Cuts in Optimization and Approximation: Generalized MIR Cutting Plane and Integrality Gap Bounds for the Planar Multicut Problem

by

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Author's Declaration

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of Contributions

The first chapter of this thesis is based on work carried out in collaboration with my advisor, Professor Ricardo Fukasawa. While I benefited from his guidance and feedback, the results and proofs presented are primarily my own contributions.

The second chapter of this thesis is based on joint work with Professor Joseph Cheriyan and Dr. Nikhil Kumar. This research has resulted in the paper [28], which appeared in the proceedings of the APPROX 2025 Conference (joint with Nikhil Kumar), as well as the paper "Improved Upper Bounds on Multiflow–Multicut Gaps in Cactus Graphs", which has been accpeted into the FSTTCS 2025 and will be appeared in the venue (joint with Joseph Cheriyan and Nikhil Kumar).

I am the primary contributor to all of the material presented in this thesis.

Abstract

This thesis has two main themes, both centered around the role of cuts in integer programming and approximation algorithms.

In the first part, we investigate the complexity of split closures in mixed-integer sets in $\mathbb{R}^3$ with two integral variables and one continuous variable. While it is known that in $\mathbb{R}^2$ the split closure admits a polynomial-size description, extending this result to the mixed-integer setting in $\mathbb{R}^3$ poses new challenges. For a rational polyhedron $P \subseteq \mathbb{R}^3$ and integer index set $I = \{1, 2\}$, we denote by $P_I := conv(P \cap (\mathbb{Z}^2 \times \mathbb{R}))$ the mixed-integer hull of P with respect to I, and by P_I^{split} the intersection of all split cuts with respect to I. We make progress toward understanding the structure of P_I^{split} by proving the existence of a polyhedron Q such that $P_I \subseteq Q \subseteq P_I^{split}$, where Q can be described by a polynomial number of inequalities relative to the input size of P. In addition, we introduce a generalized mixed-integer rounding (MIR) procedure that begins with systems of two equations, rather than a single equation as in the classical setting. This leads to a new family of valid inequalities for mixed-integer sets, whose derivation relies on structural results for split closures in $\mathbb{R}^3$.

The second part focuses on approximation algorithms for the minimum multicut problem in planar graphs and certain subclasses thereof. We analyze the integrality gap of the natural LP relaxation within the framework of small-diameter decompositions. By introducing novel tools, we improve the best known lower bound on this integrality gap from 2 to $\frac{20}{9}$ for the family of cactus graphs, which in turn yields an improved lower bound of the integrality gap for planar graphs. Complementing this, we establish an upper bound of 3.44 for cactus graphs, improving the previous bound of 4. We further obtain tight bounds in special subclasses: the integrality gap is exactly 2 for unicyclic graphs and for path cactus graphs. Extending the study to outerplanar graphs, we prove an upper bound of 8 on the integrality gap by designing a randomized rounding algorithm that transforms the optimal fractional solution of the LP relaxation into a feasible multicut whose expected cost is at most 8 times the cost of the optimal fractional solution. Together, these results sharpen our understanding of the minimum multicut problem and its LP relaxation in planar graphs.

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Dedication

To Mariia Sobchuk, the angel who once shined so brightly in my life. Her presence, support, and love helped me through some of the most formative moments of this journey. Her impact remains deeply etched in this work.

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Preface

This thesis consists of two main parts. The first part (Chapter 1) focuses on developments in integer programming. The second part (Chapter 2) turns to approximation algorithms for the minimum multicut problem. Each chapter is largely self-contained and can be read independently, though common themes and techniques connect the two.

Chapter 1

Mixed-Integer Sets with Two Integral and One Continuous Variables and Their Applications.

1.1 Introduction

Many combinatorial optimization problems can be modeled as the problem of optimizing a linear objective function over a set of feasible solutions defined by linear constraints with variables forced to be integer, known as integer programming (IP). In many cases, some variables are allowed to be continuous, which is called mixed-integer programming (MIP). In general, MIP seeks to maximize a linear objective function $c^T x$ over the set $S = \{x \in \mathbb{Z}^{|I|} \times \mathbb{R}^{|C|} \mid Ax \leq b\}$, where $I \subseteq [n]$ denotes the indices of the integral variables, and $C = [n] \setminus I$ denotes the indices of the continuous variables. Here, $[n] \coloneqq \{1, \ldots, n\}, c \in \mathbb{Z}^n$, $A \in \mathbb{Z}^{m \times n}$, and $b \in \mathbb{Z}^m$, where n is the number of variables, and m is the number of linear constraints. The special case where $I = [n]$ and $C = \varnothing$ corresponds to a pure integer program (IP).

Solving an IP (or MIP) is, in general, very hard; in fact, IP is known to be NP-Hard. However, many approaches have been developed to solve IP (MIP); we refer the interested reader to [8] for an overview. Among these, the cutting plane method has been proved to be one of the powerful ones, and this chapter of this thesis focuses on this method. The cutting plane method is based on relaxing the integrality constraints by replacing the feasible set S with the polyhedron $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$, which defines the linear

programming relaxation

$$\begin{aligned} \max\, & c^T x \\ & Ax \leq b. \end{aligned} \tag{LP}$$

While P is not necessarily pointed in general, we assume it is pointed for simplicity, as this guarantees the existence of an optimal basic solution. The method proceeds by finding the optimal solution x^* of (LP), which we assume is an extreme point of P. If $x^* \in \mathbb{Z}^{|I|} \times \mathbb{R}^{|C|}$, then it is already feasible for the original MIP, and the process terminates. Otherwise, one finds a cutting plane that separates x^* from S, and then adds it as a constraint to the system $Ax \leq b$, and repeats the process. Note that the existence of such a cutting plane is guaranteed when x^* is an extreme point. Numerous cutting plane methods have been developed over the years. We begin by reviewing some of the most important ones.

In the 1960s, Gomory [24] introduced the first cutting plane method to solve IPs, the resulting cuts are known as Gomory cuts. The method is based on solving the LP relaxation (LP) using the simplex method, and then deriving a cutting plane using the corresponding single simplex tableau row of a non-integral component of x^*. Gomory showed that this method eventually terminates. Later, he introduced the Gomory Mixed-Integer (GMI) cuts for MIPs [23]; however, the use of GMI cuts does not guarantee termination. In fact, finding a finite cutting plane method for MIPs had been open until very recently, the interested reader is referred to [27].

Around the same time, Chvátal [11] introduced another class of cutting planes for the pure integer case, now known as Chvátal-Gomory (CG) cuts. A CG cut is obtained by taking a valid inequality with integer coefficients for P and rounding down the right-hand side to the nearest integer. Formally, in the pure integer case, a CG cut takes the form:

$$\pi^T x \leq \left\lfloor \max_{x \in P} \pi^T x \right\rfloor,$$

where $\pi \in \mathbb{Z}^n$ is an arbitrary integer vector. The name Chvátal-Gomory arises because, when the non-negativity constrains are present, CG cuts coincide with Gomory cuts. For mixed-integer sets, an analogous definition of CG cuts exists. The algebraic form remains the same:

$$\pi^T x \leq \left\lfloor \max_{x \in P} \pi^T x \right\rfloor,$$

but here $\pi \in \mathbb{Z}^{|I|} \times \mathbf{0}^{|C|}$, meaning that the coefficients of continuous variables are zero. In both cases, the form of the inequality is identical; the distinction lies in the domain of valid multipliers π which must respect the integrality structure of the problem.

Later, Balas [4] introduced the concept of intersection cuts for mixed-integer sets. An intersection cut is obtained by selecting a closed convex set L whose interior contains no points of S, and deriving a valid inequality for the set $P \cap (\mathbb{R}^n \setminus L)$. It turns out that both GMI cuts and CG cuts can be viewed as a special case of this framework. In particular, if $L = \{x \in \mathbb{R}^n \mid \pi_0 \leq \pi^T x \leq \pi_0 + 1\}$ for some $\pi \in \mathbb{Z}^{|I|} \times \mathbf{0}^{|C|}, \pi_0 \in \mathbb{Z}$, then any valid inequality for $P \cap (\mathbb{R}^n \setminus L)$ is called a split cut. The class of split cuts strictly generalizes CG cuts, which correspond to the special case where $\pi_0 = \lfloor \max_{x \in P} \pi^T x \rfloor$.

In a subsequent work, Nemhauser and Wolsey [30] introduced mixed-integer rounding (MIR) cuts for the mixed-integer sets and showed that they are equivalent to both split cuts and GMI cuts, that is, all three families define the same set of inequalities. Informally, MIR cuts offer an algebraic approach to derive split cuts, in contrast with the geometric way of deriving split cuts via the intersection cut procedure.

One way to assess the quality of a cutting plane method is by how quickly a general polyhedron P converges to its integer hull $P_I = \mathit{conv}(S)$ through repeated application of the method, where $\mathit{conv}(S)$ denotes the convex hull of S. The Chvátal closure of a polyhedron P with respect to I is

$$P_I \subseteq P_I^{Ch} = \bigcap_{\pi \in \mathbb{Z}^n} (\pi^T x \leq \left\lfloor \max_{x \in P} \pi^T x \right\rfloor) \subseteq P.$$

Chvátal [11] introduced this set in the pure integer case and proved that P_I^{Ch} is indeed a polyhedron provided that P is rational. Let $P^{(1)}$ denote the Chvátal closure of P, and define $P^{(t)}$ recursively as the Chvátal closure of $P^{(t-1)}$ for $t \geq 2$. Chvátal [11] also proved that there exists $t \in \mathbb{N}$ such that $P^{(t)} = P_I$. This result shows that, in the pure integer case, applying CG cuts iteratively yields the integer hull in finitely many steps. In addition to convergence, one may ask how many inequalities are needed to describe the closure obtained after a single round of CG cuts. In a seminal work, Bockmayr and Eisenbrand [9] showed that in the pure integer case, P_I^{Ch} can be described using a polynomial number of inequalities with respect to the description-size of the system $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$, when the dimension is fixed.

Cook et al. [14] studied the following closure in the context of mixed-integer sets:

$$P_I \subseteq P_I^{split} = \bigcap_{\substack{\pi \in \mathbb{Z}^{|I|} \times \mathbf{0}^{|C|} \\ \pi_0 \in \mathbb{Z}}} conv(\{x \in P \mid \pi^T x \leq \pi_0\} \cup \{x \in P \mid \pi^T x \geq \pi_0 + 1\}) \subseteq P_I^{Ch} \subseteq P,$$

and showed that P_I^{split} is a polyhedron, referred as the split closure of P with respect to I, provided that P is rational. Let $P_{split}^{(1)}$ denote the split closure of P, and define $P_{split}^{(t)}$ recursively as the split closure of $P_{split}^{(t-1)}$ for $t \geq 2$. In contrast to the pure integer case with Chvátal closure, there does not necessarily exist $t \in \mathbb{N}$ such that $P_{split}^{(t)} = P_I$. In fact, Cook et al. [14] constructed an example of a rational polyhedron $P \subseteq \mathbb{R}^3$ with $I = \{1, 2\}$, such that P_I is always strictly contained in $P_{split}^{(t)}$ for all $t \in \mathbb{N}$. This shows that, in the mixed-integer case, iterated application of split cuts does not necessarily yield the mixed-integer hull. As in the case of Chvátal closure, one may also ask how many inequalities are needed to describe the split closure after one round. This has remained a long-standing open question. Recently, Basu et al. [6] resolved this in the pure integer case when $P \subseteq \mathbb{R}^2$, showing that P_I^{split} can be described using a polynomial number of inequalities in terms of the description-size of P.

It is noteworthy that integer programming in $\mathbb{R}^2$ has been shown to be tractable. Specifically, in the case of pure integers, Harvey [25] showed that P_I can be described using a polynomial number of inequalities with respect to the description-size of P, and that optimizing any linear function can be optimized in P_I in polynomial time.

1.1.1 Overview of Results

The failure of split cuts to obtain P_I in the aforementioned mixed-integer example of Cook et al. [14], where $P \subseteq \mathbb{R}^3$ and $I = \{1, 2\}$ suggests that mixed-integer sets with two integral variables and one continuous variable present the first challenge in extending the results from pure integer programming to mixed-integer programming. Having this point of view, one direction of this thesis is an attempt to extend the aforementioned result of Basu et al. [6] about the complexity of split closure of polyhedrons in $\mathbb{R}^2$ to the mixed-integer sets with two integral variables in $\mathbb{R}^3$. Given a rational polyhedron $P \subseteq \mathbb{R}^3$ and $I = \{1, 2\}$, the goal is to show that P_I^{split} can likewise be described by polynomial many inequalities in terms of the description-size of P. While this goal is not fully achieved, we obtain a noteworthy result in section 1.3. We are able to find a polyhedron Q such that $P_I \subseteq Q \subseteq P_I^{split}$, and that can be described with polynomial number of inequalities with respect to the description-size of P.

The other main direction of this thesis concerns the derivation of new cutting planes. In section 1.4, we study mixed-integer systems defined by equality constraints together with non-negativity constraints. The material in this section begins with a review of classical results that are well established in the literature and are not part of our contribution. Specifically, the derivation of standard cuts such as Gomory Mixed-Integer (GMI) cuts, Mixed-Integer Rounding (MIR) cuts, and split cuts typically proceeds through linear combinations of the constraints into a single equality, after which a standard cutting-plane procedure is applied (see Sections 5.1.4 and 5.1.5 in [12]). For example, GMI cuts are obtained from a single row of the simplex tableau corresponding to a basic variable with a fractional value. Once the system is reduced to a single equality with non-negativity constraints, it is a classical result of Nemhauser and Wolsey [30] that the GMI cut, the split cut, and the MIR cut all coincide. This background will be reviewed in section 1.2 before we move on to our new contributions.

We generalize this MIR procedure by reducing the system to two equalities, either through aggregation or by directly selecting two rows from the simplex tableau. This leads to a system of two equalities along with non-negativity constraints and integrality conditions on some variables, which will be detailed in section 1.4. Specifically, we model the system as a mixed-integer set in $\mathbb{R}^3$, with two integer variables and one continuous variable. We then construct a valid inequality for this set and map it back to the original formulation. Interestingly, our main result in section 1.3, namely theorem 1.3.2, plays a crucial role in the derivation of this generalized cutting plane.

1.1.2 Notation

For any point $x \in \mathbb{R}^n$ and index $i \in [n]$, we denote $x^{(i)}$ as the i-th component of x. For a set $S \subseteq \mathbb{R}^n$, let

$$conv(S) = \left\{ \sum_{i=1}^{k} \lambda_i x_i \;\middle|\; x_i \in S,\ \lambda_i \geq 0\ \forall i,\ \sum_{i=1}^{k} \lambda_i = 1,\ k \in \mathbb{N} \right\}$$

denote the convex hull of a set $S \subseteq \mathbb{R}^n$. For points $x_1, \ldots, x_k \in \mathbb{R}^n$, we simply use $conv(x_1, \ldots, x_k)$ instead of $conv(\{x_1, \ldots, x_k\})$. For any subset of indices $I = \{i_1, \ldots, i_k\} \subseteq [n]$, let

$$S_I = conv\left(S \cap \left(\mathbb{Z}^{|I|} \times \mathbb{R}^{|[n]\setminus I|}\right)\right)$$

denote the mixed-integer hull of S with respect to I. For a point $x \in \mathbb{R}^n$, let

$$proj_I(x) = (x^{(i_1)}, \ldots, x^{(i_k)}) \in \mathbb{R}^{|I|},$$

and, let

$$proj_I(S) = \{proj_I(x) \mid x \in S\} \subseteq \mathbb{R}^{|I|}.$$

For rays $r_1, \ldots, r_k \in \mathbb{R}^n$, define

$$cone(r_1, \ldots, r_k) = \left\{ \sum_{i=1}^{k} \lambda_i r_i \mid \lambda_i \geq 0 \right\},$$

that is, the set of all conic combinations of the rays $r_1, \ldots, r_k$. A vector $r \in \mathbb{R}^n$ is called a conic combination of $r_1, \ldots, r_k$ if $r = \sum_{i=1}^{k} \lambda_i r_i$ for some scalars $\lambda_i \geq 0$. All polyhedrons that we consider are assumed to be pointed. For any polyhedron $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$, we define its recession cone as

$$rec(P) = \{r \in \mathbb{R}^n \mid Ar \leq 0\}.$$

A ray $r \in rec(P)$ is called an extreme ray if it cannot be written as a nontrivial conic combination of other rays in $rec(P)$, that is, if $r = \sum_{i=1}^{k} \lambda_i r_i$ with $\lambda_i > 0$, $r_i \in rec(P)$, then all $r_i \in cone(r)$. For any real number $a \in \mathbb{R}$, define the fractional part of a as

$$f(a) = a - \lfloor a \rfloor.$$

Whenever we refer to a polynomial in the description-size of a polyhedron

$$P = \{x \in \mathbb{R}^n \mid Ax \leq b\},$$

we mean a polynomial in the parameters n, m, $\log \|A\|_\infty$, and $\log \|b\|_\infty$, where m is the number of inequalities and n is the number of variables. Here, $\|A\|_\infty$ denotes the maximum absolute value of the entries of A, and $\|b\|_\infty$ denotes the maximum absolute value of the entries of b. Thus, $\log \|A\|_\infty$ and $\log \|b\|_\infty$ measure the binary encoding length of the largest coefficients in A and b, respectively. In the fixed-dimension setting, where n is constant, a polynomial in the description-size refers to a polynomial in m, $\log \|A\|_\infty$, and $\log \|b\|_\infty$.

1.2 GMI Cut, Split Cut, MIR Cut for a Single Equality System

In this section, we review the GMI cut applied to a system consisting of a single equality together with non-negativity constraints. Reducing a system of equalities to a single equality via linear combinations is a standard approach that has long been established in the

literature. We then review the split cut and MIR procedures, and recall the classical result of Nemhauser and Wolsey [30] showing that, in this setting, all three yield the same cutting plane. This material is purely background and serves as preparation for the generalization of the MIR procedure presented in section 1.4.

Let S_1 be the following system:

$$\sum_{i=1}^{n} a^{(i)} x^{(i)} = b, \tag{1.1}$$

$$x^{(i)} \geq 0 \quad \forall i \in [n],$$

$$x^{(i)} \in \mathbb{Z} \quad \forall i \in I, \tag{1.2}$$

where $I \subseteq [n]$ denotes the set of indices corresponding to integer variables. Also, let $C = [n] \setminus I$ denote the set of indices corresponding to continuous variables. We assume that the fractional part $f(b) > 0$. We are going to derive a valid inequality for S_1. We rewrite the equality constraint of S_1 by separating integer and fractional components:

$$\sum_{\substack{f(a^{(i)}) \leq f(b) \\ i \in I}} f(a^{(i)})\, x^{(i)} + \sum_{\substack{f(a^{(i)}) > f(b) \\ i \in I}} (f(a^{(i)}) - 1)\, x^{(i)} + \sum_{i \in C} a^{(i)} x^{(i)} =$$

$$\left(\lfloor b \rfloor - \sum_{i \in I} \lfloor a^{(i)} \rfloor\, x^{(i)} - \sum_{\substack{f(a^{(i)}) > f(b) \\ i \in I}} x^{(i)} \right) + f(b),$$

and let

$$k = \lfloor b \rfloor - \sum_{i \in I} \lfloor a^{(i)} \rfloor\, x^{(i)} - \sum_{\substack{f(a^{(i)}) > f(b) \\ i \in I}} x^{(i)}.$$

k is always an integer, we now distinguish between two cases:

1. $k \leq -1$; in this case, the left-hand side of the aggregated constraint satisfies

$$\sum_{\substack{f(a^{(i)}) \leq f(b) \\ i \in I}} f(a^{(i)})\, x^{(i)} + \sum_{\substack{f(a^{(i)}) > f(b) \\ i \in I}} (f(a^{(i)}) - 1)\, x^{(i)} + \sum_{i \in C} a^{(i)} x^{(i)} \leq f(b) - 1.$$

   Dividing both sides by $f(b) - 1$ yields the inequality:

$$-\sum_{\substack{f(a^{(i)}) \leq f(b) \\ i \in I}} \frac{f(a^{(i)})}{1 - f(b)}\, x^{(i)} + \sum_{\substack{f(a^{(i)}) > f(b) \\ i \in I}} \frac{1 - f(a^{(i)})}{1 - f(b)}\, x^{(i)} - \sum_{i \in C} \frac{a^{(i)}}{1 - f(b)}\, x^{(i)} \geq 1. \quad (1.3)$$

2. $k \geq 0$; in this case, the left-hand side of the aggregated constraint satisfies:

$$\sum_{\substack{f(a^{(i)}) \leq f(b) \\ i \in I}} f(a^{(i)})\, x^{(i)} + \sum_{\substack{f(a^{(i)}) > f(b) \\ i \in I}} (f(a^{(i)}) - 1)\, x^{(i)} + \sum_{i \in C} a^{(i)} x^{(i)} \geq f(b).$$

   Dividing both sides by $f(b)$ gives:

$$\sum_{\substack{f(a^{(i)}) \leq f(b) \\ i \in I}} \frac{f(a^{(i)})}{f(b)}\, x^{(i)} + \sum_{\substack{f(a^{(i)}) > f(b) \\ i \in I}} \frac{f(a^{(i)}) - 1}{f(b)}\, x^{(i)} + \sum_{i \in C} \frac{a^{(i)}}{f(b)}\, x^{(i)} \geq 1. \quad (1.4)$$

Since both inequalities (1.3) and (1.4) have right-hand side equal to 1, and all variables are constrained to be nonnegative, we can construct a valid cut by taking the maximum coefficient for each variable across both. This yields the Gomory Mixed-Integer (GMI) cut associated with the 1-equality system S_1:

$$\sum_{\substack{f(a^{(i)}) \leq f(b) \\ i \in I}} \frac{f(a^{(i)})}{f(b)}\, x^{(i)} + \sum_{\substack{f(a^{(i)}) > f(b) \\ i \in I}} \frac{1 - f(a^{(i)})}{1 - f(b)}\, x^{(i)} + \sum_{\substack{a^{(i)} \geq 0 \\ i \in C}} \frac{a^{(i)}}{f(b)}\, x^{(i)} + \sum_{\substack{a^{(i)} \leq 0 \\ i \in C}} \frac{-a^{(i)}}{1 - f(b)}\, x^{(i)} \geq 1. \quad (1.5)$$

In the special case of pure integer programs (i.e., when $I = [n]$ and $C = \varnothing$), the GMI cut simplifies to:

$$\sum_{f(a^{(i)}) \leq f(b)} \frac{f(a^{(i)})}{f(b)}\, x^{(i)} + \sum_{f(a^{(i)}) > f(b)} \frac{1 - f(a^{(i)})}{1 - f(b)}\, x^{(i)} \geq 1. \quad (1.6)$$

In the following, we present two alternative procedures for deriving a cutting plane from the equality constraint of S_1, both of which yield the same GMI cut as in (1.5). The first is the intersection cut procedure, applied after lifting the integer variables; and the second is the MIR procedure. We begin with the first approach, which is formally outlined below.

Let $t_i \in \mathbb{Z}$ be an arbitrary integer for each $i \in N$. Starting from the equality constraint (1.1), we can rewrite it as:

$$\sum_{i \in I} (-t_i)\, x^{(i)} + \sum_{i \in I} (a^{(i)} + t_i)\, x^{(i)} + \sum_{i \in C} a^{(i)}\, x^{(i)} = b.$$

Since both $x^{(i)}$ and t_i are integers for $i \in I$, the first summation on the left-hand side is an integer. We introduce a new integer variable $x^{(n+1)}$ to represent this quantity, and relax the integrality requirements of the original variables $x^{(i)}$ for $i \in I$. This yields the new system $S_1^{'}$:

$$\begin{aligned} x^{(n+1)} + \sum_{i \in I} (a^{(i)} + t_i)\, x^{(i)} + \sum_{i \in C} a^{(i)}\, x^{(i)} &= b, \\ x^{(i)} &\geq 0 \quad \forall i \in [n], \\ x^{(n+1)} &\in \mathbb{Z}. \end{aligned}$$

Next, consider the polyhedron $S_1^{''}$ obtained by relaxing the integrality of $x^{(n+1)}$:

$$\begin{aligned} x^{(n+1)} + \sum_{i \in I} (a^{(i)} + t_i)\, x^{(i)} + \sum_{i \in C} a^{(i)}\, x^{(i)} &= b, \\ x^{(i)} &\geq 0 \quad \forall i \in [n]. \end{aligned}$$

This defines a pointed polyhedron in $\mathbb{R}^{n+1}$. One can verify that

$$\bar{x} = (0, \ldots, 0, b)$$

is the unique vertex of $S_1^{''}$, where the last coordinate corresponds to $x^{(n+1)}$. Also, it can be verified that the extreme rays of $S_1^{''}$ are given by:

$$r_i := \begin{cases} (0, \ldots, 0,\ 1,\ 0, \ldots, 0,\ -a^{(i)} - t_i) & \text{for } i \in I, \\ (0, \ldots, 0,\ 1,\ 0, \ldots, 0,\ -a^{(i)}) & \text{for } i \in C, \end{cases}$$

where the 1 appears in the i-th position (corresponding to $x^{(i)}$), and all other components are zero. We define the disjunction set:

$$L := \left\{ x \in \mathbb{R}^{n+1} \;\middle|\; \lfloor b \rfloor \leq \pi x \leq \lfloor b \rfloor + 1 \right\},$$

where $\pi := (0, \ldots, 0, 1) \in \mathbb{R}^{n+1}$. The set L contains $\bar{x}$, but no mixed-integer point of $\mathbb{R}^n \times \mathbb{Z}$ in its interior. From Theorem 6.5 of [12], the intersection cut associated with L is valid for S_1, and takes the form:

$$\sum_{i=1}^{n} \frac{x^{(i)}}{\alpha_i} \geq 1,$$

where

$$\alpha_i := \max \left\{ \alpha \geq 0 \mid \bar{x} + \alpha r_i \in S \right\}.$$

For this specific construction, the intersection cut simplifies to:

$$\sum_{\substack{a^{(i)}+t_i \geq 0 \\ i \in I}} \frac{a^{(i)} + t_i}{f(b)} x^{(i)} + \sum_{\substack{a^{(i)}+t_i < 0 \\ i \in I}} \frac{-(a^{(i)} + t_i)}{1 - f(b)} x^{(i)} + \sum_{\substack{a^{(i)} \geq 0 \\ i \in C}} \frac{a^{(i)}}{f(b)} x^{(i)} + \sum_{\substack{a^{(i)} < 0 \\ i \in C}} \frac{-a^{(i)}}{1 - f(b)} x^{(i)} \geq 1. \tag{1.7}$$

Since t_i's are arbitrary integers for $i \in I$, we can minimize each coefficient in (1.7) independently. For each $i \in I$, the minimal possible coefficient of $x^{(i)}$ in (1.7) is:

$$\min \left\{ \frac{f(a^{(i)})}{f(b)}, \frac{1 - f(a^{(i)})}{1 - f(b)} \right\},$$

which ultimately yields the same GMI cut as derived in (1.5). Now, we describe the MIR procedure. We begin by rewriting the equation (1.1):

$$\begin{aligned} &\sum_{\substack{f(a^{(i)}) \leq f(b) \\ i \in I}} \lfloor a^{(i)} \rfloor \, x^{(i)} + \sum_{\substack{f(a^{(i)}) > f(b) \\ i \in I}} \lceil a^{(i)} \rceil \, x^{(i)} \quad + \sum_{\substack{f(a^{(i)}) \leq f(b) \\ i \in I}} f(a^{(i)}) \, x^{(i)} \\ &+ \sum_{\substack{f(a^{(i)}) > f(b) \\ i \in I}} (f(a^{(i)}) - 1) \, x^{(i)} \quad + \sum_{\substack{a^{(i)} \geq 0 \\ i \in C}} a^{(i)} \, x^{(i)} + \sum_{\substack{a^{(i)} < 0 \\ i \in C}} a^{(i)} \, x^{(i)} = b, \end{aligned}$$

and let

$$w = \sum_{\substack{f(a^{(i)}) \leq f(b) \\ i \in I}} \lfloor a^{(i)} \rfloor \, x^{(i)} + \sum_{\substack{f(a^{(i)}) > f(b) \\ i \in I}} \lceil a^{(i)} \rceil \, x^{(i)},$$
$$s = \sum_{\substack{f(a^{(i)}) \leq f(b) \\ i \in I}} f(a^{(i)}) \, x^{(i)} + \sum_{\substack{a^{(i)} \geq 0 \\ i \in C}} a^{(i)} \, x^{(i)}.$$

Since

$$\sum_{\substack{f(a^{(i)}) > f(b) \\ i \in I}} \left(f(a^{(i)}) - 1 \right) \, x^{(i)} + \sum_{\substack{a^{(i)} < 0 \\ i \in C}} a^{(i)} \, x^{(i)} \leq 0,$$

then the pair (w, s) satisfies the following system:

$$\begin{aligned} w + s &\geq b \\ s &\geq 0 \\ w &\in \mathbb{Z}. \end{aligned} \tag{1.8}$$

We have the following Remark 1.2.1:

Remark 1.2.1 ([12])**.** *The inequality*

$$s \geq f(b)(\lceil b \rceil - w) \tag{1.9}$$

is valid for the system (1.8)*. (See Chapter 1.4.1 of [12]).*

Substituting back the expressions for w and s into (1.9) yields the same GMI cut as stated in (1.5).

1.3 Split Closure of Mixed-Integer Sets in $\mathbb{R}^3$ with Two Integral and One Continuous Variables

Basu et al. [6] proved the following theorem (see Theorem 20 in [6]):

Theorem 1.3.1 ([6]). *Let $Ax \leq b$ be a description of a polyhedron $P \subseteq \mathbb{R}^2$ consisting of m inequalities. Then P_I^{split}, where $I = \{1, 2\}$, admits a description whose size is polynomial in $m, \log \|A\|_\infty, \log \|b\|_\infty$.*

In this chapter, we attempt to extend this result to the case where $P \subseteq \mathbb{R}^3$ and $I = \{1, 2\}$. Although we do not succeed in proving that P_I^{split} admits a polynomial-size description, we construct a polyhedron Q such that

$$P_I \subseteq Q \subseteq P_I^{\text{split}},$$

and Q can be described using a polynomial number of inequalities in terms of the description-size of P.

We first reduce the problem to the case of translated cones in $\mathbb{R}^3$, namely polyhedron described by three independent facet-defining inequalities. This reduction is not new; it follows directly from Corollary 5.7 in [12]. In particular, by this corollary it suffices to show that the split closure of a translated cone can be described with a polynomial number of inequalities. Given such a translated cone, instead of analyzing its split closure directly, we study its mixed-integer hull. Rather than characterizing this hull by its facet-defining inequalities, we use its Minkowski-sum representation, namely, as the convex hull of its extreme points plus its recession cone. By Lemma 6.8 in [25], the set of extreme points of this mixed-integer hull is bounded polynomially (in fact, linearly) in the description-size of the translated cone. This implies that the mixed-integer hull of a translated cone has a polynomial-size description in terms of the description-size of the cone. Finally, by intersecting the mixed-integer hulls of all the translated cones derived from P, we obtain the polyhedron Q, and we will show that Q has a polynomial-size description.

We now formally describe our contribution. Let $P = \{x \in \mathbb{R}^3 \mid Ax \leq b\}$ be a non-empty pointed polyhedron such that $A \in \mathbb{Z}^{m\times 3}, b \in \mathbb{Z}^m$, and let $I = \{1, 2\}$. We have $P_I = conv(P \cap (\mathbb{Z}^2 \times \mathbb{R}))$, and since P is pointed, then $rank(A) = 3$. Let $\mathcal{B} \neq \varnothing$ denote the family of all bases of the system $Ax \leq b$, that is, all subsets $B \subseteq [m]$ such that $|B| = 3$ and the corresponding rows of A with respect to B are linearly independent. For each $B \in \mathcal{B}$, let $P(B) = \{x \in \mathbb{R}^3 \mid a_i^T x \leq b^{(i)},\ \forall i \in B\}$, in which $a_i^T x \leq b^{(i)}$ is the i-th row of $Ax \leq b$. We have the following Lemma 1.3.1, which relates the split closures of the polyhedrons $P(B)$'s to P_I^{split}:

Lemma 1.3.1 (Corollary 5.7 in [12]).

$$P_I^{split} = \bigcap_{B\in\mathcal{B}} P(B)_I^{split}$$

Since $|\mathcal{B}| \leq \binom{m}{3}$, to prove that P_I^{split} admits a polynomial description-size, it suffices to show that $P(B)_I^{split}$ admits a polynomial description-size for an arbitrary basis B, by Lemma 1.3.1. Let $B \in \mathcal{B}$ be an arbitrary basis, the polyhedron $P(B)$ is full-dimensional, and described as:

$$\begin{aligned} a_1^T x &\leq b^{(1)} \\ a_2^T x &\leq b^{(2)} \\ a_3^T x &\leq b^{(3)}, \end{aligned} \tag{1.10}$$

where $rank(a_1, a_2, a_3) = 3$. For ease of terminology, we refer to $P(B)$ as C, and call it the translated cone.

Definition 1.3.1. *Let $C = \{x \in \mathbb{R}^3 \mid a_i^T x \leq b^{(i)},\ i = 1, 2, 3\}$ be a translated cone. It can be derived that C has a unique vertex*

$$v = (v^{(1)}, v^{(2)}, v^{(3)}) \in \mathbb{R}^3,$$

where we denote it as the apex of C, and it has 3 extreme rays

$$\begin{aligned} r_1 &= (r_1^{(1)}, r_1^{(2)}, r_1^{(3)}), \\ r_2 &= (r_2^{(1)}, r_2^{(2)}, r_2^{(3)}), \\ r_3 &= (r_3^{(1)}, r_3^{(2)}, r_3^{(3)}) \in \mathbb{R}^3, \end{aligned}$$

such that

$$C = v + cone(r_1, r_2, r_3).$$

We can assume that these rays satisfy the orthogonality and normalization conditions:

$$a_i^T r_j = \begin{cases} 0 & \text{if } i \neq j, \\ -1 & \text{if } i = j, \end{cases} \quad \text{for all } i, j \in \{1, 2, 3\}.$$

For each $i \in \{1, 2, 3\}$, let

$$C_i := \{x \in C \mid a_i^T x = b^{(i)}\}.$$

be the facet of the cone C (see Figure 1.1).

