Multivariate Limit Theorems and Algebraic Generating Functions

by

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Author's Declaration

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of Contributions

This work is based on the following papers that I co-authored during my time at the University of Waterloo.

- Central Limit Theorems via Analytic Combinatorics in Several Variables, joint work with Stephen Melczer [60]
- Algebraic Generating Functions and Analytic Combinatorics in Several Variables, joint work with Torin Greenwood, Stephen Melczer and Mark Wilson (in progress)

Abstract

The field of analytic combinatorics is dedicated to the creation of effective techniques to study the large-scale behaviour of combinatorial objects. Although classical results in analytic combinatorics are mainly concerned with univariate generating functions, over the last two decades a theory of Analytic Combinatorics in Several Variables (ACSV) has been developed to study the asymptotic behaviour of multivariate sequences. This thesis provides results for two areas of ACSV: limit theorems and asymptotics of algebraic generating functions. For both, the aim is to provide readers a blueprint to apply the powerful tools of ACSV in their own work, making them more accessible to combinatorialists, probabilists, and those in adjacent fields.

First, we survey ACSV from a probabilistic perspective, illustrating how its most advanced methods provide efficient algorithms to derive limit theorems, and comparing the results to past work deriving limit theorems. Using the results of ACSV, we provide a SageMath package that can automatically compute (and rigorously verify) limit theorems for a large class of combinatorial generating functions. To illustrate the techniques involved, we also establish explicit local central limit theorems for a family of combinatorial classes whose generating functions are linear in the variables tracking each parameter. Applications covered by this result include the distribution of cycles in certain restricted permutations (proving a limit theorem conjectured in work of Chung et al. [16]), integer compositions, and n-colour compositions with varying restrictions and values tracked. Key to establishing these explicit results in an arbitrary dimension is an interesting symbolic determinant, which we compute by conjecturing and then proving an appropriate LU-factorization.

The second part of this thesis shifts focus to the calculation of asymptotics of multivariate algebraic generating functions through ACSV. So far, the methods of ACSV have largely focused on rational (or, more generally, meromorphic) generating functions, although many natural combinatorial objects have generating functions with algebraic singularities. In this part, we survey techniques for analyzing multivariate algebraic generating functions, going into detail specifically for the process of embedding an algebraic generating function into a sub-series of a rational function of more variables. Other methods mentioned include explicit singularity analysis of algebraic singularities, and manipulation of complex integrals over algebraic hypersurfaces. We give implementations of the embedding techniques in the SageMath computer algebra system, and provide examples from the combinatorics literature.

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- My co-authors Torin Greenwood and Mark C. Wilson for their contributions, support and collaboration on the second paper associated with this thesis.

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Dedication

In memory of Florence Metzger, thank you for inspiring me to follow my dreams.

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Chapter 1

Introduction

A key goal in combinatorics is enumerating mathematical objects or, rather, counting the number of elements of some finite set. In particular, the aim is to find techniques, be it through algorithms, exact methods or large-scale behaviour of objects, to determine the sequence which counts a mathematical object. There are two main goals of this thesis. The first goal is to showcase how approaches from the field of analytic combinatorics in several variables (ACSV) can be used to find the large-scale behaviour of numerous mathematical objects. The second goal is to demonstrate how the methods of ACSV can be automated, and provide implementations for future use. More precisely, we aim to showcase how implementations of ACSV techniques create black box functions that allow a user to obtain results while requiring little or no background knowledge of the field of ACSV.

1.1 Motivating Example

The mathematical objects that are studied in the field of enumerative combinatorics are vast and varied. Problems in this field include questions such as "How many permutations are there of size n", "What is the probability that a random tree of size n has $\frac{n}{3}$ leaves" and "What is the average number of comparisons that quicksort performs when sorting all possible permutations?". In this thesis, we focus on counting objects which have more than one aspect, or parameter, tracked. An example of such a question is:

How many non-empty planar rooted binary trees are there with n vertices and k leaves?

To understand what this question is asking, we first need the definition of a planar rooted binary tree.

Definition 1. A *rooted binary tree* is a tree with the following properties:

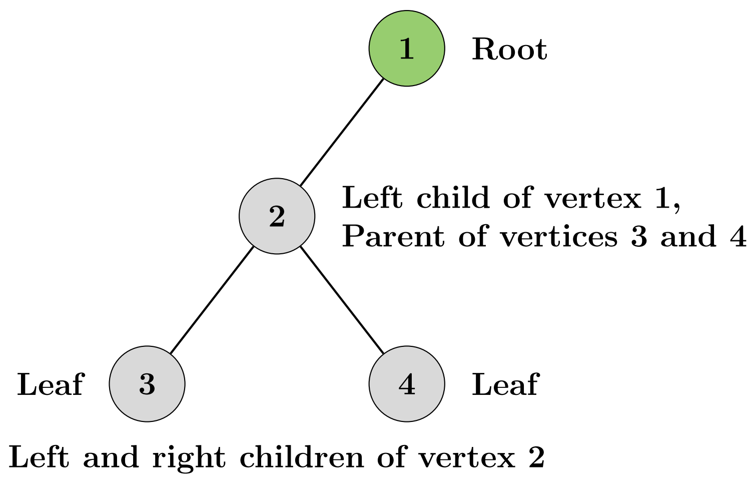


Figure 1.1: An example rooted binary tree with important characteristics labeled.

- If the tree has at least one vertex, one vertex is labeled as the root.
- Every vertex, aside from the root, has exactly one parent vertex.
- Every vertex, including the root, is connected to at most two non-parent vertices, called (if they exist) left and right children.

Vertices of a binary tree that have no children are *leaves*. A *non-empty* tree is a tree with at least one vertex. A binary tree is *planar* if we distinguish between left and right children. For example, a tree with a root vertex and one left child viewed as different from the tree with a root vertex and one right child, even though they are both graphs with two vertices and one edge connecting them. This can be seen in Figure 1.2, with the two leftmost trees being distinct from the two rightmost trees.

Figure 1.1 demonstrates the different aspects of a non-empty planar rooted binary tree in the above definitions. Figure 1.2 shows all non-empty planar rooted binary trees with 3 vertices.

We can compute a number pieces of enumerative information about such trees. For example, using classical techniques from enumerative combinatorics it can be shown that there are

$$\frac{1}{n+1}\binom{2n}{n}$$

planar rooted binary trees with n vertices (where we do not track the number of leaves). Additionally, it is possible to prove that the number of leaves among the planar rooted binary trees of size n satisfy a normal distribution with mean $\frac{n}{4}$, an example of a Central Limit Theorem (CLT). Using ACSV we can find more precise information, for instance

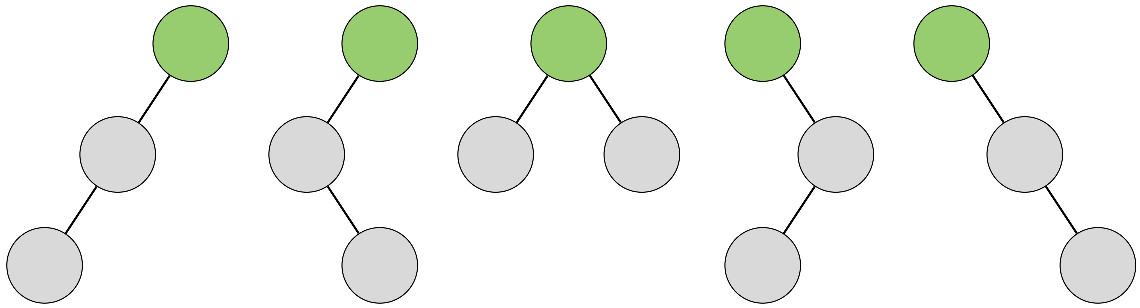


Figure 1.2: All non-empty planar rooted binary trees with 3 vertices. Note that there are four trees with one leaf and one tree with two leaves.

that as the size $n \to \infty$, the number of non-empty planar rooted binary trees with an leaves and bn total vertices tends to

$$\frac{\left(\frac{2^{-2a+b}(-2a+b)^{2a-b}b^b}{a^{2a}}\right)^n}{(a+b)n}\left(\frac{a+b}{\pi n(b-2a)^{\frac{3}{2}}\sqrt{b}}+O\left(\frac{1}{n^2}\right)\right)$$

when $b > 2a$. Binary trees have applications in the sorting and storing of data, coding theory, and more. They are one of many examples that highlight the diverse applications and importance of enumerative combinatorics. The remainder of this thesis will provide two main approaches, limit theorems and asymptotics, to study such sequences.

1.2 Organization

This thesis is a combination of work from two papers, *Central Limit Theorems via Analytic Combinatorics* [60] and *Algebraic Generating Functions and Analytic Combinatorics in Several Variables* (in progress). Chapter 2 provides the background required for both papers, specifically background for enumeration in Section 2.1, analytic combinatorics in Section 2.2, and ACSV in Section 2.3. Chapter 3 discusses results that provide Local Central Limit Theorems (LCLTs) for multivariate generating functions. Chapter 4 discusses various methods for calculating asymptotics of algebraic generating functions, including algorithms. In Chapter 5, documentation and examples are provided for SageMath implementations of automatic LCLTs and asymptotics of algebraic generating functions. This chapter is intended to be self-contained, and thus can be read independently of the remainder of this thesis. Finally, Chapter 6 provides the next steps and conclusions of this thesis.

Chapter 2

Background

This chapter provides a general background for the field of ACSV. Specifically, we discuss the required enumeration background including combinatorial classes, generating functions and multivariate notation. Then we provide details of analytic combinatorics in the univariate setting, before finally offering an overview of the field of ACSV. Additional background material that is required only for Chapter 3 or Chapter 4 will be provided within those chapters.

2.1 Enumeration

We begin by establishing definitions for the mathematical objects that we wish to count, as well as providing definitions for how we encode the information we study.

Definition 2. Let $\mathcal{C}$ represent a set of objects. A *weight function* is a function $w : \mathcal{C} \to \mathbb{N}$, which maps the elements of $\mathcal{C}$ to the natural numbers[1].

We illustrate this definition with an example.

Example 3. Let $\mathcal{C}$ represent the set of binary strings. Then the function $w_1 : \mathcal{C} \to \mathbb{N}$, where $w_1(c)$ gives the length of the binary string, and the function $w_2 : \mathcal{C} \to \mathbb{N}$, where $w_2(c)$ is the number of 1s in the string, are weight functions. ◁

Definition 4. A *combinatorial class* is a set of objects $\mathcal{C}$ paired with a weight function w such that for each $n \in \mathbb{N}$ there are a finite number of objects $c \in \mathcal{C}$ such that $w(c) = n$.

Example 5. With w_1 and w_2 as in Example 3 above, $(\mathcal{C}, w_1)$ is a combinatorial class because there are a finite number of binary strings of any given length and hence a finite number of binary strings c such that $w_1(c) = n$ for any $n \in \mathbb{N}$. However, $(\mathcal{C}, w_2)$ is not a combinatorial class as there are infinitely many binary strings with which have no 1's. In other words, there are an infinite number of binary strings c such that $w_2(c) = 0$. ◁

[1]Note that in this thesis we have that $0 \in \mathbb{N}$.

Definition 6. Let $(\mathcal{C}, w)$ be a combinatorial class. Then the *counting sequence* of $(\mathcal{C}, w)$ is the sequence (c_n) where c_n is the number of objects c in $\mathcal{C}$ such that $w(c) = n$. The *generating function* of a combinatorial class $(\mathcal{C}, w)$ with counting sequence (c_n) is the series

$$C(x) = \sum_{n \geq 0} c_n x^n.$$

The coefficient of x^n in $C(x)$, denoted $[x^n]C(x)$, is the number c_n of objects in $\mathcal{C}$ of weight n.

Example 7. If $\mathcal{C}$ is the set of binary strings, and w_1 is the length function defined in Example 3, then the counting sequence of $(\mathcal{C}, w_1)$ is $c_n = 2^n$. The generating function of $(\mathcal{C}, w_1)$ is the series

$$C(x) = \sum_{n \geq 0} 2^n x^n = \frac{1}{1 - 2x}$$

and $[x^n]C(x) = 2^n$. ◁

Example 8. If b_n denotes the number of (non-empty planar rooted) binary trees on n vertices then the *symbolic method* [29, Chapter I] allows one to translate the recursive definition of such a tree as either a single root vertex, a root followed by either a left or right subtree, or a root followed by two subtrees into the algebraic equation

$$B(x) = x + 2xB(x) + xB(x)^2 \tag{2.1}$$

for the generating function $B(x) = \sum_{n \geq 0} b_n x^n$. In particular, this recursive definition provides a *combinatorial specification*, a way of deconstructing the class of objects into smaller pieces such that the larger class can be re-created from the smaller pieces. The quadratic formula then recovers the classical *Catalan generating function* (with the constant removed)

$$B(x) = \frac{1 - \sqrt{1 - 4x}}{2x} - 1.$$

◁

A weight function allows us to track one property of a combinatorial object, and generating functions provide a method for storing the corresponding information. We can also have multivariate generating functions, where multiple properties of objects are tracked.

Definition 9. A *combinatorial class with d parameters* is a set of objects C, paired with weight function $w : \mathcal{C} \to \mathbb{N}$ (such that for each $n \in \mathbb{N}$ there are a finite number of objects $c \in \mathcal{C}$ with $w(c) = n$) and a parameter function $p : \mathcal{C} \to \mathbb{Z}^d$. The *multivariate generating function* of the combinatorial class with d parameters $(\mathcal{C}, w, p)$, is

$$C(\mathbf{z}, t) = \sum_{\sigma \in \mathcal{C}} \mathbf{z}^{p(\sigma)} t^{w(\sigma)} = \sum_{n \geq 0} \left(\sum_{\mathbf{i} \in \mathbb{Z}^d} f_{\mathbf{i},n} \mathbf{z}^{\mathbf{i}} \right) t^n$$

with $\mathbf{z} = (z_1, z_2, \ldots, z_d)$, where $\mathbf{z}^{\mathbf{i}} = z_1^{i_1} z_2^{i_2} \cdots z_d^{i_d}$ and $f_{\mathbf{i},n}$ is the number of objects in $\mathcal{C}$ such that $p(\sigma) = \mathbf{i}$ and $w(\sigma) = n$. Our notation for extracting a coefficient is similar to the univariate setting. In particular, $[\mathbf{z}^{\mathbf{i}} t^n]C(\mathbf{z}, t) = f_{\mathbf{i},n}$ and $[t^n]C(\mathbf{z}, t) = \sum_{\mathbf{i} \in \mathbb{Z}^d} f_{\mathbf{i},n} \mathbf{z}^{\mathbf{i}}$.

Remark 10. In this thesis extra parameters in a combinatorial class do not need to follow the same restriction as the weight function. The result of the map could be any integer and parameters could have an infinite number of objects that map to a certain integer. In general, we simply need that every coefficient in the resulting function would be finite under the weight and parameter functions defined. That being said, the definition above is most useful for this thesis.

Example 11. Return again to the set $\mathcal{C}$ of binary strings where $w(c)$ is the length of string c. This time, we also define $p(c)$ as the number of 1's in the binary string c. Then $(\mathcal{C}, w, p)$ is a combinatorial class with 1 parameter, whose generating function is

$$C(z,t) = \frac{1}{1-(1+z)t}.$$

◁

Example 12. Let $\mathcal{B}$ be the set of non-empty planar rooted binary trees and let w be the function $w : \mathcal{B} \to \mathbb{N}$ where $w(b)$ is the number of vertices in b. If p is the function $p : \mathcal{B} \to \mathbb{Z}$ where $p(b)$ is the number of leaves in b then $(\mathcal{B}, w, p)$ is a combinatorial class with 1 parameter. The multivariate generating function for this class is

$$B(z,t) = \sum_{k,n\geq 0} b_{k,n} z^k t^n,$$

where $b_{k,n}$ is the number of non-empty planar rooted binary trees with k leaves and n vertices. Modifying the combinatorial specification in Example 8 allows one to write $B(\mathbf{z}, t)$ in closed form. Indeed, the class of non-empty planar rooted binary trees can still be thought of as the class where every object is one of the following:

- Just a root,
- A root with a left subtree in $\mathcal{B}$ and no right child,
- A root with a right subtree in $\mathcal{B}$ and no left child, or
- A root with a left subtree in $\mathcal{B}$ and a right subtree in $\mathcal{B}$

Since the root adds one to the number of vertices in an object, and the only time we add leaves are through the subtrees or if we have only a root and no children, we may decompose $\mathcal{B}$ as $B = zt + 2tB + tB^2$. The quadratic formula can then be applied to find the generating function

$$B(z,t) = \frac{1-\sqrt{1-4t+(1-z)4t^2}}{2t} - 1,$$

. ◁

In this thesis, we typically assume that we already have access to the generating function and focus on extracting information from this generating function.

2.2 Analytic Combinatorics

Prior to diving into the multivariate case, we provide a brief discussion of the univariate theory of analytic combinatorics. A detailed treatment of the results described here can be found in the work of Flajolet and Sedgewick [29]. The methods of analytic combinatorics rely on the Cauchy integral formula.

Theorem 13 (Flajolet and Sedgewick [29, Theorem IV.4])**.** *Let $F(x) = \sum_{n\geq 0} f_n x^n$ be a function which is analytic in an open connected subset $\mathcal{D}$ of $\mathbb{C}$. Then for any simple closed curve $\gamma \in \mathcal{D}$ which surrounds the origin,*

$$f_n = \frac{1}{2\pi i}\int_\gamma \frac{F(x)}{x^{n+1}}dx.$$

This powerful result links the coefficients of a generating function to the analytic behaviour of the function that it represents. Thus, the question of determining sequence asymptotics turns to one of studying the Cauchy integral in Theorem 13. Before we continue, we require a few definitions.

Definition 14. An *isolated singularity* of a function $F(x)$ is a point such that F is analytic in a neighbourhood of the point except at the point. An isolated singularity ρ of a function $F(x)$ is a *pole* if

$$\lim_{x\to\rho}|F(x)| = \infty.$$

Isolated singularities which are not poles are called *essential singularities.* An alternative, and more common method for defining poles is to say that ρ is a pole if

$$\lim_{x\to\rho}(x-\rho)^k F(x)$$

is finite and non-zero for some $k \in \mathbb{Z}_{>0}$. We say $F(x)$ is *meromorphic* over an open connected subset of $\mathbb{C}$ if it is analytic in that subset aside from a finite set of points, each of which are poles.

Mermorphic functions can be integrated using *residues*.

Definition 15. Let $F(x)$ be a function which has a pole at ρ. Then there exists a positive integer k such that

$$F(x) = \sum_{n\geq -k} f_n(x-\rho)^n$$

converges for all x in a neighbourhood of ρ, except at ρ, where $f_{-k} \neq 0$. The *order of the pole* $z = \rho$ is k and the coefficient f_{-1} is called the *residue* of $F(x)$ at ρ.

Theorem 16 (Flajolet and Sedgewick [29, Theorem IV.3]). *Let $F(x)$ be a meromorphic function in some open connected subset $\mathcal{D}$ of $\mathbb{C}$. Then, for any simple closed curve $\gamma \in \mathcal{D}$ avoiding the poles of F,*

$$\frac{1}{2\pi i}\int_\gamma F(x)dx = \sum_{\rho\in\Lambda}\operatorname*{Res}_{x=\rho} F(x)$$

where Λ is the set of poles of F inside γ.

A few helpful results and definitions for applying Theorem 16 are the following.

Definition 17. Assume that we have two functions $\mathbb{N} \to \mathbb{R}_{\geq 0}$, $a(n)$ and $b(n)$. Then we have the following definitions.

- If $\lim_{n\to\infty} \frac{a(n)}{b(n)} = 1$ then we write $a(n) \sim b(n)$.
- If there exists two constants $c > 0$ and $N \in \mathbb{N}$ such that, for all $n \geq N$, $a(n) \leq c\cdot b(n)$ then $a(n) = O(b(n))$.
- If $\lim_{n\to\infty} \frac{a(n)}{b(n)} = 0$ then $a(n) = o(b(n))$.

Theorem 18 (Melczer [59, Lemma 2.4]). *If $F(x) = \frac{G(x)}{H(x)}$ has a pole at $x = \rho$ of order k then*

$$\operatorname*{Res}_{x=\rho} F(x) = \frac{1}{(k-1)!}\lim_{x\to\rho}\left(\frac{d^{k-1}}{dx^{k-1}}(x-\rho)^k F(x)\right).$$

In particular, if $F(x)$ has a pole of order k at $x = \rho$ then

$$\operatorname*{Res}_{x=\rho}\frac{F(x)}{x^{n+1}} = \rho^{-n}n^{k-1}\left((-1)^{k-1}\frac{kG(\rho)}{\rho^k H^{(k)}(\rho)} + O\left(\frac{1}{n}\right)\right).$$

Theorem 19 (Flajolet and Sedgewick [29, Chapter VIII.2, Equation 5]). *Let γ be a curve of finite length. Then*

$$\left|\int_\gamma F(x)dx\right| \leq length(\gamma)\cdot sup_{x\in\gamma}|F(x)|$$

when this integral exists.

Example 20. An *integer partition* of size n with parts at most 5 is a non-decreasing sequence of positive integers each less than 6 that sum to n. Consider

$$F(x) = \frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)},$$

the generating function for integer partitions with parts at most 5. We will apply Theorem 16 to determine an approximation for $[x^n]F(x)$. Note first that all singularities of this function lie on the unit circle, and the singularity of highest order is $x = 1$ with order

5. Since the poles of $F(x)$ are all on the unit circle, $F(x)$ is analytic inside the unit disk. Thus, the Cauchy integral Formula gives that $f_n = \frac{1}{2\pi i}\int_\gamma \frac{F(x)}{x^{n+1}}dx$ for $\gamma = \{|x| = \epsilon\}$ with any $0 < \epsilon < 1$. Theorem 16 then gives that $f_n = \mathrm{Res}_{x=0}\frac{F(x)}{x^{n+1}}$, since 0 is the only pole of $\frac{F(x)}{x^{n+1}}$ in γ.

Now, let Λ represent the set of poles of $F(x)$. Since all poles of $F(x)$ have (maximum) modulus 1, Theorem 16 implies that

$$\frac{1}{2\pi i}\int_{|x|=2} F(x)\frac{dx}{x^{n+1}} = \sum_{\rho\in\Lambda} \operatorname*{Res}_{x=\rho} F(x)\frac{1}{x^{n+1}} + \operatorname*{Res}_{x=0} F(x)\frac{1}{x^{n+1}}.$$

Since all poles of $F(x)$ are on the unit circle and have order lower than 5, aside from $x = 1$, we see that

$$\frac{1}{2\pi i}\int_{|x|=2} F(x)\frac{dx}{x^{n+1}} = \operatorname*{Res}_{x=1}\frac{F(x)}{x^{n+1}} + O(n^3) + \operatorname*{Res}_{x=0}\frac{F(x)}{x^{n+1}}.$$

By Theorem 18,

$$\operatorname*{Res}_{x=1}\frac{F(x)}{x^{n+1}} = 1^{-n}n^4\left((-1)^4\frac{5\cdot 1}{1^5\cdot(-14400)}\right) = -\frac{n^4}{4!5!},$$

and hence

$$f_n = \operatorname*{Res}_{x=0} F(x)\frac{dx}{x^{n+1}} = \frac{1}{2\pi i}\int_{|z|=2} F(x)\frac{dx}{x^{n+1}} + \frac{n^4}{4!5!} + O(n^3).$$

Theorem 19 provides a bound

$$\left|\frac{1}{2\pi i}\int_{|z|=2} F(x)\frac{dx}{x^{n+1}}\right| = O(2^{-n}),$$

and thus

$$f_n \sim \frac{n^4}{4!5!}.$$

◁

Another classical method to approximate integrals is the *saddle-point method*, named after points that visually look like a saddle. For an integral of the form

$$\frac{1}{2\pi i}\int_{|x|=r} F(x)\frac{dx}{x^{n+1}},$$

the saddle-point method can be broken into the following steps.

1. Pick $r > 0$ such that $x = r$ is a *saddle-point* by solving the saddle-point equation $\frac{rF'(r)}{F(r)} = n + 1$.

2. Divide $\{|x| = r\}$ into two regions, call them A and B, such that the integral over B is exponentially small. Re-parameterize to integrate over an interval in $\mathbb{R}$.

3. Prove that the integrand can be replaced by the leading terms of its series expansion at a point when integrating over A.

4. Prove that the interval we are integrating over can be extended to $\mathbb{R}$.

5. Compute the integral.

Remark 21. The saddle-point method is known to work for large classes of functions, see [29].

We illustrate the method through an example.

Example 22. Consider the integral

$$\frac{1}{2\pi i} \int_{|x|=r} \frac{1}{(1-x)^{n+1}} \frac{dx}{x^{n+1}},$$

which equals $\binom{2n}{n}$ for any $0 < r < 1$. Our first step is to find a saddle-point. For this example, the saddle-point equation gives

$$\frac{r(n+1)(1-r)^{n+1}}{(1-r)^{n+2}} = n + 1,$$

which reduces to $r = 1 - r$ and thus we take $r = \frac{1}{2}$.

Now, we divide the circle of radius $\frac{1}{2}$ around the origin into two regions,

$$A = \left\{ \frac{e^{i\theta}}{2} : \theta \in \left(-\frac{\pi}{4}, \frac{\pi}{4} \right) \right\}$$

and B the rest of the circle. For any $x \in B$, Theorem 19 gives

$$\left| \frac{1}{2\pi i} \int_B \frac{1}{(1-x)^{n+1}} \frac{dx}{x^{n+1}} \right| = O(3^n) \tag{2.2}$$

as a result of the following. First, we note that

$$\left| \frac{1}{(1-x)x} \right| \leq \frac{1}{|1-x|} \frac{1}{|x|}$$

by definition of $|\cdot|$. Then, for any $x \in B$,

$$\frac{1}{|1-x|} \leq \frac{1}{\left|1 - \frac{e^{i\pi/4}}{2}\right|} < \frac{3}{2}$$

and

$$\frac{1}{|x|} = \frac{1}{\left|\frac{e^{i\theta}}{2}\right|} = 1.$$

When we take these results to the power of $n+1$ and multiply by the length of our curve B we get our results. We will see that the integral in (2.2) is exponentially less than the integral over A. We re-parameterize A by taking $x = \frac{1}{2}e^{i\theta}$, to get

$$\frac{4^n}{2\pi}\int_{-\pi/4}^{\pi/4} \frac{1}{\left(1-\frac{e^{i\theta}}{2}\right)} \frac{d\theta}{((2-e^{i\theta})e^{i\theta})^n}.$$

It can be shown that the integrand can be replaced by the leading terms of its series expansion with sufficiently small error to approximate our integral as

$$\frac{4^n}{2\pi}\int_{-\pi/4}^{\pi/4} 2e^{-n\theta^2} d\theta.$$

Again approximating by extending this integral to $\mathbb{R}$ gives

$$\frac{4^n}{\pi}\int_{-\infty}^{\infty} e^{-n\theta^2} d\theta = \frac{4^n}{\sqrt{\pi n}}.$$

So, the saddle-point method ultimately implies

$$\binom{2n}{n} = \frac{1}{2\pi i}\int_{|x|=\frac{1}{2}} \frac{1}{(1-x)^{n+1}} \frac{dx}{x^{n+1}} \sim \frac{4^n}{\sqrt{\pi n}}.$$

◁

2.3 Analytic Combinatorics in Several Variables

While the methods of analytic combinatorics are quite strong in the univariate case, this thesis aims to consider multivariate generating functions. For these problems we look to the relatively new field of analytic combinatorics in several variables (ACSV). ACSV provides a unified framework to study the asymptotics of multivariate generating functions, and a deeper understanding of which singularities contribute to coefficient asymptotics. The results are also explicit to the point of being completely automated for large classes of combinatorial generating functions (such as through work of Hackl et al. [41] and the software packages corresponding to this thesis), allow for the computation of asymptotic expansions to arbitrary order, and work under different sets of assumptions.

In this section we provide an overview of the methods of ACSV as used to calculate asymptotics for rational generating functions. More specific background for computing LCLTs and asymptotics of algebraic generating functions can be found in Section 3.3 and

Section 4.1, respectively. Note that much of the discussion for this background comes from [60].

