Edge-disjoint Linkages in Infinite Graphs

by

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Author's Declaration

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the next page. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of Contributions

Chapters 2 and 4 are wholly my own work. The main result in Chapter 2 is a complete description of graphs with a vertex whose lifting graph has a connected complement. This is a central result in the thesis that is applied in Chapters 3 and 4.

Chapter 4 proves that a $4k$-edge-connected, locally finite, 1-ended, infinite graph has an orientation that is k-arc-connected. In this special case, this improves Thomassen's result that every $8k$-edge-connected graph has such an orientation.

Chapter 3 is almost completely my own work. The exception is Lemma 3.2.10 and the proof of Proposition 3.2.11, which were proved by Bruce Richter. I had formulated Proposition 3.2.11 and conversations between us led to Professor Richter's proof.

The main result of Chapter 3 is an application of the results of Chapter 2 to show that Huck's Theorem (for odd k, every $k + 1$-edge-connected graph is weakly k-linkable) holds for infinite graphs. This improves the result of Ok, Richter, and Thomassen that $k + 2$-edge-connected, locally finite, 1-ended infinite graph is weakly k-linkable.

Abstract

The main subject of this thesis is the infinite graph version of the weak linkage conjecture by Thomassen [24]. We first prove results about the structure of the lifting graph; Theorems 2.2.8, 2.2.24, and 2.3.1. As an application, we improve the weak-linkage result of Ok, Richter, and Thomassen [18]. We show that an edge-connectivity of $(k+1)$ is enough to have a weak k-linkage in infinite graphs in case k is odd, Theorem 3.3.6. Thus proving that Huck's theorem holds for infinite graphs. This is only one step far away from the conjecture, which has an edge-connectivity condition of only k in case k is odd. As another application, in Theorem 4.2.7 we improve a result of Thomassen about strongly connected orientations of infinite graphs [25], in the case when the infinite graph is 1-ended. This brings us closer to proving the orientation conjecture of Nash-Williams for infinite graphs [15].

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Chapter 1

Introduction

Finding disjoint paths in a graph (network) is a natural problem that is seen in many real life problems as well as in graph theoretic and discrete optimization problems. It has connections to the problems of multi-commodity network flows [8], degree bounded designs with metric costs [2], the graph homeomorphism problem [3], and VLSI designs [11], [17].

One particular version we are interested in is when we have stations $s_1, \cdots, s_k$ and destinations $t_1, \cdots, t_k$ in a given network, and we wish to find k disjoint paths, one for each i connecting s_i to t_i. An obvious condition for the existence of such paths is that the network itself in general contains many disjoint paths between any two parts of it, or in other words the network is highly connected. Does it have to be too highly connected? Or is a connectivity of strength almost k enough? This is what Thomassen conjectured four decades ago, for edge-disjoint paths, and is the main question this thesis revolves around. On the other hand a vertex-connectivity of k is very low to have a vertex-disjoint linkage. However, it was proved by Thomas and Wollan that a vertex-connectivity of $10k$ is sufficient for a graph to be k-linked [23].

In 1980 Thomassen conjectured [24] that for each odd integer k, any k-edge-connected graph is weakly k-linked, and for each even integer k, any $(k+1)$-edge-connected graph is weakly k-linked. Where a graph G is weakly k-linked if, given any set of k pairs of terminals, not necessarily distinct, s_i and t_i, $i = 1, \cdots, k$, then there is in G, for each $i \in \{1, \cdots, k\}$, a path between s_i and t_i such that this collection of k paths is edge-disjoint.

This conjecture aims to extend the famous Menger's Theorem, which is the special case when all of $s_1, \ldots, s_k$ are equal and all of $t_1, \ldots, t_k$ are equal (in this special case the parity of k does not matter; k-edge-connectivity is enough). In other words, the edge version of Menger's theorem is that a graph is k-edge-connected if and only if there are k edge-disjoint paths between any two vertices in it.

The simple case of $k = 2$ was obtained independently by Thomassen [24] and Seymour [22]. The case $k = 3$ was proved by Okamura [19], and $k = 4$ was proved by Enomoto and Saito [5]. A harder conjecture based on local connectivites between the pairs of terminals was proved by Okamura for $k = 6$ in 1987 [20].

After that, in 1988, Okamura proved that every $3k$-edge-connected graph is weakly $(2k+1)$-linked, and every $(3k-1)$-edge-connected graph is weakly $2k$-linked [21].

In 1991, Huck [9] had come within 1 of proving Thomassen's conjecture. He proved that for a graph to be weakly k-linked, $(k + 1)$-edge-connectivity is enough in the case of odd k, and $(k+2)$-edge connectivity is enough in the case of even k.

Ok, Richter, and Thomassen [18] considered the question for infinite graphs, and in 2016 they proved for odd k that if G is a $(k + 2)$-edge-connected, 1-ended, locally finite graph, then G is weakly k-linked. Where locally finite means that the number of edges incident with each vertex is finite, and 1-ended means deleting any finite set of vertices leaves only one infinite component.

To prove their result, Ok, Richter, and Thomassen first proved a theorem about lifting graphs. The lifting graph is an auxiliary graph frequently used in edge-connectivity proofs. In addition to the lifting graph, they have a connected graph on the same vertices. The fact that the complement of their lifting graph is known to be disconnected - in a special case - implies that the connected graph has an edge that is also in the lifting graph. This is central to our applications of the lifting graph.

In the thesis we have the following:

(i) Chapter 2 presents an extension of the theorem of Ok, Richter, and Thomassen characterizing the lifting graph for finite graphs. For the four possibilities of the parity of $\deg(s)$ with the parity of the edge-connectivity k of the graph, we give a detailed description of the lifting graphs that can occur.

(ii) In particular, the interesting case of a connected complement of the lifting graph is completely described: the maximal independent sets of the lifting graph form either a path or a cycle via intersections. The same (path or cycle) structure is displayed in the graph, with blobs corresponding to the intersections. Consecutive blobs are joined by precisely $(k-1)/2$ edges except in the case of when the structure is a path of length 1 (two maximal independent sets with a nonempty intersection).

(iii) The case when $\deg(s)$ is 3 is problematic as the lifting graph might consist of three isolated vertices. Section 2.3.1 is dedicated to studying that case.

(iv) In Chapter 3, we improve the linkage result of Ok, Richter, and Thomassen by reducing the connectivity condition from $k+2$ to $k+1$, and generalizing it to the case of arbitrarily many ends. In other words, we prove that Huck's theorem holds for all infinite graphs. Huck's Theorem is different from the conjecture only in that it has the connectivity requirement of $(k+1)$ instead of k for odd k.

(v) The reduction from general infinite graphs to 1-ended locally finite graphs is done using two theorems of Thomassen, proved in 2016 in [25], together with our lifting graph results for finite graphs referred to in (i).

The new lifting graph results are used to deal with odd cuts, which posed a problem in the proof technique of Ok, Richter, and Thomassen.

(vi) Recall that a directed graph is k-arc-connected if between any two vertices v and w in the graph there are k edge-disjoint directed paths from v to w and also that many paths from w to v. Nash-Williams proved that every $2k$-edge-connected finite graph has a k-arc-connected orientation [16]. He conjectured that the result also holds for infinite graphs [15]. In his paper [13], Mader proved the existence of a feasible lifting at a vertex and used it to give a simpler proof of Nash-Williams' orientation theorem for finite graphs.

In Chapter 4 we present another application of our lifting graph results. We prove that for a 1-ended infinite graph, and edge-connectivity of $4k$ suffices to have a k-arc-connected orientation. The proof involves finding in such a graph an immersion of a highly connected graph on a given finite set of vertices. This improves a result of Thomassen in [25] and brings us closer to proving the conjecture of Nash-Williams.

(vii) In Chapter 5, the Future Work chapter, we present in Section 5.1 our steps trying to prove that, assuming the truth of the conjecture for finite graphs, then it is true for locally-finite 1-ended infinite graphs, and hence also for general infinite graphs. We use the path/cycle structure found in (ii) in our proofs. The main obstacle is when the degree of the vertex s at which the lifting is performed is 3. In the remaining sections of Chapter 5 we discuss other problems related to weak linkages.

The thesis relies heavily on, and extends the results in, a paper by Ok, Richter, and Thomassen about the lifting graph and weak-linkage [18], and a paper by Thomassen about orientations in infinite graphs [25]. We developed the following new ideas.

New ideas in the thesis:

(1) Studying the structure of the lifting graph through its maximal independent sets. This appears throughout Chapter 2.

(2) The following fact, which is stated as Proposition 3.2.11. In a locally finite infinite graph, given a set of edges F of size at most the assumed edge-connectivity, in a finite edge-cut, with one side infinite and the other finite, there is a vertex on the infinite side with edge-disjoint paths contained in the infinite side from that vertex to the different edges of F. This is crucial to the proof of one of the main results of the thesis, Theorem 3.2.13.

Chapter 2

The Lifting Graph

Throughout this chapter a graph is a finite multigraph, that is parallel edges possibly exist.
In this chapter we prove results analysing the structure of the lifting graph, and we also show that we can infer a specific structure of the graph when the complement of its lifting graph is connected.

2.1 Background

In this section we present the basic definitions and theorems regarding edge-connectivity and the lifting graph, as well as the results of Ok, Richter, and Thomassen we are going to build upon.

2.1.1 Definitions

In this subsection we display some of the definitions. In the following subsections more of the classic definitions will be presented when needed. Most of the hypotheses of the theorems we have here involve a condition on the edge-connectivity of the graph, so we begin with the definition of edge-connectivity and other related definitions.

Definition 2.1.1. (k-edge-connected) A graph G - finite or infinite - is *k-edge-connected* if $|G| > 1$ and for every $F \subseteq E(G)$ with $|F| < k$, $G - F$ is connected.

Definition 2.1.2. (edge-disjoint) Two paths in a graph G *edge-disjoint* if the edge sets of the two paths are disjoint.

We use the lifting graph frequently in our proofs. Here we present the definitions of a *lift* and the *lifting graph*.

Definition 2.1.3. (lift) Let s be a vertex of a graph G and let sv and sw be two edges incident with s such that $v \neq w$. To *lift* the pair of edges sv and sw is to delete them and add the edge vw and the resulting graph $G_{v,w}$ is the *lift of G at sv and sw*, or simply a *lift at s*.

Definition 2.1.4. (feasible lift) A lift of G at s is *feasible* if the number of edge-disjoint paths between any two vertices other than s stays the same after the lift.

In this thesis we only need the local edge-connectivity between any two vertices - other than a certain fixed vertex s - not to fall below a certain target connectivity k. The vertex s is the vertex at which the lifting is performed. Ideally, we want to have all the edges incident with s lifted, except for possibly one edge in case $\deg(s)$ is odd, so that the resulting final graph is k-edge-connected in the usual sense. Then we can apply theorems that hold for k-edge-connected graphs to that final graph. This is why we need the following definition, in which the graph G is not necessarily k-edge-connected.

Definition 2.1.5. A triple (G, s, k) is a *connectivity triple* if G is a graph on at least three vertices, s is a vertex of G, and k is an integer such that $k \geqslant 2$ and any two vertices in G both different from s are joined by k pairwise edge-disjoint paths in G (that may go through s).

Note that since $k \geqslant 2$ in a connectivity triple, if $\deg(s) > 1$, then s is not incident with a cut-edge.

Definition 2.1.6. (k-liftable) Let (G, s, k) be a connectivity triple. The edges sv and sw are *k-liftable*, or they form a *k-liftable* pair, if $(G_{v,w}, s, k)$ is a connectivity triple.

Definition 2.1.7. (lifting graph) Let G be a graph, s vertex in G, and $k \geqslant 2$ an integer. The *k-lifting* graph $L(G, s, k)$ has as its vertices the edges of G incident with s and two vertices are adjacent in $L(G, s, k)$ if they form a k-liftable pair.

One thing lifting is helpful in but that we will not use here in the thesis is inductive proofs. Either induction on the number of edges in the graph or on the degree of the vertex at which lifting takes place. Unlike contraction, a set of n edge-disjoint paths in a graph that resulted from a sequence of lifting operations can be returned to a set of n edge-disjoint paths in the original graph.

2.1.2 Basic Theorems

As mentioned in Chapter 1, Thomassen's weak linkage conjecture is a generalization of Menger's Theorem for edge-connectivity. Here we present Menger's Theorem, and also the theorems of Mader and Frank about feasible lifts. Mader's theorem says that at any vertex s there is at least one feasible lift if $\deg(s) \neq 3$ and s is not incident with a cut-edge. Frank's theorem implies that there are at least $\lfloor \deg(s)/2 \rfloor$ such lifts. We are going to make use of these three theorems.

Theorem 2.1.8. *(Menger's Theorem)[14] A graph G is k-edge-connected if and only if there are k edge-disjoint paths between any two vertices in G.*

Menger's theorem is true in infinite graphs as well [7], [1].

In 1976 Lovász proved that for any vertex s in an Eulerian graph G a feasible lift exists of G at s [12]. After that Mader proved the following for any graph G, not necessarily Eulerian.

Theorem 2.1.9. *[13] (Mader 1978) Let G be a graph containing a vertex s not incident with a cut-edge such that* $\deg(s) \neq 3$*. Then there exists a feasible lift of G at s.*

Frank also proved that, under the same conditions of Mader's theorem, not only one pair lifts, but also any independent set in $L(G, s, k)$ has size at most $\lceil \deg(s)/2 \rceil$.

Theorem 2.1.10. *[6] (Frank 1992) Let G be a graph containing a vertex s not incident with a cut-edge such that* $\deg(s) \neq 3$*. Then there are* $\lfloor \deg(s)/2 \rfloor$ *pairwise disjoint pairs of edges incident with s each of which defining a feasible lift.*

In 2016 Thomassen showed that the k-lifting graph of an Eulerian graph has a disconnected complement [25]. Later in the same year, Ok, Richter, and Thomassen showed that when both $\deg(s) \geqslant 4$ and k are even, the lifting graph is a complete multipartite graph that is not a star (hence its complement is disconnected) [18]. They also characterised the structure of the lifting graph in the case when it is disconnected. The results of Ok, Richter, and Thomassen about the lifting graph are listed in Theorem 2.1.11 below.

Theorem 2.1.11. *[18] (Ok, Richter, and Thomassen 2016) Let (G, s, k) be a connectivity triple such that* $\deg(s) \geqslant 4$*. Then:*

(1) The k-lifting graph $L(G, s, k)$ has at most two components;

(2) If $\deg(s)$ *is odd and $L(G, s, k)$ has two components, then one has only one vertex and the other component is complete multipartite;*

(3) If $\deg(s)$ *is even and $L(G, s, k)$ has two components, then each component is complete multipartite with an even number of vertices and;*

(4) If $\deg(s)$ *and k are both even, then $L(G, s, k)$ is a connected, complete, multipartite graph (in particular, it has a disconnected complement).*

If either $L(G, s, k)$ is not connected or both $\deg(s)$ *and* k *are even, then any component of* $L(G, s, k)$ *with at least* 4 *vertices is not a star* $K_{1,r}$.

This theorem was proved by induction on $\deg(s)$, and the proof of the base cases, as will the proof of our extension of this theorem, relied on the concept of a *dangerous set*, to be defined soon in Definition 2.1.13, and the following standard equation 2.1.1 about two intersecting cuts. We will be dealing a lot with edge-cuts, or simply cuts.

Definition 2.1.12. (cut) A *cut* in a graph G consists of, for some partition (A, B) of $V(G)$, the set $\delta(A)$ of all edges having one end in A and one end in B.

Note that $G - \delta(A)$ is disconnected.

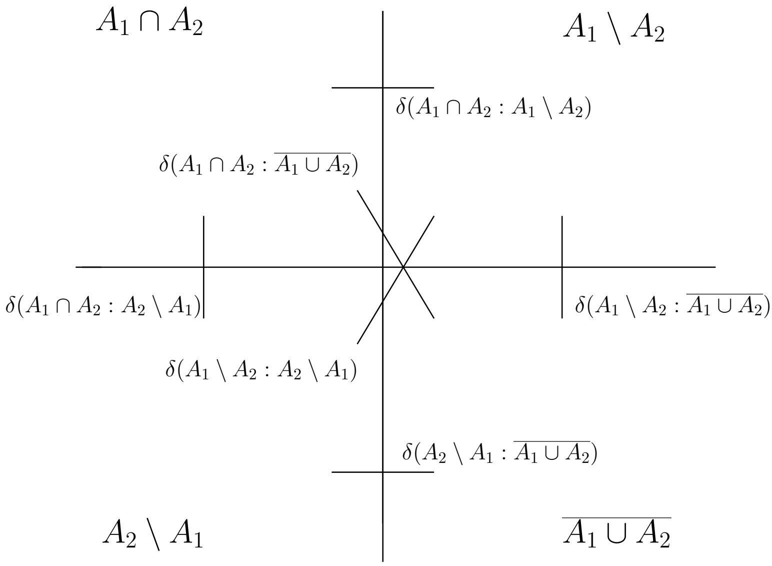


Figure 2.1: Two intersecting cuts.

The following equation is very helpful, which is about two intersecting cuts $\delta(A_1)$ and $\delta(A_2)$. For a vertex set A, let $\overline{A} := V(G) \setminus A$. If B is another set of vertices, then the set of edges with one end in A and one end in B is denoted by $\delta(A : B)$. Rudimentary counting, with the help of Figure 2.1, gives the equation.

$$2\Big[\big|\delta(A_1)\big| + \big|\delta(A_2)\big| - \Big(\big|\delta(A_1 \cap A_2 : \overline{A_1 \cup A_2})\big| + \big|\delta(A_2 \setminus A_1 : A_1 \setminus A_2)\big|\Big)\Big]$$
$$= \big|\delta(A_1 \cap A_2)\big| + \big|\delta(A_2 \setminus A_1)\big| + \big|\delta(A_1 \setminus A_2)\big| + \big|\delta(\overline{A_1 \cup A_2})\big|. \tag{2.1.1}$$

Definition 2.1.13. (dangerous set) Let (G, s, k) be a connectivity triple. A subset A of $V(G) \setminus \{s\}$ is *k-dangerous*, or simply *dangerous*, if $A \neq \emptyset$, $V(G) \setminus (A \cup \{s\}) \neq \emptyset$, and $|\delta_G(A)| \leqslant k + 1$.

Dangerous sets were also used in [2] to prove results about degree bounded network designs with metric costs. For that purpose, some propositions about dangerous sets were proved in [2] that were later developed and used by Ok, Richter, and Thomassen. The following proposition is a special case of Theorem 1.1 in their paper [18].

Proposition 2.1.14. *[18] Let G be a graph and let s be a vertex of G that does not have degree 3 and is not incident with a cut-edge. Let F be any set of at least two edges, all incident with s. Then no pair of edges in F yields a k-feasible lift if and only if there is a k-dangerous set A so that, for every $sv \in F$, $v \in A$.*

The core idea in this proposition can be seen more easily in the special case when $|F| = 2$. Simply, if a pair is not liftable, this means that lifting it results in a small cut.

Lemma 2.1.15. *Let (G, s, k) be a connectivity triple, and let su and sv be edges in G. If su and sv are not k-liftable, then there is a dangerous set A in G containing u and v.*

Proof. If su and sv are not liftable, then there is a cut $\delta(A)$ in $G_{u,v}$ of size at most $k - 1$ that separates two vertices of $G_{u,v} - s$ and such that $s \notin A$. Otherwise $\delta(A)$ is a cut in G of size $k - 1$ that is not $\delta(\{s\})$, contradicting that (G, s, k) is a connectivity triple. Both u and v have to be on the side of the cut that does not contain s, otherwise there is a cut in G of size $k-1$ that is not $\delta(\{s\})$, contradicting that (G, s, k) is a connectivity triple. That side of the cut containing u and v but not s is the desired dangerous set. $\square$

Definition 2.1.16. (independent set) A set X of vertices in a graph G is *independent* if no two vertices of X are adjacent in G.

Note that a set F of edges incident with s in G such that no pair of edges in F is k-liftable corresponds to an independent set of vertices in the lifting graph $L(G, s, k)$. The following is Lemma 3.2 in [18].

Definition 2.1.17. Let (G, s, k) be a connectivity triple, and let F be an independent set of edges in $L(G, s, k)$. A dangerous set A as given by Proposition 2.1.14 is a dangerous set *corresponding* to F.

Such a dangerous set is not necessarily unique. Consider for example a graph G that consists of a vertex s and three sets of vertices A_1, A, and A_2, where s has neighbours in both A_1 and A_2, but no neighbours in A, and there are $k+1$ edges between A_1 and A, and between A_2 and A, but no edges between A_1 and A_2. Let F_1 be the set of edges incident with s whose other end-vertices are in A_1. Then A_1 and $A \cup A_1$ are two different dangerous sets corresponding to F_1.

Lemma 2.1.18. *Let (G, s, k) be a connectivity triple such that $\deg(s) = 3$. Any set of vertices in G containing all the three neighbours of s is not dangerous.*

Proof. If A is a dangerous set, then $|\delta(A)| \leqslant k+1$. By definition, $A' = V(G) \setminus (A \cup \{s\})$ is not empty. If all neighbours of s are in A, then $|\delta(A')| = |\delta(A \cup \{s\})| \leqslant (k+1) - 3 = k-2$, contradicting the fact that (G, s, k) is a connectivity triple. $\square$

Lemma 2.1.19. *[18] Let (G, s, k) be a connectivity triple. For $i = 1, 2$, let F_i be an independent set in $L(G, s, k)$ of size r_i and suppose there is a dangerous set A_i so that $F_i = \delta(\{s\}) \cap \delta(A_i)$. Set $\alpha = |F_1 \cap F_2|$. If $\alpha > 0$, $r_1 > \alpha$, $r_2 > \alpha$, and $\overline{A_1 \cup A_2 \cup \{s\}} \neq \emptyset$, then $r_1 + r_2 \leqslant \lfloor \deg(s)/2 \rfloor + 2$.*

We will also need Lemma 3.3 from [18],

Lemma 2.1.20. *[18] Let (G, s, k) be a connectivity triple.*

(i) If $\deg(s)$ is at least 4, then every independent set in $L(G, s, k)$ has size at most $\lceil \deg(s)/2 \rceil$ and;

(ii) If $\deg(s)$ is even and at least 6, then any two distinct independent sets in $L(G, s, k)$ of size $\frac{1}{2}\deg(s)$ are disjoint.

2.2 Structure of the lifting graph

In this section we extend Theorem 2.1.11 of Ok, Richter, and Thomassen, about the structure of the lifting graph. That theorem was proved by induction on the degree of the vertex whose edges are lifted, or in other words the number of vertices of the lifting graph at that vertex. Not all the properties Ok, Richter, and Thomassen showed at the base cases of degrees 4 and 5 were carried over in the induction. We show - without induction - that all the specific structures they found in the base cases of their inductive proof hold in general. We present this mainly in Theorem 2.2.8, Corollary 2.2.24, and Proposition 2.2.26. We also show that the structure of the lifting graph has an implication on the structure of the graph, Theorem 2.3.1.

Proposition 2.2.1. *Let (G, s, k) be a connectivity triple. Assume that $L(G, s, k)$ contains an independent set of size $\lceil \deg(s)/2 \rceil$. If $\deg(s)$ is odd and at least 5, then $L(G, s, k)$ is either (see Figure 2.2)*

(i) a complete bipartite graph with one side of size $\lceil \deg(s)/2 \rceil$ and the other of size $\lfloor \deg(s)/2 \rfloor$ or

(ii) an isolated vertex plus a complete bipartite graph with both sides of size $\frac{(\deg(s)-1)}{2}$.

Proof. This is the same as the proof of Case 1 in Proposition 3.5 of [18]. Let F be a set of edges incident with s of size $\lceil \deg(s)/2 \rceil$ such that no two edges in F form a feasible pair, i.e. F is independent in $L(G, s, k)$. By

Proposition 2.1.14 there is a dangerous set A_1 containing the non-s ends of the edges in F. By definition of a dangerous set, $|\delta_G(A_1)| \leqslant k+1$.

Note that the disjoint unions of $\delta(A_1 : \overline{A_1 \cup \{s\}})$ with $\delta(\{s\} : A_1)$ and with $\delta(\{s\} : \overline{A_1 \cup \{s\}})$ respectively give $\delta(A_1)$ and $\delta(\overline{A_1 \cup \{s\}})$. By Lemma 2.1.20, F has the maximum possible size of an independent set in $L(G, s, k)$, therefore, $|\delta(\{s\} : A_1)| = |F| = \frac{deg(s)+1}{2}$, $|\delta(\overline{A_1 \cup \{s\}})| = |\delta(A_1)| - 1 \leqslant k$. Thus $\overline{A_1 \cup \{s\}}$ is dangerous, and by Proposition 2.2.1, $\delta(\{s\}) \setminus F$ is an independent set in $L(G, s, k)$. Now we will show that

(i) either every edge in F lifts with every edge in $\delta(\{s\}) \setminus F$ (this gives the complete bipartite possibility) or

(ii) there is a unique edge in F that does not lift with any edge in $\delta(\{s\})$, but any other edge of F lifts with each edge of $\delta(\{s\}) \setminus F$. This gives the isolated vertex plus complete bipartite case.

In particular we will show that if an edge $e_1 \in F$ does not lift with an edge $e_2 \in \delta(\{s\}) \setminus F$, then e_1 does not lift with any edge, and then by Frank's Theorem 2.1.10 it is the only such edge. We present this in the following claim.

Claim 2.2.2. *Let $e_1 \in F$ and $e_2 \in \delta(\{s\}) \setminus F$ be a pair of edges that is not k-liftable. Then e_1 is not k-liftable with any edge in $\delta(\{s\}) \setminus \{e_1\}$. Moreover, any edge $f \in F \setminus \{e_1\}$ is k-liftable with every edge in $\delta(\{s\}) \setminus F$.*

Proof. Suppose $e_1 \in F$ and $e_2 \in \delta(\{s\}) \setminus F$ do not form a k-liftable pair and let A_2 be a dangerous set that witnesses this (such a set exists by Lemma 2.1.15), so the non-s ends of e_1 and e_2 are in A_2. By Lemma 2.1.20 the maximum size of an independent set in $L(G, s, k)$ is $\lceil \deg(s)/2 \rceil$; consequently, at most $\lceil \deg(s)/2 \rceil$ of the edges incident with s have their non-s ends in A_2. Therefore, at least $\lfloor \deg(s)/2 \rfloor$ of the edges incident with s have their non-s ends in $\overline{A_2 \cup \{s\}}$. Now since e_2 is in $\delta(\{s\}) \setminus F$ but has its non-s end in A_2 and $|\delta(\{s\}) \setminus F| = \lfloor \deg(s)/2 \rfloor$, the set of edges incident with s whose non-s ends are in $\overline{A_2 \cup \{s\}}$ contains an edge e from F. Also since $e_1 \in F$ has its non-s end in A_2, e is in $F \setminus \{e_1\}$.

