in \mathbb{R}^n$ such that

$$Y_i + \pi_i = g_i(S) + c_i \quad \forall i \in \mathcal{N}\,,$$

and $\sum_{i=1}^n c_i = 0$.

6.3 Pareto Optima for Robust Distortion Risk Measures

In this section, we provide an explicit characterization of Pareto-optimal allocations when agents use robust distortion risk measures. These are defined below.

Definition 6.3.1. A risk measure $\varrho : \mathcal{X} \to \mathbb{R}$ is a robust distortion risk measure if there exists a set of distortion functions $\mathcal{T}$ and a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ such that for all random variables Z on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$,

$$\varrho(Z) = \sup_{T \in \mathcal{T}} \int Z\, dT \circ \mathbb{P}\,.$$

Robustness in the context of Definition 6.3.1 refers to the consideration of different distortion functions by the decision maker, not different probability distributions on the underlying measurable space. Our definition is in line with that of Wang and Xu [109],

who argue that these risk measures can be motivated by the ambiguity present in each agent's preference We note that the use of the term *robust distortion risk measure* is not consistent throughout the literature. For instance, Bernard et al. [14] uses this term to refer to the case where the robustness relates to uncertainty in the distribution of the risk. In contrast, Definition 6.3.1 assumes that the distribution of a given risk is fixed with respect to a reference probability measure.

Lemma 6.3.2. *For a robust distortion risk measure with respect to a set of distortion functions $\mathcal{T}$, we have*

$$\sup_{T\in\mathcal{T}} \int Z\, dT\circ\mathbb{P} = \sup_{T\in\overline{\mathrm{co}}(\mathcal{T})} \int Z\, dT\circ\mathbb{P}\,,$$

where $\overline{\mathrm{co}}(\mathcal{T})$ is the closed convex hull of $\mathcal{T}$ with respect to pointwise convergence.

Proof. Since the Choquet integral is linear in T, we have that

$$\sup_{T\in\mathcal{T}} \int Z\, dT\circ\mathbb{P} = \sup_{T\in\mathrm{co}(\mathcal{T})} \int Z\, dT\circ\mathbb{P}\,,$$

where $\mathrm{co}(\mathcal{T})$ denotes the convex hull of $\mathcal{T}$. It suffices to show that the Choquet integral is continuous with respect to pointwise convergence of T. To this end, let $\{T^{(k)}\}_{k=1}^{\infty}$ be a sequence of continuous functions converging pointwise to T. Then we have

$$\begin{aligned}\lim_{k\to\infty}\int Z\, dT_k\circ\mathbb{P} &= \lim_{k\to\infty}\int_0^\infty T_k(\mathbb{P}(Z>x))\,dx = \int_0^\infty \lim_{k\to\infty} T_k(\mathbb{P}(Z>x))\,dx\\ &= \int_0^\infty T(\mathbb{P}(Z>x))\,dx = \int Z\, dT\circ\mathbb{P}\,,\end{aligned}$$

where we may apply the dominated convergence theorem since

$$\int_0^\infty |T_k(\mathbb{P}(Z>x))|\,dx \le \int_0^\infty \mathbb{1}_{\{||Z||_\infty>x\}}\,dx = ||Z||_\infty < \infty\,.$$

□

We now assume that for each $i\in\mathcal{N}$, the risk measure ρ_i is a robust distortion risk measure. We allow for each agent to have heterogeneous beliefs, represented by different probability measures $\mathbb{Q}_i$ on the same measurable space. Specifically, we assume that for each $i\in\mathcal{N}$, there exists a set of distortion functions $\mathcal{T}_i$ such that

$$\rho_i(Z) = \sup_{T_i\in\mathcal{T}_i}\int Z\, dT_i\circ\mathbb{Q}_i\,, \tag{6.2}$$

for all risks $Z \in \mathcal{X}$. By Lemma 6.3.2, we may assume without loss of generality that each $\mathcal{T}_i$ is convex and closed with respect to pointwise convergence.

Note that distortion risk measures are translation invariant. For translation-invariant risk measures, a characterization of Pareto-optimal allocations is given by Proposition 5.4.3. However, in the case of robust distortion risk measures, a more explicit characterization of the set $\mathcal{S}$ is possible, as shown by the following result. The proof can be found in Section 6.6.

Theorem 6.3.3. *Suppose that for each $i \in \mathcal{N}$, the risk measure ρ_i is given by a robust distortion risk measure as in (6.2). Then the following hold:*

(i) There exists a solution to the problem

$$\sup_{\{T_i\}_{i=1}^n \in \prod_{i=1}^n \mathcal{T}_i} \int_0^\infty \min_{i \in \mathcal{N}} \{T_i(\mathbb{Q}_i(S > x))\}\, dx\,. \tag{6.3}$$

(ii) A necessary condition for the allocation $(\{Y_i^*\}_{i=1}^n, \{\pi_i^*\}_{i=1}^n)$ to be Pareto optimal is that

$$Y_i^* + \pi_i^* = g_i^*(S) + c_i^*\,,$$

where $\{c_i^*\}_{i=1}^n \in \mathbb{R}^n$ is chosen such that $\sum_{i=1}^n c_i^* = 0$ and $\{g_i^*(S) + c_i^*\}_{i=1}^n \in \mathcal{IR}$, and $\{g_i^*\}_{i=1}^n \in \mathcal{G}$ can be written in terms of the integrals of suitable functions h_i. Specifically, for each $i \in \mathcal{N}$, we can write $g_i^*(x) = \int_0^x h_i(z)\, dz$, where each $h_i : \mathbb{R}_+ \to [0,1]$ is a function such that for almost every $x \in \mathbb{R}_+$,

$$\sum_{i \in L_x} h_i(x) = 1 \text{ and } \sum_{i \in L_x^C} h_i(x) = 0\,,$$

where

$$L_x := \left\{ i \in \mathcal{N} : T_i^*(\mathbb{Q}_i(S > x)) = \min_{j \in \mathcal{N}} \{T_j^*(\mathbb{Q}_j(S > x))\} \right\},$$
$$L_x^C = \mathcal{N} \setminus L_x\,,$$

and $(T_1^*, \dots, T_n^*)$ is a solution to (6.3).

In other words, Theorem 6.3.3 provides a necessary condition for an allocation to be Pareto optimal. This result is similar to the characterization obtained by Theorem 5.4.11 in the previous chapter. However, a key difference is that the present result of Theorem

6.3.3 does not require concavity of the distortion function (or equivalently, the convexity of the risk measure). The present result also holds under a setting of heterogeneous beliefs, as represented by the different probability measures $\mathbb{Q}_i$. These stronger conclusions can be obtained in the setting of this chapter because the admissible allocations are constrained to be comonotone, which was not the case in Chapter 5.

In particular, Pareto-optimal allocations are obtained as translations of suitable functions of the aggregate risk. Part (ii) of Theorem 6.3.3 decomposes the optimal risk allocation for agent i into a constant c_i^* and a risky portion $g_i^*(S)$, which is normalized so that $g_i^*(0) = 0$. Additionally, for any fixed $\{g_i^*\}_{i=1}^n$ given by Theorem 6.3.3, it is always possible to choose $\{c_i^*\}_{i=1}^n \in \mathbb{R}^n$ such that $\{g_i^*(S) + c_i^*\}_{i=1}^n \in \mathcal{IR}$. One such choice is given by

$$c_i^* := \rho_i(X_i) - \rho(g_i^*(S)), \quad i \in \{1, \dots, n-1\}, \qquad c_n^* := -\sum_{i=1}^{n-1} c_i^*. \tag{6.4}$$

Then by translation invariance of each ρ_i, we have $\rho_i(X_i) - \rho(g_i^*(S) + c_i^*) = 0$ for $i \in \{1, \dots, n\}$. Furthermore,

$$\begin{aligned}\rho_n(X_n) - \rho_n(g_n^*(S) + c_n^*) &= \rho_n(X_n) - \rho_n(g_n^*(S)) + \sum_{i=1}^{n-1}(\rho_i(X_i) - \rho(g_i^*(S))) \\ &= \left(\sum_{i=1}^{n} \rho_i(X_i)\right) - \left(\sum_{i=1}^{n} \rho_i(g_i^*(S))\right) \geq 0,\end{aligned}$$

where the last inequality follows since $\{g_i^*(S)+c_i^*\}_{i=1}^n \in \mathcal{S}$ and the no-risk-sharing arrangement $\{X_i\}$ is individually rational.