Let $F(\mathbf{z},t) = G(\mathbf{z},t)/H(\mathbf{z},t)$ be a ratio of complex-valued functions G and H analytic in a domain $\mathcal{D} \subset \mathbb{C}^{d+1}$ containing the origin, and suppose that F has a power series expansion

$$F(\mathbf{z},t) = \sum_{(\mathbf{i},k)\in\mathbb{N}^{d+1}} f_{\mathbf{i},k}\mathbf{z}^{\mathbf{i}}t^k = \sum_{(\mathbf{i},k)\in\mathbb{N}^{d+1}} f_{i_1,\dots,i_d,k} z_1^{i_1}\cdots z_d^{i_d}t^k$$

valid in some neighbourhood of the origin in $\mathcal{D}$ (meaning, in particular, that $H(\mathbf{0},0) \neq 0$).

To discuss asymptotics in a multivariate setting, we must describe how the indices we consider go to infinity. One approach, which we use in this background, is to fix a *direction vector* $(\mathbf{r},s) \in \mathbb{R}^{d+1}_{>0}$ and determine asymptotics of the univariate $(\mathbf{r},s)$-*diagonal sequence* $(f_{n\mathbf{r},ns})_{n\geq 0}$ whose generating function is the $(\mathbf{r},s)$-*diagonal*

$$(\Delta_{(\mathbf{r},s)}F)(x) = \sum_{n\geq 0} f_{n(\mathbf{r},s)}x^n.$$

Because we deal with Laurent series expansions with at least one of the variables always non-negative, we consider all formulas for $f_{n(\mathbf{r},s)}$ in this thesis as holding only when $n(\mathbf{r},s) \in \mathbb{N}^{d+1}$, and take $f_{n(\mathbf{r},s)}$ to be undefined otherwise. The theory of ACSV shows how asymptotics often vary smoothly with $(\mathbf{r},s)$, allowing for more general asymptotic expansions and limit theorems. The most common case occurs for the *main diagonal* $(\mathbf{r},s) = \mathbf{1}$, in which case we write

$$(\Delta F)(x) = (\Delta_{\mathbf{1}}F)(x) = \sum_{n\geq 0} f_{n,\dots,n}x^n.$$

Similar to the univariate case, asymptotic arguments typically start with the Cauchy integral representation

$$f_{\mathbf{r},s} = \frac{1}{(2\pi i)^{d+1}} \int_{\mathcal{T}} F(\mathbf{z},t)\mathbf{z}^{-\mathbf{r}}t^{-s}\frac{d\mathbf{z}dt}{z_1\cdots z_d t}, \tag{2.3}$$

where $\mathcal{T}$ is any product of circles $|z_j| = |t| = \varepsilon$ sufficiently close to the origin. The methods of ACSV manipulate the domain of integration $\mathcal{T}$ to convert the Cauchy integral (2.3) into something that can be asymptotically approximated. As in the more classical univariate case, this process depends heavily on the singular set of the generating function F. Because F is a ratio, its singularities form a subset of the analytic variety $\mathcal{V} = \{(\mathbf{z},t) \in \mathbb{C}^{d+1} : H(\mathbf{z},t) = 0\}$ defined by the vanishing of the denominator H, and includes all points where H vanishes and the numerator G does not. In many applications F is a rational function, in which case we may assume that G and H are coprime polynomials and the singular set of F equals the algebraic variety $\mathcal{V}$ defined by the vanishing of H (a similar characterization holds for general meromorphic functions, but one must introduce the notion of *coprime germs of holomorphic functions*).

Univariate meromorphic functions that are not entire always have a finite set of *dominant singularities* (the singularities with minimal modulus) dictating their asymptotic behaviour, and explicit expressions for asymptotics can be determined by adding up *contributions* given by each of these points. In contrast, if F is rational but not a polynomial and the dimension $d \geq 2$ then the set $\mathcal{V}$ is infinite and the *geometry* of $\mathcal{V}$ plays a large role in determining coefficient behaviour. In order to characterize the singularities determining asymptotics, we make the following definitions.

Definition 23. Let $(\mathbf{w}, s) \in \mathbb{C}_*^{d+1} = (\mathbb{C} \setminus \{0\})^{d+1}$. We say that $(\mathbf{w}, s)$ is

- a *minimal point* if $H(\mathbf{w}, s) = 0$ and there is no element of $\mathcal{V}$ which is coordinate-wise closer to the origin, i.e., there does not exist $(\mathbf{y}, q)$ with $|y_j| < |w_j|$ for all $1 \leq j \leq d$ and $|q| < |s|$ such that $H(\mathbf{y}, q) = 0$;
- a *strictly minimal point* if it is minimal and no other point of $\mathcal{V}$ has the same coordinate-wise modulus;
- a *finitely minimal point* if it is minimal and only a finite number of points in $\mathcal{V}$ have the same coordinate-wise modulus;
- a *smooth critical point in the direction* $(\mathbf{r}, m) \in \mathbb{R}_{>0}^{d+1}$ if it satisfies the system of equations

$$H(\mathbf{w}, s) = 0, \qquad \frac{w_1}{r_1} H_{z_1}(\mathbf{w}, s) = \frac{w_2}{r_2} H_{z_2}(\mathbf{w}, s) = \cdots = \frac{w_d}{r_d} H_{z_d}(\mathbf{w}, s) = \frac{t}{m} H_t(\mathbf{w}, s) \tag{2.4}$$

  and one of these partial derivatives does not vanish (which, in fact, implies that all of the derivatives do not vanish).

Remark 24. If H and all of its partial derivatives simultaneously vanish at a point $\mathbf{w}$ then either $\mathbf{w}$ is a zero of H *with multiplicity greater than one* or $\mathcal{V}$ is not a manifold near $\mathbf{w}$. In the first case, H can be replaced by its *square-free part* near $\mathbf{w}$ to determine critical points (when H is a polynomial this means replacing it by the product of its distinct irreducible factors). In the second case, when $\mathcal{V}$ has non-smooth points, critical points can be defined by *stratifying* $\mathcal{V}$ into a finite collection of smooth manifolds that 'fit together nicely' and calculating critical points on each stratum. In general, if H is a polynomial then the critical points on each stratum are defined by a finite collection of polynomial equalities and inequalities that can be computed automatically from H (see [63, Section 8.2] and [41]). To simplify our presentation here we state our main results for the smooth case with zeroes of multiplicity one.

2.3.1 Asymptotic Results

The earliest techniques of ACSV were derived using an explicit *surgery method*. ACSV determines asymptotic behaviour by manipulating the domain of integration $\mathcal{T}$ in (2.3), splitting it into some regions where the Cauchy integral is negligible and other regions where the integral can be approximated with analytic techniques. Roughly speaking, in the smooth setting one can push out the domain of integration $\mathcal{T}$ in (2.3) to approach the set of singularities of F, take a residue in one variable to reduce to a $d-1$ dimensional integral lying 'on the singular set,' and then (hopefully) determine asymptotics of this lower dimensional integral using the saddle-point method. Minimal points are those to which $\mathcal{T}$ can be easily deformed, while critical points are those where a saddle-point analysis can be performed locally after computing a residue.

The surgery approach to ACSV (introduced for the smooth case in [62]) applies in the presence of *finitely minimal* critical points. The assumption of finite minimality allows one to make explicit residue computations by fixing the moduli of the $\mathbf{z}$ variables and varying only the modulus of the t variable. This makes the surgery method a fairly straightforward analogue of the techniques from univariate analytic combinatorics, however finite minimality is difficult to verify computationally and is a stronger condition than necessary. The surgery method is covered in detail in Melczer [59, Chapter 5], yielding asymptotic results relying on one further quantity.

Definition 25. Let $(\mathbf{w}, s) \in \mathbb{C}_*^{d+1}$ be a smooth critical point in the direction $(\mathbf{r}, m)$ and suppose $H_t(\mathbf{w}, s) \neq 0$. The *phase Hessian* $\mathcal{H}(\mathbf{w}, s)$ of H at $(\mathbf{z}, t) = (\mathbf{w}, s)$ is the $d \times d$ matrix $\mathcal{H}$ with entries

$$\mathcal{H}_{i,j} = \begin{cases} \frac{r_i r_j}{m^2} + U_{i,j} - \frac{r_j}{m} U_{i,d+1} - \frac{r_i}{m} U_{j,d+1} + \frac{r_i r_j}{m^2} U_{d+1,d+1} & : i \neq j \\ \frac{r_i}{m} + \frac{r_i^2}{m^2} + U_{i,i} - \frac{2r_i}{m} U_{i,d+1} + \left(\frac{r_i}{m}\right)^2 U_{d+1,d+1} & : i = j \end{cases} \tag{2.5}$$

where $U_{i,j} = \frac{w_i w_j H_{z_i z_j}(\mathbf{w},s)}{t H_t(\mathbf{w},s)}$ for $1 \leq i, j \leq d$ while $U_{i,d+1} = \frac{w_i H_{z_i t}(\mathbf{w},s)}{H_t(\mathbf{w},s)}$ and $U_{d+1,d+1} = \frac{t H_{tt}(\mathbf{w},s)}{H_t(\mathbf{w},s)}$. The point $(\mathbf{w}, s)$ is called *nondegenerate* if the phase Hessian matrix $\mathcal{H}(\mathbf{w}, s)$ has non-zero determinant.

Theorem 26. *Suppose that the rational function $F(\mathbf{z}, t) = G(\mathbf{z}, t)/H(\mathbf{z}, t)$ admits a nondegenerate strictly minimal smooth critical point $(\mathbf{w}, s) \in \mathbb{C}_*^{d+1}$ in the direction $(\mathbf{r}, m) \in \mathbb{R}_*^{d+1}$, such that $H_t(\mathbf{w}, s) \neq 0$. Then for any non-negative integer M there exist computable constants $C_0, \ldots, C_M$ such that*

$$f_{n(\mathbf{r},m)} = (w_1^{r_1} \cdots w_d^{r_d} s^m)^{-n} n^{-d/2} \frac{(2\pi)^{-d/2}}{\sqrt{\det(m\mathcal{H})}} \left(\sum_{j=0}^{M} C_j (mn)^{-j} + O\left(n^{-M-1}\right) \right), \tag{2.6}$$

where $\mathcal{H} = \mathcal{H}(\mathbf{w}, s)$ is the phase Hessian matrix and

$$C_0 = \frac{-G(\mathbf{w}, s)}{s\, H_t(\mathbf{w}, s)}.$$

The asymptotic expansion (2.6) holds uniformly in neighbourhoods $\mathcal{R} \subset \mathbb{R}_*^{d+1}$ of $\mathbf{r}$ where there is a smoothly varying nondegenerate strictly minimal critical point such that H_t does not vanish.

The matrix in Definition 25 is equivalent to the $\mathcal{H}$ in Theorems 30 and 34 below up to sign, with the entries now determined explicitly from evaluations of partial derivatives of H using (2.5).

Example 27. Let

$$F(z,t) = \frac{1}{1-z-t} = \sum_{i,j\geq 0} f_{i,j} z^i t^j = \sum_{i,j\geq 0} \binom{i+j}{i} z^i t^j.$$

Although we can find the asymptotics of $(f_{i,j})$ directly from Theorem 26, we sketch the techniques involved for the $(1,1)$-direction. For this example, the Cauchy integral formula implies that

$$\binom{2n}{n} = \frac{1}{(2\pi i)^2} \int_{T(a,b)} \frac{1}{1-z-t} \frac{dzdt}{z^{n+1}t^{n+1}},$$

where $(a,b) \in \{(z,t) \in \mathbb{C}^2 : |z| + |t| < 1\}$. In the $(1,1)$ direction, we solve the critical point equations

$$1 - z - t = -z + t = 0$$

to get the critical point $\left(\frac{1}{2}, \frac{1}{2}\right)$. By the triangle inequality, this point is strictly minimal. Taking $(a,b) = \left(\frac{1}{2}, \frac{1}{2}\right)$, it can be shown that with exponentially small error we can introduce a new integral

$$\begin{aligned}
\binom{2n}{n} &\sim \frac{1}{(2\pi i)^2} \int_{\substack{|z|=1/2 \\ |t|=1/4}} \frac{1}{1-z-t} \frac{dzdt}{z^{n+1}t^{n+1}} - \frac{1}{(2\pi i)^2} \int_{\substack{|z|=\frac{1}{2} \\ |t|=2}} \frac{1}{1-z-t} \frac{dzdt}{z^{n+1}t^{n+1}} \\
&= \frac{1}{2\pi i} \int_{|z|=1/2} \operatorname*{Res}_{t=1-z} \frac{1}{1-z-t} \frac{dzdt}{z^{n+1}t^{n+1}} \\
&= \frac{1}{2\pi i} \int_{|z|=1/2} \frac{dz}{z^{n+1}(1-z)^{n+1}} \\
&\sim \frac{4^n}{\sqrt{\pi n}},
\end{aligned}$$

by Example 22 above. For a general direction (r,s) we get the critical point equations

$$1 - z - t = -sz + st = 0,$$

with corresponding strictly minimal critical point $\left(\frac{r}{r+s}, \frac{s}{r+s}\right)$. Applying Theorem 26 directly then gives asymptotics

$$f_{rn,sn} \sim \left(\frac{r+s}{r}\right)^{rn} \left(\frac{r+s}{s}\right)^{sn} \frac{\sqrt{r+s}}{\sqrt{2rs\pi n}}.$$

◁

In Chapter 3 we modify such theorems to help prove LCLTs. We also discuss how to prove minimality in cases not as obvious as Example 27. For Chapter 4 we apply such techniques to study multivariate algebraic generating functions.

Chapter 3

Multivariate Limit Theorems

This chapter is modified from the paper "Central Limit Theorems via Analytic Combinatorics in Several Variables" [60]. It will provide some history of Central Limit Theorems (CLTs), background, the motivation for the automation work, results, and two applications where we quickly prove LCLTs.

3.1 Central Limit Theorems

Let (X_n) be a sequence of random variables. A *limit theorem* (or *limit law*) for X_n is an approximation of the cumulative distribution functions $\mathbb{P}(X_n \leq k)$ as $n \to \infty$. A *local limit theorem* is an approximation of the exact probabilities $\mathbb{P}(X_n = k)$ as $n \to \infty$. A *central limit theorem (CLT)* or *local central limit theorem (LCLT)* compares these probabilities to a *normal density function*

$$\phi_{\mu,\sigma}(x) = \frac{1}{\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma}.$$

The *classical CLT* states that if $A_1, A_2, \ldots$ is a sequence of independent identically distributed random variables with an expected value μ and a finite variance $\sigma^2 > 0$ then the sequence of random variables

$$X_n = A_1 + \cdots + A_n$$

converges in distribution, after rescaling, to the standard normal distribution, so that

$$\lim_{n\to\infty} \mathbb{P}\left[\frac{X_n - n\mu}{\sigma\sqrt{n}} \leq x\right] = \int_{-\infty}^{x} \phi_{0,1}(t)dt$$

for all real x. Note that we rescale the sequence X_n so that its limit is a fixed distribution instead of one that varies with n.

The classical CLT has a long history (see Section 3.2 for a brief recap), and has been generalized to weaken its assumptions, give explicit rates of convergence, and work in more abstract settings. In d-dimensions, a *multivariate (L)CLT* compares probabilities to *multivariate normal density functions*

$$\phi_{\boldsymbol{\mu},\boldsymbol{\Sigma}}(\mathbf{x}) = \frac{1}{(2\pi)^{d/2}} \exp\left(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right).$$

The *classical multivariate CLT* states that if $(\mathbf{A}_n)$ is a sequence of d-dimensional independent identically distributed random variables with an expected value vector $\boldsymbol{\mu}$ and a positive definite covariance matrix $\boldsymbol{\Sigma}$ then the sequence of d-dimensional random variables

$$\mathbf{X}_n = \frac{\mathbf{A}_1 + \cdots + \mathbf{A}_n - n\boldsymbol{\mu}}{\sqrt{n}}$$

converges in distribution to the multivariate normal distribution with density function $\phi_{0,\boldsymbol{\Sigma}}(\mathbf{x})$.

In combinatorial contexts, one often has a *combinatorial class* $(\mathcal{C}, |\cdot|)$ defined by a set of objects $\mathcal{C}$ and a *size function* $|\cdot| : \mathcal{C} \to \mathbb{N}$ such that there are only a finite number of objects of any given size. A *d-dimensional parameter* on such a class is any map $\chi : \mathcal{C} \to \mathbb{N}^d$, and the *multivariate generating function* with respect to the triple $(\mathcal{C}, |\cdot|, \chi)$ is the $(d+1)$-dimensional power series[1]

$$C(\mathbf{z}, t) = \sum_{n\geq 0} \left(\sum_{\mathbf{i}\in\mathbb{N}^d} c_{\mathbf{i},n}\mathbf{z}^{\mathbf{i}}\right) t^n$$

where $c_{\mathbf{i},n}$ denotes the number of objects in $\mathcal{C}$ with size n and parameter value $\mathbf{i}$. Well-established combinatorial theories and methods exist to derive algebraic, differential, or functional equations satisfied by the generating functions of many types of combinatorial classes (see, for instance, [29, 38, 68, 70]). Thus, one often has a multivariate generating function $C(\mathbf{z}, t)$ encoded in some way, and wants to determine the limiting behaviour of the d-dimensional array $(c_{\mathbf{i},n})_{\mathbf{i}\in\mathbb{N}^d}$ as a function of $\mathbf{i}$ when $n \to \infty$.

Strikingly, the theory of *analytic combinatorics* (see Section 2.2 for an introduction) shows how central limit theorems for many combinatorial parameters can be derived directly from a study of the singular behaviour of $C(\mathbf{z}, t)$. Flajolet and Sedgewick [29, Chapter IX] gives a detailed introduction, and Hwang [48] lists more than 25 limit laws proven by the late Philippe Flajolet using this framework, including patterns in words, cost analyses of various sorting algorithms, parameters in different kinds of trees, polynomials with restricted coefficients, and ball-urn models.

The classical theory of analytic combinatorics is chiefly concerned with the derivation of asymptotics of univariate generating functions. In typical cases, one applies *transfer theorems* to deduce the asymptotic behaviour of a generating function's coefficients from series expansions of the generating function near its singularities. For many classes of generating functions, it is essentially algorithmic to compute the singularities determining

[1] As mentioned in Chapter 2, throughout this thesis we use the multi-index notation $\mathbf{z}^{\mathbf{i}} = z_1^{i_1} z_2^{i_2} \cdots z_d^{i_d}$ for any d-dimensional vectors $\mathbf{z}$ and $\mathbf{i}$.

asymptotics, apply transfer theorems, and find asymptotic behaviour. In particular, on examples occurring in practice, there exist well established theorems, methods and techniques to compute such asymptotics. We can see an example of this with an application of Theorem 26 in Example 27. General multivariate results in analytic combinatorics can be traced back to work of Bender, Richmond, and collaborators in the 1980s (see Section 3.2 below for more details). Although such results apply to many applications in combinatorics and the analysis of algorithms, they capture only one type of multivariate singularity and require one to verify certain properties of multivariate generating functions (such as grouping singularities by moduli), which can be computationally expensive.

These issues led to the development of *analytic combinatorics in several variables (ACSV)* [59, 63] starting in the early 2000s (see Section 2.3 for more details). Following the framework of univariate analytic combinatorics, the goal is to derive effective methods to take an encoding of a multivariate generating function and return asymptotics or limit theorems of its series coefficients. Most results in ACSV focus on sequences with multivariate rational generating functions. Such sequences appear naturally in many applications, and many functions with more complicated singular behaviour can be encoded by rational functions in a higher number of variables (for instance, every algebraic function in d variables can be encoded as an explicit subseries of a rational function in $2d$ variables [19]). The crux of this chapter is to illustrate both the power of ACSV for proving limit theorems and the (in many cases completely) automatic nature of the required computations.

3.2 Analytic Methods for Central Limit Theorems

The application of analytic methods to prove limit theorems in probability and combinatorics has a long and wide-ranging history. It is impossible to give a full account of such a vast topic in this space, so the following presentation is a broad overview tailored to the context of the work of this chapter.

3.2.1 Probablistic CLTs

The proto-history of the CLT goes back at least as far[2] as a 1733 offprint *Approximatio ad summam terminorum binomii* $(a+b)^n$ *in seriem expansi* of Abraham de Moivre (printed in English in the 1738 edition of his seminal Doctrine of Chances [18]). Motivated by the computation of explicit bounds for the Law of Large Numbers, de Moivre used Stirling's approximation for $n!$ (which de Moivre independently approximated around the same time

[2]The history of the CLT is covered in great detail by Fischer [27], from which we have adapted some of our historical details.

as Stirling) to deduce[3]

$$\mathbb{P}\left[\left|X_n - \frac{n}{2}\right| \leq t\right] \sim \frac{4}{\sqrt{2\pi n}} \int_0^t e^{-2x^2/n} dx$$

for large n, where $X_n = A_1 + \cdots + A_n$ and each A_k is a random variable taking the value 0 with probability 1/2 and the value 1 with probability 1/2. Note that in the above equation we use $\approx$ to denote that the probability

Perhaps the first systematic uses of analytic methods to derive CLT-like results are due to Laplace. Laplace's approach to the CLT built off of his ground-breaking work [52] on the approximation of parameterized integrals in the 1770s. Still influential to this day, *Laplace's method* (in its classical form) is used to approximate integrals of the form

$$\int_a^b A(x) e^{-n\phi(x)} dx$$

where A and ϕ are analytic functions with ϕ minimized over $[a, b]$ at a unique point $c \in (a, b)$ such that ϕ has a Taylor expansion at $x = c$ that begins with a quadratic term. In an 1810 memoir, Laplace [53] introduced the concept of *characteristic functions* by making a change of variable $t = e^{ix}$ in integral representations for the sum of certain independent and identically distributed random variables. He then (formally) used his method to asymptotically approximate the probability that this sum lies between two factors at $\sqrt{n}$-scale by integrating a normal density. Although not working with a modern standard of rigor, Laplace's techniques continue to be influential to this day.

Many famous mathematicians in the nineteenth century worked on topics related to central limit theorems, including Gauss [35] (who derived the now sometimes-eponymous *Gaussian function* in the context of probabilistic error analysis), Poisson [64] (who stated a CLT for a normalized sum of random variables, and gave explicit conditions on characteristic functions for a CLT to hold), Dirichlet (who gave proofs with more rigor, including correct truncation bounds to prove errors arising in Laplace's method go to zero), and Cauchy [13] (who gave an updated proof of the CLT using characteristic functions). As pointed out by Fischer [27], it is also instructive from a historical perspective to reflect that Cauchy and Dirichlet were working at the time period when mathematics and probability were starting to move away from a discipline chiefly concerned by modelling observations of the physical world to a more abstract logic-based subject. A Russian school, involving mathematicians such as Hausdorff, Chebyshev, and Markov, also developed CLT-like results in the late nineteenth and early twentieth century using techniques such as the *method of moments* and *moment generating functions*. The first "modern" treatment of the CLT (as a general mathematical result not dedicated to specific applications or to illustrate analytic methods) is often considered to be work of Lyapunov [57, 58] around the turn of

[3] In historical formulas we have updated some notation; for instance, de Moivre referred to the exponential function only by its series expansion, while we use its symbolic form to align with modern presentations of the CLT.

the twentieth century. The term *central limit theorem* was likely coined by Pólya [65] (who studied various aspects of CLTs, and when sequences of distribution functions converge) in 1920. Lévy [56] and Feller [26] gave necessary and sufficient conditions for the CLT to hold in 1935.

Modern probability theory texts typically prove the CLT using the *continuity theorem* developed by Lévy [54, 55] in the 1920s, which states that if the characteristic functions

$$\varphi_n(t) = \mathbb{E}\left[e^{itX_n}\right]$$

of a sequence (X_n) of random variables converge pointwise to a continuous function $\varphi(t)$ then φ is the characteristic function of some random variable X and the sequence (X_n) converges in distribution to X (see [9, Theorem 26.3] or [23, Theorem 3.3.6] for proofs). If A_n is a sequence of independent random variables with a common characteristic function $\varphi(t)$ then the characteristic function of $X_n = A_1 + \cdots + A_n$ is $\varphi(t)^n$. Assuming that the A_j have a mean 0 and a finite variance $\sigma > 0$, it is possible to get an expansion

$$\varphi(t) = 1 + (it)\mathbb{E}[A_1] - \frac{t^2}{2}\mathbb{E}[A_1^2] + o(t^2) = 1 - \frac{\sigma^2}{2}t^2 + o(t^2)$$

as $t \to 0$, so the characteristic function of $X_n^* = \frac{X_n}{\sigma\sqrt{n}}$ satisfies

$$\varphi\left(\frac{t}{\sigma\sqrt{n}}\right)^n = \left(1 - \frac{t^2}{2n} + o\left(\frac{t^2}{n}\right)\right)^n \to e^{-t^2/2}$$

for each t as $n \to \infty$. The CLT in the mean zero case then follows from the continuity theorem, and the result for a general mean follows from a shift of the X_n.

A complete proof of the general *Lindeberg-Feller CLT* can be found in [9, Theorem 27.2] and [23, Theorem 3.4.5]. Under slightly stronger conditions (such as finiteness of the third moments of the A_j) the *Berry–Esseen theorem* [8, 25] describes the rate of convergence between the cumulative distribution functions of the X_n and the cumulative distribution function of the normal distribution (see [23, Theorem 3.4.9] for a modern presentation). Multivariate limit theorems, including the multivariate CLT, can be established from univariate results through the classical *Cramér-Wold theorem* [17] provided below (see [9, Theorem 29.4] for a proof).

Theorem 28 (Cramér and Wold [17])**.** *A sequence* $(\mathbf{X}_n)$ *of* d*-dimensional random variables converges in distribution to a random variable* $\mathbf{Y}$ *if and only if the sequence of univariate random variables* $\theta \cdot \mathbf{X}_n$ *converges to* $\theta \cdot \mathbf{Y}$ *for all* $\theta \in \mathbb{R}^d$.

3.2.2 Combinatorial CLTs

As described in Section 3.1, in combinatorial contexts one has a multivariate sequence defined by a multivariate generating function

$$C(\mathbf{z}, t) = \sum_{n \geq 0} \left(\sum_{\mathbf{i} \in \mathbb{N}^d} c_{\mathbf{i},n}\mathbf{z}^{\mathbf{i}}\right) t^n \tag{3.1}$$

tracking some parameter χ, and wishes to extract a limit theorem for the random variables $(\mathbf{X}_n)$ taking the values $\mathbf{i} \in \mathbb{N}^d$ with probabilities

$$p_n(\mathbf{i}) = \mathbb{P}\left[\chi(\sigma) = \mathbf{i} \text{ when } \sigma \text{ has size } n\right] = \frac{c_{\mathbf{i},n}}{\sum_{\mathbf{i}\in\mathbb{N}^d} c_{\mathbf{i},n}}\text{[4]}. \tag{3.2}$$

In the 1-dimensional case, a direct computation verifies

$$\mathbb{E}[X_n] = \frac{\sum_{i\geq 0} i\, c_{i,n}}{\sum_{i\geq 0} c_{i,n}} = \frac{[t^n]C_z(1,t)}{[t^n]C(1,t)},$$

where C_z denotes the partial derivative of C with respect to z. Similarly, the kth moment of X_n can be expressed in terms of the first k derivatives (with respect to z) of $C(z,t)$, and limit theorems for X_n can be established by extracting coefficients from these expressions. The theory of analytic combinatorics shows how, instead of explicitly computing coefficients, their behaviour can be approximated — to a degree often sufficient to establish limit theorems — implicitly by studying the analytic behaviour of $C(z,t)$ in t when $z \approx 1$ (for CLTs) or when $|z| \approx 1$ (for LCLTs).

Remark 29. It may seem odd that the 1-dimensional case corresponds with a bivariate generating function. This behaviour corresponds to the understanding that, for this Chapter of the thesis, we consider dimension to be the additional variables representing the parameters tracked. Thus, in this instance, we have one parameter tracked and it is hence the 1-dimensional case.

A similar approach can be used for higher-dimensional parameters, except that the translation of the analytic behaviour of C to the limit behaviour of $c_{\mathbf{i},n}$ is more delicate. Classical treatments typically proceed by requiring that the *coefficient slices*

$$\phi_n(\mathbf{z}) = [t^n]C(\mathbf{z},t) = \sum_{\mathbf{i}\in\mathbb{N}^d} c_{\mathbf{i},n}\mathbf{z}^{\mathbf{i}}$$

encoding the objects of size n are approximated by *quasi-powers*. We briefly describe some of these results, and illustrate the methods involved in their proof, to compare with the limit theorems derived using the framework of ACSV in Section 3.3.