For $i = 1, 2$, set $F_i = \delta(A_i) \cap \delta(s)$; in particular $F_1 = F$. Three of the hypotheses of Lemma 2.1.19 are satisfied: $e_1 \in F_1 \cap F_2$ ($\alpha > 0$), $e_2 \in F_2 \backslash F_1$ ($|F_2| > \alpha$), and $e \in F_1 \backslash F_2$ ($|F_1| > \alpha$). If the other hypothesis of the lemma $\overline{A_1 \cup A_2 \cup \{s\}} \neq \emptyset$ is also satisfied, then $|F_1| + |F_2| \leqslant \lfloor \deg(s)/2 \rfloor + 2$, i.e. $\lceil \deg(s)/2 \rceil + |F_2| \leqslant \lfloor \deg(s)/2 \rfloor + 2$. Since $\deg(s)$ is odd, this means that $|F_2| \leqslant 1$, a contradiction since F_2 contains both e_1 and e_2. Thus, $\overline{A_1 \cup A_2 \cup \{s\}} = \emptyset$, so all the edges of $\delta(\{s\}) \setminus F$ have their non-s ends in $A_2 \setminus A_1$.

Now since, $|\delta(\{s\}) \setminus F| = \lfloor \deg(s)/2 \rfloor$ and there is an edge from s to $A_1 \cap A_2$, $|\delta(A_2) \cap \delta(\{s\})| \geqslant \lceil \deg(s)/2 \rceil$. By Lemma 2.1.20 this is the maximum possible size of an independent set in $L(G, s, k)$, and since A_2 is dangerous, $|\delta(A_2) \cap \delta(\{s\})| = \lceil \deg(s)/2 \rceil$.

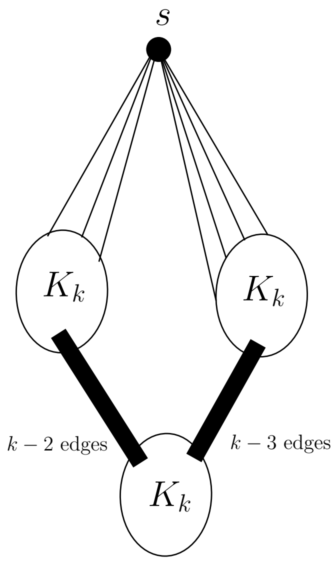
Consequently, $|\delta(\{s\} : (A_1 \cap A_2))| = 1$, and all the non-$s$ ends of the edges of F other than e_1 are in $A_1 \setminus A_2$. Since all the non-s ends of the edges incident with s other than e_1 are either in $A_1 \setminus A_2$ or $A_2 \setminus A_1$, and e_1 has its non-s end in $A_1 \cap A_2$, e_1 does not lift with any other edge and it is the only such edge by Frank's Theorem.

If f is an edge in $F \setminus \{e_1\}$ and e_2 is an edge in $\delta(\{s\}) \setminus F$, then by the uniqueness of e_1, f and e_2 is a k-liftable pair. Thus every edge in $F \setminus \{e_1\}$ lifts with every edge in $\delta(\{s\}) \setminus F$. □

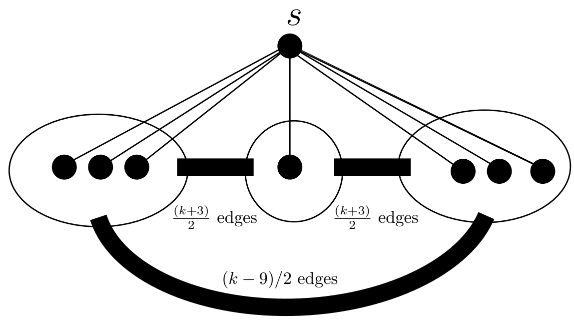
This claim gives the structure of an isolated vertex (e_1) plus a complete bipartite graph for the lifting graph. □

Proposition 2.2.1 dealt with the case when $\deg(s)$ is odd and $L(G, s, k)$ contains an independent set of the maximum possible size $\lceil \deg(s)/2 \rceil$, Figure 2.2. There it was proved that the maximal independent sets are either disjoint or intersect in exactly one vertex. Point (4) in Theorem 2.1.11 showed that if both $\deg(s)$ is even and k is even, then the maximal independent sets of $L(G, s, k)$ are disjoint (complete multipartite). The case of an isolated vertex plus complete bipartite means two maximal independent sets of $L(G, s, k)$ intersecting in exactly one vertex. Now we consider the other cases.

The new idea here is that we focus on the maximal independent sets



$L(G,s,k) = K_{3,4}$



$L(G,s,k)$ is an isolated vertex plus $K_{3,3}$

Figure 2.2: Examples with $\deg(s) = 7$ and an independent sets of size $\lceil \deg(s)/2 \rceil$ in $L(G,s,k)$.

of the lifting graph. In the following lemmas and theorem we will only see how they intersect. After that we will use this to find out what kind of structure they are arranged into, a path, a cycle, pairwise disjoint, or otherwise.

Lemma 2.2.3. *Let (G,s,k) be a connectivity triple such that $\deg(s) \geqslant 4$. Suppose that I_1 and I_2 are two maximal independent sets in $L(G,s,k)$ and let A_1 and A_2 be two dangerous sets in G corresponding to I_1 and I_2 respectively.*

(1) Then $|I_1 \cap I_2| \leqslant 1$.

(2) If $|I_1 \cap I_2| = 1$ and $I_1 \cup I_2 \neq V(L(G,s,k))$, then k is odd, and

(a) $|\delta_G(A_1)| = |\delta_G(A_2)| = k+1$;

(b) $|\delta_G(A_2 \setminus A_1 : A_1 \setminus A_2)| = 0$;

(c) $|\delta_{G-s}(A_1 \cap A_2 : \overline{A_1 \cup A_2})| = 0$; and

(d) $|\delta_G(A_2 \setminus A_1 : A_1 \cap A_2)| = |\delta_G(A_1 \cap A_2 : A_1 \setminus A_2)| = \frac{k-1}{2}$.

Proof. We follow a generalized version of the proof of Case 2 in Proposition 3.5 of [18]. Let $k_1 = |\delta(\{s\} : A_1 \setminus A_2)|$, $k_2 = |\delta(\{s\} : A_2 \setminus A_1)|$, $k_3 = |\delta(\{s\} : A_1 \cap A_2)|$, and assume that $k_3 \neq 0$.

Since A_1 and A_2 are dangerous, we have $|\delta_G(A_1)| \leqslant k+1$ and $|\delta_G(A_2)| \leqslant k+1$. In particular, $|\delta_{G-s}(A_1)| \leqslant (k+1) - (k_1 + k_3)$, and $|\delta_{G-s}(A_2)| \leqslant (k+1) - (k_2 + k_3)$.

Since I_1 and I_2 are maximal independent sets in $L(G, s, k)$, $I_1 \cup I_2$ is not independent, consequently, by Proposition 2.1.14, $A_1 \cup A_2$ is not dangerous.

Also because I_i is a maximal independent set of $L(G, s, k)$ for $i \in \{1, 2\}$, each A_i does not contain neighbours of s other than those that are end-vertices of edges in I_i.

This and the assumption that $I_1 \cup I_2 \neq V(L(G, s, k))$ imply that at least one edge incident with s has its non-s end in the set $\overline{A_1 \cup A_2 \cup \{s\}}$. Thus, $A_1 \cup A_2$ is separating in $G - s$ in the sense that $V(G-s) \setminus (A_1 \cup A_2)$ and $A_1 \cup A_2$ are both non-empty. By the definition of a dangerous set, the only way for $A_1 \cup A_2$ to not be dangerous is if $|\delta_G(A_1 \cup A_2)| \geqslant k + 2$. This means that $|\delta_{G-s}(A_1 \cup A_2)| \geqslant (k+2) - (k_1 + k_2 + k_3)$. Note that in $G - s$, s is not in $\overline{A_1 \cup A_2}$.

Also since (G, s, k) is a connectivity triple, and $A_1 \cap A_2 \neq \emptyset$ (as $k_3 \neq 0$), $|\delta_{G-s}(A_1 \cap A_2)| \geqslant k - k_3$, $|\delta_{G-s}(A_1 \setminus A_2)| \geqslant k - k_1$, and $|\delta_{G-s}(A_2 \setminus A_1)| \geqslant k - k_2$. Observe that in $G - s$:

$$
\begin{aligned}
&2\Big[\big|\delta(A_1)\big| + \big|\delta(A_2)\big| - \Big(\big|\delta(A_1 \cap A_2 : \overline{A_1 \cup A_2})\big| + \big|\delta(A_2 \setminus A_1 : A_1 \setminus A_2)\big|\Big)\Big] \\
&\leqslant 2\Big[(k+1) - (k_1 + k_3) + (k+1) - (k_2 + k_3)\Big] \\
&= 4k - 2(k_1 + k_2 + k_3) + 2 + (2 - 2k_3)
\end{aligned}
$$

and

$$
\begin{aligned}
&\big|\delta(A_1 \cap A_2)\big| + \big|\delta(A_2 \setminus A_1)\big| + \big|\delta(A_1 \setminus A_2)\big| + \big|\delta(\overline{A_1 \cup A_2})\big| \geqslant \\
&(k - k_3) + (k - k_2) + (k - k_1) + (k+2) - (k_1 + k_2 + k_3) \\
&= 4k - 2(k_1 + k_2 + k_3) + 2
\end{aligned}
$$

By Equation 2.1.1, it follows that $2 - 2k_3 \geqslant 0$, i.e. $k_3 \leqslant 1$ as desired.

If $k_3 = 1$, then inequalities throughout have to be equalities. More precisely,

(1) $|\delta_{G-s}(A_1)| = |\delta_{G-s}(A_1 \setminus A_2)| = k - k_1$, so $|\delta_G(A_1 \setminus A_2)| = k$ and $|\delta_G(A_1)| = k + 1$;

(2) $|\delta_{G-s}(A_2)| = |\delta_{G-s}(A_2 \setminus A_1)| = k - k_2$, so $|\delta_G(A_2 \setminus A_1)| = k$ and $|\delta_G(A_2)| = k + 1$;

(3) $|\delta_{G-s}(A_1 \cap A_2)| = k - 1$;

(4) $|\delta_{G-s}(\overline{A_1 \cup A_2})| = (k+2) - (k_1 + k_2 + k_3) = k + 1 - k_1 - k_2$;

(5) $|\delta_{G-s}(A_1 \cap A_2 : \overline{A_1 \cup A_2})| = |\delta_{G-s}(A_2 \setminus A_1 : A_1 \setminus A_2)| = 0$.

Together, (1) and (2) give (a), and (5) gives (b) and (c). To prove (d) note that (5) also gives (cf. Figure 2.1),
$|\delta_{G-s}(A_2)| = |\delta_{G-s}(A_2 \setminus A_1 : \overline{A_1 \cup A_2})| + |\delta_{G-s}(A_1 \cap A_2 : A_1 \setminus A_2)|$, and
$|\delta_{G-s}(A_2 \setminus A_1)| = |\delta_{G-s}(A_2 \setminus A_1 : \overline{A_1 \cup A_2})| + |\delta_{G-s}(A_2 \setminus A_1 : A_1 \cap A_2)|$.
From (2) we have $|\delta_{G-s}(A_2)| = |\delta_{G-s}(A_2 \setminus A_1)|$. Cancelling the common $|\delta_{G-s}(A_2 \setminus A_1 : \overline{A_1 \cup A_2})|$ on both sides yields

$$|\delta_{G-s}(A_2 \setminus A_1 : A_1 \cap A_2)| = |\delta_{G-s}(A_1 \cap A_2 : A_1 \setminus A_2)|.$$

Now this last equality, (3), and (5) imply that,
$k - 1 = |\delta_{G-s}(A_1 \cap A_2)| = |\delta_{G-s}(A_1 \cap A_2, A_1 \setminus A_2)| + |\delta_{G-s}(A_1 \cap A_2, A_2 \setminus A_1)|$ $=2|\delta_{G-s}(A_1 \cap A_2, A_2 \setminus A_1)|$. Thus k has to be odd and,

$$|\delta_{G-s}(A_1 \cap A_2, A_1 \setminus A_2)| = |\delta_{G-s}(A_2 \setminus A_1, A_1 \cap A_2)| = \frac{k-1}{2}.$$

Since $\deg(s) > 3$, the maximum size of an independent set is $\lceil \deg(s)/2 \rceil$ by Frank's theorem 2.1.10. Therefore if the union of two intersecting maximal independent sets is the entire vertex set of $L(G, s, k)$, then they are both of size $\lceil \deg(s)/2 \rceil$ and they intersect in exactly one vertex. The examples in Figure 2.4 show that k can be even as well as odd in that case. $\square$

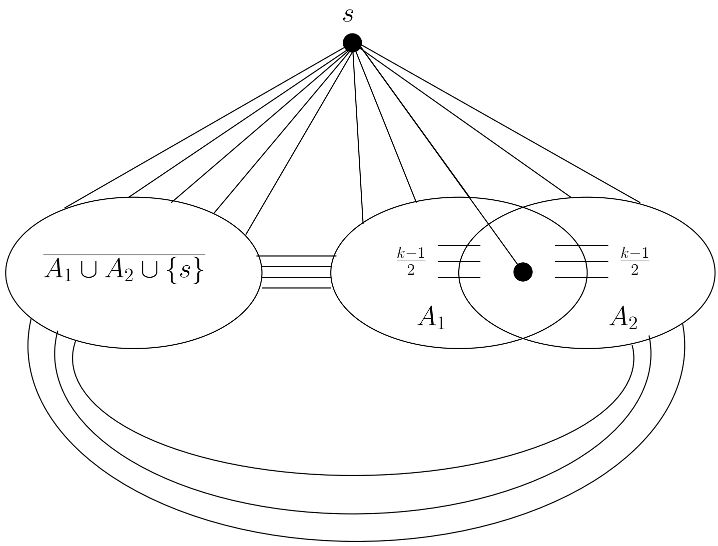
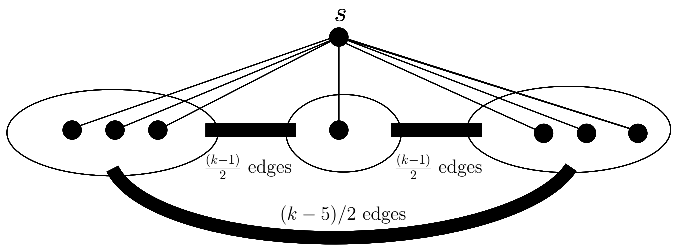


Figure 2.3: Two intersecting dangerous sets corresponding to maximal independent sets not both of size $\lceil \deg(s)/2 \rceil$.

From this Lemma we have as a corollary that when two maximal independent sets in the lifting graph - not both of the large size $\lceil \deg(s)/2 \rceil$ - intersect, then k has to be odd. But there is no restriction on the parity of $\deg(s)$ in that case. On the other hand, if two independent of size $\lceil \deg(s)/2 \rceil$ intersect, then by Lemma 2.1.20 either $\deg(s)$ is odd or is equal to 4.

In case $\deg(s)$ is odd, then $L(G, s, k)$ is an isolated vertex plus a complete bipartite graph as shown in Lemma 2.2.1. The parity of k in that case can be even as well as odd (examples in Figure 2.4 and the right drawing of Figure 2.2). In case $\deg(s) = 4$, such an intersection can only happen if k is odd as shown in the comment after Proposition 3.4 in [18], and the graph G has a specific structure in that case, illustrated in Figure 2.5.

In Lemma 2.2.3 it was also shown that the dangerous sets A_1 and A_2 in G corresponding to the two intersecting maximal independent sets of $L(G, s, k)$ have a fixed number of edges, $(k-1)/2$, between $A_1 \cap A_2$ and $A_1 \setminus A_2$ as well as between $A_1 \cap A_2$ and $A_2 \setminus A_1$, a number depending only on the connectivity. By the example illustrated in the right drawing in Figure 2.2 this does not have to be the case when the two intersecting maximal independent sets are both of size $\lceil \deg(s)/2 \rceil$. However, in the



$L(G, s, k)$ is an isolated vertex plus $K_{3,3}$

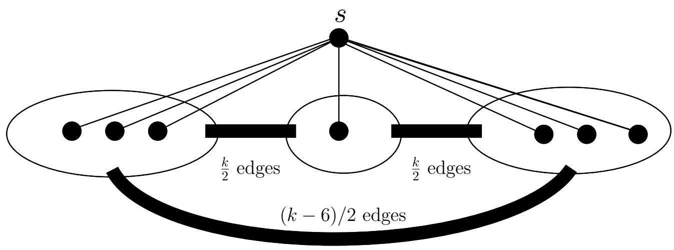


Figure 2.4: Examples with lifting graph an isolated vertex plus complete bipartite graph for even and odd k.

following lemma we will see that the numbers of edges between $A_1 \cap A_2$ and each of the sets $A_1 \setminus A_2$ still has to be equal, and its sum with the number of edges between $A_1 \setminus A_2$ and $A_2 \setminus A_1$ is a constant of the graph depending only on the connectivity and the degree of s.

Lemma 2.2.4. *Let (G, s, k) be a connectivity triple. Suppose that* $\deg(s) \geqslant 3$*,* $\deg(s) \neq 4$*, and two independent sets of size* $\lceil \deg(s)/2 \rceil$ *in* $L(G, s, k)$ *have a non-empty intersection. Then* $\deg(s)$ *is odd and if* A_1 *and* A_2 *are dangerous sets in* G *corresponding to those two independent sets in* $L(G, s, k)$*, then*

(1) there are no vertices outside $A_1 \cup A_2 \cup \{s\}$*;*

(2) $|\delta_G(A_1 \setminus A_2 : A_2 \setminus A_1)| \leqslant (k - \deg(s) + 2)/2$*;*

(3) $|\delta_G(A_2 \setminus A_1 : A_1 \cap A_2)| = |\delta_G(A_1 \cap A_2 : A_1 \setminus A_2)| \geqslant \frac{k-1}{2}$*;*

(4) $|\delta_G(A_1)| = |\delta_G(A_2)| = k + 1$*;*

(5) $|\delta_G(A_1 \setminus A_2 : A_2 \setminus A_1)| + |\delta_G(A_1 \cap A_2 : A_1 \setminus A_2)| = (k+1) - \frac{\deg(s)+1}{2}$*; and*

(6) $\deg(s) \leqslant k + 2$.

Proof. Suppose that $L(G, s, k)$ contains two independent sets of the size $\lceil \deg(s)/2 \rceil$ with non-empty intersection, and that $deg(s) \neq 4$. By Lemma 2.1.20 $\deg(s)$ is odd. By Proposition 2.2.1, if $\deg(s) \neq 3$, then the two sets intersect in exactly one vertex and they are the only maximal independent sets of $L(G, s, k)$ (this is the isolated vertex plus complete bipartite case). In case $\deg(s) = 3$ an independent set of size 2 does not have to be maximal as it is possible that all three edges incident with s form an independent set of size 3. Figure 2.14 in the next section provides examples where all three edges at s form an independent set.

By Lemma 2.1.14, in case $\deg(s) > 4$, there are two dangerous sets A_1 and A_2 in G corresponding to the two maximal independent sets. For $\deg(s) = 3$ the existence of such dangerous sets is guaranteed by Lemma 2.1.15, and by Lemma 2.1.18 each one of those dangerous sets contains exactly two of the neighbours of s (the third neighbour is outside it but in the other dangerous set). In any case, the union $A_1 \cup A_2$ is not dangerous, and hence $|\delta_G(\overline{A_1 \cup A_2})| \geqslant k + 2$ if $\overline{A_1 \cup A_2 \cup \{s\}} \neq \emptyset$.

We again use, in $G - s$, the same standard equation we used before 2.1.1. We first show that there are no vertices in $G - s$ outside the union of two dangerous sets corresponding to the two maximal independent sets of $L(G, s, k)$.

Claim 2.2.5. $A_1 \cup A_2 = V(G) \setminus \{s\}$.

Proof. Suppose by way of contradiction that $\overline{A_1 \cup A_2 \cup \{s\}} \neq \emptyset$. Then the right hand side of equation 2.1.1 applied in $G - s$ is at least

$$(k-1) + \left(k - \left(\frac{\deg(s) - 1}{2}\right)\right) + \left(k - \left(\frac{\deg(s) - 1}{2}\right)\right) + ((k+2) - \deg(s)).$$

Note that s has exactly one neighbour in $A_1 \cap A_2$, and this follows from the last paragraph in the statement of Lemma 2.2.3. This, and the k-edge-connectivity of G, give the first three terms. The last term follows from the fact that $A_1 \cup A_2$ is not dangerous.

This is a lower bound of $4k - 2\deg(s) + 2$. On the other hand, the left side has the upper bound of $2[(k+1-(\frac{\deg(s)+1}{2}))+(k+1-(\frac{\deg(s)+1}{2}))] = 4k+4-2(\deg(s)+1) = 4k-2\deg(s)+2$. Thus, both sides are equal to $4k-2\deg(s)+2$, and the individual upper and lower bounds on each term hold with equality. In particular $|\delta_{G-s}(\overline{A_1 \cup A_2})| = ((k+2)-\deg(s))$.

The set $\overline{A_1 \cup A_2}$ does not contain any neighbours of s, as A_1 and A_2 contain all the neighbours of s. Therefore, $|\delta_G(\overline{A_1 \cup A_2})| = |\delta_{G-s}(\overline{A_1 \cup A_2})|$ $= ((k+2)-\deg(s)) < k$, a contradiction. $\square$

Now, knowing that $\overline{A_1 \cup A_2 \cup \{s\}} = \emptyset$, then the lower bound on the right side of equation 2.1.1 is $(k-1)+(k-(\frac{\deg(s)-1}{2}))+(k-(\frac{\deg(s)-1}{2})) = 3k-\deg(s)$.

The upper bound on the left side is $4k-2\deg(s)+2 = (3k-\deg(s))+(k-\deg(s)+2)$. This means that, cf. equation 2.1.1,
$(|\delta_{G-s}(A_1 \cap A_2 : \overline{A_1 \cup A_2})|+|\delta_{G-s}(A_2 \setminus A_1 : A_1 \setminus A_2)|) \leqslant (k-deg(s)+2)/2$.

We know that $|\delta_{G-s}(A_1 \cap A_2 : \overline{A_1 \cup A_2})| = 0$ as $\overline{A_1 \cup A_2 \cup \{s\}} = \emptyset$. Thus, $|\delta_{G-s}(A_2 \setminus A_1 : A_1 \setminus A_2)| \leqslant (k-\deg(s)+2)/2$, and this same upper bound also holds in G.

Since G is k-edge-connected and, for $i=1,2$, A_i is dangerous,$|\delta(A_i)|$ is either k or $k+1$.

Claim 2.2.6. *For $i=1,2$, $|\delta(A_i)| = k+1$.*

Proof. If, say, $|\delta_G(A_1)| = k$, then,
$|\delta_G(A_1 \setminus A_2 : A_2 \setminus A_1)| + |\delta_G(A_1 \cap A_2 : A_2 \setminus A_1)| + |\delta(\{s\} : A_1)| = k$. It follows that $|\delta_G(A_2 \setminus A_1)| = k-1$, as s has exactly one neighbour in $A_1 \cap A_2$ (Lemma 2.2.3) and $|\delta_G(\{s\} : A_2 \setminus A_1)| = |\delta_G(\{s\} : A_1 \setminus A_2)|$ (as A_1 and A_2 correspond to maximal independent sets of the same size), a contradiction. The same argument holds for A_2. $\square$

The equality $|\delta_G(A_1)| = |\delta_G(A_2)| = k+1$ means that $|\delta_{G-s}(A_1)| = |\delta_{G-s}(A_2)| = k+1-(\frac{\deg(s)+1}{2})$, i.e.
$|\delta_G(A_1 \setminus A_2 : A_2 \setminus A_1)| + |\delta_G(A_1 \cap A_2 : A_2 \setminus A_1)| =$
$|\delta_G(A_2 \setminus A_1 : A_1 \setminus A_2)| + |\delta_G(A_1 \cap A_2 : A_1 \setminus A_2)| = k+1-(\frac{\deg(s)+1}{2})$.
In particular,

$$|\delta_G(A_1 \cap A_2, A_2 \setminus A_1)| = |\delta_G(A_1 \cap A_2, A_1 \setminus A_2)|.$$

Now since $|\delta_G(A_1 \cap A_2)| \geqslant k$ and s has exactly one neighbour in $A_1 \cap A_2$, both
$|\delta_G(A_1 \cap A_2 : A_2 \setminus A_1)|$ and $|\delta_G(A_1 \cap A_2 : A_1 \setminus A_2)|$ have to be at least $(k-1)/2$.

The lower bound $|\delta_G(A_2 \setminus A_1 : A_1 \cap A_2)| = |\delta_G(A_1 \cap A_2 : A_1 \setminus A_2)| \geqslant \frac{k-1}{2}$, and the fact that A_1 and A_2 are both dangerous and each contain $(\deg(s)+1)/2$ neighbours of s, imply that $(\deg(s)+1)/2 \leqslant (k+3)/2$. Thus, $\deg(s) \leqslant k+2$. □

Remark 2.2.7. It is possible that k is even in the previous lemma. That is, it is possible to have the lifting graph of isolated vertex plus complete bipartite with even k (only $\deg(s)$ has to be odd).
In that case $|\delta_G(A_1 \cap A_2 : A_2 \setminus A_1)|$ and $|\delta_G(A_1 \cap A_2 : A_1 \setminus A_2)|$ will be at least $k/2$. See Figure 2.4 and the right drawing of Figure 2.2.

To summarize, we have the following theorem.

Theorem 2.2.8. *Let (G, s, k) be a connectivity triple such that* $\deg(s) \geqslant 4$.

(1) Any two maximal independent sets of $L(G, s, k)$ intersect in at most one vertex.

(2) If $\deg(s) > 4$ and there exists two intersecting maximal independent sets of $L(G, s, k)$ of size $\lceil \deg(s)/2 \rceil$, then $L(G, s, k)$ consists of an isolated vertex and a balanced complete bipartite graph. In particular,

(i) these are the only two maximal independent sets of $L(G, s, k)$;

(ii) they intersect in exactly one vertex; and

(iii) $\deg(s)$ is odd.

(3) The maximal independent sets of $L(G, s, k)$ are pairwise disjoint ($L(G, s, k)$ is complete multipartite) if k is even and

(i) $\deg(s)$ is even or

(ii) at most one independent set of $L(G, s, k)$ has size $\lceil \deg(s)/2 \rceil$.

Moreover, if $L(G, s, k)$ is complete multipartite, then it is not a star.

Proof. From Lemma 2.2.3, we have (1). Suppose that $\deg(s) > 4$ and there exists two intersecting maximal independent sets of $L(G, s, k)$ of size $\lceil \deg(s)/2 \rceil$, then by Lemma 2.2.4 it follows that $\deg(s)$ is odd, and then by Proposition 2.2.1 we have that $L(G, s, k)$ consists of an isolated vertex and a balanced complete bipartite graph. This completes the proof of (2).