More generally, the role of the constants c_i^* is to determine the distribution of the aggregate welfare gain from P2P insurance among the agents in the market. We define the aggregate welfare gain as the difference between the aggregate valuation of the individuals' initial risks, given by $\sum_{i=1}^n \rho_i^*(X_i)$, and the aggregate valuation of the agents' post-transfer risks, given by $\sum_{i=1}^n \rho_i(g_i^*(S) + c_i^*)$:

$$W := \sum_{i=1}^{n} (\rho_i(X_i) - \rho_i(g_i^*(S) + c_i^*)) = \sum_{i=1}^{n} (\rho_i(X_i) - \rho_i(g_i^*(S))) .$$

Note that $\sum_{i=1}^n \rho_i(g_i^*(S)+c_i^*)$ is the value of problem (6.1), and therefore does not depend on the choice of the Pareto-optimal allocation $\{g_i^*(S)+c_i^*\}_{i=1}^n$. Hence, the aggregate welfare

gain W is well-defined, and $W \geq 0$. Now let $\{w_i\}_{i=1}^n \in \mathbb{R}^n$ be such that $\sum_{i=1}^n w_i = W$ and $w_i \geq 0$ for all $i \in \mathcal{N}$. Fix $\{g_i^*\}_{i=1}^n$ satisfying Theorem 6.3.3, and define

$$c_i^* := \rho_i(X_i) - \rho(g_i^*(S)) - w_i\,, \quad \forall i \in \mathcal{N}\,.$$

By construction, we have $\sum_{i=1}^n c_i^* = (\sum_{i=1}^n \rho_i(X_i)) - (\sum_{i=1}^n \rho(g_i^*(S))) - W = 0$. The allocation $\{g_i^*(S) + c_i^*\}$ is the risk-sharing arrangement that provides a welfare gain of w_i to each agent i, in the sense that

$$\rho_i(X_i) - \rho(g_i^*(S) + c_i^*) = w_i \geq 0\,.$$

Hence, $\{g_i^*(S) + c_i^*\} \in \mathcal{IR}$. Note that the above example (6.4) corresponds to the arrangement that allocates the entire welfare gain to agent n: i.e., $(w_1, \dots, w_n) = (0, 0, \dots, W)$.

6.3.1 Common Parametric Families of Distortion Functions

When each set of distortion functions $\mathcal{T}_i$ is a singleton, there is no robustness in the risk measure of each agent, and we recover the simpler setting of P2P risk-sharing with distortion risk measures. Suppose that $\mathcal{T}_i = \{T_i\}$ for all $i \in \mathcal{N}$. As a special case of Theorem 6.3.3, we obtain a closed-form solution for Pareto optima that is known in the literature (e.g., Liu [82, Theorem 3.3]). This result is restated below.

Corollary 6.3.4. *Suppose that for each $i \in \mathcal{N}$, the risk measure ρ_i is given by a robust distortion risk measure as in (6.2). Then an allocation $(\{Y_i^*\}_{i=1}^n, \{\pi_i^*\}_{i=1}^n)$ is Pareto optimal if and only if*

$$Y_i^* + \pi_i^* = g_i^*(S) + c_i^*\,,$$

where $\{c_i^*\}_{i=1}^n \in \mathbb{R}^n$ is chosen such that $\sum_{i=1}^n c_i^* = 0$ and $\{g_i^*(S) + c_i^*\}_{i=1}^n \in \mathcal{IR}$, and $\{g_i^*\}_{i=1}^n \in \mathcal{G}$ can be written in terms of the integrals of suitable functions h_i. Specifically, for each $i \in \mathcal{N}$, we can write $g_i^*(x) = \int_0^x h_i(z)\,dz$, where each $h_i : \mathbb{R}_+ \to [0,1]$ is a function such that for almost every $x \in \mathbb{R}_+$,

$$\sum_{i \in L_x} h_i(x) = 1 \text{ and } \sum_{i \in L_x^C} h_i(x) = 0\,,$$

where

$$L_x := \left\{ i \in \mathcal{N} : T_i(\mathbb{Q}_i(S > x)) = \min_{j \in \mathcal{N}} \{T_j(\mathbb{Q}_j(S > x))\} \right\}\,,$$
$$L_x^C = \mathcal{N} \setminus L_x\,.$$

Proof. We apply Theorem 6.3.3 in the special case where for $i \in \mathcal{N}$, each $\mathcal{T}_i$ is given by the singleton set $\{T_i\}$. However, note that we obtain a stronger result by showing sufficiency of the characterization as well as necessity. The result follows from the same proof as that of Theorem 6.3.3, after applying the fact that problems (6.7) and (6.8) are identical problems in this special case. □

In this subsection, we examine the consequences of Corollary 6.3.4 when agents' preferences are determined by common parametric families of probability distortion functions. First, suppose that each T_i is an inverse S-shaped distortion function from a parametric family, and let α_i denote the value of the parameter that determines T_i. It can be verified that certain values of the parameter α_i are "more S-shaped", which indicates a larger deviation from risk neutrality. To facilitate an economically meaningful comparison, we provide the following definition, which can be interpreted as a proxy for the level of risk aversion for a distortion function.

Definition 6.3.5 (Probabilistic Risk Aversion Index)**.** For a given distortion T_i assumed to be twice differentiable, we define the Probabilistic Risk Aversion Index of T_i to be

$$PRA_i(t) := -\frac{T_i''(t)}{T_i'(t)}\,.$$

This index is similar to the index of ambiguity aversion proposed by Carlier and Dana [30]. Indeed, if $T_2 = g \circ T_1$ where g is a concave function, then it can be verified that $PRA_2 \geq PRA_1$. Hence, T_2 is more risk averse than T_1, which is consistent with the notion that the risk aversion of distortion risk measures is due to concavity of the distortion function (see Yaari [112] for more on this interpretation). Alternatively, we may use an index of relative probabilistic risk aversion, as defined below.

Definition 6.3.6 (Relative Probabilistic Risk Aversion Index)**.** For a given distortion T_i assumed to be twice differentiable, we define the Relative Probabilistic Risk Aversion Index of T_i to be

$$RPRA_i(t) := t \times PRA_i(t) = -\frac{t\,T_i''(t)}{T_i'(t)}\,.$$

One example of inverse S-shaped distortion functions is the parametric family of Prelec-1 distortion functions introduced by Prelec [91], who showed that these distortion functions are inverse-S shaped with a unique inflection point at (e^{-1}, e^{-1}). These distortion functions are of the form

$$T_i(t) = \exp(-(-\ln(t))^{\alpha_i})$$

where $\alpha_i \in (0,1)$.

In the following, suppose that each ρ_i is a distortion risk measure with respect to a Prelec-1 distortion function with parameter α_i. Let $j,k \in \mathcal{N}$ such that $\alpha_j = \max_{i\in\mathcal{N}} \alpha_i$ and $\alpha_k = \min_{i\in\mathcal{N}} \alpha_i$.

Proposition 6.3.7. *For $t \in (0, e^{-1})$, the probabilistic risk aversion index $PRA_i(t)$ is larger for smaller values of the parameter α_i. In particular,*

$$PRA_k(t) \geq PRA_i(t) \geq PRA_j(t)\,.$$

Proof. For a given parameter $\alpha_i \in (0,1)$, we have

$$\begin{aligned}
\frac{d}{dt}\,\exp(-(-\ln(t))^{\alpha_i}) &= -\frac{\alpha_i \mathrm{e}^{-(-\ln(t))^{\alpha_i}} \cdot (-\ln(t))^{\alpha_i}}{t\ln(t)} \\
\frac{d^2}{dt^2}\,\exp(-(-\ln(t))^{\alpha_i}) &= \frac{\alpha_i \mathrm{e}^{-(-\ln(t))^{\alpha_i}} \cdot (-\ln(t))^{\alpha_i} \left(\ln(t) + \alpha_i \cdot (-\ln(t))^{\alpha_i} - \alpha_i + 1\right)}{t^2 \ln^2(t)} \\
PRA_i(t) &= \frac{\ln(t) + \alpha_i(-\ln(t))^{\alpha_i} - \alpha_i + 1}{t\ln(t)}\,.
\end{aligned}$$

Differentiating this with respect to α_i gives

$$\frac{d}{d\alpha_i}\,PRA_i(t) = \frac{(-\ln(t))^{\alpha_i} + \alpha_i \cdot \ln(-\ln(t)) \cdot (-\ln(t))^{\alpha_i} - 1}{t\ln(t)}\,.$$

For $t \in (0, e^{-1})$, we have $-\ln(t) > 1$. Therefore

$$(-\ln(t))^{\alpha_i} + \alpha_i \cdot \ln(-\ln(t)) \cdot (-\ln(t))^{\alpha_i} - 1 > 1^{\alpha_i} + \alpha_i \cdot \ln(1) \cdot 1^{\alpha_i} - 1 = 0.$$

Since $t\ln(t) < 0$, this implies that

$$\frac{d}{d\alpha_i}\,PRA_i(t) < 0\,,$$

which yields the desired result. □

This result can be interpreted in the following manner. Recall that in the context of distortion risk measures, each distortion function T_i is applied to the survival function $\mathbb{P}(S > x)$. Hence, the argument of the distortion $t \in (0, e^{-1})$ represents the probability of a tail event. Proposition 6.3.7 then implies that agent k is most averse to tail events, provided that the probability of this tail event is no more than $e^{-1} \approx 0.3679$.