Theorem 30 (Bender and Richmond [7, Theorem 1]). *Suppose that* $\mathbf{z} = (z_1, z_2, \ldots, z_d)$, $\phi_n(\mathbf{z}) \sim f(n)g(\mathbf{z})\lambda(\mathbf{z})^n$ *uniformly in a neighbourhood of* $\mathbf{z} = \mathbf{1}$ *where* $g(\mathbf{z})$ *is uniformly continuous and* $\lambda(\mathbf{z})$ *has a quadratic Taylor series expansion with error term* $O\left(\left(\sum_{k=1}^d |z_k - 1|\right)^3\right)$. *If the Hessian matrix* $\mathcal{H}$ *of* $\log\lambda(e^{\mathbf{s}})$ *at* $\mathbf{s} = \mathbf{0}$ *is non-singular then*

$$\sup_{\mathbf{a}_n}\left|\sum_{\mathbf{i}\leq\mathbf{a}_n} p_n(\mathbf{i}) - \frac{1}{(2\pi n)^{d/2}\sqrt{\det(\mathcal{H})}}\int_{\mathbf{z}\leq\mathbf{a}_n} e^{-\frac{1}{2n}(\mathbf{z}-n\mathbf{m})\mathcal{H}^{-1}(\mathbf{z}-n\mathbf{m})^T}\, d\mathbf{z}\right| = o(1),$$

where $p_n(\mathbf{i})$ *is the scaled coefficient defined in* (3.2) *above,* $\mathbf{m}$ *is the gradient of* $\log\lambda(e^{\mathbf{s}})$ *at* $\mathbf{s} = \mathbf{0}$, *and the inequalities are taken coordinate-wise.*

[4]We define σ to have size n when the weight function associated with $C(\mathbf{z},t)$ has value n at σ.

Theorem 30 and its corollaries below were first established in the 1-dimensional case by Bender [6]. Generalizations of Theorem 30 in the 1-dimensional case were given by Hwang [45, 46, 47] and Heuberger and Kropf [44] (see also the treatment in Flajolet and Sedgewick [29, Chapter IX]).

Proof Sketch. The random variable $\mathbf{X}_k$ with probability distribution function p_n has characteristic function

$$\mathbb{E}\left[e^{i(\mathbf{s}\cdot\mathbf{X}_k)}\right] = \sum_{\mathbf{k}\in\mathbb{N}^d} e^{i(\mathbf{s}\cdot\mathbf{k})} \cdot \frac{c_{\mathbf{k},n}}{\phi(\mathbf{1})} = \frac{\phi_n(e^{i\mathbf{s}})}{\phi(\mathbf{1})}.$$

Shifting $\mathbf{X}_k$ by its mean $n\mathbf{m}$ and scaling by $1/\sqrt{n}$ gives a new random variable with a characteristic function

$$f_n(\mathbf{s}) = e^{-i\sqrt{n}(\mathbf{m}\cdot\mathbf{s})} \cdot \frac{\phi_n(e^{i\mathbf{s}/\sqrt{n}})}{\phi_n(\mathbf{1})},$$

and the expansion

$$\log\lambda(e^{\mathbf{s}}) = \log\lambda(\mathbf{1}) + (\mathbf{s}\cdot\mathbf{m}) + \frac{\mathbf{s}\mathcal{H}\mathbf{s}^T}{2} + \cdots$$

combines with our assumptions on ϕ_n to give $f_n(\mathbf{s}) \sim \exp[-\frac{\mathbf{s}\mathcal{H}\mathbf{s}^T}{2}]$ for all fixed $\mathbf{s}$. The 1-dimensional random variable $\mathbf{s}\cdot\mathbf{X}_n$ thus satisfies a CLT for any $\mathbf{s}$ by the continuity theorem, meaning $\mathbf{X}_n$ satisfies a multivariate CLT by the Cramér-Wold theorem (Theorem 28). □

The key insight for combinatorial CLTs is that it is often possible to prove that ϕ_n satisfies the assumptions of Theorem 30 directly from the analytic behaviour of $C(\mathbf{z}, t)$ near some of its singularities.

Corollary 31 (Bender and Richmond [7, Corollary 1])**.** *Let $C(e^{\mathbf{s}}, t)$ be the generating function described in (3.1). Suppose that there exists $\epsilon, \delta > 0$, a number $q \in \mathbb{Q} \setminus \{-1, -2\ldots\}$, and functions $A(\mathbf{s}), B(\mathbf{s}, t)$, and $r(\mathbf{s})$ such that*

$$C(e^{\mathbf{s}}, t) = A(\mathbf{s})\left(1 - \frac{t}{r(\mathbf{s})}\right)^{-q-1} + B(\mathbf{s}, t)\left(1 - \frac{t}{r(\mathbf{s})}\right)^{-q} \tag{3.3}$$

where

- *$A(\mathbf{s})$ is continuous and non-zero,*
- *$r(\mathbf{s})$ is positive and has continuous third-order partial derivatives, and*
- *$B(\mathbf{s}, t)$ is analytic and bounded*

for $\|\mathbf{s}\| < \epsilon$ and $|t| < |r(\mathbf{0})| + \delta$. If the Hessian matrix of $\lambda(\mathbf{s}) = 1/r(\mathbf{s})$ is non-singular at $\mathbf{s} = \mathbf{0}$ then the CLT in Theorem 30 holds.

Proof. For any fixed $\mathbf{s}$ with $\|\mathbf{s}\| < \epsilon$, bounding the modulus of the Cauchy integral

$$[t^n]B(\mathbf{s},t) = \frac{1}{2\pi i}\int_{|t|=|r(\mathbf{0})|+2\delta/3} \frac{B(\mathbf{s},t)}{t^{n+1}}\,dt$$

by the length of the curve of integration times the modulus of the integrand implies

$$|[t^n]B(\mathbf{s},t)| \leq C \cdot (|r(\mathbf{0})| + 2\delta/3)^{-n}$$

for some $C > 0$. Thus, taking the coefficient of t^n in (3.3) gives, after making use of the general binomial expansion $[z^n](1-z)^{-a} = \binom{n+a-1}{a-1}$, the bound

$$\begin{aligned}
\left|\phi_n(\mathbf{s}) - A(\mathbf{s})\binom{n+q}{q} r(\mathbf{s})^{-n}\right| &= \left|[t^n]B(\mathbf{s},t)\left(1 - \frac{t}{r(\mathbf{s})}\right)^{-q}\right| \\
&\leq C \sum_{k=0}^{n} \binom{k+q-1}{q-1} |r(\mathbf{s})|^{-k} \left(|r(\mathbf{0})| + 2\delta/3\right)^{k-n} \\
&\leq C \binom{n+q-1}{q-1} |r(\mathbf{s})|^{-n} \sum_{k=0}^{n} (1 + |r(\mathbf{s})|\delta/3)^{-k}
\end{aligned}$$

if ϵ is small enough such that $|r(\mathbf{0}) - r(\mathbf{s})| < \delta/3$ whenever $\|\mathbf{s}\| < \epsilon$. This final sum is a geometric series with ratio less than 1, so the asymptotic behaviour of binomial coefficients implies

$$\phi_n(\mathbf{z}) = \phi_n(e^{\mathbf{s}}) = \binom{n+q}{q} A(\mathbf{s}) r(\mathbf{s})^{-n} \left(1 + O\left(n^{-1}\right)\right)$$

uniformly in a sufficiently small neighbourhood of $\mathbf{z} = \mathbf{1}$, and the stated CLT follows from Theorem 30. □

Remark 32. Note that the key reason that Corollary 34 uses $C(e^{\mathbf{s}}, t)$ (rather than $C(\mathbf{z}, t)$) stems from the proof of Theorem 30, where having the variables represented in this manner allows for re-writing the function as an expression involving $\exp\left(-\frac{1}{2}\mathbf{s}B\mathbf{s}^T\right)$ for some B, which is important for showing that the CLT holds.

Example 33. Bender and Richmond [7, p. 261] illustrate Corollary 31 on the *Tutte polynomials* $T_n(x,y)$ of *wheel graphs* on n vertices, which can be defined recursively by

$$T_n - (x+y+2)T_{n-1} + (xy+x+y+1)T_{n-2} - xyT_{n-3} = 0$$

for $n \geq 3$, along with the initial conditions

$$T_0 = xy - x - y + 1, \quad T_1 = xy, \quad T_2 = x^2 + y^2 + xy + x + y.$$

Solving this recurrence gives a trivariate generating function

$$C(x,y,t) = \sum T_n(x,y)t^n = \frac{(1 - x + (xy - y - 1)t)(1 - y + (xy - x - 1)t) - xyt}{(1-t)(1 - (x+y+1)t + xyt^2)}$$

which has, for positive x and y,

$$t = r(x, y) = \frac{2}{x + y + 1 + ((x - y)^2 + 2x + 2y + 1)^{1/2}}$$

as the smallest root of the denominator. Then C has a simple pole at r, as the numerator does not vanish at $t = r(x, y)$, and differentiating

$$1 - (e^{s_1} + e^{s_2} + 1) r\left(e^{s_1}, e^{s_2}\right) + e^{s_1+s_2} r\left(e^{s_1}, e^{s_2}\right)^2 = 0$$

implicitly gives that

$$-\nabla \log r\left(e^{s_1}, e^{s_2}\right)$$

is equal to

$$\left(\frac{e^{s_1} - e^{s_1+s_2} r\left(e^{s_1}, e^{s_2}\right)}{1 + e^{s_1} + e^{s_2} - 2e^{s_1+s_2} r\left(e^{s_1}, e^{s_2}\right)}, \quad \frac{e^{s_2} - e^{s_1+s_2} r\left(e^{s_1}, e^{s_2}\right)}{1 + e^{s_1} + e^{s_2} - 2e^{s_1+s_2} r\left(e^{s_1}, e^{s_2}\right)}\right).$$

At $\mathbf{s} = \mathbf{0}$, further algebraic manipulation shows Corollary 31 applies with $q = 0$ and $A(\mathbf{s}) = (1 - z/r)C$, giving a central limit theorem with

$$\mathbf{m} = \left(\frac{1}{2} - \frac{1}{2\sqrt{5}}, \frac{1}{2} - \frac{1}{2\sqrt{5}}\right) \quad \text{and} \quad \mathcal{H} = \begin{bmatrix} \frac{3}{5\sqrt{5}} & -\frac{2}{5\sqrt{5}} \\ -\frac{2}{5\sqrt{5}} & \frac{3}{5\sqrt{5}} \end{bmatrix}.$$

◁

Further CLTs of this type using the theory of singularity analysis [28] were derived (in the 1-dimensional case) by Flajolet and Soria [30, 31] and, in the multivariate setting, by Gao and Richmond [34]. In particular, the work of Gao and Richmond allows $C(\mathbf{z}, t)$ to have algebraic *and* logarithmic-type singularities near $\mathbf{z} = \mathbf{1}$. Drmota [20, 21] used an analytic approach (based on techniques such as the saddle-point method) to derive central limit theorems from generating functions defined implicitly by (systems of) functional equations; see also the treatment in Drmota [22, Section 2.2]. A detailed survey of analytic methods for the derivation of (largely 1-dimensional) CLTs is given in Flajolet and Sedgewick [29, Chapter IX].

Central limit theorems of the type described in Corollary 31 are derived from knowledge of the generating function $C(\mathbf{z}, t)$ in a neighbourhood of $\mathbf{z} = \mathbf{1}$. When further information about the behaviour of C is known for all points with the same coordinate-wise modulus, LCLTs can also be produced.

Theorem 34 (Bender and Richmond [7, Corollary 2])**.** *Let R be a compact subset of $(-\infty, \infty)^d$ and suppose that there exists $\epsilon > 0$, a number $q \in \mathbb{Q} \setminus \{-1, -2 \ldots\}$, and functions $A(\mathbf{s}), B(\mathbf{s}, t)$, and $r(\mathbf{s})$ such that*

$$C(e^{\mathbf{s}}, t) = A(\mathbf{s}) \left(1 - \frac{t}{r(\mathbf{s})}\right)^{-q-1} + B(\mathbf{s}, t) \left(1 - \frac{t}{r(\mathbf{s})}\right)^{-q} \tag{3.4}$$

where

(i) $A(\mathbf{s})$ *is continuous and non-zero in an* ϵ*-neighbourhood of* R*,*

(ii) $r(\mathbf{s})$ *is non-zero and has continuous third-order partial derivatives in an* ϵ*-neighbourhood of* R*,*

(iii) $B(\mathbf{s}, t)$ *is analytic and bounded for* $\mathbf{s}$ *in an* ϵ*-neighbourhood of* R *and* $|t| < |r(\mathbf{s})|(1+\epsilon)$*,*

(iv) the Hessian matrix $\mathcal{H}(\mathbf{s})$ *of* $\lambda(\mathbf{s}) = 1/r(\mathbf{s})$ *is non-singular for all* $\mathbf{s}$ *in an* ϵ*-neighbourhood of* R*, and*

(v) $C(e^{\mathbf{s}}, t)$ *is analytic and bounded whenever* $|t| < |r(Re(\mathbf{s}))|(1+\epsilon)$ *and* $\epsilon \leq |Im(\mathbf{s})_j| \leq \pi$ *for all* $1 \leq j \leq d$*.*

Then

$$c_{\mathbf{k},n} \sim r(\mathbf{w})^{-n}\, n^q \, \frac{e^{-\mathbf{w}\cdot\mathbf{k}^T} A(\mathbf{w})}{\Gamma(q+1)\sqrt{(2\pi n)^d \det \mathcal{H}(\mathbf{w})}}$$

uniformly for all $\mathbf{k}$ *such that* $\mathbf{k}/n = -\nabla \log r(\mathbf{w})$ *has a solution for* $\mathbf{w} \in R$*. Furthermore, there is a local limit theorem*

$$c_{\mathbf{j},n} \sim c_{\mathbf{k},n} \cdot e^{\mathbf{w}\cdot(\mathbf{k}-\mathbf{j})^T} \left(\exp\left[-\frac{1}{2}\mathbf{u}\mathcal{H}(\mathbf{w})^{-1}\mathbf{u}^T \right] + o(1) \right)$$

uniformly, where $\mathbf{u} = (\mathbf{j} - \mathbf{k})/\sqrt{n}$*.*

The key to establishing the LCLT in Theorem 34 is Condition (v), which implies that the modulus of the singularity $t = r(\mathbf{s})$ of $C(\mathbf{z}, t)$ is uniquely minimized among the points $\mathbf{z} = e^{\mathbf{s}}$ with fixed coordinate-wise modulus when $\mathbf{z}$ has positive real coordinates. For instance, in one variable, information about $C(z, t)$ near the point $z = 1$ is not sufficient to establish local limit theorems, one must also verify that $C(z, t)$ has no other singularities with $|z| = 1$ and the same value of $|t|$. The advanced methods of ACSV, however, show that the multivariate situation is more complicated: in at least two variables there can be singularities that do not form 'obstructions' to deforming domains of integration, making these singularities irrelevant to determining asymptotic behaviour. We return to this in the context of ACSV in Section 3.3 below.

Proof Sketch. The Cauchy integral formula implies

$$c_{\mathbf{i},n} = \frac{1}{(2\pi i)^d} \int_{|\mathbf{z}|=e^{\mathbf{w}}} \phi_n(\mathbf{z}) \frac{d\mathbf{z}}{z_1^{i_1+1}\cdots z_d^{i_d+1}}$$

for $\mathbf{w} \in R$, where $|\mathbf{z}| = (|z_1|, \ldots, |z_d|)$. A modification of the argument used in the proof of Corollary 31 shows that

$$\phi_n(\mathbf{z}) = \phi_n(e^{\mathbf{s}}) = \binom{n+q}{q} A(\mathbf{s}) r(\mathbf{s})^{-n} \left(1 + O\left(n^{-1}\right)\right)$$

uniformly for $\mathbf{s}$ in an ϵ-neighbourhood of R, and following the proof of Theorem 30 we shift the mean of $\mathbf{X}_k$ by $-n\nabla \log r(\mathbf{w})$ and scale by $1/\sqrt{n}$ to get a new random variable with characteristic function

$$f_n(\mathbf{s}) \sim \exp\left[-\frac{\mathbf{s}\mathcal{H}\mathbf{s}^T}{2}\right] \tag{3.5}$$

for certain values of $\mathbf{w}$. Tracing through the definitions of the characteristic functions, and making a polar change of variables, to establish the LCLT from the Cauchy integral above it becomes sufficient to prove that

$$\left| \int_{[-\pi\sqrt{n},\pi\sqrt{n}]^d} e^{-i(\mathbf{s}\cdot\mathbf{z})} f_n(\mathbf{s}) d\mathbf{s} - \int_{\mathbb{R}} e^{-i(\mathbf{s}\cdot\mathbf{z}) - \frac{1}{2}\mathbf{s}\mathcal{H}\mathbf{s}^T} \right| = o(1)$$

for $\mathbf{w}$ given in the statement of the theorem. Near the origin (up to roughly $\sqrt{n}$-scale) it can be shown that this difference is small using (3.5), and the second integral, which has an explicit integrand, is $o(1)$ when bounded sufficiently away from the origin. Roughly speaking, Condition (v) implies that $|\phi_n(\mathbf{z})|$ is small when $\mathbf{z}$ is away from the positive real axis, so the first integral is also $o(1)$ when bounded sufficiently away from the origin and the claimed limit theorem holds. □

An extension of Theorem 34 to functions $C(\mathbf{z}, t)$ with algebraic and logarithmic singularities was given by Gao and Richmond [34].

Although Theorem 34 is now classical, it still finds use today. In fact, since the work provided in this chapter was originally published, there have been two new cases appearing in the literature where we proved conjectured LCLTs quickly using this method. We provide both examples below.

Example 35. The first example stems from a paper by Ekhad and Zeilberger exploring Werner Krandick's binary tree jump statistics [24]. In particular, they study the bivariate generating function

$$H(q, x) = -\frac{-qx + x - 1 + \sqrt{q^2x^2 - 2qx^2 - 2qx + x^2 - 2x + 1}}{2qx}$$

which tracks the number of binary trees enumerated by the number of internal vertices (x) and the number of jumps in a DFS of the tree (q). Writing

$$H(e^s, t) = A(s)\left(1 - \frac{t}{r(s)}\right)^{1/2} + B(s,t)\left(1 - \frac{t}{r(s)}\right)^{3/2}$$

for $A(s) = e^{-3s/4}\left(\mathrm{e}^{s/2} + 1\right)$ and $r(s) = (e^{s/2} + 1)^{-2}$, the results of Theorem 34 imply (after some analytic bounding on B) a central limit theorem and, with a bit more work, give an LCLT

as $n \to \infty$. As a check, we plot the series coefficients $c(s) = [q^s x^{175}]H(q, x)$ compared to the expected distribution, as seen in Figure 3.1.

◁

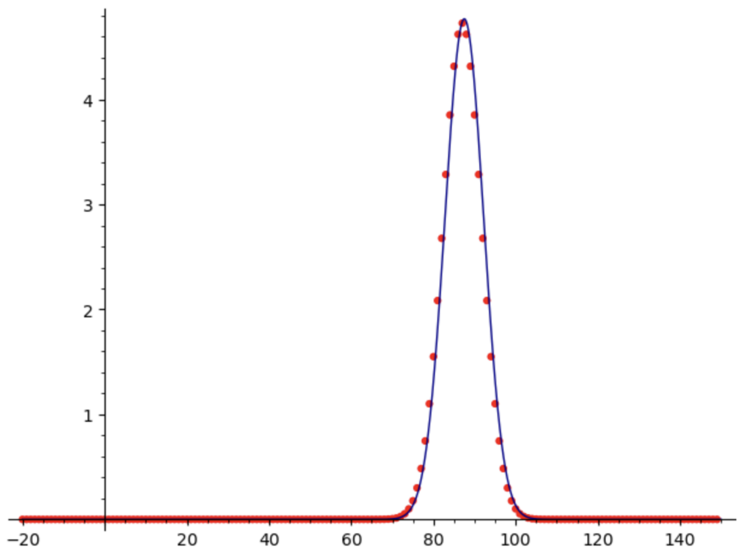


Figure 3.1: A plot of series coefficients $c(s) = [q^s x^{175}]H(q, x)$ compared to the expected distribution.

Example 36. The second example follows a similar derivation to Example 35. In this case, we examine the trivariate generating function given by Jessica Khera and Erik Lundberg in their paper on the distribution of the length of the longest path in random acyclic orientations of a complete bipartite graph [49]. Note that an acyclic orientation of a graph is a way of assigning a direction to each edge in the graph such that no directed cycles are created. For this work, the authors consider the generating function

$$F(x, y, u) = \frac{e^{x+y} - (u-1)^2(e^x - 1)(e^y - 1)}{1 - u^2(e^x - 1)(e^y - 1)},$$

which enumerates the number of acyclic orientations on the complete graph $K_{n,k}$ with length of longest path equal to ℓ, where x tracks n, y tracks k and u tracks ℓ. As with Example 35, we can re-write $F(x, y, u)$ in a way that satisfies Theorem 34. Specifically,

$$F(e^{s_0}, e^{s_1}, t) = A(\mathbf{s}) \left(1 - \frac{t}{r(\mathbf{s})}\right) + B(\mathbf{s}, t)$$

where $\mathbf{s} = (s_0, s_1)$,

$$A(\mathbf{s}) = \sqrt{(e^{(e^{s_0})} - 1)(e^{(e^{s_1})} - 1)} + \frac{1}{2}e^{(e^{s_0})} + \frac{1}{2}e^{(e^{s_1})} - 1$$

and

$$r(\mathbf{s}) = \left(\sqrt{e^{(e^{s_0 + e^{s_1}})} - e^{(e^{s_0})} - e^{(e^{s_1})} + 1}\right)^{-1}.$$

This gives an LCLT of

$$[x^n y^k u^\ell] \sim \frac{2(e-1)^{\ell+3}}{\pi \ell e(e-2)} \exp\left(\frac{2\ell(n+k)(e^2 - e) - 2(n^2 + k^2)(e^2 - 2e + 1) - \ell^2 e^2}{2\ell(e^2 - 2e)}\right)$$

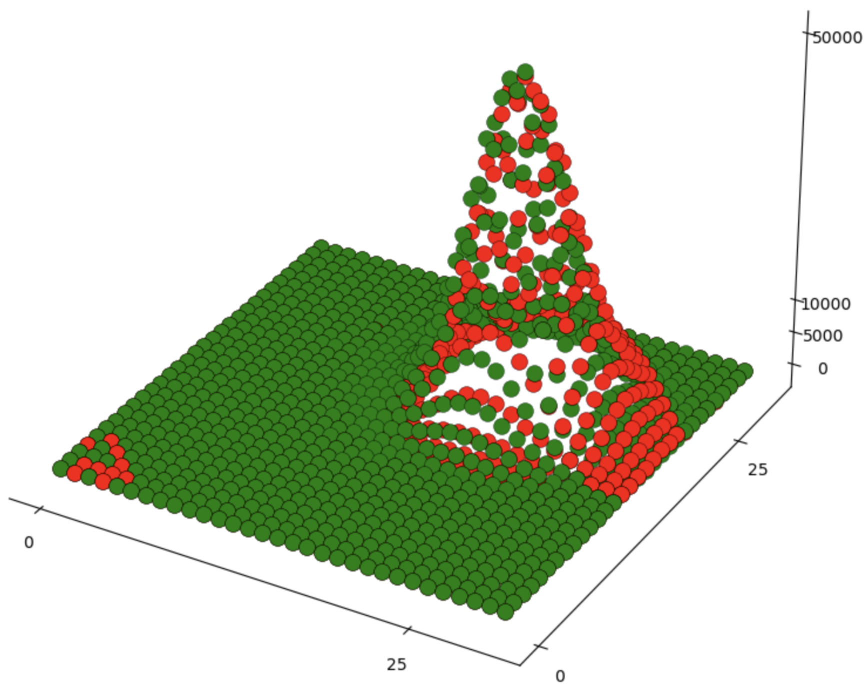


Figure 3.2: A plot of series coefficients $[x^n y^k u^{25}]F(x, y, u)$ in red compared to the expected distribution in green.

for n and k as $\ell \to \infty$. We can see these results visually in Figure 3.2.

◁

3.3 ACSV and Multivariate CLTs

The multivariate results in Section 3.2 are derived from, in the words of Flajolet and Sedgewick [29, Page 768], "a perturbative theory of one-variable complex function theory." In contrast, the methods of ACSV in its modern form are based around the theory of complex analysis *in several variables*. Although the ACSV framework draws from a much larger (and more recently developed) collection of mathematical techniques[5], it is possible to package the end results in ways that allow them to be used without understanding most of this background.

In this section we give an overview of the methods of ACSV required specifically for use for CLTs, and survey the explicit limit theorems they provide. Applications of these results are given in Section 3.5.

[5]For instance, Pemantle et al. [63] includes roughly 100 pages of appendices with (highly abridged) background material, beyond the additional background material discussed in the main text.

3.3.1 Background

We note that this background is an extension of the background given in Section 2.3, however it focuses on the background required specifically for LCLTs.

We begin by extending from Theorem 26 to LCLTs. In particular, we use the fact that the asymptotic behaviour described in Theorem 26 varies smoothly with the direction $\mathbf{r}$ under small perturbations which allows (with a small amount of extra analysis) for the derivation of LCLTs.

Remark 37. There are several natural ways to take the sequence indices of $f_{\mathbf{i},k}$ to infinity and search for limit theorems: for instance, one could examine *coefficient slices* with $i_1 + \cdots + i_d + k = n$ and take $n \to \infty$. For us it makes the most combinatorial sense to take the final index to infinity (as in the results above) and study coefficient behaviour in the first d indices, but the methods of ACSV adapt naturally to other situations. Work extending the types of limit theorems derived by our software package is ongoing.

Proposition 38 (Melczer [59, Proposition 5.10])**.** *Suppose $F(\mathbf{z}, t) = G(\mathbf{z}, t)/H(\mathbf{z}, t)$ has a power series expansion $F(\mathbf{z}, t) = \sum_{(\mathbf{i},k)\in\mathbb{N}^{d+1}} f_{\mathbf{i},k}\mathbf{z}^{\mathbf{i}}t^k$ at the origin such that $f_{\mathbf{i},k}$ is non-negative for all but a finite number of terms. Suppose further that, in some direction $(\mathbf{m}, 1)$, there is a strictly minimal critical point of the form $(\mathbf{1}, \rho)$ for some $\rho > 0$. If $H_t(\mathbf{w}, \rho)$ and $G(\mathbf{w}, \rho)$ are non-zero, and the phase Hessian $\mathcal{H}$ of H at $(\mathbf{1}, \rho)$ is non-singular, then*

$$\sup_{\mathbf{s}\in\mathbb{Z}^d} n^{d/2} \left| \rho^n f_{\mathbf{s},n} - \frac{-G(\mathbf{1}, \rho)}{\rho H_t(\mathbf{1}, \rho)} \frac{(2\pi n)^{-d/2}}{\sqrt{\det \mathcal{H}}} \exp\left[-\frac{(\mathbf{s} - n\hat{\mathbf{m}})^T \mathcal{H}^{-1} (\mathbf{s} - n\hat{\mathbf{m}})}{2n} \right] \right| \to 0. \tag{3.6}$$

The requirement of strict minimality in Proposition 38 is analogous to Condition (v) in Theorem 34 from Section 2.3.

Remark 39. Proposition 5.10 in Melczer [59] requires G and H to be polynomials to simplify its presentation, however the result continues to hold as long as G and H are analytic functions [59, Remark 5.14]. Other generalizations that can be handled with the ACSV framework include (non-power series) Laurent expansions, non-smooth geometries, and non-minimal points when additional assumptions are verified.

The most computationally expensive hypothesis to verify in Proposition 38 is strict minimality (see Melczer and Salvy [61] for a complexity analysis of smooth ACSV methods). Generically[6] $\mathcal{V}$ is smooth and the set of smooth critical point equations (2.4) admits a finite set of solutions, however verifying strict minimality of a point $\mathbf{w}$ requires examining whether $\mathcal{V}$ intersects the (always infinite) set of points with the same coordinate-wise moduli as $\mathbf{w}$. The perturbative approach in Section 3.2.2 and the surgery ACSV method both require strict minimality (or finite minimality with some extra bounding), however more advanced multivariate methods show that this can be weakened, as will be seen below.

[6] For instance, these properties hold for all polynomials H except for those whose coefficients lie in an algebraic set determined only by the degree of H.

Indeed, in the univariate setting it is the singularities of minimal modulus that contribute to dominant asymptotic behaviour, so it is tempting to assume that minimal singularities are the ones determining dominant multivariate asymptotics. In fact, as the theory of ACSV matured, its methods were re-examined through more advanced mathematical frameworks, illustrating how *critical points* are the singularities dictating asymptotic behaviour. The most explicit results still hold for minimal critical points, however one only needs to verify that the (generically finite) set of critical points has no other elements with the same coordinate-wise modulus as a candidate minimal critical point. Without this strengthening, algorithms to compute asymptotics (or rigorously establish limit theorems) would not terminate in reasonable time even for examples with relatively low dimension and degree.