If k is even and $\deg(s)$ is even, then the maximal independent sets of $L(G, s, k)$ are pairwise disjoint by (4) in Theorem 2.1.11. If at most one independent set of $L(G, s, k)$ has size $\lceil \deg(s)/2 \rceil$, then either $L(G, s, k)$ consists of two disjoint maximal independent sets of sizes $\lceil \deg(s)/2 \rceil$ and $\lfloor \deg(s)/2 \rfloor$, or for any two maximal independent sets I_1 and I_2, $I_1 \cup I_2 \neq V(L(G, s, k))$, and by Lemma 2.2.3 it follows that in case k is even, $I_1 \cap I_2 = \emptyset$.

Since $\deg(s) \geqslant 4$, then by the theorem of Frank 2.1.10, $L(G, s, k)$ cannot be a star because a star on n vertices contains a maximal independent set of size $n - 1 > \lceil \frac{n}{2} \rceil$. □

To be able to talk more neatly about a connected collection of maximal independent sets in $L(G, s, k)$, we define the *independence graph.*

Definition 2.2.9. (Independence Graph) For a graph H the independence graph $I(H)$ is the graph whose vertex set is the set of maximal independent sets of H and in which two vertices are adjacent if and only if the corresponding independent sets have a nonempty intersection.

Note that the complement of H is connected if and only if $I(H)$ is connected. Note also that when $L(G, s, k)$ is complete multipartite, $I(L(G, s, k))$ is a collection of isolated vertices.

Rereading Theorem 2.2.8 we see for example that when $L(G, s, k)$ is an isolated vertex plus a balanced complete bipartite graph, $I(L(G, s, k))$ is a path of length two, and when $L(G, s, k)$ is complete multipartite graph, $I(L(G, s, k))$ consists of singletons.

In Section 3.2 of [18] it was shown that when $\deg(s) = 4$, $L(G, s, k)$ is either a 4-cycle, a K_4, or a perfect matching. It was also proved there that the matching case does not occur when k is even. Therefore when k is even any two maximal independent sets are disjoint, otherwise, they intersect in at most one vertex.

In the matching case, which can happen only when k is odd, $I(L(G, s, k))$ is a 4-cycle. The maximal independent sets of $L(G, s, k)$ are each of size 2 and they form a 4-cycle in which every two consecutive independent sets intersect in exactly one vertex. See Figure 2.5.

Lemma 2.2.10. *[18] Let (G, s, k) be a connectivity triple. If $\deg(s) = 4$, then $L(G, s, k)$ is one of: a perfect matching; C_4; and K_4. If k is even, then $L(G, s, k)$ is not a perfect matching.*
Moreover, if $L(G, s, k)$ is a perfect matching, then G consists of the vertex s and four sets of vertices S_i, $1 \leqslant i \leqslant 4$, such that s has exactly one neighbour in each S_i, and $|\delta(S_1 : S_2) = \delta(S_2 : S_3) = \delta(S_3 : S_4) = \delta(S_4 : S_1)$.

This cyclic structure, for $\deg(s) = 4$, is the basis for the structural results we prove in the next section. We will generalize this cyclic structure to arbitrary $\deg(s)$ and show that it is one of two possible structures that happen when the maximal independent sets of the lifting graph form a connected entity, i.e. when $I(L(G, s, k))$ is connected.

2.2.1 Maximal independent sets and dangerous sets

In this subsection we will study the arrangement of the maximal independent sets of $L(G, s, k)$ and their corresponding dangerous sets in G. This is the way we will learn the relation between the structure of a graph and the structure of its lifting graph. In the previous section, we saw that, from [18], for $deg(s) = 4$, only in the case $L(G, s, k)$ is a perfect matching is $I(L(G, s, k))$ connected: it is a 4-cycle, and this case can happen only when k is odd. In this section we show that in general there are only two possibilities for $I(L(G, s, k))$ to be connected: it is either a path or a cycle. The latter generalizes the situation for $deg(s) = 4$. See Figure 2.5.

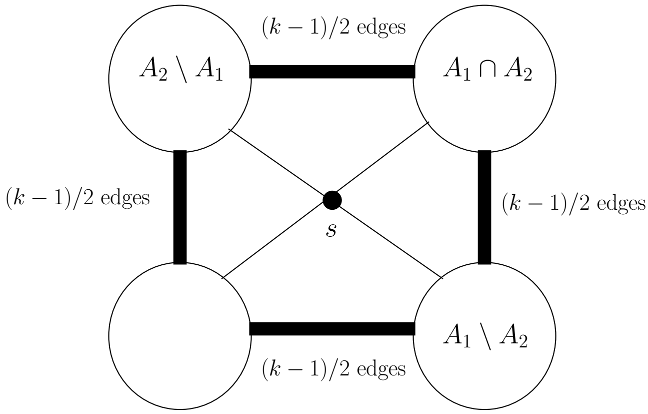


Figure 2.5: The graph G has this structure if $deg(s) = 4$ and two independent sets of size 2 in $L(G, s, k)$ have a nonempty intersection.

We will begin by showing that no three maximal independent sets of $L(G, s, k)$ have a nonempty common intersection. For this, we need first to prove the following lemma about three intersecting dangerous sets.

Lemma 2.2.11. *Let (G, s, k) be a connectivity triple such that k is odd, and let A_1, A_2, A_3 be three distinct dangerous sets such that, for any distinct $i, j \in \{1, 2, 3\}$:*

(a) $A_i \cap A_j$ contains at most one neighbour of s;

(b) $|\delta(A_i \cap A_j : A_j \setminus A_i)| = (k-1)/2$;

(c) $|\delta(A_i \setminus A_j : A_j \setminus A_i)| = 0$; and

(d) $|\delta_{G-s}(A_i \cap A_j : \overline{A_i \cup A_j})| = 0$.

Then $A_1 \cap A_2 \cap A_3 = \emptyset$.

Proof. By way of contradiction, assume that $A_1 \cap A_2 \cap A_3 \neq \emptyset$.

Claim 2.2.12. *At least one of the sets $(A_1 \cap A_2) \setminus A_3$, $(A_1 \cap A_3) \setminus A_2$, and $(A_2 \cap A_3) \setminus A_1$ is empty.*

Proof. Assume that all of these sets are non-empty. Let

- $a = \delta((A_2 \cap A_3) \setminus A_1 : A_2 \setminus (A_1 \cup A_3))$,

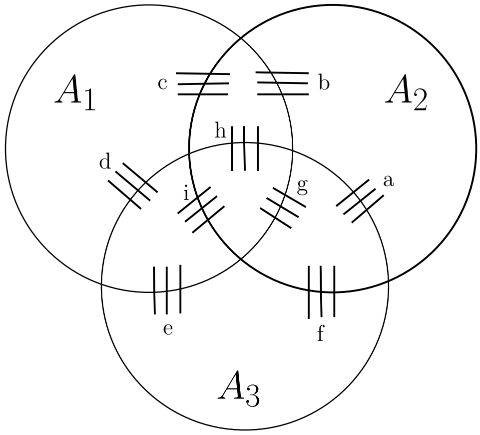


Figure 2.6: Intersection of three dangerous sets.

- $b = \delta((A_1 \cap A_2) \setminus A_3 : A_2 \setminus (A_1 \cup A_3))$,
- $c = \delta((A_1 \cap A_2) \setminus A_3 : A_1 \setminus (A_2 \cup A_3))$,
- $d = \delta((A_1 \cap A_3) \setminus A_2 : A_1 \setminus (A_2 \cup A_3))$,
- $e = \delta((A_1 \cap A_3) \setminus A_2 : A_3 \setminus (A_1 \cup A_2))$,
- $f = \delta((A_2 \cap A_3) \setminus A_1 : A_3 \setminus (A_1 \cup A_2))$,
- $g = \delta(A_1 \cap A_2 \cap A_3 : (A_2 \cap A_3) \setminus A_1)$,
- $h = \delta(A_1 \cap A_2 \cap A_3 : (A_1 \cap A_2) \setminus A_3)$, and
- $i = \delta(A_1 \cap A_2 \cap A_3 : (A_1 \cap A_3) \setminus A_2)$.

Illustrated in Figure 2.6.
These are all the different edge sets between the different parts of the intersections of A_1, A_2, and A_3 since we are given $|\delta(A_i \setminus A_j : A_j \setminus A_i)| = 0$, and $|\delta(A_i \cap A_j : \overline{A_i \cup A_j})| = 0$.

We will show that each one of g, h, and i is at least $(k-1)/3$ and then show that we have a contradiction, and as a result $(A_1 \cap A_2) \setminus A_3$, $(A_1 \cap A_3) \setminus A_2$, and $(A_3 \cap A_2) \setminus A_1$ cannot be all nonempty.

If one of g, h, or i is less than $(k-1)/3$, then by k-edge-connectivity, then one of them has to be greater than $(k-1)/3$ because $\delta(A_1 \cap A_2 \cap A_3)$ contains at most one edge incident with s (hypothesis (a)) in addition to

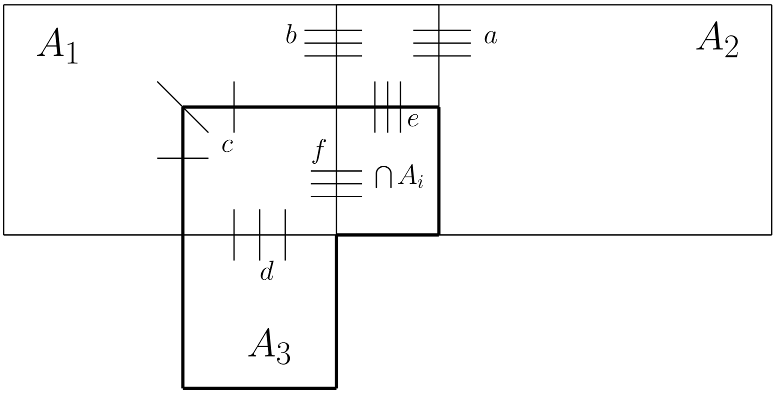


Figure 2.7: $(A_3 \cap A_2) \setminus A_1$ is empty.

the three sets of edges whose sizes are g, h, and i. Suppose without loss of generality that g is less than $(k-1)/3$, and h is greater than $(k-1)/3$. By (b), $g+b = (k-1)/2$ and $h+a = (k-1)/2$, therefore b is greater than $(k-1)/6$ and a is less than $(k-1)/6$.

Since $(A_2 \cap A_3) \setminus A_1$ is nonempty by assumption, $|\delta((A_2 \cap A_3) \setminus A_1)|$ has to be at least k. We must have $f > (k-1)/2$ because g is less than $(k-1)/3$, a is less than $(k-1)/6$, and $\delta((A_2 \cap A_3) \setminus A_1)$ contains at most one edge incident with s. This is a contradiction to the fact that $f + i = (k-1)/2$ (hypothesis (b)).

Thus each one of g, h, and i is at least $(k-1)/3$. Then, by (b), a, b, c, d, e, f are each at most $\frac{(k-1)}{6}$. Again, since $(A_2 \cap A_3) \setminus A_1$ is nonempty by assumption and contains at most one neighbour of s, now g has to be at least $k - (1 + \frac{2(k-1)}{6}) = \frac{2(k-1)}{3} > \frac{(k-1)}{2}$, a contradiction to (b).

Thus our assumption that $A_1 \cap A_2 \cap A_3$, $(A_1 \cap A_2) \setminus A_3$, $(A_1 \cap A_3) \setminus A_2$, and $(A_3 \cap A_2) \setminus A_1$ are all nonempty is false. $\square$

Claim 2.2.13. *At least two of the sets $(A_1 \cap A_2) \setminus A_3$, $(A_1 \cap A_3) \setminus A_2$, and $(A_2 \cap A_3) \setminus A_1$ are empty.*

Proof. By Claim 2.2.12, we may assume $(A_2 \cap A_3) \setminus A_1$ is empty and by way of contradiction we assume the other two are not empty. See Figure 2.7.

Note that there are no edges between $A_1 \cap A_2 \cap A_3$ and $A_2 \setminus A_1$ because $A_2 \setminus A_1$ is a subset of $\overline{A_1 \cup A_3}$ (as $(A_2 \cap A_3) \setminus A_1$ is empty).

As illustrated in Figure 2.7, let

- $a = |\delta((A_1 \cap A_2) \setminus A_3 : A_2 \setminus A_1)|$,
- $b = |\delta((A_1 \cap A_2) \setminus A_3 : A_1 \setminus (A_2 \cup A_3))|$,
- $c = |\delta((A_1 \cap A_3) \setminus A_2 : A_1 \setminus (A_2 \cup A_3))|$,
- $d = |\delta((A_1 \cap A_3) \setminus A_2 : A_3 \setminus (A_1 \cup A_2))|$,
- $e = |\delta(A_1 \cap A_2 \cap A_3 : (A_1 \cap A_2) \setminus A_3)|$, and
- $f = |\delta(A_1 \cap A_2 \cap A_3 : (A_3 \cap A_1) \setminus A_2)|$.

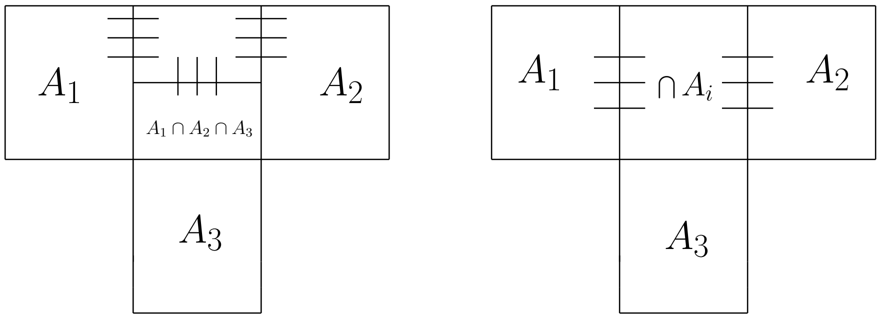
Note that $\delta(A_1 \cap A_2 \cap A_3 : A_2 \setminus A_3) = \delta(A_1 \cap A_2 \cap A_3 : (A_1 \cap A_2) \setminus A_3)$ (e in Figure 2.7) because $A_1 \cap A_2 \cap A_3 \subseteq A_1 \cap A_3$ and $\delta_{G-s}(A_1 \cap A_3 : \overline{A_1 \cup A_3}) = \emptyset$.

If $e < (k-1)/2$, then $f > (k-1)/2$ for $|\delta(A_1 \cap A_2 \cap A_3)|$ to be at least k since $(A_1 \cap A_2 \cap A_3)$ contains at most one neighbour of s. However, this is impossible because $f + b = (k-1)/2$ (hypothesis (b)). Thus, by symmetry, each one of e and f is at least $(k-1)/2$, and so is in fact equal to $(k-1)/2$ and each of b and c is equal to 0.

Since $(A_3 \cap A_2) \setminus A_1 = \emptyset$, and by hypothesis ($d$) once applied with $\{i, j\} = \{1, 2\}$ and once with $\{i, j\} = \{1, 3\}$, we have $\delta(A_1 \cap A_3 : A_3 \setminus A_1) = \delta((A_1 \cap A_3) \setminus A_2 : A_3 \setminus (A_1 \cup A_2))$ (d in Figure 2.7) and $\delta(A_1 \cap A_2 : A_2 \setminus A_1) = \delta((A_1 \cap A_2) \setminus A_3 : A_2 \setminus (A_1 \cup A_3))$ (a in Figure 2.7). Thus, by hypothesis (b), $a = d = (k-1)/2$.

Since $A_1 \cap A_2$ contains at most one neighbour of s, either $(A_1 \cap A_2) \setminus A_3$ or $A_1 \cap A_2 \cap A_3$ does not contain a neighbour of s. Therefore, either $|\delta((A_1 \cap A_2) \setminus A_3)| \leqslant a + b + e = \frac{k-1}{2} + 0 + \frac{k-1}{2} = k - 1$ or $|\delta(A_1 \cap A_2 \cap A_3)| \leqslant e + f = (k-1)$, a contradiction. $\square$

Claim 2.2.14. *All of* $(A_1 \cap A_2) \setminus A_3$, $(A_1 \cap A_3) \setminus A_2$, *and* $(A_2 \cap A_3) \setminus A_1$ *are empty.*



$(A_1 \cap A_2) \setminus A_3 \neq \emptyset$ $\quad$ $(A_1 \cap A_2) \setminus A_3 = (A_1 \cap A_3) \setminus A_2 = (A_3 \cap A_2) \setminus A_1 = \emptyset$

Figure 2.8

Proof. We may assume without loss of generality that only $((A_1 \cap A_2) \setminus A_3)$ is nonempty. This is illustrated in the left drawing of Figure 2.8. There are no edges between $A_1 \cap A_2 \cap A_3$ and any set of the form $A_i \setminus (A_j \cup A_l)$, because those will be edges of the sort $\delta_{G-s}(A_j \cap A_l : \overline{A_j \cup A_l})$, and by (d) this latter set is empty. Thus the only edges we have coming out of $A_1 \cap A_2 \cap A_3$ are to $(A_1 \cap A_2) \setminus A_3$. However, since $A_1 \cap A_2 \cap A_3 = A_2 \cap A_3$ in the case we are discussing now, there should be $(k-1)/2$ edges from $A_1 \cap A_2 \cap A_3$ to $A_3 \setminus A_2$, but $A_3 \setminus A_2 = A_3 \setminus (A_1 \cup A_2)$, a contradiction. $\square$

Now, $(A_1 \cap A_2) \setminus A_3$, $(A_1 \cap A_3) \setminus A_2$, and $(A_3 \cap A_2) \setminus A_1$ are all empty (right drawing in Figure 2.8). By (b) there are $(k-1)/2$ edges from $A_1 \cap A_2 = A_1 \cap A_2 \cap A_3$ to $A_2 \setminus A_1$. Those are also edges from $A_1 \cap A_3$ to $\overline{A_1 \cup A_3}$, and so we have a contradiction because $\delta_{G-s}(A_1 \cap A_3 : \overline{A_1 \cup A_3}) = \emptyset$ ($k \geqslant 2$ in the definition of a connectivity triple, and so $(k-1)/2 > 0$). $\square$

Lemma 2.2.15. *Let (G, s, k) be a connectivity triple such that* $\deg(s) \geqslant 4$. *Then no three distinct maximal independent sets of $L(G, s, k)$ have a nonempty common intersection.*

Proof. By way of contradiction, assume that there are three maximal independent sets I_1, I_2, and I_3 in $L(G, s, k)$ such that $I_1 \cap I_2 \cap I_3 \neq \emptyset$, and let A_1, A_2, and A_3 respectively be corresponding dangerous sets in G. By Lemma 2.2.3, $|I_1 \cap I_2 \cap I_3| = 1$, so s has exactly one neighbour in $A_1 \cap A_2 \cap A_3$.

At most one of I_1, I_2, and I_3 has size $\lceil deg(s)/2 \rceil$: If two intersecting independent sets have this size, then either $\deg(s) = 4$ and four independent sets of size 2 each form a cycle, Lemma 2.2.10, or this is the isolated vertex plus complete bipartite case and there is no third maximal independent set, (2) in Theorem 2.2.8.

By Lemma 2.2.3, k is odd since two independent sets not both of size $\lceil \deg(s)/2 \rceil$ have a nonempty intersection, and we have $|\delta(A_i \cap A_j : A_i \setminus A_j)| = |\delta(A_i \cap A_j : A_j \setminus A_i)| = (k-1)/2$.

Also from Lemma 2.2.3 we know that for each i and j in $\{1, 2, 3\}$, $|I_i \cap I_j| = 1$, $|\delta_{G-s}(A_i \cap A_j : \overline{A_i \cup A_j})| = 0$, and $|\delta(A_i \setminus A_j : A_j \setminus A_i)| = 0$. In particular, the only sets of edges between the different parts of the intersections of A_1, A_2, and A_3 are the ones illustrated in Figure 2.6.

All the conditions of Lemma 2.2.11 are satisfied, therefore, $\bigcap_{i=1}^{3} A_i = \emptyset$. This is a contradiction since $\bigcap_{i=1}^{3} A_i = \emptyset$ contains the non-s end-vertex of the common edge between the three maximal independent sets. $\square$

The following lemma tells us that for each pair of disjoint maximal independent sets in $L(G, s, k)$ there exist a pair of disjoint dangerous sets corresponding to them in G.

Lemma 2.2.16. *Let (G, s, k) be a connectivity triple such that $\deg(s) \geqslant 4$. If I_1 and I_2 are disjoint maximal independent sets in $L(G, s, k)$, and A_1 and A_2 are corresponding dangerous sets in G, then either $A_1 \setminus A_2$ or $A_2 \setminus A_1$ is dangerous.*

Proof. Consider two disjoint maximal independent sets I_1 and I_2 of $L(G, s, k)$ and let A_1 and A_2 be corresponding dangerous sets in G such that $A_1 \cap A_2 \neq \emptyset$. Define k_1 to be $|\delta(\{s\} : A_1 \setminus A_2)|$, $k_2 = |\delta(\{s\} : A_2 \setminus A_1)|$, and $k_3 = |\delta(\{s\} : A_1 \cap A_2)|$. By the assumption of the independent sets being disjoint, $k_3 = 0$, so k_1 and k_2 are the sizes of the two independent sets.

Now, in $G - s$,

(1) $|\delta_{G-s}(A_1)| \leqslant (k+1) - k_1$,

(2) $|\delta_{G-s}(A_2)| \leqslant (k+1) - k_2$,

(3) $|\delta_{G-s}(A_1 \cap A_2)| \geqslant k$,

(4) $|\delta_{G-s}(A_2 \setminus A_1)| \geqslant k - k_2$,

(5) $|\delta_{G-s}(A_1 \setminus A_2)| \geqslant k - k_1$, and

(6) if $\overline{A_1 \cup A_2 \cup \{s\}} \neq \emptyset$, then $|\delta_{G-s}(\overline{A_1 \cup A_2})| \geqslant (k+2) - (k_1 + k_2)$ because $A_1 \cup A_2$ is not dangerous.

In the first case, $\overline{A_1 \cup A_2 \cup \{s\}} = \emptyset$. Then either $k_1 = k_2 = \deg(s)/2$, in case $\deg(s)$ is even, or one of k_1 and k_2, say k_1, is $\frac{(\deg(s)+1)}{2}$ and the other is $\frac{(\deg(s)-1)}{2}$ in case $\deg(s)$ is odd (Frank's theorem 2.1.10). Since $\overline{A_1 \cup A_2 \cup \{s\}} = \emptyset$, we have,

$$|\delta(A_1)| = k_1 + |\delta(A_1 \cap A_2 : A_2 \setminus A_1)| + |\delta(A_1 \setminus A_2 : A_2 \setminus A_1)|$$

and

$$|\delta(A_2 \setminus A_1)| = k_2 + |\delta(A_1 \cap A_2 : A_2 \setminus A_1)| + |\delta(A_1 \setminus A_2 : A_2 \setminus A_1)|.$$

Thus, $0 \leqslant |\delta(A_1)| - |\delta(A_2 \setminus A_1)| \leqslant 1$, so $A_2 \setminus A_1$ is dangerous, and we are done.

We will show that, in case $\overline{A_1 \cup A_2 \cup \{s\}} \neq \emptyset$, either $\delta_{G-s}(A_1 \setminus A_2) \leqslant (k+1) - k_1$ or $\delta_{G-s}(A_2 \setminus A_1) \leqslant (k+1) - k_2$, i.e. $\delta_G(A_1 \setminus A_2) \leqslant (k+1)$ or $\delta_G(A_2 \setminus A_1) \leqslant (k+1)$. That is either $(A_1 \setminus A_2)$ or $(A_2 \setminus A_1)$ is dangerous in G. Then we can replace A_1 by $A_1 \setminus A_2$ or replace A_2 by $A_2 \setminus A_1$ and consequently have disjoint dangerous sets.

In the remaining case, $\overline{A_1 \cup A_2 \cup \{s\}} \neq \emptyset$. Let $\varepsilon_i = |\delta_{G-s}(A_i \setminus A_{3-i})| - (k - k_i)$, for $i = 1, 2$. Then (1) – (5) applied to the standard equation 2.1.1 in $G - s$ gives,

$$4k - 2(k_1 + k_2) + 2 + \varepsilon_1 + \varepsilon_2 \leqslant RHS = LHS \leqslant 4k - 2(k_1 + k_2) + 4$$

Therefore $\varepsilon_1 + \varepsilon_2 \leqslant 2$, so, for some $i \in \{1, 2\}$, $\varepsilon_i \leqslant 1$. For such an i, $A_i \setminus A_{3-i}$ is dangerous, as required. $\square$

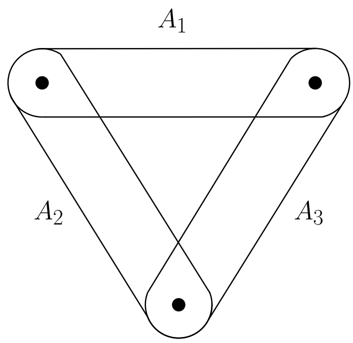


Figure 2.9: Cycle of three dangerous sets.

Lemma 2.2.17. *Let (G, s, k) be a connectivity triple such that $\deg(s) \geqslant 4$. Then for any three maximal independent sets I_1, I_2, I_3 in $L(G, s, k)$ there exist corresponding dangerous sets A_1, A_2, A_3 in G such that $A_1 \cap A_2 \cap A_3 = \emptyset$. In particular, if I_1, I_2, I_3 form a cycle in $I(L(G, s, k))$, then A_1, A_2, A_3 can be chosen such that for $\{i, j, l\} = \{1, 2, 3\}$, A_i intersects $A_j \cup A_l$ exactly in $A_j \setminus A_l$ and $A_l \setminus A_j$, see Figure 2.9.*

Proof. By Lemma 2.2.15 $I_1 \cap I_2 \cap I_3 = \emptyset$. Up to symmetry there are four cases to consider:

(1) $I_1 \cap I_2 \neq \emptyset$, $I_2 \cap I_3 \neq \emptyset$, but $I_1 \cap I_3 = \emptyset$, that is $I_1 I_2 I_3$ is a path in $I(L(G, s, k))$,

(2) I_1, I_2, and I_3 make a cycle in $I(L(G, s, k))$,

(3) $I_1 \cap I_2 \neq \emptyset$ but I_3 is disjoint from $I_1 \cup I_2$,

(4) I_1, I_2, and I_3 are pairwise disjoint.

By Lemma 2.1.14 there are dangerous sets A_1, A_2, and A_3 in G corresponding to I_1, I_2, and I_3 respectively. By Lemma 2.2.16, in cases (1), (3), and (4), A_1, A_2, and A_3 can be chosen such that at least two of them are disjoint. Then $A_1 \cap A_2 \cap A_3 = \emptyset$.