Proposition 6.3.8. *Suppose each ρ_i is distortion risk measure given by a Prelec-1 distortion function with parameter α_i, and let $\alpha_j = \max_{i\in\mathcal{N}} \alpha_i$ and $\alpha_k = \min_{i\in\mathcal{N}} \alpha_i$. Then a Pareto-optimal allocation $(\{Y_i^*\}_{i=1}^n, \{\pi_i^*\}_{i=1}^n)$ is given by*

$$Y_i^* + \pi_i^* = g_i^*(S) + c_i^*\,,$$

where $\{c_i\}_{i=1}^n \in \mathbb{R}^n$ is chosen so that the allocation is $\mathcal{IR}$, and

$$\begin{aligned} g_k^*(x) &= \min\{x, d^*\}\,,\\ g_j^*(x) &= \max\{x - d^*, 0\}\,,\\ g_i^*(x) &= 0,\ \textit{for } i \in \mathcal{N}\setminus\{j,k\}\,. \end{aligned}$$

where $d^* = \mathrm{VaR}_{e^{-1}}(S)$.

Proof. We first show that

$$\begin{aligned} j &\in L_x\,,\ \text{if } \mathbb{P}(S > x) \le e^{-1}\,,\\ k &\in L_x\,,\ \text{if } \mathbb{P}(S > x) > e^{-1}\,. \end{aligned}$$

Suppose first that $\mathbb{P}(S > x) \le e^{-1}$. Then we have

$$\begin{aligned} \mathbb{P}(S > x) &\le e^{-1}\\ \ln(\mathbb{P}(S > x)) &\le -1\\ -\ln(\mathbb{P}(S > x)) &\ge 1\,. \end{aligned}$$

It follows that for all $i \in \mathcal{N}$,

$$\begin{aligned} (-\ln(\mathbb{P}(S > x)))^{\alpha_j} &\ge (-\ln(\mathbb{P}(S > x)))^{\alpha_i}\\ \exp(-(-\ln(\mathbb{P}(S > x)))^{\alpha_j}) &\le \exp(-(-\ln(\mathbb{P}(S > x)))^{\alpha_i})\,, \end{aligned}$$

which implies that $j \in L_x$. The second case is similar. Suppose that $\mathbb{P}(S > x) > e^{-1}$. Then we have

$$-\ln(\mathbb{P}(S > x)) < 1\,,$$

from which it is straightforward to conclude that $k \in L_x$.

Therefore, defining

$$\begin{aligned} h_j(x) &:= \mathbb{1}_{\{x\ge d^*\}}\,,\\ h_k(x) &:= \mathbb{1}_{\{x< d^*\}}\,,\\ h_i(x) &:= 0,\ \text{for } i \in \mathcal{N}\setminus\{j,k\}\,, \end{aligned}$$

satisfies the conditions of Theorem 6.3.3. The result follows by taking $g_i^*(x) = \int_0^x h_i(z)\,dz$ for all $i \in \mathcal{N}$. □

Consequently, when all agents use a Prelec-1 distortion function, a possible PO allocation is a translate of the contract where agent k provides full coverage of the aggregate risk up to a limit d^*, and the excess aggregate loss beyond the level d^* is covered by agent j. By Proposition 6.3.7, agent k exhibits the most risk aversion toward tail events. This is reflected by the form of Pareto optima in this case, since agent k does not cover losses beyond the deductible. Instead, the tail risk is assumed by agent j, who exhibits the least risk aversion toward tail events.

Prelec [91] also defines a two-parameter family of distortion functions (Prelec-2 distortion functions), by

$$T_i(t) = \exp(-\beta_i(-\ln(t))^{\alpha_i}),$$

where $\alpha_i \in (0,1)$ and $\beta_i > 0$. It can be verified that these distortion functions have an inflection point at $(e^{-1}, e^{-\beta_i})$. In this case, we have a result similar to that of Proposition 6.3.7.

Proposition 6.3.9. *If each ρ_i is a distortion risk measure given by a Prelec-2 distortion function with parameters α_i, β_i, then we have the following:*

(i) For a fixed α_i and for all $t \in (0,1)$, the probabilistic risk aversion index $PRA_i(t)$ is larger for smaller values of the parameter β_i.

(ii) For a fixed β_i and for all $t \in (0,\varepsilon)$ where $\varepsilon > 0$, the probabilistic risk aversion index $PRA_i(t)$ is larger for smaller values of the parameter α_i.

Proof. Similarly to the calculation in the proof of Proposition 6.3.7, we have

$$\begin{aligned} PRA_i(t) &= \frac{\ln(t) + \alpha_i\beta_i(-\ln(t))^{\alpha_i} - \alpha_i + 1}{t\ln(t)}, \\ \frac{d}{d\alpha_i} PRA_i(t) &= \frac{\beta_i(-\ln(t))^{\alpha_i} + \alpha_i\beta_i \cdot \ln(-\ln(t)) \cdot (-\ln(t))^{\alpha_i} - 1}{t\ln(t)}, \\ \frac{d}{d\beta_i} PRA_i(t) &= \frac{\alpha_i(-\ln(t))^{\alpha_i}}{t\ln(t)}. \end{aligned}$$

We see that $\frac{d}{d\beta_i} PRA_i(t) < 0$ for all $t \in (0,1)$, which shows (i). For (ii), if we take

$$\varepsilon = \min\left\{ e^{-1}, \exp\left(-\left(\frac{1}{\alpha_i}\right)^{\left(\frac{1}{\beta_i}\right)} \right) \right\},$$

then for $t \in (0, \varepsilon)$, we have

$$\begin{aligned} -\ln(t) &> 1\,, \\ \beta_i(-\ln(t))^{\alpha_i} &> 1\,. \end{aligned}$$

Therefore

$$\beta_i(-\ln(t))^{\alpha_i} + \alpha_i\beta_i \cdot \ln(-\ln(t)) \cdot (-\ln(t))^{\alpha_i} - 1 > 1 + 0 - 1 = 0.$$

Since $t\ln(t) < 0$, this implies

$$\frac{d}{d\alpha_i} PRA_i(t) < 0\,,$$

which yields the desired result. □

Hence, if there exists an agent k such that $\alpha_k = \min_{i\in\mathcal{N}} \alpha_i$ and $\beta_k = \min_{i\in\mathcal{N}} \beta_i$, then this agent exhibits the most risk aversion to tail events. A similar result to that of Proposition 6.3.8 is given below.

Proposition 6.3.10. *Suppose each ρ_i is distortion risk measure given by a Prelec-2 distortion function with parameters α_i and β_i.*

i) Suppose that $\beta_1 = \beta_2 = \ldots = \beta_n := \beta$. Let $\alpha_j = \max_{i\in\mathcal{N}} \alpha_i$ and $\alpha_k = \min_{i\in\mathcal{N}} \alpha_i$. Then a Pareto-optimal allocation $(\{Y_i^*\}_{i=1}^n, \{\pi_i^*\}_{i=1}^n)$ is given by

$$Y_i^* + \pi_i^* = g_i^*(S) + c_i^*\,,$$

where $\{c_i\}_{i=1}^n \in \mathbb{R}^n$ is chosen so that the allocation is $\mathcal{IR}$, and

$$\begin{aligned} g_k^*(x) &= \min\{x, d^*\}\,, \\ g_j^*(x) &= \max\{x - d^*, 0\}\,, \\ g_i^*(x) &= 0, \text{ for } i \in \mathcal{N} \setminus \{j, k\}\,. \end{aligned}$$

where $d^* = \mathrm{VaR}_{e^{-1}}(S)$.

ii) Suppose that $\alpha_1 = \alpha_2 = \ldots = \alpha_n := \alpha$. Let $\beta_j = \max_{i\in\mathcal{N}} \beta_i$. Then a Pareto-optimal allocation $(\{Y_i^*\}_{i=1}^n, \{\pi_i^*\}_{i=1}^n)$ is given by

$$Y_i^* + \pi_i^* = g_i^*(S) + c_i^*\,,$$

where $\{c_i\}_{i=1}^n \in \mathbb{R}^n$ is chosen so that the allocation is $\mathcal{IR}$, and

$$\begin{aligned} g_j^*(x) &= x\,, \\ g_i^*(x) &= 0, \text{ for } i \in \mathcal{N} \setminus \{j\}\,. \end{aligned}$$

Proof. Similarly to the proof of Proposition 6.3.8, we show that

$$\begin{aligned} j \in L_x\,, \text{ if } \mathbb{P}(S > x) \leq e^{-1}\,, \\ k \in L_x\,, \text{ if } \mathbb{P}(S > x) > e^{-1}\,. \end{aligned}$$

Suppose first that $\mathbb{P}(S > x) \leq e^{-1}$. Then we have

$$\begin{aligned} \mathbb{P}(S > x) &\leq e^{-1} \\ \ln(\mathbb{P}(S > x)) &\leq -1 \\ -\ln(\mathbb{P}(S > x)) &\geq 1\,. \end{aligned}$$

It follows that for all $i \in \mathcal{N}$,

$$\begin{aligned} (-\ln(\mathbb{P}(S > x)))^{\alpha_j} &\geq (-\ln(\mathbb{P}(S > x)))^{\alpha_i} \\ \beta(-\ln(\mathbb{P}(S > x)))^{\alpha_j} &\geq \beta(-\ln(\mathbb{P}(S > x)))^{\alpha_i} \\ \exp(-(-\ln(\mathbb{P}(S > x)))^{\alpha_j}) &\leq \exp(-(-\ln(\mathbb{P}(S > x)))^{\alpha_i})\,, \end{aligned}$$

which implies that $j \in L_x$. The second case is similar. Suppose that $\mathbb{P}(S > x) > e^{-1}$. Then we have

$$-\ln(\mathbb{P}(S > x)) < 1\,,$$

from which it is straightforward to conclude that $k \in L_x$. Applying Theorem 6.3.3 gives (i).