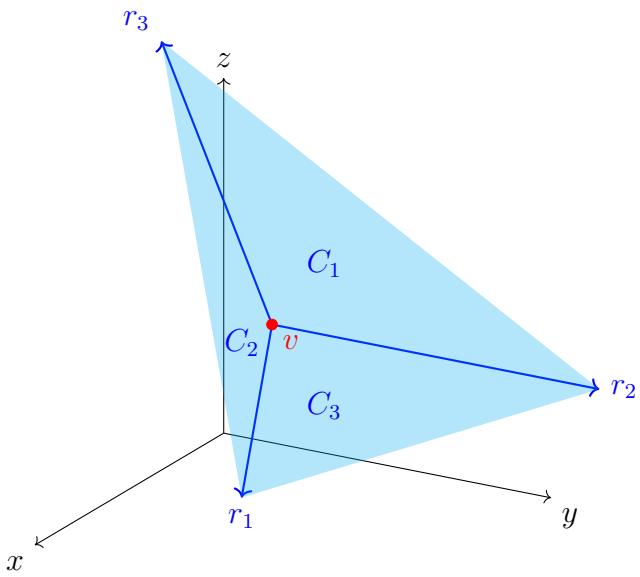


Figure 1.1: $C = v + cone(r_1, r_2, r_3)$

Note that r_1, r_2, r_3 are linearly independent; otherwise, C is not full-dimensional. We also have the following Lemma 1.3.2:

Lemma 1.3.2. *Let*

$$C = \{x \in \mathbb{R}^3 \mid a_i^T x \leq b^{(i)},\ i = 1, 2, 3\}$$

be a translated cone with vertex v. Denote its facets by C_1, C_2, C_3 and its extreme rays by r_1, r_2, r_3, as defined in Definition 1.3.1. Then

$$\begin{aligned} C_1 &= v + cone(r_2, r_3), \\ C_2 &= v + cone(r_1, r_3), \\ C_3 &= v + cone(r_1, r_2). \end{aligned}$$

Proof. We only show $C_1 = v + cone(r_2, r_3)$. Let $x = v + \mu_2 r_2 + \mu_3 r_3$, where $\mu_2, \mu_3 \geq 0$. v is the unique vertex of C, which means that $a_i^T v = b^{(i)}$ for $i = 1, 2, 3$. Since $a_1^T r_2 = a_1^T r_3 = 0$, then one can derive that $a_1^T x = b^{(1)}$. Since $a_i^T r_2, a_i^T r_3 \leq 0$, then $x \in C$. Thus, $x \in C_1$. Now, let $x' \in C_1$. Since $x' \in C$, then there exists $\mu_1', \mu_2', \mu_3' \geq 0$ such that $x' = v + \mu_1' r_1 + \mu_2' r_2 + \mu_3' r_3$. Since $a_1^T r_2 = a_1^T r_3 = 0$ and $a_1^T r_1 < 0$, then one can derive that $a_1^T x < b^{(1)}$ if $\mu_1' > 0$. Thus, $x' = v + \mu_2' r_2 + \mu_3' r_3$, which completes the proof. □

Instead of directly describing the split closure C_I^{split}, we aim to describe the set C_I by leveraging the structure of the integer hulls of its facets. $(C_i)_I$ is a polyhedron for $i = 1, 2, 3$, see Theorem 4.30 in [12].

Definition 1.3.2. *Let C be a translated cone with facets by C_1, C_2, C_3 as defined in Definition 1.3.1. Let V_i denote the set of extreme points of the integer hull $(C_i)_I$, for $i = 1, 2, 3$.*

Note that each of V_1, V_2, V_3 could be an empty set; however, we later establish in Lemma 1.3.5 that the union $V_1 \cup V_2 \cup V_3$ is non-empty.

Remark 1.3.1 (Theorem 4.30 in [12])**.** *$rec((C_i)_I) = rec(C_i)$ for $i = 1, 2, 3$. So,*

$$\begin{aligned}(C_1)_I &= conv(V_1) + cone(r_2, r_3)\\ (C_2)_I &= conv(V_2) + cone(r_1, r_3)\\ (C_3)_I &= conv(V_3) + cone(r_1, r_2)\end{aligned}$$

The following lemma will be useful in our subsequent arguments; see Lemma 6.8 in [25].

Lemma 1.3.3 ([25])**.** *Let $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ be a system consisting of two inequalities. The P_I can be described using at most*

$$O\left(\max\left\{\log \|A\|_\infty,\ \log \|b\|_\infty\right\}\right)$$

inequalities.

We apply Lemma 1.3.3 to show that $|V_i|$ is bounded linearly in the bit-length of the input defining the cone C, for each $i = 1, 2, 3$.

Lemma 1.3.4. *Let*

$$C = \{x \in \mathbb{R}^3 \mid a_i^T x \leq b^{(i)},\ i = 1, 2, 3\}$$

be a translated cone with facets C_1, C_2, C_3, and fix $I = \{1, 2\}$. For each $i \in \{1, 2, 3\}$, let V_i denote the set of extreme points of $(C_i)_I$. Then the cardinality of V_i is bounded linearly in the bit-length of the input defining the cone C.

Proof. We prove the Lemma for $i = 3$; the cases $i = 1$ and $i = 2$ follow analogously. The facet C_3 is defined by the following system:

$$\begin{aligned}a_1^T x &\leq b^{(1)},\\ a_2^T x &\leq b^{(2)},\\ a_3^T x &= b^{(3)}.\end{aligned}$$

If $V_3 = \varnothing$, the result is trivial. Otherwise, we proceed by case analysis on the value of $a_3^{(3)}$, the third component of a_3:

1. $a_3^{(3)} = 0$; let $x' = (p, q, \ell) \in V_3$, with $p, q \in \mathbb{Z}$. We claim the following:

   Claim 1.3.1. *At least one of the inequalities $a_1^T x \leq b^{(1)}$ or $a_2^T x \leq b^{(2)}$ must be tight at x'.*

   Proof of Claim. Suppose not. Then both inequalities are strict at x', i.e., $a_1^T x' < b^{(1)}$ and $a_2^T x' < b^{(2)}$. Since $a_3^{(3)} = 0$, the equation $a_3^T x = b^{(3)}$ is independent of the third variable. Thus, there exists $\epsilon > 0$ such that

   $$x_1 = (p, q, \ell + \epsilon), \quad x_2 = (p, q, \ell - \epsilon)$$

   both satisfy the constraints of C_3, and since $p, q \in \mathbb{Z}$, we have $x_1, x_2 \in (C_3)_I$. But then $x' = \frac{1}{2}(x_1 + x_2)$, contradicting extremality. ▲

   Now, we show $|V_3| \leq 4$. Suppose for contradiction that $|V_3| \geq 5$. Since each point in V_3 must satisfy one of the two inequalities, $a_1^T x \leq b^{(1)}, a_1^T x \leq b^{(1)}$, with equality, then by the Pigeonhole Principle, there exist 3 points x_1, x_2, x_3 on the same hyperplane. WLOG, assume

   $$a_1^T x_j = b^{(1)} \quad \text{for } j = 1, 2, 3.$$

   As all three points also satisfy $a_3^T x = b^{(3)}$, they lie on the affine line

   $$\{x \in \mathbb{R}^3 \mid a_1^T x = b^{(1)},\ a_3^T x = b^{(3)}\}.$$

   Hence, one of x_1, x_2, x_3 lies in the convex hull of the others, violating extremality. Thus, $|V_3| \leq 4$ in this case.

2. Case $a_3^{(3)} \neq 0$; for any $x \in C_3$, the third component is uniquely determined by the first two via:

   $$x^{(3)} = \frac{b^{(3)} - a_3^{(1)} x^{(1)} - a_3^{(2)} x^{(2)}}{a_3^{(3)}}.$$

   Substituting this expression into the first two inequalities yields a projected description of $\widetilde{C} = proj_I(C_3) \subseteq \mathbb{R}^2$ as follows:

   $$\begin{aligned} (a_3^{(3)} a_1^{(1)} - a_3^{(1)}) x^{(1)} + (a_3^{(3)} a_1^{(2)} - a_3^{(2)}) x^{(2)} &\leq a_3^{(3)} b^{(1)} - b^{(3)}, \\ (a_3^{(3)} a_2^{(1)} - a_3^{(1)}) x^{(1)} + (a_3^{(3)} a_2^{(2)} - a_3^{(2)}) x^{(2)} &\leq a_3^{(3)} b^{(2)} - b^{(3)}. \end{aligned} \tag{1.11}$$

   Claim 1.3.2. *For any point $x' \in \widetilde{C}$, there exists a unique point $x \in C_3$ such that $proj_I(x) = x'$.*

Proof of Claim. Suppose for contradiction that two points $x_1 = (p, q, \ell_1), x_2 = (p, q, \ell_2) \in C_3$ project to the same $(p, q) \in \widetilde{C}$. Then $a_3^T x_1 = a_3^T x_2 = b^{(3)}$. Since $a_3^{(3)} \neq 0$, it follows that $\ell_1 = \ell_2$, contradicting the assumption that $x_1 \neq x_2$. ▲

This shows that there is a one-to-one correspondence between points in C_3 and their projections in $\widetilde{C}$. We now relate extreme points.

Claim 1.3.3. *Let $x \in C_3$, and let $x' = proj_I(x)$. Then $x \in V_3$ if and only if x' is an extreme point of $(\widetilde{C})_I$.*

Proof of Claim. Suppose $x = (p, q, \ell) \in V_3$ with $p, q \in \mathbb{Z}$, and assume for contradiction that $x' = (p, q)$ is not an extreme point of $(\widetilde{C})_I$. Then there exist integer points $x'_1, \ldots, x'_t \in \widetilde{C}$ such that

$$x' \in conv(x'_1, \ldots, x'_t).$$

For each $x'_i = (p_i, q_i)$, define the lifted point

$$x_i = \left(p_i, q_i, \frac{b^{(3)} - a_3^{(1)} p_i - a_3^{(2)} q_i}{a_3^{(3)}} \right) \in C_3.$$

Then $x_i \in (C_3)_I$, and x lies in the convex hull of these x_i, contradicting the extremality of x. The reverse direction follows similarly. ▲

Thus, the cardinality of V_3 is equal to the number of extreme points of $(\widetilde{C})_I$, where $\widetilde{C} \subseteq \mathbb{R}^2$ is defined by two linear inequalities. By Lemma 1.3.3, the number of inequalities required to describe $(\widetilde{C})_I$, and hence the number of its extreme points of $(\widetilde{C})_I$), is linearly bounded in the bit-length of the description of $\widetilde{C}$. Moreover, since $\widetilde{C}$ is obtained from C_3 via rational projection operations, its bit-length is at most a constant factor (e.g., at most 16 times) the bit-length of the description of C_3. Therefore, the number of extreme points of $(\widetilde{C})_I$, and thus $|V_3|$, is linearly bounded in the description-size of C.

□

We now show that $V_1 \cup V_2 \cup V_3 \neq \varnothing$, a result essential for our subsequent analysis.

Lemma 1.3.5. *Let*

$$C = \{x \in \mathbb{R}^3 \mid a_i^T x \leq b^{(i)},\ i = 1, 2, 3\}$$

be a translated cone with facets C_1, C_2, C_3, and fix $I = \{1, 2\}$. For each $i \in \{1, 2, 3\}$, let V_i denote the set of extreme points of $(C_i)_I$. Then, $V_1 \cup V_2 \cup V_3 \neq \varnothing$.

Proof. Since the vectors a_1, a_2, a_3 are linearly independent, at least one of the entries $a_1^{(3)}, a_2^{(3)}, a_3^{(3)}$ must be non-zero. Without loss of generality, assume $a_3^{(3)} \neq 0$. From the proof of Lemma 1.3.4, we know that there exists a one-to-one correspondence between the set V_3 and the set of extreme points of the integer hull of the cone $\widetilde{C} = proj_I(C_3) \subseteq \mathbb{R}^2$. The cone $\widetilde{C}$ is rational, full-dimensional (as the system describing it—see 1.11—has rank two), and pointed, and hence it contains integer points. Therefore, the integer hull $(\widetilde{C})_I$ has at least one extreme point, implying that $V_3 \neq \varnothing$. Thus, $V_1 \cup V_2 \cup V_3 \neq \varnothing$. □

We now show how C_I can be reconstructed from the sets V_1, V_2, V_3 of and the recession directions of the original cone.

Theorem 1.3.2. *Let*

$$C = \{x \in \mathbb{R}^3 \mid a_i^T x \leq b^{(i)},\ i = 1, 2, 3\}$$

be a translated cone with apex v and extreme rays r_1, r_2, r_3, so that

$$C = v + cone(r_1, r_2, r_3).$$

Let C_1, C_2, C_3 denote the three facets of C, and let $I = \{1, 2\}$ be the index set of integer-constrained variables. For each $i \in \{1, 2, 3\}$, let V_i be the set of extreme points of $(C_i)_I$. Define

$$\widetilde{C} := conv(V_1 \cup V_2 \cup V_3) + cone(r_1, r_2, r_3).$$

Then,

$$C_I = \widetilde{C}.$$

Proof. It is clear that $\widetilde{C} \subseteq C_I$, since all $V_i \subseteq (C_i)_I \subseteq C_I$. We now prove the reverse inclusion $C_I \subseteq \widetilde{C}$ by contradiction. Suppose there exists a point $\widetilde{x} \in C \cap (\mathbb{Z}^2 \times \mathbb{R})$ such that $\widetilde{x} \notin \widetilde{C}$. Then, there exists an inequality $a^T x \leq b$ such that

$$a^T \widetilde{x} > b \quad \text{and} \quad a^T x \leq b \quad \text{for all } x \in \widetilde{C}.$$

Claim 1.3.4. *$a^T r_i \leq 0$ for each $i = 1, 2, 3$.*

Proof of Claim. Let $z \in V_1 \cup V_2 \cup V_3$ be an arbitrary point. By Lemma 1.3.5, such a point exists. Suppose, for contradiction, that $a^T r_1 > 0$. Then for any sufficiently large scalar $k > 0$, the point $z + kr_1 \in \widetilde{C}$, but

$$a^T(z + kr_1) = a^T z + k a^T r_1 > b,$$

contradicting the assumption that $a^T x \leq b$ for all $x \in \widetilde{C}$. ▲

Claim 1.3.5. $a^T v > b$.

Proof of Claim. Suppose instead that $a^T v \leq b$. Since $\widetilde{x} \in C$, we can write

$$\widetilde{x} = v + \mu_1 r_1 + \mu_2 r_2 + \mu_3 r_3$$

for some $\mu_1, \mu_2, \mu_3 \geq 0$. Then,

$$a^T \widetilde{x} = a^T v + \mu_1 a^T r_1 + \mu_2 a^T r_2 + \mu_3 a^T r_3 \leq b,$$

since $a^T v \leq b$ and $a^T r_i \leq 0$ for each i. This contradicts the assumption that $a^T \widetilde{x} > b$. ▲

We have $a_1^T \widetilde{x} < b_1$, $a_2^T \widetilde{x} < b_2$, and $a_3^T \widetilde{x} < b_3$, since otherwise $\widetilde{x}$ would lie on one of the facets C_i, and hence be contained in $(C_i)_I$, which would imply $\widetilde{x} \in \widetilde{C}$, contradicting our assumption. Now consider the third components of the vectors a_1, a_2, a_3, denoted by $a_1^{(3)}, a_2^{(3)}, a_3^{(3)}$. WLOG, assume that

$$a_1^{(3)} \leq a_2^{(3)} \leq a_3^{(3)}.$$

We analyze the situation through the following four cases, and derive a contradiction in all:

1. $a_1^{(3)} = a_2^{(3)} = a_3^{(3)} = 0$; this contradicts the assumption that C is full-dimensional cone.

2. $0 \leq a_1^{(3)}$ and $a_3^{(3)} > 0$; let $r = (0, 0, -1)$. Then for sufficiently large $k > 0$, the point

   $$x_1 = \widetilde{x} - kr = \widetilde{x} + k(0, 0, 1)$$

   has the same first two coordinates as $\widetilde{x}$, and hence $proj_I(x_1) = proj_I(\widetilde{x}) \in \mathbb{Z}^2$. Since $\widetilde{x} \in C$, and $a_i^T \widetilde{x} < b_i$ for all $i = 1, 2, 3$, and all third components $a_i^{(3)}$ are non-negative with at least one of them strictly positive, it follows that for sufficiently large $k > 0$, the point x_1 satisfies $a_i^T x_1 \leq b_i$ for some $i \in \{1, 2, 3\}$. Thus, $x_1 \in C_i$, and since its first two coordinates are integer, we have $x_1 \in (C_i)_I \subseteq \widetilde{C}$. Therefore,

   $$a^T x_1 \leq b.$$

   However, since $x_1 = \widetilde{x} - kr$, and we have assumed $a^T \widetilde{x} > b$, it follows that

   $$b \geq a^T x_1 = a^T \widetilde{x} - k a^T r > b - k a^T r,$$

which implies $a^T r > 0$. Now, consider the point $x_2 = \widetilde{x} + k'r$ for some $k' \geq 0$. Since $r = (0, 0, -1)$ and $\widetilde{x} \in C$, we claim that $x_2 \in C$ as well. This follows from the fact that $a_i^T r \leq 0$ for all $i = 1, 2, 3$. Therefore, $x_2 \in C$, and thus there exist non-negative scalars $\mu_1, \mu_2, \mu_3 \geq 0$ such that

$$x_2 = v + \mu_1 r_1 + \mu_2 r_2 + \mu_3 r_3.$$

Applying the linear functional a^T to both sides, we get

$$a^T x_2 = a^T \widetilde{x} + k' a^T r = a^T v + \mu_1 a^T r_1 + \mu_2 a^T r_2 + \mu_3 a^T r_3.$$

Since $a^T r_i \leq 0$ for all $i \in \{1, 2, 3\}$ (as shown previously) and $a^T v$ is fixed, the right-hand side is bounded above by $a^T v$. However, the left-hand side grows unbounded as $k' \to \infty$ due to $a^T r > 0$. Thus, we encountered a contradiction in this case as well.

3. $a_1^{(3)} < 0$ and $a_3^{(3)} \leq 0$; let $r = (0, 0, 1)$. This case is symmetric to the previous case. Repeating the same reasoning, it can be derived that $a^T r > 0$, and this leads a similar contradiction.

4. $a_1^{(3)} < 0$ and $a_3^{(3)} > 0$; let $r = (0, 0, 1)$.

   Since the third components of the facet normals have mixed signs, we can move both upward and downward along the direction r while remaining inside the cone. Define

   $$x_1 = \widetilde{x} + kr, \quad x_2 = \widetilde{x} - k'r$$

   for sufficiently large $k, k' > 0$. Then, due to the signs of $a_i^{(3)}$, it follows that $x_1 \in C_i$ and $x_2 \in C_j$ for some $i \neq j \in \{1, 2, 3\}$. Furthermore, since $proj_I(x_1) = proj_I(x_2) = proj_I(\widetilde{x}) \in \mathbb{Z}^2$, both points belong to their respective mixed-integer hulls:

   $$x_1 \in (C_i)_I, \quad x_2 \in (C_j)_I.$$

   Hence, $x_1, x_2 \in \widetilde{C}$, and since $a^T x_1 \leq b$ and $a^T x_2 \leq b$, we conclude that every point on the line segment between them satisfies $a^T x \leq b$. But $\widetilde{x} \in conv(x_1, x_2)$, and we assumed $a^T \widetilde{x} > b$, which leads to a contradiction in this final case.

□

Corollary 1.3.1. *Let*

$$C = \{x \in \mathbb{R}^3 \mid a_i^T x \leq b^{(i)},\ i = 1, 2, 3\}$$

be a translated cone, and fix $I = \{1, 2\}$. The set C_I can be described using a polynomial number of inequalities in terms of the description-size of C.

Proof. Let C_1, C_2, C_3 be the facets of C. For each $i \in \{1,2,3\}$, let V_i be the set of extreme points of $(C_i)_I$. By Lemma 1.3.4, the total number of points in the union $V_1 \cup V_2 \cup V_3$ is linear in the bit-length of the input defining the cone C. Each facet of the polytope $conv(V_1 \cup V_2 \cup V_3)$ is determined by three affinely independent extreme points. Hence, the number of such facets—and consequently, the number of inequalities needed to describe this convex hull is polynomial in the input size. Let $Q_1 = \{x \in \mathbb{R}^3 \mid Dx \leq d\}$ be a description of the polytope $conv(V_1 \cup V_2 \cup V_3)$, where the number of rows in D is polynomial $p(D)$. Now, let

$$Q_2 := \{x \in \mathbb{R}^3 \mid a_1^T x \leq 0,\ a_2^T x \leq 0,\ a_3^T x \leq 0\}$$

denote the $cone(r_1, r_2, r_3)$. The Minkowski sum $C_I = Q_1 \oplus Q_2$ can be represented as a projection of a higher-dimensional polyhedron. Specifically, define

$$Q_3 := \left\{(z, x, y) \in \mathbb{R}^9 \;\middle|\; Dx \leq d,\ a_1^T y \leq 0,\ a_2^T y \leq 0,\ a_3^T y \leq 0,\ z = x + y\right\}.$$

Then $C_I = proj_{\{1,2,3\}}(Q_3)$, the projection of Q_3 onto the z-coordinates. The polyhedron Q_3 is described using a polynomial number of inequalities $p(D) + 4$, and projection onto a 3-dimensional subspace can be computed via Fourier–Motzkin elimination. Since the number of eliminated variables is at most 6 (eliminating $x \in \mathbb{R}^3$ and $y \in \mathbb{R}^3$), and each step of Fourier–Motzkin introduces at most a quadratic blow-up in the number of inequalities, the total number of resulting inequalities is at most $(p(D) + 4)^{64}$. Therefore, C_I admits a description with a polynomial number of inequalities in terms of the description-size of C. □

Corollary 1.3.2. *Let P be a pointed polyhedron in $\mathbb{R}^3$ with basis family $\mathcal{B}$ and fix $I = \{1,2\}$. Define*

$$Q = \bigcap_{B \in \mathcal{B}} P(B)_I.$$

$P_I \subseteq Q \subseteq P_I^{split}$, and Q has polynomial description-size in terms of the description-size of P.

One direction for further exploration is to determine whether there exists an example in which P_I is strictly contained in Q. We have not found such an example. It would be surprising if no such polyhedron exists, since this would immediately imply that P_I itself admits a polynomial-size description. The question of whether the split closure P_I^{split} admits a description using a polynomial number of inequalities remains open, even in the restricted setting of translated cones.

For a structural characterization of translated cones in $\mathbb{R}^3$, we refer the interested reader to Section 1.3.1. Although not directly related to our main development, this discussion provides useful geometric insights that support the broader theory.

1.3.1 A Characterization of Translated Cones in $\mathbb{R}^3$

As a brief deviation from the main thread of this chapter, we present a structural classification of translated cones in $\mathbb{R}^3$, which may be of independent geometric interest. Here, we distinguish between three types of translated cones in $\mathbb{R}^3$. Let $C = v + cone(r_1, r_2, r_3)$ be a translated cone with apex $v \in \mathbb{R}^3$ and extreme rays $r_1, r_2, r_3 \in \mathbb{R}^3$. Let $I = \{1, 2\}$, and let $C' = proj_I(C) = v' + cone(r'_1, r'_2, r'_3)$ be the projection of C into the (x, y)-space such that $v' = proj_I(v)$ and $r'_i = proj_I(r_i)$ for $i = 1, 2, 3$. Note that $dim(r'_1, r'_2, r'_3) = 2$.

Lemma 1.3.6. *Let*

$$C = \{x \in \mathbb{R}^3 \mid a_i^T x \leq b^{(i)},\ i = 1, 2, 3\}$$

be a translated cone with vertex v. Denote its facets by C_1, C_2, C_3 and its extreme rays by r_1, r_2, r_3, as defined in Definition 1.3.1. Let $v' = proj_I(v)$ and $r'_i = proj_I(r_i)$ for $i = 1, 2, 3$. Exactly one of the following holds:

1. *One of the r'_i is contained in the cone generated by the other two, i.e., exactly one of the following holds:*
   - $r'_3 \in cone(r'_1, r'_2)$;
   - $r'_2 \in cone(r'_1, r'_3)$;
   - $r'_1 \in cone(r'_2, r'_3)$.

   We say that C is of type 1 in this case. (Figure 1.2)
2. *$C' = \mathbb{R}^2$; we say that C is of type 2 in this case. (Figure 1.3)*
3. *Exactly two vectors among r'_1, r'_2, r'_3's are linearly dependent; we say that C is of type 3 in this case. (Figure 1.4)*

Proof. Suppose that none of the r_i's lies in the cone generated by the other two. This implies that $r'_1, r'_2, r'_3 \neq \mathbf{0}$. Suppose that $C' = v' + cone(r'_1, r'_2, r'_3) \neq \mathbb{R}^2$. This means that there exists a vector $\mu = (\mu^1, \mu^2) \in \mathbb{R}^2$ such that $\mu \notin cone(r'_1, r'_2, r'_3)$. So, the system

$$\begin{pmatrix} r_1^1 & r_2^1 & r_3^1 \\ r_1^2 & r_2^2 & r_3^2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$$
$$\lambda_1, \lambda_2, \lambda_3 \geq 0$$

is infeasible. Using the Farkas' Lemma (See Theorem 3.5 in [12]), there exists $y = (y^1, y^2) \in \mathbb{R}^2$, such that $y^T r'_i \leq 0$ for $i = 1, 2, 3$, and $y^T \mu > 0$. Note that $y \neq \mathbf{0}$, since $y^T \mu > 0$. We have two cases,

1. At least two of the values $y^T r'_1, y^T r'_2, y^T r'_3$ are equal to 0; without loss of generality assume that $y^T r'_1 = y^T r'_2 = 0$, then r'_1, r'_2 are linearly dependent.

2. At most one of the values $y^T r'_1, y^T r'_2, y^T r'_3$ is equal to 0; then, there exist $\tilde{y} \in \mathbb{R}^2$ such that

$$\tilde{y}^T r'_1, \tilde{y}^T r'_2, \tilde{y}^T r'_3 < 0.$$

   The reason is that either

$$y^T r'_1, y^T r'_2, y^T r'_3 < 0,$$

   in which $\tilde{y} = y$ works, or exactly one of the values of $y^T r'_1, y^T r'_2, y^T r'_3$ is equal to 0. Without loss of generality, suppose $y^T r'_1 = 0$ and $y^T r'_2, y^T r'_3 < 0$. Since $r'_1 \neq 0$, we can slightly change y to obtain $\tilde{y}$ that satisfies the condition.

In the remaining assume there exists a vector $\tilde{y} \in \mathbb{R}^2$ such that $\tilde{y}^T r'_i < 0$, and we are going to show that this leads to a contradiction. Without loss of generality assume that

$$\tilde{y}^T r'_1 \leq \tilde{y}^T r'_2 \leq \tilde{y}^T r'_3 < 0. \tag{1.12}$$

Since $span(r'_1, r'_3) = \mathbb{R}^2$, then there exists $\lambda_1, \lambda_3 \in \mathbb{R}$, such that

$$r'_2 = \lambda_1 r'_1 + \lambda_3 r'_3.$$

We have multiple cases, and each leads to a contradiction:

1. $\lambda_1, \lambda_3 < 0$; rewriting (1.12) gives

$$\tilde{y}^T r'_1 \leq \lambda_1 \tilde{y}^T r'_1 + \lambda_3 \tilde{y}^T r'_3 \leq \tilde{y}^T r'_3 < 0.$$

   Since $\tilde{y}^T r'_1, \tilde{y}^T r'_3 < 0$, then $\tilde{y}^T r'_1 + \lambda_3, \tilde{y}^T r'_3 > 0$, which is a contradiction to $\tilde{y}^T r'_1 + \lambda_3 \tilde{y}^T r'_3 < \tilde{y} r'_3 < 0$.

2. $\lambda_1 < 0, \lambda_3 \geq 0$; then $r'_2 - \lambda_1 r'_1 = \lambda_3 r'_3$. $\lambda_3 \neq 0$; otherwise, r'_2 and r'_1 are linearly dependent. Thus,

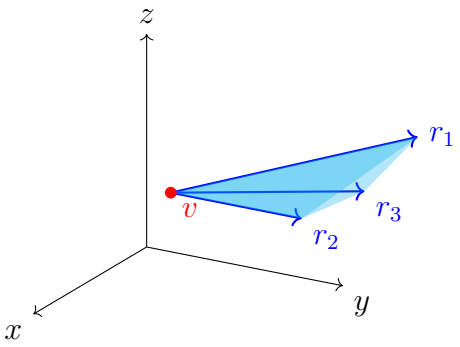
$$r'_3 = \frac{1}{\lambda_3} r'_2 + \frac{-\lambda_1}{\lambda_3} r'_1,$$

   which means $r'_3 \in cone(r'_1, r'_2)$, contradiction.

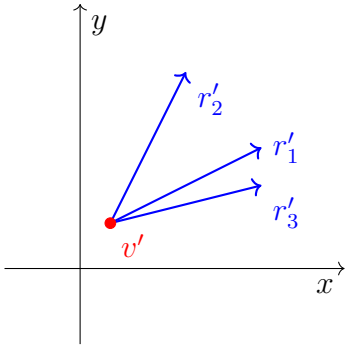
3. $\lambda_1 \geq 0, \lambda_3 < 0$; similar to the previous case, it can be derived that $r'_1 \in cone(r'_2, r'_3)$, contradiction.

4. $\lambda_1, \lambda_3 \geq 0$; this means that $r'_2 \in cone(r'_1, r'_3)$, contradiction.

□

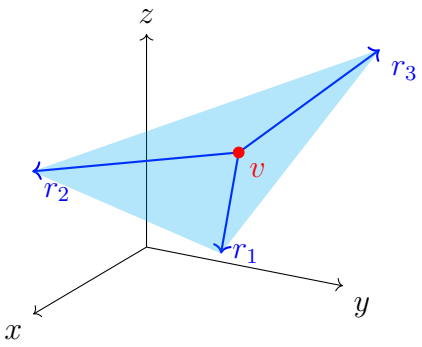


(a) C

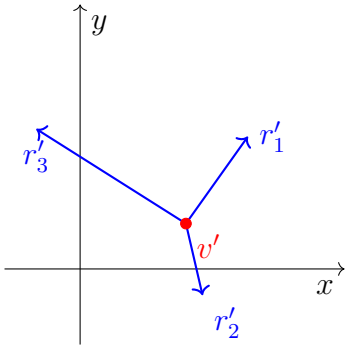


(b) The projection of C into the x, y-space

Figure 1.2: C is of type 1, since $r'_1 \in cone(r'_2, r'_3)$.

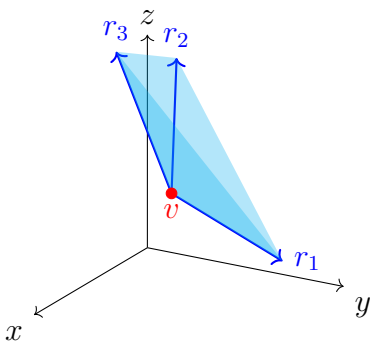


(a) C

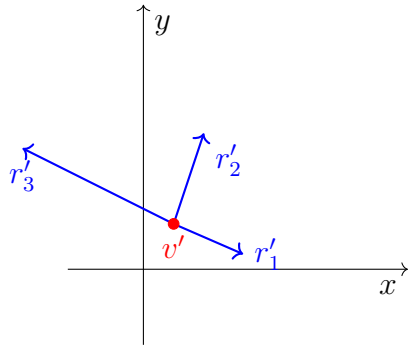


(b) The projection of C into the x, y-space

Figure 1.3: C is of type 2, since $C' = \mathbb{R}^2$.



(a) C



(b) The projection of C into the (x, y)-space

Figure 1.4: C is of type 3, since r'_1, r'_3 are linearly dependent.

1.4 The MIR Procedure for Systems with Two Equalities

In section 1.2, we reviewed the classical result that the GMI cut, the split cut, and the MIR cut coincide when applied to a system consisting of a single equality constraint together with non-negativity conditions, a result well established in the literature. In this section, we present our contribution: an extension of the MIR procedure to systems with two equality constraints, still under non-negativity and mixed-integer restrictions. In section 1.4.1, we model such systems as a mixed-integer set in $\mathbb{R}^3$, involving two integer variables and one continuous variable. Then, in section 1.4.2, we study the convex hull of this set and apply Theorem 1.3.2 from section 1.3 to derive the description of a facet of this convex hull, yielding a new valid inequality. Finally, in section 1.4.3, we project this inequality back to the original space, thereby obtaining a generalization of the MIR cut.

We consider the following system for S_2, consisting of two linear equalities:

$$\begin{aligned}
\sum_{i=1}^{n} a_1^{(i)} x^{(i)} &= b_1, \\
\sum_{i=1}^{n} a_2^{(i)} x^{(i)} &= b_2, \\
x^{(i)} &\geq 0 \quad \forall i \in [n], \\
x^{(i)} &\in \mathbb{Z} \quad \forall i \in I,
\end{aligned}$$

where $I \subseteq [n]$ denotes the index set of variables constrained to be integer, and $C := [n] \setminus I$ denotes the set of continuous variables. We assume throughout that $f(b_1), f(b_2) > 0$.