Theorem 40 (Pemantle, Wilson, and Melczer [63, Theorem 9.12])**.** *Suppose that the rational function $F(\mathbf{z},t) = G(\mathbf{z},t)/H(\mathbf{z},t)$ admits a nondegenerate minimal smooth critical point $(\mathbf{w},s) \in \mathbb{C}_*^{d+1}$ in the direction $\mathbf{r} \in \mathbb{R}_*^{d+1}$, such that $H_t(\mathbf{w},s) \neq 0$ and no other critical point has the same coordinate-wise modulus as $(\mathbf{w},s)$. Then the conclusions of Theorem 26 hold, where the expansion (2.6) now holds uniformly over neighbourhoods where there is a smoothly varying nondegenerate minimal critical point such that H_t does not vanish and no other critical point has the same coordinate-wise modulus.*

Theorem 40 was first discussed in Pemantle and Baryshnikov [5] using *cones of hyperbolicity*. To briefly summarize, if $(\mathbf{z},t)$ is a minimal point then the tangent plane to the modified function $H(e^{x_1},\dots,e^{x_d},e^q)$ at $(\log(\mathbf{z}),\log(t))$ is defined by a normal vector that is a multiple of a real vector $\mathbf{v}$. If $(\mathbf{z},t)$ is not critical then $\mathbf{v}$ is not parallel to the direction vector $\mathbf{r}$ and this can be used to locally deform a domain of integration near $(\mathbf{z},t)$ into a region of complex space where it can be bounded and shown to be negligible. The framework of hyperbolic cones shows that these local deformations can be done in a consistent manner (away from critical points) and also generalizes to non-smooth cases.

Example 41. The rational function

$$F(x,t) = \frac{1}{(1+x)(2-x-t)}$$

admits $(x,t) = (1,1)$ as a minimal critical point in the direction $\mathbf{r} = (1,1)$, however this point is not finitely minimal as $(-1,t)$ is a singularity for any $t \in \mathbb{C}$, and none of Theorem 34, Theorem 26, or Proposition 38 directly apply. Since there are no other critical points with the same coordinate-wise modulus as $(1,1)$, Theorem 40 does apply and we can compute

$$f_{n,n} = n^{-1/2}\left(\frac{1}{4\sqrt{\pi}} + O\left(n^{-1}\right)\right).$$

In fact, the local central limit theorem

$$\sup_{s\in\mathbb{Z}} n^{1/2}\left|f_{s,n} - \frac{1}{4}\frac{1}{\sqrt{\pi n}}\exp\left[-\frac{(s-n)^2}{4n}\right]\right| \to 0$$

holds. ◁

When dealing with combinatorial generating functions, the importance of critical points can be seen directly. Indeed, the idea underpinning cones of hyperbolicity (that the tangent space near any smooth minimal point has a normal vector that is a multiple of a real vector) implies that every minimal point is critical in some direction. When the series under consideration has non-negative coefficients, this implies that any minimal point with the same coordinate-wise modulus as a critical point with positive coordinates is also critical [59, Corollary 5.5], giving the following result on which we base our algorithm for automatically finding and verifying LCLTs.

Proposition 42. *Suppose $F(\mathbf{z},t) = G(\mathbf{z},t)/H(\mathbf{z},t)$ has a power series expansion $F(\mathbf{z},t) = \sum_{(\mathbf{i},k)\in\mathbb{N}^{d+1}} f_{\mathbf{i},k}\mathbf{z}^{\mathbf{i}}t^k$ at the origin such that $f_{\mathbf{i},k}$ is non-negative for all but a finite number of terms. Suppose further that, in some direction $(\mathbf{m},1)$, there is a minimal critical point of the form $(\mathbf{1},\rho)$ for some $\rho > 0$ and no other critical point has the same coordinate-wise modulus. If $H_t(\mathbf{w},\rho)$ and $G(\mathbf{w},\rho)$ are non-zero, and the phase Hessian $\mathcal{H}$ of H at $(\mathbf{1},\rho)$ is non-singular, then the LCLT (3.6) holds.*

Going even further, Baryshnikov, Pemantle, and Melczer [3] show how asymptotic behaviour can be characterized in the absence of minimal critical points using techniques from *stratified Morse theory*. This requires introducing the notion of *critical points at infinity* and gives asymptotics as a linear combination of asymptotic expansions with (generally unknown) integer coefficients, so here we stick to the explicit case of (not necessarily strictly) minimal critical points covered by Theorem 40.

Finally, although it is beyond the scope of this thesis, we give a non-smooth example from forthcoming work that illustrates the differences from the smooth cases discussed above.

Example 43. Consider the generating function

$$F(x,t) = \sum_{n,k\in\mathbb{N}} f_{n,k}x^nt^k = \frac{6}{(1-3t(1+x))\,(1-2t(2+x))}.$$

The methods of ACSV for *transverse multiple points* (see [59, Chapter 9] or [63, Chapter 10]) imply that the asymptotic behaviour of $f_{\lambda n,n}$ varies with $\lambda \in (0,1)$ in a computable manner. Indeed, if $0 < \lambda < 1/3$ or $1/2 < \lambda < 1$ then the dominant asymptotic behaviour of $f_{\lambda n,n}$ is dictated by a smooth minimal critical point vanishing on only one of the denominator factors of F, and $f_{\lambda n,n} \to 0$ exponentially quickly. In contrast, when $\lambda \in (1/3, 1/2)$ then the dominant asymptotic behaviour is determined by the non-smooth minimal critical point $(1,1)$ where both denominator factors vanish, and the methods of ACSV imply $f_{\lambda n,n} \to 6$.

Thus, unlike the smooth case where a central limit theorem is obtained, here we have a range of indices where the same minimal critical point determines asymptotics, giving a limiting distribution with a flat plateau (see the middle plot in Figure 3.3). A minor modification of the arguments in [4, Section 6.2] – developed there only for generating

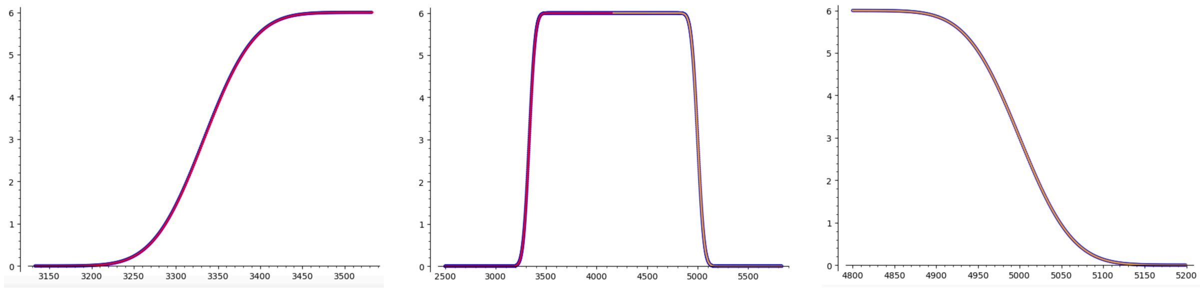


Figure 3.3: The coefficients $c(k) = f_{k,N}$ (in blue) compared to their limiting behaviour (in red and orange) when $N = 1000$. The transitions in behaviour near $N/3$ and $N/2$ are shown on intervals of length $2\sqrt{N}$ in the left and right plots.

functions whose denominator factors are linear – shows how to capture the transition between the different limiting regimes, which occurs on a square-root scale. Indeed, if

$$\Phi(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-s^2} ds$$

is the *Gaussian error function* then

$$f_{n/3+t\sqrt{n},\, n} \sim 3 + 3\Phi(3t/2) \quad \text{and} \quad f_{n/2+t\sqrt{n},\, n} \sim 3 - 3\Phi(\sqrt{2}\, t)$$

when t grows sufficiently slower than $\sqrt{n}$ (for instance, when t is bounded). See Figure 3.3 for an illustration. ◁

3.3.2 Verifying Minimality

Because verifying criticality is easier than verifying minimality, most ACSV algorithms for asymptotics fix a direction, compute critical points in this direction, and then study the critical points to determine which are minimal points. In contrast, because Proposition 38 requires a minimal critical point of the form $(\mathbf{1}, t)$, to prove an LCLT it is often easiest to use (2.4) to discover a direction $\mathbf{r} = (\mathbf{m}, 1)$ with critical points of this form and then verify the required conditions.

Determining minimality is easier for points with positive coefficients when $F(\mathbf{z})$ has only a finite number of non-negative coefficients, as it does under our assumptions. The following result should be seen as a multivariate generalization of the well-known Vivanti-Prinsheim theorem in the univariate case, and the approach to proving an LCLT that it suggests when combined with Proposition 42 is summarized in Figure 3.4.

Lemma 44 (Melczer [59, Lemma 5.7])**.** *Suppose $F(\mathbf{z}, t) = G(\mathbf{z}, t)/H(\mathbf{z}, t)$ has a power series expansion at the origin with (at most) a finite number of negative coefficients. Then $(\mathbf{w}, \rho) \in \mathbb{R}^d_{>0}$ is minimal if the line segment from the origin to $(\mathbf{w}, \rho)$ contains no roots of $H(\mathbf{z}, t)$, i.e., if*

$$H(sw_1, \ldots, sw_d, s\rho) \neq 0 \text{ for all } s \in (0, 1).$$

1. Characterize the smallest positive solution $t = \rho$ of $H(\mathbf{1}, t) = 0$ and define $\mathbf{m} = \left(\frac{H_{z_1}(\mathbf{1},\rho)}{\rho H_t(\mathbf{1},\rho)}, \dots, \frac{H_{z_d}(\mathbf{1},\rho)}{\rho H_t(\mathbf{1},\rho)}\right)$

↓

2. Show that $H(s, \dots, s, s\rho) \neq 0$ for all $0 < s < 1$

↓

3. Show that no other critical point has the same coordinate-wise modulus as $(\mathbf{1}, \rho)$

↓

4. Prove that the determinant of the phase Hessian $\mathcal{H}$ is non-zero at $(\mathbf{1}, \rho)$ in the direction $\mathbf{r} = (\mathbf{m}, 1)$

↓

5. Prove that $H_t(\mathbf{1}, \rho)$ and $G(\mathbf{1}, \rho)$ are non-zero and read off the local central limit theorem from (3.6)

Figure 3.4: A schema to prove LCLTs using Proposition 42.

Although proving strict minimality is difficult in general, there is one case arising in some combinatorial applications where it is automatic.

Definition 45. A power series $S(\mathbf{z}) = \sum_{\mathbf{n}\in\mathbb{N}^d} p_{\mathbf{n}}\mathbf{z}^{\mathbf{n}}$ is called *aperiodic* if every element of $\mathbb{Z}^d$ can be written as an integer linear combination of the exponents $\{\mathbf{n} \in \mathbb{N}^d : p_{\mathbf{n}} \neq 0\}$ appearing in S.

Proposition 46 (Melczer [59, Proposition 5.5])**.** *Suppose $F(\mathbf{z}) = G(\mathbf{z})/H(\mathbf{z})$ is a ratio of analytic functions G and H. If $H(\mathbf{z}) = 1 - S(\mathbf{z})$ for some aperiodic power series S with non-negative coefficients then every minimal point that is within the domain of convergence of the power series expansion of $F(\mathbf{z})$ is strictly minimal and has positive real coordinates.*

Proof. Suppose that $\mathbf{w}$ is a minimal point and for each $1 \leq j \leq d$ write $w_j = x_j e^{i\theta_j}$ with $x_j > 0$ and $\theta_j \in \mathbb{R}$. Let $s_{\mathbf{n}}$ denote the coefficient of $z^{\mathbf{n}}$ in $S(\mathbf{z})$. Then

$$1 = |S(\mathbf{w})| = \left|\sum_{\mathbf{n}\in\mathbb{N}^d} s_{\mathbf{n}}\mathbf{x}^{\mathbf{n}}e^{i(\mathbf{n}\theta)}\right| \leq \sum_{\mathbf{n}\in\mathbb{N}^d} s_{\mathbf{n}}\mathbf{x}^{\mathbf{n}}$$

since $\mathbf{w}$ is within the domain of convergence of G and H and hence $|S(\mathbf{w})| \leq S(|w_1|, \dots, |w_d|)$. □

3.4 Motivating Problem

In this section, and the next two, we will illustrate how to apply the approach of Figure 3.4 to prove LCLTs. We begin with the example that originally motivated the work of this chapter.

Definition 47. For any $d \in \mathbb{N}$, let $\mathcal{F}_d(n)$ be the set of permutations σ on $\{1, \dots, n\}$ such that $i - d \leq \sigma(i) \leq i + 1$ for all i. Let $\mathcal{F}_d$ be the union of sets $\mathcal{F}_d(n)$ for all $n \in \mathbb{N}$.

Note that every element of $\mathcal{F}_d$, when written in disjoint cycle notation, has cycles of length at most $d + 1$.

Proposition 48 (Chung et al. [16, Theorem 1])**.** *The number of permutations in $\mathcal{F}_d(n)$ with i_k cycles of length k equals the coefficient $[z_1^{i_1} \cdots z_{d+1}^{i_{d+1}} t^n] F(\mathbf{z}, t)$ in the power series expansion of the rational function*

$$F(\mathbf{z}, t) = F(z_1, \dots, z_{d+1}, t) = \frac{1}{1 - z_1 t - z_2 t^2 - \cdots - z_{d+1} t^{d+1}}.$$

Chung et al. [16] prove Proposition 48 by considering the set of perfect matchings of the graph associated with $\mathcal{F}_d$ and using this to find an explicit recurrence that can be manipulated. One of the motivations for their study of this family is a relationship to the determination of sample sizes required for sequential importance sampling of certain random perfect matchings in classes of bipartite graphs.

Conjecture 49 (Chung et al. [16, Page 45])**.** For fixed d the joint limiting distribution of the number of k-cycles approaches a multivariate normal distribution as $n \to \infty$.

Experimentally checking the properties of Proposition 42 in low dimension using a computer algebra system, we were initially surprised to find that some of the properties did not hold! To better understand the behaviour of the coefficients we thus plotted the coefficients of $[t^{150}]F(\mathbf{z}, t)$ for $d = 1$, shown on the left of Figure 3.5.

Figure 3.5 shows the problem establishing a limit theorem on the coefficients of $F(\mathbf{z}, t)$: the coefficients approach a normal distribution, but the distribution is supported on a d-dimensional slice of $\mathbb{R}^{d+1}$. Indeed, if the size n of one of our restricted permutations is fixed, and the number i_k of k cycles it contains is specified for all $k \geq 2$, then its number of one cycles (i.e., fixed points) is uniquely determined as $i_1 = n - 2i_2 - \cdots - (d+1)i_{d+1}$. We thus prove that Conjecture 49 is true by setting $z_1 = 1$ and applying the techniques of ACSV to the coefficients of $F(1, z_2, \dots, z_d, t)$.

Theorem 50. *Let $h(t) = 1 - t - \cdots - t^{d+1}$ and let $\rho > 0$ be the smallest positive root of $h(t)$. As $n \to \infty$, the maximum coefficient of $[t^n]F(\mathbf{z}, t)$ as a polynomial in $z_2, \dots, z_{d+1}$ approaches*

$$A_n = \frac{\rho^{-n} n^{-d/2}}{-\rho h'(\rho)(2\pi)^{d/2}} \sqrt{\frac{(1 + 2\rho + \cdots + (d+1)\rho^d)^{d+2}}{(1 + \rho + \cdots + \rho^d)\rho^{\frac{d(d+1)}{2}}}}.$$

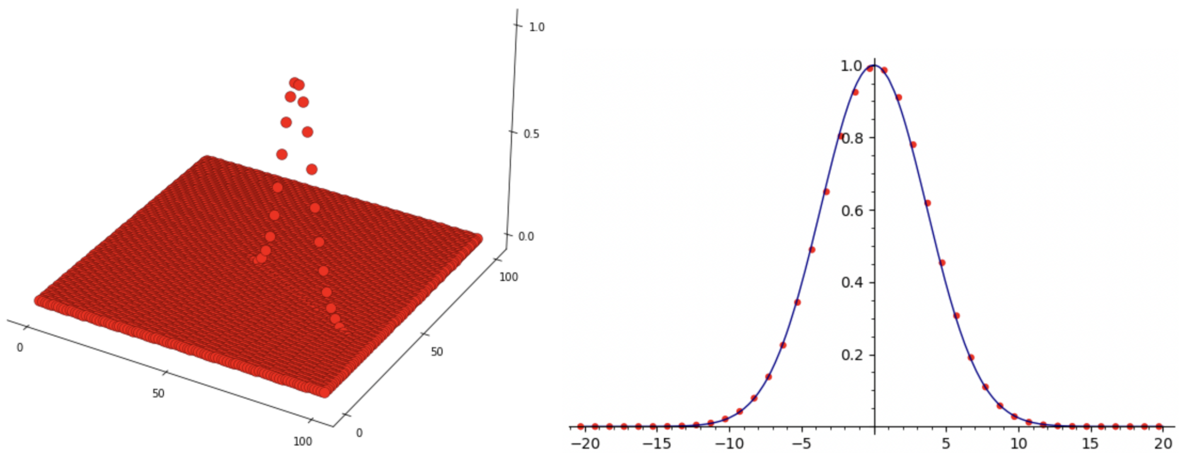


Figure 3.5: *Left:* The coefficients of $[t^{150}]F(\mathbf{z}, t)$ when $d = 1$, divided by the maximum coefficient and shifted to put the maximum at the origin. *Right:* The coefficients of $[t^{150}]F(1, z_2, t)$ compared to their limiting normal distribution and shifted to put the maximum at the origin.

Furthermore,

$$\sup_{s_2,\ldots,s_{d+1}\in\mathbb{N}} \left| \frac{[z_2^{s_2}\cdots z_{d+1}^{s_{d+1}} t^n]F(1, z_2, \ldots, z_{d+1}, t)}{A_n} - v_n(s_2, \ldots, s_{d+1}) \right| \to 0 \tag{3.7}$$

where

- $v_n(\mathbf{s}) = \exp\left[-\frac{(\mathbf{s}-n\mathbf{m})\mathcal{H}^{-1}(\mathbf{s}-n\mathbf{m})^T}{2n}\right]$,
- $\mathbf{m} = \left(-\frac{\rho^2}{h'(\rho)}, \ldots, -\frac{\rho^{d+1}}{h'(\rho)}\right)$,
- *and $\mathcal{H}$ is the non-singular matrix with entries*

$$\mathcal{H}_{i,j} = \begin{cases} \frac{\rho^{i+j+1}h''(\rho)-\rho^{i+j}(1+i+j)h'(\rho)}{h'(\rho)^3} & i \neq j \\ \frac{\rho^{i+j+1}h''(\rho)-\rho^{i+j}(1+i+j)h'(\rho)-\rho^i h'(\rho)^2}{h'(\rho)^3} & i = j \end{cases}.$$

Theorem 50 follows directly from Proposition 42 after some direct calculations, the most difficult of which is computing a closed form for the determinant of the Hessian matrix $\mathcal{H}$. We complete these calculations in Section 3.6, using a guess-and-check method for symbolic determinants which is a useful tool for establishing limit theorems such as these with parameterized dimension. In fact, we will prove a more general result in Section 3.5. Note that Conjecture 49 is a direct result of Theorem 50!

3.5 Family of Limit Theorems

In this section we will give an automated version of an LCLT theorem which will be highlighted with a number of examples. Note that Theorem 50 is an example application for this automated theorem. The proof for this theorem can be found in Section 3.6.

Theorem 51 (Automatic LCLT Theorem). *Let* $F(\mathbf{z},t) = \frac{G(\mathbf{z},t)}{H(\mathbf{z},t)}$ *be a ratio of functions where*

$$H(\mathbf{z},t) = 1 - q(t) - \sum_{k=1}^{d} q_k(t) z_k,$$

let $P(t) = H(\mathbf{1},t)$*, and let* ρ *be the smallest positive root of* $P(t)$*. Suppose that*

- *each of the* $q_i(t)$ *is a non-zero polynomial vanishing at the origin and* $q(t)$ *is a complex-valued analytic function for* $|t| \le \rho$ *that vanishes at the origin,*
- *the power series expansion of* $S(\mathbf{z},t) = 1 - H(\mathbf{z},t) = q(t) + \sum_k q_k(t) z_k$ *at the origin has non-negative coefficients,*
- *the exponents appearing in the power series* $q(t)$ *have greatest common divisor* 1*, and*
- $G(\mathbf{1},\rho)$ *is non-zero.*

As $n \to \infty$*, the maximum coefficient of* $[t^n]F(\mathbf{z},t)$ *as a polynomial in* $z_1, \ldots, z_d$ *approaches*

$$A_n = \rho^{-n} n^{-d/2} \frac{G(\mathbf{1},\rho)}{-\rho P'(\rho)(2\pi)^{d/2}\sqrt{\det \mathcal{H}}},$$

where $\mathcal{H}$ *is the non-singular* $d \times d$ *matrix*

$$\mathcal{H}_{i,j} = \begin{cases} \dfrac{\rho q_i(\rho) q_j(\rho) P''(\rho) - (q_j(\rho) q_i'(\rho)\rho + q_i(\rho) q_j'(\rho)\rho - q_i(\rho) q_j(\rho)) P'(\rho)}{\rho^2 P'(\rho)^3} & i \neq j \\ \dfrac{\rho q_j(\rho)^2 P''(\rho) - (2 q_j(\rho) q_j'(\rho)\rho - q_j(\rho)^2) P'(\rho) - q_j(\rho)\rho P'(\rho)^2}{\rho^2 P'(\rho)^3} & i = j \end{cases} \tag{3.8}$$

whose determinant

$$\det \mathcal{H} = \frac{(-1)^d \left(\prod_{k=1}^{d} q_k(\rho)\right) \left[(q(\rho) - 1)\left(\rho P''(\rho) + P'(\rho) + \rho \sum_{k=1}^{d} \frac{q_k'(\rho)^2}{q_k(\rho)}\right) + q'(\rho)^2 \rho\right]}{P'(\rho)^{d+2} \rho^{d+1}} \tag{3.9}$$

is non-zero. Furthermore,

$$\sup_{s_1,\ldots,s_d \in \mathbb{N}} \left| \frac{[z_1^{s_1} \cdots z_d^{s_d} t^n] F(z_1, \ldots, z_d, t)}{A_n} - v_n(s_1, \ldots, s_d) \right| \to 0$$

where

$$v_n(\mathbf{s}) = \exp\left[-\frac{(\mathbf{s} - n\mathbf{m})\mathcal{H}^{-1}(\mathbf{s} - n\mathbf{m})^T}{2n} \right] \qquad \text{for} \qquad \mathbf{m} = \left(\frac{-q_1(\rho)}{\rho P'(\rho)}, \ldots, \frac{-q_d(\rho)}{\rho P'(\rho)} \right).$$

Remark 52. The SageMath package accompanying this article automatically proves central limit theorems for all explicit rational functions satisfying the conditions of Proposition 42, which includes all those satisfying the conditions of Theorem 51.

Although the form of H in Theorem 51 might seem restrictive, generating functions of this form appear quite frequently when tracking parameters in combinatorial classes using, for instance, the *symbolic method framework* described in Flajolet and Sedgewick [29]. We end this section by describing some other combinatorial limit theorems it captures.

3.5.1 Strings with Tracked Letters

Let $\mathcal{A} = \{a_1, \dots, a_\ell\}$ be an alphabet with ℓ letters and let $\Omega = \{\omega_1, \dots, \omega_d\} \subset \mathcal{A}$ be a subset of d letters we wish to track. The multivariate generating function enumerating such strings is

$$F(\mathbf{z}, t) = \frac{1}{1 - (z_1 + z_2 + \ldots + z_d)t - (\ell - d)t},$$

which trivially satisfies the hypotheses of Theorem 51 when $\ell > d$ (if $\ell = d$ then all coefficients live in a d-dimensional hyperplane of $\mathbb{R}^{d+1}$, because adding the number of occurrences of each letter gives the length of the string). Thus, as $n \to \infty$, the maximum coefficient of $[t^n]F(\mathbf{z}, t)$ as a polynomial in $z_1, \dots, z_d$ approaches

$$A_n = \ell^n n^{-d/2} \frac{1}{(2\pi)^{d/2}\sqrt{\frac{\ell-d}{\ell^{d+1}}}}$$

and

$$\sup_{s_1,\dots,s_d \in \mathbb{N}} \left| \frac{[z_1^{s_1} \cdots z_d^{s_d} t^n] F(z_1, \dots, z_d, t)}{A_n} - \exp\left[-\frac{(\mathbf{s} - n\mathbf{m})\mathcal{H}^{-1}(\mathbf{s} - n\mathbf{m})^T}{2n} \right] \right| \to 0,$$

where $\mathbf{m} = \left(\frac{1}{\ell}, \dots, \frac{1}{\ell}\right)$ and $\mathcal{H}$ is the non-singular $d \times d$ matrix with off-diagonal entries $-\frac{1}{\ell^2}$ and diagonal entries $\frac{\ell-1}{\ell^2}$. Note that tracking the number of 1s in binary strings (where $\ell = 2$ and $d = 1$) recovers the classical central limit theorem,

$$\binom{n}{s} = [z^s t^n] \frac{1}{1 - (1+z)t} \approx 2^n \sqrt{\frac{2}{\pi n}} e^{-(n-2s)^2/2n}.$$

3.5.2 Compositions with Tracked Summands

Fix a positive integer d and recall that an *(integer) composition* of size n is an ordered tuple of positive integers which sum to n. If $\mathcal{C}$ is the class of compositions, enumerated by size and the number of times each element of $\{1, \ldots, d\}$ occurs, then $\mathcal{C}$ has the multivariate generating function

$$\mathcal{C}(\mathbf{z}, t) = \frac{1}{1 - S(\mathbf{z}, t)} = \frac{1}{1 - z_1 t - z_2 t^2 - \cdots - z_d t^d - \frac{t^{d+1}}{1-t}},$$

where

$$\begin{aligned} S(\mathbf{z}, t) &= z_1 t + z_2 t^2 + \cdots + z_d t^d + \sum_{k \geq d+1} t^k \\ &= z_1 t + z_2 t^2 + \cdots + z_d t^d + \frac{t^{d+1}}{1-t} \end{aligned}$$

is the multivariate generating function of positive integers where z_k tracks the number of occurrences of k. Once again, it is easy to verify the hypotheses of Theorem 51 hold with $\rho = 1/2$, the smallest positive root of $P(t) = H(\mathbf{1}, t) = 1 - \frac{t}{1-t}$. Figure 3.6 illustrates the corresponding local central limit theorem when $d = 1$ and $d = 2$.

Theorem 53. *As $n \to \infty$ the maximum coefficient of $[t^n]C(\mathbf{z}, t)$ as a polynomial in $z_1, \ldots, z_d$ approaches*

$$A_n = 2^n n^{-d/2} \frac{2^{\frac{d^2}{4} + \frac{7d}{4} + 1}}{2(2\pi)^{d/2}\sqrt{(d^2 + 4d + 6)2^d - 2}},$$

and

$$\sup_{s_1, \ldots, s_d \in \mathbb{N}} \left| \frac{[z_1^{s_1} \cdots z_d^{s_d} t^n]\mathcal{C}(z_1, \ldots, z_d, t)}{A_n} - \exp\left[-\frac{(\mathbf{s} - n\mathbf{m})\mathcal{H}^{-1}(\mathbf{s} - n\mathbf{m})^T}{2n} \right] \right| \to 0$$

where $\mathbf{m} = \left(\frac{1}{4}, \frac{1}{8}, \ldots, \frac{1}{2^{d+1}}\right)$ and $\mathcal{H}$ is the $d \times d$ matrix with off-diagonal entries $\mathcal{H}_{i,j} = -2^{-i-j-2}(i + j - 3)$ and diagonal entries $\mathcal{H}_{j,j} = 2^{-2(j+1)}(2^{j+1} - 2j + 3)$.

The flexibility of Theorem 51 allows us to modify the restrictions on the elements appearing or tracked among the compositions under consideration. For instance, for any finite set $\Omega \subset \mathbb{Z}_{>0}$ the multivariate generating function enumerating compositions by size and number of times each element of $\Omega = \{\omega_1, \ldots, \omega_d\}$ occurs is

$$F(\mathbf{z}, t) = \frac{1}{1 - \sum_{k=1}^{d}(z_k - 1)t^{\omega_k} - \frac{t}{1-t}},$$

where z_k tracks the number of occurrences of ω_k. The hypotheses of Theorem 51 still hold, so a local central limit theorem applies (although, of course, the parameters of the limiting distribution will depend on which summands are tracked).