Now suppose that I_1, I_2, I_3 form a cycle in $I(L(G, s, k))$, and let A_1, A_2, and A_3 be corresponding dangerous sets. By Lemma 2.2.3, each I_i and I_j intersect in exactly one vertex, so s has exactly one neighbour in

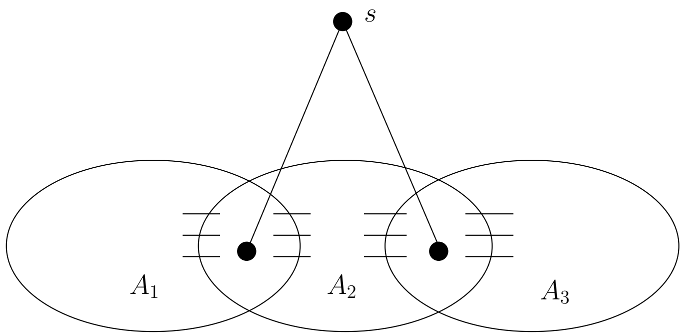


Figure 2.10

$A_i \cap A_j$. For distinct $i, j \in \{1, 2, 3\}$, $I_i \cup I_j \neq V(L(G, s, k))$ because the only case in which the union of two maximal independent sets of $L(G, s, k)$ equals $V(L(G, s, k))$ is the case when $L(G, s, k)$ is an isolated vertex plus a complete balanced bipartite graph, and there are only two maximal independent sets in that case, whereas here we have at least three maximal independent sets. Thus, by Lemma 2.2.3, hypotheses $(a) - (d)$ of Lemma 2.2.11 hold for any distinct $i, j \in \{1, 2, 3\}$. Therefore, $A_1 \cap A_2 \cap A_3 = \emptyset$. $\square$

Lemma 2.2.18. *Let (G, s, k) be a connectivity triple such that $\deg(s) \geqslant 4$. Let A_1, A_2, A_3 be, in order, dangerous sets corresponding to consecutive vertices I_1, I_2, I_3 in a path in $I(L(G, s, k))$. Then (see Figure 2.10),*

(1) $A_2 \setminus (A_1 \cup A_3) = \emptyset$;

(2) s has exactly two neighbours in A_2;

(3) for $i = 1, 3$, $|\delta(\{s\} : A_i \cap A_2)| = 1$; and

(4) $|\delta(A_1 \cap A_2 : A_2 \cap A_3)| = (k - 1)/2$.

It is possible in this statement that I_1 is adjacent to I_3 in $I(L(G, s, k))$.

Proof. Let A_1, A_2, and A_3 be three dangerous sets as described above. The assumption that $I(L(G, s, k))$ has three vertices, i.e. $L(G, s, k)$ contains three maximal independent sets, means that the union of any two maximal independent sets in $L(G, s, k)$ is a proper subset of $V(L(G, s, k))$. Thus Lemma 2.2.3 applies to any two of A_1, A_2, and A_3. See Figure 2.10.

In particular, $|\delta(A_1 \cap A_2 : A_1 \setminus A_2)| = |\delta(A_3 \cap A_2 : A_3 \setminus A_2)| = (k-1)/2$, and $|\delta(\{s\} : A_1 \cap A_2)| = |\delta(\{s\} : A_3 \cap A_2)| = 1$. This is a total of $1 + 1 + \frac{(k-1)}{2} + \frac{(k-1)}{2} = k + 1$ edges in $\delta(A_2)$. There cannot be more since A_2 is dangerous. Therefore, the two edges from s to $A_1 \cap A_2$ and $A_2 \cap A_3$ are the only edges between s and A_2, and $|\delta_{G-s}(A_2 : \overline{A_1 \cup A_2 \cup A_3})| = 0$.

Now we show that $A_2 \setminus (A_1 \cup A_3) = \emptyset$. By way of contradiction suppose not. As shown in the previous paragraph, $|\delta(\{s\} : A_2 \setminus (A_1 \cup A_3))| = 0$ and $|\delta(A_2 \setminus (A_1 \cup A_3) : \overline{A_1 \cup A_2 \cup A_3})| = 0$. By Lemma 2.2.3, $|\delta(A_2 \setminus (A_1 \cup A_3) : (A_1 \setminus A_2) \cup (A_3 \setminus A_2))| = 0$. Thus, $|\delta(A_2 \setminus (A_1 \cup A_3))| = |\delta(A_2 \setminus (A_1 \cup A_3) : A_1 \cap A_2)| + |\delta(A_2 \setminus (A_1 \cup A_3) : A_3 \cap A_2)|$. This is less than or equal to $|\delta(A_2 \setminus A_1 : A_1 \cap A_2)| + |\delta(A_2 \setminus A_3 : A_3 \cap A_2)| = \frac{(k-1)}{2} + \frac{(k-1)}{2} = k-1$, a contradiction since (G, s, k) is a connectivity triple.

Now, since $A_2 \setminus (A_1 \cup A_3) = \emptyset$, $A_2 \setminus A_1 = A_2 \cap A_3$. Thus from Lemma 2.2.3, part (2)-(d), we have $|\delta(A_1 \cap A_2 : A_2 \cap A_3)| = (k-1)/2$. $\square$

Lemmas 2.2.15, 2.2.17, and 2.2.18 together give us the following.

Lemma 2.2.19. *Let (G, s, k) be a connectivity triple such that* $\deg(s) \geqslant 4$. *Then every component of $I(L(G, s, k))$ is either a path (possibly a singleton) or a cycle. Moreover, there exist corresponding dangerous sets in G that respectively make a path or a cycle via intersections.*

Proof. We will show that $I(L(G, s, k))$ has no vertex of degree at least 3. Therefore, each component of $I(L(G, s, k))$ is either a cycle or a path. Suppose for a contradiction that $I(L(G, s, k))$ contains a vertex of degree 3. Then there are maximal independent sets I_1, I_2, I_3, I_4 in $L(G, s, k)$ such that I_1 intersects each one of I_2, I_3, and I_4. By Lemma 2.2.15, no three of I_2, I_3, and I_4 have a nonempty common intersection, therefore, I_4 intersects I_1 in $I_1 \setminus (I_2 \cup I_3)$. Let A_1, A_2, A_3, and A_4 be corresponding dangerous sets. Then the non-s end of the unique edge in $I_4 \cap (I_1 \setminus (I_2 \cup I_3))$ belongs to $(A_4 \cap A_1) \setminus (A_2 \cup A_3)$, contradicting Lemma 2.2.18 as $A_1 \setminus (A_2 \cup A_3)$ should be empty.

We can choose dangerous sets corresponding to non-adjacent vertices of $I(L(G,s,k))$ to be disjoint. Therefore, this collection of dangerous sets corresponding to a component of $I(L(G,s,k))$ will form either a path or a cycle via intersections. Each one of those intersections contains exactly one neighbour of s (but may contain other vertices).
In particular, if a component of $I(L(G,s,k))$ is a 3-cycle with vertices I_1, I_2, and I_3, then by Lemma 2.2.17 there are corresponding dangerous sets A_1, A_2, and A_3 such that $A_1 \cap A_2 \cap A_3 = \emptyset$, forming a 3-cycle of dangerous sets. □

Lemma 2.2.20. *Let (G,s,k) be a connectivity triple such that $\deg(s) \geqslant 4$. If A_1 and A_2 are minimal dangerous sets in G corresponding to disjoint maximal independent sets in $L(G,s,k)$, then A_1 and A_2 are disjoint.*

Proof. By Lemma 2.2.16, either $A_1 \setminus A_2$ or $A_2 \setminus A_1$ is dangerous. If $A_1 \cap A_2 \neq \emptyset$, then this contradicts the minimality of either A_1 or A_2. □

By choosing dangerous sets corresponding to maximal independent sets to be minimal, we have the path and cycle structures of the components of $I(L(G,s,k))$ reflected in the arrangement of dangerous sets.

From Lemma 2.2.19 that each component of $I(L(G,s,k))$ is either a path or a cycle, thus any spanning tree of any component is a path.

Lemma 2.2.21. *Let (G,s,k) be a connectivity triple such that $\deg(s) \geqslant 4$. If $A_1, \cdots, A_m$ are, in order, minimal dangerous sets corresponding to a spanning path of a component C of $I(L(G,s,k))$, then*

(1) $\delta_{G-s}(A_1 \cup \cdots \cup A_m) = \emptyset$ if C is a cycle, and

(2) $\delta_{G-s}(A_1 \cup \cdots \cup A_m) = \delta_{G-s}((A_1 \setminus A_2) \cup (A_m \setminus A_{m-1}) : \overline{A_1 \cup \cdots \cup A_m})$ if C is a path.

Proof. The dangerous sets in a cycle or the interior of a path are saturated. That is we have already identified $k+1$ edges incident with each one of them, as follows. If A_2 is between A_1 and A_3, then there are $(k-1)/2$ edges to $A_1 \setminus A_2$, $(k-1)/2$ edges to $A_3 \setminus A_2$, one edge from s to $A_2 \cap A_1$, and one edge from s to $A_2 \cap A_3$.

Thus if the component C is a cycle, then the corresponding cycle of dangerous sets in G does not have any edges to vertices outside their union except to s. If C is a path, then only the first and last sets A_1 and A_m of the corresponding path of dangerous sets in G can have edges to vertices in $\overline{A_1 \cup \cdots \cup A_m}$ different from s. $\square$

Lemma 2.2.22. *Let (G, s, k) be a connectivity triple such that $\deg(s) \geqslant 4$. For each maximal independent set of $L(G, s, k)$ fix one minimal corresponding dangerous set in G, and let $\mathcal{D}$ be the collection of these dangerous sets. If every component of $I(L(G, s, k))$ has at least two vertices, and at most one component of $I(L(G, s, k))$ is a path, then there are no vertices in $G - s$ outside the union of all the sets in $\mathcal{D}$.*
Moreover, if exactly one of the components of $I(L(G, s, k))$ is a path, and this component is not a singleton, and $A_1, \cdots, A_m$ is the corresponding path of dangerous sets in G, then

(1) $\delta(A_1 \cup \cdots \cup A_m) \subseteq \delta(\{s\})$,

(2) $\delta(A_1 : \overline{\{s\} \cup A_2 \cup A_m}) = \emptyset$, *and*

(3) $\delta(A_m : \overline{\{s\} \cup A_{m-1} \cup A_1}) = \emptyset$.

Proof. Let $B = V(G - (\{s\} \cup \bigcup \mathcal{D}))$. Suppose by way of contradiction that at most one component of $I(L(G, s, k))$ is a path, this component is not a singleton, and $B \neq \emptyset$. Then since (G, s, k) is a connectivity triple, $|\delta(B)| \geqslant k$. All the neighbours of s are in $\bigcup \mathcal{D}$, therefore, $\delta(B : \{s\}) = \emptyset$, i.e. $\delta(B) = \delta(B : \bigcup \mathcal{D})$.

If C is a cycle component of $I(L(G, s, k))$ and $D_1, \cdots, D_n$ is a corresponding cycle of dangerous sets, then $\delta_{G-s}(D_1 \cup \cdots \cup D_n) = \emptyset$ (Lemma 2.2.21). Thus for any such component $\delta(B : D_1 \cup \cdots \cup D_n) = \emptyset$, so if all the components of $I(L(G, s, k))$ are cycles, there cannot be vertices in $G - s$ outside $\bigcup \mathcal{D}$.

Now let us consider the case when exactly one of the components is a path. By hypothesis this component is not a singleton. Let $A_1, \cdots, A_m$ be the corresponding path of dangerous sets. Then $m \geqslant 2$ and $\delta(B) =$

$\delta(B : A_1 \cup \cdots \cup A_m)$. Again by Lemma 2.2.21, if $m \geqslant 3$, then $\delta(B : A_i) = \emptyset$ for any $1 < i < m$. Thus, $\delta(B) = \delta(B : A_1 \cup A_m)$.

By Lemma 2.2.3 $|\delta(A_{m-1} \cap A_m : A_{m-1} \setminus A_m)| = \frac{(k-1)}{2}$ and also that $|\delta(A_2 \cap A_1 : A_2 \setminus A_1)| = \frac{(k-1)}{2}$. There is one edge from s to $A_{m-1} \cap A_m$, and one edge from s to $A_2 \cap A_1$, and there is at least one edge from s to $A_1 \setminus A_2$ and at least one edge to $A_m \setminus A_{m-1}$. By Lemma 2.2.3, (b) and (c), these are the only edges that leave A_1, or A_m.

Therefore, since A_1 and A_m are dangerous, both $|\delta(A_1 : B)|$ and $|\delta(A_m : B)|$ are at most $(k+1) - (1+1+\frac{(k-1)}{2}) = \frac{(k-1)}{2}$. Thus if there are any vertices in $G - s$ outside $\bigcup \mathcal{D}$, then $|\delta(B)| \leqslant k-1$, a contradiction.

Now it can be easily seen that any edges coming out of A_1 (or A_m) other than the edges it sends to s or $A_2 \setminus A_1$ (respectively $A_{m-1} \setminus A_m$) have their other ends in A_m (respectively A_1). $\square$

Remark 2.2.23. One needed to exclude the case when a path component of $I(L(G, s, k))$ is a singleton because the one maximal independent set corresponding to that component could possibly be of size 1. In that case, if A_1 is a dangerous set corresponding to that maximal independent set, then $|\delta(B : A_1)| = k$, because there is exactly one edge from s to A_1, and there is no contradiction in that case. However if the singleton set contains more than one neighbour of s, then the theorem holds for A_1, and B has to be empty.

Now we present one of our main results in the following theorem. It tells us the structure of the lifting graph when its complement is connected, equivalently when $I(L(G, s, k))$ is connected (i.e. the maximal independent sets of the lifting graph form a connected entity). In particular it lets us know that one possibility is that the complement of the lifting graph is a Hamilton cycle, hence generalizing what Ok, Richter, and Thomassen proved at $\deg(s) = 4$.

The other possibility is that the complement is two cliques of the same size with a path between them (possibly of length 0), hence generalizing an example of Ok, Richter, and Thomassen found at base case $\deg(s) = 6$, where the lifting graph was $K_{3,3}$ minus an edge. Illustrated in Fig. 4 of [18] it can be seen that the maximal independent sets of the lifting graph are of sizes 3, 2, and 3 and they form a path via intersections of size 1. This means the complement is two cliques of size 3 with a path of length one (the lost edge of $K_{3,3}$) between them. See Figures 2.11 and 2.12.

Theorem 2.2.24. *Let (G, s, k) be a connectivity triple such that $\deg(s) \geqslant 4$.*

If $I(L(G, s, k))$ is connected, then:

(a) $I(L(G, s, k))$ is either a cycle or a path;

(b) any vertex of $I(L(G, s, k))$ of degree 2 corresponds to a maximal independent set of size 2;

(c) the complement of $L(G, s, k)$ is either a Hamilton cycle or two cliques of the same size, at most $(k + 3)/2$, joined by a path; and

(d) if $\mathcal{D}$ is a collection consisting of one minimal dangerous set corresponding to each maximal independent set of $L(G, s, k)$, then the union of the dangerous sets in $\mathcal{D}$ is $V(G) \setminus \{s\}$.

Proof. If $I(L(G, s, k))$ is connected, then by Lemma 2.2.19 it is either a path or a cycle. This proves (a).

By Lemma 2.2.18, s only has neighbours in the intersections of the dangerous sets and also in the first and last dangerous sets in the path case. From this it follows that every maximal independent set in the cycle case has size 2 and every maximal independent set that is represented by an interior vertex in the path case is also of size 2. This gives (b).

From this it follows that in case $I(L(G, s, k))$ is a cycle, $L(G, s, k)$ consists of a cycle of maximal independent sets of size 2 each. Therefore, the complement of $L(G, s, k)$ is a cycle in that case (each maximal independent set of size 2 gives an edge of the cycle in the complement). Also it follows that in case $I(L(G, s, k))$ is a path, then the complement of $L(G, s, k)$ consists of a path between two cliques (the complements of the edge-less subgraphs induced by the first and last maximal independent sets).

The graph $I(L(G,s,k))$ consists of one component by assumption. This component is not a singleton because $L(G,s,k)$ consists of more than one maximal independent sets by Frank's theorem 2.1.10 as $\deg(s) > 3$ and (G,s,k) is a connectivity triple. Therefore, by Lemma 2.2.22, there are no vertices other than s outside the union of the dangerous sets in $\mathcal{D}$. This proves (d).

If $I(L(G,s,k))$ is a path, then it is not a singleton as explained above. If $A_1, \cdots, A_m$ is a path of minimal dangerous sets corresponding to the maximal independent sets of $L(G,s,k)$, then $m \geqslant 2$ since $I(L(G,s,k))$ is not a singleton, and by Lemma 2.2.22 any edges coming out of A_1 or A_m, other than the edges that connect them to s, A_2, or A_{m-1}, have their other ends in A_m and A_1 respectively.
By Lemma 2.2.3,

$$|\delta(A_1 \cap A_2 : A_2 \setminus A_1)| = |\delta(A_{m-1} \cap A_m : A_{m-1} \setminus A_m)| = (k-1)/2,$$

and

$$|\delta_G(A_1)| = |\delta_G(A_m)| = k+1.$$

Thus, $|\delta(\{s\} : A_1)| = |\delta(\{s\} : A_m)| = (k+1) - (\frac{k-1}{2}) - |\delta(A_1 : A_m)| = (\frac{k+3}{2}) - |\delta(A_1 : A_m)|$.
If we denote this number by l, then $l \leqslant (\frac{k+3}{2})$ and $|\delta(A_1 : A_m)| = (\frac{k+3}{2}) - l$ as desired. This is the size of the first and last maximal independent sets in the path, or the size of the two cliques in the complement of $L(G,s,k)$. $\square$

Corollary 2.2.25. *Let (G,s,k) be a connectivity triple such that $\deg(s) \geqslant 4$ and assume that $I(L(G,s,k))$ is connected.*

1. *If $I(L(G,s,k))$ is a cycle, then k is odd. This is the case when the complement of $L(G,s,k)$ is a cycle.*

2. *If $I(L(G,s,k))$ is a path and $A_1, A_2, \ldots, A_m$ are minimal dangerous sets corresponding to the vertices of $I(L(G,s,k))$, then:*

   (a) if $m = 2$, then $L(G,s,k)$ is an isolated vertex plus a balanced complete bipartite graph. In this case the complement

of $L(G,s,k)$ is two cliques of size $(\deg(s)+1)/2$ intersecting in a vertex (that is they are joined by a path of length 0);

(b) if $m > 2$, then the complement of $L(G,s,k)$ is two cliques of size $|\delta(\{s\}:A_1)|$ joined by a path of length $m-2$, in particular:

- *k is odd;*
- *$|\delta(\{s\}:A_1)| = |\delta(\{s\}:A_m)|$;*
- *$|\delta(\{s\}:A_1)| \leqslant \frac{k+3}{2}$; and*
- *$|\delta(A_1:A_m)| = \frac{k+3}{2} - l$, where $l = |\delta(\{s\}:A_1)| = |\delta(\{s\}:A_m)|$.*

Proof. If k is even and $\deg(s)$ is even, then by Theorem 2.1.11 $L(G,s,k)$ is complete multipartite, i.e. $I(L(G,s,k))$ is disconnected.

If k is even, and the size of every independent set of $L(G,s,k)$ is at most $\lfloor \deg(s)/2 \rfloor$, then any two maximal independent sets of $L(G,s,k)$ are disjoint by (ii) in (3) of Theorem 2.2.8. This means that $I(L(G,s,k))$ is a collection of singletons, so it is disconnected.

Thus the only case in which $I(L(G,s,k))$ is connected while k is even is when $\deg(s)$ is odd and there is a maximal independent set in $L(G,s,k)$ of size $\lceil \deg(s)/2 \rceil$. Then by Proposition 2.2.1 either $L(G,s,k)$ is a complete bipartite graph or an isolated vertex plus a balanced complete bipartite graph. In the first case $I(L(G,s,k))$ is disconnected, and in the second case $I(L(G,s,k))$ is a path of length 1 and the conclusions in $2-(a)$ hold.

If $\deg(s) = 4$, then by Lemma 2.2.10 the only case in which $I(L(G,s,k))$ is connected is when $L(G,s,k)$ is a perfect matching and its complement is a 4-cycle [18], i.e. the maximal independent sets of $I(L(G,s,k))$ make a 4-cycle of independent sets of size 2 each, and k is odd in that case.

We can now conclude that there is no case in which $I(L(G,s,k))$ is a cycle and k is even. Thus proving the first item in this corollary.

The claims in $2-(b)$ follow from Theorem 2.2.24. □

The path and cycle structures are illustrated in Figures 2.11 and 2.12. Note that each blob in the figures is not a dangerous set but the inter-

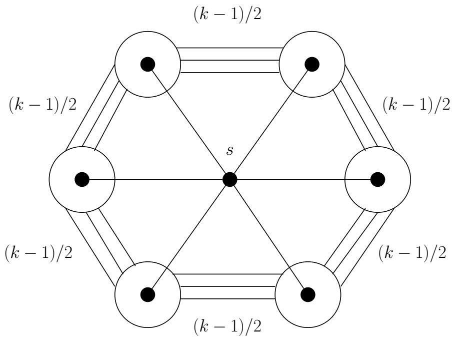


Figure 2.11: Cycle of intersections of dangerous sets.

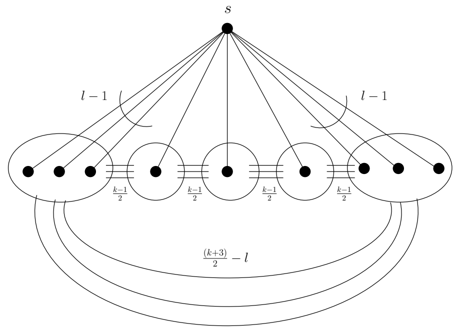


Figure 2.12: Path of intersections of dangerous sets.

section of two dangerous sets. The first and last blobs in Figure 2.12 are $A_1 \setminus A_2$ and $A_m \setminus A_{m-1}$.

Corollary 2.2.26. *Let* (G, s, k) *be a connectivity triple such that* $\deg(s) \geqslant 5$. *The only case in which* $L(G, s, k)$ *is disconnected, is when it is an isolated vertex plus a complete balanced bipartite graph.*

Proof. If $I(L(G, s, k))$ is not connected, then the complement of $L(G, s, k)$ is not connected, and $L(G, s, k)$ is connected in that case.

If $I(L(G, s, k))$ is connected and $L(G, s, k)$ is not an isolated vertex plus a complete bipartite graph, then $L(G, s, k)$ contains at least three maximal independent sets and these sets are arranged in a path or in a cycle by Theorem 2.2.24. The complement of the lifting graph in those two cases is either a Hamilton cycle or two cliques of the same size at most $(k + 3)/2$ with a path between them of length at least 1 (each maximal independent set of $L(G, s, k)$ of size 2 gives one edge in the path).

If the complement of $L(G, s, k)$ is a cycle, then $L(G, s, k)$ is connected (since it contains at least 5 vertices). The other possibility is that the complement is two cliques with a path between them of length $\geqslant 1$.

If the path between the two cliques is of length 1, then $L(G, s, k)$ is in fact a complete bipartite graph minus an edge, which is connected since the two cliques are of the same size at least 2 as $\deg(s) \geqslant 5$.

If the path is of length at least 2, then first and last maximal independent sets in the path (whose complements are the two cliques), induce a complete bipartite graph in $L(G, s, k)$. Also any vertex outside the two cliques is a neighbour to all the vertices in the two cliques except possibly one vertex from each clique. Thus, $L(G, s, k)$ is connected in that case too, in fact there is a path of length at most 2 between any two vertices in it. $\square$

2.3 Structure of a graph from the structure of its lifting graph

The results of the previous subsection about maximal independent sets in the lifting graph and their corresponding dangerous sets in G give us the following theorem about the structure of G. See Figures 2.11 and 2.12.

Theorem 2.3.1. *Let (G, s, k) be a connectivity triple such that $\deg(s) \geqslant 4$. If $L(G, s, k)$ contains at least three maximal independent sets, and $I(L(G, s, k))$ is connected, then*

(i) ***Cycle case:*** *in case $I(L(G, s, k))$ is a cycle, G consists of the vertex s and disjoint sets of vertices $S_0, \dots, S_{d-1}$, where $d = deg(s)$, such that the neighbours $s_0, \dots, s_{d-1}$ of s satisfy $s_i \in S_i$, and there are $(k-1)/2$ edges between S_i and S_{i+1}, for every $i \in \mathbb{Z}_d$, and,*

(ii) ***Path case:*** *in case $I(L(G, s, k))$ is a path, G consists of the vertex s and disjoint sets of vertices $S_1, \dots, S_m$ that contain the neighbours of s such that the numbers of the neighbours of s in S_1 and S_m are the same, and s has exactly one neighbour in each S_i for $i \notin \{1, m\}$. If $m \geqslant 4$, then there are $(k-1)/2$ edges between S_i and S_{i+1}, for every $i \in \{1, \dots, m-1\}$, and $\frac{k+3}{2} - l$ edges between S_1 and S_m where $l - 1$ is the numbers of neighbours of s in S_1. If $m = 3$, then $L(G, s, k)$ is an isolated vertex plus a balanced complete bipartite graph, and the number of edges from S_2 to each of S_1 and S_3 is at least $(k-1)/2$.*

Proof. By Theorem 2.2.24 when $I(L(G, s, k))$ is connected, it is either a path or a cycle. Let $A_1, \cdots, A_n$ be minimal dangerous sets corresponding, in order, to the vertices of a spanning path of $I(L(G,s,k))$. By Lemma 2.2.22, $V(G) = \{s\} \cup A_1 \cup \cdots A_n$. If $I(L(G,s,k))$ is a cycle, then by Lemma 2.2.19, they make a cycle or a path via intersections. By Lemma 2.2.3 there is exactly one neighbour of s in each intersection. Therefore, in the cycle case, $n = d = \deg(s)$. For each $i \in \mathbb{Z}_d$ define S_i to be the intersection $A_i \cap A_{i+1}$.

Note that $n \geqslant 2$ as $L(G,s,k)$ contains at least two maximal independent sets (it is not a collection of singletons) by Frank's theorem 2.1.10 as $\deg(s) > 3$ and (G,s,k) is a connectivity triple.

In the path case, for $i \in \{2, \cdots, n\}$, define S_i to be equal to the intersection $A_{i-1} \cap A_i$. Let $m = n+1$ and define $S_m := A_n \setminus A_{n-1}$, and $S_1 := A_1 \setminus A_2$. By Lemma 2.2.3, for every $i \in \{2, \cdots, m-1\}$, S_i contains exactly one neighbour of s, and there are $(k-1)/2$ edges between S_i and S_{i+1}, for every $i \in \{1, \ldots, m-1\}$.