1.4.1 Modeling into $\mathbb{R}^3$

We begin by relaxing S_2 by discarding all non-positive terms of the form $a_1^{(i)}x^{(i)} \leq 0$ and $a_2^{(i)}x^{(i)} \leq 0$ for continuous variables ($i \in C$) with coefficients $a_1^{(i)}, a_2^{(i)} \leq 0$. Since all variables satisfy $x^{(i)} \geq 0$. We retain all positive continuous-variable terms, and keep all terms for integer variables unchanged. This yields the following relaxed system, denoted by S_2':

$$\sum_{i \in I} a_1^{(i)} x^{(i)} + \sum_{\substack{i \in C \\ a_1^{(i)} > 0}} a_1^{(i)} x^{(i)} \geq b_1, \tag{1.14a}$$

$$\sum_{i \in I} a_2^{(i)} x^{(i)} + \sum_{\substack{i \in C \\ a_2^{(i)} > 0}} a_2^{(i)} x^{(i)} \geq b_2, \tag{1.14b}$$

$$x^{(i)} \geq 0 \quad \forall i \in [n],$$
$$x^{(i)} \in \mathbb{Z} \quad \forall i \in I.$$

Let $N_2 = \{i \in I \mid f(a_2^{(i)}) > 0\}$, and let $T \subseteq N_2$ be an arbitrary subset. We rewrite (1.14b) as:

$$\begin{aligned} &\sum_{i \in T} \lfloor a_2^{(i)} \rfloor x^{(i)} + \sum_{i \in N_2 \setminus T} \lceil a_2^{(i)} \rceil x^{(i)} + \sum_{i \in T} f(a_2^{(i)}) x^{(i)} \\ &+ \sum_{i \in N_2 \setminus T} \Big(f(a_2^{(i)}) - 1 \Big) x^{(i)} + \sum_{i \in I \setminus N_2} a_2^{(i)} x^{(i)} + \sum_{\substack{a_2^{(i)} > 0 \\ i \in C}} a_2^{(i)} x^{(i)} \geq b_2. \end{aligned} \tag{1.15}$$

Observe that the term

$$\sum_{i \in N_2 \setminus T} \Big(f(a_2^{(i)}) - 1 \Big) x^{(i)} \leq 0$$

since $f(a_2^{(i)}) \in [0, 1)$ and $x^{(i)} \geq 0$. Thus, removing this term yields a valid relaxation of (1.14b):

$$\sum_{i\in T}\lfloor a_2^{(i)}\rfloor x^{(i)} + \sum_{i\in N_2\setminus T}\lceil a_2^{(i)}\rceil x^{(i)} + \sum_{i\in T} f(a_2^{(i)})x^{(i)} + \sum_{i\in I\setminus N_2} a_2^{(i)}x^{(i)} + \sum_{\substack{a_2^{(i)}>0\\ i\in C}} a_2^{(i)}x^{(i)} \geq b_2. \quad (1.16)$$

Similarly, we rewrite (1.14a) grouping by $T \subseteq N_2$, and obtain:

$$\begin{aligned}&\sum_{i\in T}\lfloor a_1^{(i)}\rfloor\, x^{(i)} + \sum_{i\in N_2\setminus T}\lceil a_1^{(i)}\rceil\, x^{(i)} + \sum_{i\in T} f(a_1^{(i)})\, x^{(i)}\\ &+ \sum_{i\in N_2\setminus T}\left(a_1^{(i)} - \lceil a_1^{(i)}\rceil\right) x^{(i)} + \sum_{i\in I\setminus N_2}\lceil a_1^{(i)}\rceil\, x^{(i)}\\ &+ \sum_{i\in I\setminus N_2}\left(a_1^{(i)} - \lceil a_1^{(i)}\rceil\right) x^{(i)} + \sum_{\substack{a_1^{(i)}>0\\ i\in C}} a_1^{(i)}\, x^{(i)} \geq b_1.\end{aligned} \quad (1.17)$$

Again, the terms

$$\sum_{i\in N_2\setminus T}\left(a_1^{(i)} - \lceil a_1^{(i)}\rceil\right) x^{(i)} \quad \text{and} \quad \sum_{i\in I\setminus N_2}\left(a_1^{(i)} - \lceil a_1^{(i)}\rceil\right) x^{(i)}$$

are non-positive for the same reason (since $x^{(i)} \geq 0$, and $a_1^{(i)} \leq \lceil a_1^{(i)}\rceil$). Dropping them leads to a valid relaxation of (1.14a):

$$\sum_{i\in T}\lfloor a_1^{(i)}\rfloor\, x^{(i)} + \sum_{i\in N_2\setminus T}\lceil a_1^{(i)}\rceil\, x^{(i)} + \sum_{i\in T} f(a_1^{(i)})\, x^{(i)} + \sum_{i\in I\setminus N_2}\lceil a_1^{(i)}\rceil\, x^{(i)} + \sum_{\substack{a_1^{(i)}>0\\ i\in C}} a_1^{(i)}\, x^{(i)} \geq b_1. \quad (1.18)$$

Let

$$\mu = \max\left\{\max_{i\in T}\left(\frac{f(a_1^{(i)})}{f(a_2^{(i)})}\right), \max_{\substack{a_2^{(i)}>0\\ i\in C}}\left(\frac{a_1^{(i)}}{a_2^{(i)}}\right), 1\right\}. \quad (1.19)$$

Note that $\mu \geq 1$ by definition. We now reinterpret the valid relaxations (1.16) and (1.18) as a system in $\mathbb{R}^3$, using the variables $w_1, w_2 \in \mathbb{Z}$ and $s \in \mathbb{R}_{\geq 0}$. These variables are defined as follows:

$$
\begin{aligned}
w_1 &= -\lfloor b_1 \rfloor + \sum_{i \in T} \lfloor a_1^{(i)} \rfloor \, x^{(i)} + \sum_{i \in N_2 \setminus T} \lceil a_1^{(i)} \rceil \, x^{(i)} + \sum_{i \in I \setminus N_2} \lceil a_1^{(i)} \rceil \, x^{(i)}, \\
w_2 &= -\lfloor b_2 \rfloor + \sum_{i \in T} \lfloor a_2^{(i)} \rfloor \, x^{(i)} + \sum_{i \in N_2 \setminus T} \lceil a_2^{(i)} \rceil \, x^{(i)} + \sum_{i \in I \setminus N_2} a_2^{(i)} \, x^{(i)}, \\
s &= \sum_{i \in T} f(a_2^{(i)}) \, x^{(i)} + \sum_{\substack{a_2^{(i)} > 0 \\ i \in C}} a_2^{(i)} \, x^{(i)} + \sum_{\substack{a_2^{(i)} \leq 0 \\ a_1^{(i)} > 0 \\ i \in C}} \frac{a_1^{(i)}}{\mu} x^{(i)}.
\end{aligned}
\tag{1.20}
$$

Lemma 1.4.1. *The quantities w_1, w_2, s defined in (1.20) satisfy the following system:*

$$
\begin{aligned}
w_1 + \mu s &\geq f(b_1), && \text{(1.21a)} \\
w_2 + s &\geq f(b_2), && \text{(1.21b)} \\
s &\geq 0, \\
w_1, w_2 &\in \mathbb{Z}.
\end{aligned}
$$

Here, μ is defined in (1.19).

Proof. The non-negativity of s follows directly from the fact that each term in its definition involves non-negative coefficients and variables: for $i \in T \subseteq N_2$, $f(a_2^{(i)}) > 0$; for $i \in C$, we only keep terms with $a_2^{(i)} > 0$; and all variables satisfy $x^{(i)} \geq 0$. The integrality of w_1 and w_2 is immediate, noting that $a_2^{(i)} \in \mathbb{Z}$ for all $i \in I \setminus N_2$, and $x^{(i)} \in \mathbb{Z}$ for all $i \in I$ by assumption. Finally, the inequality (1.16) can be rewritten in terms of the variables w_2 and s as:

$$
w_2 + s - \sum_{\substack{a_2^{(i)} \leq 0 \\ a_1^{(i)} > 0 \\ i \in C}} \frac{a_1^{(i)}}{\mu} x^{(i)} \geq f(b_2),
$$

which directly implies that

$$
w_2 + s \geq f(b_2),
$$

since the term

$$
\sum_{\substack{a_2^{(i)} \leq 0 \\ a_1^{(i)} > 0 \\ i \in C}} \frac{a_1^{(i)}}{\mu} x^{(i)}
$$

is non-negative. Now, we show that

$$w_1 + \mu s \geq f(b_1).$$

Bringing $\lfloor b_1 \rfloor$ from the right-hand side to the left in (1.18) and substituting the definition of w_1, we can rewrite (1.18) as:

$$w_1 + \sum_{i \in T} f(a_1^{(i)})\, x^{(i)} + \sum_{\substack{a_1^{(i)} > 0 \\ i \in C}} a_1^{(i)}\, x^{(i)} \geq f(b_1).$$

Thus, to prove $w_1 + \mu s \geq f(b_1)$, it suffices to show that:

$$\mu s = \sum_{i \in T} \mu f(a_2^{(i)})\, x^{(i)} + \sum_{\substack{a_2^{(i)} > 0 \\ i \in C}} \mu a_2^{(i)}\, x^{(i)} + \sum_{\substack{a_2^{(i)} \leq 0 \\ a_1^{(i)} > 0 \\ i \in C}} a_1^{(i)}\, x^{(i)} \geq \sum_{i \in T} f(a_1^{(i)})\, x^{(i)} + \sum_{\substack{a_1^{(i)} > 0 \\ i \in C}} a_1^{(i)}\, x^{(i)}.$$

Canceling the common term

$$\sum_{\substack{a_2^{(i)} \leq 0 \\ a_1^{(i)} > 0 \\ i \in C}} a_1^{(i)}\, x^{(i)}$$

from both sides, we are left with showing:

$$\sum_{i \in T} \mu f(a_2^{(i)})\, x^{(i)} + \sum_{\substack{a_2^{(i)} > 0 \\ i \in C}} \mu a_2^{(i)}\, x^{(i)} \geq \sum_{i \in T} f(a_1^{(i)})\, x^{(i)} + \sum_{\substack{a_2^{(i)} > 0 \\ a_1^{(i)} > 0 \\ i \in C}} a_1^{(i)}\, x^{(i)}.$$

By the definition of μ, we have:

$$\mu \geq \max \left\{ \max_{i \in T} \left(\frac{f(a_1^{(i)})}{f(a_2^{(i)})} \right), \max_{\substack{a_2^{(i)} > 0 \\ i \in C}} \left(\frac{a_1^{(i)}}{a_2^{(i)}} \right) \right\} \geq 1,$$

and since all $x^{(i)} \geq 0$, each term on the left-hand side dominates its corresponding term on the right-hand side. Hence, the inequality holds.

□

We define the system (1.21) as Q_3, which is obtained by relaxing the original mixed-integer system S_2 in a structured manner. In the next section, we derive a valid cutting plane for the convex hull $conv(Q_3)$ of this relaxed system. The goal is to use the geometric structure of this three-dimensional relaxation to identify an inequality that separates infeasible fractional points from $conv(Q_3)$. Later, in section 1.4.3, we show how this cutting plane for $conv(Q_3)$ can be lifted back to the original space to obtain a valid cutting plane for the original set S_2.

1.4.2 Finding Valid Cutting Planes for $conv(Q_3)$

We now focus on deriving valid inequalities for the convex hull of the mixed-integer set Q_3 defined in (1.21). As a starting point, we recall that certain inequalities can be immediately obtained by applying the MIR procedure to appropriate projections.

Remark 1.4.1. *Let $(w_1, w_2, s) \in Q_3$ be an arbitrary mixed-integer point in $\mathbb{Z}^2 \times \mathbb{R}$. Consider the pair (w_2, s), which satisfies:*

$$\begin{aligned} w_2 + s &\geq f(b_2), \\ s &\geq 0, \\ w_2 &\in \mathbb{Z}. \end{aligned}$$

Then, by Remark 1.2.1, the inequality

$$f(b_2)w_2 + s \geq f(b_2) \tag{1.22}$$

is valid for Q_3. Similarly, the pair (w_1, s), satisfying

$$\begin{aligned} w_1 + \mu s &\geq f(b_1), \\ s &\geq 0, \\ w_1 &\in \mathbb{Z}, \end{aligned}$$

leads to the valid inequality

$$\frac{f(b_1)}{\mu} w_1 + s \geq \frac{f(b_1)}{\mu}. \tag{1.23}$$

Although the inequalities (1.22) and (1.23) are valid for $conv(Q_3)$, they arise from applying MIR independently to lower-dimensional projections. As such, they do not fully capture the structure of $conv(Q_3)$. To make further progress, we relax the integrality constraints and consider the continuous relaxation of Q_3.

Definition 1.4.1. *Let $Q'_3 \subseteq \mathbb{R}^3$ denote the full-dimensional pointed cone obtained from Q_3 by allowing both w_1 and w_2 to be continuous. Then one can see that Q'_3 is a translated cone and is defined by the system:*

$$\begin{aligned} w_1 + \mu s &\geq f(b_1), \\ w_2 + s &\geq f(b_2), \\ s &\geq 0. \end{aligned}$$

The unique vertex of Q'_3, which we refer to as the apex, is given by

$$v = (f(b_1), f(b_2), 0).$$

One can verify that the extreme rays of Q'_3 are:

$$\begin{aligned} \rho_1 &= (1, 0, 0), \\ \rho_2 &= (0, 1, 0), \\ \rho_3 &= (-\mu, -1, 1). \end{aligned}$$

See Figure 1.5.

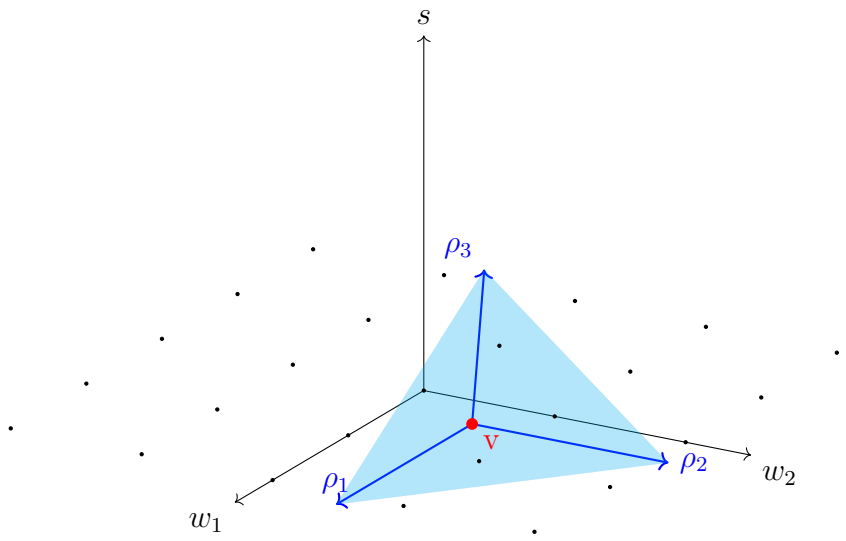


Figure 1.5: Q'_3 in $\mathbb{R}^3$. The black points represents the integer points in the (w_1, w_2)-plane.

Let $I' = \{1, 2\}$. One can verify that $conv(Q_3) = (Q'_3)_{I'}$. Therefore, from now on we only focus on $(Q'_3)_{I'}$ instead of $conv(Q_3)$. We apply Theorem 1.3.2 to obtain a valid inequality

for $(Q_3')_{I'}$. To do so, we must first identify the extreme points of the mixed-integer hulls of the facets of the cone Q_3'.

Definition 1.4.2. *Let the three facets of Q_3' be defined as:*

$$\begin{aligned} C_1 &= \{x \in Q_3' \mid w_1 + \mu s = f(b_1)\}, \\ C_2 &= \{x \in Q_3' \mid w_2 + s = f(b_2)\}, \\ C_3 &= \{x \in Q_3' \mid s = 0\}. \end{aligned}$$

Also, let V_i denote the set of extreme points of the mixed-integer hull $(C_i)_{I'}$ for each $i = 1, 2, 3$.

Theorem 1.3.2 implies that:

$$(Q_3')_{I'} = conv(V_1 \cup V_2 \cup V_3) + cone(\rho_1, \rho_2, \rho_3).$$

To study the sets V_1, V_2, V_3, we analyze the projections of the facets C_i onto the (w_1, w_2)-space. Let $C_i' = proj_{I'}(C_i)$ for $i = 1, 2, 3$, and define the projections of the rays:

$$\begin{aligned} \rho_1' &= proj_{I'}(\rho_1) = (1, 0), \\ \rho_2' &= proj_{I'}(\rho_2) = (0, 1), \\ \rho_3' &= proj_{I'}(\rho_3) = (-\mu, -1). \end{aligned}$$

Let $v' = proj_{I'}(v) = (f(b_1), f(b_2))$ be the projection of the apex of Q_3', and consider the four integer points surrounding v' in $\mathbb{R}^2$:

$$\begin{aligned} B_1' &= (0, 0), \quad & B_2' &= (1, 0), \\ B_3' &= (1, 1), \quad & B_4' &= (0, 1). \end{aligned}$$

While these points themselves are not necessarily significant, it is important to determine which facet of the shell of Q_3' (i.e., the union $C_1 \cup C_2 \cup C_3$) contains the corresponding preimage of each point. To analyze this structure, we distinguish between the following three cases, which are illustrated in Figure 1.9:

$$\mu < \frac{f(b_1)}{f(b_2)}, \quad \mu = \frac{f(b_1)}{f(b_2)}, \quad \mu > \frac{f(b_1)}{f(b_2)}.$$

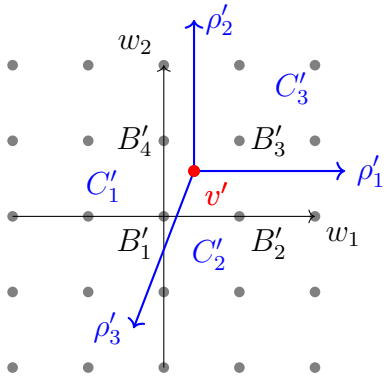


Figure 1.6: C'_1, C'_2, C'_3 when $\mu < \frac{f(b_1)}{f(b_2)}$.

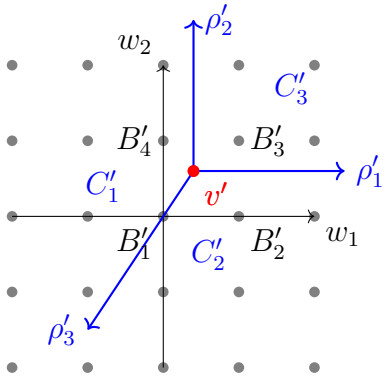


Figure 1.7: C'_1, C'_2, C'_3 when $\mu = \frac{f(b_1)}{f(b_2)}$.

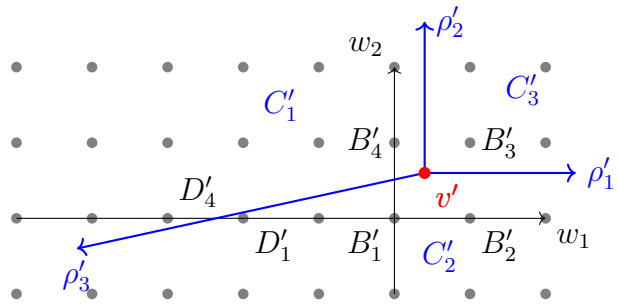


Figure 1.8: C'_1, C'_2, C'_3 when $\mu > \frac{f(b_1)}{f(b_2)}$. D'_1, D'_4 are closest integer points on $w_2 = 0$ to the ray ρ'_3.

Figure 1.9: $proj_{I'}(Q'_3)$ under different assumptions on μ.

Regardless of the structural differences across the three cases, the set $(C_3)_{I'}$ admits a particularly simple description. Define the point $B_3 = (1, 1, 0)$, and observe that its projection onto the index set I' is given by $B'_3 = proj_{I'}(B_3) = (1, 1)$. The following Remark 1.4.2 is useful in analyzing the structure of V_1, V_2, and V_3:

Remark 1.4.2. *Consider Q'_3 together wit its extreme rays ρ_1, ρ_2, ρ_3 defined in Definition 1.4.1. Let C_1, C_2, C_3 be the facets of Q'_3, and let V_i denote the set of extreme points of C_i as defined in Definition 1.4.2. One can verify that:*

$$(C_1)_{I'} = conv(V_1) + cone(\rho_2, \rho_3),$$
$$(C_2)_{I'} = conv(V_2) + cone(\rho_1, \rho_3),$$
$$(C_3)_{I'} = conv(V_3) + cone(\rho_1, \rho_2).$$

Lemma 1.4.2. *Consider the facet C_3 of Q_3' together with the set of extreme points V_3 of its integer hull with respect to I', as defined in Definition 1.4.2. Then V_3 consists of a single point:*

$$V_3 = \{B_3\}, \quad \text{where } B_3 = (1, 1, 0).$$

Proof. Note that $B_3 \in (C_3)_{I'}$, which immediately implies $V_3 \neq \varnothing$. Let $x \in (C_3)_{I'}$, so $x = (a, b, 0)$ for some $a, b \in \mathbb{Z}$. Since $x \in Q_3'$ and $x^{(3)} = 0$, we must have $a \geq f(b_1)$ and $b \geq f(b_2)$. As both $0 < f(b_1), f(b_2) < 1$, we conclude $a, b \geq 1$. Comparing the coordinates of $B_3 = (1, 1, 0)$ and $x = (a, b, 0)$, we obtain:

$$x = B_3 + (a-1)\rho_1 + (b-1)\rho_2,$$

where $\rho_1 = (1, 0, 0)$ and $\rho_2 = (0, 1, 0)$ are extreme rays of $(C_3)_{I'}$. Now, consider the point:

$$\tilde{x} = B_3 + 2(a-1)\rho_1 + 2(b-1)\rho_2,$$

which is contained in $(C_3)_{I'}$, as $B_3 \in (C_3)_{I'}$. However, $x = \frac{B_3 + \tilde{x}}{2}$. Thus, x is not an extreme point of $(C_3)_{I'}$ unless $a = b = 1$, i.e., $x = B_3$. Since $V_3 \neq \varnothing$, we conclude that $V_3 = \{B_3\}$. □

Now, we explore V_1. We define the point

$$D_1 = (\lceil f(b_1) - \mu f(b_2) \rceil, \ 0, \ f(b_2)) \in (C_2)_{I'},$$

whose projection $D_1' = proj_{I'}(D_1)$ appears in Figure 1.8. Specifically, $D_1 = (0, 0, f(b_2))$ when $\mu = \frac{f(b_1)}{f(b_2)}$; $D_1 = (1, 0, f(b_2))$ when $\mu < \frac{f(b_1)}{f(b_2)}$, and has a non-positive integer first coordinate when $\mu > \frac{f(b_1)}{f(b_2)}$. In all cases, we will show that $D_1 \in V_2$ in the following Lemma 1.4.3:

Lemma 1.4.3. *Consider the facet C_2 of Q_3' together with the set of extreme points V_2 of its integer hull with respect to I', as defined in Definition 1.4.2. Then the point*

$$D_1 = (\lceil f(b_1) - \mu f(b_2) \rceil, \ 0, \ f(b_2))$$

belongs to V_2.

Proof. Note that $D_1 \in (C_2)_{I'}$, which implies that $V_2 \neq \varnothing$. We first show that the second coordinate of any point in $conv(V_2)$ is non-positive:

Claim 1.4.1. *For any $x \in conv(V_2)$, we have $x^{(2)} \leq 0$.*

Proof of Claim. We prove the claim for the elements of V_2, which implies the result for all points in $conv(V_2)$ by convexity. Let $x \in V_2$. Since $x \in C_2$, we have $x^{(3)} \geq 0$ and $x^{(2)} + x^{(3)} = f(b_2)$. As $x^{(2)} \in \mathbb{Z}$ and $f(b_2) < 1$, it follows that $x^{(2)} \leq 0$. ▲

Suppose, for contradiction, that $D_1 \notin V_2$. Then by Remark 1.4.2, there exists $\tilde{x} \in conv(V_2)$ and $\lambda_1, \lambda_3 \geq 0$ such that

$$D_1 = \tilde{x} + \lambda_1 \rho_1 + \lambda_3 \rho_3.$$

From the previous claim, $\tilde{x}^{(2)} \leq 0$. Comparing the second coordinate on both sides, we find:

$$0 = D_1^{(2)} = \tilde{x}^{(2)} + \lambda_1 \rho_1^{(2)} + \lambda_3 \rho_3^{(2)}.$$

Since $\rho_1^{(2)} = 0$ and $\rho_3^{(2)} = -1 < 0$, this forces $\lambda_3 = 0$. Thus, we conclude:

$$D_1 = \tilde{x} + \lambda_1 \rho_1.$$

From this, we obtain that $\tilde{x}^{(2)} = 0$. Since $\tilde{x} \in C_2$, it follows that

$$\tilde{x}^{(2)} + \tilde{x}^{(3)} = \tilde{x}^{(3)} = f(b_2).$$

Because $\tilde{x} \in conv(V_2)$, any convex combination representing $\tilde{x}$ uses only points from V_2 with second coordinate equal to zero.

Claim 1.4.2. *For any $x \in V_2$ with $x^{(2)} = 0$, we have*

$$x^{(1)} \geq \lceil f(b_1) - \mu f(b_2) \rceil .$$

Proof of Claim. Since $x \in C_2$, we have $x^{(2)} + x^{(3)} = f(b_2)$, so $x^{(3)} = f(b_2)$. Also, since $x \in Q_3$, it must satisfy

$$x^{(1)} + \mu x^{(3)} = x^{(1)} + \mu f(b_2) \geq f(b_1).$$

So, $x^{(1)} \geq f(b_1) - \mu f(b_2)$. Since $x^{(1)} \in \mathbb{Z}$, we conclude that

$$x^{(1)} \geq \lceil f(b_1) - \mu f(b_2) \rceil .$$

▲

Since $\tilde{x}$ is a convex combination of elements of V_2 with second coordinate equal to zero, claim 1.4.2 implies that

$$\tilde{x}^{(1)} \geq \lceil f(b_1) - \mu f(b_2) \rceil .$$

Comparing the first coordinate of both sides in $D_1 = \tilde{x} + \lambda_1 \rho_1$, we get:

$$\lceil f(b_1) - \mu f(b_2) \rceil = D_1^{(1)} = \tilde{x}^{(1)} + \lambda_1.$$

Since $\tilde{x}^{(1)} \geq \lceil f(b_1) - \mu f(b_2) \rceil$, this forces $\lambda_1 = 0$, and hence $\tilde{x} = D_1$, which contradicts the assumption that $D_1 \notin V_2$. □

Finally, we analyze the set V_1. To this end, we define the following points:

$$B_1 = (0, 0, f(b_2)),$$
$$B_4 = \left(0, 1, \frac{f(b_1)}{\mu}\right).$$

Note that $B_i' = proj_{I'}(B_i)$ for $i = 1, 4$. In Lemma 1.4.4 below, we show that the membership of these points in V_1 depends on the value of μ relative to $\frac{f(b_1)}{f(b_2)}$. Specifically, if $\mu \leq \frac{f(b_1)}{f(b_2)}$, then $B_1 \in V_1$. And, if $\mu > \frac{f(b_1)}{f(b_2)}$, then $B_4 \in V_1$. This dichotomy arises from the relative positions of the points with respect to the facet structure of C_1, C_2.

Lemma 1.4.4. *Consider the facet C_1 of Q_3' together with the set of extreme points V_1 of its integer hull with respect to I', as defined in Definition 1.4.2. Let $B_1 = \left(0, 0, \frac{f(b_1)}{\mu}\right)$ and $B_4 = \left(0, 1, \frac{f(b_1)}{\mu}\right)$.*

- *If $\mu \leq \frac{f(b_1)}{f(b_2)}$, then $B_1 \in V_1$.*
- *If $\mu > \frac{f(b_1)}{f(b_2)}$, then $B_4 \in V_1$.*

Proof. Note that $B_4 \in (C_1)_{I'}$, which implies that $V_1 \neq \varnothing$. We begin by showing that the first coordinate of any point in $conv(V_1)$ is non-positive:

Claim 1.4.3. *For any $x \in conv(V_1)$, we have $x^{(1)} \leq 0$.*

Proof of Claim. We prove the claim for all $x \in V_1$, which suffices by convexity. Since $x \in C_1$, we have $x^{(3)} \geq 0$ and $x^{(1)} + \mu x^{(3)} = f(b_1)$. Given that $f(b_1) < 1$ and $x^{(1)} \in \mathbb{Z}$, it follows that $x^{(1)} \leq 0$. ▲

We now show the existence of a point in V_1 with first coordinate equal to zero:

Claim 1.4.4. *There exists an extreme point $x' \in V_1$ such that $(x')^{(1)} = 0$.*

Proof of Claim. Suppose, for the sake of contradiction, that $x^{(1)} < 0$ for all $x \in V_1$. Then every point in $conv(V_1)$ also satisfies $x^{(1)} < 0$. Since $B_4 \in (C_1)_{I'}$, by Remark 1.4.2, there exist $\tilde{x} \in conv(V_1)$ and $\lambda_2, \lambda_3 \geq 0$ such that

$$B_4 = \tilde{x} + \lambda_2 \rho_2 + \lambda_3 \rho_3.$$

But the first coordinate of the left-hand side is 0, while that of the right-hand side is strictly negative: $\tilde{x}^{(1)} < 0$, $\rho_2^{(1)} = 0$, and $\rho_3^{(1)} = -\mu < 0$. This contradiction implies the existence of an extreme point $x' \in V_1$ with $(x')^{(1)} = 0$. ▲

Let $x' \in V_1$ be such that $(x')^{(1)} = 0$. Since $x' \in C_1$, we have

$$\mu(x')^{(3)} = f(b_1) \quad \Rightarrow \quad (x')^{(3)} = \frac{f(b_1)}{\mu}.$$

Since $\mu \geq 1$, it follows that $(x')^{(3)} \leq f(b_1) < 1$. As $x' \in C$, we have

$$(x')^{(2)} + (x')^{(3)} \geq f(b_2) \quad \Rightarrow \quad (x')^{(2)} \geq f(b_2) - \frac{f(b_1)}{\mu} > f(b_2) - 1 > -1.$$

Given $(x')^{(2)} \in \mathbb{Z}$, we conclude $(x')^{(2)} \geq 0$.

Claim 1.4.5. *We have* $(x')^{(2)} \leq 1$, *hence* $(x')^{(2)} \in \{0, 1\}$.

Proof of Claim. Suppose $(x')^{(2)} \geq 2$. Then,

$$x' = B_4 + ((x')^{(2)} - 1)\rho_2.$$

Define $\tilde{x} = B_4 + 2((x')^{(2)} - 1)\rho_2 \in (C_1)_{I'}$. Since

$$x' = \frac{1}{2}(B_4 + \tilde{x}),$$

it follows that $x' \notin V_1$, contradicting the extremality of x'. Thus, $(x')^{(2)} \leq 1$. ▲

If $(x')^{(2)} = 0$, then $x' = B_1$, and if $(x')^{(2)} = 1$, then $x' = B_4$. We now distinguish two cases depending on the relation between μ and $\frac{f(b_1)}{f(b_2)}$:

- $\mu > \frac{f(b_1)}{f(b_2)}$; suppose $(x')^{(2)} = 0$. Then,

$$(x')^{(2)} + (x')^{(3)} = \frac{f(b_1)}{\mu} < f(b_2),$$

which violates the constraint $x' \in C$. Therefore, we must have $(x')^{(2)} = 1$, and conclude that

$$x' = B_4 = \left(0, 1, \tfrac{f(b_1)}{\mu}\right) \in V_1.$$

- $\mu \leq \frac{f(b_1)}{f(b_2)}$; in this case, $(x')^{(2)} = 0$ is feasible, since

$$(x')^{(2)} + (x')^{(3)} = \frac{f(b_1)}{\mu} \geq f(b_2).$$

Thus, we may take

$$x' = \left(0, 0, \tfrac{f(b_1)}{\mu}\right) \in C_1 \quad \text{and therefore } x' \in (C_1)_{I'}.$$

Moreover, $x' \in V_1$, since the only other possibility $(x')^{(2)} = 1$ corresponds to $x' = B_4 = B_1 + \rho_2$, and both B_1 and B_4 lie in $(C_1)_{I'}$.

□

Thus, we have identified the following extreme points in $V_1 \cup V_2 \cup V_3$:

1. $B_3 = (1, 1, 0) \in V_3$,
2. $D_1 = (\lceil f(b_1) - \mu f(b_2) \rceil, 0, f(b_2)) \in V_2$,
3. $B_1 = \left(0, 0, \tfrac{f(b_1)}{\mu}\right) \in V_1$, when $\mu \leq \frac{f(b_1)}{f(b_2)}$,
4. $B_4 = \left(0, 1, \tfrac{f(b_1)}{\mu}\right) \in V_1$, when $\mu > \frac{f(b_1)}{f(b_2)}$.

Before proceeding, however, observe that when $\mu = \frac{f(b_1)}{f(b_2)}$, the structure of $conv(Q_3) = (Q_3')_{I'}$ becomes particularly tractable, as stated in the following Lemma 1.4.5. Note that in this case we have:

$$B_1 = D_1 = (0, 0, f(b_2)) = \left(0, 0, \frac{f(b_1)}{\mu}\right).$$

Lemma 1.4.5. *Consider the sets of extreme points V_1 and V_2 as defined in Definition 1.4.2. Assume that $\mu = \frac{f(b_1)}{f(b_2)}$, and let $\tilde{B}_1 = (0, 0, f(b_2))$. Then*

$$V_1 = V_2 = \{\tilde{B}_1\} = \{B_1\} = \{D_1\}.$$

Proof. We already know $\tilde{B}_1 = B_1 \in V_1$ and $\tilde{B}_1 = D_1 \in V_2$. We prove that $V_2 = \{\tilde{B}_1\}$; the proof for $V_1 = \{\tilde{B}_1\}$ is analogous. Let $x = (w_1^*, w_2^*, s^*) \in V_2$ be an arbitrary extreme point such that $x \neq \tilde{B}_1$.

Claim 1.4.6. $s^* \geq f(b_2)$.

Proof of Claim. Suppose, for contradiction, that $s^* < f(b_2)$. Since $x \in C_2$, it satisfies $w_2^* + s^* = f(b_2)$. Because $w_2^* \in \mathbb{Z}$, we must have:

$$w_2^* = f(b_2) - s^* > 0.$$

Thus, $w_2^* \geq 1$, which contradicts the fact that $f(b_2) \in (0, 1)$, since $w_2^* + s^* = f(b_2)$ and $s^* \geq 0$. Hence, $s^* \geq f(b_2)$.

▲

Now consider the point

$$\tilde{x} := (2w_1^*,\, 2w_2^*,\, 2s^* - f(b_2)).$$

Claim 1.4.7. *We have $\tilde{x} \in (C_2)_{I'}$.*

Proof of Claim. The first two coordinates of $\tilde{x}$ are integers, thus we only need to verify $\tilde{x} \in C_2$. Since $x \in C_2$, we have $w_2^* + s^* = f(b_2)$, so:

$$2w_2^* + (2s^* - f(b_2)) = 2f(b_2) - f(b_2) = f(b_2),$$

which satisfies the second constraint. Also, since $s^* \geq f(b_2)$, we have $2s^* - f(b_2) \geq 0$. For the first inequality:

$$2w_1^* + \mu(2s^* - f(b_2)) = (w_1^* + \mu s^*) + (w_1^* + \mu s^* - \mu f(b_2)).$$

Since $x \in Q_3'$, the first term is at least $f(b_1)$, and the second term is nonnegative due to $f(b_1) = \mu f(b_2)$, so the inequality is satisfied. ▲

Observe that

$$x = \frac{\tilde{B}_1 + \tilde{x}}{2} \quad \text{and} \quad \tilde{B}_1, \tilde{x} \in (C_2)_{I'}$$

so x is not an extreme point of $(C_2)_{I'}$. Thus, $\tilde{B}_1$ is the only element of V_2. □

Lemma 1.4.5 implies that $conv(Q_3) = conv(B_3, \tilde{B}_1) + cone(\rho_1, \rho_2, \rho_3)$ when $\mu = \frac{f(b_1)}{f(b_2)}$. In this case, we obtain the following characterization of $conv(Q_3)$.

Theorem 1.4.1. *Assume $\mu = \frac{f(b_1)}{f(b_2)}$, and define the polyhedron $\tilde{Q} \subseteq \mathbb{R}^3$ by the following system:*

$$\begin{aligned} f(b_2)w_1 + s &\geq f(b_2), && (1.24)\\ f(b_2)w_2 + s &\geq f(b_2), && (1.25)\\ w_1 + \mu s &\geq f(b_1),\\ w_2 + s &\geq f(b_2),\\ s &\geq 0. \end{aligned}$$

Then $conv(Q_3) = (Q_3')_{I'} = \tilde{Q}$, where Q_3 is the system defined in (1.21) and Q_3' is its relaxation stated in Definition 1.4.1.

Proof. Substituting $\mu = \frac{f(b_1)}{f(b_2)}$, we see that the inequalities (1.24) and (1.25) correspond exactly to the MIR-type inequalities (1.23) and (1.22). Thus, the inequality description of $\tilde{Q}$ contains all valid inequalities already known for Q_3, implying $conv(Q_3) \subseteq \tilde{Q}$.

Additionally, all constraints in $\tilde{Q}$ are satisfied by the continuous relaxation Q_3', so $\tilde{Q} \subseteq Q_3'$. Since $conv(Q_3) = (Q_3')_{I'}$, we have:

$$rec(conv(Q_3)) = rec(Q_3') = cone(\rho_1, \rho_2, \rho_3),$$

by Theorem 4.30 in [12]. It follows that:

$$rec(\tilde{Q}) = rec(Q_3') = cone(\rho_1, \rho_2, \rho_3).$$

Therefore, $\tilde{Q}$ and $conv(Q_3)$ share the same recession cone. To complete the proof, we observe that $\tilde{Q}$ has exactly two extreme points: $\tilde{B}_1$ and B_3. This can be verified by examining all active constraint combinations that define extreme points. Hence, since the extreme point and recession cone structure of $\tilde{Q}$ matches that of $conv(Q_3)$, we conclude $conv(Q_3) = \tilde{Q}$. □

We now analyze the two cases $\mu < \frac{f(b_1)}{f(b_2)}$ and $\mu > \frac{f(b_1)}{f(b_2)}$, in which the description of $conv(Q_3)$ becomes more intricate. In the first case $\mu < \frac{f(b_1)}{f(b_2)}$, we construct a hyperplane passing through the three linearly independent points B_3, D_1, B_1. In the second case $\mu > \frac{f(b_1)}{f(b_2)}$, we pass a hyperplane through the points B_3, D_1, B_4. The motivation behind this construction is that, in each case, the involved points are candidate extreme points of $(Q'_3)_{I'}$, since they belong to the set $V_1 \cup V_2 \cup V_3$, and from our structural characterization, we know that

$$(Q'_3)_{I'} = conv(V_1 \cup V_2 \cup V_3) + cone(\rho_1, \rho_2, \rho_3).$$

Therefore, if our goal is to identify a facet-defining inequality for $conv(Q_3)$, it is natural to look for hyperplanes that pass through three linearly independent extreme points. This justifies our approach and shows why it geometrically sounds.