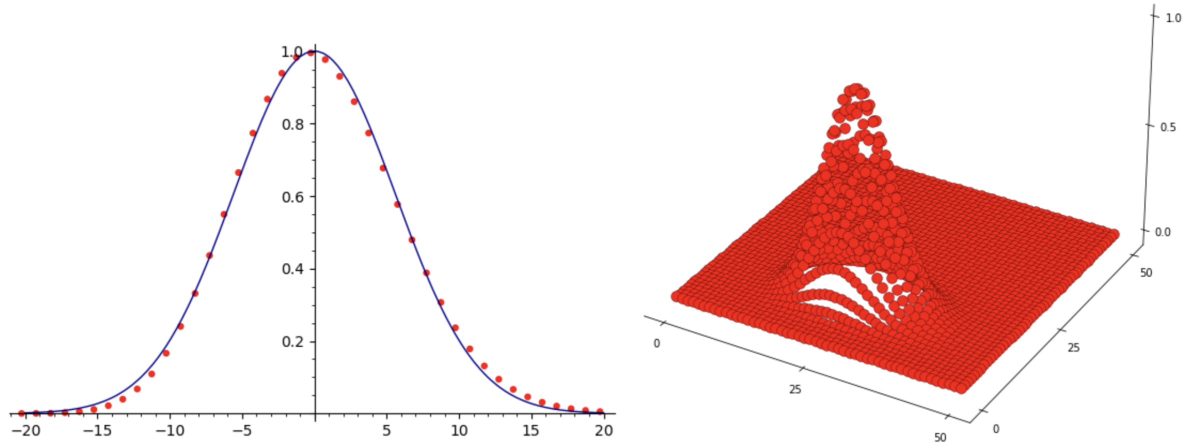


Figure 3.6: *Left:* The coefficients of $[t^{100}]\,(1 - z_1t - t^2/(1-t))^{-1}$ divided by the projected maximum coefficient compared to their limiting normal distribution and shifted to put the maximum at the origin. *Right:* The coefficients of $[t^{100}]\,(1 - z_1t - z_2t^2 - t^3/(1-t))^{-1}$ divided by the projected maximum coefficient and shifted to put the maximum at the origin.

Even more generally, we can restrict our compositions to summands in some set $\Lambda \subset \mathbb{Z}_{>0}$ and track occurrences of elements in a finite subset $\Omega = \{\omega_1, \ldots, \omega_d\} \subset \Lambda$. The relevant multivariate generating function becomes

$$F(\mathbf{z}, t) = \frac{1}{1 - \sum_{k=1}^{d}(z_k - 1)t^{\omega_k} - \sum_{k\in\Omega} t^k}.$$

When the elements of Ω are coprime, Theorem 51 applies, so a local central limit theorem holds (when the elements of Ω have greatest common divisor larger than 1 then some intervention is needed to determine which singularities determine asymptotic behaviour).

Remark 54. Restricting compositions to use only the numbers $1, \ldots, d+1$ and tracking occurrences of those numbers gives the same generating function as the family of restricted permutations discussed above, whose behaviour is described in Theorem 50.

3.5.3 n-Colour Compositions with Tracked Summands

Finally, we consider the class of *n-colour compositions*, introduced by Agarwal [1] and studied in Gibson et al. [37], where each integer i is coloured by one of i available colours (each summand must be coloured and different colourings give distinct n-colour compositions).

Example 55. The twenty-one n-colour compositions of 4 are

$$\begin{aligned}
&4_1,\ 4_2,\ 4_3,\ 4_4,\\
&3_11_1,\ 3_21_1,\ 3_31_1,\ 1_13_1,\ 1_13_2,\ 1_13_3,\\
&2_12_1,\ 2_12_2,\ 2_22_1,\ 2_22_2,\\
&2_11_11_1,\ 2_21_11_1,\ 1_12_11_1,\ 1_12_21_1,\ 1_11_12_1,\ 1_11_12_2,\\
&1_11_11_11_1,
\end{aligned}$$

where the subscripts denote the colours assigned to each summand.

Once again, we fix a positive integer d and track the number of times each element of $\{1,\ldots,d\}$ occurs. If $\mathcal{S}_n$ is the class of positive integers where each integer i is coloured with one of i possible colours then

$$\begin{aligned}
\mathcal{S}_n(\mathbf{z},t) &= z_1t + 2z_2t^2 + \cdots + dz_dt^d + (d+1)t^{d+1} + (d+2)t^{d+2} + \cdots\\
&= z_1t + 2z_2t^2 + \cdots + dz_dt^d + t\frac{d}{dt}\left(\frac{t^{d+1}}{1-t}\right)\\
&= z_1t + 2z_2t^2 + \cdots + dz_dt^d + \frac{dt^{d+1}}{1-t} + \frac{t^{d+1}}{(1-t)^2}
\end{aligned}$$

is the multivariate generating function of positive integers where z_k tracks the number of occurrences of k, so the corresponding multivariate generating function for n-colour compositions is

$$C_n(\mathbf{z},t) = \frac{1}{1 - z_1t - 2z_2t^2 - \cdots - dz_dt^d - \frac{dt^{d+1}}{1-t} - \frac{t^{d+1}}{(1-t)^2}}.$$

As expected, the hypotheses of Theorem 51 hold, and we get a (messier) LCLT whose parameters are given explicitly in the SageMath notebooks corresponding to this paper. Similar to our last example, we may further restrict which elements (or colours!) are allowed or tracked, and immediately get LCLTs. The proofs for such extensions follow similarly to those for integer compositions.

3.6 Family of Limit Theorems Proof

We now prove Theorem 51 by applying the outline in Figure 3.4 to verify the conditions of Proposition 42. To that end, assume the hypotheses of Theorem 51 hold.

Remark 56. Note that the proof provided in this Section is not only a proof of Theorem 51, but also an example of how the outline for an LCLT proof provided in Figure 3.4 may be applied.

Step 1: Finding the correct direction.

Following the outline of Figure 3.4, we substitute $\mathbf{w} = (\mathbf{1}, \rho)$ into the smooth critical point equations (2.4) to find that $\mathbf{w}$ is a critical point in the direction $(\mathbf{m}, 1)$ where

$$\mathbf{m} = \left(\frac{-q_1(\rho)}{\rho P'(\rho)}, \dots, \frac{-q_d(\rho)}{\rho P'(\rho)} \right).$$

Steps 2 and 3: Establishing minimality.

Suppose $H(s, \dots, s, t) = 0$ for $0 < s < 1$ and $0 < t < \rho$. Then $s \sum_{k=1}^{d} q_k(t) = 1 - q(t)$, and we cannot have $\sum_{k=1}^{d} q_k(t) = 0$ since this implies $q(0) = 1$ which would violate the fact that q has no constant, so

$$s = \frac{1 - q(t)}{\sum_{k=1}^{d} q_k(t)}.$$

Our assumptions imply that the polynomial $\sum_{k=1}^{d} q_k(t)$ and series expansion of $q(t)$ have non-negative coefficients and $0 < q(\rho) < 1$, which implies that $|s|$ increases as $0 < t < \rho$ decreases. Thus $H(s, \dots, s, s\rho) \neq 0$ for all $0 < s < 1$, and $\mathbf{w}$ is minimal by applying Lemma 44 to the function $1/H(\mathbf{z}, t) = 1/(1 - S(\mathbf{z}, t))$ whose series expansion at the origin has non-negative coefficients. Our assumptions imply that $S(\mathbf{z}, t)$ is an aperiodic power series with non-negative coefficients, so Proposition 46 implies that no other singularities (including no other critical points) have the same coordinate-wise modulus as $\mathbf{w}$.

Step 4a: Computing an LU-factorization.

We now prove that the phase Hessian matrix defined by (2.5), which simplifies to (3.8) in our case, has non-zero determinant. To compute this symbolic determinant, we originally used the SageMath computer algebra system to compute and factor the Hessian determinant for the permutation generating function described by Proposition 48 in small dimensions. Observing a pattern in the factors, we were able to conjecture, and then *a posteriori* prove, an LU-factorization for the Hessian matrix in that case, and then extend to the general case. This approach immediately gives the Hessian determinant: if $\mathcal{H}U = L$ for a lower triangular matrix L and an upper triangular matrix U with diagonal elements equal to 1 then $\det \mathcal{H}$ is simply the product of the diagonal elements of L.

Remark 57. When the dimension d is fixed, the matrix equation $\mathcal{H}U = L$ defines an explicit system of equations in terms of the entries of U and L, so it is often possible to computationally determine suitable U and L in low dimension, conjecture their general form, then prove this inference. This approach to symbolic determinants is described in the well-known treatise of Krattenthaler [51], which attributes the popularization of such a 'guess-and-check' LU-factorization to George Andrews after he used it to great effect in a variety of papers starting in the 1970s.

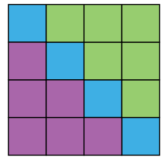


Figure 3.7: The three cases, distinguished by colour, for $d = 4$.

A companion SageMath notebook to this paper gives a procedure to calculate U and L for any fixed d. Studying the numerator and denominator of the rational function entries we are able to deduce certain patterns, such as the denominators being constant down columns, which leads us to conjecture that, in general dimension, $\mathcal{H}U = L$ where

$$\mathrm{U}_{ij} = \begin{cases} \frac{q_j(\rho)g_{ij}}{r_j} & i < j \\ 1 & i = j \\ 0 & i > j \end{cases} \qquad \text{and} \qquad L_{ij} = \begin{cases} 0 & i < j \\ \frac{-q_j(\rho)r_{j+1}}{P'(\rho)\rho r_j} & i = j \\ \frac{q_j(\rho)q_i(\rho)s_{ij}}{P'(\rho)\rho r_j} & i > j \end{cases}$$

for

$$r_j = P'(\rho)^2\rho - P''(\rho)\rho A_j + 2P'(\rho)\rho B_j - P'(\rho)A_j - \rho\left(A_jD_j - B_j^2\right)$$

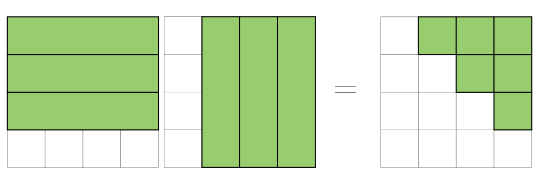
$$g_{ij} = P'(\rho) + P''(\rho)\rho - \rho\left(\frac{q_i'(\rho)}{q_i(\rho)} + \frac{q_j'(\rho)}{q_j(\rho)}\right)(P'(\rho) + B_j - q_i'(\rho)) + \rho\left(D_j - \frac{q_i'(\rho)^2}{q_i(\rho)}\right)$$
$$+ \frac{\rho q_i'(\rho)q_j'(\rho)}{q_i(\rho)q_j(\rho)}(A_j - q_i(\rho))$$

$$s_{ij} = P''(\rho)\rho + \rho D_j - \rho\left(\frac{q_i'(\rho)}{q_i(\rho)} + \frac{q_j'(\rho)}{q_j(\rho)}\right)(P'(\rho) + B_j) + P'(\rho) + \left(\frac{\rho q_i'(\rho)q_j'(\rho)}{q_i(\rho)q_j(\rho)}\right)A_j$$

with

$$A_j = \sum_{k=1}^{j-1} q_k(\rho), \qquad B_j = \sum_{k=1}^{j-1} q_k'(\rho), \qquad D_j = \sum_{k=1}^{j-1} \frac{q_k'(\rho)^2}{q_k(\rho)}.$$

Although these formulas are quite involved, we note that by keeping $P'(\rho)$ and $P''(\rho)$ as symbolic parameters the entries of U and L are independent of d, which allows us to algorithmically verify that $\mathcal{H}U = L$ with the aid of a computer algebra system. We break this verification into three cases depending on the behaviour of the entries of U and L: see Figure 3.7 for an illustration of the different cases on a 4×4 matrix.



Case 1: $j > i$

Case 1 ($j > i$): As U is upper-triangular, we have

$$(\mathcal{H}U)_{ij} = \sum_{a=1}^{j} \mathcal{H}_{ia}U_{aj} = \sum_{\substack{1\leq a<j \\ a\neq i}} \mathcal{H}_{ia}U_{aj} + \mathcal{H}_{ii}U_{ij} + \mathcal{H}_{ij}$$

where we split the sum in such a way that the summands in the indefinite series have a uniform definition. Using SageMath for algebraic manipulations, we see that

$$\mathcal{H}_{ia}U_{aj} = \frac{\alpha_{ij}\left(\frac{q_a'(\rho)^2}{q_a(\rho)}\right) + \beta_{ij}q_a(\rho) + \gamma_{ij}q_a'(\rho)}{\rho^2 P'(\rho)^3 r_j}$$

where

$$\alpha_{ij} = \left(P'(\rho)q_j(\rho) - q_j'(\rho)A_j + q_j(\rho)B_j\right) P'(\rho)q_i(\rho)\rho^2 \tag{3.10}$$

$$\begin{aligned}\beta_{ij} = &\Big[P'(\rho)\left(q_j'(\rho)\rho - q_j(\rho)\right) - P''(\rho)q_j(\rho)\rho + q_j'(\rho)\rho B_j - q_j(\rho)\rho D_j\Big] \\ &\cdot \Big[(P'(\rho)\left(q_i'(\rho)\rho - q_i(\rho)\right) - P''(\rho)q_i(\rho)\rho\Big]\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}\gamma_{ij} =&\Big[q_j'(\rho)q_i(\rho) + q_i'(\rho)q_j(\rho)\Big]\rho^2 P'(\rho)^2 - 2\Big[P'(\rho) + P''(\rho)\rho\Big]\rho P'(\rho)q_i(\rho)q_j(\rho) \\ &+ \Big[P''(\rho)q_j'(\rho)q_i(\rho)\rho - P'(\rho)q_i'(\rho)q_j'(\rho)\rho + P'(\rho)q_j'(\rho)q_i(\rho)\Big]\rho A_j \\ &+ \Big[q_j'(\rho)q_i(\rho)\rho + q_i'(\rho)q_j(\rho)\rho - q_i(\rho)q_j(\rho)\Big]\rho P'(\rho)B_j - P''(\rho)q_i(\rho)q_j(\rho)\rho^2 B_j \\ &- P'(\rho)q_i(\rho)q_j(\rho)\rho^2 D_j.\end{aligned} \tag{3.12}$$

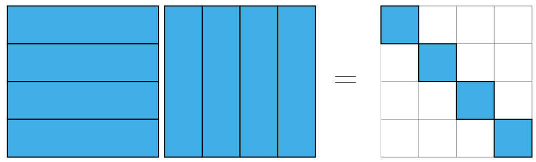
Since α_{ij}, β_{ij}, γ_{ij}, and the denominator in our above expression for $\mathcal{H}_{ia}U_{aj}$ are not dependent on a,

$$\begin{aligned}\sum_{\substack{1\leq a<j \\ a\neq i}} \mathcal{H}_{ia}U_{aj} &= \sum_{\substack{1\leq a<j \\ a\neq i}} \frac{\alpha_{ij}\left(\frac{q_a'(\rho)^2}{q_a(\rho)}\right) + \beta_{ij}q_a(\rho) + \gamma_{ij}q_a'(\rho)}{\rho^2 P'(\rho)^3 r_j} \\ &= \frac{1}{\rho^2 P'(\rho)^3 r_j}\left[\alpha_{ij}\sum_{\substack{1\leq a<j \\ a\neq i}} \frac{q_a'(\rho)^2}{q_a(\rho)} + \beta_{ij}\sum_{\substack{1\leq a<j \\ a\neq i}} q_a(\rho) + \gamma_{ij}\sum_{\substack{1\leq a<j \\ a\neq i}} q_a'(\rho)\right] \\ &= \frac{1}{\rho^2 P'(\rho)^3 r_j}\left[\alpha_{ij}\left(D_j - \frac{q_i'(\rho)^2}{q_i(\rho)}\right) + \beta_{ij}\left(A_j - q_i(\rho)\right) + \gamma_{ij}\left(B_j - q_i'(\rho)\right)\right].\end{aligned}$$

Substitution of our above formulas then gives, after some algebraic simplification by SageMath, that

$$\sum_{\substack{1\le a<j\\ a\neq i}} \mathcal{H}_{ia}U_{aj} + \mathcal{H}_{ii}U_{ij} + \mathcal{H}_{ij} = 0$$

in this case, as required.



Case 2: $i = j$

Case 2 ($i = j$): Since U is upper-triangular we again have

$$(\mathcal{H}U)_{jj} = \sum_{a=1}^{d} \mathcal{H}_{ja}U_{aj} = \sum_{a=1}^{j} \mathcal{H}_{ja}U_{aj} = \sum_{a=1}^{j-1} \mathcal{H}_{ja}U_{aj} + \mathcal{H}_{jj},$$

where we separate the case $a = j$ since the entries of U have a different definition on the diagonal and above the diagonal. It is therefore sufficient to prove

$$\sum_{a=1}^{j-1} \mathcal{H}_{ja}U_{aj} + \mathcal{H}_{jj} = L_{jj}$$

using the definitions above. Analogously to Case 1,

$$\mathcal{H}_{ja}U_{aj} = \frac{\alpha_{jj}\left(\frac{q_a'(\rho)^2}{q_a(\rho)}\right) + \beta_{jj}q_a(\rho) + \gamma_{jj}q_a'(\rho)}{\rho^2 P'(\rho)^3 r_j}$$

where α_{jj}, β_{jj} and γ_{jj} are defined in (3.10), (3.11) and (3.12) respectively. Since α_{jj}, β_{jj}, γ_{jj}, and the denominator in this expression for $\mathcal{H}_{ja}U_{aj}$ are not dependent on a, we can algebraically simplify to get

$$\begin{aligned}\sum_{a=1}^{j-1} \mathcal{H}_{ja}U_{aj} &= \sum_{a=1}^{j-1}\left[\frac{\alpha_{jj}\left(\frac{q_a'(\rho)^2}{q_a(\rho)}\right) + \beta_{jj}q_a(\rho) + \gamma_{jj}q_a'(\rho)}{\rho^2 P'(\rho)^3 r_j}\right]\\ &= \frac{\alpha_{jj}D_j + \beta_{jj}A_j + \gamma_{jj}B_j}{\rho^2 P'(\rho)^3 r_j}.\end{aligned}$$

Substitution of our above formulas then gives, after some algebraic simplification by SageMath, that

$$\sum_{a=1}^{j-1} \mathcal{H}_{ja}U_{aj} + \mathcal{H}_{jj} - L_{jj} = 0$$

in this case, as desired.

Case 3 ($j < i$): At this point, we have proven that $\mathcal{H}U$ is lower-triangular and described its diagonal entries, which gives the desired determinant. Although not needed to establish Theorem 51, for completeness we prove the claimed expression above for the below diagonal entries in the companion SageMath notebook.

Step 4b: Computing the Hessian determinant

Our LU-factorization expresses the Hessian determinant as the product of the diagonal entries of the lower-triangular matrix L. In fact, the form of these diagonal entries causes cancellation of many terms, leading to a relatively compact final answer,

$$\det \mathcal{H} = \prod_{m=1}^{d} L_{mm} = \prod_{m=1}^{d} \frac{-q_m(\rho) r_{m+1}}{P'(\rho)\rho r_m} = \frac{(-1)^d \left(\prod_{k=1}^{d} q_k(\rho)\right) r_{d+1}}{P'(\rho)^d \rho^d r_1}$$

$$= \frac{(-1)^d \left(\prod_{k=1}^{d} q_k(\rho)\right) r_{d+1}}{P'(\rho)^{d+2} \rho^{d+1}}.$$

Furthermore,

$$P(\rho) = 1 - q(\rho) - \sum_{k=1}^{d} q_k(\rho) = 0 \quad \text{and} \quad P'(\rho) = -q'(\rho) - \sum_{k=1}^{d} q_k'(\rho)$$

so that $A_{d+1} = \sum_{k=1}^{d} q_k(\rho) = 1 - q(\rho)$ and $B_{d+1} = \sum_{k=1}^{d} q_k'(\rho) = -q'(\rho) - P'(\rho)$. Making these substitutions into our definition of r_{d+1} gives, after some algebraic simplification, that the Hessian determinant has the form (3.9).

Step 4c: Non-singularity of the determinant

It remains to show that the expression in (3.9) is non-zero under our assumptions. Each $q_k(\rho) \neq 0$ because $\rho > 0$ and the q_k have non-negative coefficients, so it is sufficient to prove that

$$(q(\rho) - 1)\left(\rho P''(\rho) + P'(\rho) + \rho \sum_{k=1}^{d} \frac{q_k'(\rho)^2}{q_k(\rho)}\right) + q'(\rho)^2 \rho > 0.$$

First, we note that $q'(\rho)^2 \rho > 0$ (as $q(t)$ is non-constant with non-negative coefficients and $\rho > 0$) and $q(\rho) - 1 \leq 0$, so it is enough to prove that

$$\rho P''(\rho) + P'(\rho) + \rho \sum_{k=1}^{d} \frac{q_k'(\rho)^2}{q_k(\rho)} \leq 0.$$

Because

$$\rho P''(\rho) + P'(\rho) + \rho \sum_{k=1}^{d} \frac{q_k'(\rho)^2}{q_k(\rho)} = \rho\Big[- q''(\rho) - \sum_{k=1}^{d} q_k''(\rho)\Big] - q_k'(\rho) - \sum_{k=1}^{d} q_k'(\rho) + \rho \sum_{k=1}^{d} \frac{q_k'(\rho)^2}{q_k(\rho)}$$

and $q'(\rho)$ and $\rho q''(\rho)$ are non-negative, this holds if

$$\rho \sum_{k=1}^{d} \frac{q_k'(\rho)^2}{q_k(\rho)} \leq \rho \sum_{k=1}^{d} q_k''(\rho) + \sum_{k=1}^{d} q_k'(\rho).$$

Now, if $f(z) = a_1 z + a_2 z^2 + \cdots + a_s z^s$ is any non-zero polynomial vanishing at the origin with non-negative coefficients then

$$\begin{aligned} z f'(z)^2 &= z \left(a_1 + 2a_2 z + \cdots + s a_s z^{s-1}\right)^2 \\ &\leq z \left(a_1 + a_2 z + \cdots + a_s z^{s-1}\right) \left(a_1 + 2^2 a_2 z + \cdots + s^2 a_s z^{s-1}\right) \\ &= f(z) \left(z f'(z)\right)' \\ &= f(z) \left(z f''(z) + f'(z)\right) \end{aligned}$$

for any $z > 0$, where the first inequality holds because each term in the expansion is non-negative and $2ij \leq i^2 + j^2$ for all $i, j \in \mathbb{N}$. Thus,

$$\frac{\rho q_k'(\rho)^2}{q_k(\rho)} \leq \rho q_k''(\rho) + q_k'(\rho)$$

for all k, and summing over k gives the desired inequality. The Hessian determinant is therefore non-zero.

Remark 58. When each of the q_k are monomials this inequality becomes an equality.

Step 5: Checking final hypotheses

To conclude Theorem 51 from Proposition 38 it remains only to note that $G(\mathbf{1}, \rho) \neq 0$, which is one of our assumptions, and that $H_t(\mathbf{w}) = P'(\rho) \neq 0$ because none of the $q_k(t)$ and $q(t)$ have constant terms and we assume the denominator is not constant.

Chapter 4

Asymptotics of Algebraic Generating Functions

This chapter is modified from the upcoming work "Algebraic Generating Functions and Analytic Combinatorics in Several Variables" with Torin Greenwood, Stephen Melczer and Mark C. Wilson. We provide an exploration of various methods to embed multivariate algebraic generating functions as rational diagonals for the purpose of finding asymptotics and conclude with a brief discussion of other methods for approaching finding asymptotics of algebraic generating functions.

4.1 Rational Embeddings

Let $F(\mathbf{x})$ be a generating function with power series expansion

$$F(\mathbf{x}) = \sum_{\mathbf{i}\in\mathbb{N}^d} f_{\mathbf{i}}\mathbf{x}^{\mathbf{i}} = \sum_{i_1,\ldots,i_d\geq 0} f_{i_1,\ldots,i_d} x_1^{i_1}\cdots x_d^{i_d}.$$

In additional to the $\mathbf{r}$-diagonals discussed in Section 2.3, we also consider diagonals in blocks of variables. An *elementary diagonal* of F is any series $\Delta_{x_i=x_j}F$ in one less variable obtained by taking only the terms where the power of x_i matches the power of x_j, and a *block diagonal* is any series obtained by repeatedly taking elementary diagonals. We often group elementary diagonals together, for instance if

$$F(x_1,x_2,x_3,x_4) = \sum_{i_1,i_2,i_3,i_4\geq 0} f_{i_1,i_2,i_3,i_4} x_1^{i_1}\cdots x_4^{i_4}$$

then we can write

$$\left(\Delta_{(x_1,x_2)=(x_3,x_4)}F\right)(x_1,x_2) = \left(\Delta_{x_1=x_3}\Delta_{x_2=x_4}F\right)(x_1,x_2) = \sum_{i_1,i_2\geq 0} f_{i_1,i_2,i_1,i_2} x_1^{i_1}x_2^{i_2}.$$

An *algebraic generating function* is a generating function $F(\mathbf{x})$ such that $F(\mathbf{x})$ is a root $P(\mathbf{x}, F(\mathbf{x})) = 0$ for a polynomial $P(\mathbf{x}, Y) \in \mathbb{Z}[\mathbf{x}, Y]$. Any such P of minimal degree in Y is called the *minimal polynomial* of F. The minimal polynomial is unique up to multiplication by a constant with respect to Y.

Example 59. The Catalan generating function

$$f(x) = \frac{1 - \sqrt{1 - 4x}}{2x}$$

is a root of the integer polynomial

$$P(x, Y) = xY^2 - Y + 1.$$

◁

The main technique highlighted in this thesis to analyze an algebraic generating function is to systematically embed it as a sub-series of a higher-dimensional rational function. This approach is possible due to two facts. First, much work has been done in the quest to find asymptotics of rational generating functions including the work that automates asymptotics under some conditions [41]. Second, there is a deep connection between algebraic functions and rational diagonals that dates back at least to work of Pólya [66] in the 1920s, who noted that the diagonal $(\Delta F)(t) = \sum_n a_{nn} t^n$ of any bivariate rational generating function $F(x, y) = \sum_{r,s} a_{rs} x^r y^s$ is algebraic (see [33, 42] for analytic arguments of this result, [36, 68] for algebraic proofs holding over any field of characteristic zero, Bousquet-Mélou [12, Section 3.4] for a combinatorial proof, and Bostan et al. [11] for effective algorithms and complexity results). In the 1960s, Furstenberg noted a partial converse: any univariate algebraic generating function that is the unique branch of its minimal polynomial vanishing at the origin can be written as the diagonal of an explicit bivariate rational function.

Lemma 60 (Furstenberg [33])**.** *If the minimal polynomial $P(x, Y)$ of a univariate algebraic generating function $f(x)$ that vanishes at the origin satisfies $P_Y(0, 0) \neq 0$ then*

$$f(x) = \Delta\left(\frac{Y^2 P_Y(xY, Y)}{P(xY, Y)}\right). \tag{4.1}$$

Proof. The hypotheses imply $P(x, Y) = (Y - f(x))u(x, Y)$ for some analytic function u near the origin with $u(0, 0) \neq 0$, so

$$\frac{Y^2 P_Y(xY, Y)}{P(xY, Y)} = \frac{Y^2}{Y - f(xY)} + \frac{Y^2 u_Y(xY, Y)}{u(xY, Y)}.$$

The main diagonal of the first summand is $f(x)$, while the main diagonal of the second vanishes. ☐

If $f(x) \neq 0$ but no other branch of $P(x, Y)$ takes the value $f(0)$ at $x = 0$ then Lemma 60 can still be applied to, for instance, $f(x) - f(0)$ or $xf(x)$ to determine a rational embedding. Different *preprocessing steps* (like subtracting constants or multiplying by powers of x) can lead to different diagonal expressions.