For (ii), note that if $\beta_j \geq \beta_i$, then

$$\exp(-\beta_j(-\ln(t))^{\alpha}) \leq \exp(-\beta_i(-\ln(t))^{\alpha})\,.$$

Therefore $j \in L_x$ for all $x \in \mathbb{R}_+$. The result follows by applying Theorem 6.3.3. □

Part (i) of Proposition 6.3.10 is a similar result to that of Proposition 6.3.8 in the case where every agent has the same second parameter. That is, the agent with the largest aversion to tail events provides coverage up to a deductible. Part (ii) states that if every agent uses the same first parameter α, then the agent with the least aversion to tail events provides full insurance.

Similar results can be obtained for the family of S-shaped distortion functions first introduced by Kahneman and Tversky [70, 105]:

$$T_i(t) = \frac{t^{\gamma_i}}{(t^{\gamma_i} + (1-t)^{\gamma_i})^{1/\gamma_i}}\,,$$

where $\gamma_i \in (0.279, 1]$. Let $\gamma_j = \max_{i \in \mathcal{N}} \gamma_i$ and $\gamma_k = \min_{i \in \mathcal{N}} \gamma_i$.

Remark 6.3.11. *It can be numerically verified that for any $t \in (0, \varepsilon)$ for a sufficiently small $\varepsilon > 0$, the probabilistic risk aversion index $PRA_i(t)$ is larger for smaller values of the parameter γ_i. The probabilistic risk aversion index is plotted in Figure 6.1b. In particular,*

$$PRA_k(t) \geq PRA_i(t) \geq PRA_j(t)\,.$$

Again, since small values of t correspond to tail events, we see that agent k exhibits the most aversion to the risk of tail events, whereas agent j exhibits the least aversion.

It can be shown using numerical methods that there exists $t^* \in (0, 1)$ for which

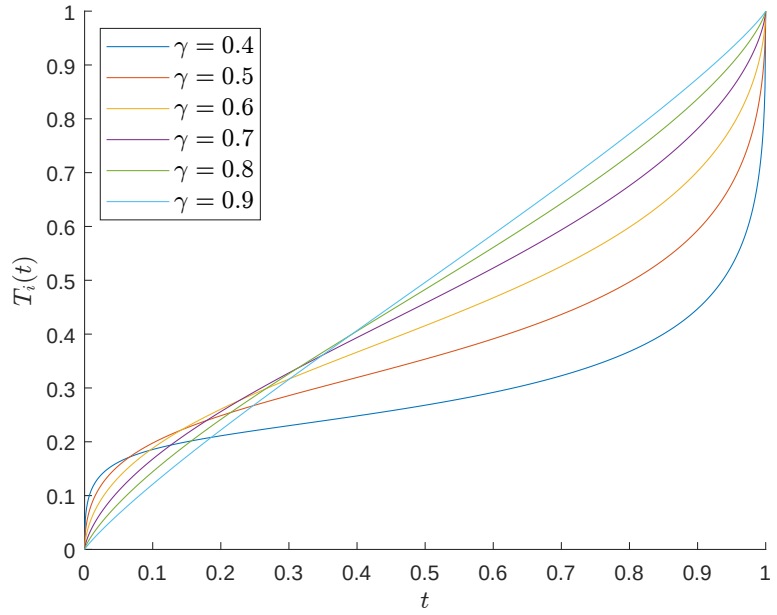
$$\begin{aligned} j &\in L_x\,, \text{ if } \mathbb{P}(S > x) \leq t^*\,, \\ k &\in L_x\,, \text{ if } \mathbb{P}(S > x) > t^*\,. \end{aligned}$$

Distortion functions for different values of the parameter γ are shown in Figure 6.1a. It is clear from the figure that the minimum of the distortion functions is obtained by either the agent with the largest parameter (agent j), or by the agent with the smallest parameter (agent k). Thus, by the same arguments as those of Proposition 6.3.8, it follows that a PO allocation of a similar form exists. That is, consider the contract such that agent k provides full coverage of the aggregate risk up to a limit d^*, and the excess aggregate loss beyond the level d^* is covered by agent j. Then a suitable translate of this contract is PO.

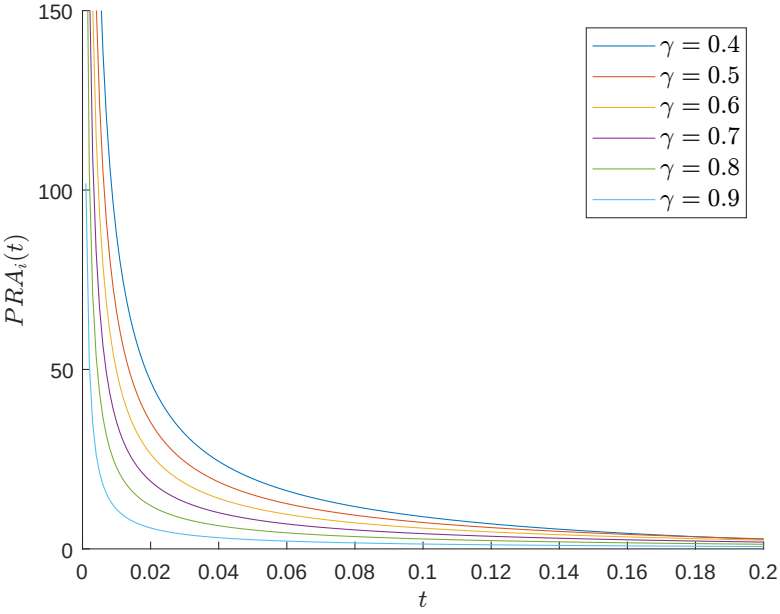
6.4 Centralized Versus Decentralized Insurance for Flood Risk

As an application of our explicit characterization of Pareto-optimal risk-sharing contracts, we reexamine the problem of flood risk insurance in the United States. This setting has been examined in detail in Chapter 3 and by Boonen et al. [17], who consider a market where each State may insure their flood risk with the federal government. We refer to this structure as a *centralized insurance market*, namely, one where insurance is only provided by a limited set of entities in the market.

In contrast, the main result of the present chapter characterizes Pareto optima in a *decentralized insurance market* or *peer-to-peer insurance market*. This market structure allows for agents to ensure each other's risk through participation in a risk sharing pool, which is not possible in a centralized insurance market. In the context of flood risk insurance, a decentralized insurance market is one where individual agents (communities,



(a) Distortion Functions



(b) Probability Risk Aversion Index.

Figure 6.1: Kahneman-Tversky Inverse S-Shaped Distortion Functions

municipalities, or State-level agencies) agree to combine their flood risk exposures and each cover a portion of the aggregate risk exposure. For the purposes of this section, we assume that the individual agents are States that wish to pool State-level flood risk. This is mainly for ease of comparison with the centralized setting. However, the conclusions of our analysis are also applicable to agents at a higher level of granularity (e.g., individual communities, etc.). Feng et al. [54] also apply their peer-to-peer risk sharing results to the flood risk insurance in the United States, but they consider a market where only quota-share risk sharing rules (i.e., proportional insurance) are available.

The purpose of the numerical illustrations presented in this section is to study the differences between centralized and decentralized insurance markets for flood risk, and to assess the economic implications thereof. For this reason, we use distortion risk measures for each agent and the associated characterization result of Corollary 6.3.4, to facilitate a direct comparison with the previous work, which did not consider the effect of robustness. The effects of introducing robustness of distortion risk measures are therefore not explored in the current study.

6.4.1 A Centralized Insurance Market

First, we revisit the scenario of a centralized insurance market, in which a single monopolistic insurer is the sole provider of insurance. In the context of flood risk in the United States, this insurer is the National Flood Insurance Program (NFIP)[1] overseen by the Federal Emergency Management Agency (FEMA). Under the NFIP, the federal government of the United States is ultimately responsible for financial gains and losses.

A comprehensive numerical study of flood risk insurance in this centralized market is performed by Boonen et al. [17], using a public dataset of claims statistics dating back to the year 1978. They find that by aggregating the flood risk across geographically distant regions, a centralized market is able to realize a significant welfare gain. In the following, we provide a similar but updated analysis of a centralized market. Firstly, the public NFIP dataset has recently been updated to include the amount of financial loss for each flood event, instead of only including the claim amount. As a result, we no longer need to use the claim amount as a proxy for the loss amount. Additionally, by the results of Chapter 3, Pareto optima can be characterized without requiring that each agent uses a coherent risk measure. As a result, we are able to identify Pareto-optimal contracts under more general

[1] A comprehensive overview of the NFIP can be found on the website of the Congressional Research Service: https://sgp.fas.org/crs/homesec/R44593.pdf.

conditions Finally, in our example, the risk measure of the central agent will be given by the Expected Shortfall, which is the most popular coherent risk measure in practice.