Note that in the case $\mu < \frac{f(b_1)}{f(b_2)}$, $D_1 = (1, 0, f(b_2))$, we have the following Theorem 1.4.2:

Theorem 1.4.2. *Assume $\mu < \frac{f(b_1)}{f(b_2)}$. The hyperplane $\mathcal{P}$ passing through the points B_3, D_1, B_1 is given by*

$$\left(\frac{f(b_1)}{\mu} - f(b_2)\right) w_1 + f(b_2)w_2 + s = \frac{f(b_1)}{\mu},$$

where

$$B_3 = (1, 1, 0) \in V_3, \quad D_1 = (\lceil f(b_1) - \mu f(b_2) \rceil, 0, f(b_2)) \in V_2, \quad B_1 = \left(0, 0, \frac{f(b_1)}{\mu}\right) \in V_1,$$

and V_i is the set of extreme points of $(C_i)_{I'}$ for $i = 1, 2, 3$ as defined in Definition 1.4.2. The corresponding inequality

$$\left(\frac{f(b_1)}{\mu} - f(b_2)\right) w_1 + f(b_2)w_2 + s \geq \frac{f(b_1)}{\mu} \tag{1.26}$$

is valid for Q_3 (See (1.21) for the definition), and cuts off the fractional point v of Q'_3 (v is the apex of Q'_3, see Definition 1.4.1).

Proof. Let $(w_1, w_2, s) \in Q_3$. Then, by remark 1.4.1, the following inequalities hold:

$$f(b_2)w_2 + s \geq f(b_2), \tag{1.27}$$

$$\frac{f(b_1)}{\mu} w_1 + s \geq \frac{f(b_1)}{\mu}. \tag{1.28}$$

Rewriting (1.26), we get:

$$\frac{f(b_1)}{\mu} w_1 + s + (w_2 - w_1) f(b_2) \geq \frac{f(b_1)}{\mu}. \tag{1.29}$$

If $w_2 - w_1 \geq 0$, then this inequality follows directly from (1.28). Thus, assume $w_2 \leq w_1 - 1$, and suppose for contradiction that (1.26) does not hold. Then,

$$\left(\frac{f(b_1)}{\mu} - f(b_2) \right) w_1 + f(b_2) w_2 + s < \frac{f(b_1)}{\mu},$$

which is equivalent to

$$s < \frac{f(b_1)}{\mu} + \left(f(b_2) - \frac{f(b_1)}{\mu} \right) w_1 - f(b_2) w_2.$$

Adding w_2 to both sides yields:

$$w_2 + s < \frac{f(b_1)}{\mu} + \left(f(b_2) - \frac{f(b_1)}{\mu} \right) w_1 + (1 - f(b_2)) w_2.$$

The left-hand side is at least $f(b_2)$ by (1.27). Using $w_2 \leq w_1 - 1$ and $1 - f(b_2) > 0$, we obtain:

$$\begin{aligned} f(b_2) \leq w_2 + s &< \frac{f(b_1)}{\mu} + \left(f(b_2) - \frac{f(b_1)}{\mu} \right) w_1 + (1 - f(b_2))(w_1 - 1) \\ &= \left(\frac{f(b_1)}{\mu} - 1 \right) + \left(1 - \frac{f(b_1)}{\mu} \right) w_1 + f(b_2). \end{aligned}$$

Subtracting $f(b_2)$ from both sides implies:

$$0 < \left(1 - \frac{f(b_1)}{\mu} \right) (w_1 - 1).$$

Since $\mu \geq 1$, it follows that $\frac{f(b_1)}{\mu} < f(b_1) < 1$, hence $1 - \frac{f(b_1)}{\mu} > 0$, so the inequality above implies $w_1 > 1$.

Now observe that

$$\left(\frac{f(b_1)}{\mu} - f(b_2) \right) + f(b_2) w_2 + s \leq \left(\frac{f(b_1)}{\mu} - f(b_2) \right) w_1 + f(b_2) w_2 + s,$$

and by assumption, the right-hand side is less than $\frac{f(b_1)}{\mu}$, hence:

$$f(b_2)w_2 + s < f(b_2),$$

which contradicts (1.27). Therefore, the inequality (1.26) is valid for all points in Q_3. □

Note that when $\mu < \frac{f(b_1)}{f(b_2)}$, (1.26) defines a facet defining inequality for $conv(Q_3)$, as it passes 3 linearly independent points of Q_3, B_3, D_1, B_1. See the following example:

Remark 1.4.3. *Let $f(b_1) = \frac{4}{9}, f(b_2) = \frac{1}{15}$, and $\mu = 2.4$. Q_3 is defined as:*

$$\begin{aligned} w_1 + 2.4s &\geq \frac{4}{9} \\ w_2 + s &\geq \frac{1}{15} \\ s &\geq 0 \\ w_1, w_2 &\in \mathbb{Z}. \end{aligned}$$

Note that $\mu < \frac{f(b_1)}{f(b_2)}$. Consider the point $\tilde{x} = (\frac{16}{25}, 0, \frac{1}{15})$. One can verify that $\tilde{x} \in Q_3'$ and satisfies (1.22),(1.23), but does not satisfy (1.26).

Now, assume $\mu > \frac{f(b_1)}{f(b_2)}$. As mentioned earlier, we pass a hyperplane through the points B_3, B_4, D_1 and show that it defines a valid inequality for Q_3, under the assumption

$$(\lceil f(b_1) - \mu f(b_2) \rceil - 1) + \mu f(b_2) \leq 0.$$

We will see in remark 1.4.4 why this assumption is necessary. Let

$$t = \lceil f(b_1) - \mu f(b_2) \rceil - 1.$$

Note that we do not use $\lfloor f(b_1) - \mu f(b_2) \rfloor$ for this quantity, since in the case $f(b_1) - \mu f(b_2) \in \mathbb{Z}$, we have $\lceil f(b_1) - \mu f(b_2) \rceil - 1 \neq \lfloor f(b_1) - \mu f(b_2) \rfloor$.

Theorem 1.4.3. *Assume $\mu > \frac{f(b_1)}{f(b_2)}$, and let $t = \lceil f(b_1) - \mu f(b_2) \rceil - 1$. The hyperplane $\mathcal{P}$ passing through the points B_3, D_1, B_4 is*

$$\frac{f(b_1)}{\mu} w_1 + \left(f(b_2) + \frac{f(b_1)}{\mu} t \right) w_2 + s = f(b_2) + \frac{f(b_1)}{\mu}(t+1),$$

where

$$B_3 = (1,1,0) \in V_3, \quad D_1 = (\lceil f(b_1) - \mu f(b_2) \rceil, 0, f(b_2)) \in V_2, \quad B_4 = \left(0, 1, \frac{f(b_1)}{\mu}\right) \in V_1,$$

and V_i is the set of extreme points of $(C_i)_{I'}$ for $i = 1, 2, 3$ as defined in Definition 1.4.2. The corresponding cutting plane

$$\frac{f(b_1)}{\mu} w_1 + \left(f(b_2) + \frac{f(b_1)}{\mu} t\right) w_2 + s \geq f(b_2) + \frac{f(b_1)}{\mu}(t+1) \tag{1.30}$$

is valid for Q_3 (See (1.21) for the definition) if $t + \mu f(b_2) \leq 0$, and cuts off the fractional point v of Q_3'(v is the apex of Q_3', see Definition 1.4.1).

Proof. Let $(w_1, w_2, s) \in Q_3$. Then, by remark 1.4.1, the following inequalities hold:

$$f(b_2) w_2 + s \geq f(b_2), \tag{1.31}$$

$$\frac{f(b_1)}{\mu} w_1 + s \geq \frac{f(b_1)}{\mu}. \tag{1.32}$$

We rewrite (1.30) as

$$f(b_2) w_2 + s + \frac{f(b_1)}{\mu}(w_1 + t w_2) \geq f(b_2) + \frac{f(b_1)}{\mu}(t+1).$$

If $w_1 + t w_2 \geq t + 1$, then combining this with (1.31) implies that (1.30) holds. So assume instead that

$$w_1 \leq t(1 - w_2).$$

Suppose for contradiction that (1.30) does not hold. Then:

$$s < f(b_2) + \frac{f(b_1)}{\mu}(t+1) - \frac{f(b_1)}{\mu} w_1 - \left(f(b_2) + \frac{f(b_1)}{\mu} t\right) w_2.$$

Multiplying both sides by μ and adding w_1 to both sides gives:

$$w_1 + \mu s < \mu f(b_2) + f(b_1)(t+1) + (1 - f(b_1)) w_1 - (\mu f(b_2) + f(b_1) t) w_2.$$

The left-hand side is at least $f(b_1)$, so:

$$\begin{aligned} f(b_1) \leq w_1 + \mu s &< \mu f(b_2) + f(b_1)(t+1) + (1 - f(b_1)) w_1 - (\mu f(b_2) + f(b_1) t) w_2 \\ &\leq \mu f(b_2) + f(b_1)(t+1) + (1 - f(b_1)) t(1 - w_2) - (\mu f(b_2) + f(b_1) t) w_2 \\ &= f(b_1) + t + \mu f(b_2) - (t + \mu f(b_2)) w_2. \end{aligned}$$

Thus:

$$0 < (t + \mu f(b_2))(1 - w_2).$$

Since $t + \mu f(b_2) \leq 0$, it follows that $1 - w_2 < 0$, i.e., $w_2 > 1$.

Now observe:

$$\frac{f(b_1)}{\mu} w_1 + \left(f(b_2) + \frac{f(b_1)}{\mu} t \right) + s \leq \frac{f(b_1)}{\mu} w_1 + \left(f(b_2) + \frac{f(b_1)}{\mu} t \right) w_2 + s.$$

Combining with (1.32), the left-hand side is greater than

$$f(b_2) + \frac{f(b_1)}{\mu}(t + 1),$$

which contradicts the assumption that the inequality is violated. Therefore, the inequality (1.30) must hold. □

Remark 1.4.4. *In theorem 1.4.3, the condition $t + \mu f(b_2) \leq 0$ is necessary for (1.30) to be valid for Q_3. Consider the example where $f(b_1) = 0.9$, $f(b_2) = 0.3$, and $\mu = 9$. One can verify that the point $(8, 0, \frac{89}{90}) \in Q_3$, but it does not satisfy (1.30). Furthermore, in this example we have $t + \mu f(b_2) > 0$.*

Thus, we have obtained valid inequalities for the system Q_3, corresponding to the cases $\mu < \frac{f(b_1)}{f(b_2)}$ and $\mu > \frac{f(b_1)}{f(b_2)}$. As discussed earlier, both inequalities (1.26) and (1.30) pass through three linearly independent points of Q_3, implying that they define facet-defining inequalities for $conv(Q_3)$. Furthermore, their descriptions differ from the two MIR-type inequalities (1.24) and (1.25). Therefore, when $\mu < \frac{f(b_1)}{f(b_2)}$, the inequality (1.26) is a new facet-defining inequality for $conv(Q_3)$. Likewise, in the case $\mu > \frac{f(b_1)}{f(b_2)}$, provided that (1.30) is valid, it too constitutes a new facet-defining inequality for $conv(Q_3)$.

In the next section, we map the inequalities (1.26) and (1.30) back to obtain valid cutting planes for the mixed-integer system S_2 defined in (1.13). The main open question in this section is whether there exists a facet-defining inequality for $conv(Q_3) = (Q_3')_{I'}$ in the case $\mu > \frac{f(b_1)}{f(b_2)}$ that holds unconditionally. While such an inequality must exist, explicitly identifying it remains a key challenge in fully characterizing $conv(Q_3) = (Q_3')_{I'}$ in this regime. Additionally, characterizing the full description of $conv(Q_3) = (Q_3')_{I'}$ in the regimes $\mu < \frac{f(b_1)}{f(b_2)}$ and $\mu > \frac{f(b_1)}{f(b_2)}$ remains an intriguing and challenging direction, especially in light of the complete characterization already obtained in the degenerate case $\mu = \frac{f(b_1)}{f(b_2)}$.

1.4.3 Mapping the Valid Inequalities for Q_3 Back to the Original System

Recall that the system describing S_2 is:

$$\begin{aligned}
\sum_{i=1}^{n} a_1^{(i)} x^{(i)} &= b_1, \\
\sum_{i=1}^{n} a_2^{(i)} x^{(i)} &= b_2, \\
x^{(i)} &\geq 0 \quad \forall i \in [n], \\
x^{(i)} &\in \mathbb{Z} \quad \forall i \in I,
\end{aligned}$$

and the variables w_1, w_2, s are defined as follows:

$$\begin{aligned}
w_1 &= -\lfloor b_1 \rfloor + \sum_{i \in T} \lfloor a_1^{(i)} \rfloor \, x^{(i)} + \sum_{i \in N_2 \setminus T} \lceil a_1^{(i)} \rceil \, x^{(i)} + \sum_{i \in I \setminus N_2} \lceil a_1^{(i)} \rceil \, x^{(i)}, \\
w_2 &= -\lfloor b_2 \rfloor + \sum_{i \in T} \lfloor a_2^{(i)} \rfloor \, x^{(i)} + \sum_{i \in N_2 \setminus T} \lceil a_2^{(i)} \rceil \, x^{(i)} + \sum_{i \in I \setminus N_2} a_2^{(i)} \, x^{(i)}, \\
s &= \sum_{i \in T} f(a_2^{(i)}) \, x^{(i)} + \sum_{\substack{a_2^{(i)} > 0 \\ i \in C}} a_2^{(i)} \, x^{(i)} + \sum_{\substack{a_2^{(i)} \leq 0 \\ a_1^{(i)} > 0 \\ i \in C}} \frac{a_1^{(i)}}{\mu} x^{(i)}.
\end{aligned}$$

1. $\mu < \frac{f(b_1)}{f(b_2)}$; the inequality (1.26):

$$\left(\frac{f(b_1)}{\mu} - f(b_2) \right) w_1 + f(b_2) w_2 + s \geq \frac{f(b_1)}{\mu}$$

translates into:

$$\sum_{i\in T}\left[\frac{f(b_1)}{\mu}\lfloor a_1^{(i)}\rfloor+\left(f(b_2)+\frac{f(b_1)}{\mu}t\right)\lfloor a_2^{(i)}\rfloor+f(a_2^{(i)})\right]x^{(i)}$$
$$+\sum_{i\in N_2\setminus T}\left[\frac{f(b_1)}{\mu}\lceil a_1^{(i)}\rceil+\left(f(b_2)+\frac{f(b_1)}{\mu}t\right)\lceil a_2^{(i)}\rceil\right]x^{(i)}$$
$$+\sum_{i\in I\setminus N_2}\left[\frac{f(b_1)}{\mu}\lceil a_1^{(i)}\rceil+\left(f(b_2)+\frac{f(b_1)}{\mu}t\right)a_2^{(i)}\right]x^{(i)}+\sum_{\substack{a_2^{(i)}>0\\ i\in C}}a_2^{(i)}x^{(i)}$$
$$+\sum_{\substack{a_2^{(i)}\le 0\\ a_1^{(i)}>0\\ i\in C}}\frac{a_1^{(i)}}{\mu}x^{(i)}\ge f(b_2)+\frac{f(b_1)}{\mu}(t+1)+\frac{f(b_1)}{\mu}\lfloor b_1\rfloor+\left(f(b_2)+\frac{f(b_1)}{\mu}t\right)\lfloor b_2\rfloor.$$

2. $\mu>\frac{f(b_1)}{f(b_2)}$; define:
$$t=\lceil f(b_1)-\mu f(b_2)\rceil-1.$$
Then the inequality (1.30):
$$\frac{f(b_1)}{\mu}w_1+\left(f(b_2)+\frac{f(b_1)}{\mu}t\right)w_2+s\ge f(b_2)+\frac{f(b_1)}{\mu}(t+1)$$
translates into:

$$\sum_{i\in T}\left[\frac{f(b_1)}{\mu}\lfloor a_1^{(i)}\rfloor+\left(f(b_2)+\frac{f(b_1)}{\mu}t\right)\lfloor a_2^{(i)}\rfloor+f(a_2^{(i)})\right]x^{(i)}$$
$$+\sum_{i\in N_2\setminus T}\left[\frac{f(b_1)}{\mu}\lceil a_1^{(i)}\rceil+\left(f(b_2)+\frac{f(b_1)}{\mu}t\right)\lceil a_2^{(i)}\rceil\right]x^{(i)}$$
$$+\sum_{i\in I\setminus N_2}\left[\frac{f(b_1)}{\mu}\lceil a_1^{(i)}\rceil+\left(f(b_2)+\frac{f(b_1)}{\mu}t\right)a_2^{(i)}\right]x^{(i)}+\sum_{\substack{a_2^{(i)}>0\\ i\in C}}a_2^{(i)}x^{(i)}$$
$$+\sum_{\substack{a_2^{(i)}\le 0\\ a_1^{(i)}>0\\ i\in C}}\frac{a_1^{(i)}}{\mu}x^{(i)}\ge f(b_2)+\frac{f(b_1)}{\mu}(t+1)+\frac{f(b_1)}{\mu}\lfloor b_1\rfloor+\left(f(b_2)+\frac{f(b_1)}{\mu}t\right)\lfloor b_2\rfloor.$$

To conclude, we emphasize that while all facet-defining inequalities of $conv(S_2)$ can be generated through the intersection cut procedure, as established by Andersen et al. [2], the explicit inequalities we derived—namely (1.26) and (1.30)—offer concrete analytical expressions that can be directly interpreted and potentially utilized in computational frameworks. Although the intersection cut framework inherently yields a vast collection of valid inequalities (including all facet-defining ones), identifying which specific cuts are most effective remains a significant practical challenge. Our explicit constructions contribute to this space by highlighting cases where meaningful facet-defining inequalities can be written in closed form, thus advancing both theoretical understanding and algorithmic applicability. Related results were also obtained by Cornuéjols and Margot [15], who characterized all facet-defining inequalities of systems similar to $conv(S_2)$. Further work by Basu et al. [5] and Awate et al. [3] studied closures defined by splits, triangles, and quadrilaterals, comparing the strength of these families of inequalities and analyzing how well each approximates the $conv(S_2)$. However, as in Andersen et al., these approaches rely on the entire infinite family of lattice-free convex sets to generate the closure and therefore do not yield a practical or explicit description of a good cutting plane.

In addition to the theoretical contributions, several computational directions emerge: one may study the practical performance of these inequalities in cutting-plane algorithms, evaluate their MIR rank, and investigate how they can be recovered via intersection cut procedures. Extending this approach to uncover other meaningful inequalities for S_2, both analytically and algorithmically, represents a promising direction for future research.

Chapter 2

Minimum Multicut

2.1 Introduction

Let $G = (V(G), E(G))$ be an undirected graph with edge costs given by $c : E(G) \to \mathbb{Z}_+$. We are also given a collection of k source–sink pairs $\{(s_1, t_1), \ldots, (s_k, t_k)\} \subseteq V \times V$. For each $i \in [k]$, let $\mathcal{P}_i$ denote the set of all paths between s_i and t_i in G, and let $\mathcal{P} = \bigcup_{i=1}^{k} \mathcal{P}_i$. A multicut is a subset of edges $F \subseteq E(G)$ such that every path $P \in \mathcal{P}$ contains at least one edge of F, that is,

$$P \cap F \neq \emptyset \quad \text{for all } P \in \mathcal{P}.$$

Equivalently, the removal of F from G disconnects each source s_i from its corresponding sink t_i for all $i \in [k]$. The cost of a multicut F is defined as $c(F) = \sum_{e \in F} c(e)$. Thus, the minimum multicut problem can be written as

$$\min \big\{ c(F) : F \subseteq E(G),\ P \cap F \neq \emptyset \text{ for all } P \in \mathcal{P} \big\}.$$

This problem generalizes the classical minimum s–t cut problem (the case $k = 1$) and has been extensively studied. Notably, computing the minimum multicut is NP-hard, even on simple graph classes such as trees [22].

Closely related to the multicut problem is the multicommodity flow problem (or multiflow), where the objective is to maximize the total flow that can be simultaneously routed between the source–sink pairs. In fact, the natural linear programming (LP) relaxation of the minimum multicut problem is the dual of the LP formulation of the multiflow problem.

For a path $P \in \mathcal{P}$, let

$$x(P) \coloneqq \sum_{e \in P} x(e).$$

The LP relaxation of the minimum multicut problem is:

$$\begin{aligned} \min \quad & \sum_{e \in E(G)} c(e)\, x(e) \\ \text{s.t.} \quad & x(P) \;\geq\; 1 \qquad \forall\, P \in \mathcal{P}, \\ & x(e) \;\geq\; 0 \qquad \forall\, e \in E(G). \end{aligned} \tag{2.1}$$

Even though there are an exponential number of constraints, it is well known that the optimal solution to (2.1) can be computed in polynomial time [21]. We denote the optimal solution of the minimum multicut problem by OPT_{IP}, and the optimal solution of its linear relaxation (2.1) by OPT_{LP}. This thesis focuses on the integrality gap of the multicut LP, defined as

$$\frac{\mathrm{OPT}_{IP}}{\mathrm{OPT}_{LP}}.$$

Before presenting our contributions, we briefly review some of the known results. The celebrated max-flow min-cut theorem [19] states that the integrality gap is exactly 1 when $k = 1$, i.e., for a single source–sink pair. A classic result of Hu [26] extends this and shows that the gap remains 1 when $k = 2$. However, the equality fails when $k \geq 3$, even on very simple graphs (see [22] for an example).

Garg, Vazirani, and Yannakakis [21] established a tight bound of $\Theta(\log k)$ on the integrality gap in general graphs. For trees, the gap is exactly 2 [22]. For K_r-minor-free graphs, Tardos and Vazirani [32] used the decomposition theorem of Klein, Plotkin, and Rao [29] to obtain an upper bound of $\mathcal{O}(r^3)$. Building on a long sequence of results [17, 1, 20, 18], Conroy and Filtser [13] recently proved an asymptotically tight bound of $\Theta(\log r)$ on the integrality gap for K_r-minor-free graphs.

The primary motivation behind the aforementioned works was to establish asymptotic bounds on the integrality gap (as a function of r), without focusing on optimizing the constants involved. However, for specific graph families, such as planar graphs, the constants obtained from these results are quite large (close to 100). Thus, determining the exact integrality gap remains an intriguing open question. It is known that the integrality gap is at least 2 for trees, and consequently for planar graphs as well. Tighter upper and lower bounds for this problem remain elusive, serving as the primary motivation for this thesis. In particular, we provide partial progress toward identifying the integrality gap of the minimum multicut problem in planar graphs.

2.1.1 Overview of Results

We first state and review the small diameter decomposition (SDD) problem in section 2.3 and describe its relation to the integrality gap of the minimum multicut problem. This connection forms the foundation of most of our results.

The previously best known lower bound on the integrality gap of the minimum multicut problem was 2, established already for trees [22]. We improve this by showing that the gap is at least $\frac{20}{9}$ for cactus graphs in section 2.4 and section 2.5. Recall that in a cactus graph, any two simple cycles are either vertex-disjoint or share exactly one vertex. Since cactus graphs are a subclass of planar graphs, this also implies that the gap for planar graphs is strictly larger than 2. Our proof is initially non-constructive, but it provides the structural insight needed to build an explicit cactus instance with gap $\frac{20}{9}$ (see section 2.5.1).

Complementing this lower bound, we prove that the integrality gap in cactus graphs is at most 3.44, improving upon the previous best upper bound of 4 due to Bentz [7] (see section 2.6). We then consider structured subclasses of cactus graphs and obtain tight bounds. In section 2.7, we show that the gap is exactly 2 for unicyclic graphs (graphs containing exactly one cycle). This implies that adding a single extra edge to a tree does not increase the integrality gap, which is somewhat surprising. Insection 2.8, we prove the same tight bound of 2 for path cactus graphs, where all cycles in the cactus lie along a single path.

All of the above results rely on the connection between the integrality gap of the minimum multicut problem and SDD. In section 2.9, we turn to outerplanar graphs (planar graphs in which all vertices lie on the outer face) and prove an upper bound of 8. For this case, we work directly with the minimum multicut problem. Given an optimal fractional solution x^* of the LP relaxation (2.1), we design a randomized rounding algorithm that, in polynomial time, produces a multicut of cost at most $8 \cdot \mathrm{OPT}_{LP}$.

2.2 Preliminaries

Given a graph G, we denote its vertex and edge sets by $V(G)$ and $E(G)$, respectively. We will use K_r to denote the complete graph on r vertices. In this paper, we will only be concerned with planar graphs. A graph G is planar if it does not contain K_5 or $K_{3,3}$ as a minor. Equivalently, a graph is planar if it can be drawn in the plane without any of its edges crossing. Graphs in which every edge is contained in at most one cycle are called cactus graphs. Cactus graphs are a subclass of series-parallel and planar graphs,

and are arguably the simplest family of planar graphs after trees and cycles. Cactus and series-parallel graphs do not contain K_4 as a minor.

Given two arbitrary vertices $u, v \in V(G)$, we use $d_G(u, v)$ to denote the shortest path distance between u and v in G, if G is clear in the context then we use $d(u, v)$ for simplicity. The diameter of G is the maximum distance between a pair of vertices in G, i.e., $\text{diam}(G) = \max_{u,v \in V(G)} d_G(u, v)$. We use $d_G(v, e)$ to denote the distance of a vertex v from an edge $e = (x, y)$, i.e., $d_G(v, e) = \min\{d_G(v, x), d_G(v, y)\}$.

For $F \subseteq E(G)$, we use $G \setminus F$ to denote the remaining graph after the removal of F from G. For any $v \in V(G)$, we use $C_F(v)$ to denote the connected component of $G \setminus F$ containing v. We overload notation and also use $C_F(v)$ to denote the set of vertices in the connected component containing v. We define the radius of v with respect to F as the distance of the farthest vertex from v in $C_F(v)$, i.e., $\text{rad}_F(v) = \max_{u \in C_F(v)} d_G(v, u)$. In addition, the diameter of F is the maximum diameter of a connected component after the removal of F from G, i.e., $\text{diam}(F) = \max_{v \in V(G)} \text{diam}(C_F(v))$. Given $t \in \mathbb{R}_{\geq 0}$ as a parameter, we say that F forms a t-diameter decomposition if $\text{diam}(F) < t$. We denote the set of all t-diameter decompositions of G by $\mathcal{F}_t(G)$. Note that when referring to the distance between two vertices u, v in a component C, $d_G(u, v)$ denotes their distance in G, rather than in the subgraph induced by C, i.e., $G[C]$. Given a cycle C and a path P, we denote by $\text{len}(C)$ and $\text{len}(P)$ the number of edges in C and P, respectively.

Definition 2.2.1. *Let $P = (v_1, \ldots, v_n)$ be a path on n vertices. For any $i \neq j$, we denote by $P[v_i, v_j]$ the subpath of P between v_i and v_j (inclusive).*

Definition 2.2.2. *Let $P = (v_1, \ldots, v_n)$ and $Q = (v_n, \ldots, v_{n+m})$ be two distinct paths such that they share the vertex v_n. We denote by $P \oplus Q$ the path from v_1 to v_{n+m} obtained by concatenating P and Q.*

We recall a few definitions from the graph theory textbook by Diestel [16]. A *block* is a maximal connected subgraph without a cut-vertex (of that subgraph). Thus, every block is either a maximal 2-connected subgraph, or a cut-edge (with its ends), or an isolated vertex. The block graph of a graph G is a bipartite graph on $A \cup B$, where A is the set of cut-vertices of G and B is the set of blocks of G, and the block graph has an edge ab (where $a \in A$, $b \in B$) iff $a \in V(B)$.

A cactus is a connected graph such that every block is a cycle or else a cut-edge (with its ends). A path cactus is a cactus such that the block graph is a path, and there are no cut-edges. A unicyclic graph is a connected graph that has at most one cycle. We will denote these classes of graphs by CACTUS, PATH CACTUS and UNICYCLIC respectively.

2.3 Integrality Gap and Small Diameter Decomposition

We now formally define the integrality gap of the minimum multicut problem on a family of graphs.

Definition 2.3.1. *Let $\mathcal{G}$ be a family of graphs, and let $\mathcal{M}(\mathcal{G})$ denote the family of all instances of the minimum multicut problem on $\mathcal{G}$, obtained by assigning arbitrary costs to the edges and selecting a set of source–sink pairs. The integrality gap $\alpha_{\mathcal{M}(\mathcal{G})}$ of the minimum multicut problem on $\mathcal{M}(\mathcal{G})$ is defined as*

$$\alpha_{\mathcal{M}(\mathcal{G})} := \max_{M \in \mathcal{M}(\mathcal{G})} \frac{\mathrm{OPT}_{IP}(M)}{\mathrm{OPT}_{LP}(M)},$$

where $\mathrm{OPT}_{IP}(M)$ is the optimal value of the integer program and $\mathrm{OPT}_{LP}(M)$ is the optimal value of its linear relaxation (2.1).

As mentioned in the introduction, $\alpha_{\mathcal{M}(\textsc{tree})} = 2$ [22], where TREE denotes the family of all trees, and $\alpha_{\mathcal{M}(\textsc{planar})} = \mathcal{O}(1)$ [29], where PLANAR denotes the family of all planar graphs.

Let $\mathcal{G}$ be a family of graphs closed under taking minors and under edge subdivisions, and let $\mathcal{M}(\mathcal{G})$ denote the corresponding family of minimum multicut instances. Theorem 2.3.1, which is a direct application of the work of Carr and Vempala [10] to the minimum multicut problem, shows that any feasible fractional solution to the LP relaxation can be approximately represented as a convex combination of feasible multicuts.

Theorem 2.3.1. *Suppose we are given an instance $M \in \mathcal{M}(\mathcal{G})$. Let $\mathcal{F} \subseteq 2^{E(G)}$ be the set of all feasible multicuts for M, and let x be a feasible fractional solution to the LP relaxation. Then there exists a probability distribution y over $\mathcal{F}$ such that*

$$\sum_{\substack{F \in \mathcal{F} \\ e \in F}} y_F \;\leq\; \alpha_{\mathcal{M}(\mathcal{G})} \cdot x(e) \quad \forall e \in E(G).$$

Proof. Suppose the statement does not hold. Then the following linear system (2.2) is infeasible.

$$\sum_{F \in \mathcal{F}} y_F = 1 \tag{2.2}$$

$$\sum_{\substack{F \in \mathcal{F} \\ e \in F}} y_F \leq \alpha \cdot x(e) \qquad \forall e \in E(G) \tag{2.3}$$

$$y_F \geq 0 \tag{2.4}$$

This implies that the following system (2.7) is infeasible as well. The reason is that if the system below is feasible, then we can scale down the feasible solution appropriately and obtain a feasible solution for the system above.

$$\sum_{F \in \mathcal{F}} y_F \geq 1, \tag{2.5}$$

$$\sum_{\substack{F \in \mathcal{F} \\ e \in F}} y_F \leq \alpha \cdot x(e) \quad \forall\, e \in E(G), \tag{2.6}$$

$$y_F \geq 0 \quad \forall\, F \in \mathcal{F}. \tag{2.7}$$

By reversing the inequality (2.5), we obtain that the following system (2.8) is also infeasible,

$$\sum_{F \in \mathcal{F}} (-y_F) \leq -1, \tag{2.8}$$

$$\sum_{\substack{F \in \mathcal{F} \\ e \in F}} y_F \leq \alpha \cdot x(e) \quad \forall\, e \in E(G), \tag{2.9}$$

$$y_F \geq 0 \quad \forall\, F \in \mathcal{F}. \tag{2.10}$$

Now, we use the following variant of Farkas Lemma (See [31] for a proof).

Lemma 2.3.1. *$\{x \in \mathbb{R}^n | Ax \leq b, x \geq 0\} = \emptyset$ iff there exists a vector u such that $A^T u \geq 0, u \geq 0$ and $b^T u < 0$.*

For a feasible multicut F, let $\chi_F \in \{0,1\}^E$ denote its indicator vector. By Lemma 2.3.1, there exists $u \geq 0$ and $c \geq 0$ such that $c^T \chi_F - u \geq 0$ for all $F \in \mathcal{F}$ and $-u + \alpha \cdot c^T x < 0$. This means that $c^T \chi_F \geq u$ for all $F \in \mathcal{F}$, and $\alpha \cdot c^T x < u$. Thus, $c^T \chi_F \geq u > \alpha \cdot c^T x$ for all $F \in \mathcal{F}$. Therefore with respect to the cost function c, $\mathrm{OPT}_{LP} \leq \frac{u}{\alpha}$ and $\mathrm{OPT}_{IP} > u$. This implies that integrality gap of the multicut instance M is $> \alpha$, a contradiction. □

To connect this with the notion of small diameter decompositions, we now give the formal definition.

Definition 2.3.2 (Small Diameter Decomposition (SDD))**.** *Given an unweighted graph G, an integer parameter $k \in \mathbb{N}$, and a probability parameter $0 < p < 1$, the small diameter decomposition (SDD) problem asks whether there exists a probability distribution*

$\mathcal{D} = \{y_F\}_{F \in \mathcal{F}_k(G)}$ *over* $\mathcal{F}_k(G)$*, the family of* k*-diameter decompositions of* G*, such that every edge* $e \in E(G)$ *is included in a random* k*-diameter decomposition sampled from* $\mathcal{D}$ *with probability at most* p*, that is,*

$$\sum_{\substack{F \in \mathcal{F}_k(G) \\ e \in F}} y_F \leq p \quad \text{for all } e \in E(G).$$

If such a distribution exists, we denote it by $SDD(G, k, p)$ *(See Figure 2.1). Moreover, a family of graphs* $\mathcal{G}$ *is said to be* $SDD(k, p)$*-acceptable if for every* $G \in \mathcal{G}$ *there exists an* $SDD(G, k, p)$*.*

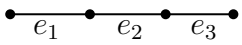


Figure 2.1: G is a simple path with 3 edges, and $k = 2$. Let $F_1 = \{e_1, e_2\}, F_2 = \{e_2\}, F_3 = \{e_2, e_3\}, F_4 = \{e_1, e_3\}$. One can see that $\mathcal{F}_2(G) = \{F_1, F_2, F_3, F_4, E(G)\}$. Let $y_{F_1} = y_{F_3} = y_{E(G)} = 0$, and $y_{F_2} = y_{F_4} = \frac{1}{2}$. This distribution is a $SDD(G, 2, \frac{1}{2})$

.

In the following Theorem 2.3.2, we make explicit the relation between the integrality gap $\alpha_{\mathcal{M}(\mathcal{G})}$ and the existence of suitable SDDs for graph families $\mathcal{G}$ closed under minors and edge subdivisions. The forward implication is a direct consequence of Theorem 2.3.1, which itself follows straightforwardly from the work of Carr and Vempala [10] in the context of the minimum multicut problem, and is therefore not our contribution. The backward implication similarly relies on the simple subdivision technique used in [32], and again is not part of our new contributions. Theorem 2.3.2 merely states this relation explicitly.