Example 61. The Catalan generating function

$$f(x) = \frac{1 - \sqrt{1-4x}}{2x} = 1 + x + 2x + 5x^3 + \cdots$$

has a non-zero constant term. Applying Lemma 60 to $f(x) - 1$ gives the representation

$$f(x) = \Delta \left(\frac{Y\left(1 - 2Y^2x - 2Yx\right)}{1 - x(Y+1)^2} \right),$$

while applying Lemma 60 to $xf(x)$ gives the representation

$$xf(x) = \Delta \left(\frac{Y(1-2Y)}{1 - Y - x} \right).$$

An asymptotic expansion beginning

$$[x^n]f(x) = 4^n \left(\frac{1}{n^{3/2}\sqrt{\pi}} + O\left(\frac{1}{n}\right) \right)$$

can be derived completely automatically from either of these expressions, using the smooth minimal critical point $(1/4, 1)$ for the first representation and the smooth minimal critical point $(1/2, 1/2)$ for the second. ◁

An analogous argument gives a higher-dimensional generalization of Lemma 60.

Lemma 62. *If the multivariate algebraic generating function $f(\mathbf{x})$ has no constant term, and its minimal polynomial $P(\mathbf{x}, Y)$ can be written $P(\mathbf{x}, Y) = (Y - f(\mathbf{x}))u(\mathbf{x}, Y)$ for an analytic function $u(\mathbf{x}, Y)$ with $u(\mathbf{0}, 0) \neq 0$, then*

$$[\mathbf{x}^\mathbf{r}]f(\mathbf{x}) = \left[\mathbf{x}^\mathbf{r} Y^{|\mathbf{r}|}\right] \left(\frac{Y^2 P_Y(Y\mathbf{x}, Y)}{P(Y\mathbf{x}, Y)} \right), \tag{4.2}$$

where $|\mathbf{r}| = r_1 + \cdots + r_d$.

Equation (4.2) can be expressed in several equivalent ways. For instance, under the hypotheses of Lemma 62 examining the coefficients in Y that are extracted gives the *constant term expression*

$$f(\mathbf{x}) = \left[Y^0\right] \left(\frac{Y^2 P_Y(\mathbf{x}, Y)}{P(\mathbf{x}, Y)} \right). \tag{4.3}$$

If the hypotheses of Lemma 62 are strengthened so that x_k divides $f(\mathbf{x})$ for some k (i.e., f vanishes identically when $x_k = 0$) then we also have the elementary diagonal expression

$$[\mathbf{x}^\mathbf{r}]f(\mathbf{x}) = [\mathbf{x}^\mathbf{r} Y^{r_k}] \left(\left. \frac{Y^2 P_Y(\mathbf{x}, Y)}{P(\mathbf{x}, Y)} \right|_{x_k = Yx_k} \right). \tag{4.4}$$

Remark 63. These modifications can give different embeddings. For instance, if $f(x,y)$ is the only branch of its minimal polynomial to take the value $f(0,0)$ at the origin then we can apply (4.2) to $f(x,y) - f(0,0)$, apply (4.4) to $yf(x,y)$, or apply (4.4) to $xf(x,y)$ to obtain different embeddings.

4.1.1 Non-Simple Roots

In the 1980s, Denef and Lipshitz used high-level algebraic arguments to show that rational embeddings exist even if multiple branches of the minimal polynomial collide.

Theorem 64 (Denef and Lipshitz [19, Theorem 6.2]). *Let $f(\mathbf{x})$ be any algebraic power series in d variables.*

(i) There exists an algebraic power series $\phi(\mathbf{x})$ vanishing at the origin and a rational function $W(\mathbf{x}, Y)$ such that $W(\mathbf{x}, \phi(\mathbf{x})) = f(\mathbf{x})$ and the minimal polynomial $Q(\mathbf{x}, Y)$ of ϕ satisfies $Q_Y(\mathbf{0}, 0) \neq 0$. If

$$R(\mathbf{x}, Y) = \frac{Y \cdot W(Y\mathbf{x}, Y) Q_Y(Y\mathbf{x}, Y)}{Q(Y\mathbf{x}, Y)} \tag{4.5}$$

for any such rational function W then $[\mathbf{x}^\mathbf{r}]f(\mathbf{x}) = [\mathbf{x}^\mathbf{r} Y^{|\mathbf{r}|}]R(\mathbf{x}, Y)$ for all $\mathbf{r} \in \mathbb{N}^d$.

(ii) There exists a rational power series $S(\mathbf{x}, \mathbf{Y})$ in $2d$ variables such that $f(\mathbf{x}) = (\Delta_{\mathbf{x}=\mathbf{Y}} S)(\mathbf{x})$ for all $\mathbf{r} \in \mathbb{N}^d$.

Denef and Lipshitz prove the existence of ϕ and W by exploiting the fact that the ring defined by adjoining f to the local ring $\mathbb{Q}[\mathbf{x}]_{(x_1,\ldots,x_d)}$ lies in a *finite étale extension* of $\mathbb{Q}[\mathbf{x}]_{(x_1,\ldots,x_d)}$. This argument does not give a method to construct ϕ, but (as will be seen in examples below) we can often guess a good choice for ϕ and then verify the existence of W computationally. Assuming the existence of ϕ and W, Theorem 64(i) follows from algebraic manipulations similar to the proof of Lemma 62 above, and Theorem 64(ii) follows from repeated applications of Theorem 64(i) in an explicit manner.

Remark 65. The series ϕ is not unique, and different choices of ϕ yield different embeddings with varying properties that can make it easier (or harder) to apply the techniques of ACSV. The preprocessing steps discussed above (subtracting off a constant or multiplying by a variable) can be viewed as different choices of ϕ.

Example 66. Consider the algebraic generating function $f(x,y) = x\sqrt{1-x-y}$. If we take $\phi(x,y) = 1 - \sqrt{1-x-y}$ then $f(x,y) = W(x,y,\phi(x))$ for the rational function $W(x,y,Y) = x(1-Y)$. The minimal polynomial $Q(x,y,Y) = (1-Y)^2 - (1-x-y)$ of ϕ satisfies $Q_Y(0,0,0) = -2 \neq 0$, so

$$[x^r y^s]f(x,y) = [x^r y^s Y^{r+s}] \left(\frac{2xY(Y-1)^2}{2-x-y-Y} \right). \tag{4.6}$$

We can also calculate $S(x,y,Y_1,Y_2)$ from our original embedding. If $R(x,y,Y)$ is the rational function in (4.6) then

$$\begin{aligned} S(x,y,Y_1,Y_2) &= \frac{Y_1R(x,y,Y_1) - Y_2R(x,y,Y_2)}{Y_1 - Y_2} \\ &= \frac{2xY_1^2(Y_1-1)^2(2-x-y-Y_2) - 2xY_2^2(Y_2-1)^2(2-x-y-Y_1)}{(Y_1-Y_2)(2-x-y-Y_1)(2-x-y-Y_2)} \end{aligned}$$

and $f(x,y) = (\Delta_{x=Y_1,y=Y_2}S)(x,y)$. In general, for more variables, the process is continued with $R_k = \frac{Y_{k-1}R_{k-1}(\mathbf{x},Y_1,\ldots,Y_{k-1}) - Y_kR_{k-1}(\mathbf{x},Y_1,\ldots,Y_{k-2},Y_k)}{Y_{k-1}-Y_k}$.

We can obtain asymptotics by applying the results of ACSV to (4.6), giving

$$[x^{nr}y^{ns}]f(x,y) = -\frac{\left(\frac{(r+s)^{r+s}}{r^rs^s}\right)^n}{n(r+s)} \cdot \left(\frac{\sqrt{2r}}{4\pi n(r+s)\sqrt{s}} + O\left(\frac{1}{n^2}\right)\right).$$

Similarly, to lift $g(x,y) = x/\sqrt{1-x-y}$ we can again take $\phi(x,y) = 1-\sqrt{1-x-y}$ where now $W(x,y,Y) = x/(1-Y)$, so that

$$[x^ry^s]g(x,y) = [x^ry^sY^{r+s}]\left(\frac{2xY}{2-x-y-Y}\right),$$

and

$$[x^{nr}y^{ns}]g(x,y) = \frac{\left(\frac{(r+s)^{r+s}}{r^rs^s}\right)^n}{n(r+s)}\left(\frac{\sqrt{2r}}{2\pi\sqrt{s}} + O\left(\frac{1}{n}\right)\right).$$

◁

Remark 67. As noted in Remark 65, the embeddings that could result from the application of Theorem 64 can vary greatly in complexity depending on the ϕ chosen. This fact is both a challenge and advantage to the Denef and Lipshitz method, as there is no known method for choosing a ϕ that will give a 'nicer' embedding. We see an example of the difference that ϕ can make in following example.

Example 68. Consider again the Catalan generating function

$$f(x) = \frac{1-\sqrt{1-4x}}{2x}.$$

Applying Theorem 64 with $\phi(x) = f-1$ and hence $Q(x,Y) = Y^2x + 2Yx - Y + x$ and $W(x,Y) = Y+1$ gives an embedding using

$$R(x,Y) = \frac{(1-2Y^2x-2Yx)(Y+1)}{1-Y^2x-2Yx-x}$$

which is essentially combinatorial and thus relatively simple to work with. However, using $\phi(x) = -x(f-1)$ and hence $Q(x,Y) = Y^2 - 2Yx + x^2 + Y$ and $W(x,Y) = -Y/x + 1$ gives an embedding using

$$R(x,Y) = \frac{(2Yx - 2Y - 1)(1-x)}{(Yx^2 - 2Yx + Y + 1)x}$$

which is not combinatorial (the series coefficients are not all non-negative). This distinction is important as combinatorial functions have much more results and automations available for computing their asymptotics. ◁

Although there is no known method to find a valid choice of ϕ in general, there are methods to determine ϕ for generating functions with minimal polynomials of certain degrees.

Example 69. Let $f(\mathbf{x})$ be an algebraic generating function vanishing at the origin with quadratic minimal polynomial $P(\mathbf{x}, Y) = a(\mathbf{x})Y^2 + b(\mathbf{x})Y + c(\mathbf{x})$ and discriminant $D(\mathbf{x}) = b(\mathbf{x})^2 - 4a(\mathbf{x})c(\mathbf{x}) = C(\mathbf{x})^2\tilde{D}(\mathbf{x})$ factored such that $\tilde{D}(0) = 1$ and $\tilde{D}$ is squarefree. Assume first that $a(\mathbf{0}) \neq 0$ and define

$$\phi(\mathbf{x}) = 1 - \sqrt{\tilde{D}(\mathbf{x})}\,,$$

so that $\phi(\mathbf{0}) = 0$, the minimal polynomial $Q(\mathbf{x}, Y) = (Y-1)^2 - \tilde{D}(\mathbf{x})$ of ϕ has $Q_Y(\mathbf{0}, 0) \neq 0$, and $f(\mathbf{x}) = W(\mathbf{x}, \phi(\mathbf{x}))$ where

$$W(\mathbf{x}, Y) = \frac{-b(\mathbf{x})(1-Y) \pm C(\mathbf{x})\tilde{D}(\mathbf{x})}{2a(\mathbf{x})(1-Y)}.$$

Theorem 64 then gives $[\mathbf{x}^{\mathbf{r}}]f(\mathbf{x}) = [\mathbf{x}^{\mathbf{r}}Y^{|\mathbf{r}|}]R(\mathbf{x}, Y)$ where

$$R(\mathbf{x}, Y) = Y \cdot \frac{2(b(Y\mathbf{x})(1-Y) \pm C(Y\mathbf{x})\tilde{D}(Y\mathbf{x}))}{a(Y\mathbf{x})(\tilde{D}(Y\mathbf{x}) - (1-Y)^2)}. \tag{4.7}$$

Regardless of whether $a(\mathbf{0}) = 0$, an analogous construction applied to $a(\mathbf{x})f(\mathbf{x})$ implies $[\mathbf{x}^{\mathbf{r}}]a(\mathbf{x})f(\mathbf{x}) = [\mathbf{x}^{\mathbf{r}}Y^{|\mathbf{r}|}]S(\mathbf{x}, Y)$ where

$$S(\mathbf{x}, Y) = Y \cdot \frac{2(b(Y\mathbf{x})(1-Y) \pm C(Y\mathbf{x})\tilde{D}(Y\mathbf{x}))}{\tilde{D}(Y\mathbf{x}) - (1-Y)^2}. \tag{4.8}$$

Remark 70. Note that the signs in R and S should be chosen so that $W(\mathbf{x}, \phi(\mathbf{x})) = f(\mathbf{x})$.

◁

Algebraic functions of degree two have the large benefit that they can be expressed explicitly using the quadratic formula. Of course, this does not occur for most algebraic functions of large degree.

Example 71. In their 2024 paper on an implicit method for finding asymptotics of algebraic generating functions (described briefly below), Baryshnikov, Jin, and Pemantle [2] provide a variety of examples. One such example is that of the 0-2-5 trees or rather trees where every vertex has zero, two or five children. We may create a bivariate generating function based on such trees $F(x, y)$ where y gives the number of vertices in the tree and x gives the number of vertices with five children. In particular,

$$F(x, y) = 1 + y[((F(x, y) - 1)^2 + 1) + x(F(x, y) - 1)^5]$$

and so

$$P(x, y, Y) = 1 - Y + y[((Y - 1)^2 - 1) + x(Y - 1)^5].$$

We note that in this example F is not known explicitly, but we can still obtain an embedding. Specifically, we know that $F(0, 0) = 1$. Thus, using $\phi = F - 1$ with

$$Q(x, y, Y) = -Y + y(Y^2 - 1 + xY^5)$$

and $W(x, y, Y) = Y + 1$ gives an embedding with

$$R(x, y, Y) = \frac{(5Y^6xy + 2Y^2y - 1)(Y + 1)}{Y^6xy + Y^2y - y - 1}.$$

◁

With regards to ACSV, the form of the denominator in (4.5) implies that the smooth critical point equations (2.4) for R with respect to the direction $(\mathbf{r}, |\mathbf{r}|)$ are equivalent to the polynomial system

$$Q(Y\mathbf{x}, Y) = Q_Y(Y\mathbf{x}, Y) = 0, \qquad \frac{x_1}{r_1}Q_{x_1}(Y\mathbf{x}, Y) = \cdots = \frac{x_d}{r_d}Q_{x_d}(Y\mathbf{x}, Y). \tag{4.9}$$

The solutions to this system are thus either smooth critical points, or points where the gradient of Q vanishes (including all non-smooth points of its zero set). Even for quadratic generating functions, it is possible that asymptotics will be determined by non-smooth critical points.

Example 72. Consider the function $f(x, y) = \sqrt{(1 - y)/(1 - x)}$ with minimal polynomial $P(x, y, Y) = (1 - x)Y^2 - (1 - y)$. Example 69 implies that we have a rational embedding $[x^r y^s]f(x, y) = [x^r y^s Y^{r+s}]R(x, y, Y)$ for

$$R(x, y, Y) = \frac{2(1 - yY)}{2 - Y - x - y + xyY}.$$

For any direction $(r, s, r+s)$ the only solution to the smooth critical point equations (2.4) is $(x, y, Y) = (1, 1, 1)$, which is a non-smooth point (reflecting the fact that the discriminant of P factors as $4(1 - x)(1 - y)$). The expansion

$$2 - Y - x - y + xyY = (1-x)(1-Y) + (1-Y)(1-y) + (1-x)(1-y) - (1-y)(1-Y)(1-x)$$

shows that $(1, 1, 1)$ is a *cone singularity.* ◁

4.1.2 Effective Embeddings

As discussed in the last section, the method provided by Denef and Lipshitz does not provide a constructive approach to finding an appropriate ϕ. We now turn our attention to a method from Safonov [67] which provides an algorithm for finding a rational embedding.

Theorem 73 (Safonov [67, Theorem 1])**.** *If $f(\mathbf{x})$ is an algebraic generating function in d variables that vanishes at the origin then Algorithm 1 computes a rational function $R(\mathbf{x}, Y)$ in $d+1$ variables and unimodular matrix $A \in \mathbb{N}^{d\times d}$ such that*

$$[\mathbf{x}^{\mathbf{r}}]f(\mathbf{x}) = \left[\mathbf{x}^{A\mathbf{r}}Y^{|A\mathbf{r}|}\right] R(\mathbf{x}, Y)$$

for all $\mathbf{r} \in \mathbb{N}^d$.

Proof Sketch. Suppose that the algebraic generating function $f(\mathbf{x})$ has minimal polynomial $P(\mathbf{x}, Y)$, so that we can write $P(\mathbf{x}, Y) = (Y - f(\mathbf{x}))u(\mathbf{x}, Y)$ for an analytic function $u(\mathbf{x}, Y)$ with $u(\mathbf{x}, f(\mathbf{x}))$ not identically zero. If $u(\mathbf{0}, 0) \neq 0$ then f is the unique branch of P that goes through the origin, and the desired result follows from Lemma 62.

Difficulty arises when $u(\mathbf{0}, 0) = 0$, so that multiple branches of P go through the origin. In this case, let $\mu > 0$ denote the lowest degree of the terms in the series expansion of $u(\mathbf{x}, f(\mathbf{x}))$ at the origin and let $S_q(\mathbf{x})$ be the polynomial containing the terms in the power series expansion of $f(\mathbf{x})$ at the origin with degree at most $q = \mu(\mu+1)/2$. Safonov performs an explicit *resolution of singularities* by constructing a polynomial change of variables

$$Y = S_q\left(\boldsymbol{\zeta}^{\mathbf{v}_1}, \ldots, \boldsymbol{\zeta}^{\mathbf{v}_d}\right) + w\boldsymbol{\zeta}^{\mathbf{k}} \qquad \text{and} \qquad z_j = \boldsymbol{\zeta}^{\mathbf{v}_j} \qquad (1 \leq j \leq d), \tag{4.10}$$

with rational inverse, such that

$$\tilde{P}(\boldsymbol{\zeta}, w) = P\left(\boldsymbol{\zeta}^{\mathbf{v}_1}, \ldots, \boldsymbol{\zeta}^{\mathbf{v}_d}, S_q\left(\boldsymbol{\zeta}^{\mathbf{v}_1}, \ldots, \boldsymbol{\zeta}^{\mathbf{v}_d}\right) + w\boldsymbol{\zeta}^{\mathbf{k}}\right) = \boldsymbol{\zeta}^{\mathbf{m}}(w - h(\boldsymbol{\zeta}))\tilde{u}(\boldsymbol{\zeta}, w)$$

for some $\mathbf{m} \in \mathbb{Z}^d$ and analytic functions h and $\tilde{u}$ with $h(\boldsymbol{\zeta}, 0) \equiv 0$ and $\tilde{u}(\mathbf{0}, 0) \neq 0$. Since the roots of $\tilde{P}$ are separated at the origin, we can write $[\boldsymbol{\zeta}^{\mathbf{r}}]h(\boldsymbol{\zeta}) = [\boldsymbol{\zeta}^{\mathbf{r}}w^{r_d}]\tilde{R}(\boldsymbol{\zeta}, w)$ where $\tilde{R}$ is the rational function defined by replacing P with $\tilde{P}$ on the right-hand side of (4.2). By construction, there exists a vector $\ell \in \mathbb{N}^d$ such that

$$f(\boldsymbol{\zeta}^{\mathbf{v}_1}, \ldots, \boldsymbol{\zeta}^{\mathbf{v}_d}) = \boldsymbol{\zeta}^{\ell}h(\boldsymbol{\zeta}) + S_q(\boldsymbol{\zeta}^{\mathbf{v}_1}, \ldots, \boldsymbol{\zeta}^{\mathbf{v}_d}),$$

so

$$[\mathbf{x}^{\mathbf{r}}]f(\mathbf{x}) = \left[\mathbf{x}^{A\mathbf{r}}\right] f(\mathbf{x}^{\mathbf{v}_1}, \ldots, \mathbf{x}^{\mathbf{v}_d}) = \left[\mathbf{x}^{A\mathbf{r}}Y^{|A\mathbf{r}|}\right] \left(\frac{\mathbf{x}^{\ell}Y^2\tilde{P}_Y(\mathbf{x}, Y)}{\tilde{P}(\mathbf{x}, Y)} + S_q(\mathbf{x}^{\mathbf{v}_1}, \ldots, \mathbf{x}^{\mathbf{v}_d})\right)\Bigg|_{x_i = Yx_i \forall i} \tag{4.11}$$

where A is the matrix with rows $\mathbf{v}_1, \ldots, \mathbf{v}_d$. □

The key to making Theorem 73 effective is Safonov's description of the change of variables (4.10). See Algorithm 1 for a full description of the change of variables used.

Algorithm 1: SafonovEmbed

Input: The minimal polynomial $P(\mathbf{x}, Y) \in \mathbb{Z}[\mathbf{x}, Y]$ of an algebraic series $f(\mathbf{x})$ that vanishes at the origin and an oracle to compute the series coefficients of f up to any desired order.

Output: Rational $R(\mathbf{x}, Y)$ and unimodular matrix A such that $[\mathbf{x}^{\mathbf{r}}]f(\mathbf{x}) = [\mathbf{x}^{A\mathbf{r}} Y^{|A\mathbf{r}|}]R(\mathbf{x}, Y)$.

Let $k = 0$, $S_k(\mathbf{x}) = 0$ and $A = I_d$
while $P_Y(\mathbf{x}, S_k(\mathbf{x}))$ *vanishes to order* k **do**
  Set $k = k + 1$
  Let $S_k(\mathbf{x})$ be the polynomial defined by the terms in f of degree at most k
end
Let μ be the lowest degree of a term in $P_Y(\mathbf{x}, S_k(\mathbf{x}))$ and define $q = \mu(\mu + 1)/2$
Let $S_q(\mathbf{x})$ be the polynomial defined by the terms in f of degree at most q
Let $\tilde{P}(\boldsymbol{\zeta}, w) = P(\boldsymbol{\zeta}, w)$
while $\mu > 0$ **do**
  Let ζ_i be a variable of maximum degree among the lowest degree terms of $\tilde{P}_w(\boldsymbol{\zeta}, S_q(\boldsymbol{\zeta}))$
  Make the change of variables $w = w\zeta_i^\mu$ and $\zeta_j = \zeta_i\zeta_j$ for $j \neq i$ (with ζ_i unchanged)
  Let $A_{temp} = I_d$
  Update A_{temp} so that $A_{ki} = 1$ for all k
  Set $A = A \cdot A_{temp}$
  Factor ζ_i^μ out of $\tilde{P}_w(\boldsymbol{\zeta}, w)$ to find $P^{(1)}(\boldsymbol{\zeta}, w)$
  Let μ be the order of vanishing of $P^{(1)}(\boldsymbol{\zeta}, S_q(\boldsymbol{\zeta}))$
  Set $\tilde{P}(\boldsymbol{\zeta}, w) = P^{(1)}(\boldsymbol{\zeta}, w)$
end
Remove the monomial factors from $\tilde{P}$
if $\tilde{P}$ *is not divisible by* ζ_d **then**
  Make the change of variables $\zeta_j = \zeta_j\zeta_d$ for all $1 \leq j < d$
end
Set $R(\mathbf{x}, Y) = \frac{Y^2\tilde{P}_Y}{\tilde{P}}$, where all variables in $\tilde{P}$ are multiplied by Y
Return R and A

Example 74. A bivariate refinement of the Catalan generating function is the Narayana generating function

$$f(x,y) = \frac{1}{2x}\Big(1 - x(y-1) - \sqrt{1-2x(y+1)+x^2(y-1)^2}\Big),$$

which enumerates non-crossing partitions by set size and number of blocks, rooted ordered trees by edges and leaves, Dyck paths by semi-length and number of peaks, and more. Note that $f(x,y)-1$ satisfies the hypotheses of Theorem 73, and hence embeds into

$$R(x,y,Y) = \frac{(1-(2Y^2x+Y^2xy+Yx))\,Y}{1-(Y^2x+Y^2xy+Yx+Yxy)}$$

with $A = I$. Note that by making the substitution $y = y/Y$ and setting $y = 1$ recovers one of the rational functions encoding the Catalan generating function in Example 61. Using this embedding we can obtain asymptotics of

$$[x^{rn}y^{sn}]f(x,y) = \frac{\left(\frac{(r-s)^{2s-2r}r^{2r}}{s^{2s}}\right)^n}{n(r+s)}\left(\frac{r+s}{2\pi n(r-s)^2} + O\left(\frac{1}{n^2}\right)\right).$$

If we instead multiply by x before embedding, we obtain

$$R(x,y,Y) = \frac{Y\,(1-2Y-Yx(yY-1))}{1-x-Y-Yx(yY-1)},$$

again with $A = I$, which specializes to the other rational function in Example 61 when we substitute $y = 1/Y$. ◁

Example 75. A *polygon dissection* is a non-crossing configuration whose vertices are connected in sequence to form a polygon. If $\Delta = \{\delta_1,\ldots,\delta_m\}$ is a collection of 2-connected graphs then Velona [69] proves that algebraicity of the generating function $F_\Delta(x_1,\ldots,x_m,y)$ for the number of polygon dissections where y marks the size of the dissection and x_k marks the number of occurrences of δ_k as a pattern (subgraph up to relabelling of vertices), and gives a method to compute its minimal polynomial.

For instance, the generating function enumerating dissections by size and number of 3-cycles is a root of the polynomial

$$P(x,y,Y) = (1-x)Y^3 + (x+1)yY^2 - Yy^2(1+y) + y^4.$$

We can then apply Safonov's algorithm to get an embedding using $R = G/H$ such that

$$\begin{aligned} G =& ((3Y^4+7Y^3+6Y^2+3Y+1)xy^3Y^3 - (3Y^4+7Y^3+6Y^2+3Y+1 \\ &+ (2Y^4+3Y^3+2Y^2+1)xY)y^2Y - (2Y^2+2Y+1)yY + Y + 1)y^2Y^2 \end{aligned}$$

and

$$H = (Y^3+3Y^2+3Y+1)xy^3Y^3 - (Y^3+3Y^2+3Y+(Y^2+2Y+1)xY+1)y^2Y - (Y+1)yY + 1$$

with corresponding matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Note that this example would not have been able to use Lemma 62 directly and required Safonov's algorithm to loop. ◁

Example 76. If $P(\mathbf{x}, Y) = a_2(\mathbf{x})Y^2 + a_1(\mathbf{x})Y + a_0(\mathbf{x})$ is quadratic then Safonov's algorithm only performs its loop if the discriminant $d(\mathbf{x}) = a_1(\mathbf{x})^2 - 4a_0(\mathbf{x})a_2(\mathbf{x})$ vanishes at the origin. Because $f(\mathbf{0}) = 0$, we know that $a_0(\mathbf{0}) = 0$, so the loop runs if and only if $a_1(\mathbf{0}) = 0$. In general, we can achieve r loops of Safonov's algorithm with any generating function of the form

$$\left(\frac{p(\mathbf{x}) - b(\mathbf{x})\sqrt{m(\mathbf{x})}}{q(\mathbf{x})}\right) x_1^{k_1} \dots x_r^{k_r}$$

such that $k_1, \dots, k_r \geq 1$. ◁

4.2 Examples

This section provides a few more examples of algebraic generating functions and the application of methods by Denef and Lipshitz and Safonov. The notebook corresponding to this work can be found at

https://github.com/Tia1300/masters_thesis/.

Example 77. We may return again to our example of enumerating the number of non-empty planar rooted binary trees with n vertices and k leaves. As seen in Example 12, we have that

$$B(z, t) = \frac{1 - \sqrt{1 - 4t + (1 - z)4t^2}}{2t} - 1$$

which has corresponding minimal polynomial $P(z, t, Y) = tY^2 + (2t - 1)Y + zt$. Since the class we are counting is non-empty we know that $B(z, t)$ has no constant term. We also note that $P_Y(0, 0, 0) = -1 \neq 0$. Thus, this is an application of Furstenberg and hence, either Denef and Lipshitz can be applied with $\phi = F$, $Q = P$ and $W = Y$ or Safonov can be applied with no loops to achieve an embedding of

$$R(z, t, Y) = \frac{(2Y^2t + 2Yt - 1)Y}{Y^2t + Ytz + 2Yt - 1)}$$

where $[z^{an}t^{bn}]B(z, t) = [z^{an}t^{bn}Y^{(a+b)n}]R(z, t, Y)$.