As in Chapter 3, we assume throughout that the distribution of the financial losses arising from floods in a given month adheres to the historical data. Specifically, we assume a discrete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω has 553 states, corresponding to each of the months from January 1978 to January 2024. The measure $\mathbb{P}$ assigns an equal probability to each of the states in Ω. The risk measures used by each agent are distortion risk measures on this probability space. That is, for all $Z \in \mathcal{X}$ and $i \in \{1, \dots, n\}$,

$$\rho_i(Z) = \int Z \, dT_i \circ \mathbb{P} \,,$$

where each T_i is a distortion function. Let $\nu_i := T_i \circ \mathbb{P}$.

In the context of centralized insurance markets with n agents and a central insurer, an insurance contract is a pair $(\{I_i\}_{i=1}^n, \{\pi_i\}_{i=1}^n) \in \mathcal{I}^n \times \mathbb{R}^n$, where $\mathcal{I}$ denotes an admissible set of indemnity functions (usually monotone and 1-Lipschitz). Here, I_i represents the amount of coverage (i.e. the indemnity) that agent i receives from the central insurer, in exchange for a premium of the amount π_i. In other words, agent i cedes the risk $I_i(X_i)$ to the central insurer. We denote the retained risk of agent i by $R_i(X_i) := X_i - I_i(X_i)$. A characterization of Pareto-optimal contracts in a centralized market is provided by the following.

Proposition 6.4.1 (Pareto Optima in a Centralized Market)**.** *Suppose that all agents use distortion risk measures, and that the insurer's risk measure is given by an Expected Shortfall at level $\alpha \in (0,1)$. Define the set of measures*

$$\mathcal{Q} := \left\{ \mathbb{Q} \text{ a probability measure} : \mathbb{Q} \ll \mathbb{P}, \, \frac{d\mathbb{Q}}{d\mathbb{P}} \leq \frac{1}{\alpha} \right\} .$$

Then a necessary condition for $(\{I_i^*\}_{i=1}^n, \{\pi_i^*\}_{i=1}^n)$ to be Pareto optimal is that for every solution $\mathbb{Q}^*$ to

$$\sup_{\mathbb{Q} \in \mathcal{Q}} \sum_{i=1}^n \int_0^M \min \left\{ \mathbb{Q}(X_i > t), \, \nu_i(X_i > t) \right\} dt \,, \tag{6.5}$$

we have, for each $i \in \{1, \dots, n\}$ and for almost all $t \in (0, M)$,

$$(I_i^*)'(t) = \begin{cases} 1, & \text{if} \quad \mathbb{Q}^*(X_i > t) < \nu_i(X_i > t) \,, \\ h_i(t), & \text{if} \quad \mathbb{Q}^*(X_i > t) = \nu_i(X_i > t) \,, \\ 0, & \text{if} \quad \mathbb{Q}^*(X_i > t) > \nu_i(X_i > t) \,, \end{cases} \tag{6.6}$$

where each h_i is a $[0,1]$-valued measurable function and $\{\pi_i^*\}_{i=1}^n$ are chosen so that the contract $(\{I_i^*\}_{i=1}^n, \{\pi_i^*\}_{i=1}^n)$ is individually rational.

Proof. It is well-known (e.g., McNeil et al. [89, Theorem 8.14]) that the Expected Shortfall satisfies

$$\mathrm{ES}_\alpha^{\mathbb{P}}(Z) = \max\left\{\mathbb{E}^{\mathbb{Q}}[Z] : \mathbb{Q} \ll \mathbb{P},\ \frac{d\mathbb{Q}}{d\mathbb{P}} \leq \frac{1}{\alpha}\right\},$$

for all $Z \in \mathcal{X}$. The result then follows from Corollary 3.3.7. □

We now apply the result of Proposition 6.4.1 to obtain Pareto-optimal contracts in the centralized insurance market. We consider a market with three agents: the states of California ($i = 1$), New York ($i = 2$), and Texas ($i = 3$). The correlation between the monthly losses of these states is given by the matrix

$$\begin{bmatrix} 1 & -0.0094 & -0.0109 \\ -0.0094 & 1 & -0.0044 \\ -0.0109 & -0.0044 & 1 \end{bmatrix},$$

which indicates very little correlation. Some summary statistics for the monthly losses for these states are shown in Table 6.1, in which all values are dollar amounts. We see that the financial losses are right-skewed, and that it is possible to experience losses far larger than the average.

	California	New York	Texas
Average	1.6038×10^6	1.1005×10^7	3.3259×10^7
Median	2.7647×10^4	1.4873×10^5	4.7270×10^5
$\mathrm{VaR}_{5\%}$	4.9501×10^6	7.1506×10^6	4.6885×10^7
Maximum	1.1725×10^8	4.3537×10^9	9.4550×10^9
Standard Deviation	8.6640×10^6	1.8625×10^8	4.2063×10^8

Table 6.1: Summary Statistics for Monthly Losses Due to Floods.

We assume that the distortion functions T_1 and T_2 are both inverse S-shaped distortion functions, as in Kahneman and Tversky [70, 105]. Specifically, for $i = 1, 2$, we take

$$T_i(t) = \frac{t^{\gamma_i}}{(t^{\gamma_i} + (1-t)^{\gamma_i})^{\frac{1}{\gamma_i}}},$$

with $\gamma_1 = 0.4$ and $\gamma_2 = 0.5$. A parameter value of approximately 0.5 has recently been estimated by Rieger et al. [96] to be descriptive of actual behaviour. We assume a power distortion for the remaining agent. Specifically,

$$T_3(t) = t^{\gamma_3},$$

with $\gamma_3 = 0.4$. Finally, we assume that the central insurer uses the Expected Shortfall with parameter $\alpha = 15\%$. While this parameter may seem large, this is to allow for the central insurer's risk measure to be display enough risk tolerance that it will accept a significant portion of risk from the agents in the market. If this parameter is too low, then the central insurer would be too conservative and avoid taking on any flood risk, leading to a market where no insurance is possible. Indeed, it can be verified that under these parameters, no insurance is provided if $\alpha \approx 2.5\%$.

The retained risk of each agent in a Pareto-optimal contract is shown in Figure 6.2. Note that in this figure and subsequent figures, the amount of premium is not shown, for visual clarity and to facilitate easier comparison between plots. In other words, each plot has been normalized so that an experienced loss of \$0 corresponds to a retention of \$0 as well. We see that in each state prefers to cede a portion of their tail risk to the central insurer, while choosing to retain some of the smaller financial losses. This suggests that a deductible contract is best in this situation, with losses beyond the deductible fully covered by the insurer. The value of this deductible is approximately 1.9047×10^7 for California, 4.0645×10^7 for New York, and 5.1287×10^7 for Texas.

As a measure of the effectiveness of this contract, we calculate the aggregate welfare gain that results from this arrangement. The welfare gain for each agent i is the difference between their evaluation of the initial risk and their evaluation of the post-contract position:

$$\rho_i(X_i) - \rho_i(R_i(X_i) + \pi_i).$$

On the other hand, the welfare gain for the central insurer is the sum of premia received less their evaluation of their assumed risk:

$$\sum_{i=1}^{n} \pi_i - \mathrm{ES}_\alpha^{\mathbb{P}}\left(\sum_{i=1}^{n} I_i(X_i)\right).$$

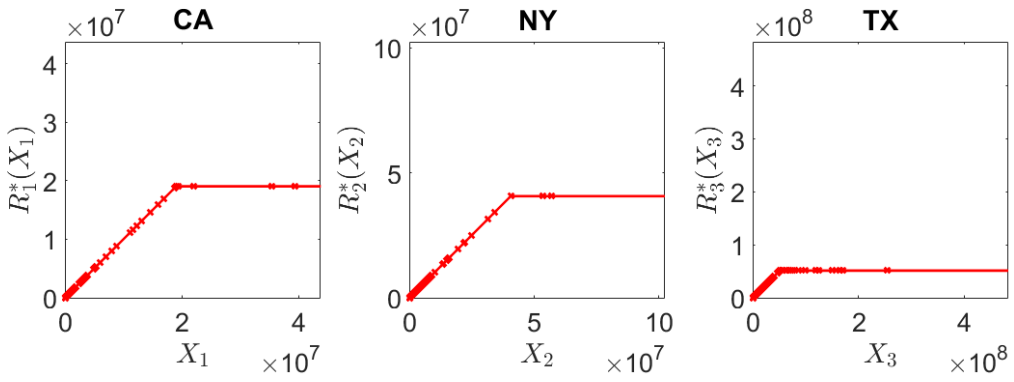


Figure 6.2: Centralized Pareto-Optimal Retention, CA/NY/TX

The aggregate welfare gain is defined to be the sum of the welfare gain of each agent in the market, including the central insurer. Under our assumptions, the aggregate welfare gain is maximized in a Pareto-optimal allocation (see, e.g., Boonen et al. [17, Theorem 2.1]). In the example above, the maximum aggregate welfare gain achievable through centralized insurance is 7.9785×10^8. The average gain per agent is in this example is 1.9946×10^8. Note that including the centralized insurer, there are four agents in the market in this example.