Theorem 2.3.2. *Let* $\mathcal{G}$ *be a family of graphs closed under minors and edge subdivisions, and let* α *be a parameter. Then*

$$\alpha_{\mathcal{M}(\mathcal{G})} \leq \alpha \quad \Longleftrightarrow \quad \forall G \in \mathcal{G},\ \forall w \in \mathbb{N},\ \exists\, SDD\big(G, 2w, \tfrac{\alpha}{2w}\big)\,.$$

Proof. Forward direction. Assume $\alpha_{\mathcal{M}(\mathcal{G})} \leq \alpha$, and let $G \in \mathcal{G}$ be arbitrary and $w \in \mathbb{N}$. We define a multicut instance M on G and apply Theorem 2.3.1. Let

$$S = \{(u, v) \in V(G) \times V(G) \mid d_G(u, v) \geq 2w\}$$

be the set of source–sink pairs, and assign unit costs to all edges. Define the fractional solution x by setting $x(e) = \frac{1}{2w}$ for all $e \in E(G)$. This is easily seen to be a feasible solution

to the LP relaxation (2.1). By Theorem 2.3.1, there exists a probability distribution y over feasible multicuts of M such that each edge $e \in E(G)$ is cut with probability at most

$$\alpha_{\mathcal{M}(\mathcal{G})} \cdot \tfrac{1}{2w} \;\leq\; \tfrac{\alpha}{2w}.$$

Finally, note that a set of edges is a feasible multicut for this instance if and only if it defines a $2w$-diameter decomposition of G. This yields the desired $SDD(G, 2w, \frac{\alpha}{2w})$.

Backward direction. Assume that for every $G \in \mathcal{G}$ and $w \in \mathbb{N}$ there exists $SDD(G, 2w, \frac{\alpha}{2w})$. Let M be an arbitrary instance of the minimum multicut problem on $G \in \mathcal{G}$ with edge costs $c : E(G) \to \mathbb{Z}_+$ and source–sink pairs $\{(s_i, t_i)\}_{i=1}^{k}$. We will show that

$$\frac{\mathrm{OPT}_{IP}(M)}{\mathrm{OPT}_{LP}(M)} \leq \alpha.$$

Let $x = \{x_e\}_{e \in E(G)}$ be an optimal fractional solution to the LP relaxation (2.1). Since LP solutions are rational, we may assume $x_e \in \mathbb{Q}$ for all e. Define the support of x as

$$A = \{e \in E(G) \mid x_e > 0\}.$$

Choose $w \in \mathbb{N}$ such that $2wx_e$ is an integer for every $e \in A$. Construct a new graph $S(G, M)$ by contracting every edge $e \notin A$ and replacing each edge $e \in A$ with a path P_e of length $2wx_e$. Since $\mathcal{G}$ is closed under minors and edge subdivisions, we have $S(G, M) \in \mathcal{G}$.

By assumption, there exists $SDD(S(G, M), 2w, \frac{\alpha}{2w})$. This means there is a probability distribution y over $\mathcal{F}_{2w}(S(G, M))$ (the family of edge sets inducing components of diameter at most $2w - 1$) such that

$$\sum_{\substack{F \in \mathcal{F}_{2w}(S(G,M)) \\ e \in F}} y_F \;\leq\; \tfrac{\alpha}{2w}, \quad \forall e \in E(S(G, M)).$$

Claim 2.3.1. *For the multicut instance M, we have*

$$\frac{\mathrm{OPT}_{IP}(M)}{\mathrm{OPT}_{LP}(M)} \leq \alpha.$$

Proof of Claim. For each $F \in \mathcal{F}_{2w}(S(G, M))$, define

$$g(F) = \{e \in E(G) \mid F \cap E(P_e) \neq \varnothing\}.$$

$g(F)$ is a feasible multicut for the instance M. The reason is as follows. Suppose, for contradiction, that there exists a source-sink pair (s_1, t_1) and a path $P \in \mathcal{P}_1$ connecting

them in $G \setminus g(F)$ such that $E(P) \cap g(F) = \emptyset$. Since the LP solution x satisfies $\sum_{e \in E(P)} x_e \geq 1$, the corresponding path P' in $S(G, M)$ (formed by replacing each $e \in E(P)$ with the path P_e) has length at least $2w$. The assumption $E(P) \cap g(F) = \emptyset$ implies that $E(P') \cap F = \emptyset$, and thus s_1 and t_1 remain connected in $S(G, M) \setminus F$. This contradicts the assumption that $F \in \mathcal{F}_{2w}(S(G, M))$, as no connected component in the decomposition can have diameter $\geq 2w$.

Now, let $B = \{g(F) \mid F \in \mathcal{F}_{2w}(S(G, M))\}$, and for each $b \in B$, define

$$y'_b = \sum_{\substack{F \in \mathcal{F}_{2w}(S(G,M)) \\ g(F)=b}} y_F.$$

Then $y' = \{y'_b\}_{b \in B}$ is a probability distribution over multicuts in G. We now analyze the expected cost of a multicut drawn from this distribution. For any edge $e \in E(G)$, if $x_e = 0$, then e was contracted and does not appear in $S(G, M)$, so:

$$\sum_{a \ni e,\, b \in B} y'_b = 0.$$

If $e \in A$, then:

$$\sum_{b \ni e,\, b \in B} y'_b = \sum_{e' \in P_e} \sum_{\substack{F \ni e' \\ F \in \mathcal{F}_{2w}(S(G,M))}} y_F \leq |E(P_e)| \cdot \frac{\alpha}{2w} = \alpha x_e.$$

Therefore, in expectation, each edge e appears in a randomly sampled multicut with probability at most αx_e. This implies there exists a multicut $b \in B$ such that

$$\sum_{e \in b} c_e \leq \alpha \sum_{e \in E(G)} c_e x_e = \alpha \cdot OPT_{LP}(M),$$

which completes the proof. ▲

□

The transition to the SDD framework eliminates the dependence on the specific placement of source–sink pairs and edge costs, which could otherwise be arbitrary, and instead provides a uniform way of analyzing the integrality gap. Theorem 2.3.2 serves as the bridge between SDDs and the integrality gap: its forward direction will be used in section 2.5 to establish lower bounds for cactus graphs, while its backward direction will be applied in

section 2.6, section 2.7, and section 2.8 to derive upper bounds for cactus, unicyclic, and path cactus graphs.

In the backward direction, it is sufficient to guarantee the existence of $SDD(S(G,M), 2w, \frac{\alpha}{2w})$ for the particular choice of w arising in the construction. For our upper-bound analysis, we select w so that each quantity wx_e is not only integral but also even. This ensures that, after transforming G into $S(G,M)$, the distance between any two original vertices of G is even—a technical simplification that will play an important role in the arguments that follow.

To prove the existence of the required SDDs, we explicitly design randomized algorithms. Each of these algorithms produces a $2w$-diameter decomposition with carefully controlled probabilities of cutting edges. This in turn certifies the existence of an appropriate SDD and, by Theorem 2.3.2, yields the desired bounds on the integrality gap. In the following, we present the outlines of these algorithms and explain how they establish the corresponding bounds.

- **Algorithm 1:** Given a cactus graph G in which every cycle has even length, the algorithm removes each edge $e \in E(G)$ with probability at most $\left(1 + \frac{1}{2\ln 2}\right) \cdot \frac{1}{w} \approx 1.721 \cdot \frac{1}{w}$. This shows that there exists an $SDD(G, 2w, 1.721 \cdot \frac{1}{w})$. By Theorem 2.3.2, this implies that

$$\alpha_{\text{CACTUS}} < 3.45.$$

- **Algorithm 2:** For an unicyclic graph G in which the unique cycle has even length, the algorithm removes each edge $e \in E(G)$ with probability at most $\frac{1}{w}$. Thus there exists an $SDD(G, 2w, \frac{1}{w})$, and Theorem 2.3.2 gives

$$\alpha_{\text{UNICYCLIC}} \leq 2.$$

  Since unicyclic graphs contain trees as subgraphs (for which the gap is known to be 2), it follows that $\alpha_{\text{UNICYCLIC}} = 2$ [22].

  Remark 2.3.1. *In section 2.7, we do not explicitly construct an $SDD(G, 2w, \frac{1}{w})$, but instead prove an equivalent statement that implies its existence. The details will be elaborated in that section.*

- **Algorithm 3:** For a path cactus graph G in which all cycles have even length, the algorithm removes each edge $e \in E(G)$ with probability at most $\frac{1}{w}$. Hence there exists an $SDD(G, 2w, \frac{1}{w})$, and Theorem 2.3.2 yields

$$\alpha_{\text{PATH CACTUS}} \leq 2.$$

In section 2.8, we provide an explicit instance of the minimum multicut problem showing that the gap is at least 2 in path cactus graphs, and therefor $\alpha_{\text{PATH CACTUS}} = 2$

As mentioned earlier, $\alpha_{\mathcal{M}(\text{TREE})} = 2$. By Theorem 2.3.2, this implies that for any tree T and any integer $w \in \mathbb{N}$, there exists

$$SDD\big(T, 2w, \tfrac{2}{2w} = \tfrac{1}{w}\big)\,.$$

In other words, the family of trees is $SDD(2w, \frac{1}{w})$-acceptable for any $w \in \mathbb{N}$. Moreover, without directly appealing to Theorem 2.3.2, we can explicitly construct such an $SDD(T, 2w, \frac{1}{w})$. This explicit construction will serve as a foundation for developing intuition regarding structured small-diameter decompositions in the next section 2.4.

2.3.1 Small Diameter Decomposition for Trees

Theorem 2.3.3. *Let T be a tree. Then for every integer $w \in \mathbb{N}$, there exists*

$$SDD\big(T, 2w, \tfrac{1}{w}\big)\,.$$

Equivalently, there exists a probability distribution $\mathcal{D} = \{y_F\}_{F \in \mathcal{F}_{2w}(T)}$ over $\mathcal{F}_{2w}(T)$ such that

$$\sum_{\substack{F \in \mathcal{F}_{2w}(T) \\ e \in F}} y_F \;\leq\; \frac{1}{w} \quad \forall e \in E(T). \tag{2.11}$$

Proof. Root the tree T at an arbitrary vertex $r \in V(T)$. For $i = 0, \ldots, w-1$, define

$$F_i = \{\, e \in E(T) \mid d(r, e) = i + kw \text{ for some } k \in \mathbb{Z}_{\geq 0} \,\}.$$

Set $y_{F_i} = \frac{1}{w}$ for each $i = 0, \ldots, w-1$, and $y_F = 0$ otherwise. Note that the sets F_i partition $E(T)$: we have $E(T) = \bigcup_{i=0}^{w-1} F_i$ and $F_i \cap F_j = \emptyset$ for $i \neq j$. Thus,

$$\sum_{\substack{F \in \mathcal{F}_{2w}(T) \\ e \in F}} y_F = \sum_{\substack{i=0 \\ e \in F_i}}^{w-1} y_{F_i} = \tfrac{1}{w}, \quad \forall e \in E(T).$$

It remains to show that each F_i is a valid $2w$-diameter decomposition. Fix F_i, and consider a pair of vertices (u, v) with $d(u, v) \geq 2w$. Let q be the lowest common ancestor of u and v.

The unique u–v path consists of the u–q path and the q–v path. Since $d(u,v) \geq 2w$, one of these subpaths has length at least w. Without loss of generality, suppose $d(q,v) \geq w$, and denote this path by $Q = e_0, e_1, \ldots, e_p$. Because q is an ancestor of v, we have $d(r,e_i) = d(r,e_{i-1}) + 1$ for $i = 1, \ldots, p$. Hence there exists some $e_j \in Q$ such that $d(r,e_j) \equiv i \pmod{w}$, i.e., $e_j \in F_i$. Removing F_i therefore separates u and v, as required. This shows that F_i defines a $2w$-diameter decomposition, and hence $\mathcal{D}$ is a valid $SDD(T, 2w, 1/w)$. □

The $2w$-diameter decompositions $F_0, \ldots, F_{w-1}$ described in the proof of Theorem 2.3.3 will be useful in the remainder of the thesis, so we record a formal definition.

Definition 2.3.3. *Let $w \in \mathbb{N}$, and let T be a tree with a distinguished root vertex $r \in V(T)$. For each $i = 0, 1, \ldots, w-1$, define*

$$F_w^i(T,r) := \{e \in E(T) \mid d_T(r,e) \equiv i \pmod{w}\},$$

where $d_T(r,e)$ denotes the distance from r to the closer endpoint of e. Then $\{F_w^i(T,r)\}_{i=0}^{w-1}$ forms a partition of $E(T)$. Moreover, each $F_w^i(T,r)$ defines a $2w$-diameter decomposition of T, and the connected component containing the root r has radius at most i from r, that is,

$$\mathrm{rad}_{F_w^i(T,r)}(r) \leq i \leq w-1.$$

Thus, the $SDD(T, 2w, 1/w)$ constructed in the proof of Theorem 2.3.3 also satisfies this useful structural property, which we highlight next.

Observation 2.3.1. *For the $2w$-diameter decompositions $F_0, \ldots, F_{w-1}$ described in the proof of Theorem 2.3.3, the following properties hold:*

1. *For every $i = 0, \ldots, w-1$, we have*

   $$rad_{F_i}(r) \leq i \leq w-1.$$

   Equivalently,

   $$\sum_{F : rad_F(r) \leq w-1} y_F = 1.$$

2. *For all $1 \leq k \leq w$, we have $rad_{F_i}(r) \leq k-1$ with probability $\frac{k}{w}$. More precisely,*

   $$\sum_{F_i : rad_{F_i}(r) \leq k-1} y_{F_i} = \sum_{i=0}^{k-1} y_{F_i} = \frac{k}{w} \geq 1 - \frac{2}{2w}(w-k) = 1 - \frac{\alpha_{\mathcal{M}(\text{TREE})}}{2w}(w-k).$$

This observation shows that the $SDD(T, 2w, 1/w)$ established in Theorem 2.3.3 not only meets the basic edge condition of equation (2.11), but also enjoys additional structural properties. These strengthened features are captured in the following Corollary 2.3.1. In the next section 2.4, we will extend this idea and show that a similar phenomenon holds for families of graphs closed under the 1-sum operation, a property that the family of trees also satisfies.

Corollary 2.3.1. *Let T be a tree, and let $r \in V(T)$ be an arbitrary root. For the family $\mathcal{F}_{2w}(T)$ of all $2w$-diameter decompositions, there exists an $SDD(T, 2w, 1/w)$, i.e., a distribution $\mathcal{D} = \{y_F\}_{F \in \mathcal{F}_{2w}(T)}$, with the following additional properties:*

$$\sum_{F: rad_F(r) \leq w-1} y_F = 1,$$

$$\sum_{F: rad_F(r) \leq k-1} y_F \geq 1 - \frac{\alpha_{\mathcal{M}(\text{TREE})}}{2w}(w-k) \quad \forall k = 1, \ldots, w.$$

2.4 A Structural Result for Small Diameter Decompositions

We now define the 1-sum operation on graphs, which will play a crucial role going forward. Let $G_1, \ldots, G_k$ be non-empty graphs, and let $r_i \in V(G_i)$ for $i = 1, \ldots, k$. The graph G^S is obtained by taking the disjoint union of $G_1, G_2, \ldots, G_k$, and identifying the vertices $r_1, r_2, \ldots, r_k$. We say that G^S is obtained by performing the 1-sum of the G_i's at the vertices r_i's. The vertex $r = r_1 = \cdots = r_k$ is called the main vertex of G^S. See Figure 2.2 for an illustration.

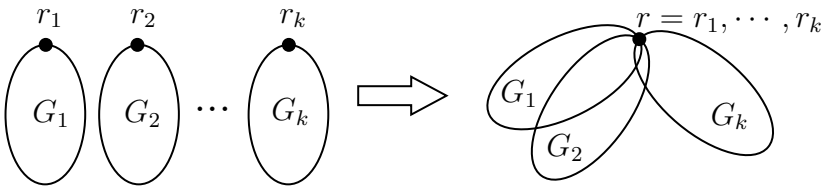


Figure 2.2: An illustration of the 1-sum operation

Let $\mathcal{G}$ be a family of graphs. We say that $\mathcal{G}$ is closed under the 1-sum operation if for any $G_1, \ldots, G_k \in \mathcal{G}$ and $r_i \in G_i$, the graph obtained by taking 1-sum of $G_1, \ldots, G_k$ at

$r_1, r_2, \ldots, r_k$ is a graph in $\mathcal{G}$. Many natural classes of family are closed under the 1-sum operation, such as trees, cactus graphs and planar graphs. Note that 1-sum is a special case of a well known and a more general notion of clique-sums.

In the remainder of this section, let $w \in \mathbb{N}$ and $0 < p < 1$ be fixed parameters, and assume that the graph family $\mathcal{G}$ is closed under minors, subdivisions, and the 1-sum operation, and is $SDD(2w, p)$-acceptable. Note that the existence of an $SDD(G, 2w, p)$ implies that one can sample a $2w$-diameter decomposition of G in which each edge is included with probability at most p.

We note down a few more definitions before stating the main theorem of this section. Let $G \in \mathcal{G}$ be a graph and $r \in V(G)$ be an arbitrary vertex. Recall that $\mathcal{F}_{2w}(G)$ denotes the set of all $2w$-diameter decompositions of G. For $k \in \{1, \ldots, w\}$, we use $\mathcal{F}^k_{2w}(G, r)$ to denote the set of all $2w$-diameter decompositions of G such that every vertex in the connected component containing r is within distance strictly less than k from it. More precisely,

$$\mathcal{F}^k_{2w}(G, r) = \{F \in \mathcal{F}_{2w}(G) \mid rad_F(r) < k\}.$$

In Theorem 2.4.1, we show that if $\mathcal{G}$ is closed under the 1-sum operation and is $SDD(2w, p)$-acceptable, then for any $G \in \mathcal{G}$ and $r \in V(G)$, there exists an $SDD(G, 2w, p)$ over the family of $2w$-diameter decompositions $\mathcal{F}_{2w}(G)$ such that, when sampling a decomposition $F \in \mathcal{F}_{2w}(G)$ from this distribution, we are guaranteed that $\mathrm{rad}_F(r) \leq w - 1$, i.e., $F \in \mathcal{F}^w_{2w}(G, r)$. This condition is directly analogous to the first item of Observation 2.3.1.

Theorem 2.4.1. *Suppose that $\mathcal{G}$ is closed under the 1-sum operation and is $SDD(2w, p)$-acceptable. Let $G \in \mathcal{G}$ and let $r \in V(G)$ be an arbitrary vertex. Then there exists an $SDD(G, 2w, p)$, i.e., a distribution $y = \{y_F \mid F \in \mathcal{F}_{2w}(G)\}$, such that*

$$\sum_{F \in \mathcal{F}^w_{2w}(G,r)} y_F = 1,$$

meaning that every sampled $2w$-diameter decomposition F from this distribution satisfies $rad_F(r) \leq w - 1$.

Proof. Let $\mathcal{F}_{2w} = \mathcal{F}_{2w}(G)$ and $\mathcal{F}^w_{2w}(r) = \mathcal{F}^w_{2w}(G, r)$ for simplicity. It is sufficient to show that the following LP (2.12) is feasible and has optimal value 0.

$$\begin{aligned} \min \quad & \sum_{F \in \mathcal{F}_{2w} \setminus \mathcal{F}^w_{2w}(r)} y_F \\ & \sum_{\substack{F \in \mathcal{F}_{2w} \\ e \in F}} y_F \leq p \quad \forall e \in E(G) \\ & \sum_{F \in \mathcal{F}_{2w}} y_F = 1 \\ & y_F \geq 0 \quad \forall F \in \mathcal{F}_{2w} \end{aligned} \tag{2.12}$$

The above LP (2.12) is feasible since $\mathcal{G}$ is $SDD(2w, p)$-acceptable. For the sake of contradiction, assume that the optimal value of the above LP is $z > 0$. Let $m > \frac{1}{z}$ be a natural number. Let $G_1, \ldots, G_m$ be m disjoint copies of G and r_i be the vertex of G_i which corresponds to r. Let G' be formed by taking 1-sum of $G_1, \ldots, G_m$ at $r_1, \ldots, r_m$. See the following Figure 2.3 for an illustration.

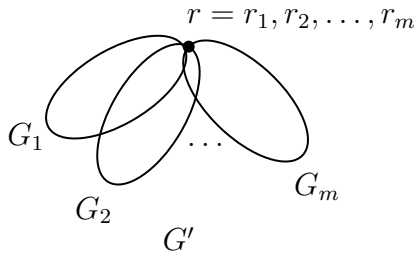


Figure 2.3: Illustration for construction of G' by taking 1-sum of $G_1, G_2, \ldots, G_m$ at $r_1, r_2, \ldots, r_m$.

Note that $G' \in \mathcal{G}$ since $\mathcal{G}$ is closed under the 1-sum operation. Let $\mathcal{F}'_{2w} = \mathcal{F}_{2w}(G')$ be the set of all $2w$-diameter decompositions of G'. Since $\mathcal{G}$ is $SDD(2w, p)$-acceptable, then G' has a $SDD(G', 2w, p)$. Let $\{g_{F'}\}_{F' \in \mathcal{F}_{2w}(G')}$ denote such a distribution for G'. Let $G_i = (V_i, E_i)$ and $\mathcal{F}_{2w}(G_i)$ be the set of all $2w$-diameter decompositions of G_i for $i = 1, 2, \ldots, m$.[1] The distribution g over $\mathcal{F}'_{2w}$ defines a distribution g^i over $\mathcal{F}_{2w}(G_i)$ as follows:

$$g^i_F = \sum_{\substack{F' \in \mathcal{F}'_{2w} \\ F' \cap E_i = F}} g_{F'} \quad \text{for all} \quad F \in \mathcal{F}_{2w}(G_i).$$

[1]Note that $\bigcap_{i=1}^m V_i = \{r\}$.

Since g^i is the projection of g onto G_i, and g is an $SDD(G', 2w, p)$, it follows that g^i is also an $SDD(G_i, 2w, p)$. The following claim states this more generally.

Claim 2.4.1. *Let G be a graph and let $x = \{x_{F'}\}_{F' \in \mathcal{F}_{2w}(G)}$ be an $SDD(G, 2w, p)$. Let H be a subgraph of G. For any $F \in \mathcal{F}_{2w}(H)$, define*

$$y_F \;=\; \sum_{\substack{F' \in \mathcal{F}_{2w}(G) \\ F' \cap E(H) = F}} x_{F'}.$$

Then $y = \{y_F\}_{F \in \mathcal{F}_{2w}(H)}$ is an $SDD(H, 2w, p)$.

Proof of Claim. For any edge $e \in E(H)$, we have

$$\sum_{\substack{F \in \mathcal{F}_{2w}(H) \\ e \in F}} y_F = \sum_{\substack{F \in \mathcal{F}_{2w}(H) \\ e \in F}} \sum_{\substack{F' \in \mathcal{F}_{2w}(G) \\ F' \cap E(H) = F}} x_{F'} = \sum_{\substack{F' \in \mathcal{F}_{2w}(G) \\ e \in F'}} x_{F'} \;\leq\; p.$$

Moreover, $y \geq 0$ and y is a probability distribution since

$$\sum_{F \in \mathcal{F}_{2w}(H)} y_F = \sum_{F \in \mathcal{F}_{2w}(H)} \sum_{\substack{F' \in \mathcal{F}_{2w}(G) \\ F' \cap E(H) = F}} x_{F'} = \sum_{F' \in \mathcal{F}_{2w}(G)} x_{F'} = 1.$$

Thus, y defines an $SDD(H, 2w, p)$. ▲

Since each G_i is an identical copy of G, and g^i is a feasible solution to the extended LP (2.12) above, we have

$$\sum_{F \in \mathcal{F}_{2w}(G_i) \setminus \mathcal{F}^w_{2w}(G_i, r)} g^i_F \;\geq\; z \quad \text{for all } i = 1, 2, \ldots, m.$$

Recall that $\mathcal{F}^w_{2w}(G_i, r)$ denotes the set of all $2w$-diameter decompositions of G_i in which the distance of every vertex in the connected component containing r is at most $w-1$ from it. Let T_i be the event that, when sampling $F' \in \mathcal{F}'_{2w}$ according to the distribution g, the intersection $F' \cap E_i$ does not belong to $\mathcal{F}^w_{2w}(G_i, r)$. From the above discussion, it follows that $\Pr[T_i] \geq z$. Since $m > \frac{1}{z}$, we have

$$\sum_{i=1}^{m} \Pr[T_i] \geq z \cdot m > 1.$$

This implies that the events $T_1, \ldots, T_m$ are not disjoint, and there exist indices i, j such that $\Pr[T_i \cap T_j] \neq \emptyset$. Therefore, there exists a $F' \in \mathcal{F}'_{2w}$, and vertices $u \in V_i$, $v \in V_j$ such that:

1. $g_{F'} > 0$,
2. u and v are in the connected component containing r in $G' - F'$,
3. the distance of u and v from r is at least w.

But then, the diameter of F' is at least $2w$, which contradicts the fact that $F' \in \mathcal{F}'_{2w}$. This implies that $z = 0$, and completes the proof of the theorem. □

We say that a graph G together with a vertex $r \in V(G)$ has an $SDD(G, 2w, p, r)$ if there exists an $SDD(G, 2w, p)$ distribution $y = \{y_F\}_{F \in \mathcal{F}_{2w}(G)}$ such that

$$\sum_{F \in \mathcal{F}^w_{2w}(G,r)} y_F \;=\; 1.$$

In this case, every sampled $2w$-diameter decomposition F from y satisfies $\mathrm{rad}_F(r) \leq w - 1$. We say that a graph class $\mathcal{G}$ is strongly $SDD(w, p)$-acceptable if for every $G \in \mathcal{G}$ and every $r \in V(G)$, there exists an $SDD(G, 2w, p, r)$.

In Theorem 2.4.1, we proved that if a graph class $\mathcal{G}$, closed under the 1-sum operation, is $SDD(2w, p)$-acceptable, then it is also strongly $SDD(w, p)$-acceptable. In Theorem 2.4.2, we extend this result to arbitrary radii. For the proof, we additionally assume that $\mathcal{G}$ contains K_2, the complete graph on two vertices (i.e., a single edge).

Theorem 2.4.2. *Suppose that $\mathcal{G}$ is closed under the 1-sum operation, contains K_2, and is strongly $SDD(w, p)$-acceptable. Let $G \in \mathcal{G}$, $r \in V(G)$, and $k \in \{1, \ldots, w\}$. Then there exists an $SDD(G, 2w, p, r)$ distribution $y = \{y_F \mid F \in \mathcal{F}_{2w}(G)\}$ such that*

$$\sum_{F \in \mathcal{F}^k_{2w}(G,r)} y_F \;\geq\; 1 - p(w - k).$$

Proof. Let $\mathcal{F}_{2w} = \mathcal{F}_{2w}(G), \mathcal{F}^w_{2w}(r) = \mathcal{F}^w_{2w}(G, r)$ and $\mathcal{F}^k_{2w}(r) = \mathcal{F}^k_{2w}(G, r)$ for simplicity. It is sufficient to show that the following LP (2.13) is feasible and has optimal value 0.

$$
\begin{aligned}
\min \quad & \sum_{F \in \mathcal{F}_{2w} \setminus \mathcal{F}^{w}_{2w}(r)} y_F \\
& \sum_{F \in \mathcal{F}^{k}_{2w}(r)} y_F \geq 1 - p(w-k) \\
& \sum_{\substack{F \in \mathcal{F}_{2w} \\ e \in F}} y_F \leq p \quad \forall e \in E(G) \\
& \sum_{F \in \mathcal{F}_{2w}} y_F = 1 \\
& y_F \geq 0 \quad \forall F \in \mathcal{F}_{2w}
\end{aligned}
\tag{2.13}
$$

For now, assume that the LP (2.13) is feasible and its optimal value is $z > 0$. The proof that LP (2.13) is feasible will also follow from the discussion below. Let $m > \frac{1}{z}$ and $G_1, \ldots, G_m$ be m disjoint copies of G. Let r_i be the vertex of G_i which corresponds to r and G' be formed by taking 1-sum of $G_1, \ldots, G_m$ at $r_1, \ldots, r_m$. We construct H by adding a path of length $w-k$ to G' at r. Let $P = \{e_1, e_2, \ldots, e_{w-k}\}$ be the set of edges on this path, where $e_1 = (r, v_1), e_2 = (v_1, v_2), \ldots, e_{w-k} = (v_{w-k-1}, v)$. See the following Figure 2.4 for an illustration.

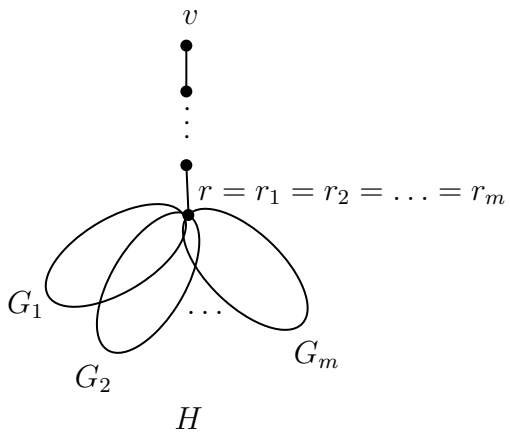


Figure 2.4: Illustration of the construction of H

Since $\mathcal{G}$ is closed under the 1-sum operation and $K_2 \in \mathcal{G}$, we conclude that $H \in \mathcal{G}$. By the assumption that $\mathcal{G}$ is strongly $SDD(2w, p)$-acceptable, there exists an $SDD(H, 2w, p, r)$ distribution

$$x = \{x_{F'} \mid F' \in \mathcal{F}_{2w}(H)\},$$

with properties

$$\sum_{F' \in \mathcal{F}^w_{2w}(H,r)} x_{F'} = 1 \quad \text{and} \quad \sum_{\substack{F' \in \mathcal{F}_{2w}(H) \\ e \in F'}} x_{F'} \leq p \quad \text{for all } e \in E(H).$$

Let $A = \{F' \in \mathcal{F}^w_{2w}(H,v) \mid F' \cap E(P) \neq \emptyset\}$ and $B = \{F' \in \mathcal{F}^w_{2w}(H,v) \mid F' \cap E(P) = \emptyset\}$. Let $A_i = \{F' \in A \mid e_i \in F'\}$ for $i = 1, \ldots, w-k$. Note that $\sum_{F' \in A} x_{F'} + \sum_{F' \in B} x_{F'} = 1$. Since $\sum_{F' \in A_i} x_{F'} \leq p$, and $A = \cup_{i=1}^{w-k} A_i$, it follows that:

$$\sum_{F' \in A} x_{F'} \leq (w-k)p \quad \Rightarrow \quad \sum_{F' \in B} x_{F'} \geq 1 - (w-k)p.$$

Let $G_i = (V_i, E_i)$ and $\mathcal{F}_{2w}(G_i)$ be the set of all $2w$-diameter decompositions of G_i for $i = 1, 2, \ldots, m$. Recall that $\mathcal{F}^w_{2w}(G_i, r), \mathcal{F}^k_{2w}(G_i, r)$ is the set of $2w$-diameter decompositions of G_i in which the distance of every vertex in the connected component containing r is at most $w-1$ and $k-1$ from it, respectively. For $F \in \mathcal{F}_{2w}(G_i)$, let

$$y^i_F = \sum_{\substack{F' \in \mathcal{F}^w_{2w}(H,v) \\ F' \cap E_i = F}} x_{F'}.$$

In other words, y^i_F is the projection of x to the graph G_i. The next claim shows that the projection of the distribution x onto G_i is a feasible solution to the LP (2.13).

Claim 2.4.2. *$y^i = \{y^i_F \mid F \in \mathcal{F}_{2w}(G_i)\}$ is a feasible solution to the LP (2.13).*

Proof of Claim. As shown in Claim 2.4.1, y^i is a $SDD(G_i, 2w, p)$ for G_i. Let $B_i = \{F' \cap E_i \mid F' \in B\}$. Observe that $B_i \in \mathcal{F}_{2w}(G_i)$. Let $F \in B_i$. Then there exists $F' \in B$ such that $F = F' \cap E_i$. Furthermore, for any $F' \in B$, we have $F' \in \mathcal{F}^w_{2w}(H,v)$ and $F' \cap E(P) = \emptyset$. Hence we can conclude that $F \in \mathcal{F}^k_{2w}(G_i, r)$. Thus,

$$\sum_{F \in \mathcal{F}^k_{2w}(G_i,r)} y^i_F \geq \sum_{F \in B_i} y^i_F = \sum_{F \in B_i} \sum_{\substack{F' \in B \\ F' \cap E_i = F}} x_{F'} = \sum_{F' \in B} x_{F'} \geq 1 - (w-k) \cdot p.$$

▲

Since G_i is an identical copy of G and $\mathcal{F}_{2w}(G_i)$ is also an identical copy of $\mathcal{F}_{2w} = \mathcal{F}_{2w}(G)$, Claim 2.4 shows that y^i is a feasible solutions for LP (2.13) for G_i, it follows that LP (2.13) is feasible. Since z is the optimal solution of LP (2.13), for each y^i, we have,

$$\sum_{F \in \mathcal{F}_{2w}(G_i) \setminus \mathcal{F}^w_{2w}(G_i,r)} y^i_F \geq z.$$

This implies that,

$$\sum_{i=1}^{m} \sum_{F \in \mathcal{F}_{2w}(G_i) \setminus \mathcal{F}^w_{2w}(G_i,r)} y^i_F \geq mz > 1.$$

Using same argument as in the proof of Theorem 2.4.1, we can show that there exists $1 \leq i \neq j \leq m$ and $F' \in \mathcal{F}^w_{2w}(H, v)$ such that

$$y_{F'} > 0, \ F' \cap E_i \notin \mathcal{F}^w_{2w}(G_i, r) \text{ and } F' \cap E_j \notin \mathcal{F}^w_{2w}(G_j, r).$$

This means that there exists a vertex $a \in V_i, b \in V_j$ such that the distance of a and b from r is at least w, and they are both included in the connected component containing r in $H - F'$. Hence u and v are at least $2w$ distance apart, and this contradicts the fact that F' is a $2w$-diameter decomposition. Hence $z = 0$ and this completes the proof of the theorem.

□

So far, we have shown that SDDs for any graph class closed under the 1-sum operation are consistent with those constructed for trees in observation 2.3.1. We now state a simple Lemma 2.4.1, which will be useful in proving the lower bound on the integrality gap for cactus graphs in section 2.5. From this point onward, by Theorem 2.4.1 and Theorem 2.4.2, we may restrict our attention to $\mathcal{F}^w_{2w}(G, r)$ instead of $\mathcal{F}_{2w}(G)$, for a given graph G and vertex $r \in V(G)$.

Lemma 2.4.1. *Let $G \in \mathcal{G}$ and let $r \in V(G)$ be an arbitrary vertex. Let P be any shortest path of length w starting at r, and denote its other endpoint by r'. If $y = \{y_F \mid F \in \mathcal{F}^w_{2w}(G, r)\}$ is an $SDD(G, 2w, p, r)$, then:*

$$F \cap E(P) \neq \emptyset \quad \textit{for all } F \in \mathcal{F}^w_{2w}(G, r),$$

and

$$\sum_{\substack{F \in \mathcal{F}^w_{2w}(G,r) \\ |F \cap E(P)| \geq 2}} y_F \ \leq \ (w \cdot p) - 1.$$

Proof. If there exists $F \in \mathcal{F}^w_{2w}(G,r)$ such that $F \cap E(P) = \emptyset$, then r, r' are within the same connected component in $G - F$, which contradicts the fact that $F \in \mathcal{F}^w_{2w}(G,r)$. Let,

$$A = \{F \in \mathcal{F}^w_{2w}(G,r) \mid |F \cap E(P)| \geq 2\} \text{ and } B = \{F \in \mathcal{F}^w_{2w}(G,r) \mid |F \cap E(P)| = 1\}.$$

Note that A, B forms a partition of $\mathcal{F}^w_{2w}(G,r)$. Using the definition of A and B, and the fact that $\sum_{F\in A} y_F + \sum_{F\in B} y_F = 1$, we can derive the statement of the theorem as follows:

$$\sum_{F\in A} y_F + 1 = 2 \cdot \sum_{F\in A} y_F + \sum_{F\in B} y_F \leq \sum_{e\in E(P)} \sum_{\substack{F\in\mathcal{F}^w_{2w}(G,r)\\ e\in F}} y_F \leq \sum_{e\in E(P)} p \;\leq\; w \cdot p.$$

Note that the first inequality can be derived by showing that y_F appears at least twice in the right hand side if $F \in A$, and exactly once if $F \in B$. □

We are now equipped to prove the lower bound on the integrality gap for cactus graphs, which will be the focus of the next section 2.5.