By Corollary 2.2.25, A_1 and A_n contain the same number of neighbours l, and if $n \geqslant 3$, then there are $\frac{k+3}{2} - l$ edges between them. Since $A_1 \cap A_2$ and $A_{n-1} \cap A_n$ contain one neighbour of s each, S_1 and S_m contain the same number, $l-1$, of neighbours of s. By Lemma 2.2.3 there are no edges between $A_1 \cap A_2$ and $\overline{\{s\} \cup A_1 \cup A_2}$, so there are no edges between $A_1 \cap A_2$ and A_n if $n \geqslant 3$ (equivalently $m \geqslant 4$), and similarly no edges between $A_{n-1} \cap A_n$ and A_1. This means that the $\frac{k+3}{2} - l$ edges between A_1 and A_n are in fact between $A_1 \setminus A_2$ and $A_n \setminus A_{n-1}$, i.e. between S_1 and S_m. $\square$

We have a few further remarks on the structures we found.

Remark 2.3.2. Each one of the sets S_i in the cycle case, and each of the middle sets S_i in the path case induces a $(\frac{k+1}{2})$-edge-connected subgraph. The two sets S_1 and S_m at the beginning and end in the path case each induce a $(\frac{k-1}{2})$-edge-connected subgraph. This can be seen by noticing that of the k edge-disjoint paths between two vertices from such a set at most $(k-1)/2$ or $(k+1)/2$ go out if $|\delta(S_i)| = k$ or $k+1$ respectively.

Remark 2.3.3. For each i, in both the cycle and path cases, $|\delta(S_i)| = k$ and for every vertex v in S_i there are k edge-disjoint paths from v to the

k edges of $\delta(S_i)$. To see this, note that for every S_i in the cycle case, and every interior S_i in the path case, S_i has $(k-1)/2$ edges to each of the two neighbouring blobs, and one edge to s. In the path case, S_1 and S_m satisfy the following:

- $|\delta(\{s\} : S_1)| = |\delta(\{s\} : S_m)| = l - 1$,
- $|\delta(S_{m-1} : S_m)| = |\delta(S_1 : S_2)| = (k-1)/2$,
- $|\delta(S_1 : S_m)| = \frac{k+3}{2} - l$.

This is a total of exactly k edges incident with each one of S_1 and S_m.

Now for any vertex $v \in S_i$ there are k edge-disjoint paths between it and every vertex outside S_i other than s since (G, s, k) is a connectivity triple. Thus there are k edge-disjoint paths from v to the k edges of $\delta(S_i)$ as they are the only exits out of S_i.

Remark 2.3.4. In case $\deg(s)$ is odd, successively lifting pairs of edges in the path case from the inside going out brings us to the isolated vertex plus complete bipartite case as illustrated in Figure 2.13. More precisely if for every $i \in \{2, \cdots, m-1\}$, s_i is the unique neighbour of s in S_i, then successively lifting the pairs of edges $(ss_{(m-1)/2}, ss_{(m+3)/2})$, $\cdots$,$(\mathrm{ss}_2, ss_{m-1})$ preserves the path structure and at the end leaves a graph whose lifting graph at s is an isolated vertex plus $K_{(l-1),(l-1)}$.

Remark 2.3.5. Lifting a pair of edges in case the lifting graph is an isolated vertex plus a complete bipartite graph, results in a graph whose lifting graph is also an isolated vertex - the same vertex - plus a complete bipartite graph. Also when the lifting graph is complete multipartite its structure is kept through lifting. This can be seen from the structure of the dangerous sets corresponding to the maximal independent sets of the lifting graph.

Remark 2.3.6. In case $\deg(s) \geqslant 4$, lifting a pair of edges incident with s in case G has the cyclic structure in (i) of Theorem 2.3.1 that have their non-s ends in S_{i-1} and S_{i+1} for some i (indices modulo $\deg(s)$) gives the same cyclic structure again on a cycle of length less by 2. The union of

S_{i-1}, S_i, and S_{i+1} gives one new blob. All the other blobs remain the same.

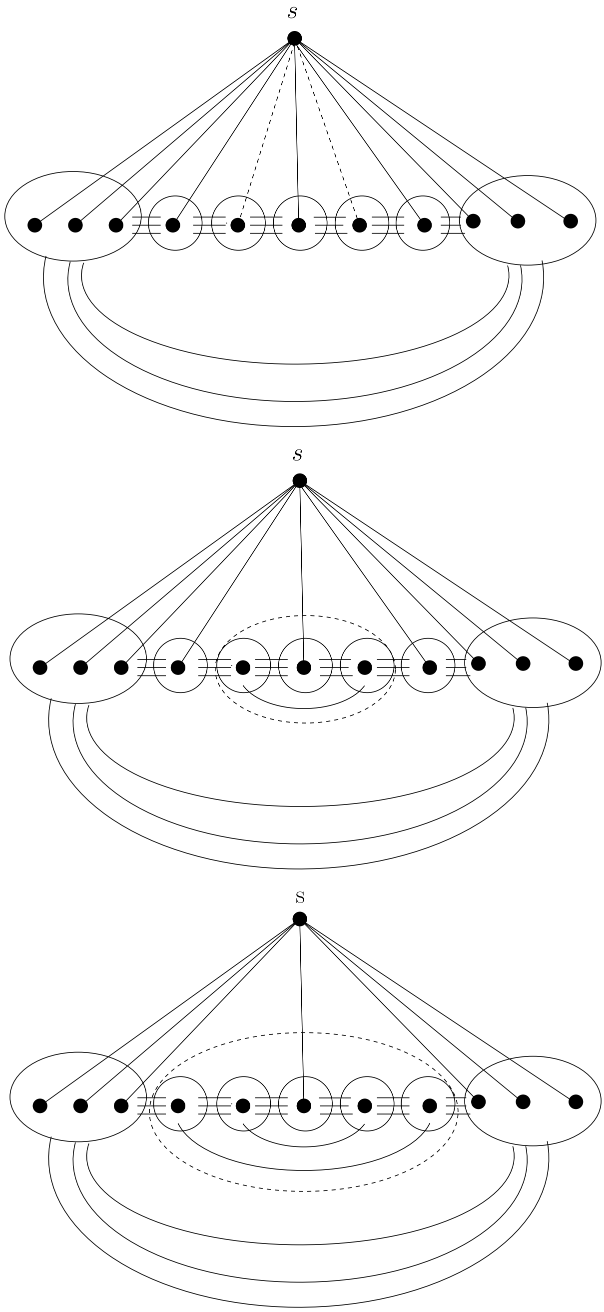


Figure 2.13: Successively lifting pairs of edges until we reach a lifting graph that is an isolated vertex plus a complete bipartite graph.

2.3.1 Degree Three

Degree three needs a special treatment as it is the excluded case in the theorems of Mader 2.1.9 and Frank 2.1.10, and also in the dangerous sets lemma by Ok, Richter, and Thomassen 2.1.14. Here we show that if $\deg(s) = 3$ and no pair of edges incident with s is liftable, then G has a specific structure. Figure 2.14 shows the situation arising from (3) of the following proposition, Par (2) is the $\deg(s) = 3$ case for $I(L(G, s, k))$ being a cycle.

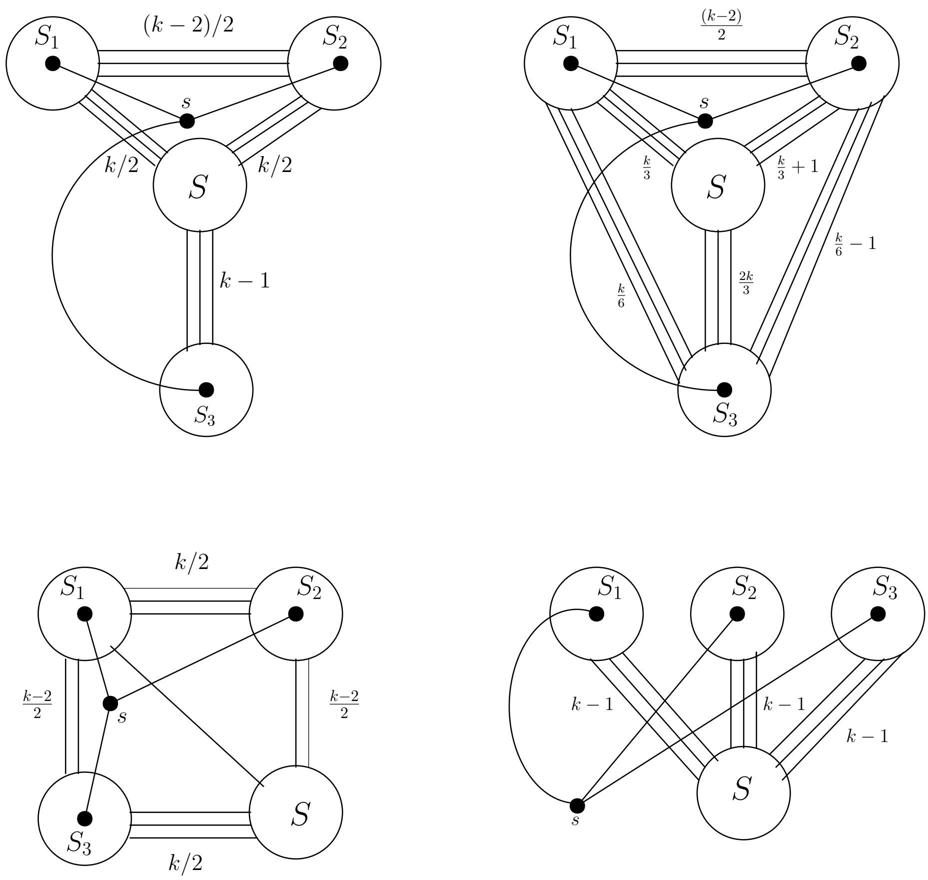


Figure 2.14: Examples of graphs with $deg(s) = 3$ where no pair of edges incident with s is liftable.

Proposition 2.3.7. *Let (G, s, k) be a connectivity triple such that* $\deg(s) = 3$. *Let s_1, s_2, and s_3 be the neighbours of s. If no pair of edges incident with s is liftable, then there are sets of vertices S_1, S_2, and S_3 in $G - s$*

such that:

(1) for each i, $S_i \cap \{s_1, s_2, s_3\} = \{s_i\}$;

(2) if $S_1 \cup S_2 \cup S_3 \cup \{s\} = V(G)$, then k is odd and for distinct $i, j \in \{1,2,3\}$, $|\delta(S_i : S_j)| = (k-1)/2$; and

(3) if $S_1 \cup S_2 \cup S_3 \cup \{s\} \neq V(G)$, then, for each $i \in \{1,2,3\}$, $|\delta_G(S_i)| = k$ (the parity of k is not determined and $|\delta(S_i : S_j)|$ is not necessarily $(k-1)/2$ for $j \neq i$).

Proof. By Lemmas 2.1.18 and 2.1.15 for distinct $i, j \in \{1, 2, 3\}$ there is a dangerous set $A_{i,j} = A_{j,i}$ such that $A_{i,j}$ contains s_i and s_j but not s_l for $l \in \{1,2,3\} \setminus \{i, j\}$.
By Lemma 2.2.4, noting that it is valid for $\deg(s) = 3$, we have the following for distinct $i, j, l \in \{1, 2, 3\}$,

(a) there are no vertices outside $A_{i,l} \cup A_{j,l}$ other than s;

(b) $|\delta(A_{j,l} \setminus A_{i,l} : A_{i,l} \setminus A_{j,l})| \leqslant (k-1)/2$ and

Define $S_1 = A_{1,3} \setminus A_{2,3}$, $S_2 = A_{1,2} \setminus A_{1,3}$, and $S_3 = A_{2,3} \setminus A_{1,2}$. Then the only neighbour of s contained in S_i is s_i, for $1 \leqslant i \leqslant 3$. By (b) $|\delta(S_i : S_j)| \leqslant (k-1)/2$ for any distinct $i, j \in \{1,2,3\}$. If $S_1 \cup S_2 \cup S_3 \cup \{s\} = V(G)$, then this has to be an equality for (G, s, k) to be a connectivity triple.

If $S_1 \cup S_2 \cup S_3 \cup \{s\} \neq V(G)$, then let $S = V(G) \setminus (S_1 \cup S_2 \cup S_3 \cup \{s\})$. Note that S is also equal to $A_{1,3} \cap A_{2,3} \cap A_{1,2}$. By (a), $S \subseteq A_{i,l} \cup A_{j,l} = (A_{i,l} \setminus A_{j,l}) \cup (A_{i,l} \cap A_{j,l}) \cup (A_{j,l} \setminus A_{i,l}) = S_i \cup (A_{i,l} \cap A_{j,l}) \cup S_j$. Thus, $S \subseteq (A_{i,l} \cap A_{j,l})$ for every distinct $i, j, l \in \{1, 2, 3\}$. Note that by the definitions of S_i and S_j they are disjoint from $(A_{i,l} \cap A_{j,l})$. Thus, $(A_{i,l} \cap A_{j,l}) \subseteq S \cup S_l$, so $(A_{i,l} \cap A_{j,l}) = S \cup S_l$. Thus, $A_{i,j} = S_i \cup S \cup S_j$.

Thus $|\delta(A_{i,j})| = 2 + |\delta(S_i : S_l)| + |\delta(S : S_l)| + |\delta(S_j : S_l)| = 2 + |\delta_{G-s}(S_l)|$, so $|\delta_{G-s}(S_l)| \leqslant k - 1$ since $A_{i,j}$ is dangerous. Recall that S_l contains exactly one neighbour of s, and that (G, s, k) is a connectivity triple. Therefore $\delta(S_l) = k$ as desired. $\square$

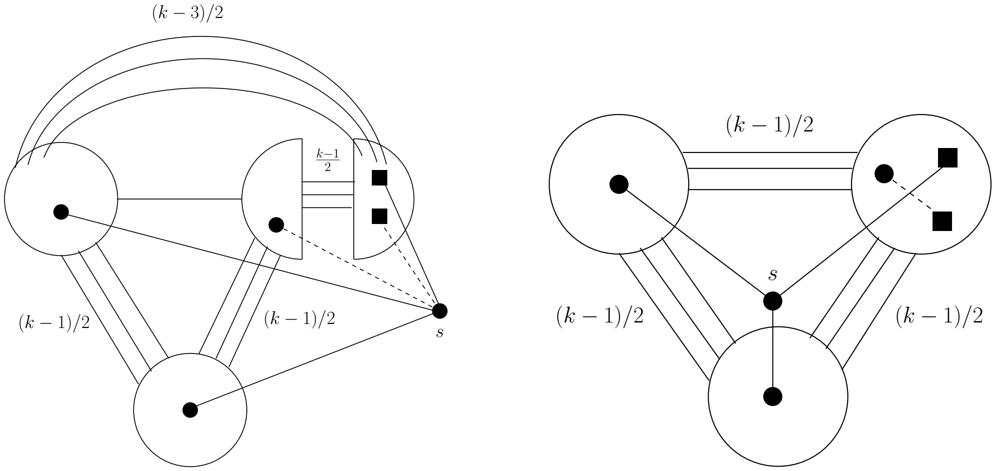


Figure 2.15: A graph with $deg(s) = 5$ and lifting graph $K_{2,3}$ such that when the dashed pair of edges is lifted no remaining pair of edges incident with s is liftable.

Note that for distinct $i, j \in \{1, 2, 3\}$ the dangerous set containing s_i and s_j consists of the union $S_i \cup S \cup S_j$, see Figure 2.14.

Remark 2.3.8. Note that when all three edges incident with s are pairwise non-liftable it is possible that k is even as well as odd.

Remark 2.3.9. The intersections $A_{1,3} \cap A_{2,3}$, $A_{2,3} \cap A_{1,2}$, and $A_{1,2} \cap A_{1,3}$ each contain a cut of size at most $k-1$ in the subgraphs induced by each one of them. For distinct $i, j, l \in \{1, 2, 3\}$, a cut of size at most $k-1$ in the subgraph induced by $A_{i,j} \cap A_{i,l}$ is the set of edges between S and S_i. See Figure 2.14.

Remark 2.3.10. The lifting graph on three vertices and without any edges, that is when $deg(s) = 3$ and no pair of edges incident with it is liftable, can come from different lifting graph structures for $deg(s) = 5$ as illustrated in Figures 2.15 and 2.16.

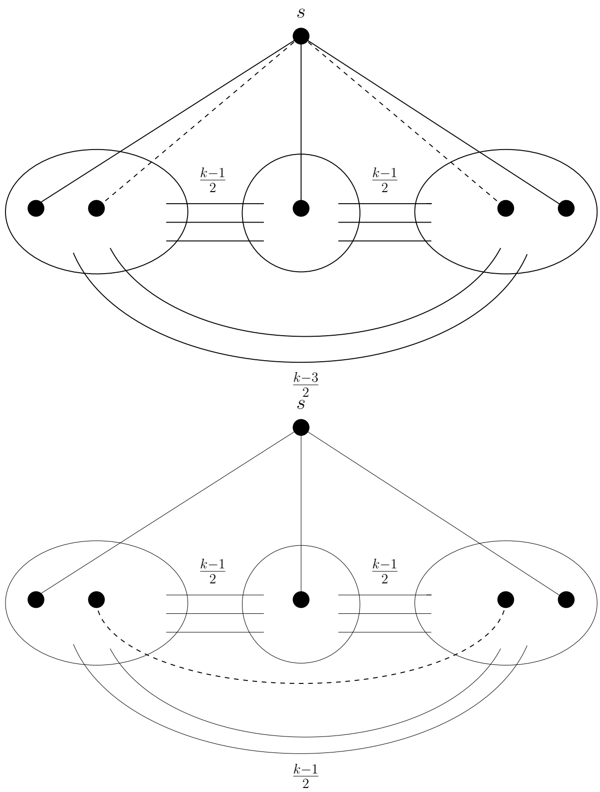


Figure 2.16: A graph with $deg(s) = 5$ and lifting graph isolated vertex plus $K_{2,2}$ such that when the dashed pair of edges is lifted no remaining pair of edges incident with s is liftable.

Chapter 3

Extending Huck's theorem to infinite graphs

Throughout this chapter a graph is infinite unless stated otherwise. We prove that Huck's theorem holds for infinite graphs. We start with the locally finite case, and in later sections reduce the general case to the locally finite case.
As mentioned earlier in the introduction, Huck's theorem differs by only 1 in the connectivity condition from Thomassen's weak linkage conjecture. Before we present the proof, here are some definitions, and the statement of the conjecture.

3.1 Preliminaries

Here we introduce the necessary definitions needed to understand the statement of the Weak Linkage Conjecture, and the related theorems in finite and infinite graphs.

Definition 3.1.1. (weak linkage) A graph G is *weakly k-linked* if given any set of k pairs of terminals $(s_1, t_1), \cdots, (s_k, t_k)$ in G, not necessarily distinct, there is a path P_i between s_i and t_i for every $i \in \{1, \cdots, k\}$ such that $P_1, \cdots, P_k$ are edge-disjoint. Such a set of paths is called a *weak linkage* or *edge-disjoint linkage.*

Conjecture 3.1.2. *(Thomassen 1980 [24]) If k is odd, then every k-edge-connected graph is weakly k-linked, and if k is even, then every $(k+1)$-edge-connected graph is weakly k-linked.*

The conjecture for even numbers follows from the conjecture for odd numbers. An edge-connectivity of k is not enough for even k as shown in an example by Thomassen [24] of $k/2$ cycles on the same set of $2k$ vertices in the order $s_1, \cdots, s_k, t_1, \cdots, t_k$. In this order all the pairs are overlapping. There is no edge-disjoint linkage in this graph from s_i to t_i for $i \in \{1, \cdots, k\}$.

For example, a cycle is 2-edge-connected, however two overlapping pairs of vertices on the cycle cannot be linked by two edge-disjoint paths. To prove the general case, suppose for a contradiction that a linkage exists, and let $s_1, s_2, \cdots, s_k, t_1, t_2, \cdots, t_k$ be the forward direction. If $i_1 < i_2 < \cdots < i_r$ are such that $s_{i_j} t_{i_j}$-paths are forward, then all use the forward edge from s_{i_r}. Therefore, $i \leqslant k/2$. Likewise there are at most $k/2$ paths in the reverse direction. Thus, there are $k/2$ paths in each direction. If the $s_1 t_1$-path is in the forward direction, then the edge $s_1 s_2$ is in this path and the $k/2$ reverse direction paths. Otherwise $t_1 t_2$ is in the $s_1 t_1$-path and all the $k/2$ forward direction paths. In both cases we get a contradiction as there are exactly $k/2$ edges between any two consecutive vertices on the cycle.

The best known result for finite graphs is:

Theorem 3.1.3. *(Huck 1991 [9]) Let k be an odd positive integer. If G is a $(k+1)$-edge-connected finite graph, then G is weakly k-linked.*

Ok, Richter, and Thomassen [18] used this theorem, with their lifting graph theorem 2.1.11 to prove a weak linkage result for infinite graphs under a connectivity condition of $(k+2)$. Before we state their theorem some definitions are needed.

Definition 3.1.4. (locally finite) An infinite graph G is *locally finite* if the degree of each vertex is finite.

Definition 3.1.5. A *ray* is one-way infinite path. A ray subgraph of a ray is a *tail* of it.

Definition 3.1.6. (end) An *end* of a graph G is an equivalence class of rays within G, where two rays are in the same class (*end*) if there are infinitely many vertex disjoint paths between them in G.

Definition 3.1.7. (1-ended) An infinite graph G is 1-*ended* if for every finite set of vertices X, only one of the components of $G - X$ is infinite.

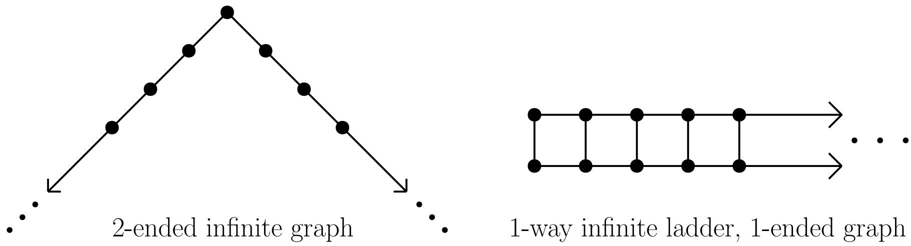


Figure 3.1

For example, a ray is a 1-ended graph, and so is a 1-way ladder. See Figure 3.1. A double ray, or a two way infinite path, is a two-ended graph. The binary tree has uncountably many ends. There is a ray corresponding to each infinite sequence of 0's and 1's, and any two different such sequences can be separated in the binary tree by deleting the vertex, which exists in both rays, corresponding to the first digit on which the two sequences differ.

Other examples: The 1-way infinite ladder has vertex set $\{0, 1, 2, \cdots\} \times \{0, 1\}$ and an edge joining any two vertices that have one coordinate equal and other coordinate differing by 1. See Figure 3.1. This is 1-ended as is the infinite integer grid in the plane (same rule determining the edges). The 2-way infinite ladder is defined analogously and has two ends.

Theorem 3.1.8. *(Ok, Richter, and Thomassen 2016 [18]) Let k be an odd positive integer. If G is a $(k + 2)$-edge-connected, 1-ended, locally finite graph, then G is weakly k-linked.*

3.2 1-ended locally finite graphs

In this section we reduce the connectivity condition in the result of Ok, Richter, and Thomassen 3.1.8 from $k + 2$ to $k + 1$. That is we show that Huck's theorem holds for 1-ended locally finite graphs. We follow the same steps of the proof of Ok, Richter, and Thomassen and change things where needed.

We will need the same proof idea again in Chapter 4. To be able to reference the steps of the proof, we break them down into definitions and lemmas here.

The following is a standard fact about locally-finite 1-ended infinite graphs. A more general form of it is Theorem 3.3.5 by Thomassen [25], to be presented in Section 3.3.3. This special case can be proved by finding a sequence of disjoint cuts of non-decreasing sizes such that any two cuts in the sequence are not separated by a cut of a smaller size, then joining each two consecutive cuts in the sequence by edges-disjoint paths such that each edge of the smaller cut is the first edge of one of those paths. Together these segments of paths give rays going through the sequence of cuts.

Lemma 3.2.1. *If G is a connected, 1-ended, locally finite graph, then given any finite set of vertices T, there is a finite set of vertices S containing it such that each edge of $\delta(S)$ is the first edge of a ray in an edge-disjoint collection of rays $\mathcal{P}$.*

The finite set of vertices T to which we will apply this lemma, as in the proof of the linkage result, Theorem 3.1.8, by Ok, Richter, and Thomassen, is the set of terminals to be linked. The work is in a connected, 1-ended, locally finite graph. In application fix a finite set $S \supseteq T$ of vertices containing the terminals as described in Lemma 3.2.1, and let $\mathcal{P}$ be the associated set of rays. Then we contract the infinite side of the cut $\delta(S)$ to a single vertex s, and apply the lifting graph results from Chapter 2 to the finite graph $G/(G-S)$ at the vertex s.

We may assume without loss of generality that $G - S$ consists of only the unique infinite component of this 1-ended subgraph, any finite component can be added to S. The size of the cut $\delta(S)$ is finite assuming that G is locally finite. So when we say the infinite side of the cut $\delta(S)$, this means the unique infinite component of $G - S$.

Observe that the set S and the vertex s defined as described above satisfy $\deg(s) = |\delta(S)|$. Note also that if G is k-edge-connected, then $G/(G-S)$ is k-edge-connected as well.

Definition 3.2.2. (The end graph) Let G be a locally finite graph with a finite set of vertices S such that every edge in $\delta(S)$ is the first edge of a ray in an edge-disjoint collection of rays $\mathcal{P}$ from one end. The *end graph* $\mathcal{E}_G(\mathcal{P}, S)$, simply $\mathcal{E}$, is the graph with vertex set $\delta(S)$ and edge set defined as follows. Distinct edges e and e' from $\delta(S)$ are adjacent in $\mathcal{E}$ if there are infinitely many vertex-disjoint paths in $G - S$ that:

(i) join the two rays in $\mathcal{P}$ containing e and e', and

(ii) are edge-disjoint from all the other rays in $\mathcal{P}$.

Remark 3.2.3. The end graph $\mathcal{E}$ is connected because the rays of $\mathcal{P}$ are in one end.

Note that the definition of $\mathcal{E}_G(\mathcal{P}, S)$ does not require the graph to be 1-ended as in Lemma 3.2.1. The set S and its associated rays $\mathcal{P}$ are separate from Lemma 3.2.1 even though they satisfy the statements in it. They, however, come from a more general form of Lemma 3.2.1 for multiple ends, which was proved by Thomassen in 2016 [25] and is going to be presented and used in Section 3.3.3.

The following lemma is a cornerstone of the proof of our new linkage result. It shows that in case the connectivity is even there is a sequence of lifts of pairs of edges from $\delta(S)$ such that these pairs of edges can be linked by edge-disjoint paths in $G - S$ and the sequence ends by having all the edges of $\delta(S)$ lifted if $|\delta(S)|$ is even (proved by Ok, Richter, and Thomassen in [18]) or having three of them remaining if it is odd. Figure 3.2 illustrates the proof idea. We need the following definition.