However, while a significant welfare gain is possible in a centralized market, the results of Chapter 3 warn that a monopolistic insurer with a first-mover's advantage can absorb the entirety of the welfare gain by increasing premium prices. In particular, the central insurer has an incentive to increase prices and ultimately leave each policyholder with no welfare gain at an equilibrium. The amount of premium paid by each state in a Stackelberg equilibrium in our example is shown in Table 6.2. These amounts are extremely high, even exceeding the Value-at-Risk at the 5% level. In this case, the entire aggregate welfare gain of 7.9785×10^8 is allocated only to the central insurer.

California	9.1543×10^6
New York	1.7795×10^8
Texas	8.6323×10^8

Table 6.2: Premium Paid in a Stackelberg Equilibrium, CA/NY/TX.

6.4.2 Peer-to-Peer Risk Sharing

As a partial solution to the concerns arising from the Stackelberg setting, the present chapter proposes a peer-to-peer risk sharing scheme among the agents themselves without the presence of a central agency. In the context of flood risk, this is the scenario where the individual state governments of California, New York, and Texas choose to pool their flood risk and each assume a portion of the aggregate financial loss.

Applying the result of Theorem 6.3.3 to the present example yields the Pareto-optimal allocation shown in Figure 6.3. The States of California and New York end up with all of the variability of the risk, since the risk measure of Texas is more risk averse under our chosen parameters. However, this does not mean that Texas is receiving free insurance, since each state must pay a premium for participation in the risk-sharing pool. Both Figure 6.3 and Figure 6.2 show that Texas prefers retaining less risk than the other agents.

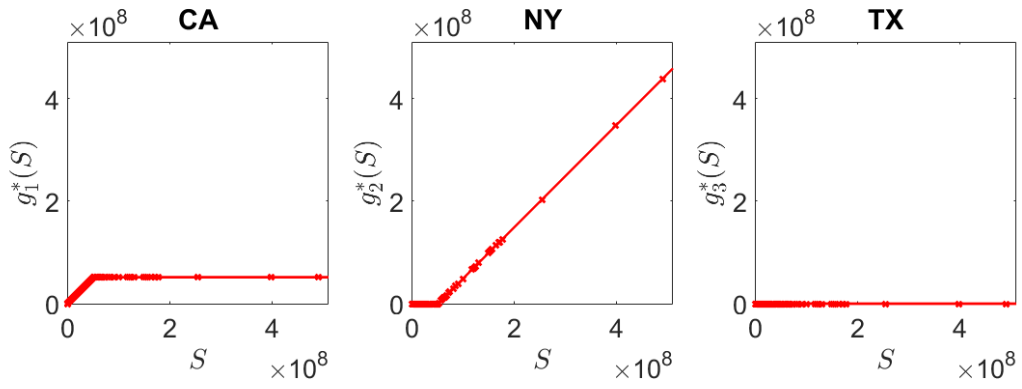


Figure 6.3: Decentralized Pareto-Optimal Distribution, CA/NY/TX.

A comparison between the centralized and decentralized insurance schemes is shown in Figure 6.4, which also includes the scenario of no insurance. The vertical axis represents the retained monthly loss in dollars, and the horizontal axis represents the 553 states of the world $\omega \in \Omega$, which are sorted by the value of $S(\omega)$, the realized aggregate loss in state $\omega \in \Omega$. For ease of comparison, the plots have been normalized so that each agent is indifferent between each retention structure. In the case of centralized insurance, we see that each agent pays a premium to insure their risk beyond a deductible. This results in significantly less variance in retained risk, at a cost of a monthly premium. On the other hand, the structure of the Pareto-optimal risk-sharing arrangement is different for

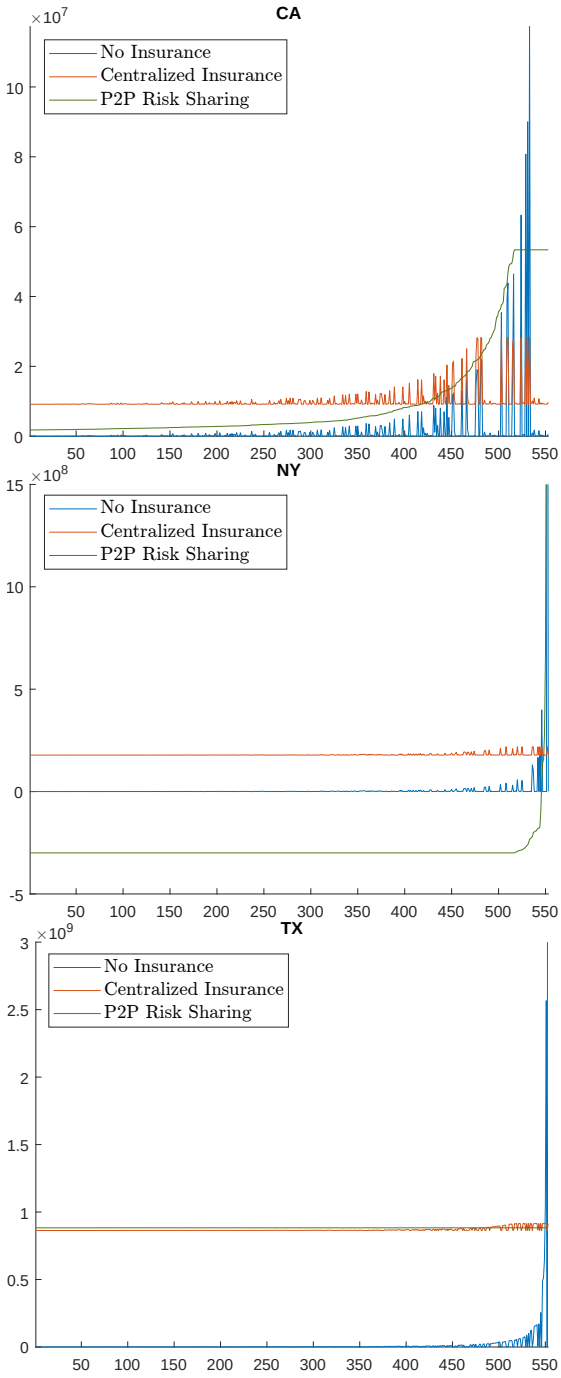


Figure 6.4: Retained Loss, CA/NY/TX.

each agent. The plots show that California's retention is similar to that under centralized insurance, in the sense that variation is reduced and extreme losses are insured. However, extreme losses are transferred to New York, who accepts a payment of a premium as compensation. Since there is no exogenous insurer in the decentralized market, all flood risk must be retained among the three agents in the market. The plot in Figure 6.4 shows that New York is the agent that accepts the extreme tail risk in this example. Finally, Texas elects to pay a premium to transfer all the flood risk to the other two agents, and so its retention structure is a flat horizontal line. This suggests that Texas is the most risk-averse in this scenario.

By Theorem 6.3.3, both the aggregate welfare gain and the average welfare gain are maximized by Pareto-optimal allocations. Hence, evaluating the effectiveness of the peer-to-peer risk sharing scheme is possible by using the average welfare gain as a metric. Recall that the welfare gain for agent i is defined as the improvement in the risk-sharing scheme over the status quo:

$$\rho_i(X_i) - \rho_i(Y_i + \pi_i)\,.$$

The average welfare gain for the peer-to-peer risk sharing scheme is the arithmetic average of the welfare gains of the three agents in the market. In this example, the average welfare gain is 1.9512×10^8. While this is a significant gain, this is not quite as large as the average gain of 1.9946×10^8 that is possible from a centralized insurance market with the monopolistic insurer. Hence, we see the peer-to-peer risk sharing scheme as a compromise compared to the centralized structure. While a larger welfare gain is ultimately possible in the centralized setting, this carries the risk that the welfare gain is absorbed by only the monopolistic insurer through aggressive pricing schemes. On the other hand, the decentralized market admits no such risk, since the welfare gain is necessarily distributed among the agents participating in the risk-sharing pool. However, the trade-off is that the maximum possible average welfare gain is potentially lower in the peer-to-peer market.

In Table 6.3, we show the average welfare gain per agent in both centralized and decentralized markets for varying values of the parameter γ_3 in the distortion function of the third agent T_3. It can be verified that power distortion functions have a constant relative index of probabilistic risk aversion. As γ_3 increases, the $RPRA$ of the third agent decreases, and the hence the third agent eventually prefers to retain more of the flood risk in the market. In the extreme case, we see that the peer-to-peer market can actually dominate that of the centralized market in terms of possible welfare gain, since the decentralized scheme allows for the third agent to insure others' risk whereas the centralized scheme does not. Hence, it is possible for the decentralized market's average welfare gain to exceed that of the centralized market. However, in practical settings, we would expect

that the policyholders are more risk averse than the central authority, which would lead to an advantage for the centralized insurance market in general.