2.5 Lower Bound on the Integrality Gap For Cactus Graphs

We are now ready to prove our main theorem. Suppose that w is a fixed even integer. We will apply the tools developed in the previous sections to the family of cactus graphs. Let $\mathcal{G}$ be the family of cactus graphs, and define $\alpha = \alpha_{\mathcal{M}(\mathcal{G})}$. Since $\mathcal{G}$ is closed under minors and subdivisions, Theorem 2.3.2 implies that $\mathcal{G}$ is $SDD\big(2w, \frac{\alpha}{2w}\big)$-acceptable. Let $p = \frac{\alpha}{2w}$. Since $\mathcal{G}$ is also closed under the 1-sum operation, Theorem 2.4.1 ensures that $\mathcal{G}$ is strongly $SDD(2w,p)$-acceptable. In Theorem 2.5.1, we show that if $\mathcal{G}$ is strongly $SDD(2w,p)$-acceptable, then $w \cdot p \geq \frac{10}{9}$. This implies

$$w \cdot p \;=\; w \cdot \frac{\alpha}{2w} \;\geq\; \frac{10}{9},$$

and hence $\alpha \geq \frac{20}{9}$.

Theorem 2.5.1. *Let $\mathcal{G}$ be the family of cactus graphs. If $\mathcal{G}$ is strongly $SDD(2w,p)$-acceptable, then $w \cdot p \geq \frac{10}{9}$.*

Proof. Let H be a cycle of length $2w$. Let r, u, r', and v be four vertices of the cycle in anti-clockwise order, such that $d(r,u) = d(u,r') = d(r',v) = d(v,r) = \frac{w}{2}$. We construct G from H by attaching paths $u - u'$ and $v - v'$ of length $\frac{w}{2}$ from u and v, respectively. We denote the path of length $\frac{w}{2}$ from r to u by P_u, from u to r' by P_a, from r' to v by P_b, and from v to r by P_v. We denote the path from u to u' by P_c and the path from v to v' by P_d. See the Figure 2.5 for an illustration. Note that $G \in \mathcal{G}$.

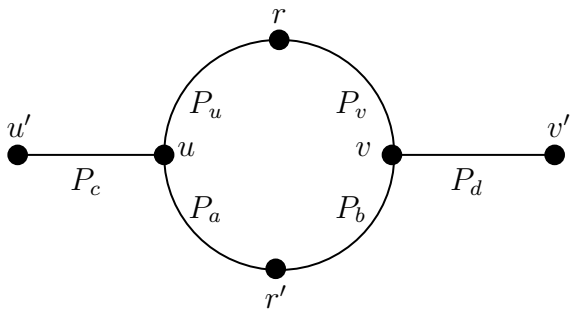


Figure 2.5: G

For the sake of contradiction, assume that $w \cdot p < \frac{10}{9}$. Let $k = \frac{w}{2}$, $\mathcal{F} = \mathcal{F}^w_{2w}(G,r)$ and $A = \mathcal{F}^k_{2w}(G,r)$. Since $K_2 \in \mathcal{G}$ and $\mathcal{G}$ is closed under 1-sum, we can use Theorem 2.4.1 and Theorem 2.4.2 to conclude that there exists $SDD(G, 2w, p, r)$ $y = \{y_F \mid F \in \mathcal{F}^w_{2w}(G,r)\}$ such that,

$$\sum_{F \in A} y_F = \sum_{F \in \mathcal{F}^k_{2w}(r)} y_F \geq 1 - (w-k) \cdot p = 1 - \frac{w}{2} \cdot p = 1 - \frac{w \cdot p}{2}.$$

Suppose that $F \in A$. Since $d(u,r) = k = \frac{w}{2}$ and $l(v,r) = k = \frac{w}{2}$, we have that $F \cap E(P_v) \neq \emptyset$ and $F \cap (P_u) \neq \emptyset$. The next claim shows that under the assumption that $w \cdot p < \frac{10}{9}$, there exists $F \in A$ which does not pick any edges from P_a, P_b, P_c, P_d. This will lead to a contradiction, as u' and v' are $2w$ distance apart, and since F is a $2w$-diameter decomposition, they should have been separated by F. This implies that $w \cdot p \geq \frac{10}{9}$ and completes the proof of the theorem.

Claim 2.5.1. *There exists $F \in A$ such that $F \cap E(P_a), F \cap E(P_b), F \cap E(P_c), F \cap E(P_d) = \emptyset$.*

Proof of Claim. Let P'_a, P'_b be the two paths between r, r' containing P_a, P_b respectively. Also, let P'_u, P'_v be the unique shortest paths starting at r and ending at u', v', respectively.

Let $A_a = \{F \in A \mid F \cap E(P_a) \neq \emptyset\}$ and $F \in A_a$. This implies that $|F \cap E(P'_a)| \geq 2$. Since P'_a is a shortest path with length w from r, using Lemma 2.4.1, we have,

$$\sum_{F \in \mathcal{F}, |F \cap E(P'_a)| \geq 2} y_F \leq (w \cdot p) - 1 \;\Rightarrow\; \sum_{F \in A_a} y_F \leq (w \cdot p) - 1.$$

Similarly, we define,

$$A_b = \{F \in A \mid F \cap E(P_b) \neq \emptyset\}, A_c = \{F \in A \mid F \cap E(P_c) \neq \emptyset\}, A_d = \{F \in A \mid F \cap E(P_d) \neq \emptyset\},$$

and doing the same argument for P'_b, P'_u, P'_v, we obtain,

$$\sum_{F \in A_b} y_F \leq (w \cdot p) - 1; \; \sum_{F \in A_c} y_F \leq (w \cdot p) - 1; \; \sum_{F \in A_d} y_F \leq (w \cdot p) - 1.$$

Let $A^* = A \setminus (A_a \cup A_b \cup A_c \cup A_d)$. From the discussion above, it follows that,

$$\sum_{F \in A^*} y_F = \sum_{F \in A} y_F - \sum_{F \in A_a \cup A_b \cup A_c \cup A_d} y_F \geq \left(1 - \frac{w \cdot p}{2}\right) - 4 \cdot (w \cdot p - 1) = 5 - \frac{9 \cdot w \cdot p}{2} > 0.$$

This shows that $A^* \neq \emptyset$ and completes the proof of the claim. ▲

□

2.5.1 An Explicit Construction

Inspired by the proof presented in the previous sections, we now give an explicit example on a cactus graph where the integrality gap is at least $\frac{20}{9}$. In particular, given an $\epsilon > 0$, we are going to give an instance of the minimum multicut problem M on a cactus graph G such that,

$$\frac{\text{OPT}_{IP}(M)}{\text{OPT}_{LP}(M)} \geq \frac{20}{9} - \epsilon.$$

We will construct the graph G by using two $1-sum$ operations. Let H be the following graph,

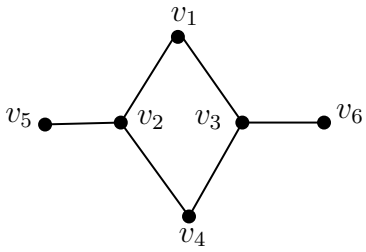


Figure 2.6: H

and k be a sufficiently large natural number. Let $H_1, \ldots, H_k$ be k disjoint copies of H. Let v_1^i be the vertex corresponding to v_1 in H. We construct H' by taking 1-sum of $H_1, H_2, \ldots, H_k$ at $v_1^1, v_1^2, \ldots, v_1^k$, respectively. Let $v_1 = v_1^1 = v_1^2 = \ldots = v_1^k$. We obtain H'' by adding an edge (v_1, v_0) to H'. See the Figure 2.7 below for an illustration.

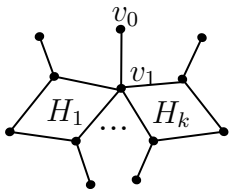


Figure 2.7: H''

Let $H_1'', \ldots, H_k''$ be k disjoint copies of H''. Let v_0^i be the vertex of $H_i^{''}$ corresponding to the vertex v_0 of H''. Let $G = (V, E)$ be the graph obtained by taking 1-sum of $H_1'', \ldots, H_k''$ be at $v_0^1, \ldots, v_0^k$, respectively. Let $v_0 = v_0^1 = \ldots = v_0^k$. Let v_i be the unique neighbor of v_0 in $H_i^{''}$. See the Figure 2.8 below for an illustration.

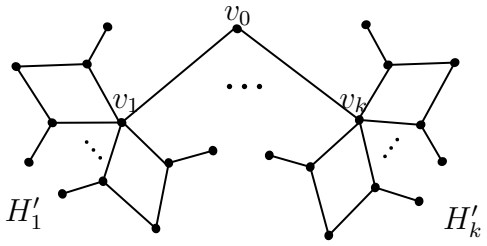


Figure 2.8: G

We partition the set of edges E based on their distance from v_0 as follows:

$$E_1 = \{e \in E \mid l(v_0, e) = 0\},\ E_2 = \{e \in E \mid l(v_0, e) = 1\},\ E_3 = \{e \in E \mid l(v_0, e) = 2\}.$$

Note that E_1 is the set of incident edges to v_0, and E_1, E_2, E_3 is a partition of E. Now we are going to define an instance of the minimum multicut problem on G. We first assign costs to edges as follows:

$$c(e) = \begin{cases} k & e \in E_1 \\ 2 & e \in E_2 \\ 1 & e \in E_3 \end{cases}$$

be the cost function, and let $S = \{(u, v) \in V \times V \mid d(u, v) \geq 4\}$ be the set of source-sink pairs. We will denote this multicut instance by M. It is easy to see that $x = \{x_e = \frac{1}{4} \mid e \in E\}$ is a feasible fractional solution to M with cost

$$\frac{k \cdot k + 2 \cdot k^2 \cdot 2 + 4k^2 \cdot 1}{4} = \frac{9 \cdot k^2}{4}.$$

We will now show that the cost of any feasible multicut (i.e. an integral solution) is at least $5k^2 - 9k$. This will imply that the integrality gap for this instance is at least $\frac{20}{9} - \frac{4}{k}$, which can be arbitrarily close to $\frac{20}{9}$. More precisely, for any $\epsilon > 0$, we can set $k > \frac{4}{\epsilon}$ to obtain a lower bound of $\frac{20}{9} - \epsilon$.

Recall that for a graph H, we use $V(H)$ and $E(H)$ to denote the set of vertices and edges in H, and for $E' \subseteq E(H)$, we use $c(E')$ to denote the total cost of edges in E'. Let F be a feasible multicut solution to M. Let $t = |E_1 \cap F|$. For now assume that $0 < t < k$. Without loss of generality, assume that $(v_0, v_i) \in F$ for $i = 1, \ldots, t$.

Claim 2.5.2. *$c(F \cap E(H_i'')) \geq 4 \cdot (k - 1) + k$ for $i = 1, 2, \ldots, t$.*

Proof of Claim. We will prove the claim for $i = 1$. The proof for other values of i is identical. Denote $H_1, \ldots, H_k$ as the k copies of H incident at v_1. For each $1 \leq i \leq k$, we call H_i good if $rad_F(v_1) \leq 1$, and we call it bad otherwise. It is not too difficult to see that there is at most one bad H_i. Suppose that H_1, H_2 are bad graphs, then there exists $a \in V(H_1), b \in V(H_2)$ such that $d(a, v_1), d(b, v_1) \geq 2$. But this is a contradiction since a and b are within the same component as v_1 after the removal of F, and are at a distance 4 apart, i.e. they are a source-sink pair in the multicut instance M. Thus, there are at least $k - 1$ good graphs attached to v_1. By doing a simple case analysis, it can be verified that $c(F \cap E(H_i)) \geq 4$ if H_i is good. Combining the above with the fact that the edge between v_0, v_1 is included in F, we obtain the statement of the claim. ▲

Note that $v_{t+1}, \ldots, v_k$ are within the same component as v_0 after the removal of F. For each $t + 1 \leq j \leq k$, we call H_j'' good if $rad_F(v_0) \leq 1$ in H_j'', otherwise we call it bad. Using

a similar argument as in the proof of Claim 2.5.1, one can show that there at most 1 bad H''_j. Thus, we have at least $k-t-1$ good H''_j's.

Claim 2.5.3. *$c(F \cap E(H''_j)) \geq 5 \cdot k$ if H''_j is good for $t+1 \leq j \leq k$.*

Proof of Claim. Since the edge between v_0, v_j is not included in F, $E_2 \cap E(H''_j) \subseteq F$ or equivalently, all the edges with cost 2 of H''_j are included in F. On the other hand, let H_i be one of the attached copies of H to v_j. Note that we have already showed that $E_2 \cap E(H_i) \in F$. If $E_3 \cap E(H_i) \cap F = \emptyset$, then there is a source-sink pair at distance 4 in H_i which is not disconnected. Therefore, in each H_i, F picks edges of total weight at least 5. ▲

Therefore we obtain,

$$c(F) \geq t \cdot (5k-4) + (k-1-t) \cdot (5k) = 5k^2 - 5k - 4t \geq 5k^2 - 9k.$$

Even when $t = 0, k$, one can use the same arguments as above to obtain the same lower bound on the cost of F. This concludes the proof.

2.6 Upper Bound on the Integrality Gap for Cactus Graphs

Let $w \in \mathbb{N}$ and G be a cactus with distinct cycles $C_1, \ldots, C_k$ such that $len(C_i)$ is an even number for $i = 1, \ldots, k$. For simplicity, we denote the distance function d_G as d throughout this section. Let $r \in V(G)$ be an arbitrary vertex, which we denote as the root. For any cycle C_i, let $r_{C_i} \in V(C_i)$ be the unique vertex such that $d(r, r_{C_i}) = \min_{v \in V(C_i)} d(r, v)$, and let

$$\beta_i = 2 \cdot \sum_{j=1}^{len(C_i)/2} \frac{1}{len(C_i) - j},$$

and let $\mathcal{D}_i$ be the probability distribution on $E(C_i)$ defined by

$$\Pr_{\mathcal{D}_i}(e) = \frac{1}{\beta_i} \cdot \frac{1}{len(C_i) - d(r_{C_i}, e) - 1}, \quad \forall e \in E(C_i),$$

where β_i is the normalizing constant ensuring that the total probability sums to 1. To see this note that for each $j = 1, \ldots, \frac{len(C_i)}{2}$, we have $|\{e \in E(C_i) \mid d(r_{C_i}, e) = j-1\}| = 2$. The algorithm is described in detail in Algorithm 1.

Algorithm 1 Cactus Algorithm

1: Let $F = \varnothing$.
2: **for** $i = 1, \ldots, k$ **do**
3: Choose an edge $e_i \in E(C_i)$ randomly according to the distribution $\mathcal{D}_i$. Add e_i to F, and remove e_i from C_i.
4: **end for**
5: Let T denote the resulting graph, which is a tree.
6: Pick $\rho \in \{0, 1, \ldots, w-1\}$ uniformly at random.
7: Add $F_w^\rho(T, r)$ to F, and remove $F_w^\rho(T, r)$ from T.
8: **for** $i = 1, \ldots, k$ **do**
9: **if** the endpoints of e_i are within the same connected component in $G \setminus F$ **then**
10: Remove e_i from F, and keep e_i in G.
11: **end if**
12: **end for**
13: Return F.

Lemma 2.6.1. $F \in \mathcal{F}_{2w}(G)$.

Proof. Let F' denote the set of edges in F immediately before the execution of the line 8 of Algorithm 1. At that point, we have $F' = (E(G) \setminus E(T)) \cup F_w^\rho(T, r)$, which means that $F' \in \mathcal{F}_{2w}(G)$, as $F_w^\rho(T, r) \in \mathcal{F}_{2w}(T)$ mentioned in Definition 2.3.3. During the execution of the lines 8 to 12 of Algorithm 1, each edge e_i is removed from F' only if its endpoints are still connected in $G \setminus F'$. This means that no two previously disconnected components become reconnected. Thus, $F \in \mathcal{F}_{2w}(G)$. □

Now, we show that each edge $e \in E(G)$ is removed by Algorithm 1 with probability at most $(1 + \frac{1}{2 \ln 2}) \cdot \frac{1}{w}$.

Remark 2.6.1. *The choice of ρ in the line 6 is independent of the choice of the edges in the lines 2 to 4 of Algorithm 1.*

Remark 2.6.1 implies that each cut-edge $e \in E(G)$ is removed by Algorithm 1 with probability $\frac{1}{w}$. For edges that lie within cycles, we have the following Lemma 2.6.2.

Lemma 2.6.2. *Let $e \in E(C_i)$ for some $i = 1, \ldots, k$.* $\Pr[e \in F] = (1 + \frac{1}{\beta_i}) \cdot \frac{1}{w}$.

Proof. Let A be the event that $e \in F$, and let B be the event that e is chosen as e_i in line 3 of Algorithm 1. Then,

$$\Pr[A] = \Pr[A \mid B] \cdot \Pr[B] + \Pr[A \mid \bar{B}] \cdot \Pr[\bar{B}].$$

By Remark 2.6.1, we have

$$\Pr[A \mid \bar{B}] \cdot \Pr[\bar{B}] \leq \frac{1}{w} \cdot 1 = \frac{1}{w}.$$

To compute $\Pr[A \mid B]$, let P_1 and P_2 denote the shorter and longer paths, respectively, between r_{C_i} and e in C_i (see Figure 2.9). Note that

$$\text{len}(P_1) = d(r_{C_i}, e) < \text{len}(P_2) = \text{len}(C_i) - d(r_{C_i}, e) - 1.$$

Since r_{C_i} is the least common ancestor (LCA) of P_1 and P_2 in the tree T rooted at r (as defined in line 5), it follows that if $E(P_1) \cap F_w^\rho(T, r) \neq \varnothing$, then $E(P_2) \cap F_w^\rho(T, r) \neq \varnothing$ as well. Each edge in $E(P_2)$ is removed independently with probability $\frac{1}{w}$. Therefore, by the union bound,

$$\Pr[A \mid B] \leq \frac{\text{len}(P_2)}{w}.$$

Now, observe that

$$\Pr[A \mid B] \cdot \Pr[B] = \frac{\text{len}(C_i) - d(r_{C_i}, e) - 1}{w} \cdot \frac{1}{\beta_i} \cdot \frac{1}{\text{len}(C_i) - d(r_{C_i}, e) - 1} = \frac{1}{\beta_i w}.$$

Combining both parts, we conclude:

$$\Pr[A] = \frac{1}{\beta_i w} + \frac{1}{w} = \left(1 + \frac{1}{\beta_i}\right) \cdot \frac{1}{w}.$$

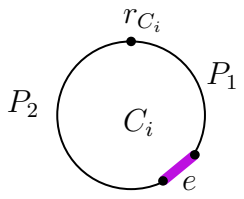


Figure 2.9: P_1, P_2, e in C_i

□

Lemma 2.6.3. *$\beta_i \geq 2 \ln 2$ for all $i = 1, \ldots, k$.*

Proof. It is sufficient to prove $f(k) = \frac{1}{k} + \cdots + \frac{1}{2k-1} \geq \ln 2$ for any $k \in \mathbb{N}$. $f(k)$ is decreasing in $\mathbb{N}$. We have

$$\sum_{i=k}^{2k-1} \frac{1}{i} \geq \int_{k}^{2k} \frac{1}{x}\, dx = \ln(2k) - \ln(k) = \ln\left(\frac{2k}{k}\right) = \ln 2.$$

□

Thus, the probability that any edge is removed by Algorithm 1 is at most $\left(1 + \frac{1}{2\ln 2}\right) \cdot \frac{1}{w}$. By Theorem 2.3.2, which connects the existence of SDDs to integrality gap upper bounds, together with the lower bound established in section 2.5, we obtain

$$2.22 \;=\; \tfrac{20}{9} \;\leq\; \alpha_{\text{CACTUS}} \;\leq\; 2 \cdot \left(1 + \tfrac{1}{2\ln 2}\right) \;\leq\; 3.442.$$

It remains an interesting open problem to close this gap.

2.7 Tight Integrality Gap for Unicyclic Graphs

Let $w \in \mathbb{N}$, and let G be a unicyclic graph with central cycle C, where C has even length. For each $j = 1, \ldots, n$, let T_j denote the tree attached to the vertex v_j on the cycle. See Figure 2.10 for an illustration. Algorithm 2 returns a (random) subset of edges $F \in \mathcal{F}_{2w}(G)$ such that $\Pr[e \in F] \leq \frac{2}{2w-1}$ for all $e \in E(G)$. However, this bound is slightly larger than the probability claimed in 2.3.1, since $\frac{1}{w} < \frac{2}{2w-1}$. Fortunately, we can resolve this discrepancy as follows.

Let $k \in \mathbb{N}$ be arbitrary, and let G^k be the graph obtained from G by replacing each edge $e \in E(G)$ with a path P_e of length k. Note that G^k is also unicyclic. Define $w' := kw$. There is a natural relationship between $\mathcal{F}_{2w'}(G^k)$ and $\mathcal{F}_{2w}(G)$. For each $F' \in \mathcal{F}_{2w'}(G^k)$, define the projection

$$g(F') := \{e \in E(G) \mid F' \cap E(P_e) \neq \varnothing\}.$$

Claim 2.7.1. $g(F') \in \mathcal{F}_{2w}(G)$.

Proof of Claim. Suppose, for contradiction, that there exist two vertices $u, v \in V(G)$ that lie in the same connected component of $G \setminus g(F')$ and satisfy $d_G(u, v) \geq 2w$. Then the corresponding distance in G^k is at least $d_{G^k}(u, v) \geq 2wk = 2w'$. Let Q be the u–v path in $G \setminus g(F')$. Since $F' \cap \left(\bigcup_{e \in E(Q)} E(P_e)\right) = \emptyset$, the replacement of Q in G^k remains connected in $G^k \setminus F'$. This contradicts the assumption that $F' \in \mathcal{F}_{2w'}(G^k)$. ▲

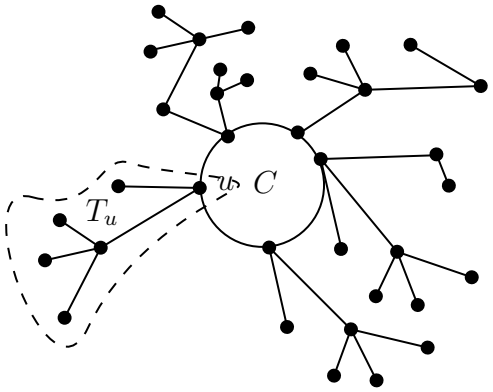


Figure 2.10: Unicyclic graph with the central cycle C. T_u is the attached tree to u.

Now, we apply Algorithm 2 to G^k with parameter $w' = kw$. It outputs a (random) subset $F' \in \mathcal{F}_{2w'}(G^k)$ such that

$$\Pr[e' \in F'] \leq \frac{2}{2w' - 1} = \frac{2}{2kw - 1}.$$

By projecting the output via $g(F')$, we obtain a random subset $g(F') \in \mathcal{F}_{2w}(G)$. Moreover, for each edge $e \in E(G)$,

$$\Pr[e \in g(F')] \leq \sum_{e' \in E(P_e)} \Pr[e' \in F'] \leq k \cdot \frac{2}{2kw - 1} = \frac{2k}{2kw - 1},$$

where we used the union bound and the fact that each path P_e has length k. Therefore, for any $k \in \mathbb{N}$, we obtain an algorithm that returns a $2w$-diameter decomposition of G in which each edge is removed with probability at most $\frac{2k}{2kw-1}$. Taking the limit as $k \to \infty$ recovers the bound claimed in 2.3.1.

We are now ready to describe Algorithm 2. The algorithm involves two sources of randomness. First, the algorithm selects an edge $e \in E(C)$ uniformly at random and temporarily removes it, transforming the cycle C into a path $P := C \setminus e$.

The path P is then extended at both ends, which we refer to as the extended path P'. Next, the algorithm selects a sequence of roots R along P' such that consecutive roots are spaced at distance exactly $2w - 1$ from each other. For each pair of consecutive roots $r_1, r_2 \in R$, the algorithm removes the middle edge of the subpath $P'[r_1, r_2]$. Then, for each vertex $u \in V(P)$, it removes the set $F_w^{l(u)}(T(u), u)$ from the tree $T(u)$ attached at u, where $0 \leq l(u) \leq w - 1$ is chosen appropriately.

Finally, if the algorithm ends up removing an additional edge from P (other than the initially chosen edge e), then e is permanently removed. Otherwise, if no other edge from P is removed, the edge e is retained.

Algorithm 2 Unicyclic Algorithm

1: Let $F = \emptyset$.
2: Choose an edge $e \in E(C)$ uniformly at random. Denote the path $C \setminus e$ as P, and let $H = G \setminus e$ (see figure 2.11).
3: Let P_{left}, P_{right} be two new paths, each of length $2w-1$. Construct a new graph G' by gluing P_{left} to one endpoint of P and P_{right} to the other endpoint of P in the graph H. Define the extended path $P' = P_{left} \oplus P \oplus P_{right} = (v_0, \ldots, v_k)$, where v_0, v_k are the left endpoint and the right endpoint of P', respectively. (see figure 2.12)
▷ Note that $P_{left} = (v_0, \ldots, v_{2w-1})$.
4: Pick $\rho \in \{1, \ldots, 2w-1\}$ uniformly at random.
5: Let $R = \{u \in V(P') \mid d_{G'}(v_\rho, u) = 0 \mod 2w-1\}$ denote the set of roots.
▷ Every two consecutive roots in R are at distance $2w-1$ of each other.
6: **for** each two consecutive $r_1, r_2 \in R$ **do**
7: Add the middle edge of $P'[r_1, r_2]$ to F, and remove it from P'.
8: **end for**
▷ Since all edges of P' are cut-edges in G', then each connected component of $G' \setminus F$, except the ones containing the endpoints of P', has a unique root at this stage of the algorithm.
9: **for** $u \in V(P)$ **do**
10: Let $r(u) \in R$ be the unique root within the same connected component as u in $G' \setminus F$. Let $l(u) = w - 1 - d_{G'}(r(u), u)$, and add $F_w^{l(u)}(T(u), u)$ to F, and remove $F_w^{l(u)}(T(u), u)$ from $T(u)$.
▷ One can verify that $0 \leq d_{G'}(r(u), u) \leq w-1$, which implies $0 \leq l(u) \leq w-1$. So, the set $F_w^{l(u)}(T(u), u)$ is well-defined. See Remark 2.7.1.
11: **end for**
12: **if** the endpoints of e are within different connected component in $G' \setminus F$ **then**
13: Add e to F, and remove e from G.
14: **end if**
15: $F \to F \setminus (E(P_{left}) \cup E(P_{right}))$
16: Return F.

Let F' denote the set of edges in F immediately before the execution of the line 12 of the algorithm 2.

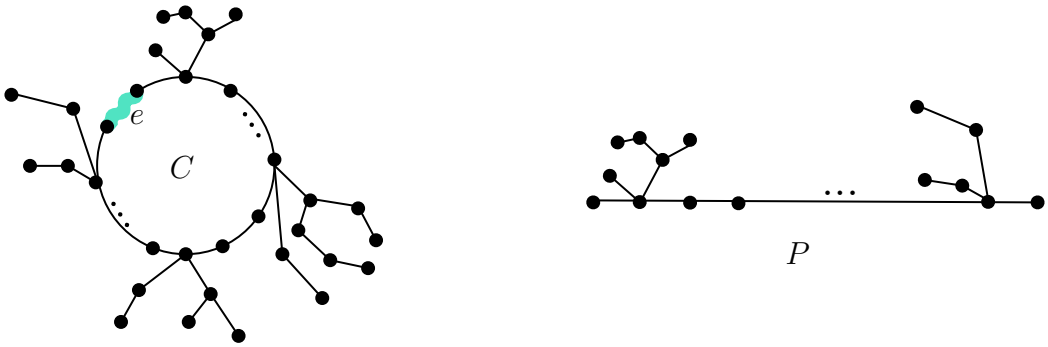


Figure 2.11: The left figure shows the random chosen edge $e \in E(C)$, and the right figure shows the graph $H = G \setminus e$.

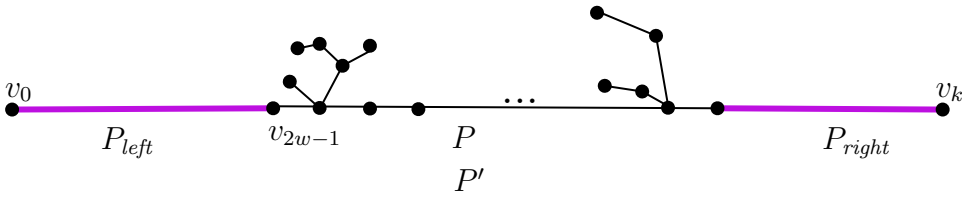


Figure 2.12: The extended graph G' and the extended path $P' = P_{left} \oplus P \oplus P_{right}$

Remark 2.7.1. *Each two consecutive roots in R are at distance $2w-1$ of each other, and the middle edge of the path between each two consecutive roots is contained in F'. This means that on P', any two consecutive edges in F' are at distance $2w-2$ of each other, meaning there are exactly $2w-2$ number of edges of P' between them. Since $len(P_{left}), len(P_{right}) = 2w-1$, then $|F' \cap E(P_{left})| = |F' \cap E(P_{right})| = 1$. This means that any vertex $v \in V(P)$ is between two consecutive edges in $F' \cap E(P')$. Moreover, there are exactly $2w-2$ edges of the path P' between these two edges, which creates the path Q. The middle vertex of this sub-path Q of P' is the unique root $r(v) \in R$ denoted in the line 10 of the algorithm 2. So, it can be derived that $0 \leq d_{G'(r(v),v)} \leq w-1$. Also, since each two consecutive roots of R are at distance $2w-1$ of each other, then $r(v)$ is the unique closest root to v in G'.*

Lemma 2.7.1. *$F \in F_{2w}(G)$.*

Proof.

Claim 2.7.2. *It is enough to show $F' \in \mathcal{F}_{2w}(G')$.*

Proof of Claim. Assume $F' \in \mathcal{F}_{2w}(G')$. Note that $G'+e$ is a super graph of G, i.e., G is an induced sub-graph of $G+e$. So, we have $F' \cup \{e\} \in \mathcal{F}_{2w}(G'+e)$. Based on the if condition in the line 12 of the algorithm 2, there are two cases to consider:

1. The endpoints of e are within different connected component in $G' \setminus F'$; then the line 13 is executed by the algorithm 2, which means that $F = F' \cap \{e\}$ right after the execution of the line 14.

2. The endpoints of e are within the same connected component in $G' \setminus F'$; then the line 13 is not executed by the algorithm 2, which means that $F = F'$ right after the execution of the line 14, and $F' \in \mathcal{F}_{2w}(G'+e)$.

Thus, in both cases, $F \in \mathcal{F}_{2w}(G')$ right after the execution of the line 14. In the line 15 the edges of P_{left}, P_{right} are being excluded from F. Thus, $F \in \mathcal{F}_{2w}(G)$ after the execution of the algorithm 2.

▲

So, we prove $F' \in \mathcal{F}_{2w}(G')$ in the remaining. As stated in the Remark 2.7.1, $|F' \cap E(P_{left})| = |F' \cap E(P_{right})| = 1$, which means that the connected components containing the endpoints of P' in $G' \setminus F'$ are simple paths with length at most $2w-2$. Now, consider the other connected components in $G' \setminus F'$. Let D be a connected component not containing the endpoints of P'. If D is a subgraph of T_u for some $u \in V(C)$, then $diam(D) \leq 2w-1$, since $F_w^{l(u)}(T(u), u) \subseteq F'$, and $F_w^{l(u)}(T(u), u)$ is a $2w$-diameter decomposition for $T(u)$, as stated in the Definition 2.3.3.

Thus, assume that D is not a sub-graph of any $T(u)$. This means that $V(D) \cap V(P') \neq \varnothing$. Since $F' \cap E(P')$ must consists of edges along P' spaced exactly $2w-2$ apart using Remark 2.7.1, and D does not contain the endpoints of P', then the sub-graph induced by D on the vertices $V(P')$, denoted $D[V(P')]$, forms a contiguous sub-path Q of P' of length $2w-2$. Moreover, the unique middle vertex of Q is a root $r' \in R$. Note that the only vertex in $V(D)$ that belongs to the root set R is this single, central vertex r'.

Claim 2.7.3. *$d_{G'}(r', u) \leq w-1$ for all $u \in V(D)$.*

Proof of Claim. If $u \in V(Q)$, then since $len(Q) = 2w-2$, and r' is the middle vertex of Q, then $d_{G'}(r', u) \leq w-1$. So, assume $u \notin V(Q)$. This means that there exists a vertex $v \in V(Q)$ such that $u \in V(T_v)$. We have $l(v) = w-1-d_{G'}(r', v)$. As stated in the Definition 2.3.3, $rad_{F_w^{l(v)}(T(u),u)}(v) \leq l(v)$, which means that $d_{G'}(v, u) \leq l(v)$. The triangle inequality implies that

$$d_{G'}(r', u) \leq d_{G'}(r', v) + d_{G'}(v, u) \leq w - 1 - l(v) + l(v) = w - 1.$$

▲

This shows that $diam(D) \leq 2w - 2$. Thus, $F' \in \mathcal{F}_{2w}(G')$. □

Now, we are going to show that each edge $e \in E(G)$ is removed by the algorithm 2 with probability at most $\frac{2}{2w-1}$.