Definition 3.2.4. (lifting sequence of graphs) Let G_0 be a finite graph, and assume that (G_0, s, k) is a connectivity triple. Let $I = \{0, 1, \cdots, n\}$ for some positive integer n. A *lifting sequence* $\{G_i\}_{i \in I}$ at s is a sequence of graphs beginning with G_0 such that for every $i > 0$ in I, there is a

k-liftable pair of edges incident with s in G_{i-1}, and G_i is obtained from G_{i-1} by lifting such a pair.

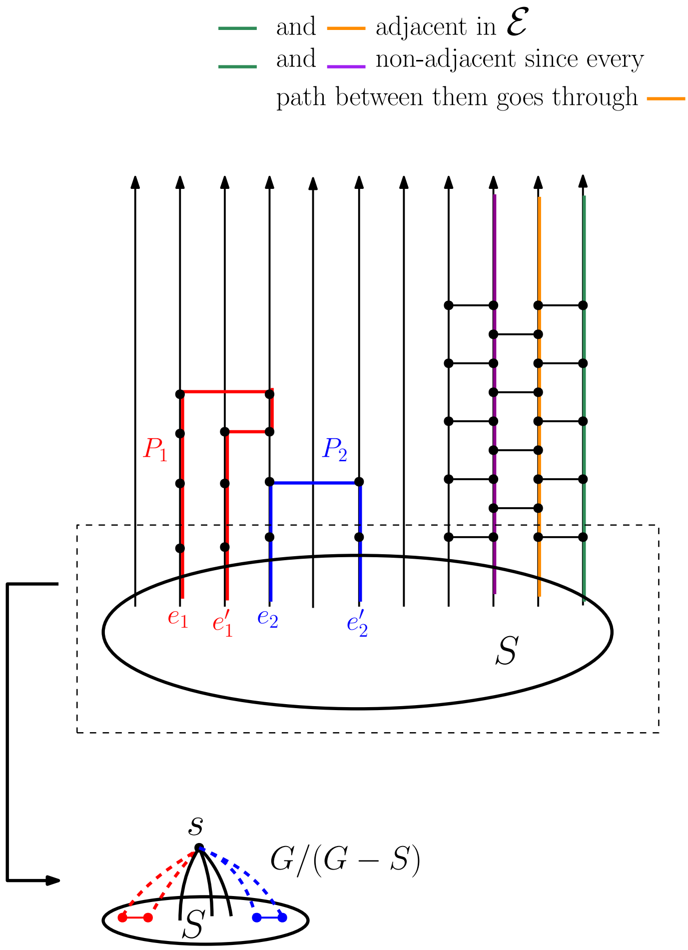
In the following lemma, we are going to find a lifting sequence in $G/(G-S)$, where $G-S$ is the infinite side of a cut $\delta(S)$. For each $i > 0$, G_i is obtained by lifting a pair of edges e_i and e'_i incident with the contraction vertex (that is edges from the cut $\delta(S)$), and a path P_i is going to be found in $G-S$ between the end-vertices of e_i and e'_i in $G-S$ such that the paths P_i are edge-disjoint. These paths are going to be used in the linkage as follows. The terminals live in S, and they can be regarded as terminals in the last graph of the lifting sequence G_I. A linkage of the terminals is found in this finite graph using the linkage theorems for finite graphs. Any path of the linkage that goes through an edge that resulted from lifting, is not an actual path in G. If it goes through the lifting edge that resulted from lifting the pair e_i and e'_i, then the actual path in G goes through $e_iP_ie'_i$. See Figure 3.2.

Lemma 3.2.5. *Let G be a k-edge-connected locally finite graph, and suppose that there exists a finite set of vertices S such that every edge of $\delta(S)$ is the first edge of a ray in an edge-disjoint collection of rays $\mathcal{P}$ in one end of G. Let s be the contraction vertex in $G/(G-S)$. If* $\deg(s) > 3$ *is odd, let $I = \{0, \cdots, (\deg(s)-3)/2\}$, and if it is even, let $I = \{0, \cdots, \deg(s)/2\}$. Then there is a lifting sequence $\{G_i\}_{i\in I}$ at s such that $G_0 := G/(G-S)$, and one of the following holds.*

(1) For every $i \in I$, $i > 0$, G_i is obtained from G_{i-1} by lifting a pair of edges e_i and e'_i incident with s such that there is a path P_i joining e_i and e'_i in $G-S$ that is edge-disjoint from $P_1, \ldots, P_{i-1}$ and from all the rays in $\mathcal{P}$ containing the edges of $\delta(S)$ that are not yet lifted.

(2) There is an $i^* \in I$ such that G_{i^*} has one of structures (i) or (ii) in Theorem 2.3.1, and such that if $i^* > 0$, then for every $i < i^*$, the sequence $G_0, \cdots, G_i$ satisfies (1) above. This case can only happen if k is odd.

Moreover, the lifting sequence, and the paths $\{P_i\}_{i\in I}$ can be chosen so that in case $\deg(s)$ *is odd and for some i the lifting graph $L(G_i, s, k)$ is* *an isolated vertex (e^*) plus a complete balanced bipartite graph, then for the three edges e, e^*, e' that remain of $\delta(S)$ at the end of the sequence of lifts, there is a vertex $w \in G - S$ and edge-disjoint paths W, W^*, W' from w to e, e^*, e' that are also disjoint from $\{P_j\}_{j \in I}$.*



Red pair and blue pair lifted, and linked by two edge-disjoint paths in $G - S$ that avoid all the rays of $\mathcal{P}$ that are not yet used in linkage.

Figure 3.2

Proof. Suppose we have the pairs $\{e_1, e_1'\}, \ldots, \{e_{i-1}, e_{i-1}'\}$ and the paths $P_1, \ldots, P_{i-1}$. We now show how to find $\{e_i, e_i'\}$ and P_i as described above.

Let $\delta_i(S) := \delta(S) \setminus \{e_1, e_1', \ldots, e_{i-1}, e_{i-1}'\}$, and let $\mathcal{P}_i$ denote the set of rays in $\mathcal{P}$ that do not contain any of the edges in $\{e_1, e_1', \ldots, e_{i-1}, e_{i-1}'\}$.

Definition 3.2.6. The graph $\mathcal{L}_i$ is the k-lifting graph for s in G_{i-1}, that is $\mathcal{L}_i = L(G_{i-1}, s, k)$.

On the same vertex set as $\mathcal{L}_i$, we have the following end graph.

Definition 3.2.7. Let $\mathcal{E}_i := \mathcal{E}_{G_{i-1}}(\mathcal{P}_i, S)$ (See Definition 3.2.2).

If $\deg_{G_{i-1}}(s) > 3$, then by Theorem 2.2.24 and Theorem 2.3.1, one of the following holds.

(a) the complement of $\mathcal{L}_i$ is disconnected,

(b) $\mathcal{L}_i$ is an isolated vertex plus a complete balanced bipartite graph, or

(c) G_{i-1} has structure (i) or structure (ii) from Theorem 2.3.1. This can only happen if k is odd and gives (2) in the statement of this lemma. Once one of these two structures is reached, the sequence terminates.

Definition 3.2.8. In case (b), let e^* be the unique edge not liftable with any edge in $\delta_i(S)$, and let R^* be the unique ray in $\mathcal{P}_i$ that begins with e^*.

We find a pair of edges e_i and e_i' to lift and obtain G_i. Where this pair is chosen as explained in the discussion below. We consider the following cases.

Case 1. *The complement of $\mathcal{L}_i$ is disconnected,*

Because $\mathcal{E}_i$ is connected, there is an edge e_ie_i' that exists in both $\mathcal{E}_i$ and $\mathcal{L}_i$. This is the pair of edges that we lift to get G_i in that case.

Case 2. *The lifting graph $\mathcal{L}_i$ is an isolated vertex e^* plus a balanced complete bipartite graph, i.e. $\mathcal{L}_i - e^*$ is a balanced complete bipartite graph (See Proposition 2.2.1).*

Note that R^* is not used yet: It is edge-disjoint from the finite construction consisting of the paths $P_1, \cdots P_{i-1}$ as it is a ray in $\mathcal{P}_i$.

Subcase (i): *All the neighbours of e^* in $\mathcal{E}_i$ are on one side of the bipartite graph $\mathcal{L}_i - e^*$.*

The end graph $\mathcal{E}_i$ is connected, therefore an edge e_i (different from e^*) on one side of $\mathcal{L}_i - e^*$ has to be a neighbour in $\mathcal{E}_i$ of an edge e_i' on the other side. This is the pair of edges we lift in that case, and it is good in that there are infinitely many disjoint paths avoiding R^* between the ray that begins with e_i and the ray that begins with e'_i.

Subcase (ii): *If e^* has neighbours on both sides of the bipartite graph $\mathcal{L}_i - e^*$,*

Let e and e' be such neighbours and let R and R' be the rays of $\mathcal{P}_i$ that begin with them. There are infinitely many disjoint paths between R and R^* that are edge-disjoint from the other rays in $\mathcal{P}_i$ and infinitely many disjoint paths between R^* and R' that are edge-disjoint from the other rays in $\mathcal{P}_i$. Pick a path P between R and R^* that is edge-disjoint from the other rays in $\mathcal{P}_i$ and a path P' between R^* and R' that that are edge-disjoint from the other rays in $\mathcal{P}_i$ such that these two paths are edge-disjoint from the paths $P_1, \cdots, P_{i-1}$. Let the end-vertex of P on R^* be w and the end-vertex on R^* of P' be w'. Assume without loss of generality that w is closer to S on R^* from w', and let Q be the path contained in R^* between w' and w. Let W^* be the path contained in the ray R^* from w to e^*, let W be the path from w along P and then down on R to e, and let W' be the path from w up along Q to w' then along P' then down on R' to e'. Then W, W^*, and W' are edge-disjoint from the rays in $\mathcal{P}_i \setminus \{R, R', R^*\}$ and from the paths $P_1, \cdots, P_{i-1}$

Now consider $\mathcal{E}_{G_{i-1}-\{e,e^*,e'\}}(\mathcal{P}_i \setminus \{R, R', R^*\}, S)$. This is a connected graph, and so there is a pair of edges e_i and e'_i from two different sides of the bipartite graph $\mathcal{L}_i - \{e, e^*, e'\}$, in $\delta(S) \setminus \{e_1, e'_1 \cdots, e_{i-1}, e'_{i-1}, e, e^*, e'\}$, that is separated in $\mathcal{E}_{G_{i-1}}(\mathcal{P}_i, S)$ only possibly by e, e^*, e'. This is the pair of edges to lift in this case. It is liftable as long as $\deg(s) > 3$ since it is from two different sides of the complete bipartite lifting graph.

Let Q and Q' be the rays in $\mathcal{P}$ containing e_i and e'_i respectively. There are infinitely many vertex-disjoint paths in $G - S$ joining Q and Q' that are edge-disjoint from the other rays in $\mathcal{P}_i$ (or $\mathcal{P}_i \setminus \{R, R', R^*\}$). Choose one of those paths, P, that is also disjoint from all of the finitely many finite paths $P_1, \ldots, P_{i-1}$ (and W, W^*, W'). Then $Q \cup P \cup Q'$ contains a path P_i that begins with e_i and ends with e'_i.

Note that we did not lift the pair e, e' that is in the rays R and R'

which are direct neighbour of R^*. If Subcase (ii) recurs, for the m-th time after i, we lift a pair of edges e_{j_m} and e'_{j_m} on two different sides of the bipartite lifting graph that are the first edges of rays R_m and R'_m. The infinitely many disjoint paths between R_m and R'_m may go through the edges of the rays $R_{m-1}, \cdots, R_1, R, R^*, R, R'_1, \cdots, R_{m-1}$ (these rays begin with the edges $e_{j_{m-1}}, \cdots, e_{j_1}, e, e^*, e, e'_{j_1}, \cdots, e_{j_{m-1}}$ respectively). We continue the lifting process until only e, e^*, e' are left. We join e_{j_m} and e'_{j_m} by a path Q_m that is edge-disjoint from the rays of $\mathcal{P}$ not yet used, and also is edge-disjoint from all the paths already used in connecting lifted pairs of edges, and from $\{W, W^*, W'\}$ (this path may go through the rays $R_{m-1}, \cdots, R_1, R, R^*, R, R'_1, \cdots, R_{m-1}$ at a higher point than any one of the previously constructed paths).

By Theorem 2.2.24 only cases (a) and (b) can hold if k is even. If $\deg(s)$ is even and k is even, then the lifting graph is complete multipartite that is not a star (disconnected complement) by Theorem 2.1.11, and we can have all the edges of $\delta(S)$ paired and lifted. This can be done by choosing at every step a pair of edges that is adjacent in the connected graph $\mathcal{E}_i$ but not adjacent in the disconnected complement of $\mathcal{L}_i$. If $\deg(s)$ is odd, and k is even, we can lift until s has degree 3. Otherwise, if k is odd, the sequence may stop early if we have structures (i) or (ii) from Theorem 2.3.1. $\square$

Note that we terminate the sequence when we have the path or cycle structures from Theorem 2.3.1 because in this case the complement of the lifting graph is either a Hamilton cycle or two cliques with a nontrivial path between them. If the connected end graph is a subgraph of such a connected complement of the lifting graph, then there is no appropriate pair of edges to lift.

Note that when $\deg(s)$ is odd, we may not be able to lift all the edges incident with s. In this case, consider applying Lemma 3.2.5 with connectivity $k+1$ for odd k. If we continue lifting until $\deg(s) = 3$, then $G_{|I|}$ is not a $(k+1)$-edge-connected graph, so we cannot apply Huck's theorem 3.1.3 to it. However, if we stop lifting at $\deg(s) = k+2$, say this happens at $i = n$, then G_n is $(k+1)$-edge-connected.

If we find a linkage in G_n using Huck's theorem, then a path that goes through s uses exactly two of the edges incident with it. Thus the paths of the linkage will define a pairing on an even subset of $|\delta(S)|$, but we do not know this pairing beforehand. Therefore it is not clear if these paths can be replaced with edge-disjoint paths in G. Unlike the paths that go through the edges that resulted from lifting, we know exactly which pair of edges this edge represents and the pairs where chosen such that they are linkable by edge-disjoint paths in G.

Now note that at most $k+1$ (even number) edges of the $k+2$ edges incident with s can be used in a linkage in G_n. If for any subset of such $k+1$ edges we could find a vertex in $G-S$ that has $k+1$ edge-disjoint paths to these edges that also avoid the finite construction consisting of the paths $P_1, \cdots P_n$ of Lemma 3.2.5, then our problem is solved. This is what we are going to prove in Proposition 3.2.11. To prove this proposition we need a couple of lemmas first.

Lemma 3.2.9. *(Kőnig's Lemma 1927 [10], [4]) Let $V_0, V_1, \cdots$ be an infinite sequence of disjoint nonempty finite sets, and let G be an infinite graph whose vertex set is the union of the sets $V_0, V_1, \cdots$. Assume that every vertex v in a set V_n with $n \geqslant 1$ has a neighbour $f(v)$ in V_{n-1}. Then G contains a ray $v_0, v_1, \cdots$ with $v_n \in V_n$ for all n.*

Kőnig's Lemma will be needed to turn path segments into a ray, and the following lemma will be used in finding disjoint paths between an old and a new collection of rays such that the path between two rays does not go through preceding rays. This lemma, and the proof idea of Proposition 3.2.11 are due to Bruce Richter.

Lemma 3.2.10. *Let T be a finite tree and let $A, B \subseteq V(T)$ be disjoint and $|A| = |B|$. Then there are paths $P_1, P_2, \ldots, P_{|A|}$ in T such that each P_i connects a vertex a_i in A and a vertex b_i in B and,*

- *the a_i and b_i are all distinct,*
- *each a_i does not occur in $P_{i+1}, \ldots, P_{|A|}$ and,*
- *each b_i does not occur in P_j for any $j < i$.*

Proof. The proof is by induction on $|V(T)|$. Let $W = V(T) \setminus (A \cup B)$. We may readily contract any edge with both ends in W without affecting the result, so we may assume W induces an independent set in T.

Let e be any edge of T and let R_e, S_e denote the vertex sets of the two components of $T - e$. Evidently $|R_e \cap A| \geqslant |R_e \cap B|$ if and only if $|S_e \cap A| \leqslant |S_e \cap B|$.

If there is an edge e of G such that $|R_e \cap A| = |R_e \cap B|$, then inductively solve the problem separately on the smaller trees induced by R_e and S_e, respectively. Therefore, we may assume that there is no such e.

In the remaining case, for each edge e of G, $|R_e \cap A| \neq |R_e \cap B|$. Orient e towards the one of R_e and S_e that has more B vertices than A vertices. This orients all the edges of T and there is a vertex v of T such that all the edges incident with v are directed out of v.

We claim that $v \in A$ and v is a leaf of T. The orientation rule above shows that, for every edge e incident with v, there are more B vertices than A in the component of $G - e$ that does not contain v.

All vertices of $(A \cup B) \setminus \{v\}$ are in these components, so $|B \setminus \{v\}| > |A \setminus \{v\}|$. These two sets differ in size by precisely one, so $v \in A$ and there is only one component in $T - v$, as claimed.

Let Q be any path in T starting at v and ending at a vertex b in B such that b is the only vertex of B in Q. Set $T' = T - v$, $A' = A \setminus \{v\}$, $B' = B \setminus \{b\}$, and $W' = W \cup \{v\}$. Inductively, there is a solution for (T', A', B') and addition of Q to this solution yields a solution for (T, A, B). $\square$

The following proposition is for locally finite graphs with possibly more than one end. However the set of rays in the proposition are all from the same end. It will therefore be applicable in Section 3.2.

Proposition 3.2.11. *Let G be a locally finite k-edge-connected graph. Let S be a finite set of vertices in G, and let $e_1, \cdots, e_n$ be the edges of $\delta(S)$. Suppose that for each $i \in \{1, \cdots, n\}$ there is a ray P_i that begins with e_i such that $P_1, \cdots, P_n$ are edge-disjoint and are all in the same end. Suppose that $m \in \{1, \cdots, n\}$ is such that $|\{m, \cdots, n\}| \leqslant k$, and let*

α denote $|\{m, \cdots, n\}|$. If a finite set of edges $X \supseteq \{e_i : i < m\}$ disjoint from $P_m, \cdots, P_n$ is deleted from G, then there is a vertex v in $G - S$ that has α edge-disjoint paths in $G - X$ to S ending with $e_m, \cdots, e_n$.

Proof. Let $\mathcal{P} = \{P_1, \cdots, P_n\}$. These rays are from one end, so there is a sequence of infinitely many vertex-disjoint finite connected subgraphs of G, $L_0, L_1, \cdots$, such that for each L_i and each $j \in \{1, \cdots, n\}$ there is a vertex from P_j in L_i, and such that for each $i \in \mathbb{N}$, the vertices of L_{i+1} are at a bigger distance of S than the vertices of L_i.

For each $i \in \mathbb{N}$, let v_i be a vertex in L_i that is on a ray from $\mathcal{P}$, and let $\mathcal{W}_i$ be a collection of α pairwise edge-disjoint paths in the graph G from v_i to the edges $e_m, \cdots, e_n$. These paths may go through X. Such paths exist because the graph is k-edge-connected and $\alpha \leqslant k$.

For each $j \in \{m, \cdots, n\}$ let $W_{i,j} \in \mathcal{W}_i$ be the path from v_i to e_j, and consider the graph consisting of the union of $W_{i,j}$ over $i \in \mathbb{N}$. Apply Kőnig's Lemma 3.2.9 to the distance sets of vertices having distance exactly i from e_j. This gives a ray W_j starting at e_j such that infinitely often the initial segments of W_j are also initial segments of $W_{i,j}$, $i \in \mathbb{N}$. In this way, we have a collection $\{W_j\}_{j=m}^n$ of edge-disjoint rays in the same end as the rays $\mathcal{P}$ such that for every $i \in \mathbb{N}$, there is an edge-disjoint fan F_i from v_i to $\{W_j\}_{j=m}^n$. Let I be an infinite subset of $\mathbb{N}$ such that for each i in I, the layer L_i intersects all the rays $\{W_j\}_{j=m}^n$. Then renumber these layers, and the associated vertices and fans, such that the index i is in $\mathbb{N}$. Now for each $i \in \mathbb{N}$ add the fan F_i to the layer L_i.

We work with the 2α rays $\{W_j\}_{j=m}^n$ and $\{P_j\}_{j=m}^n$. For each $i \in \mathbb{N}$, layer L_i intersects all the rays in $\{W_j\}_{j=m}^n$ and in $\mathcal{P}$. However, the rays $\{W_j\}_{j=m}^n$ do not necessarily go straight from a layer to the following like the rays of $\mathcal{P}$. They may go down to a lower layer then back up.

For every $i \in \mathbb{N}$ the ith *connection graph* C_i has as its vertices the rays $\{W_j\}_{j=m}^n$ and $\{P_j\}_{j=m}^n$. Distinct vertices Q, Q' of C_i are connected by an edge if and only if there is a path Z in L_i having one end in Q and one end in Q' such that Z is edge-disjoint from every one of the rays of $(\{W_j\}_{j=m}^n \cup \{P_j\}_{j=m}^n) \setminus \{Q, Q'\}$. Since L_i is connected, C_i is connected. Let T_i be a spanning tree of C_i.

There is a tree T and an infinite set I of indices, such that, for each $i \in I$, $T_i = T$ (because there are only finitely many possibilities for a tree on a fixed finite number of vertices). We apply Lemma 3.2.10 to T with $A = \{W_j\}_{j=m}^{n}$ and $B = \{P_j\}_{j=m}^{n}$. Then, for each $j \in \{m, \cdots, n\}$, there is a W_j, P_j-path in T that does not go through $W_m, \cdots, W_{j-1}$, nor through $P_{j+1}, \cdots, P_n$.

Picking any layer L_{i_0}, such that C_{i_0} has spanning tree is T, in which to start, we construct the desired α paths so that the only portions of the paths below L_{i_0} are in the rays $\{P_j\}_{j=m}^{n}$. First we connect W_m and P_m by a path Z_m in L_{i_0}. We want the connections we use to connect P_j and W_j to be in distinct layers. To formalize this, we prove the following claim.

Claim 3.2.12. *Let R be any ray in the same end as the rays of $\mathcal{P}$, and let v be the unique vertex of degree 1 in R. For any layer L_i, there is an integer $J = (i, R)$ such that, if $j \geqslant J$ and e is an edge of R in L_i, then the shortest path $P_e \subseteq R$ containing v and e is disjoint from L_j.*

Proof. The layer L_i is finite, so $J(i) = i + max\{|E(P_e)| : e \in E(R \cap L_i)\}$ is a suitable choice. ☐

We will choose α many layers, $L_{i_0}, \cdots, L_{i_{\alpha-1}}$, one layer will be used to link one pair W_j and P_j for $j \in \{m, \cdots, n\}$. For $t \in \{1, \cdots, \alpha - 1\}$, we choose i_t such that $i_t \geqslant J(i_{t-1}, W_j)$ (Claim 3.2.12) for every $j \in \{m, \cdots, n\}$. This will guarantee that for each $j \in \{m, \cdots, n\}$, L_{i_t} and each layer beyond is disjoint from the segment of W_j from e_j to its last edge in $L_{i_{t-1}}$ (such last edge exists because $L_{i_{t-1}}$ is finite). In particular, for every $j \in \{m, \cdots, n\}$, the tail of W_j beginning at layer L_{i_t} is disjoint from the layers $L_{i_0}, \cdots, L_{i_{t-1}}$, and so is disjoint from any path in one of those layers used as a connection between rays as shown below.

In layer L_{i_t}, we use the $W_{t+m}P_{t+m}$-path in T to connect W_{t+m} to P_{t+m} by a path Z_{t+m} in L_{i_t}. The ordering of the paths in T implies that no edge of Z_{t+m} is in any of $W_m, \cdots, W_{t+m-1}$. The choice of i_t guarantees that the tail of W_{t+m} starting at the intersection with Z_{t+m} is disjoint from all of $Z_m, \cdots, Z_{m+t-1}$.

For each $j \in \{m, \cdots, n\}$, define the ray S_j to be as follows: from the starting edge e_j it is P_j up to the intersection with Z_j, then Z_j to W_j, and then the tail of W_j from the intersection with Z_j.

Each S_j becomes identical with W_j after a certain layer. We consider the highest such layer for $j \in \{m, \cdots, n\}$. More precisely, let M be such that for every $l \geqslant M$, L_l is disjoint from X, and every $S_j \cap L_l$ is contained in W_j. There is a sufficiently high vertex v_i, for some $i \geqslant M$, such that the paths $\{W_{i,j}\}_{j=m}^{n}$ from v_i to $\{e_j\}_{j=m}^{n}$ contain the W_j-portion of S_j from layer L_M up to the last layer where $W_{i,j}$ and S_j intersect. These truncations of the S_j yield the desired α paths.

□

Now we can prove our new result, that Huck's theorem extends to 1-ended locally finite graphs.

Theorem 3.2.13. *Let k be an odd positive integer. If G is a $(k+1)$-edge-connected, 1-ended, locally finite graph, then G is weakly k-linked.*

Proof. By Lemma 3.2.1 there is a finite set S of vertices containing the given finite set of k pairs of terminals in G such that each edge of $\delta(S)$ is the first edge of a ray in an edge disjoint collection of rays $\mathcal{P}$. Then $|\delta(S)| \geqslant k+1$ as G is $(k+1)$-edge-connected.

The idea is to find a linkage in the finite graph $G/(G-S)$ such that the paths of the linkage that go through s are replaceable with actual paths in G.

By successive lifting in $G/(G-S)$ of appropriate pairs of edges from $\delta(S)$ as in Lemma 3.2.5 we will reduce the problem to finite graphs, and then apply Huck's Theorem to the resulting $(k+1)$-edge-connected finite graph. Since $k+1$ is even, then by Lemma 3.2.5 applied to $(k+1)$ instead of k, we can lift all the edges of $\delta(S)$ in case $\deg(s)$ is even, and if $\deg(s)$ is odd we can lift until $\deg(s)$ is 3.

Suppose that $\deg(s)$ is odd and that for some i the lifting graph of G_i is an isolated vertex plus a complete bipartite graph. By (6) of Lemma 2.2.4, $\deg_{G_i}(s) \leqslant (k+1)+2 = k+3$. Since $\deg_{G_i}(s)$ is odd as $\deg(s)$ is, $\deg_{G_i}(s) \leqslant k+2$. Stop lifting at i_0 such that $\deg_{G_{i_0}}(s) = k+2$. Any i

for which G_i is an isolated vertex plus a complete bipartite graph satisfies $i \geqslant i_0$ (because if $i < i_0$, then $\deg_{G_i}(s) > k+2$, i.e. $\deg_{G_i}(s) \geqslant k+4$, contradicting (6) of Lemma 2.2.4). At this step, i_0, the ray R^* (See Definition 3.2.8) is not used yet, it is edge-disjoint from the paths already found which link edges e_i and e'_i for $i < i_0$.