◁

Example 78. An *assembly tree* $\mathcal{T}(G)$ for a connected graph G is a rooted tree whose nodes are labelled by subsets of the vertex set $V(G)$ of G such that the leaves of $\mathcal{T}(G)$ are labelled by the singleton sets $\{v\}$ for $v \in V(G)$, the root of $\mathcal{T}(G)$ is labelled with $V(G)$, and each non-leaf node in $\mathcal{T}(G)$ has at least two children and is labelled by the union of the labels on its children. Assembly trees are named because they encode a process that builds G from its individual vertices. An *edge gluing rule* for an assembly tree adds the property that each non-leaf node has exactly two children, labelled by U_1 and U_2, such that there exists an edge between a vertex from U_1 and a vertex from U_2 in the graph G. Bóna and Vince [10] study the generating functions of assembly trees with the edge gluing rule for certain families of graphs. In particular, they prove that the (exponential) generating function for the number of assembly trees of the complete bipartite graph $K_{k,n}$ is

$$F(x,y) = \sum_{k,n\geq 0} f_{k,n}x^k y^n = 1 - \sqrt{(1-x)^2 + (1-y)^2 - 1}.$$

The minimal polynomial for this generating function is

$$P(x,y,Y) = (Y-1)^2 - ((1-x)^2 + (1-y)^2 - 1)$$

and thus $P_Y(0,0,0) = -2 \neq 0$. This is an application of Furstenberg and therefore, either Denef and Lipshitz can be applied with $\phi = F$, $Q = P$ and $W = Y$ or Safonov can be applied with no loops to achieve an embedding of

$$R(x,y,Y) = \frac{2(1-Y)Y}{Yx^2 + Yy^2 - Y - 2x - 2y + 2}$$

where $[x^{rn}y^{sn}]F(x,y) = [x^{rn}y^{sn}Y^{(r+s)n}]R(x,y,Y)$.

◁

Example 79. We say that a leaf in a rooted plane tree is *old* if it is the leftmost child of its parent, and *young* otherwise. Chen et al. [15] refine the Narayana numbers by enumerating the number $f_{k,\ell,n}$ of plane trees with k old leaves, ℓ young leaves, and n edges, showing that the generating function of this sequence satisfies

$$F(x,y,z) = \sum_{k,\ell,n\geq 0} f_{k,\ell,n}x^k y^\ell z^n = 1 + \frac{z(F(x,y,z) - 1 + x)}{1 - z(F(x,y,z) - 1 + y)}.$$

The minimal polynomial of this function is

$$((Y+1)-1)(1 - z((Y+1) - 1 + y)) - z((Y+1) - 1 + x)$$

and, as with the previous example, $P_Y(0,0,0) \neq 0$. Therefore, either Denef and Lipshitz can be applied with $\phi = F$, $Q = P$ and $W = Y$ or Safonov can be applied with no loops to achieve an embedding of

$$R(x,y,z,Y) = \frac{Y^2yz + 2Y^2z + Yz - 1)Y}{Y^2yz + Y^2z + Yxz + Yz - 1}$$

where $[x^{rn}y^{sn}z^{tn}]F(x,y) = [x^{rn}y^{sn}z^{tn}Y^{(r+s+t)n}]R(x,y,z,Y)$. We highlight this example as it demonstrates that both Denef and Lipshitz as well as Safonov's algorithm indeed work well in non-bivariate settings. ◁

Example 80. An RNA *secondary structure* is a two-dimensional approximation of the three dimensional structure of an RNA strand, where each nucleotide in the strand is paired to at most one other nucleotide, and no two pairings can cross when drawn as arcs in a line. RNA secondary structures can be represented as dot-bracket sequences such as .((...)..) where dots represent unpaired nucleotides and pairs of parenthesis represent paired nucleotides (in this example the third and seventh nucleotides are paired, as are the second and last nucleotides).

While the most prominent method of predicting the secondary structure corresponding to an RNA strand is to use an energy model, an alternative method is to use a *stochastic context-free grammar*. One particularly successful grammar is the `KH99` grammar from Knudsen and Hein [50] defined by

$$S \to LS \text{ with probability } p_1 \text{ or } L \text{ with probability } q_1 \tag{4.12}$$

$$L \to (F) \text{ with probability } p_2 \text{ or } . \text{ with probability } q_2 \tag{4.13}$$

$$F \to (F) \text{ with probability } p_3 \text{ or } LS \text{ with probability } q_3, \tag{4.14}$$

where $q_i = 1 - p_i$. Heitsch and Poznanovic [43] used the bivariate generating function

$$F(x,y) = \frac{(1-p_1q_2x)(1-p_3z^2y) - \sqrt{(1-p_1q_2x)^2(1-p_3z^2y)^2 - 4p_2q_1q_2q_3x^3y(1-p_3z^2y)}}{2p_2q_3z^2y},$$

with x tracking number of nucleotides and y tracking number of base pairs, to show that many different RNA structural features are normally distributed according to this grammar, and find interesting relationships between the means of the features. While the minimal polynomial of this example,

$$P(x,y,Y) = Yp_1p_3q_2yx^3 - Y^2p_2q_3yx^2 + p_3q_1q_2yx^3 - Yp_3yx^2 - Yp_1q_2x - q_1q_2x + Y,$$

does satisfy that $P_Y(0,0,0) \neq 0$, we highlight this example to demonstrate that both the methods and code of Denef and Lipshitz and Safonov work well with parameters. In either case, we achieve the embedding

$$R(x,y,Y) = \frac{(Y^4p_1p_3q_2x^3y - 2Y^4p_2q_3x^2y - Y^3p_3x^2y - Yp_1q_2x + 1)Y}{Y^4p_1p_3q_2x^3y + Y^3p_3q_1q_2x^3y - Y^4p_2q_3x^2y - Y^3p_3x^2y - Yp_1q_2x - q_1q_2x + 1}$$

where $[x^{rn}y^{sn}]F(x,y) = [x^{rn}y^{sn}Y^{(r+s)n}]R(x,y,Y)$. ◁

Example 81. Recall that a Dyck path P is a path from $(0,0)$ to $(2n,0)$ for some integer n where P is comprised of $2n$ unit steps from $\{\nearrow, \searrow\}$. Additionally, P never crosses below the line $y = 0$ (although it may touch the line). Then, the semi-length $\ell(P)$ of such a path is the total number of up steps in the path, or n. We also define $\rho(P)$ to be the number of peaks of the path P — in other words, the number of $\nearrow$ steps immediately followed by a $\searrow$ step. A valley is defined similarly as a $\searrow$ step immediately follows by a $\nearrow$ step.

In [32], the authors consider the y-coordinates of all of the valleys, listed in an array $\nu = (\nu_1, \nu_2, \ldots, \nu_k)$. A *restricted d-Dyck path* is defined as any path where $\nu_i - \nu_{i-1} \geq d$ for $2 \leq i \leq k$, and the set of such paths is denoted $\mathcal{D}_d$. It turns out that generating functions encoding restricted d-Dyck paths for fixed $d \geq 0$ are rational and they are algebraic when $d < 0$.

Among other results, the authors derive that when $d = -1$, the generating function

$$L(x,y) = \sum_{P \in \mathcal{D}_{-1}} x^{\ell(P)} y^{\rho(P)}$$

satisfies the minimal polynomial

$$P = Y^2x^2y + Yx^2y^2 + Y^2x^2 - 2Y^2xy + 2Yx^2y - Yxy^2 + x^2y^2 - 2Y^2x - 3Yxy - xy^2 + Y^2 + Yy.$$

We note that this example does not have that $P_Y(0,0,0) \neq 0$. However, we can apply the results of Example 69 to find $\widetilde{D}(x,y) = (x^3y^2 - x^2y^2 + 2x^2y + 2xy + x - 1)(x-1)$ and thus $Q(x,y,Y) = (Y-1)^2 - \widetilde{D}(x,y)$ and $W = \frac{-B(1-Y)+y\widetilde{D}(x,y)}{2A(1-Y)}$. Using these inputs, we can get an embedding of

$$R(x,y,Y) = \frac{G(x,y,Y)}{H(x,y,Y)}$$

where

$$G(x,y,Y) = (Y^4x^3y^2 - Y^3x^2y^2 + 2Y^2x^2y + Y^2xy + Yxy + 2Yx - x - 1)(Yx-1)Y^2y$$

and

$$\begin{aligned} H(x,y,Y) = (&Y^5x^4y^2 - 2Y^4x^3y^2 + 2Y^3x^3y + Y^3x^2y^2 + Yx^2 - 2Yxy - Y - 2x + 2) \\ &\cdot(Y^3x^2y + Y^2x^2 - 2Y^2xy - 2Yx + 1). \end{aligned}$$

We can also apply Safonov's algorithm directly to this example as $f(0,0) = 0$. Using Safonov's algorithm gives an embedding of

$$R(x,y,Y) = \frac{G(x,y,Y)}{H(x,y,Y)}$$

where

$$\begin{aligned} G(x,y,Y) = (&2Y^6x^2y^3 + Y^5x^2y^3 + 2Y^5x^2y^2 + 2Y^4x^2y^2 - 4Y^4xy^2 \\ &-Y^3xy^2 - 4Y^3xy - 3Y^2xy + 2Y + 1)Y^2y \end{aligned}$$

and

$$H(x, y, Y) = (Y^6x^2y^3 + Y^5x^2y^3 + Y^5x^2y^2 + 2Y^4x^2y^2 - 2Y^4xy^2 + Y^3x^2y^2 \\ -Y^3xy^2 - 2Y^3xy - 3Y^2xy - Yxy + Y + 1)$$

with

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

◁

Example 82. In their 2024 paper, Ekhad and Zeilberger [24] explore Werner Krandick's binary tree jump statistics. In particular, we are interested in their result for the generating function of binary trees enumerated by the number of internal vertices (x), the depth of the right-most leaf (y) and the number of jumps in a DFS of the tree (z). They provide a generating function

$$F(x, y, z) = -\frac{-xyz + xy + \sqrt{z^2y^2x^2 - 2zy^2x^2 - 2zy^2x + y^2x^2 - 2y^2x + y^2} + y - 2}{2(xyz + y^2x - xy - y + 1)}.$$

Note that most of the paper focuses just on the jumps and internal vertices, setting $y = 1$. That being said, we can find results for $F(x, y, z) - 1$. Specifically, we can get

$$P(x, y, z, Y) = (Y^2xy^2 + Y^2xyz - Y^2xy + 2Yxy^2 + Yxyz - Y^2y - Yxy + xy^2 + Y^2 - Yy) \\ \cdot(xy^2 + xyz - xy - y + 1).$$

We note that again this example does not have that $P_Y(0, 0, 0) \neq 0$. However, we can apply the results of Example 69 to find $\widetilde{D}(x, y, z) = x^2z^2 - 2x^2z + x^2 - 2xz - 2x + 1$ and thus $Q(x, y, Y) = (Y - 1)^2 - \widetilde{D}(x, y, z)$ and $W = \frac{-B(1-Y)+((xy^2+xyz-xy-y+1)y)\widetilde{D}(x,y,z)}{2A(1-Y)}$. Using these inputs, we can get an embedding of

$$R(x, y, z, Y) = \frac{G(x, y, z, Y)}{H(x, y, z, Y)}$$

where

$$G(x, y, z, Y) = (Y^4x^2z^2 - 2Y^3x^2z + 2Y^3xy + Y^3xz + Y^2x^2 \\ -2Y^2xy - 3Y^2xz - Y^2x - Yx - Y + 2)Yy$$

and

$$H(x, y, z, Y) = (Y^3x^2z^2 - 2Y^2x^2z + Yx^2 - 2Yxz - Y - 2x + 2) \\ \cdot(Y^3xy^2 + Y^3xyz - Y^2xy - Yy + 1).$$

We can also apply Safonov's algorithm directly to this example as $f(0,0,0) - 1 = 0$. Using Safonov's algorithm gives an embedding of

$$R(x,y,z,Y) = \frac{G(x,y,z,Y)}{H(x,y,z,Y)}$$

where

$$\begin{aligned} G(x,y,z,Y) = (2Y^6xy^3z + 2Y^5xy^3 + Y^4xy^2z - 2Y^4xy^2 + 2Y^3xy^2 \\ -Y^2xy - 2Y^2y + 2Y - 1)Y^2y \end{aligned}$$

and

$$\begin{aligned} H(x,y,z,Y) = Y^6xy^3z + Y^5xy^3 + Y^4xy^2z - Y^4xy^2 + 2Y^3xy^2 - Y^2xy \\ -Y^2y + Yxy + Y - 1 \end{aligned}$$

with

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

◁

4.3 Other Methods

Although embedding methods work for all algebraic generating functions, the downside of this approach is that, once embedded, the rational generating function obtained may be quite difficult to find asymptotics from. The Denef and Lipshitz method does provide a way to attempt to obtain embeddings which allow for easy computation of asymptotics (namely through changing what ϕ is used as seen in Example 68), however finding a simpler embedding is a process of trial and error. As such, it is beneficial to consider other methods to find asymptotics of algebraic generating functions. In this section we briefly outline a few such methods.

Bivariate contour manipulations.

The most obvious method of attack on such a problem is to try to mimic the custom contour of integration used by Flajolet and Odlyzko but in more dimensions. This was done in the bivariate case in the PhD thesis work of Greenwood [39]. For the results of that work to apply, an explicit formula for the generating function in the form $H^{-\beta}$ is required for some $\beta \notin \mathbb{Z}_{<0}$. The dominant singularity of $1/H$ must be a smooth minimal critical point, H

must be analytic near the origin, and H must satisfy some partial derivative constraints at the critical point. One benefit of the proof approach is that it can be extended to functions that include logarithms [40], which remains a challenge for the other methods described here. On the other hand, the results in [39] are currently limited to bivariate generating functions. While it should be possible to extend this proof method to generating functions with any number of variables, technical obstructions have blocked this extension so far.

Delta Analytic Functions

A direct generalization of the work of Flajolet and Odlyzko to multivariate problems was made by Chen [14]. Generalizing the Δ-domain ("Camembert domain") of [28] to any product of such sets, one for each variable, Chen derives detailed asymptotic transfer theorems valid for large sets of directions. However, the definition of admissible domain is very restrictive: "In particular, it implies that the dominant singularity (for any reasonable definition of the term) is unique and independent of the [direction] taken. This is in stark contrast with the case of rational functions, where the dominant singularities (a.k.a. contributing critical points) generically depend on the direction of the diagonal limit taken." [14, Section 4] Under this assumption, the exponential growth rate is constant across directions, which is unlikely to occur in practice.

Implicit Integration

Instead of lifting to higher dimensions — which can destroy nice properties of generating functions such as sparsity and coefficient positivity — or using explicit contour deformations — which can be hard to generalize — Baryshnikov, Jin, and Pemantle [2] propose a method to compute asymptotics of algebraic functions by integrating over curves in algebraic varieties. Indeed, if F is an algebraic function satisfying the polynomial equation $P(\mathbf{x}, F(\mathbf{x})) = 0$ then the Cauchy integral expression

$$f_{\mathbf{i}} = \frac{1}{(2\pi i)^d} \int_T F(\mathbf{x}) \frac{d\mathbf{x}}{\mathbf{x}^{\mathbf{i}+\mathbf{1}}}$$

for the power series coefficients of F, where the *torus* T is a product of arbitrarily small circles $|x_k| = \epsilon$, can be *lifted* to an integral expression

$$f_{\mathbf{i}} = \frac{1}{(2\pi i)^d} \int_{\widetilde{T}} Y \frac{d\mathbf{x}}{\mathbf{x}^{\mathbf{i}+\mathbf{1}}} \tag{4.15}$$

where $\widetilde{T} = \{(\mathbf{x}, Y) : \mathbf{x} \in T \text{ and } Y = F(\mathbf{x})\}$ is the *lifted torus* lying in the d-dimensional subset $\widetilde{V} \subset \mathbb{C}^{d+1}$ defined by the vanishing of P. The integral in (4.15) is simpler than the Cauchy integral because it does not contain the singularities or branch cuts of F, however the domain of integration $\widetilde{T}$ can only be deformed within $\widetilde{V}$ (or another complex d-manifold which contains it) without a priori changing the value of the integral. In practice we deform

$\widetilde{T}$ by deforming T in $\mathbb{C}^d$ and lifting to $\mathbb{C}^{d+1}$, which requires accounting for the geometric structure of $\widetilde{V}$ and thus brings back a consideration of the algebraic properties of P. The benefit of this method is that, when it can be applied, asymptotics will be obtained. In ongoing work related to this chapter, the authors are creating an implementation of this method so that it may also be compared to the embedding methods.

Chapter 5

SageMath Code

SageMath (https://www.sagemath.org) is a Python-based open-source computer algebra system which incorporates many other open-source packages such as NumPy, matplotlib, Maxima, etc. It is an extremely useful tool for not only quick mathematics calculations, but also more complex results. The open-source nature allows for the development of many new packages for the software, and there is already a package by Hackl et al. [41] which computes asymptotics of rational generating functions under certain constraints. In this chapter we describe how to use the code associated with Chapter 3 and Chapter 4 to obtain further results. Note that we will briefly redefine various aspects to make this chapter self-contained. All notebooks associated to this chapter can be found at

https://github.com/Tia1300/masters_thesis/.

The code provided in these notebooks contains the functions required to use the methods from Chapter 3 and Chapter 4 as well as companion examples and notes for those chapters.

5.1 Multivariate Limit Theorems

The goal of this work is to aid in proving multivariate Local Central Limit Theorems (LCLTs). We note that in d-dimensions, a *multivariate LCLT* compares probabilities to *multivariate normal density functions*

$$\phi_{\boldsymbol{\mu},\boldsymbol{\Sigma}}(\mathbf{x}) = \frac{1}{(2\pi)^{d/2}} \exp\left(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right).$$

The *classical multivariate CLT* states that if $(\mathbf{A}_n)$ is a sequence of d-dimensional independent identically distributed random variables with an expected value vector $\boldsymbol{\mu}$ and a positive definite covariance matrix $\boldsymbol{\Sigma}$ then the sequence of d-dimensional random variables

$$\mathbf{X}_n = \frac{\mathbf{A}_1 + \cdots + \mathbf{A}_n - n\boldsymbol{\mu}}{\sqrt{n}}$$

converges in distribution to the multivariate normal distribution with density function $\phi_{0,\mathbf{\Sigma}}(\mathbf{x})$.

The code associated with this section provides three main functions. The first two prove LCLTs using the following result, which is a restatement of Theorem 51.

Theorem 83 (Automatic LCLT Theorem)**.** *Let $F(\mathbf{z},t) = \frac{G(\mathbf{z},t)}{H(\mathbf{z},t)}$ be a ratio of functions where*

$$H(\mathbf{z},t) = 1 - q(t) - \sum_{k=1}^{d} q_k(t) z_k,$$

let $P(t) = H(\mathbf{1},t)$, and let ρ be the smallest positive root of $P(t)$. Suppose that

- *each of the $q_i(t)$ is a non-zero polynomial vanishing at the origin and $q(t)$ is a complex-valued analytic function for $|t| \leq \rho$ that vanishes at the origin,*
- *the power series expansion of $S(\mathbf{z},t) = 1 - H(\mathbf{z},t) = q(t) + \sum_k q_k(t) z_k$ at the origin has non-negative coefficients,*
- *the exponents appearing in the power series $q(t)$ have greatest common divisor 1, and*
- *$G(\mathbf{1},\rho)$ is non-zero.*

As $n \to \infty$, the maximum coefficient of $[t^n]F(\mathbf{z},t)$ as a polynomial in $z_1, \ldots, z_d$ approaches

$$A_n = \rho^{-n} n^{-d/2} \frac{G(\mathbf{1},\rho)}{-\rho P'(\rho)(2\pi)^{d/2}\sqrt{\det \mathcal{H}}},$$

where $\mathcal{H}$ is the non-singular $d \times d$ matrix

$$\mathcal{H}_{i,j} = \begin{cases} \dfrac{\rho q_i(\rho) q_j(\rho) P''(\rho) - (q_j(\rho) q_i'(\rho)\rho + q_i(\rho) q_j'(\rho)\rho - q_i(\rho) q_j(\rho)) P'(\rho)}{\rho^2 P'(\rho)^3} & i \neq j \\ \dfrac{\rho q_j(\rho)^2 P''(\rho) - (2 q_j(\rho) q_j'(\rho)\rho - q_j(\rho)^2) P'(\rho) - q_j(\rho)\rho P'(\rho)^2}{\rho^2 P'(\rho)^3} & i = j \end{cases} \tag{5.1}$$

whose determinant

$$\det \mathcal{H} = \frac{(-1)^d \left(\prod_{k=1}^d q_k(\rho)\right) \left[(q(\rho) - 1)\left(\rho P''(\rho) + P'(\rho) + \rho \sum_{k=1}^d \frac{q_k'(\rho)^2}{q_k(\rho)}\right) + q'(\rho)^2 \rho\right]}{P'(\rho)^{d+2} \rho^{d+1}} \tag{5.2}$$

is non-zero. Furthermore,

$$\sup_{s_1,\ldots,s_d \in \mathbb{N}} \left| \frac{[z_1^{s_1} \cdots z_d^{s_d} t^n] F(z_1, \ldots, z_d, t)}{A_n} - v_n(s_1, \ldots, s_d) \right| \to 0$$

where

$$v_n(\mathbf{s}) = \exp\left[-\frac{(\mathbf{s} - n\mathbf{m})\mathcal{H}^{-1}(\mathbf{s} - n\mathbf{m})^T}{2n}\right] \qquad \textit{for} \qquad \mathbf{m} = \left(\frac{-q_1(\rho)}{\rho P'(\rho)}, \ldots, \frac{-q_d(\rho)}{\rho P'(\rho)}\right).$$

1. Characterize the smallest positive solution $t = \rho$ of $H(\mathbf{1}, t) = 0$ and define $\mathbf{m} = \left(\frac{H_{z_1}(\mathbf{1},\rho)}{\rho H_t(\mathbf{1},\rho)}, \ldots, \frac{H_{z_d}(\mathbf{1},\rho)}{\rho H_t(\mathbf{1},\rho)}\right)$

↓

2. Show that $H(s, \ldots, s, s\rho) \neq 0$ for all $0 < s < 1$

↓

3. Show that no other critical point has the same coordinate-wise modulus as $(\mathbf{1}, \rho)$

↓

4. Prove that the determinant of the phase Hessian $\mathcal{H}$ is non-zero at $(\mathbf{1}, \rho)$ in the direction $\mathbf{r} = (\mathbf{m}, 1)$

↓

5. Prove that $H_t(\mathbf{1}, \rho)$ and $G(\mathbf{1}, \rho)$ are non-zero and read off the local central limit theorem from (3.6)

ù

Figure 5.1: A schema to prove LCLTs.

The third function attempts to automatically prove LCLTS for rational generating functions using the schema in Figure 5.1.

The code for applying Theorem 83 has two key functions. The first, `getLCLT`, applies Theorem 83 to a specific generating function. To call this function, assume that we wish to apply the theorem to

$$F(\mathbf{z}, t) = \frac{G}{1 - q(t) - \sum_{k=1}^{d} q_k(t) z_k}.$$

`getLCLT` takes in four parameters:

- G, the numerator,
- q_k, a list of our $q_k(t)$ in order from $k = 1$ to $k = d$,
- q_t, the analytic function $q(t)$ in closed form,
- vars, a list of the variables $z_1, \ldots, z_d$ and t used in our generating function, with t in the last position.

The function `getLCLT` verifies the majority of requirements for a LCLT, with the following caveats:

1. It does not check that the coefficients of the analytic function $q(t)$ are non-negative (ie that the analytic function is combinatorial).

```
sage: var('z t')
(z, t)
sage: A_n, m, ν_n, hes = getLCLT(1, [t], t, [z, t])
sage: show(A_n)
```

$$\frac{\sqrt{2}}{\sqrt{\pi}\left(\frac{1}{2}\right)^n\sqrt{n}}$$

```
sage: show(m)
```

(0.500000000000000)

```
sage: show(ν_n)
```

$$e^{\left(-\frac{1}{2}n+2s_0-\frac{2s_0^2}{n}\right)}$$

Figure 5.2: An example of applying getLCLT on $\frac{1}{1-t-zt}$.

2. It assumes that $q(t)$ is analytic for $|t| \leq \rho$.

3. It attempts to certify aperiodicity of $q(t)$, but does not prove if something is periodic. If the certification for aperiodicity fails, a warning is printed.

4. It assumes that the variable orderings correspond to q_k and that the last variable is the one representing t.

If the function fails one of its verifications (one of the assumptions of Theorem 83 bar those listed in the caveats), an error is printed and the function returns. However, assuming all checks pass, `getLCLT` returns four values, A_n, $\mathbf{m}$, $\nu_n(\mathbf{s})$ and $\mathcal{H}$, all as defined in Theorem 83.

Example 84. Consider applying `getLCLT` to the function

$$F(z,t) = \frac{1}{1-t-zt}$$

to recover a classical CLT. Then the numerator is $G = 1$. Since we have only $q_1(t) = t$ we get `q_k = [t]`. As well, since $q(t) = t$ and we have two variables z and t, our two remaining inputs are `q_t = t` and `vars = [z, t]`. Thus, we call `getLCLT(1, [t], t, [z, t])` and get the output displayed in Figure 5.2. ◁

The second key function for applying Theorem 83, `getSymLCLT`, is meant for quickly proving that families of generating functions satisfy an LCLT, and finding such an LCLT. The set up for this function is similar to that of `getLCLT`. In particular, we no longer assume that d is known, but rather treat it as a symbolic parameter. We again assume that we have

$$F(\mathbf{z},t) = \frac{G}{1-q(t)-\sum_{k=1}^{d} q_k(t)z_k},$$

however this time we may assume that we know $q_k(t)$ as a function of k, creating a family of generating functions. `getSymLCLT` takes in nine parameters:

- G_{sub}, the numerator with all variables set to 1 except t,
- q_k, the polynomial $q_k(t)$ as a symbolic expression in k,
- q_t, the analytic function $q(t)$ in closed form,
- the variables t, d and k, and
- either variables or values which provide ρ, P' and P'' as defined in Theorem 83.

The function `getSymLCLT` makes the same caveats as `getLCLT`, with the addition that it is assumed that the q_k pattern produces polynomials. This is assumed because patterns are allowed to have other variables which may be representing various values depending on the combinatorial context. As with `getLCLT`, `getSymLCLT` prints an error and returns should a verification fail. Assuming all checks pass, `getSymLCLT` returns four values:

- A_n,
- a symbolic expression for determining the entries of $\mathbf{m}$ in terms of k,
- a symbolic expression for determining the entries $\mathcal{H}_{i,j}$ in terms of i, j and k, and
- a symbolic expression for determining the entries $\mathcal{H}_{j,j}$ in terms of j and k.

Example 85. Consider applying `getSymLCLT` on the family of generating functions

$$F(\mathbf{z}, t) = \frac{1}{1 - (z_1 + z_2 + \ldots + z_d)t - (\ell - d)t},$$

which count the number of strings in an alphabet of size ℓ with d letters tracked. Then our numerator, no matter the substitution, is 1. Each of our $q_k(t) = t$ and thus our function $q_k = t$. We have $q(t) = (\ell - d)t$ in our family and thus the input for q_t is $(\ell - d)t$. Our variables are t, d, k and from our definitions, $\rho = \frac{1}{\ell}$, $P' = -\ell$ and $P'' = 0$. Thus, we call `getSymLCLT`$(1, t, (\ell - d)t, t, d, k, \frac{1}{\ell}, -\ell, 0)$ and see the output in Figure 5.3. ◁

5.1.1 LCLT Package

Our next code aims to prove LCLTs more generally. In particular, all that is required is a rational generating function which is combinatorial (i.e., all coefficients are non-negative) and the `getLCLT` function of this package will either return an LCLT (thus proving that the LCLT holds and is correct) or will raise an error. In order to use this package, the sage_acsv package must be installed [41].

`getLCLT` requires two inputs and has three optional parameters:

- F, the rational function G/H in d variables,

```
sage: var('t k d l')
(t, k, d, l)
sage: A_n, mpat, H_ij, H_jj = getSymLCLT(1, t, (l-d)*t, t, d, k, 1/l, -l, 0)
sage: show(A_n)
```

$$\frac{\sqrt{(-l)^d}\, l^{n+\frac{1}{2}}}{(2\pi n)^{\frac{d}{2}} \sqrt{(-1)^d}\sqrt{-d+l}}$$

```
sage: show(m)
```

$\frac{1}{l}$

```
sage: show(H_ij)
```

$-\frac{1}{l^2}$

```
sage: show(H_jj)
```

$\frac{l-1}{l^2}$

Figure 5.3: An example of applying `getSymLCLT` on $\frac{1}{1-(z_1+z_2+\ldots+z_d)t-(\ell-d)t}$.

```
sage: var('z t')
(z, t)
sage: show(getLCLT(1/(1-z*t-t), t, as_symbolic = True))
```

$$\frac{\sqrt{2}2^n e^{\left(-\frac{(n-2s_0)^2}{2n}\right)}}{\sqrt{\pi n}}$$

Figure 5.4: A limit theorem for the number of 0s in binary strings of length n.