Definition 2.7.1. *For any $z \in \mathbb{Z}$, let $m_w(z) \in \{-(w-1), \ldots, 0, \ldots, w-1\}$ be the unique integer such that $z = m_w(z) \mod 2w - 1$. Note that $0 \leq |m_w(z)| \leq w - 1$, and $m_w(-z) = -m_w(z)$.*

Lemma 2.7.2. *Fix the random edge $e \in E(C)$ in the line 2, and let G' and $P_{left} = (v_0, \ldots, v_{2w-1})$ be as stated in the line 3 and ρ stated in the line 4 of the algorithm 2. For any $v \in V(C) = V(P)$, we have $d_{G'}(r(v), v) = |m_w(d_{G'}(v_0, v) - \rho)|$.*

Proof. We have $d_{G'}(v_0, r_v) = d_{G'}(v_0, v_\rho) + d_{G'}(v_\rho, r(v)) = \rho + d_{G'}(v_\rho, r(v))$, which implies $d_{G'}(v_0, r_v) = \rho \mod 2w - 1$. There are two cases to consider:

1. $r(v) \in V(P'[v_0, v])$; this means that
$$d_{G'}(r(v), v) = d_{G'}(v_0, v) - d_{G'}(v_0, r(v)),$$
which implies
$$d_{G'}(r(v), v) = d_{G'}(v_0, v) - \rho \mod 2w - 1.$$
Since $0 \leq d_{G'}(r(v), v) \leq w - 1$, then $0 \leq m_w(d_{G'}(v_0, v) - \rho)$, which implies
$$d_{G'}(r(v), v) = m_w(d_{G'}(v_0, v) - \rho) = |m_w(d_{G'}(v_0, v) - \rho)|.$$

2. $v \in V(P'[v_0, r(v)])$; this means that
$$d_{G'}(r(v), v) = d_{G'}(v_0, r(v)) - d_{G'}(v_0, v),$$
which implies
$$d_{G'}(r(v), v) = -(d_{G'}(v_0, v) - \rho) \mod 2w - 1.$$
Since $0 \leq d_{G'}(r(v), v) \leq w-1$, then $0 \leq m_w(-(d_{G'}(v_0, v)-\rho)) = -m_w(d_{G'}(v_0, v)-\rho)$, which implies
$$d_{G'}(r(v), v) = -m_w(d_{G'}(v_0, v) - \rho) = |m_w(d_{G'}(v_0, v) - \rho)|.$$

□

Remark 2.7.2. *The choice of the edge e in the line 2 is independent from the choice of ρ in the line 4 of the algorithm 2.*

Lemma 2.7.3. *Let $u \in V(C) = V(P)$ be an arbitrary vertex, and let $k \in \{0, \dots, w-1\}$ be an arbitrary number.*

$$\Pr[d_{G'}(r(u), u) = k] = \begin{cases} \frac{1}{2w-1} & \text{if} \quad k = 0 \\ \frac{2}{2w-1} & \text{if} \quad k = 1, \dots, w-1. \end{cases}$$

Proof. Fix k. Let e be the random edge denoted in the line 2 in the algorithm 2. By conditional probability, we have

$$\Pr[d_{G'}(r(u), u) = k] = \sum_{f \in E(C)} \Pr[d_{G'}(r(u), u) = k \mid e = f] \Pr[e = f].$$

$\Pr[e = f] = \frac{1}{len(C)}$ for all $f \in E(C)$. Now, to compute $\Pr[d_{G'}(r(u), u) = k \mid e = f]$, we fix $e = f$. Lemma 2.7.2 implies that $d_{G'}(r(u), u) = |m_w(d_{G'}(v_0, v) - \rho)|$, where ρ is the random number denoted in the line 4 of the algorithm 2. Note that $d_{G'}(v_0, v)$ is a fixed value assuming $e = f$ is fixed. If $k = 0$, then $d_{G'}(r(u), u) = 0$ iff $m_w(d_{G'}(v_0, v) - \rho) = 0$, and this is equivalent to $d_{G'}(v_0, v) = \rho \mod 2w - 1$. It can be derived that $\Pr[\rho = d_{G'}(v_0, v) \mod 2w - 1] = \frac{1}{2w-1}$. Thus, $\Pr[d_{G'}(r(u), u) = 0 \mid e = f] = \frac{1}{2w-1}$. If $k \in \{1, \dots, w-1\}$, then $d_{G'}(r(u), u) = k$ iff $m_w(d_{G'}(v_0, v) - \rho) \in \{k, -k\}$. It can be derived that the events $m_w(d_{G'}(v_0, v) - \rho) = k$ and $m_w(d_{G'}(v_0, v) - \rho) = -k$ are disjoint from each other, and each has probability $\frac{1}{2w-1}$. Thus, $\Pr[d_{G'}(r(u), u) = k \mid e = f] = \frac{2}{2w-1}$. This completes the proof. □

Lemma 2.7.4. *Let $f \in E(G) \setminus E(C)$, i.e., $f \in E(T_u)$ for some $u \in V(u)$. $\Pr[f \in F] \leq \frac{2}{2w-1}$.*

Proof. As stated in the Definition 2.3.3, $\{F_k^i(T(u), u)\}_{i=0}^{k-1}$ is a partition of $E(T(u))$. Assume $f \in F_w^j(T(u), u)$ for some $j = 0, \dots, w-1$. Line 10 implies that $f \in F$ iff $j = l(u) = w - 1 - d_{G'}(r(u), u)$. So, $e \in F$ iff $d_{G'}(r(u), u) = w - 1 - j$. Lemma 2.7.3 implies that

$$\Pr[d_{G'}(r(u), u) = w - 1 - j] = \begin{cases} \frac{1}{2w-1} & \text{if} \quad j = w-1 \\ \frac{2}{2w-1} & \text{if} \quad j = 0, \dots, w-2. \end{cases}$$

□

Lemma 2.7.5. *Let $f \in E(C)$ be an arbitrary edge.* $\Pr[f \in F] \leq \frac{2}{2w-1}$.

Proof. Let A denote the event that $e = f$, where e is the random edge denoted in the line 2 of the algorithm 2. Denote $\bar{A}$ as the complement of A. We compute $\Pr[f \in F]$ by conditioning on A. We have

$$\Pr[f \in F] = \Pr[f \in F|A]P[A] + \Pr[f \in F|\bar{A}]P[\bar{A}].$$

We know that $\Pr[A] = \frac{1}{len(C)}$ and $\Pr[\bar{A}] = \frac{len(C)-1}{len(C)}$. Now, if A happens, meaning that $e = f$, then f is added to the set F' in the line 13 iff the end-points of $e = f$ are in different connected components in $G' \setminus F'$. Note that the endpoints of $e = f$ are indeed the endpoints of the path P. Since G' is a tree, then $e = f$ is added to F' iff $F' \cap E(P') \neq \varnothing$. We have two cases to consider:

1. $len(C) \geq 2w$; then $len(P) \geq 2w - 1$, which means that $F' \cap E(P') \neq \varnothing$ by Remark 2.7.1. Thus, $\Pr[f \in F|A] = 1 \leq \frac{len(C)-1}{2w-1}$ in this case.
2. $len(C) < 2w$; Remark 2.7.1 implies that each edge of P' is contained in F' with probability $\frac{1}{2w-1}$ assuming $e = f$ is fixed. Using union bound, we have $\Pr[F' \cap E(P) \neq \varnothing] \leq \sum_{f' \in E(P)} \Pr[f' \in F'] = \frac{len(C)-1}{2w-1}$. Thus, $\Pr[f \in F|A] = \frac{len(C)-1}{2w-1}$.

Now, we compute $\Pr[f \in F|\bar{A}]$. Assuming $\bar{A}$ means $f \in E(P)$, and as mentioned before each edge of P' is present in F' with probability $\frac{1}{2w-1}$ assuming $e \neq f$ is fixed. So, $\Pr[f \in F|\bar{A}] = \frac{1}{2w-1}$. Thus,

$$\Pr[f \in F] = \frac{len(C) - 1}{2w - 1} \times \frac{1}{len(C)} + \frac{1}{2w - 1} \times \frac{len(C) - 1}{len(C)} \leq \frac{2}{2w - 1}.$$

□

Thus, with the discussion provided at the beginning of the section, the probability that any edge is removed by Algorithm 2 is at most $\frac{1}{w}$. By Theorem 2.3.2, this implies that

$$\alpha_{\text{UNICYCLIC}} \leq 2.$$

Moreover, since the family UNICYCLIC contains all trees, and it is known that $\alpha_{\text{TREE}} = 2$, we obtain

$$2 \leq \alpha_{\text{UNICYCLIC}}.$$

Combining these inequalities gives

$$\alpha_{\text{UNICYCLIC}} = 2.$$

2.8 Tight Integrality Gap for Path Cactus Graphs

We first review some properties of path cactus graphs and introduce the necessary notions.

Observation 2.8.1. *Let G be a cactus. Denote by $CE(G) \subseteq E(G)$ the set of cut-edges of G, and let $\bigcup_{i=1}^{k} E(C_i)$ be the set of all non-cut-edges, where $C_1, \ldots, C_k$ are the distinct cycles of G. Moreover, for any $1 \leq i < j \leq k$, we have $|V(C_i) \cap V(C_j)| \leq 1$.*

Definition 2.8.1. *A path cactus is a cactus graph G such that $CE(G) = \varnothing$ and each cycle shares a common vertex with at most two other cycles. Equivalently, the cycles of G can be arranged in a path of consecutive cycles (see Figure 2.13). Throughout this section, we restrict attention to path cactus graphs in which all cycles have even length.*

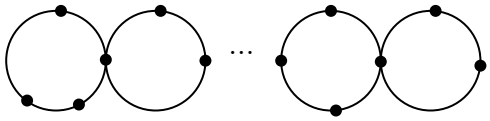


Figure 2.13: Path Cactus

Definition 2.8.2. *Let $P = (v_1, \ldots, v_n)$ be an even-length path. We denote the mid-vertex of P by $v_{\lceil \frac{n+1}{2} \rceil}$.*

Definition 2.8.3. *Let $P = (v_1, \ldots, v_n)$ be a path. Let $f = (v_i, v_{i+1})$ be an arbitrary edge and let $v_j \in V(P)$ be an arbitrary vertex. The distance between v_j, e is denoted as*

$$d_P(v_j, e) = \begin{cases} d_P(v_j, v_i), & \text{if } j \leq i \\ d_P(v_j, v_{i+1}), & \text{otherwise} \end{cases}$$

Definition 2.8.4. *Assume $n \geq 3$. Let $P = (v_1, \ldots, v_n)$ be a path. Let $e_1 = (v_i, v_{i+1}), e_2 = (v_j, v_{j+1}) \in E(P)$ be two distinct edges such that $i < j$. The distance between e_1, e_2 in P is denoted as $d_P(e_1, e_2) = j - i - 1$. Note that e_1, e_2 are consecutive edges in P iff $j = i + 1$.*

Definition 2.8.5. *Let $C = (v_1, \ldots, v_n, v_1)$ be an even-length cycle. For any vertex v_i, we denote the opposite vertex of v_i in C as*

$$\begin{cases} v_{i+\frac{n}{2}}, & \text{if } i \leq \frac{n}{2} \\ v_{i-\frac{n}{2}}, & \text{otherwise} \end{cases}$$

Let G be a path cactus with consecutive cycles $C_1, \ldots, C_k$, where

$$V(C_i) \cap V(C_{i+1}) = \{u_i\}, \quad i = 1, \ldots, k-1.$$

Fix an arbitrary vertex $u_0 \neq u_1$ in C_1 and an arbitrary vertex $u_k \neq u_{k-1}$ in C_k. For each $i = 1, \ldots, k$, the vertices u_{i-1} and u_i are joined by exactly two simple paths in the cycle C_i. Denote by S_i the shorter of the two and by L_i the longer one (see Figure 2.14). As we discussed in section 2.3, we can assume that S_i, L_i have even length.

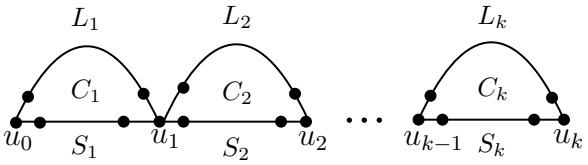


Figure 2.14: S_i and L_i

Let $P = S_1 \oplus \cdots \oplus S_k$ with endpoints u_0, u_k. We refer $v_1 = u_0, \ldots, v_n = u_k$ as the ordered vertices of P. To perform the Algorithm 3, we need to consider a super graph of G. Let $P_{left} = (v_{-2w}, \ldots, v_0), P_{right} = (v_{n+1}, \ldots, v_{n+2w})$ be two new paths. Extend G to G' by adding an edge (v_0, v_1), and another edge (v_n, v_{n+1}). Let $P' = P_{left} \oplus (v_0, v_1) \oplus P \oplus (v_n, v_{n+1}) \oplus P_{right}$. (Figure 2.15)

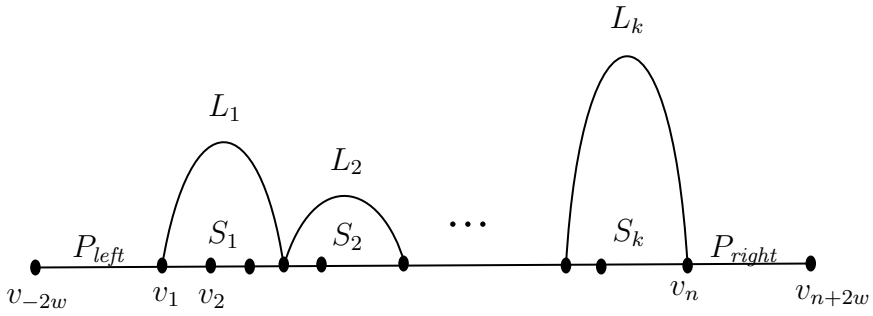


Figure 2.15: G' and P'

Remark 2.8.1. *$d_{G'}(u,v) = d_G(u,v)$ for any two vertices $u, v \in V(G)$. Moreover, $\{u_i\}_{i=0}^{k}$ are cut-vertices of G'.*

Let $w \in \mathbb{N}$ be a fixed natural number for the rest of the section. We are going to randomly pick consecutive edges on P' with distance $2w-1$ of each other, and we will refer to them as the blue edges. Next, based on which edges are picked as the blue edges, we are going to remove edges from G', and arguing that each connected component has at most $2w-1$, and each edge is removed with probability at most $\frac{1}{w}$. Note that since G is a subgraph of G', the same properties hold for G as well.

In order to explain the Algorithm 3, we need to distinguish between the cycles $C_1, \ldots, C_k$.

Definition 2.8.6. *We say that a cycle C_i is long if $len(L_i) > 2w-1$. For each long cycle C_i, let a_i be the mid-vertex of L_i. Let $Q_i = L_i[u_{i-1}, a_i], Q'_i = L_i[a_i, u_i]$. Note that $len(Q_i) = len(Q'_i) \geq w$. (Figure 2.16)*

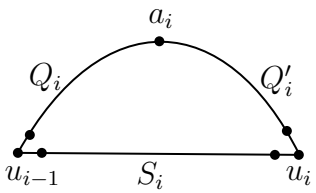


Figure 2.16: Long Cycle

Definition 2.8.7. *We say that a cycle C_i is short if $len(L_i) \leq 2w-1$. For each short cycle C_i, let a_i, b_i be the opposite points of u_{i-1}, u_i in C_i, respectively. Let $Q_i = L_i[u_{i-1}, a_i], Q'_i = L_i[u_i, b_i]$. Also, let $\tilde{Q}_i = L_i[b_i, a_i]$ be the common sub-path of Q_i, Q'_i. (Figure 2.17)*

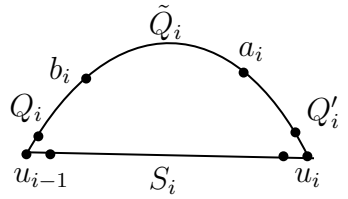


Figure 2.17: Short Cycle

Remark 2.8.2. *Let C_i be a short cycle.*

$$len(Q_i) = len(Q'_i) = \frac{len(C_i)}{2} = \frac{len(L_i) + len(S_i)}{2} \leq 2w - 1.$$

Also,

$$len(Q_i[u_{i-1}, b_i]) = len(Q'_i[u_i, a_i]) = \frac{len(L_i) - len(S_i))}{2}.$$

We need to define two more important vertices for the sake of short cycles.

Definition 2.8.8. *Let C_i be a short cycle. Let $a'_i \in V(P'[u_i, v_{n+2w}])$ be the unique vertex with $d(u_{i-1}, a'_i) = \frac{len(C_i)}{2}$. Similarly, let $b'_i \in V(P'[v_{-2w}, u_{i-1}])$ be the unique vertex with $d(u_i, b'_i) = \frac{len(C_i)}{2}$. Moreover, let $P_i = P'[a'_i, b'_i]$. (Figure 2.18)*

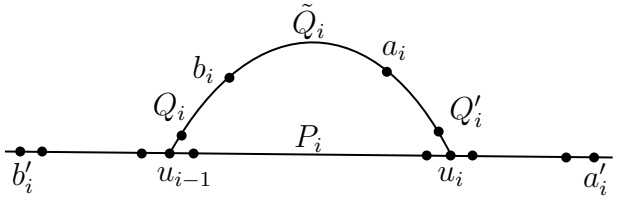


Figure 2.18: Short Cycle

Remark 2.8.3. *Let C_i be a short cycle. Since $len(P_{left}), len(P_{right}) \geq 2w - 1$, the vertices a'_i, b'_i are well-defined.*

$$d(u_{i-1}, a'_i) = d(u_i, b'_i) = \frac{len(C_i)}{2} = \frac{len(L_i) + len(S_i)}{2} \leq 2w - 1.$$

Also,

$$d(u_{i-1}, b'_i) = d(u_i, a'_i) = \frac{len(L_i) - len(S_i))}{2}.$$

Finally, $len(P_i) = len(L_i) \leq 2w - 1$.

We are now ready to describe Algorithm 3. The algorithm removes each edge of G with probability at most $\frac{1}{w}$, and we will show that every connected component in the remaining graph has diameter at most $2w - 1$. This establishes the existence of an $SDD(G, 2w, \frac{1}{w})$.

Algorithm 3 Path Cactus Algorithm

Pick uniformly random edges from P' such that the distance of every consecutive edges is $2w-1$, and call them blue edges.
for each long cycle C_i **do**
    Let o_i be the distance of the closest blue edge to u_{i-1}. Pick edges with distance $o_i + k'w (k' = 0, \ldots, \lfloor \frac{len(Q_i)-o_i}{w} \rfloor)$ from u_{i-1} in Q_i.
    Let z_i be the distance of the closest blue edge to u_i. Pick edges with distance $z_i + k'w (k' = 0, \ldots, \lfloor \frac{len(Q'_i)-z_i}{w} \rfloor)$ from u_i in Q'_i.
    Remove all the picked edges from C_i.
end for
for each short cycle C_i **do**
    if P_i contains a blue edge, **then**
        Let f_i be the blue edge in P_i.
        Let $o_i = d_{P'}(u_{i-1}, f_i)$. Pick the unique edge f with $d_{Q_i}(u_{i-1}, f) = o_i$ in Q_i.
        Let $z_i = d_{P'}(u_i, f_i)$. Pick the unique edge f' with $d_{Q'_i}(u_i, f') = z_i$ in Q'_i.
        Remove f, f' from C_i.
    end if
end for
Remove the blue edges.

Note that since we extend G to G', then at least one blue edge in both P_{left}, P_{right} is picked by the algorithm 3, which implies that each vertex in P has both a left and a right blue edge. We first prove that each edge is removed with probability at most $\frac{1}{w}$ by the algorithm 3.

Lemma 2.8.1. *Let v_j be an arbitrary vertex in P. The algorithm 3 always picks at least one blue edge from $P'[v_{n-2w}, v_j]$ and at least one blue edge from $P'[v_j, v_{n+2w}]$. Let o be the distance of the closest blue edge to v_j. Let $t \in \{0, \ldots, w-1\}$ then, $\mathbb{P}[o = t] = \frac{1}{w}$.*

Proof. Let $f \in E(P'[v_{n-2w}, v_j])$ be the unique edge such that $d(v_j, f) = t$, also let $f' \in E(P'[v_j, v_{n+2w}])$ be the unique edge such that $d(v_j, f') = t$. Since $d(f, f') = 2t < 2w - 1$, the algorithm does not pick both f, f' as blue edges, simultaneously.

Claim 2.8.1. *$o = t$ iff exactly one of the edges f, f' is picked as a blue edge.*

Proof of Claim. If $o = t$, then at least one of the edges f, f' is picked as a blue edge, and since f, f' cannot be picked as a blue edge simultaneously, then exactly one of them is picked as a blue edge. For the reverse direction, WLOG assume f is picked as a blue edge.

Then $o \leq t$. Suppose on the contrary that $o < t$. This means that there exists a blue edge $e \in P'$ picked by the algorithm 3, such that $d(v_j, e) = o < t$. We have two cases,

- $e \in E(P'[v_{n-2w}, v_j])$; then $d(e, f) = t - o - 1 < 2w - 1$.
- $e \in E(P'[v_j, v_{n+2w}])$; then $d(e, f) = t + o < 2w - 1$.

Both cases lead to contradiction, since e, f are picked as blue edges and their distance must be at least $2w - 1$ based on the line 1 of the algorithm 3. ▲

Now, since f, f' cannot be picked as blue edges at the same time, the probability that exactly one of f, f' is picked as a blue edge is the sum of the probability of each being picked as a blue edge, which is $\frac{1}{2w} + \frac{1}{2w} = \frac{1}{w}$. □

Theorem 2.8.1. *Let G' be the extended version of the path cactus G with $C_1, \ldots, C_k$ as its cycles. The algorithm 3 removes each edge $e \in E(G')$ with probability at most $\frac{1}{w}$.*

Proof. If $e \in E(P')$, then it is removed iff it is a blue edge that has probability $\frac{1}{2w}$. Thus, suppose $e \in E(L_i)$ for some $1 \leq i \leq k$. We have two cases; C_i is either a long cycle or a short one. WLOG, assume $e = (x, y) \in E(Q_i)$, and $x \in V(Q_i[u_{i-1}, y])$.

- C_i is a long cycle; then, e is removed in the line 3 of the algorithm 3. This happens iff $o_i = d_{Q_i}(u_{i-1}, e) = d_{Q_i}(u_{i-1}, x) \mod w$. Lemma 2.8.1 implies that o_i is a uniform random variable with values $\{0, 1, \ldots, w-1\}$, which implies that $Pr[o_i = d_{Q_i}(u_{i-1}, x) \mod w] = \frac{1}{w}$. (Figure 2.19)

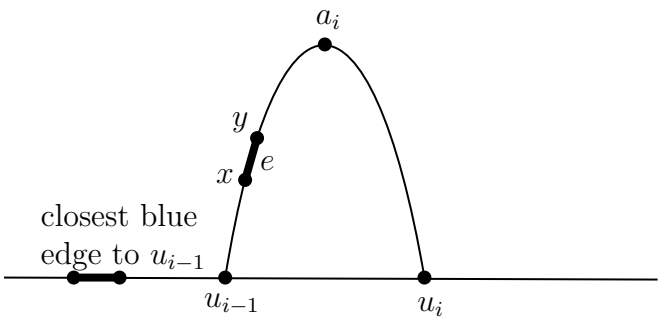


Figure 2.19: C_i is a long cycle

- C_i is a short cycle; then, the algorithm 3 removes edges from L_i iff P_i contains a blue edge. Note that P_i contains at most one blue edge, since $l(P_i) = l(L_i) \leq 2w - 1$ and blue edges are picked with distance $2w - 1$ of each other. We have two cases,

  - $e \in E(Q_i) \setminus E(\tilde{Q}_i)$; this means that e cannot be removed in the line 11 of the algorithm 3. Thus, e is removed in the line 10 of the algorithm 3. This means e is removed iff $o_i = d_{Q_i}(u_{i-1}, e) = d_{Q_i}(u_{i-1}, x)$. Lemma 2.8.1 implies that $Pr[o_i = d_{Q_i}(u_{i-1}, x)] = \frac{1}{w}$. (Figure 2.20)

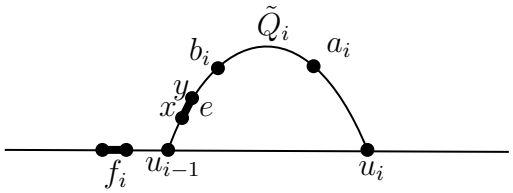


Figure 2.20: $e \in E(Q_i) \setminus E(\tilde{Q})$

  - $e \in E(\tilde{Q}_i)$; then e is either removed in the line 10 or 11 of the algorithm 3. We show that each has probability $\frac{1}{2w}$, which implies that the probability of the removal of e is $\frac{1}{w}$. Suppose e is removed in the line 10 of the algorithm 3. This implies that e is removed iff $o_i = d_{Q_i}(u_{i-1}, e) = d_{Q_i}(u_{i-1}, x)$. Since $e \in E(\tilde{Q}_i)$,

$$d_{Q_i}(u_{i-1}, x) \geq len(Q_i[u_{i-1}, b_i]) = \frac{len(L_i) - len(S_i)}{2} = d(u_{i-1}, b'_i)$$

    Consider the blue edge $f_i \in P_i$ in the line 9. Since,

$$d(u_{i-1}, f_i) = o_i = d(u_{i-1}, x) \geq d(u_{i-1}, b'_i)$$

    then $f_i \notin E(P'[b'_i, u_{i-1}])$ which implies that $f_i \in P'[u_{i-1}, a'_i]$. Thus, e is removed in the line 10 of the algorithm 3 iff f_i is the unique edge in $P'[u_{i-1}, a'_i]$ with distance o_i from u_{i-1}. This happens with probability $\frac{1}{2w}$. (Figure 2.21)

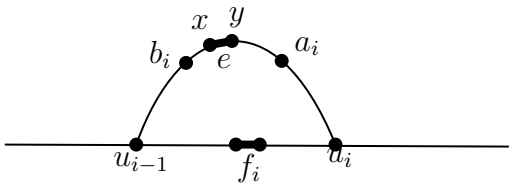


Figure 2.21: $e \in E(\tilde{Q}_i)$

□

Now we prove that the diameter of the connected components of G' after performing the algorithm 3 is bounded by $2w-1$. From now on, we will use the notation B whenever we refer to the connected component we are arguing about. The following Remark 2.8.4 is useful for the arguments.

Remark 2.8.4. *Let B be a connected component of G' after performing the algorithm 3. If there exists two vertices $u, v \in V(B)$ such that $u \in V(L_i), v \in V(L_j)$ for some $1 \leq i < j \leq k$, then $u_i, u_{j-1} \in V(B)$.*

Proof. u_i is a cut-vertex of G'. Removing u_i from G' disconnects any vertex $u' \in V(C_{i'})$ such that $i' \leq i$ from any vertex $v' \in V(C_{j'})$ such that $j' \geq i+1$. The same argument is correct for u_j. (Figure 2.22)

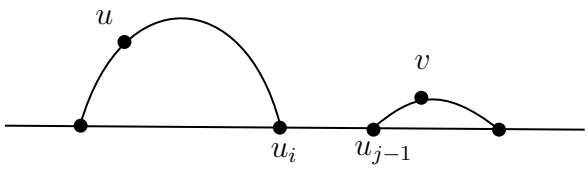


Figure 2.22: Description

□

Consider a connected component B after performing the algorithm 3. Remark 2.8.4 implies that either $B \cap V(P') \neq \varnothing$ or B is a subgraph of L_i for a specific $1 \leq i \leq k$. The latter case means that B is a simple path completely contained in L_i since, L_i is a path and B is a connected subgraph of L_i.
The above discussion implies that we have four types of connected components. The ones that share vertices with P', those that contain a_i for some long cycle C_i, simple paths with length $w-1$ that are attained from long cycles, and simple paths with length at most $2w-1$ that are attained from short cycles. We formalize this in the following Lemma 2.8.2.

Lemma 2.8.2. *Let B be a connected component of G' after performing the algorithm 3. Then B is of one of the following types of connected components, (Figure 2.23)*

- *$V(B) \cap V(P') \neq \varnothing$. (Type 1)*

- *B is a sub-path of L_i for a specific $1 \leq i \leq k$. Moreover, C_i is a long cycle and $a_i \in V(B)$. (Type 2)*
- *B is a sub-path of L_i for a specific $1 \leq i \leq k$. Moreover, C_i is a long cycle and B is either a sub-path of Q_i or a sub-path of Q'_i. (Type 3)*
- *B is a sub-path of L_i for a specific $1 \leq i \leq k$. Moreover, C_i is a short cycle. (Type 4)*

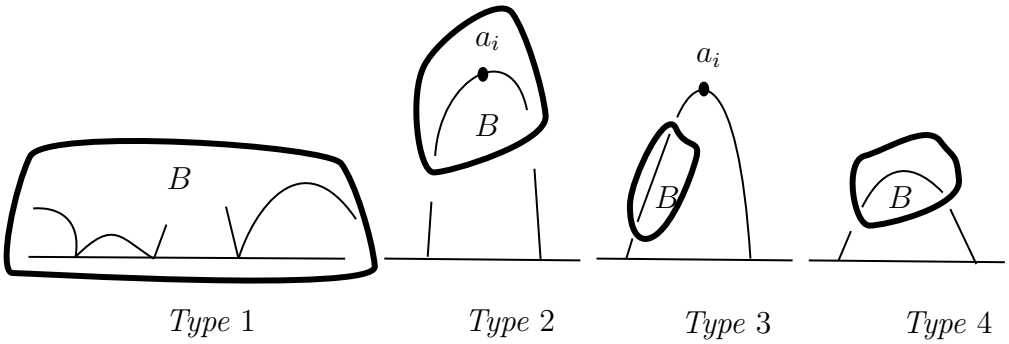


Figure 2.23: Types of Connected Components

Proof. Suppose that B is not a type 1 connected component. This means that B is a sub-path of L_i for a specific $1 \leq i \leq k$. C_i is either a long cycle or a short one.

- C_i is a long cycle; in this case either $a_i \in V(B)$ or $a_i \notin V(B)$.
  - $a_i \in V(B)$; this means that B is a type 2 connected component.
  - $a_i \notin V(B)$; note that $L_i = Q_i \oplus Q'_i$ and $V(Q_i) \cap V(Q'_i) = \{a_i\}$ and B is a sub-path of L_i. This means that B is either a sub-path of Q_i or sub-path of Q'_i, as otherwise $a_i \in V(B)$.
- C_i is a short cycle; this means that B is a type 4 connected component.

□

Lemma 2.8.3. *Let B be a connected component of G' after performing the algorithm 3. If B is of one of the types $2, 3, 4$ then, $diam(B) \leq 2w - 1$.*

Proof. B is a path. We consider the following cases,

- B is type 2 connected component; we can assume that $B = B_1 \oplus B_2$ such that B_1 is a sub-path of Q_i and B_2 is a sub-path of Q'_i. Line 3 of the algorithm 3 implies that at least one edge is removed from any w consecutive edges in Q_i which means that $len(B_1) \leq w-1$. Similarly, $len(B_2) \leq w-1$. Thus, $len(B) = len(B_1) + len(B_2) \leq 2w-2$.
- B is type 3 connected component; WLOG assume that B is a sub-path of Q_i. Line 3 of the algorithm 3 implies that at least one edge is removed from any w consecutive edges in Q_i. This means that $len(B) \leq w-1$. More precisely, $len(B) = w-1$.
- B is type 4 connected component; since C_i is a short cycle then, $len(L_i) \leq 2w-1$. Since B is a sub-path of L_i then, $len(B) \leq len(L_i) \leq 2w-1$.

□

Now, it is only remained to argue about type 1 connected components. In the following, assume that B is a type 1 connected component.

Remark 2.8.5. *If $v_{-2w} \in V(B)$ or $v_{n+2w} \in V(B)$, then B is a sub-path of P_{left} in the first case and it is a sub-path of P_{right} in the latter case. Moreover, $len(B) \leq 2w-1$ in both of the cases.*

Proof. Suppose $v_{-2w} \in V(B)$. Since $len(P_{left}) = 2w > 2w-1$, at least one blue edge is picked from P_{left}. Also, the distance of each two consecutive blue edges is $2w-1$, which implies that $len(B) \leq 2w-1$. The same argument is correct for the other case. □

Remark 2.8.6. *Let C_i be a cycle. If the algorithm 3 removes an edge from S_i, then it also removes an edge from L_i.*

Proof. If C_i is a long cycle, then the algorithm 3 removes at least 2 edges from L_i. If C_i is a long cycle, then the removed edge from S_i is picked as a blue edge by the algorithm 3 and since S_i is a sub-path of P_i then lines 10,11 remove edges from L_i. □

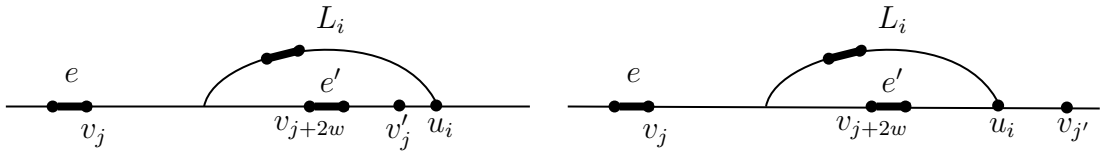
Now, we can assume that $v_{-2w}, v_{n+2w} \notin V(B)$. We have the following crucial Lemma 2.8.4.

Lemma 2.8.4. *If $v_{-2w}, v_{n+2w} \notin V(B)$, then there exists an index $-2w < j, j+2w < n+2w$ such that $V(B) \cap V(P') = \{v_j, v_{j+1}, \ldots, v_{j+2w}\}$. Moreover, the edges $(v_{j-1}, v_j), (v_{j+2w}, v_{j+2w+1})$ are picked as blue edges by the algorithm 3.*

Proof. Since B is a type 1 connected component then, $V(B) \cap V(P') \neq \varnothing$. Let $v_j, v_{j'} \in V(B) \cap V(P')$ be the leftmost and the rightmost vertices of $V(B) \cap V(P')$. Since $v_{-2w} \notin V(B)$, then $j > -2w$. This implies that the edge $e = (v_{j-1}, v_j)$ is picked as a blue edge and removed by the algorithm 3. If $j + 2w \geq n + 2w$, then there does not exist any blue edges between the edge e and the vertex v_{n+2w} which implies that $v_{n+2w} \in V(B)$, contradiction. Thus, $j + 2w < n + 2w$.

Lines 1,15 of the algorithm 3 implies that the edges $e = (v_{j-1}, v_j), e' = (v_{j+2w}, v_{j+2w+1})$ are picked as blue edges and they are removed by the algorithm 3, and none of the edges between them in P' are not removed by the algorithm 3. Consider the edge e'. There are three cases. Either $e' \in E(P_{left})$, or $e' \in e' \in E(P_{right})$, or $e' \in E(C_i)$ for some $1 \leq i \leq k$. In the first two cases, since e' is a cut edge, then v_{j+2w} is the rightmost edge in $V(B) \cap V(P')$ and we are done.

Thus, suppose $e' = (v_{j+2w}, v_{j+2w+1}) \in E(C_i)$ for some $1 \leq i \leq k$. Suppose, on the contrary, that there exists a vertex $v_{j'} \in V(B) \cap V(P')$ such that $j' > j + 2w$. Since $e' \in E(S_i)$ is removed by the algorithm 3, Remark 2.8.6 implies that at least one edge is also removed from L_i. Thus, u_i gets disconnected from v_{j+2w}, which means $u_i \notin V(B)$. Now, either $v_{j'} \in V(P'[v_{j+2w}, u_i])$, or $v_{j'} \in V(P'[u_i, v_{n+2w}])$. (Figure 2.24)



First Case Second Case

Figure 2.24: Description of Cases

In both cases, since u_i is a cut-vertex, then $v_{j'}$ gets disconnected from v_{j+2w}, which means $v_{j'} \notin V(B)$, contradiction. □

Thus, if $v_{-2w}, v_{n+2w} \notin V(B)$, then $V(B) \cap V(P') = \{v_j, \ldots, v_{j+2w}\}$ for some index $-2w < j < n$ and the edges $(v_{j-1}, v_j), (v_{j+2w}, v_{j+2w+1})$ are picked as blue edges by the algorithm 3. We refer v_j as x and v_{j+2w} as y in the remaining. Note that $d_{P'}(x, y) = 2w - 1$. In the following Lemma 2.8.5, we show $d(u, x), d(u, y) \leq 2w - 1$ for any $u \in B$. Figure 2.25 shows how B looks,

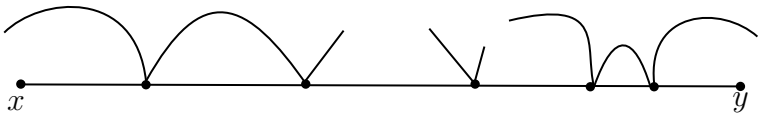


Figure 2.25: Type 1 Connected Component's Structure

Lemma 2.8.5. *Suppose* $v_{-2w}, v_{n+2w} \notin V(B)$. *Let* $u \in V(B)$, *then* $d_{G'}(u,x), d_{G'}(u,y) \leq 2w-1$.