The graph G_{i_0} is $(k+1)$-edge-connected as the degree of s is $k+2$, and because by construction there are $(k+1)$ edge-disjoint paths in it between any two vertices different from s. There is an edge-disjoint k-linkage in G_{i_0} by Huck's theorem 3.1.3.

A linkage in the final graph in the lifting sequence can be turned into a linkage in G as follows. If a path in a linkage goes through an edge that resulted from lifting a pair of edges from $\delta(S)$, then in $G/G-S$ it goes through e_i, s, e'_i for some i, where e_i and e'_i are edges as in (i) and (ii) of Lemma 3.2.5. We can replace this by P_i. Thus if all edges of $\delta(S)$ are paired and lifted, then it is clear that any linkage of the final G_j can be turned into a linkage of G.

The other possibility is that not all the edges are lifted and there are paths that go through pairs of edges incident with s. This can happen only if $\deg(s)$ is odd and in that case the lifting sequence terminates with G_{i_0} as defined above. We will use Proposition 3.2.11. Note that the paths of the linkage of G_{i_0} can use at most $k+1$ of the edges incident with s since $k+2$ is odd. Let F be the subset of $\delta(S)$ used in the linkage. Then $|F| \leqslant k+1$.

By Proposition 3.2.11, there is a vertex v in $G-S$ that has $|F|$ edge-disjoint paths to the distinct edges of F that avoid the paths $P_1, \cdots, P_{i_0}$ (X in the proposition is the set of edges of these paths). Now any path of the linkage that goes through s using a pair of edges from F can be replaced by a path in G that goes through v. $\square$

Note that the ray R^* can become problematic only after step i_0, at which $\deg_{G_{i_0}}(s) = k+2$. Particularly only after the isolated vertex plus complete bipartite structure of lifting graph shows up. The paths P_i in Lemma 3.2.5 defined after the emergence of this structure until $\deg(s)$ becomes 3 may go through the ray R^* at different levels. It will not be possible then to find a path that goes down on R^* from a higher level. This is a second reason (the first is keeping the degree of s from going below $k+2$) why we do not continue the constructions of Lemma 3.2.5 here after step i_0.

In the proof of their weak-linkage result (Theorem 3.1.8), Ok, Richter, and Thomassen used point (4) of their lifting graph result Theorem 2.1.11. They only knew that the complement of the lifting graph is disconnected when the connectivity is even and $\deg(s)$ is even. They did not know this is also the case when $\deg(s)$ is odd if the connectivity is even and at most one independent set of the lifting graph is of size $(\deg(s)+1)/2$.

For this reason when $\deg(s)$ was odd, they deleted an edge from $\delta(S)$, and hence had the connectivity reduced from $k+2$ to $k+1$. This is exactly why they needed connectivity $k+2$. This approach - of deleting an edge - is not helpful if we have more than one end. If we have cuts $\Delta_1, \cdots, \Delta_n$ such each Δ_i is the beginning of a set of rays defining an end, and we deleted one edge from each such cut, the connectivity is significantly reduced.

Therefore, when dealing with more than one end, we will again use the approach we used above in the proof. If a cut has odd size, then we will stop lifting edges from it when it reaches size $k+2$.

3.3 General infinite graphs

In this section we prove that Huck's theorem extends to general infinite graphs, with possibly uncountably many ends. The same general steps we use as exhibited in the coming subsections to reduce the question in general infinite graphs to locally finite graphs, and to generalize results from 1-ended graphs to multiple ends, were used by Thomassen in [25].

3.3.1 Reduction to countably infinite graphs

We first show that it is enough to extend Huck's theorem to countably infinite graphs in order to extend it to general infinite graphs.

Given a k-edge-connected infinite graph G, and a finite set of terminals T, we can construct a countable k-edge-connected subgraph G_c containing T as follows. The proof is straightforward, and was presented by Thomassen in [25], but we explain it here again. Let $G_0 := G[T]$. For each $i \geqslant 1$, define G_i to be the graph obtained by taking the union of the subgraphs $H_{\{x,y\}}$ over all $\{x, y\} \subseteq V(G_{i-1})$, where $H_{\{x,y\}}$ is a subgraph that consists of the union of k edge-disjoint paths between x and y in G. Now define G_c to be the union of all the graphs G_i. Then G_c is k-edge-connected, and since G_i is finite for each i, G_c is countable.

This shows that it is enough to consider the question for countably infinite graphs.

3.3.2 Reduction to locally finite graphs

In the previous subsection we showed that it is enough to consider the question for countably infinite graphs. Now we show that locally finite is enough. To go from countably infinite to locally finite, we will need the following theorem.

Definition 3.3.1. [25] A *splitting* of a graph G is a graph G' which is obtained from G by replacing each vertex v by a set of vertices V_v such that G' has no edges joining two vertices in V_v and such that the identification of all vertices of V_v into a single vertex, for each vertex v in G, results in G.

Definition 3.3.2. A *block* of a graph G is a maximal connected subgraph B of G such that, for every vertex v of B, $B - v$ is connected.

Theorem 3.3.3. *[25] Let k be a natural number, and let G be a countably infinite k-edge-connected graph. Then G has a splitting such that the resulting graph is k-edge-connected, and each block of the resulting graph is locally finite.*

Let G be a countably infinite k-edge-connected graph. By Theorem 3.3.3, there is a splitting G' of G that is k-edge-connected, and each block of it is locally finite.

A linkage in a splitting of a graph G naturally gives a linkage in G since identifying the vertices of each one of the independent sets V_v will keep the paths of the linkage edge-disjoint. If a graph is k-edge-connected, then so is every block of it. Also given a k-linkage problem on a graph, it can be solved by solving smaller linkage problems on a finite number of its blocks. This means that it is enough to consider our question for locally finite graphs only.

Note that the graph G' that results from the splitting as found by this theorem is itself countable, not only its blocks, because the edges incident with V_v are exactly the edges that were incident with v in G. There is only countably many of those edges as G is countable. If V_v were uncountable, then some of its vertices will be isolated in the splitting G'. The splitting graph G' is connected (in fact k-edge-connected) according to the theorem, and so it cannot have isolated vertices.

Now we can assume we are working on a graph that is locally finite with arbitrarily many ends (possibly uncountably many).

3.3.3 Main Result

We are now working in a locally finite graph with arbitrarily many ends. To be able to use some of the results about 1-ended graphs in graphs with arbitrarily many ends, we will need the following theorem.

Definition 3.3.4. (Boundary-linked)[25] A vertex set A - possibly infinite - in a graph G is *boundary-linked* if there is a collection $\mathcal{P}$ of pairwise edge-disjoint rays, all in one end, such that the set formed by taking the first edge of each one of the rays in $\mathcal{P}$ is $\delta(A)$, and all the other edges of the rays in $\mathcal{P}$ are contained in $G[A]$. The cut $\delta(A)$ is called the *boundary* of A.

Theorem 3.3.5. *[25] Let G be a connected locally finite graph. If A_0 is a vertex set such that $\delta(A_0)$ is finite, then $V(G) \setminus A_0$ can be partitioned into finitely many pairwise disjoint vertex sets each of which is either a singleton or a boundary-linked vertex set with finite boundary.*

Now we can prove our main result, that Huck's theorem is true for general infinite graphs. We do this by proving it is true for locally finite graphs with arbitrarily many ends.

Theorem 3.3.6. *Let k be an odd positive integer. If G is a $(k+1)$-edge-connected infinite graph, then G is weakly k-linked.*

Proof. By Sections 3.3.1 and 3.3.2 we may assume that G is locally finite. Let A be the finite set of terminals. Since G is locally finite, $\delta(A)$ is finite. By Theorem 3.3.5, $V(G) \setminus A$ can be partitioned into finitely many pairwise disjoint vertex sets that are either singletons or boundary-linked sets with finite boundary.

Adding the singletons to A, we get a finite set S containing the terminals such that $V(G) \setminus S$ is partitioned into finitely many pairwise disjoint sets $A_1, \cdots, A_n$ such that each one of them is boundary-linked with finite boundary. Having a finite boundary implies that there are only finitely many edges between the sets $A_1, \cdots, A_n$. The cuts $\delta(A_1), \cdots, \delta(A_n)$ are not necessarily disjoint.

Each edge in $\delta(A_i)$ is the first edge of a ray in an edge-disjoint collection of rays $\mathcal{P}_i$ that belong to one end E_i of G.

The set $\delta(S)$ is partitioned into the $\delta(S) \cap \delta(A_i)$, $1 \leqslant i \leqslant n$. The set $\delta(S) \cap \delta(A_i)$ has the property that each edge of it is the first edge of a ray in $\mathcal{P}_i$. These sets are not necessarily cuts as there could be edges between the different sets $A_1, \cdots, A_n$.

First we contract each A_i into a single vertex s_i. The resulting finite graph is $(k+1)$-edge-connected as G is. We then take the ends $E_1, \cdots, E_n$ one after the other and apply the same steps of the proof of Theorem 3.2.13 to it as clarified below. What plays the role of $\delta(S)$ for the end E_i here is the boundary $\delta(A_i)$ (although A_i may contain arbitrarily many ends, but the set of rays fixed for it beginning with the edges of $\delta(A_i)$ are all in E_i).

Let G^i denote the graph obtained by contracting each A_j for $j \neq i$ into the vertex s_j. This graph is not necessarily 1-ended, but the fixed collection of rays we work with in A_i is from one end (even if A_i branches into arbitrarily many ends).

If $\delta(A_i)$ is even, then all of its edges get lifted. If it is odd, then we stop lifting when the size of $\delta(A_i)$ becomes $k+2$. In any case the finite graph we have is $(k+1)$-edge-connected. Again we use Huck's theorem 3.1.3 to find a linkage in the remaining finite graph which could still have some of the vertices $s_1, \cdots, s_n$ with degree $k+2$ each.
Recall that Proposition 3.2.11 is true for graphs with more than one end. Thus we can find for each i a vertex v_i in $G^i - (S \cup \{s_j : j \neq i\})$ such that some of the paths of the linkage that go through s_i can be replaced with paths that go through v_i. Note that a path going through s_i may be pass from A_i to an A_j with $j \neq i$ as some of the edges of $\delta(A_i)$ could be edges between A_i and A_j. $\square$

Chapter 4

Strongly connected orientations

4.1 Introduction

In 1960 Nash-Williams proved that an edge-connectivity of $2k$ is sufficient for a finite graph to have a k-arc-connected orientation [16]. He then conjectured that the same is true for infinite graphs [15]. In his paper [13], Mader introduced his theorem proving the existence of a feasible lifting at a vertex and used it to give a simpler proof of Nash-Williams' orientation theorem for finite graphs.

Conjecture 4.1.1. *(Nash-Williams [15]) Every $2k$-edge-connected graph admits a k-arc-connected orientation.*

Thomassen proved in 2016 that every $8k$-edge-connected infinite graph has a k-arc-connected orientation [25].

Theorem 4.1.2. *(Thomassen [25]) Let k be a natural number, and let G be an $8k$-edge-connected graph. Then G has a k-arc-connected orientation.*

In this chapter we present another application of our lifting graph results of Chapter 2. We prove that in a locally finite 1-ended graph, an edge-connectivity of $4k$ is enough for the graph to have a k-arc-connected orientation. This is a good new step towards the conjecture of Nash-Williams 4.1.1.

One can generalize - as done in Chapter 3 in the linkage result - from locally finite graphs to countable graphs using the splitting theorem of Thomassen 3.3.1. Then one can also generalize to uncountable graphs. The method of generalization is explained in Section 8 of Thomassen's paper [25].

4.2 k-arc connected orientations in 1-ended $4k$-edge-connected graphs

In this section we prove our new orientation result, following the ideas presented by Thomassen in [25]. To prove his result Thomassen first proved that for a finite set of vertices V in a $4k$-edge-connected graph G, there is an immersion (defined below in Definition 4.2.1) in G of a finite Eulerian $2k$-edge-connected graph with vertex set A. Again, as in Chapter 3, the property of being Eulerian (every vertex has even degree) with the even connectivity was needed to use the fact that the lifting graph has a disconnected complement in that case. Now we know more about the lifting graph when the degree of the vertex at which the lifting takes place is odd, so Eulerian graphs will not show up in our work.

Here we find an immersion inside a $4k$-edge-connected graph of a $4k$-edge-connected finite graph. We only do this for 1-ended graphs, unlike the result of Thomassen, which holds for graphs with multiple ends.

Definition 4.2.1. For a finite graph G, $\mathcal{P}(G)$ denotes the set of paths in G. An *immersion* of a graph H in G consists of an injection $\phi : V(H) \to V(G)$ and a function $\theta : E(H) \to \mathcal{P}(G)$ such that, for $uv \in E(H)$,

(1) $\theta(uv)$ is a $\phi(u)\phi(v)$-path in G,

(2) for every $v \in V(H)$, the vertex $\phi(v)$ is not an interior vertex of a path in $\theta(E(H))$, and

(3) the paths in $\theta(E(H))$ are pairwise edge-disjoint.

The graph H is *immersed* in G and the subgraph $(\phi(V(H)), X)$ is an *immersion of* H *in* G, where X is the set of all the edges in the paths in $\theta(E(H))$.

We have the following result about the existence of an immersion.

Theorem 4.2.2. *Let G be a $2k$-edge-connected locally finite 1-ended graph, and let A_0 be a finite vertex set in G. Then G contains an immersion of a finite $(2k-1)$-edge-connected graph with vertex set $S \supseteq A_0$.*

Proof. By Lemma 3.2.1, let S be a finite set containing A_0 such that each edge of $\delta(S)$ is the first edge of a ray in an edge-disjoint collection of rays $\mathcal{P}$, and let s be the vertex that results from identifying all the vertices of $G - S$.

Because $2k$ is even, Lemma 3.2.5 applied with connectivity $2k$ implies there is a sequence of lifts of pairs of edges from $\delta(S)$ such that these pairs of edges are linkable in $G - S$ and the sequence ends by having all the edges of $\delta(S)$ lifted if $|\delta(S)|$ is even or having three of them remaining if it is odd.

The last graph in the sequence satisfies that for any two vertices x and y in it different from s there are $2k$ edge-disjoint paths between them. Denote this graph by G^*. Note that in any case, regardless of whether $|\delta_G(S)|$ is even or odd, $\deg_{G^*}(s) \leqslant 3$. Thus for any two vertices x and y in $G^* - s$ at most one of the edge-disjoint paths between them in G^* goes through s. Thus $G^* - s$ is a $(2k-1)$-edge-connected graph with vertex set S. Any edge in $G^* - s$ is either an edge of G or an edge that resulted from lifting, that is an edge corresponding to a path in G and this collection of paths is edge-disjoint (the paths $\{P_i\}_{i \in I}$ of Lemma 3.2.5). This gives the desired immersion. $\square$

Recall the example in Figure 2.16, the lifting graph on 5 edges incident with s is an isolated vertex plus a $K_{2,2}$. That is there are only 4 liftable pairs and they are all symmetric. Lifting any one of these liftable pairs gives a lifting graph on 3 edges incident with s such that no pair of edges is liftable. This means that in this case we cannot continue lifting until we have only one edge incident with s. If we could do so then we would have the immersion of a $2k$-edge-connected graph instead of $2k-1$ and will also be able to do the multiple end case. In other words, connectivity of $(2k-1)$ is the best one can get out of the proof approach of successive lifting in case the structure of isolated vertex plus a $K_{2,2}$ shows up. However, some discussions Bruce Richter and me had after the defense of this thesis give hope that we can choose the set S containing A_0 such that the edges of $\delta(S)$ can be entirely paired and lifted.

Definition 4.2.3. (a) An edge e is *directed* if one of its incident vertices is designated its tail and the other its *head* and its *direction* is from the tail to the head. A directed edge is also call an *arc*.

(b) An path P is *directed* if every internal vertex of P is incident with the head of one edge of P and the tail of another.

(c) A *directed* cycle is defined similarly.

(d) A directed path P is directed from x to y if x is the end of P that is the tail of some edge of P and y is the end of P that is the head of some edge of P.

(e) A path P is *mixed* if each of its edges is directed.

Definition 4.2.4. A graph is *directed* if every edge of the graph is directed. An assignment of directions to all the edges of the graph is an *orientation* of the graph. It is *strongly connected* if for any two vertices x and y it has a directed path from x to y (and also from y to x). A directed graph is *k-arc-connected* if the deletion of any set of fewer than k arcs results in a strongly connected directed graph.

By Menger's theorem, being k-arc-connected is equivalent to the existence of k arc-disjoint directed paths from x to y for any two vertices x and y.

In his paper [25], Thomassen presented the following algorithmic way for defining a k-arc-connected orientation of a $(4k-2)$-edge-connected finite graph.

Theorem 4.2.5. *[25] Let k be a positive integer, and let G be a finite $(4k-2)$-edge-connected graph. Successively perform either of the following operations:*

O1: Select a cycle in which no edge has a direction and make it into a directed cycle.

O2: Select two vertices u, v joined by $2k-1$ pairwise edge-disjoint mixed paths, and identify u, v into one vertex.

When neither of these operations can be performed the resulting oriented graph has only one vertex. The edge-orientations of G obtained by O1 result in a k-arc-connected directed graph.

If we cannot perform operation $O1$, then the set of edges without direction forms a forest, and so there are at most $(n-1)$ of them if $n \geqslant 2$ is the number of vertices in G. If we cannot perform operation $O2$, then it can easily be shown that there are at most $(2k-2)(n-1)$ directed edges. Thus if we cannot perform any of the two operations, then G has at most $(2k-1)(n-1)$ edges, contradicting the assumption that it is $(4k-2)$-edge-connected. Therefore, the only case in which none of the two operations can be performed is when the graph consists of one vertex.

The directed graph obtained at the end of the algorithm is k-arc-connected because $O1$ gives directions to the edges of any cut in a balanced way, and $O2$ can only be applied to vertices u and v if there are k-arc-disjoint directed paths from u to v and k-arc-disjoint directed paths from v to u. To see the latter, note that, since each cut is balanced, then if a cut has at most $(k-1)$ edges in one direction, then it has at most $(k-1)$ edges in the other direction. By Menger's theorem, this means that there are at most $(2k-2)$ mixed paths between u and v, and so $O2$ cannot be applied to u and v.

From this it is straightforward to see that the following lemma is also true.

Lemma 4.2.6. *Let k be a positive integer, and let G be a finite $(4k-2)$-edge-connected graph. Let H be a subgraph of G with an orientation obtained using operations O1 and O2. Then the orientation of H can be extended to an orientation of G which is k-arc-connected.*

Now we present our new result.

Theorem 4.2.7. *Let k be a positive integer, and let G be a $4k$-edge-connected locally finite 1-ended graph. Then G has a k-arc-connected orientation.*

Proof. The proof is very similar to the proof of Theorem 7 in [25]. It differs only in that it does not use Eulerian subgraphs.

Let $e_0, e_1, \cdots$ be the edges of G. We construct a nested sequence of finite directed subgraphs $\{W_n\}_{n\in\mathbb{N}}$ using operations $O1$ and $O2$ such that each orientation is an extension of the previous, W_n contains e_n, and has the following property: for any two vertices x and y in $V(W_n)$ there are k arc-disjoint paths from x to y in W_{n+1}.

The graph G has an edge-connectivity of $4k > 1$, therefore it contains a cycle containing e_0. Using $O1$ give this cycle an orientation and let W_0 be this directed cycle. This defines the first subgraph in the sequence. Note that W_0 is not required to be k-arc-connected.

Assume that W_n is defined. To get W_{n+1}, let A_0 be the union of $V(W_n)$ and the two end-vertices of e_{n+1}. By Theorem 4.2.2, G contains as a subgraph an immersion H_{n+1} of a finite $(4k-1)$-edge-connected graph G_{n+1} such that $A_0 \subseteq V(G_{n+1}) \subseteq V(G)$. Note that $V(G_{n+1}) \subseteq V(H_{n+1}) \subseteq V(G)$ but G_{n+1} is not necessarily a subgraph of G.

By construction, $V(W_n) \subseteq A_0 \subseteq V(G_{n+1}) \subseteq V(H_{n+1})$. Thus any edge in H_{n+1} between two vertices in $V(W_n)$ is an edge whose end-vertices are both in $\phi(V(G_{n+1}))$ (cf. Definition 4.2.1). Regarded as a path, by point (2) in Definition 4.2.1, any edge from W_n in H_{n+1} cannot be a part of a longer path in $\theta(E(G_{n+1}))$. Thus, such an edge is itself a path in $\theta(E(G_{n+1}))$. This means that any such edge is also an edge in G_{n+1}.

Let G'_{n+1} be the graph obtained from G_{n+1} by adding the edges of W_n that are not in G_{n+1}, and H'_{n+1} be the graph obtained from H_{n+1} by adding the edges of W_n that are not in H_{n+1}. Note that, by the previous paragraph, we have that $E(H'_{n+1}) \setminus E(H_{n+1}) \subseteq E(G'_{n+1}) \setminus E(G_{n+1})$.

The graph W_n was oriented using $O1$ and $O2$ and is a subgraph of the $(4k-1)$-edge-connected graph G'_{n+1}. Thus by Lemma 4.2.6 this orientation can be extended to a k-arc-connected orientation of G'_{n+1}. Note that in this extension, the edges of $E(G'_{n+1}) \setminus E(G_{n+1})$ inherit their orientation from W_n. An orientation of H_{n+1} can be naturally obtained from an orientation of G_{n+1} by giving each path of the immersion the direction of the edge representing it in G_{n+1}. As for the edges of $E(H'_{n+1}) \setminus E(H_{n+1})$, they have the same orientation they have in W_n.

We define W_{n+1} to be the directed graph H'_{n+1}. This graph contains W_n as we obtained it from H_{n+1} by adding the edges of W_n not in H_{n+1}. Since H_{n+1} and W_n are both subgraphs of G, then so is $W_{n+1} = H'_{n+1}$. Now it only remains to show that for any two vertices x and y in $V(W_n)$ there are k arc-disjoint paths from x to y in W_{n+1}. There are such paths from x to y in G'_{n+1}. Replacing each edge of these paths that is in $E(G_{n+1})$ with its image under θ (cf. Definition 4.2.1) gives k arc-disjoint paths from x to y in H'_{n+1}, i.e. in W_{n+1}. To see this, note that the image of any edge in $E(G_{n+1})$ under θ is one of two possibilities. One possibility is that it is its own image under θ, and in that case arc-disjointness does not change because nothing was changed. The second possibility is that its image is a path of length greater than one, and in that case, none of the edges of this path is in $E(W_n)$ as $V(W_n) \subseteq V(G_{n+1})$, so the path is disjoint from all the edges that were added to G_{n+1} to obtain G'_{n+1}, and particularly the ones that may appear in the k arc-disjoint paths from x to y in G'_{n+1}. Moreover, the paths of $\theta(E(G_{n+1}))$ are edge-disjoint by definition.

The union of the directed graphs W_n, $n \in \mathbb{N}$, defines an orientation of G. For any two vertices x and y of G, there exists $n \geqslant 1$ such that x and y are in W_n. To see this consider any path between x and y in G. For some sufficiently large n, W_n contains all the edges of this path, and so also contains x and y. Then there are k arc-disjoint directed paths from x to y in W_{n+1}. Since this is true for every x and y in G, the orientation of G is k-arc-connected. $\square$

Chapter 5

Future Work and Related Questions

This chapter has several sections. The first and longest describes major obstacles occurring in our attempt to prove that the Weak Linkage Conjecture for finite graphs implies the same conjecture for infinite graphs. Here, in the first section, we focus on the simplest case: locally-finite 1-ended graphs. In the remaining sections, we present some open problems and related questions.

5.1 The weak linkage conjecture for infinite graphs

In this section we present our steps trying to prove that the weak linkage conjecture for finite graphs implies the conjecture for infinite graphs. Throughout the section we assume Conjecture 3.1.2 is true for finite graphs. We also assume that G is a locally finite 1-ended graph that is k-edge-connected, and k is odd.

We follow the same steps presented in Section 3.2. Since the connectivity is now odd, Theorem 2.2.24 shows that there are three possible outcomes when the complement of $\mathcal{L}_i$ is connected (See Definition 3.2.6 in the proof of Lemma 3.2.5). Reordering those outcomes for convenience

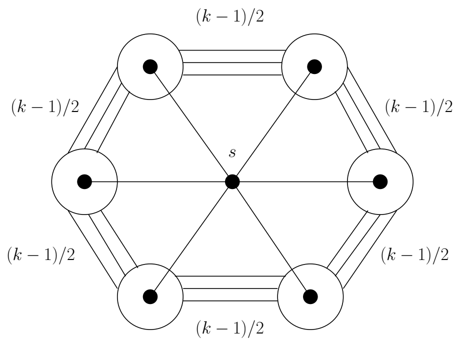


Figure 5.1: Cycle of intersections of dangerous sets.

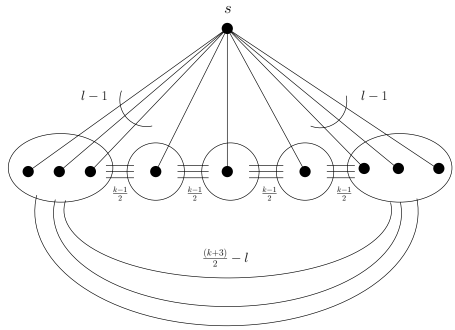


Figure 5.2: Path of intersections of dangerous sets.

here, the complement is one of:

1. a Hamilton cycle;
2. two cliques of the same size with a path of length at least 1 between them; and
3. two cliques of the same size with a single common vertex.

In case of the third outcome we have a special ray R^* (See Definition 3.2.8) corresponding to the isolated vertex, and this can be dealt with as seen in the proof of Lemma 3.2.5 (allowing the paths linking pairs of rays to go through R^* until three rays, including R^*, remain). In first two cases G_i (cf. (2) in Lemma 3.2.5) has one of the structures (i) or (ii) of Theorem 2.3.1. In both cases, we may assume without loss of generality that each one of the sets (we called them blobs) S_i from the statement of Theorem 2.3.1 contains terminals as shown in the following proposition. We add the figures illustrating the two structures here again for convenience, Figures 5.1 and 5.2.

Proposition 5.1.1. *A graph obtained from a graph G, with structure (i) or (ii) of Theorem 2.3.1, by deleting one of the intermediate blobs S_i (and hence the edge connecting it to s) and replacing it with $(k-1)/2$ edges between S_{i-1} and S_{i+1} (indices modulo $deg(s)$ in case (i)) is k-edge-connected as long as the number of edges incident with s stays at least k.*

Proof. This can be easily seen by noticing that the size of any cut - other than the cut that has s alone on one side - did not change. $\square$

Remark 5.1.2. The main conclusion of the previous proposition is that we may assume without loss of generality that each blob contains terminals. However, if most of the terminals are centered in a small number of blobs, then the above procedure of replacing blobs by $(k-1)/2$ edges may result in a graph in which s has small degree, but the subgraph obtained by deleting s is $(k-1)$-edge-connected.