$RPRA_3$	Centralized Market	Decentralized Market	Percent Decrease
0.60	1.9946×10^8	1.9512×10^8	2.1750%
0.55	1.4460×10^8	1.2112×10^8	16.2383%
0.50	1.0415×10^8	6.6055×10^7	36.5774%
0.45	7.4503×10^7	4.8805×10^7	34.4927%
0.40	5.3190×10^7	5.0507×10^7	5.0434%
0.35	3.8565×10^7	5.2590×10^7	−36.3683%
0.30	3.0251×10^7	5.4502×10^7	−80.1628%

Table 6.3: Average Welfare Gain with Varying Parameter γ_3.

6.5 Conclusion

In this chapter, we obtain a novel characterization of Pareto-optimal allocations in risk sharing markets when agents use robust distortion risk measures. As a special case of our result, we recover the characterization of optima under distortion risk measures, and we provide some general results for risk-sharing using common families of distortion functions. We find that the form of allocations depends primarily on the risk attitude of each agent to tail events, which we quantify through the probabilistic risk aversion index.

As an application of our results, we reexamine the setting of flood risk in the United States from the perspective of a decentralized market. While this market currently operates under a centralized structure with the federal government as the central authority, this comes at a risk associated with Stackelberg equilibria, namely that the central insurer has an incentive to raise prices and eliminate any welfare gain for the policyholders. We therefore argue that the decentralized structure is an alternative that avoids the Stackelberg situation, and we characterize optimal decentralized allocations using historical data. Our results show that there is less welfare gain possible in the decentralized market compared to the centralized one, which we interpret as a necessary compromise.

6.6 Proofs of Main Results

6.6.1 Proof of Theorem 6.3.3

(i) Since S is essentially bounded by $||S||_\infty < \infty$, we have

$$\sup_{\{T_i\}_{i=1}^n \in \prod_{i=1}^n \mathcal{T}_i} \int_0^\infty \min_{i\in\mathcal{N}} \{T_i(\mathbb{Q}_i(S > x))\}\, dx := V \le ||S||_\infty < \infty\,.$$

Let $\{(T_1^{(k)}, \ldots, T_n^{(k)}\}_{k=1}^\infty$ be a sequence such that

$$\int_0^\infty \min_{i\in\mathcal{N}} \{T_i(\mathbb{Q}_i(S > x))\}\, dx \ge V - \frac{1}{k}\,.$$

Then by repeatedly applying Helly's compactness theorem (e.g., [50, pp. 165-166]), there exists a subsequence $\{(T_1^{(k_j)}, \ldots, T_n^{(k_j)}\}_{j=1}^\infty$ that converges pointwise to a limit $(T_1^*, \ldots, T_n^*) \in \prod_{i=1}^n \mathcal{T}_i$. By the proof of Lemma 6.3.2, the objective of (6.3) is continuous with respect to pointwise convergence of the distortion functions. Hence,

$$\begin{aligned} V &\ge \int_0^\infty \min_{i\in\mathcal{N}} \{T_i^*(\mathbb{Q}_i(S > x))\}\, dx = \lim_{j\to\infty} \int_0^\infty \min_{i\in\mathcal{N}} \{T_i^{k_j}(\mathbb{Q}_i(S > x))\}\, dx \\ &\ge V - \lim_{j\to\infty} \frac{1}{k_j} = V\,, \end{aligned}$$

which implies that $(T_1^*, \ldots, T_n^*)$ solves (6.3).

(ii) Let $(\{Y_i\}_{i=1}^n, \{\pi_i\}_{i=1}^n) \in \mathcal{A}$. By Lemma 5.4.10, this allocation can be written in terms of functions $\{g_i\}_{i=1}^n \in \mathcal{G}$ and constants $\{c_i\}_{i=1}^n \in \mathbb{R}^n$ where $\sum_{i=1}^n c_i = 0$. Conversely, if $\{g_i\}_{i=1}^n \in \mathcal{G}$ and $\{c_i\}_{i=1}^n \in \mathbb{R}^n$, then $(\{g_i(S)\}_{i=1}^n, \{c_i\}_{i=1}^n) \in \mathcal{A}$. Therefore

$$\begin{aligned} \inf_{(\{Y_i\}_{i=1}^n, \{\pi_i\}_{i=1}^n)\in\mathcal{IR}\cap\mathcal{A}} \left\{\sum_{i=1}^n \rho_i\left(Y_i + \pi_i\right)\right\} &= \inf_{(\{g_i\}_{i=1}^n, \{c_i\}_{i=1}^n)\in\mathcal{IR}\cap(\mathcal{G}\times\mathbb{R}^n)} \left\{\sum_{i=1}^n \rho_i\left(g_i(S) + c_i\right)\right\} \\ &= \inf_{(\{g_i\}_{i=1}^n, \{c_i\}_{i=1}^n)\in\mathcal{IR}\cap(\mathcal{G}\times\mathbb{R}^n)} \left\{\sum_{i=1}^n \rho_i\left(g_i(S)\right)\right\}, \end{aligned}$$

where we write $(\{g_i\}_{i=1}^n, \{c_i\}_{i=1}^n) \in \mathcal{IR}$ when the allocation $(\{g_i(S)\}_{i=1}^n, \{c_i\}_{i=1}^n) \in \mathcal{IR}$. This problem is solved by $(\{g_i^*\}_{i=1}^n, \{c_i^*\}_{i=1}^n) \in \mathcal{G}\times\mathbb{R}^n$ if and only if $\{g_i^*\}_{i=1}^n$ solves

$$\inf_{\{g_i\}_{i=1}^n\in\mathcal{G}} \left\{\sum_{i=1}^n \rho_i(g_i(S))\right\}, \tag{6.7}$$

and the constants c_i^* are chosen such that $(\{g_i^*(S)\}_{i=1}^n, \{c_i^*\}_{i=1}^n) \in \mathcal{IR}$. It remains to show that solutions to (6.7) are of the given form.

Since each ρ_i is a robust distortion risk measure, (6.7) can be written as

$$\inf_{\{g_i\}_{i=1}^n \in \mathcal{G}} \left\{ \sum_{i=1}^n \sup_{T_i \in \mathcal{T}_i} \int g_i(S)\, dT_i \circ \mathbb{Q}_i \right\} = \inf_{\{g_i\}_{i=1}^n \in \mathcal{G}} \sup_{\{T_i\}_{i=1}^n \in \prod_{i=1}^n \mathcal{T}_i} \left\{ \sum_{i=1}^n \int g_i(S)\, dT_i \circ \mathbb{Q}_i \right\}.$$

We now verify that the minimax equality holds for this problem. To this end, let $\mathcal{C}([0, M])$ denote the set of continuous functions on $[0, M]$, which is a Banach space under the supremum norm. Let $\mathcal{D} := \mathbb{R}^{[0,1]}$ denote the space of functions from $[0, 1] \to \mathbb{R}$, which is a topological vector space under the topology of pointwise convergence. With the embeddings $\mathcal{G} \subseteq \mathcal{C}([0, M])^n$ and $T_i \subseteq \mathcal{D}$, the objective function above can be viewed as a function from $\mathcal{C}([0, M])^n \times \mathcal{D}^n$ to $\mathbb{R}$.

Note that the sets $\mathcal{G}$ and $\mathcal{T}_i$ are closed and convex. Since $\mathcal{G}$ is a closed subset of a product of 1-Lipschitz functions on the interval $[0, M]$, it is compact by the Arzela-Ascoli Theorem (e.g., Dunford and Schwartz [51, IV.6.7]). Furthermore, the objective function is linear in both g_i and T_i. Finally, the objective is continuous with respect to the topologies above, since the Choquet integral is continuous with respect to the supremum norm[2], and it is also sequentially continuous with respect to pointwise convergence of T_i as shown by Lemma 6.3.2.

Therefore by Sion's minimax theorem, the minimax equality holds. Exchanging the order of the infimum and supremum yields the problem

$$\sup_{\{T_i\}_{i=1}^n \in \prod_{i=1}^n \mathcal{T}_i} \inf_{\{g_i\}_{i=1}^n \in \mathcal{G}} \left\{ \sum_{i=1}^n \int g_i(S)\, dT_i \circ \mathbb{Q}_i \right\}. \tag{6.8}$$

By standard results on minimax problems (e.g., Barbu and Precupanu [10, Section 2.3]), a necessary condition for $\{g_i^*\}_{i=1}^n$ to solve (6.7) is that for every $(T_1^*, \ldots, T_n^*)$ solving (6.8), we have

$$\sum_{i=1}^n \int g_i^*(S)\, dT_i^* \circ \mathbb{Q}_i = \inf_{\{g_i\}_{i=1}^n \in \mathcal{G}} \left\{ \sum_{i=1}^n \int g_i(S)\, dT_i^* \circ \mathbb{Q}_i \right\}.$$

[2] In fact, it is Lipschitz continuous.