- `main_var`, the variable that marks the "size" of the objects (so that the limit theorem holds as the exponent of `main_var` goes to infinity), and
- `as_symbolic`, an optional parameter, set by default to `False`. If it is `True`, `getLCLT` returns the limit theorem as an expression from the symbolic ring in the variable n. If it is `False`, `getLCLT` returns a tuple (a, n^b, π^b, C, D, v) such that the local central limit theorem is specified by the function

$$f(\mathbf{s}) = a^n \cdot n^b \cdot \pi^b \cdot C \cdot \exp\left(-\frac{(\mathbf{s}-nv)D(\mathbf{s}-nv)^T}{2n}\right).$$

Remark 86. The calculations in `getLCLT` rely on methods from sage_acsv which use Gröbner basis calculations. If Macauly2 or msolve is installed, the back-end settings of sage_acsv can be modified to use either of these packages and speed up the calculations.

Figure 5.4, Figure 5.5, Figure 5.6 and Figure 5.7 show a variety of examples of applying `getLCLT`.

```
sage: var('x y t')
(x, y, t)
sage: F = 1/(1-x*t-y*t^2-t^3/(1-t))
sage: show(getLCLT(F, t, as_symbolic = True))
```

$$\frac{8\sqrt{\frac{1}{35}}2^n e^{\left(-\frac{7(n-4s_0)^2+5(n-8s_1)^2}{70n}\right)}}{\pi n}$$

Figure 5.5: A limit theorem for the number s_0 of 1s and s_1 of 2s in compositions of length n.

```
sage: var('x y')
(x, y)
sage: m = 2
sage: F = (1-x^m)/(1-x-x^m*(1-x^m)*y)
sage: show(getLCLT(F, x, as_symbolic = True))
```

$$\frac{3.907129588084731? \cdot 1.324717957244746?^n e^{\left(-\frac{(-0.4114955886626458?n+s_0)(-24.09277082519922?n+58.54928093761673?s_0)}{2n}\right)}}{\sqrt{\pi n}}$$

Figure 5.6: Applying `getLCLT` to a generating function for ways to maximally pack a path of length m in a path of length n to get a limit theorem for the number of copies of the smaller path, using the generating function from [63]. We note that this example has a maximum in a direction with irrational coordinates.

```
sage: var('x y')
(x, y)
sage: F = (x*y*(1-x^3))/((1-x)^4-x*y*(1-x-x^2+x^3+x^2*y))
sage: show(getLCLT(F, x, as_symbolic = True))
```

$$\frac{0.3450475264519037? \cdot 3.205569430400590?^n e^{\left(-\frac{(-0.4530745716375183?n+s_0)(-3.296142343669530?n+7.275054814390697?s_0)}{2n}\right)}}{\sqrt{\pi n}}$$

Figure 5.7: A limit theorem for the number of rows in horizontally convex polyominoes of size n using the generating function from [70].

5.2 Algebraic Generating Functions

The goal of the work from this section is to provide methods for finding asymptotics of algebraic generating functions. In particular, we look at multivariate functions

$$F(\mathbf{x}) = \sum_{\mathbf{i}\in\mathbb{Z}^d} f_{\mathbf{i}}\mathbf{x}^{\mathbf{i}}$$

such that there exists some minimal polynomial $P(\mathbf{x}, Y)$ with $P(\mathbf{x}, F) = 0$. There are two main implementations of methods for finding asymptotics of algebraic generating functions that have been developed for the work presented in this thesis. Both methods involve embedding the algebraic generating function as a sub-series of a higher dimensional rational function.

5.2.1 Denef and Lipshitz

The first such methods was provided by Denef and Lipshitz in the 1980s. Note that the theorem below is a re-statement of Theorem 64(i).

Theorem 87 (Denef and Lipshitz [19, Theorem 6.2])**.** *Let $f(\mathbf{x})$ be any algebraic power series in d variables. There exists an algebraic power series $\phi(\mathbf{x})$ vanishing at the origin and a rational function $W(\mathbf{x}, Y)$ such that $W(\mathbf{x}, \phi(\mathbf{x})) = f(\mathbf{x})$ and the minimal polynomial $Q(\mathbf{x}, Y)$ of ϕ satisfies $Q_Y(\mathbf{0}, 0) \neq 0$. If*

$$R(\mathbf{x}, Y) = \frac{Y \cdot W(Y\mathbf{x}, Y)Q_Y(Y\mathbf{x}, Y)}{Q(Y\mathbf{x}, Y)} \tag{5.3}$$

for any such rational function W then $[\mathbf{x}^{\mathbf{r}}]f(\mathbf{x}) = [\mathbf{x}^{\mathbf{r}}Y^{|\mathbf{r}|}]R(\mathbf{x}, Y)$ for all $\mathbf{r} \in \mathbb{N}^d$, where $|\mathbf{r}| = r_1 + \cdots + r_d$.

We note that although Denef and Lipshitz did not provide a constructive method for finding ϕ, we provide code that will perform the embedding once such a ϕ is determined and both Q and W have been calculated from ϕ. Specifically, the function `embed_algebraic_DL` is provided, which embeds a d-variable algebraic series $f(\mathbf{x})$ as a diagonal of a $(d + 1)$-variable rational function $R(\mathbf{x}, Y)$. This function requires specific input whose existence is guaranteed by results of Denef and Lipshitz. The function `embed_algebraic_DL` takes in five parameters:

- P, a polynomial in $\mathbb{Q}[\mathbf{x}, Y]$ such that $P(\mathbf{x}, f(\mathbf{x})) = 0$,
- Q, a polynomial in $\mathbb{Q}[\mathbf{x}, Y]$ such that $Q(\mathbf{0}, 0) = 0$, $Q_Y(\mathbf{0}, 0) \neq 0$ and vanishing at the origin such that $Q(\mathbf{x}, \phi(\mathbf{x})) = 0$,

```
sage: var('x Y')
(x, Y)
sage: P = Y^2*x-Y+1
sage: Q = Y^2*x+2*Y*x-Y+x
sage: W = Y+1
sage: ratDL = embed_algebraic_DL(P, Q, W, Y=Y)
sage: show(ratDL)
```
$\frac{(2Y^2x+2Yx-1)(Y+1)}{Y^2x+2Yx+x-1}$

Figure 5.8: An example of embedding $\frac{1-\sqrt{1-4x}}{2x}$ using Denef and Lipshitz's method.

- W, a rational function in $\mathbb{Q}(\mathbf{x}, Y)$ such that $W(\mathbf{x}, \phi(\mathbf{x})) = f(\mathbf{x})$ for a power series $\phi(\mathbf{x})$,
- Y, an optional parameter which is the variable of $P(\mathbf{x}, Y)$ for which $Y = f(\mathbf{x})$ is a root. By default, the final variable returned by `P.variables()`, and
- `params`, an optional parameter which is a list of symbolic variables appearing in P that are considered parameters, meaning they remain after coefficient extraction.

This function outputs a $(d+1)$-variable rational function $R(\mathbf{x}, Y)$ such that $[\mathbf{x}^\mathbf{r}]f(\mathbf{x}) = [\mathbf{x}^\mathbf{r} Y^{|\mathbf{r}|}]R(\mathbf{x}, Y)$ for all $\mathbf{r}$ in $\mathbb{N}^d$. The function returns an error if P, Q, or W do not satisfy the required conditions.

Remark 88. The code verifies that $W(\mathbf{x}, \phi(\mathbf{x})) = m(\mathbf{x})$ for some m with $P(\mathbf{x}, m(\mathbf{x})) = 0$ but does not verify that $m(\mathbf{x})$ is the specific root $f(\mathbf{x})$ of P. An optional argument to check this will be added later.

Example 89. Consider applying `embed_algebraic_DL` to the Catalan generating function,

$$f(x) = \frac{1 - \sqrt{1-4x}}{2x}.$$

We note that the Catalan generating function has $f(0) = 0$. However, if we use $\phi(x) = \frac{1-\sqrt{1-4x}}{2x} - 1$ and hence $Q(x, Y) = Y^2x + 2Yx - Y + x$ and $W(x, Y) = Y + 1$, this is valid input for Denef and Lipshitz. Thus, we can call `embed_algebraic_DL(Y^2*x-Y+1, Y^2*x+2*Y*x-Y+x, Y+1, Y)` and get the output displayed in Figure 5.8. ◁

5.2.2 Safonov

Our second method for finding the asymptotics of algebraic generating functions is completely automatic, relying on the following result, which is a re-statement of Theorem 73.

Theorem 90 (Safonov [67, Theorem 1]). *If $f(\mathbf{x})$ is an algebraic generating function in d variables that vanishes at the origin then Algorithm 1 computes a rational function $R(\mathbf{x}, Y)$*

in $d+1$ variables and unimodular matrix $A \in \mathbb{N}^{d\times d}$ such that

$$[\mathbf{x}^{\mathbf{r}}]f(\mathbf{x}) = \left[\mathbf{x}^{A\mathbf{r}}Y^{|A\mathbf{r}|}\right] R(\mathbf{x},Y)$$

for all $\mathbf{r} \in \mathbb{N}^d$.

The function corresponding to Theorem 90 is `embed_algebraic_Safonov`, which embeds a d-variable algebraic series $f(\mathbf{x})$ as an "A-diagonal" of a $(d+1)$-variable rational function $R(\mathbf{x},Y)$ following the algorithm of Safonov. The function `embed_algebraic_Safonov` takes in four parameters:

- P, a polynomial in $\mathbb{Q}[x,Y]$ such that $P(\mathbf{x}, f(\mathbf{x})) = 0$,
- `terms`, a function that takes a natural number n and returns the series terms of $f(\mathbf{x})$ up to order n,
- Y, an optional parameter which is the variable of $P(\mathbf{x},Y)$ for which $Y = f(\mathbf{x})$ is a root. By default, the final variable returned by `P.variables()`, and
- `params`, an optional parameter which is a list of symbolic variables appearing in P that are considered parameters, meaning they remain after coefficient extraction.

This function outputs a $(d+1)$-variable rational function $R(\mathbf{x},Y)$ and $(d\times d)$-matrix A such that $[\mathbf{x}^{\mathbf{r}}]f(\mathbf{x}) = [\mathbf{x}^{\mathbf{r}}Y^{|A\mathbf{r}|}]R(\mathbf{x}x,Y)$ for all $\mathbf{r}$ in $\mathbb{N}^d$.

Example 91. Consider applying `embed_algebraic_Safonov` to the Narayana generating function

$$G(x,y) = \frac{1}{2x}\left(1 - x(y-1) - \sqrt{1-2x(y+1)+x^2(y-1)^2}\right),$$

which enumerates non-crossing partitions by set size and number of blocks, rooted ordered trees by edges and leaves, Dyck paths by semi-length and number of peaks, and more. Note that $G(x,y)-1$ satisfies the hypotheses of Theorem 90. So, we can call `embed_algebraic_Safonov` and get output as displayed in Figure 5.9. ◁

Remark 92. We note that in the above example, Safonov's algorithm did not loop, meaning that A was the identity. This occurred as the input to Safonov's algorithm had the additional property that $P_Y(\mathbf{0},0) \neq 0$. If the minimal polynomial has this property and $f(\mathbf{x})$ vanishes at the origin, we can obtain the same embedding as Safonov's algorithm using Denef and Lipshitz with $Q = P$ and $W = Y$.

Below, we provide a further example of applying Safonov's algorithm. In this Example, we see the power of the algorithm to overcome the complication of having $P_Y(\mathbf{0},0) = 0$.

Example 93. A *polygon dissection* is a non-crossing configuration whose vertices are connected in sequence to form a polygon. If $\Delta = \{\delta_1,\ldots,\delta_m\}$ is a collection of 2-connected graphs then Velona [69] proves that algebraicity of the generating function $F_\Delta(x_1,\ldots,x_m,y)$ for the number of polygon dissections where y marks the size of the dissection and x_k marks

```
sage: var('x y Y')
(x, y, Y)
sage: f = (1 + x - x*y - sqrt(1-2*x*(y+1)+x^2*(y-1)^2))/(2*x) - 1
sage: P = Y^2*x+Y*x*y + Y*x + x*y - Y
sage: def terms(n): return f.taylor([x, y], 0, n)
sage: ratSaf, A = embed_algebraic_Safonov(P, terms, Y=Y)
sage: show(ratSaf)
```

$$\frac{((Y^3y+2Y^3+Y^2)x-Y)}{((Y^2+(Y^2+Y)y+Y)x-1)}$$

```
sage: show(A)
```

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Figure 5.9: An example of embedding $\frac{1}{2x}\left(1-x(y-1)-\sqrt{1-2x(y+1)+x^2(y-1)^2}\right)-1$ using Safonov's algorithm.

the number of occurrences of δ_k as a pattern (subgraph up to relabelling of vertices), and gives a method to compute its minimal polynomial.

For instance, the generating function enumerating dissections by size and number of 3-cycles is a root of the polynomial

$$P(x, y, Y) = (1-x)Y^3 + (x+1)yY^2 - Yy^2(1+y) + y^4.$$

We can then apply Safonov using `embed_algebraic_Safonov` as seen in Figure 5.10. Note that in this instance we do not have f solved for directly so use the expansion of f up to degree 9 for y to create the `terms` function. ◁

```
sage: var('x y Y')
(x, y, Y)
sage: f =  (429*x^7 + 1287*x^5 + 495*x^4 + 1155*x^3 + 540*x^2 + 309*x +
    64)*y^9 + (132*x^6 + 330*x^4 + 120*x^3 + 216*x^2 + 80*x + 25)*y^8 +
    (42*x^5 + 84*x^3 + 28*x^2 + 35*x + 8)*y^7 + (14*x^4 + 21*x^2 + 6*x +
    4)*y^6 + (5*x^3 + 5*x + 1)*y^5 + (2*x^2 + 1)*y^4 + x*y^3 + y^2
sage: P = -x*Y^3 + x*Y^2*y - Y*y^3 + y^4 + Y^3 + Y^2*y - Y*y^2
sage: def terms(n): return f.taylor([x, y], 0, n)
sage: ratSaf, M = embed_algebraic_Safonov(P, terms, Y=Y)
sage: show(ratSaf)
```

$$\frac{((3Y^9+7Y^8+6Y^7+3Y^6+Y^5)xy^5-(3Y^7+7Y^6+6Y^5+3Y^4+Y^3+(2Y^7+3Y^6+2Y^5+Y^4)x)y^4-(2Y^5+2Y^4+Y^3)y^3+(Y^3+Y^2)y^2)}{((Y^6+3Y^5+3Y^4+Y^3)xy^3-(Y^4+3Y^3+3Y^2+(Y^4+2Y^3+Y^2)x+Y)y^2-(Y^2+Y)y+1)}$$

```
sage: show(M)
```

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Figure 5.10: An example of embedding $P(x, y, Y) = (1-x)Y^3+(x+1)yY^2-Yy^2(1+y)+y^4$ using Safonov's algorithm.

Chapter 6

Conclusion

The methods of analytic combinatorics in several variables, while perhaps daunting to some outside users due to its reliance on a wide breadth of mathematical techniques, provide some of the most powerful tools for the study of multivariate generating functions.

In the context of proving LCLTs, although Bender and Richmond [7] already provided techniques for a variety of combinatorial generating functions, verifying required conditions on analytic regions for the generating functions is too expensive to implement in a general, practical, algorithm. Using results of ACSV, a better understanding of the singular sets of multivariate generating functions yields such an algorithm, in addition to providing a framework for further generalizations. The goal of the work provided for LCLTs is that by putting the results of ACSV into context with past probabilistic work, giving a simple outline of how to apply the results, implementing the results in a computer algebra package, and using the results to prove a family of limit theorems, readers are inspired to look further into this growing area of combinatorics. The next step for this area of research is to extend results to other generating functions, such as the non-smooth or degenerate cases.

With regard to asymptotics of algebraic generating functions, this thesis provides a detailed description and implementations of two methods to embed algebraic generating functions as diagonals of rational generating functions. Specifically, it provides an implementation of Safonov's algorithm, which can be applied to all algebraic generating functions and constructively finds an embedding, and a method of Denef and Lipshitz, which allows for more user variation to help find an embedding. The next steps in this area of research are to finish implementations of the implicit method, search for and implement other methods to help encapsulate all asymptotics of algebraic generating functions and find ways to easily choose the best method for a given generating function.

References

[1] Ashok K. Agarwal. n-colour compositions. *Indian J. Pure Appl. Math.*, 31(11):1421–1427, 2000.

[2] Yuliy Baryshnikov, Kaitian Jin, and Robin Pemantle. Coefficient asymptotics of algebraic multivariable generating functions. *La Matematica*, 3(1):293–336, 2024.

[3] Yuliy Baryshnikov, Stephen Melczer, and Robin Pemantle. Stationary points at infinity for analytic combinatorics. *Found. Comput. Math.*, 22(5):1631–1664, 2022.

[4] Yuliy Baryshnikov, Stephen Melczer, and Robin Pemantle. Asymptotics of multivariate sequences IV: Generating functions with poles on a hyperplane arrangement. *Annals of Combinatorics*, 28:169–221, 2024.

[5] Yuliy Baryshnikov and Robin Pemantle. Asymptotics of multivariate sequences, part III: Quadratic points. *Adv. Math.*, 228(6):3127–3206, 2011.

[6] Edward A. Bender. Central and local limit theorems applied to asymptotic enumeration. *J. Combinatorial Theory Ser. A*, 15:91–111, 1973.

[7] Edward A. Bender and L. Bruce Richmond. Central and local limit theorems applied to asymptotic enumeration. II. Multivariate generating functions. *J. Combinatorial Theory Ser. A*, 34(3):255–265, 1983.

[8] Andrew C. Berry. The accuracy of the gaussian approximation to the sum of independent variates. *Transactions of the American Mathematical Society*, 49(1):122–136, 1941.

[9] Patrick Billingsley. *Probability and measure*. Wiley Series in Probability and Mathematical Statistics. John Wiley & Sons, Inc., New York, third edition, 1995.

[10] Miklós Bóna and Andrew Vince. The number of ways to assemble a graph. In *2013 Proceedings of the Tenth Workshop on Analytic Algorithmics and Combinatorics (ANALCO)*, pages 8–17. SIAM, 2013.

[11] Alin Bostan, Louis Dumont, and Bruno Salvy. Algebraic diagonals and walks: algorithms, bounds, complexity. *J. Symbolic Comput.*, 83:68–92, 2017.

[12] Mireille Bousquet-Mélou. Rational and algebraic series in combinatorial enumeration. In *International Congress of Mathematicians. Vol. III*, pages 789–826. Eur. Math. Soc., Zürich, 2006.

[13] Augustin Louis Cauchy. Mémoire sur les résultats moyens d'un très-grand nombre des observations. *Comptes rendus hebdomadaires des séances de l'Académie des Sciences*, 37, 1853.

[14] Linxiao Chen. A transfer theorem for multivariate delta-analytic functions with a power-law singularity. *arXiv preprint arXiv:2201.03539*, 2022.

[15] William Y. C. Chen, Emeric Deutsch, and Sergi Elizalde. Old and young leaves on plane trees. *European Journal of Combinatorics*, 27:414–427, 2006.

[16] Fan Chung, Persi Diaconis, and Ron Graham. Permanental generating functions and sequential importance sampling. *Adv. in Appl. Math.*, 126:101916, 50, 2021.

[17] Harald Cramér and Herman Wold. Some theorems on distribution functions. *Journal of the London Mathematical Society*, s1-11(4):290–294, 1936.

[18] Abraham de Moivre. *The Doctrine of Chances or a Method of Calculating the Probabilities of Events in Play, 2nd edition.* London, 1738.

[19] Jan Denef and Leonard Lipshitz. Algebraic power series and diagonals. *J. Number Theory*, 26(1):46–67, 1987.

[20] Michael Drmota. Asymptotic distributions and a multivariate Darboux method in enumeration problems. *J. Combinatorial Theory Ser. A*, 67(2):169–184, 1994.

[21] Michael Drmota. A bivariate asymptotic expansion of coefficients of powers of generating functions. *European Journal of Combinatorics*, 15(2):139–152, 1994.

[22] Michael Drmota. *Random trees: an interplay between combinatorics and probability.* Springer Science & Business Media, 2009.

[23] Rick Durrett. *Probability: theory and examples*, volume 31 of *Cambridge Series in Statistical and Probabilistic Mathematics*. Cambridge University Press, Cambridge, fourth edition, 2010.

[24] Shalosh B. Ekhad and Doron Zeilberger. Exploring Werner Krandick's binary tree jump statistics. *arXiv preprint arXiv:2407.12218*, 2024.

[25] Carl-Gustav Esseen. On the Liapunoff limit of error in the theory of probability. *Arkiv för Matematik, Astronomi och Fysik*, A28:1–19, 1942.

[26] Willy Feller. Über den zentralen Grenzwertsatz der Wahrscheinlichkeitsrechnung. *Mathematische Zeitschrift*, 40:521–559, 1935.

[27] Hans Fischer. *A history of the central limit theorem.* Sources and Studies in the History of Mathematics and Physical Sciences. Springer, New York, 2011.

[28] Philippe Flajolet and Andrew Odlyzko. Singularity analysis of generating functions. *SIAM J. Discrete Math.*, 3(2):216–240, 1990.

[29] Philippe Flajolet and Robert Sedgewick. *Analytic Combinatorics.* Number 140 in Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2009.

[30] Philippe Flajolet and Michèle Soria. Gaussian limiting distributions for the number of components in combinatorial structures. *J. Combinatorial Theory Ser. A*, 53(2):165–182, 1990.

[31] Philippe Flajolet and Michèle Soria. General combinatorial schemas: Gaussian limit distributions and exponential tails. *Discrete Math.*, 114(1-3):159–180, 1993.

[32] Rigoberto Flórez, Toufik Mansour, José L. Ramírez, Fabio A. Velandia, and Diego Villamizar. Restricted dyck paths on valleys sequence, 2021.

[33] Harry Furstenberg. Algebraic functions over finite fields. *J. Algebra*, 7:271–277, 1967.

[34] Zhicheng Gao and L. Bruce Richmond. Central and local limit theorems applied to asymptotic enumeration. IV. Multivariate generating functions. *J. Comput. Appl. Math.*, 41(1-2):177–186, 1992.

[35] Carl Friedrich Gauss. Theoria combinationis observationum erroribus minimis obnoxiae, pars prior. *Commentationes societatis Regiae scientiarum Goettingensis recentiores*, 5, 1823.

[36] Ira M. Gessel. A factorization for formal Laurent series and lattice path enumeration. *J. Combinatorial Theory Ser. A*, 28(3):321–337, 1980.

[37] Meghann Moriah Gibson, Daniel Gray, and Hua Wang. Combinatorics of n-color cyclic compositions. *Discrete Math.*, 341(11):3209–3226, 2018.

[38] Ian P. Goulden and David M. Jackson. *Combinatorial enumeration.* A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1983.

[39] Torin Greenwood. Asymptotics of bivariate analytic functions with algebraic singularities. *Journal of Combinatorial Theory, Series A*, 153:1–30, 2018.

[40] Torin Greenwood and Tristan Larson. Asymptotics of bivariate algebraico-logarithmic generating functions. *Séminaire Lotharingien de Combinatoire*, 2024.

[41] Benjamin Hackl, Andrew Luo, Stephen Melczer, Jesse Selover, and Elaine Wong. Rigorous analytic combinatorics in several variables in SageMath. *Séminaire Lotharingien de Combinatoire*, 89B:Art. 90, 12, 2023.

[42] Maló L. J. Hautus and David Anthony Klarner. The diagonal of a double power series. *Duke Math. J.*, 38:229–235, 1971.

[43] Christine Heitsch and Svetlana Poznanović. Combinatorial insights into RNA secondary structure. In *Discrete and topological models in molecular biology*, pages 145–166. Springer, 2014.

[44] Clemens Heuberger and Sara Kropf. Higher dimensional quasi-power theorem and Berry-Esseen inequality. *Monatsh. Math.*, 187(2):293–314, 2018.

[45] Hsien-Kuei Hwang. Large deviations for combinatorial distributions. I. Central limit theorems. *Ann. Appl. Probab.*, 6(1):297–319, 1996.

[46] Hsien-Kuei Hwang. Large deviations of combinatorial distributions. II. Local limit theorems. *Ann. Appl. Probab.*, 8(1):163–181, 1998.

[47] Hsien-Kuei Hwang. On convergence rates in the central limit theorems for combinatorial structures. *European Journal of Combinatorics*, 19(3):329–343, 1998.

[48] Hsien-Kuei Hwang. Introduction to Gaussian limit laws in analytic combinatorics. In *Collected Works of Philippe Flajolet, in progress*. 2024.

[49] Jessica Khera and Erik Lundberg. The distribution of the length of the longest path in random acyclic orientations of a complete bipartite graph. *arXiv preprint arXiv:2408.12716*, 2024.

[50] Bjarne Knudsen and Jotun Hein. RNA secondary structure prediction using stochastic context-free grammars and evolutionary history. *Bioinformatics*, 15(6):446–454, 06 1999.

[51] Christian Krattenthaler. Advanced determinant calculus. *Séminaire Lotharingien de Combinatoire*, 42:Art. B42q, 67, 1999. The Andrews Festschrift (Maratea, 1998).

[52] Pierre-Simon Laplace. Mémoire sur la probabilité des causes par les événements. *Mémoires de Mathématique et de Physique, Presentés à l'Académie Royale des Sciences, par divers Savans & lus dans ses Assemblées*, 6, 1774.

[53] Pierre-Simon Laplace. Mémoire sur les approximations des formules qui sont fonctions de très grands nombres et sur leur application aux probabilités. *Mémoires de l'Académie Royale des Sciences de Paris, année 1809*, pages 353–415, 1810.

[54] Paul Lévy. Sur la détermination des lois de probabilité par leurs fonctions charactéristiques. *Comptes rendus hebdomadaires de l'Académie des Sciences de Paris*, 175:854–856, 1922.

[55] Paul Lévy. *Calcul des probabilités*. Paris: Gauthier–Villars, 1925.

[56] Paul Lévy. Propriétés asymptotiques des sommes des variables aléatoires indépendantes ou enchaînées. *Journal de mathématiques pures et appliquées*, 14:347–402, 1935.

[57] Aleksandr Mikhailovich Lyapunov. Sur une proposition de la théorie des probabilités. *Bulletin de l'Académie Impériale des Sciences de St.-Pétersbourg*, 5(13):359–386, 1900.

[58] Aleksandr Mikhailovich Lyapunov. Nouvelle forme du théorème sur la limite de probabilité. *Mémoires de l'Académie Impériale des Sciences de St.-Pétersbourg VIIIe Série, Classe Physico-Mathématique*, 12:1–24, 1901.

[59] Stephen Melczer. *An Invitation to Analytic Combinatorics: From One to Several Variables.* Texts and Monographs in Symbolic Computation. Springer International Publishing, 2021.

[60] Stephen Melczer and Tiadora Ruza. Central limit theorems via analytic combinatorics in several variables. *Electron. J. Combin.*, 31(2):Paper No. 2.27, 2024.

[61] Stephen Melczer and Bruno Salvy. Effective coefficient asymptotics of multivariate rational functions via semi-numerical algorithms for polynomial systems. *J. Symbolic Comput.*, 103:234–279, 2021.

[62] Robin Pemantle and Mark C. Wilson. Asymptotics of multivariate sequences. I. Smooth points of the singular variety. *J. Combinatorial Theory Ser. A*, 97(1):129–161, 2002.

[63] Robin Pemantle, Mark C. Wilson, and S. Melczer. *Analytic combinatorics in several variables, 2nd Edition.* Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2024.

[64] Siméon Denis Poisson. Suite du mémoire sur la probabilité des résultats moyens des observations, inséré dans la connaissance des tems de l'année 1827. *Connaissance des tems pour l'an 1832*, pages 3–22, 1829.

[65] Georg Pólya. Über den zentralen Grenzwertsatz der Wahrscheinlichkeitsrechnung und das Momentenproblem. *Mathematische Zeitschrift*, 8:171–181, 1920.

[66] Georg Pólya. Sur les séries entières, dont la somme est une fonction algébrique. *L'Enseignement Mathématique*, 22:38–47, 1921.

[67] Konstantin Vladimirovich Safonov. On power series of algebraic and rational functions in $\mathbf{C}^n$. *J. Math. Anal. Appl.*, 243(2):261–277, 2000.

[68] Richard P. Stanley. *Enumerative combinatorics. Vol. 2*, volume 62 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 1999. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin.

[69] Vasiliki Velona. Encoding and avoiding 2-connected patterns in polygon dissections and outerplanar graphs. *Discrete Math.*, 341(12):3402–3414, 2018.

[70] Herbert S. Wilf. *generatingfunctionology.* A K Peters, Ltd., Wellesley, MA, third edition, 2006.