Remark 5.1.3. Let G be a graph that has structure (i) or (ii) of Theorem 2.3.1 and suppose that we replaced a blob that does not contain terminals with $(k-1)/2$ edges between its neighbouring blobs. If we have a linkage in the new graph, then we have a linkage in the original graph because all we need to do is replace the $(k-1)/2$ edges with an edge-disjoint $\frac{(k-1)}{2}$-linkage inside the deleted blob, and such a linkage exists because the blob is $\frac{(k+1)}{2}$-edge-connected as remarked in 2.3.2. Note that the existence of such a linkage follows from the assumption that the conjecture is true in finite graphs and not from Huck's theorem as we do not know the parity of $(k-1)/2$.

The k-terminals do not have to be evenly distributed between the blobs. They might all be in one blob. Concerns arise if the number of pairs of terminals inside one blob is more than $\frac{(k-1)}{2}$ because we can only link at most $\frac{(k-1)}{2}$ pairs by going outside and then back inside the blob, but also we do not know if each of the terminals that will be linked in this way is incident with an edge that goes outside the blob. So we might need to have an inner $2(\frac{(k-1)}{2})$-linkage to go out and back in in this way. On top of that we need to find a linkage disjoint from this one between the remaining pairs of terminals not yet linked. In total, this means we might need the blob to be at least k-edge-connected, or at least more than $\frac{(k+1)}{2}$-edge-connected.

If there are at least k blobs and each one contains terminals, then each blob contains at most $(k+1)$ terminals since the total number of terminals is at most $2k$. In that case, if there is a blob that contains $(k+1)$

terminals, then it is the only such blob and each other blob contains only one terminal.

5.1.1 Wall-like structure

Definition 5.1.4. (Wall-like structure) A finite set of edge-disjoint rays in one end defines a *wall-like structure* if its end graph is a path or a cycle.

Recall that two rays (or their first edges) are adjacent in $\mathcal{E}$ if there are infinitely many vertex-disjoint paths between them that do not intersect in edges any other rays of $\mathcal{P}$.

Now if $\mathcal{E}$ is a path $P_1, P_2, P_3, \ldots, P_d$, or a cycle $P_1, P_2, P_3, \ldots, P_d, P_1$, then P_1 and P_3 being non-adjacent means that there do not exist infinitely many vertex-disjoint paths between them that are edge-disjoint from those in $\mathcal{P} \setminus \{P_1, P_3\}$. Thus, if we fix an infinite set $\mathcal{P}_1$ of vertex-disjoint paths that join P_1 and P_2, which are edge-disjoint from $\mathcal{P} \setminus \{P_1, P_2\}$, and an infinite set $\mathcal{P}_3$ of vertex-disjoint paths that join P_2 and P_3, which are edge-disjoint from $\mathcal{P} \setminus \{P_2, P_3\}$, then all but finitely many of the paths of $\mathcal{P}_1$ and $\mathcal{P}_3$ have to alternate on P_2 in the sense that only finitely many of the paths of $\mathcal{P}_1$ and $\mathcal{P}_3$ have a common vertex on P_2. If infinitely many of the paths of $\mathcal{P}_1$ and $\mathcal{P}_3$ have a common vertex on P_2, then they are infinitely many vertex-disjoint paths between P_1 and P_3 that are edge-disjoint from P_2 and the other rays, and this contradicts the assumption that P_1 and P_2 are not adjacent in the end graph. That is, $\mathcal{P}_1$ and $\mathcal{P}_3$ result in two infinite (subdivided) ladders between P_1 and P_2 and between P_2 and P_3 and their steps are alternating except for finitely many. This is why we call it a wall-like structure. Note that it is a cylindrical wall in the cycle case.

Note also that if the end graph is a path or a cycle, then all but finitely many of the connections (paths) between the rays have to be ladder-confined in the following sense. There can only finitely many paths between two rays P_i and P_k that are edge-disjoint from the other rays if $|i-k| > 1$. The infinitely many paths between P_i and P_{i+1} do not have to give an exact ladder, they only have to alternate with the paths between P_{i-1} and P_i, but they can make big diagonal jumps between the two rays.

Proposition 5.1.5. *Let G be a k-edge-connected locally finite graph, S a finite set of vertices in G, and s the vertex of contraction of $G/(G-S)$. If:*

- *every edge in $\delta(S)$ is the first edge of a ray in an edge-disjoint collection of rays $\mathcal{P}$ from one end;*
- *the complement of $L(G/(G-S), s, k)$ is connected;*
- *and $\mathcal{E}(\mathcal{P}, S)$ is a subgraph of the complement of $L(G/(G-S), s, k)$,*

then one of the following holds:

(1) $\mathcal{P}$ defines a wall-like structure or,

(2) $\mathcal{P}$ is partitioned into three sets $\mathcal{P}_1$, $\mathcal{P}'$, $\mathcal{P}_2$ such that $\mathcal{P}'$ defines a wall-like structure and separates $\mathcal{P}_1$ and $\mathcal{P}_2$ in the following sense: for any rays $P_1 \in \mathcal{P}_1$ and $P_2 \in \mathcal{P}_2$, there does not exist an infinite collection of vertex-disjoint paths between P_1 and P_2 that is edge-disjoint from all the rays in $\mathcal{P}'$.

Proof. First recall that $L(G/(G-S), s, k)$ and $\mathcal{E}(\mathcal{P}, S)$ have the same vertex set, that is $\delta(S)$. If the complement of $L(G/(G-S), s, k)$ is connected, then it is a Hamilton cycle, or two cliques with a path between them. A connected subgraph of a cycle or a path is a cycle or a path. Therefore, if the complement of $L(G/(G-S), s, k)$ is a Hamilton cycle, then $\mathcal{E}(\mathcal{P}, S)$ is a path or a cycle, and hence forms a wall-like structure (possibly a cylindrical wall on the cycle).

Now suppose that the complement of $L(G/(G-S), s, k)$ consists of two cliques with a path between them. Recall that the edges of $\delta(S)$ are the vertices of $L(G/(G-S), s, k)$. Let C_1 and C_2 be the sets of edges in $\delta(S)$ whose corresponding vertices form the two cliques in $L(G/(G-S), s, k)$, $x_1 \in C_1$ and $x_2 \in C_2$ be the two edges of $\delta(S)$ whose corresponding vertices in $L(G/(G-S), s, k)$ are the end-vertices of the path between the two cliques, and $X = \delta(S) \setminus (C_1 \cup C_2)$ the set of edges in $\delta(S)$ whose corresponding vertices in $L(G/(G-S), s, k)$ are the interior vertices of the path between the two cliques.

Define $\mathcal{P}'$ to be the set of rays in $\mathcal{P}$ whose first edges (from $\delta(S)$) are the edges in X, and $\mathcal{P}_1$ and $\mathcal{P}_2$ to be the sets of rays in $\mathcal{P}$ whose first edges are the edges in $C_1 \setminus \{x_1\}$ and $C_2 \setminus \{x_2\}$ respectively.

Since $\mathcal{E}(\mathcal{P}, S)$ is a subgraph of $L(G/(G-S), s, k)$ on the same vertex set, the subgraph of $\mathcal{E}(\mathcal{P}, S)$ induced by X is a path, and so the rays of $\mathcal{P}'$ form a wall-like structure. For any rays $P_1 \in \mathcal{P}_1$ and $P_2 \in \mathcal{P}_2$, their first edges e_1 and e_2 are not adjacent in $\mathcal{E}(\mathcal{P}, S)$ because they are not adjacent in $L(G/(G-S), s, k)$. Moreover, any path between e_1 and e_2 in $L(G/(G-S), s, k)$ has to go through the path between the two cliques, so all but finitely many of the paths between P_1 and P_2 has to go through $\mathcal{P}'$. $\square$

The construction done by Ok, Richter, and Thomassen, and which we also used in Chapters 3 and 4 fails when the end graph is a subgraph of the complement of the lifting graph. As shown above, in that case, the rays of $\mathcal{P}$ together with the infinitely many vertex disjoint paths between them give a wall-like structure.

The main reason why things worked in Chapter 3 but are difficult here is the parity of the connectivity. The connectivity we assumed in Theorem 3.2.13 is $k+1$, that is even. Here we work with a connectivity of k, odd. Thus in the successive lifting procedure we might get stuck when for some i, G_i has structure (i) or (ii) of Theorem 2.3.1. In this case the connected end graph may be a subgraph of the connected complement of the lifting graph (cycle or two cliques with a path between them).

In case the complement of the lifting graph is two cliques with a path between them, we can - by Remark 2.3.4 - lift pairs of edges from the inside going out until we reach the isolated vertex plus complete bipartite case. This is possible even when we have a wall-like structure on the infinite side as we ignore a central ray (corresponding to the isolated vertex) by allowing paths to go through it and only lift pairs of edges that lie on two different sides of it. This is why we focus on the case when the complement of the lifting graph is a Hamilton cycle.

There are two ways to go. One way is to find a linkage in $G/(G-S)$ and then try to extend it into the infinite side of $\delta(S)$ to have a linkage in G. Another way is to contract the blobs and treat them as if they were the terminals, find a linkage of the blobs, then expand them again and try to extend the linkage inside them to have a linkage in G.

In the first way it is not guaranteed that the pairs of edges of $\delta(S)$ that the linkage goes through are linkable on the infinite side of the cut by edge-disjoint paths. In the second way, the $(k-1)/2$ edges between the blobs could be used in the linkage we found of the blobs as terminals, as well as the edge connecting the blob to s. This means that we may need to find a weak k-linkage from the actual terminals inside the blob to the end-vertices in the blob of the edges coming out of it. This is not guaranteed as each blob is only $(\frac{k+1}{2})$-edge-connected at best. This is only easy if each blob contains exactly one terminal as there are k-edge-disjoint paths from any vertex inside the blob to the k edges coming out of it.

Nevertheless, we consider the following linkage problem obtained by contracting each blob into a vertex in case the blobs form a cycle with $(k-1)/2$ edges between any two consecutive blobs.

5.1.2 Special case of the linkage problem

Here we assume that we have a wall-like structure on the infinite side and contract each blob into a single vertex and think of them as terminals to be linked. We may assume without loss of generality that all the pairs of terminals (contracted blobs) overlap. That is, we assume they are in the order $s_1, \cdots, s_k, t_1, \cdots, t_k$. If two pairs (s_i, t_i) and (s_j, t_j) do not overlap in their order on the cycle, then we can have a 2-linkage of them on one cycle, delete that cycle, have the connectivity reduced by 2, and apply induction on k. This is the worst case for the linkage problem, having a wall-like structure with all overlapping pairs of terminals.

Description of a Linkage: A particular linkage is illustrated in Figure 5.3 for $k = 7$. Thanks to Ali Assem Mahmoud for describing this linkage in the case $k = 3$.

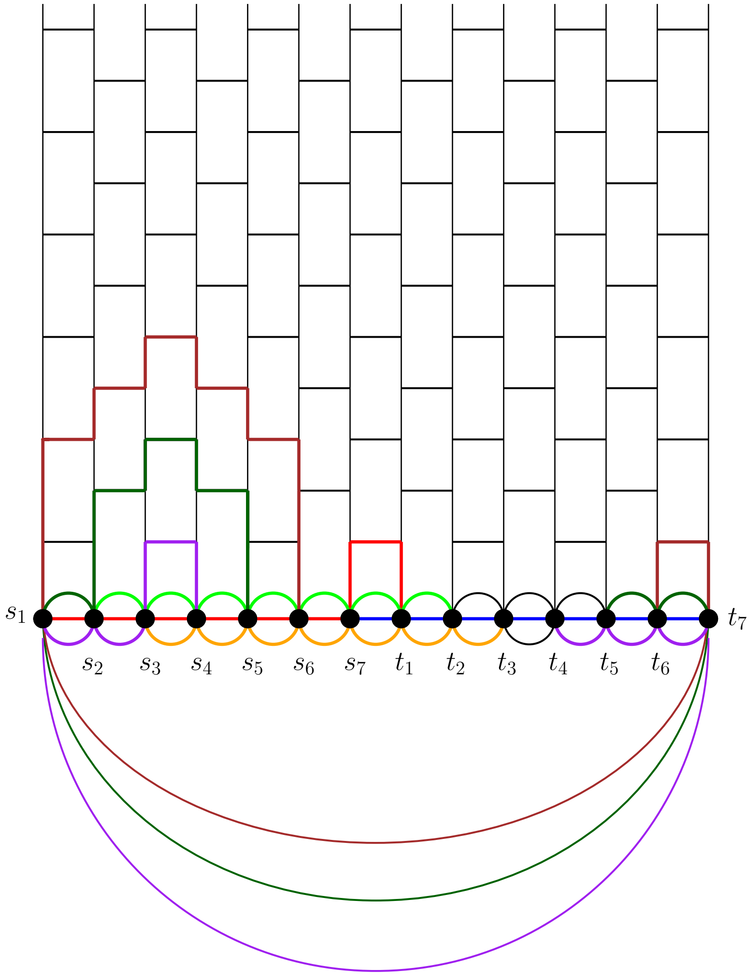


Figure 5.3: $k = 7$, the infinite side has a wall-like structure on $\mathcal{P}$, and the terminals are all-overlapping on a cycle.

Go directly from s_1 to s_k through $s_2, \ldots, s_{k-1}$, then up and down one brick to t_1. Go directly from s_k to t_k through $t_1, \ldots, t_{k-1}$. For each $i \in 2, \ldots, (k-1)/2$, connect s_i directly to t_i going through increasing indices of s then t_1 then increasing indices of t.

Now we have linked $2 + \frac{(k-3)}{2} = \frac{(k+1)}{2}$ pairs, and it remains to link the $(k-1)/2$ pairs s_i to t_i for $i \in \{\frac{(k+1)}{2}, \ldots, k-1\}$.

The remaining unused (partially parallel) segments are the $(k-3)/2$ segments $[t_2, t_3, \ldots, t_k]$, $[t_3, t_4, \ldots, t_k]$, $\ldots$, $[t_{(k-1)/2}, t_{(k+1)/2}, \ldots, t_k]$, the

$(k-3)/2$ segments $[s_1, s_2]$, $[s_1, s_2, s_3]$, ... , $[s_1, s_2, \ldots, s_{(k-3)/2}, s_{(k-1)/2}]$, and the $(k-1)/2$ parallel edges $s_1 t_k$.

Now make the $(k-1)/2$ nested linkage of s_i to s_{k-i} in the grid. That is link $s_{(k-1)/2}$ to $s_{(k+1)/2}$ first (through the grid outside the thick cycle), above it $s_{(k-3)/2}$ to $s_{(k+3)/2}$, ... , above this s_2 to s_{k-2}, then finally on the top s_1 to s_{k-1}.

Now to connect s_{k-1} to t_{k-1} we take the following path. From s_{k-1} to s_1 through the grid as described above. Then an edge of the $(k-1)/2$ parallel edges $s_1 t_k$, then up and down a brick to t_{k-1}.

For each $i \in \{2, \ldots, (k-1)/2\}$, s_{k-i} is linked to s_i through the grid as described above, then we can go from s_i to s_1 through the segment $[s_i, s_{i-1}, \ldots, s_2, s_1]$, then from s_1 to t_k using one of the $(k-1)/2$ parallel edges $s_1 t_k$, then from t_k to t_{k-i} through $[t_k, t_{k-1}, \ldots, t_{k-i+1}, t_{k-i}]$.

Note that this linkage only uses a 3-edge-connected grid, and does not need it to be infinite but only to be of height $(k-1)/2$. This looks promising for both the finite case and the general infinite case, because all we need to find is such a thin short grid accessible from the collection of terminals.

5.1.3 Another collection of rays

If the collection of rays $\mathcal{P}$ we started with turned out not to be helpful, for example its end graph gives a wall-like structure, we may try to use another collection of rays whose end graph has a better connectivity than a path or a cycle. It is possible that no such collection exists even in a k-edge-connected graph as illustrated in Figure 5.4. If along each ladder there are induced C_6 subgraphs a little far apart (to avoid small cuts in one case), then there are no rays parallel to those of $\mathcal{P}$. The growth of rays is obstructed by those induced C_6 as illustrated in Figure 5.4. The connections between the first and last vertical rays in Figure 5.4 are not drawn to make the drawing simple, but this is meant to be a cylindrical wall (this is why the empty boxes at the first and last ladders do not conflict with k-edge-connectivity).

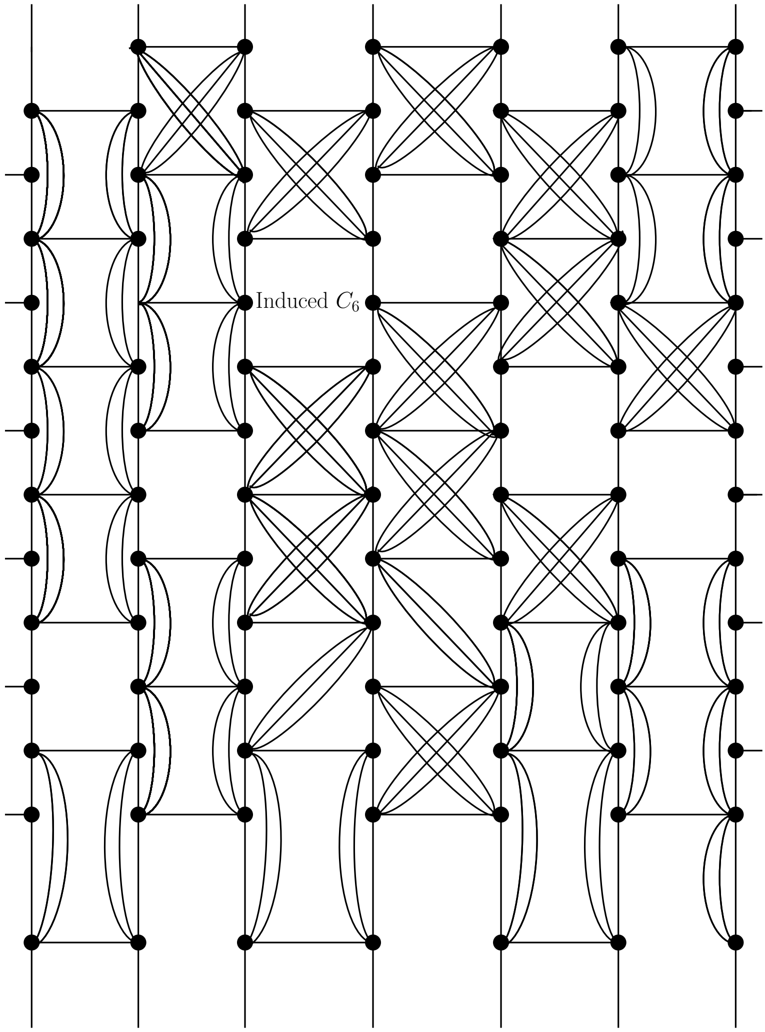


Figure 5.4: Induced C_6 subgraphs preventing the existence of an alternative collection of rays whose end graph is better connected.

5.2 One special vertex

As seen in many proofs in the thesis, in case $\deg(s)$ was odd, we stopped lifting at the smallest degree not smaller than the assumed connectivity k (or $k+1$). We did not continue lifting until we reached degree 3 or any other degree smaller than k (or $k+1$). The reason for this was to be able to use Huck's theorem, or the assumption of the truth of the conjecture for finite graphs, on the resulting finite graph. There is not a weak-linkage result on a finite graph with a vertex of small degree that is otherwise well connected. This motivates us to suggest the following conjecture.

Conjecture 5.2.1. *Let k be an odd integer and G a finite graph with a vertex s such that* $\deg(s) \leqslant k$. *If there are $k+1$ edge-disjoint paths between any two vertices different from s in G, then there is a weak k-linkage in G between any given k pairs of terminals in $G - s$.*

Note that this conjecture is harder than Huck's Theorem, and could be harder than the original conjecture. However, when $\deg(s) = 3$, it is easier than the original conjecture. At most one of the $k+1$ edge-disjoint paths between two vertices can go through s if $\deg(s) = 3$. This means that $G - s$ is k-edge-connected. Therefore, if the weak-linkage conjecture is true, then $G - s$ contains a weak k-linkage, and so does G.

Finding an example where the vertex s is necessary for the existence of a k-linkage, means finding a counterexample to the Weak Linkage Conjecture.

One can also consider the following even harder conjecture.

Conjecture 5.2.2. *Let k be an odd number and G a graph with a vertex s such that* $\deg(s) \leqslant k$. *If there are k edge-disjoint paths between any two vertices different from s in G, then there is a weak k-linkage in G between any given k pairs of terminals in $G - s$.*

Remark 5.2.3. In the finite problem, the existence of a vertex such as s is not guaranteed. It is possible that all the vertices of G are terminals.

Remark 5.2.4. Note that the weak-linkage conjecture for even numbers could be easier than the same conjecture for odd numbers. If one finds a counterexample for odd numbers, it is still possible that for even k every $(k + 1)$-edge-connected graph is weakly k-linked even if it is not $(k + 1)$-linked.

5.3 Edge-cut structure meets end structure

As seen in Chapter 2, the lifting graph $L(G, s, k)$, which is closely related to edge-connectivity, imposes a structure on the graph G. This structure is most clear when the complement of $L(G, s, k)$ is connected. By Theorem 2.3.1 and Lemma 2.2.4 has a cyclic structure around s. More precisely, the intersections and differences of the dangerous sets in G, we call them blobs, corresponding to the maximal independent sets either form a cycle with $(k-1)/2$ edges between any two consecutive blobs, Figure 2.11, or they form a path, Figure 2.12.

If the blobs form a path, it may be a path of three blobs with exactly one neighbour of s in the middle blob and $(\deg(s)-1)/2$ neighbours in each of the two other blobs, at least $(k-1)/2$ edges from the middle blob to each of the other two blobs, and at most $(k-\deg(s)+2)/2$ edges between the first and last blobs. It may also be a longer path, with exactly one neighbour of s in each of the interior blobs, and the same number l of neighbours of s in the first and last blobs, $(k-1)/2$ edges between any two consecutive blobs, except the first and the last blobs which have $\frac{(k+3)}{2} - l$ edges between them.

Note that one type of structure happens when the complement of $L(G, s, k)$ is a cycle, and the other happens when this complement is two cliques of the same size with a path between them (possibly a single-vertex path).

If these types of structures can recur as we go higher in the end, then we expect the end to either have a cylindrical grid-like structure on its rays or to have a separating grid-like structure in the middle. In both cases the end is not expected to have a set of rays as used in the proofs in this thesis for which the end graph is highly connected.

To make this more precise, in a 1-ended locally finite graph there are vertex sets $S_1 \subseteq S_2 \subseteq \cdots$ whose union is $V(G)$ such that, for each i, each edge of $\delta(S_i)$ is the first edge of a ray in an edge-disjoint collection of rays. If for infinitely many i the complement of the lifting graph of $G/(G - S_i)$ at the contraction vertex is either a cycle or two cliques with a path between them, then we expect the rays of G to make a cylindrical grid-like structure or to contain a separating grid-like structure in the middle, and is expected to happen if and only if there is not a subset of the rays of a certain big size that has a highly connected ray graph.

We think that studying the structure of the end in this way can be helpful for the conjecture in infinite graphs because when we are stuck in a situation where the end graph, for the rays we have chosen, is a subgraph of the connected complement of the lifting graph, it may be helpful to know that the end contains another collection of rays that gives a better connected end graph that has enough edges not to be a subgraph of the complement of the lifting graph.

There are also other variations of the edge-disjoint linkage question, as seen in the following sections.

5.4 Local connectivity

Let T be a set of pairs of vertices in a graph G. Then G *is* (T, k)*-connected* if, for each pair $\{s, t\}$ in T, there are k pairwise edge-disjoint st-paths in G. If T is the set of all pairs of vertices in G, then (T, k)-connectivity is the same as k-edge-connectivity. If G is a graph with a vertex s, and T is the set of all pairs of vertices in $G - s$, then G being (T, k)-connected is equivalent to that (G, s, k) is a connectivity triple.

Okamura considered the case where T is the set of k pairs of terminals in the weak linkage question. She proved in this case that if the total number of distinct terminals is at most 6, then (T, k)-connected implies the existence of a weak-linkage for T.

Note that we can have k pairs with $k > 6$ if the size of the set of terminals is 6 since the definition of weak-linkage does not require the vertices to be distinct.

The question of whether a weak k-linkage of T exists when G is (T, k)-connected is harder than Thomassen's weak linkage conjecture.

An interesting question to consider is whether Okamura's result holds also for infinite graphs.

5.5 Number of terminals

The number of terminals instead of the number of pairs was considered before by Paul Seymour. He proved that the weak-linkage conjecture is true when the number of terminals is at most 3 [22]. This was proved independently from Thomassen's paper where the conjecture was suggested [24].

Assuming that the number of terminals is bounded by something smaller than $2k$ might be an easier problem than the conjecture. One may consider the bound to be either a constant number or a fraction of k.

5.6 More on the lifting graph

We would like to know exactly which graphs can or cannot be lifting graphs. We showed in Chapter 2 that when the complement of the lifting graph is connected then it is either a Hamilton cycle or two cliques of the same size with a path between them. We also know that when the complement is disconnected one possibility for the lifting graph is that it is a complete multipartite graph. In this case if the degree of the vertex s at which the lifting takes place is odd, then the multipartite lifting graph is a complete bipartite graph with one side of size $(\deg(s)+1)/2$ and the other of size $(\deg(s)-1)/2$.

When the connectivity is even and $\deg(s)$ is even, we only know that the lifting graph is complete multipartite. In fact, every complete multipartite graph, in which the size of every cluster is at most the minimum of k and $\lceil \deg(s)/2 \rceil$, is the k-lifting graph of some graph. For example we will show how to construct a graph G such that $L(G,s,k) = K_{k_1,\cdots,k_m}$ if every k_i is at most $\lceil \deg(s)/2 \rceil$. We are going to describe a graph that is a generalized version of the graph in the left drawing of Figure 2.2. Let $A, A_1, \cdots, A_m$ be sets of size k (for $k \geqslant k_i$ for every i). Let s have k_i distinct neighbours in A_i, and let N_i denote the set of neighbours of s in A_i. Let the set A have $k+1-k_i$ edges to distinct vertices in $A_i \setminus N_i$.

Now add edges such that each A_i forms a clique of size k. The resulting graph G is such that (G, s, k) is a connectivity triple, and each A_i is a dangerous set ($|\delta(A_i)| = k + 1$). Thus for each i, $\delta(s : A_i)$ is a maximal independent set in $L(G, s, k)$ of size k_i.

We don't know much about the lifting graph when it is not complete multipartite but its complement is disconnected. Note that this case can happen only when the connectivity k is odd. This is a question to be considered.

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