To complete the proof, it suffices to show that for every fixed choice of distortion functions $(T_1, \ldots, T_n)$, the functions $\{g_i^*\}_{i=1}^n$ solve the problem

$$\inf_{\{g_i\}_{i=1}^n \in \mathcal{G}} \left\{ \sum_{i=1}^n \int g_i(S)\, dT_i \circ \mathbb{Q}_i \right\} \tag{6.9}$$

if and only if $\{g_i^*\}_{i=1}^n$ are of the given form, and that the value of problem (6.9) is

$$\int_0^\infty \min_{i \in \mathcal{N}} \{T_i(\mathbb{Q}_i(S > x))\}\, dx\,.$$

To this end, we first check that $\{g_i^*\}_{i=1}^n \in \mathcal{G}$. For all $x \in \mathbb{R}_+$, we have

$$\sum_{i=1}^n g_i^*(x) = \sum_{i=1}^n \int_0^x h_i(z)\, dz = \int_0^x \sum_{i=1}^n h_i(z)\, dz = \int_0^x 1\, dz = x\,.$$

Furthermore, since the derivative of g_i^* is non-negative, g_i^* is increasing, and so $\{g_i^*\}_{i=1}^n \in \mathcal{G}$ and $g_i^*(S)$ is comonotone with S. Now let $\{\tilde{g}_i\}_{i=1}^n \in \mathcal{G}$. We have

$$\begin{aligned}
\sum_{i=1}^n \rho_i(\tilde{g}_i(S)) &= \sum_{i=1}^n \int_0^\infty T_i(\mathbb{Q}_i(S > x))\tilde{g}_i'(x)\, dx = \int_0^\infty \sum_{i=1}^n T_i(\mathbb{Q}_i(S > x))\tilde{g}_i'(x)\, dx \\
&\geq \int_0^\infty \sum_{i=1}^n \min_{i=1,\ldots,n} \{T_i(\mathbb{Q}_i(S > x))\}\tilde{g}_i'(x)\, dx \qquad (6.10) \\
&= \int_0^\infty \min_{i=1,\ldots,n} \{T_i(\mathbb{Q}_i(S > x))\} \sum_{i=1}^n \tilde{g}_i'(x)\, dx \\
&= \int_0^\infty \min_{i=1,\ldots,n} \{T_i(\mathbb{Q}_i(S > x))\}\, dx \\
&= \int_0^\infty \min_{i=1,\ldots,n} \{T_i(\mathbb{Q}_i(S > x))\} \cdot 1 + \sum_{i \in L_x^C} T_i(\mathbb{Q}_i(S > x)) \cdot 0\, dx \\
&= \int_0^\infty \min_{i=1,\ldots,n} \{T_i(\mathbb{Q}_i(S > x))\} \cdot \sum_{i \in L_x} h_i(x) + \sum_{i \in L_x^C} T_i(\mathbb{Q}_i(S > x)) \cdot h_i(x)\, dx \\
&= \int_0^\infty \sum_{i \in L_x} \min_{i=1,\ldots,n} \{T_i(\mathbb{Q}_i(S > x))\} \cdot h_i(x) + \sum_{i \in L_x^C} T_i(\mathbb{Q}_i(S > x)) \cdot h_i(x)\, dx \\
&= \int_0^\infty \sum_{i \in L_x} T_i(\mathbb{Q}_i(S > x)) \cdot h_i(x) + \sum_{i \in L_x^C} T_i(\mathbb{Q}_i(S > x)) \cdot h_i(x)\, dx
\end{aligned}$$

$$= \int_0^\infty \sum_{i=1}^n T_i(\mathbb{Q}_i(S > x)) h_i(x)\, dx$$
$$= \sum_{i=1}^n \int_0^\infty T_i(\mathbb{Q}_i(S > x)) h_i(x)\, dx = \sum_{i=1}^n \rho_i(g_i^*(S))\,.$$

Therefore $\{g_i^*\}_{i=1}^n$ solves (6.7).

To show the converse, let $(\{Y_i\}_{i=1}^n, \{\pi_i\}_{i=1}^n) \in \mathcal{A}$. By Lemma 5.4.10, we may write

$$Y_i + \pi_i = \tilde{g}_i(x) + c_i\,,$$

for some $\{\tilde{g}_i\}_{i=1}^n \in \mathcal{G}$ and $\{c_i\}_{i=1}^n \in \mathbb{R}^n$ such that $\sum_{i=1}^n c_i = 0$. Suppose that $\tilde{g}_i$ are not of the specified form. Namely, $\sum_{i \in L_x^C} \tilde{g}_i'(x) > 0$ on a set $\mathbf{A}$ of positive measure. Then for every x in $\mathbf{A}$, we have

$$\begin{aligned}
&\sum_{i=1}^n T_i(\mathbb{Q}_i(S > x)) \tilde{g}_i'(x) \\
&\qquad = \sum_{i \in L_x} T_i(\mathbb{Q}_i(S > x)) \tilde{g}_i'(x) + \sum_{i \in L_x^C} T_i(\mathbb{Q}_i(S > x)) \tilde{g}_i'(x) \\
&\qquad = \min_{i=1,\ldots,n} \{T_i(\mathbb{Q}_i(S > x))\} \tilde{g}_i'(x) + \sum_{i \in L_x^C} T_i(\mathbb{Q}_i(S > x)) \tilde{g}_i'(x) \\
&\qquad > \min_{i=1,\ldots,n} \{T_i(\mathbb{Q}_i(S > x))\} \tilde{g}_i'(x) + \sum_{i \in L_x^C} \min_{i=1,\ldots,n} \{T_i(\mathbb{Q}_i(S > x))\} \tilde{g}_i'(x) \\
&\qquad = \min_{i=1,\ldots,n} \{T_i(\mathbb{Q}_i(S > x))\} = \sum_{i=1}^n \min_{i=1,\ldots,n} \{T_i(\mathbb{Q}_i(S > x))\} \tilde{g}_i'(x)\,,
\end{aligned}$$

where the strict inequality follows because L_x^C is non-empty and $\tilde{g}_i'(x)$ are not all zero for $i \in L_x^C$. Taking the integral over the set $\mathbf{A}$ of positive measure gives

$$\int_{\mathbf{A}} \sum_{i=1}^n T_i(\mathbb{Q}_i(S > x)) \tilde{g}_i'(x)\, dx > \int_{\mathbf{A}} \sum_{i=1}^n \min_{i=1,\ldots,n} \{T_i(\mathbb{Q}_i(S > x))\} \tilde{g}_i'(x)\, dx\,.$$

Therefore in this case, the inequality (6.10) is strict, and so $\tilde{g}_i(x)$ does not solve (6.7). □

Chapter 7

Conclusion and Future Work

The main contributions of this thesis are to answer economically pertinent questions in different structures of insurance markets. Chapters 3 and 4 examine centralized insurance markets. Recent literature has shown that in markets where a single insurer has a first-mover advantage, the only equilibria induce no welfare gain to the policyholder. The theoretical results in Chapter 3 show that this is true even when multiple policyholders are present in the market, hence identifying monopoly as the central issue. In an effort to capture value for the policyholders, we formally introduce multiple suppliers of insurance in an insurance market model. The main message of Chapter 4 is that price competition within an insurance market can ultimately result in benefits to the policyholders. As technical contributions, the work in this thesis shows that important economic phenomena can be modelled with modern tools of quantitative risk management. Furthermore, the extension of the Stackelberg model in Chapter 4 and the introduction of SPNEs in an insurance market are key contributions to this literature.

On the other hand, Chapters 5 and 6 examine decentralized markets of insurance. Chapter 5 investigates a risk-sharing market from the perspective of maximizing risk-averse utilities. Recent mathematical advances in this field are consolidated, hence providing formal motivation for the notion of a comonotone market when agents are risk-averse. The central theoretical contribution is a characterization of Pareto-optimal allocations for the case of risk-averse monetary utilities, as well as an algorithm for the calculation of these allocations. As a consequence, we obtain new results on the inf-convolution of coherent risk measures that are not necessarily comonotone additive. These theoretical results are reinterpreted in Chapter 6 in the context of robust distortion risk measures.

The results of this thesis are all explored through numerical illustrations implemented in

MATLAB. In Chapters 3 and 6, we use extensively the public data on flood risk insurance made available by the Federal Emergency Management Agency of the USA. Our results facilitate a direct comparison between centralized markets and decentralized markets, and the benefits and disadvantages of the resulting allocations are analyzed.

This thesis suggests several avenues for future research. The research presented in this thesis demonstrates the feasibility of answering economic questions in modern quantitative models of insurance markets. These techniques can be applied to different structures of insurance markets as they emerge in practice. As an immediate next step, the settings of Chapters 3 and 4 can be combined in a model of a centralized insurance market with n policyholders and m insurers. This would require addressing the technical difficulties that result from harmonizing the different assumptions on agent preferences used in these two chapters. There is also recent interest in insurance markets with two tiers of risk exchange: for example, a group of policyholders engage in a risk-sharing pool, and the administrator of the pool purchases insurance on behalf of its members. This is both a decentralized market at the policyholder level and a centralized market at the aggregate level, and can be interpreted as a combination of the settings of Chapters 3 and 6. Perhaps this market could reap the benefits of risk exchange at both levels, leading to increased agent welfare and a reduction of uninsurable risks. Characterizing optimal allocations and assessing the efficacy of these markets would be of significant theoretical and practical importance. Finally, the numerical illustrations in this thesis rely on flood risk insurance data, primarily for availability reasons. Applications of our results to different datasets may also be of practical relevance.

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