Proof. If $u \in V(P'[x,y])$, we are done. Thus, assume $u \in V(L_i)$ for some $1 \leq i \leq k$ and $u \in V(Q_i)$. We have three cases. Either C_i is either a long cycle, C_i a short cycle and the algorithm 3 does not remove an edge from L_i, or C_i is a short cycle and the algorithm 3 removes at least 1 edge from L_i.

- C_i is a long cycle; then at least one edge is removed from both Q_i, Q_i' by the algorithm 3. This implies that $u_{i-1} \in V(P'[x,y])$. Line 3 of the algorithm 3 implies that

  $$d(u_{i-1},u) = len(Q_i[u_{i-1},u]) \leq o_i = \min\{d_(u_{i-1},x), d(u_{i-1}y)\}.$$

  Using triangle inequality we have,

  $$d(u,x) \leq d(u_{i-1},x) + d(u_{i-1},u) \leq d(u_{i-1},x) + o_i$$

  Since, $d(u_{i-1}x) + d(u_{i-1},y) = 2w-1$, we can conclude that $d(u,x) \leq 2w-1$. Similar argument implies that $d(u,y) \leq 2w-1$. (Figure 2.26)

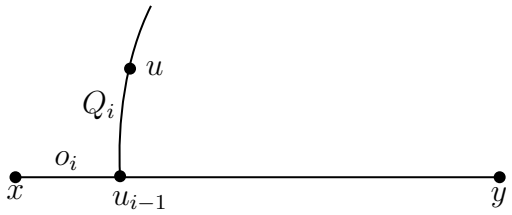


Figure 2.26: C_i is a long cycle

- C_i is a short cycle and no edges are removed from L_i by the algorithm 3; this means that P_i does not contain any blue edges, and $b_i', u_{i-1}, u_i, a_i' \in V(P'[x,y])$. So, we have $b_i' \in V(P'[u_{i-1},x])$ and $a_i' \in V(P'[u_i,y])$. (Figure 2.27)

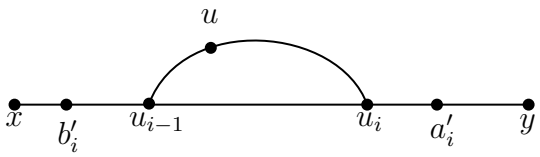


Figure 2.27: C_i is a short cycle and it is intact

Thus,

$$d(u_{i-1}, x) = (2w-1) - d(u_{i-1}, y) \leq (2w-1) - d(u_{i-1}, a'_i) = (2w-1) - \frac{len(C_i)}{2}.$$

Since $u, u_{i-1} \in V(C_i)$ then, $d(u_{i-1}, u) \leq \frac{len(C_i)}{2}$. Using triangle inequality we have,

$$d(u, x) \leq d(u_{i-1}, x) + d(u_{i-1}, u) \leq (2w-1) - \frac{len(C_i)}{2} + \frac{len(C_i)}{2} = 2w - 1.$$

Similar argument implies that $d(u, y) \leq 2w - 1$.

- C_i is a short cycle and the algorithm 3 removes at least 1 edge from L_i; this means P_i contains a blue edge f. We have three cases. Either $f \in E(P'[u_{i-1}, u_i])$, $f \in E(P'[u_{i-1}, b'_i])$, or $f \in E(P'[u_i, a'_i])$.
  - $f \in E(P'[u_{i-1}, u_i])$; We have two cases here. Either $x \in V(P'[u_{i-1}, u_i])$, or $y \in V(P'[u_{i-1}, u_i])$.
    - $x \in V(P'[u_{i-1}, u_i])$; since $x, u \in V(C)$ and $len(C_i) \leq 2(2w-1)$ then $d(u, x) \leq 2w - 1$. Line 11 of the algorithm 3 implies that $d(u_i, u) \leq d(u_i, f) = d(u_i, x)$, since f together with at least 1 edge from Q'_i are removed. Using triangle inequality we have,

      $$d(u, y) \leq d(u_i, y) + d(u_i, u) \leq d(u_i, y) + d(u_i, x) = 2w - 1.$$

      See the following, (Figure 2.28)

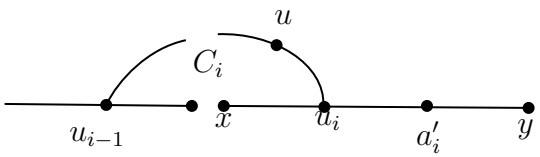


Figure 2.28: $f \in E(P'[u_{i-1}, u_i]), x \in V(P'[u_{i-1}, u_i])$

* $y \in V(P'[u_{i-1}, u_i])$; the argument is similar to the previous case.

– $f \in E(P'[b'_i, u_{i-1}])$; this means $x \in V(P'[b'_i, u_{i-1}])$. This implies that $d(u_{i-1}, x) \leq d(u_{i-1}, b'_i) = \frac{len(L_i)-len(S_i)}{2}$. Since $u_{i-1}, u \in V(C_i)$ then, $d(u_{i-1}, u) \leq \frac{len(C_i)}{2}$. Using triangle inequality we have,

$$d(u, x) \leq d(u_{i-1}, x) + d(u_{i-1}, u) \leq \frac{len(L_i) - len(S_i)}{2} + \frac{len(C_i)}{2} = len(L_i) \leq 2w - 1.$$

Line 10 of the algorithm 3 implies that exactly 1 edge is removed from $Q_i[u_{i-1}, b_i] = Q_i \setminus \tilde{Q}_i$ and line 10 of the algorithm 3 implies that exactly 1 edge is removed from Q'_i. We have two cases. Either $u \in V(Q_i[u_{i-1}, b_i])$, or $u \in V(Q'_i)$.

* $u \in V(Q_i[u_{i-1}, b_i])$; this implies that $d(u_{i-1}, u) \leq d(u_{i-1}, x)$. Using triangle inequality we have,

$$d(u, y) \leq d(u_{i-1}, u) + d(u_{i-1}, y) \leq d(u_{i-1}, x) + d(u_{i-1}, y) = 2w - 1.$$

See the following. (Figure 2.29)

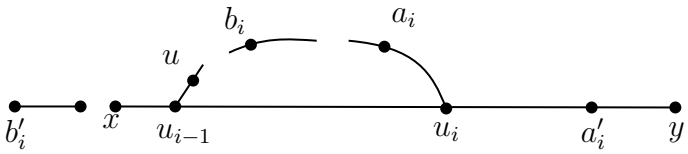


Figure 2.29: $f \in E(P'[b'_i, u_{i-1}]), u \in V(Q_i[u_{i-1}, b_i])$

* $u \in V(Q'_i)$; this implies that $d(u_i, u) \leq d(u_i, x)$. Using triangle inequality we have,

$$d(u, y) \leq d(u_i, u) + d(u_i, y) \leq d(u_i, x) + d(u_i, y) = 2w - 1.$$

See the following. (Figure 2.30)

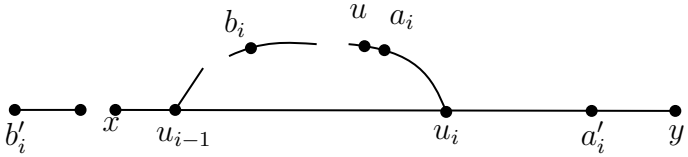


Figure 2.30: $f \in E(P'[b'_i, u_{i-1}]), u \in V(Q'_i)$

– $f \in E(P'[u_i, a'_i])$; this means $y \in V(P'[u_i, a'_i])$. This implies that $d(u_i, y) \leq d(u_i, a'_i) = \frac{len(L_i - len(S_i))}{2}$. Since $u_i, u \in V(C_i)$, then $d(u_i, u) \leq \frac{len(C_i)}{2}$. Using triangle inequality we have,

$$d(u, y) \leq d(u_i, y) + d(u_i, u) \leq \frac{len(L_i) - len(S_i)}{2} + \frac{len(C_i)}{2} = len(L_i)$$
$$\leq 2w - 1.$$

Line 11 of the algorithm 3 implies that exactly 1 edge is removed from $Q'_i[u_i, a_i] = Q'_i \setminus \tilde{Q}_i$ and line 11 of the algorithm 3 implies that exactly 1 edge is removed from Q_i. Since we assumed that $u \in V(Q_i)$, then $d(u_{i-1}, u) \leq d(u_{i-1}, y)$. Using triangle inequality we have,

$$d(u, x) \leq d(u_{i-1}, u) + d(u_{i-1}, x) \leq d(u_{i-1}, y) + d(u_{i-1}, x) = 2w - 1.$$

See the following. (Figure 2.31)

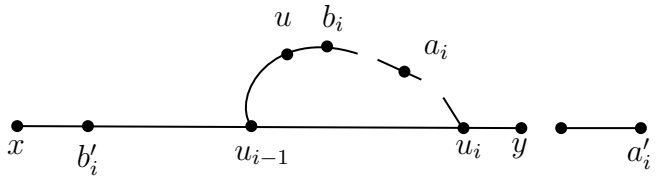


Figure 2.31: $f \in E(P'[u_i, a'_i])$

□

Now, equipped with Lemma 2.8.5, we will prove that if $u, v \in V(B)$ and $v_{-2w}, v_{n+2w} \notin V(B)$, then $d_{G'}(u, v) \leq 2w - 1$.

Lemma 2.8.6. *Suppose $v_{-2w}, v_{n+2w} \notin V(B)$ and $V(B) \cap V(P') = \{x = v_j, \ldots, v_{j+2w} = y\}$ for some index j. Let $u, v \in V(B)$, then $d(u, v) \leq 2w - 1$.*

Proof. Lemma 2.8.5 implies that,

$$d(u, x), d(u, x) \leq 2w - 1$$
$$d(v, y), d(v, y) \leq 2w - 1.$$

We have two cases. Either $u, v \in V(C_i)$ for some index i, or $u \in V(C_i), v \in V(C_{i'})$ and $i < i'$.

- $u, v \in V(C_i)$ for some index i; if C_i is a short cycle then, $d(u,v) \leq \frac{len(C_i)}{2} \leq 2w-1$. Thus, suppose C_i is a long cycle. We have three case.
  - $u, v \in V(S_i)$; then $d(u,v) \leq 2w-1$ as S_i is a sub-path of $P'[x,y]$. (Figure 2.32)

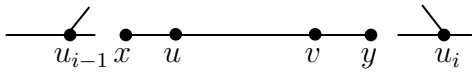


Figure 2.32: $u, v \in V(C_i), u, v \in V(S_i)$

  - $u, v \in V(Q_i)$; then line 3 of the algorithm 3 implies that $d(u_{i-1}, u), d(u_{i-1}, v) \leq o_i \leq w-1$, and u_{i-1}, u, v are in the path of length at most $w-1$. This implies that $d(u,v) \leq w-1$. (Figure 2.33)

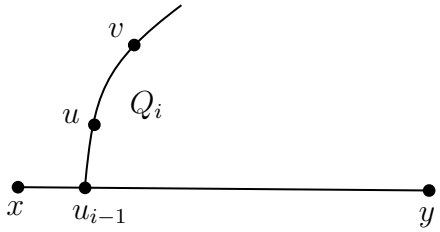


Figure 2.33: $u, v \in V(C_i), u, v \in V(Q_i)$

  - $u \in V(Q_i), v \in V(Q'_i)$; then if $len(C_i) \leq 2(2w-1)$ we are done. Otherwise, $u_{i-1}, u_i \in V(P'[x,y])$, and Lemma 2.8.5 implies that the shortest path between u, y, which is $Q_i[u, u_{i-1}] \oplus P'[u_{i-1}, u_i] \oplus P'[u_i, y]$ has length at most $2w-1$, since

$$len(Q_i[u, u_{i-1}]) + len(P'[u_{i-1}, u_i]) + len(P'[u_i, y]) \leq 2w-1. \tag{2.14}$$

Similarly, the shortest path between v, x, which is $Q'_i[v, u_i] \oplus P'[u_i, u_{i-1}] \oplus P'[u_{i-1}, x]$ has length at most $2w-1$, since

$$len(Q'_i[v, u_i]) + len(P'[u_i, u_{i-1}]) + len(P'[u_{i-1}, x]) \leq 2w-1. \tag{2.15}$$

Since,

$$len(P'[x, u_{i-1}]) + len(P'[u_{i-1}, u_i]) + len(P'[u_i, y]) = len(P'[x,y]) = 2w-1,$$

summing the inequalities 2.14,2.15 implies that,

$$len(Q_i[u, u_{i-1}]) + len(P'[u_{i-1}, u_i]) + len(Q'_i[u_i, v]) \leq 2w-1.$$

The left-hand side is the length of the path between u, v. (Figure 2.34)

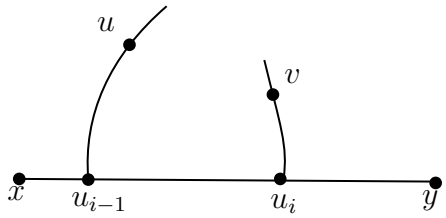


Figure 2.34: $u, v \in V(C_i), u \in V(Q_i), v \in V(Q'_i)$

- $u \in V(C_i), v \in V(C_{i'})$ and $i < i'$; then $u_i, u_{i'-1} \in V(P'[x, y])$ and $u'_{i-1} \in V(P'[u_i, y])$. Let P_u be the shortest path between u, y. Since removing u_i from G' disconnects u, y then, $P_u = Q_u \oplus P'[u_i, y]$, which Q_u is the shortest path between u, u_i. Lemma 2.8.5 implies that,

$$len(Q_u) + len(P'[u_i, u_{i'-1}]) + len(P'[u_{i'-1}, y]) \leq 2w - 1. \tag{2.16}$$

Similarly, let P_v be the shortest path between v, x. Since removing $u_{i'-1}$ from G' disconnects v, y, then $P_v = P'[x, u_{i'-1}] \oplus Q_v$, which Q_v is the shortest path between $v, u_{i'-1}$. Lemma 2.8.5 implies that,

$$len(P'[x, u_i]) + len(P'[u_i, u_{i'-1}]) + len(Q_v) \leq 2w - 1. \tag{2.17}$$

Since,

$$len(P'[x, u_i]) + len(P'[u_i, u_{i'-1}]) + len(P'[u_{i'-1}, y]) = 2w - 1,$$

then,

$$len(Q_u) + len(P'[u_i, u_{i'-1}]) + len(Q_v) \leq 2w - 1.$$

The left-hand side is the length of the path between u, v. (Figure 2.35)

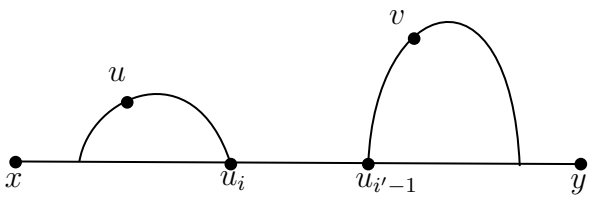


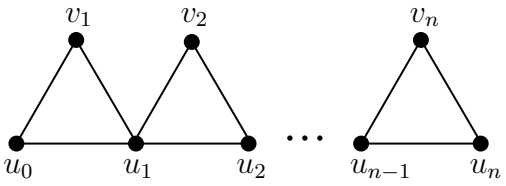
Figure 2.35: $u \in V(C_i), v \in V(C_{i'})$

□

Thus, the randomized algorithm we design always outputs a $2w$-diameter decomposition, and each edge is removed with probability at most $\frac{1}{w}$. By Theorem 2.3.2, this certifies the existence of an $SDD(G, 2w, \frac{1}{w})$ and implies that

$$\alpha_{\text{PATH CACTUS}} \leq 2.$$

Remark 2.8.7. *To show that this bound is tight, we construct an explicit instance of the minimum multicut problem on a path cactus graph whose integrality gap is at least 2. Consider a graph consisting of n triangles $\Delta_1, \ldots, \Delta_n$ arranged in a path. Two consecutive triangles Δ_i and Δ_{i+1} share a vertex u_i, and each triangle Δ_i has three vertices u_{i-1}, u_i, v_i. For $i = 1, \ldots, n$, the edges (u_i, v_i) and (u_{i-1}, v_i) each have cost 1, while all other edges are assigned arbitrarily large cost. The source–sink pairs are all pairs (v_i, v_j) with $1 \leq i < j \leq n$.*



Assigning a value of $\frac{1}{2}$ to each edge (u_i, v_i) and (u_{i-1}, v_i) for $i = 1, \ldots, n$ yields a feasible fractional solution to the LP (2.1) of cost n. However, any integral multicut must include both edges (u_i, v_i) and (u_{i-1}, v_i) for at least $n-1$ triangles in order to separate all source–sink pairs. Thus, the cost of any feasible multicut is at least $2n-2$. It follows that the integrality gap for this instance is

$$\frac{2n-2}{n}.$$

As $n \to \infty$, this ratio approaches 2. Hence,

$$\alpha_{\text{PATH CACTUS}} \geq 2.$$

Combining with the upper bound, we conclude that

$$\alpha_{\text{PATH CACTUS}} = 2.$$

2.9 Upper Bound on the Integrality Gap for Outer-Planar Graphs

In this section, we propose a randomized algorithm for the minimum multicut problem in outer-planar graphs, using the optimal solution of the LP relaxation (2.1). The expected cost of the multicut produced by our algorithm is at most $8 \cdot \mathrm{OPT}_{LP}$. We restrict our attention to maximal outer-planar graphs throughout this section; however, the arguments can be adapted with minor modifications to handle the non-maximal case as well. To prepare for the analysis, we first review some useful structural properties of maximal outer-planar graphs.

For notation, let $d_H(v)$ denote the degree of a vertex v in a graph H. For a subset of vertices $S \subseteq V(H)$, define

$$\delta_H(S) := \{\, uv \in E(H) \mid u \in S,\ v \in V(H) \setminus S \,\},$$

the set of edges with one endpoint in S and the other in its complement. We will work with an LP solution x. For any graph H and vertices $u, v \in V(H)$, let $d^x_H(u, v)$ denote the shortest-path distance between u and v in H induced by the edge lengths x. For a subgraph $H_1 \subseteq H$, we further define

$$diam^x_H(H_1) := \max_{u,v \in V(H_1)} d^x_H(u, v)$$

Remark 2.9.1. *Let G be a graph with at least two vertices. Then G is a maximal outer-planar graph if and only if it excludes K_4 and $K_{2,3}$ as minors, satisfies $|E(G)| = 2|V(G)|-3$, has exactly $|V(G)|-1$ faces, and in every planar drawing the outer face is a cycle of length $|V(G)|$ while every internal face is a triangle.*

Remark 2.9.2. *Let G be a maximal outer-planar graph, and let H be a Hamiltonian induced subgraph of G, then H is a maximal outer-planar graph.*

Lemma 2.9.1. *Let G be a maximal outer-planar graphs, and let H be a $2-$vertex connected induced subgraph of G with at least 3 vertices. H is Hamiltonian, and thus it is a maximal outer-planar graph using Remark 2.9.2.*

Proof. Suppose it is not correct. Let C be the longest cycle in H. $|V(C)| < |V(H)|$, let H_1 be one of the connected components of $H \setminus V(C)$. Since H is 2-vertex connected there exists 2 vertices $u, v \in V(C)$ such that they both have a neighbor in H_1(not necessarily same neighbors). Let $u_1, v_1 \in V(H_1)$ be the neighbors of u, v, respectively. Let P_1 be the path between u_1, v_1 in H_1. We have 2 cases,

- u, v are neighbors, then the cycle $u, \ldots, v, v_1, P_1, u_1, u$ is a larger cycle than C in H. Contradiction.
- There is at least 1 vertex between the both paths between u, v in C, let P_2, P_3 be these paths. By contracting H_1 to a single vertex, and P_2, P_3 to paths of length 2 we obtain $K_{2,3}$. Contradiction. (Figure 2.36)

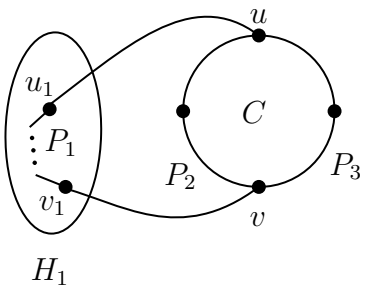


Figure 2.36: The description of the last case

□

Now we are going to orient the edges of G and elaborate on some properties of a directed maximal outer-planar graph. We will use the following Lemma 2.9.2, which is very useful (See Therese Biedl's course notes on Graph-Theoretic Algorithms, Lemma 8.7).

Lemma 2.9.2. *Let G be a maximal outer-planar graph with $n \geq 3$ vertices, and consider a planar drawing of G. Let (v_1, v_2) be an arbitrary edge on the outer face. Then G has a vertex ordering $v_1, v_2, \ldots, v_n$ such that every vertex v_i for $i \geq 3$ has exactly two predecessors that are adjacent to each other.*

Proof. We proceed by induction on the number of vertices.

Base case. If $n = 3$, then G is a triangle, and the statement is immediate.
Inductive step. Let G be a maximal outer-planar graph on $n \geq 4$ vertices, and fix an outer-face edge (v_1, v_2). By Remark 2.9.1, G has $n - 2$ internal triangular faces.

Claim 2.9.1. *There exists an internal face T such that (v_1, v_2) is not an edge of T, and at least two edges of T are edges of the outer face.*

Proof of Claim. Suppose for contradiction that every internal triangle contains at most one edge of the outer face (besides possibly (v_1, v_2)). The outer face has $n-1$ edges other than (v_1, v_2), and each such edge must be shared with an internal triangle. Since there are only $n-2$ internal triangles in total, this is impossible by the pigeonhole principle. Hence such a triangle T exists. ▲

Let T be the triangle guaranteed by the claim, and let v_n be the vertex of T that is incident with the two edges lying on the outer face. Let a, b be the other two vertices of T. Then v_n is adjacent only to a and b in G. It follows that $G' = G \setminus \{v_n\}$ is a maximal outer-planar graph with $n-1$ vertices. Moreover, since T does not contain the edge (v_1, v_2), the edge (v_1, v_2) remains on the outer face of G'. By the induction hypothesis, G' admits a vertex ordering $v_1, v_2, \ldots, v_{n-1}$ such that every vertex from v_3 to v_{n-1} has two adjacent predecessors. Appending v_n at the end of this ordering preserves the property, since the only neighbors of v_n are a and b, which are adjacent to each other. Thus we obtain the desired ordering for G. This completes the proof. □

Now let G be a fixed maximal outer-planar graph. Using the vertex ordering from Lemma 2.9.2, define

$$G_i := G[\{v_1, \ldots, v_i\}], \quad i = 2, \ldots, n,$$

and for $i = 3, \ldots, n$ set

$$E_i := E(G_i) \setminus E(G_{i-1}).$$

It follows that the edge set of G admits the partition

$$E(G) = \{(v_1, v_2)\} \uplus \bigcup_{i=3}^{n} E_i,$$

and moreover $|E_i| = 2$ for all $i = 3, \ldots, n$. See Figure 2.37.

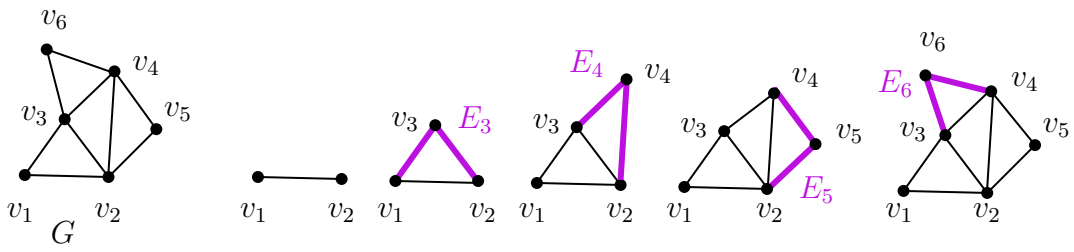


Figure 2.37: G is outer-planar and $v_1, v_2, v_3, v_4, v_5, v_6$ is its corresponding ordering in Lemma 2.9.2.

Let $r = v_1$. We are going to direct the edges of G iteratively, first direct (v_1, v_2) from v_1 to v_2. Assuming that $E(G_i)$ is directed, let $E_{i+1} = \{e, f\}$, and $e = (v_{i+1}, v_j), f = (v_{i+1}, v_k)$ such that $k, j \leq i$. By Lemma 2.9.2, $g = (v_j, v_k) \in E(G_i)$. If g is directed from v_j to v_k, direct e from v_j to v_{i+1}, and direct f from v_{i+1} to v_k (figure 2.38). If g is directed from v_k to v_j direct f from v_k to v_{i+1}, and direct e from v_{i+1} to v_j. Let $\overrightarrow{G_i}$ be the directed version of G_i. Note that $\overrightarrow{G_n} = \overrightarrow{G}$.

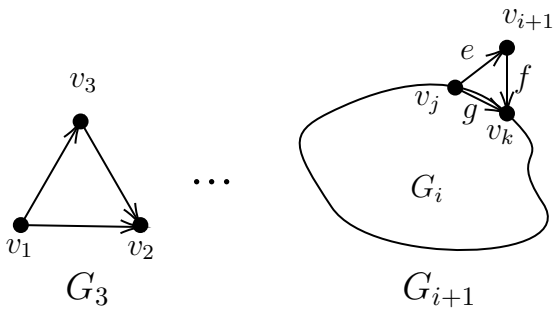


Figure 2.38: The direction of edges

Lemma 2.9.3. *There is exactly one shortest directed path from r to v_j in $\overrightarrow{G_i}$, and this path is contained in $\overrightarrow{G_j}$ for all $j \leq i, i = 2, \ldots, n$.*

Proof. The proof is based on induction on i. The base case is $k = 2$, which is correct. Suppose that statement is correct for $k < n$, meaning that there is exactly one shortest directed path from r to any v_j in $\overrightarrow{G_k}, j \leq k$, and it is contained in $\overrightarrow{G_j}$. Consider the graph $\overrightarrow{G_{k+1}}$, wlog assume that there is exactly one directed edge $e = (v_{j_1}, v_{k+1})$, and one directed edge $f = (v_{k+1}, v_{j_2})$. Since all the directed paths from r to v_{k+1} must use e, and by induction there is only one directed shortest path from r to v_{j_1}, then there is exactly one directed shortest path from r to v_{k+1}, and clearly it is contained in $\overrightarrow{G_{k+1}}$. Suppose by contradiction there exists a vertex $v_j, j < k+1$ such that there are at least 2 directed shortest path from r for in $\overrightarrow{G_{k+1}}$. At least one of these paths should use vertex v_{k+1}, so the path should use e and then use f, but this is a contradiction to being the shortest path as by replacing (v_{j_1}, v_{j_2}) with e, f there is a shorter path to v_j. With the same logic, it can be derived that the shortest path from r to v_j is contained in $\overrightarrow{G_k}$, so it is contained in $\overrightarrow{G_j}$ using the induction assumption. □

Remark 2.9.3. *Note that the proof of Lemma 2.9.3 implies that v_{j_1} is the previous vertex in the shortest directed path from r to v_{k+1} before v_{k+1}.*

Lemma 2.9.3 implies that there is exactly one directed shortest path from r to v_i in $\vec{G}$ for all $i = 2, \ldots, n$. Denote this path by P_{v_i} for each $i = 2, \ldots, n$. We now record the following useful observation.

Remark 2.9.4. *Let $v_i \in V(\vec{G})$, and let $v_j \in V(P_{v_i})$. Then P_{v_j} coincides with the subpath of P_{v_i} from r to v_j, i.e.,*

$$P_{v_j} \;=\; P_{v_i}[r, v_j].$$

Lemma 2.9.4. *Let H be an induced subgraph of $\vec{G_i}$ ($i \geq 3$) such that it is 2-vertex connected. Then there exists a vertex $v_l \in V(H)$ such that P_{v_j} contains v_l, and the sub-path of P_{v_j} starting at v_l is totally contained in H, for all v_j in $V(H)$.*

Proof. We prove this using induction on i. If $i = 3$, then the only 2−vertex of G_i is G_i itself, and $v_1 \in V(G_i)$. Now, suppose the statement is correct for i, and we want to prove it for $i+1$. Suppose that H is a 2-vertex connected induced subgraph of $\overrightarrow{G_{i+1}}$, using Lemma 2.9.1, H is a maximal outer-planar graph. If $v_{i+1} \notin V(H)$, then H is a subgraph of $\vec{G_i}$, and the statement is correct using the induction hypothesis. Thus, $v_{i+1} \in V(H)$. Since H is 2-vertex connected then $E_{i+1} \in E(H)$, or equivalently $\delta_H(v_{i+1}) = E_{i+1} = \{e, f\}$. WLOG, assume that $e = (v_{j_1}, v_{i+1}), f = (v_{i+1}, v_{j_2})$. If $V(H) = \{v_{i+1}, v_{j_1}, v_{j_2}\}$, then the statement is correct, as H is a triangle, and $v_{j_1} = v_l$. Otherwise, we have the following,

Claim 2.9.2. *$H' = H \setminus v_{i+1}$ is a 2-vertex connected induced subgraph of G_i.*

Proof of Claim. Since H is Hamiltonian, and $d_H(v_{i+1}) = 2$, then v_{j_1}, v_{j_2} are the neighbors of v_{i+1} in the outer-face of H. Since H is a maximal outer-planar graph, then $(v_{j_1}, v_{j_2}) \in E(H)$. This implies that the vertices of H' are on a cycle. Thus H' is 2-vertex connected. ▲

Using induction hypothesis on G_i, H', there exists a vertex $v_l \in V(H')$ such that for all of the vertices $v_j \in V(H')$, P_{v_j} contains v_l, and the subpath of v_j starting at v_l is totally included in H'. Let $j = j_1$, we know that $P_{i+1} = p_{v_{j_1}} \oplus e$, by 2.9.3. This completes the proof. □

Let $x = \{x(e)\}_{e \in E(G)}$ be an optimal fractional solution to the LP (2.1). Since the edge costs are non-negative, we may assume without loss of generality that x satisfies the triangle inequality in G. Indeed, suppose there exists a triangle with edges e, f, g such that $x(e) > x(f) + x(g)$. Define a new solution x^* by setting $x^*(e) := x(f) + x(g)$ and $x^*(h) := x(h)$ for all other edges $h \neq e$. For every path $P \in \mathcal{P}$ containing e, consider the modified path

$$P_1 := (P \setminus \{e\}) \cup \{f, g\},$$

which also belongs to $\mathcal{P}$. Then

$$x^*(P) \;\geq\; x^*(P_1) \;=\; x(P_1).$$

Since feasibility requires $x(P_1) \geq 1$, it follows that $x^*(P) \geq 1$ as well. Thus the modified solution x^* satisfies all constraints of the LP. Moreover, since edge costs are non-negative, the objective value does not increase. Therefore x^* is also an optimal solution. We are going design a randomized rounding algorithm, that always outputs a valid multicut. The algorithm does the following. Given a parameter Δ, it will randomly remove edges from G such that each edge is removed with probability at most $\frac{2x_e}{\Delta}$ for all $e \in E(G)$, and the diameter of connected components is strictly upper-bounded by 4Δ. The algorithm has 2 stages.

In the first stage, it will remove edges of G to obtain the graph H with 2 properties. Each edge e is removed with probability at most $\frac{x(e)}{\Delta}$, and the diameter of each block of H is upper-bounded by 2. In the second stage, it will remove edges from H such that each edge is removed with probability at most $\frac{x(e)}{\Delta}$, and eventually each connected component has diameter at most 4Δ after the algorithm is done. Here is the description of the first stage:

Algorithm 4 First Stage

1: Let $t(v_i) = x(P_{v_i})$ for all $i =, \ldots, n$, and also let $t(v_1) = 0$.
2: Choose $\rho \in (0, \Delta]$ uniformly at random.
3: Remove edge $e = (u, v) \in E(\vec{G})$ if and only if the interval $(t(u), t(u) + x(e))$ contains the number $\rho + l\Delta$ for some integer number $l \geq 0$.

Remark 2.9.5. *Let $v_i \in V(\vec{G})$, and let $v_j \in V(P_{v_i})$. By Remark 2.9.4, P_{v_j} is the subpath of P_{v_i} from r to v_j. Therefore,*

$$t(v_i) - t(v_j) \;=\; x(P_{v_i}) - x(P_{v_j}) \;=\; x(P_{v_i}[v_j, v_i])\,,$$

where $P_{v_i}[v_j, v_i]$ denotes the subpath of P_{v_i} from v_j to v_i.

Lemma 2.9.5. *For every edge $e \in E(\vec{G})$, the probability that e is removed is at most*

$$\min\left\{\tfrac{x(e)}{\Delta},\, 1\right\}.$$

Proof. Fix an edge $e = (u, v)$. If $x(e) \geq \Delta$, then the bound $\min\{\frac{x(e)}{\Delta}, 1\} = 1$ is trivial. So assume $x(e) < \Delta$. We show that the probability that e is removed is at most $\frac{x(e)}{\Delta}$. By Algorithm 4, the edge e is removed if and only if the interval $(t(u),\, t(u) + x(e))$ contains a point of the form $\rho + l\Delta$ for some integer $l \geq 0$. There are two cases to consider:

1. **Case 1.** Suppose there exists $l_1 \geq 0$ such that the interval $(t(u),\, t(u) + x(e))$ lies entirely within $(l_1\Delta, (l_1+1)\Delta)$. In this case, e is removed if and only if

$$\rho \in (t(u) - l_1\Delta,\ t(u) + x(e) - l_1\Delta),$$

an interval of length $x(e)$. Since ρ is uniform in $(0, \Delta]$, the probability is $\frac{x(e)}{\Delta}$.

2. **Case 2.** Suppose instead that $(t(u),\, t(u)+x(e))$ straddles a multiple of Δ, i.e., there exists $l_2 \geq 0$ with

$$(l_2 - 1)\Delta < t(u) \leq l_2\Delta < t(u) + x(e) < (l_2 + 1)\Delta.$$

Then e is removed if either

$$\rho \in (t(u) - (l_2 - 1)\Delta, \Delta) \quad \text{or} \quad \rho \in (0,\, t(u) + x(e) - l_2\Delta).$$

The first interval has length $l_2\Delta - t(u)$, and the second has length $t(u) + x(e) - l_2\Delta$. Thus the total probability is

$$\frac{l_2\Delta - t(u)}{\Delta} + \frac{t(u) + x(e) - l_2\Delta}{\Delta} = \frac{x(e)}{\Delta}.$$

□

Let H be the graph returned by Algorithm 4 when applied to $\vec{G}$, and let $\vec{H}$ be its directed version.

Theorem 2.9.1. *Let B be an arbitrary block of $\vec{H}$. Then $G[V(B)] = B$, that is, no edge of $G[V(B)]$ is removed by the algorithm.*

Proof. We prove this for $\overrightarrow{G_k}$ by using induction. The statement is for a fixed value of ρ, $G_i[V(B)] = B$ for any block $B \in \overrightarrow{H_i}$ after performing the algorithm on $\overrightarrow{G_i}$. The base case is correct. For the induction step, suppose that the statement is correct for $G_k, k < n$, now consider the graph $\overrightarrow{G_{k+1}}$ together with H_{k+1}. Clearly $V(H_{k+1}) = V(H_k) \cup \{v_{k+1}\}$, and $|E(H_{k+1}) \setminus E(H_k)| \leq 2$. There are multiple cases,

- $|E(H_{k+1}) \setminus E(H_k)| = 0, 1$. In this case either v_{k+1} is an isolated vertex, or $H_{k+1} = H_k + e$ such that e is an edge-cut with v_{k+1} as one of its endpoints. We can see that the block decomposition of H_{k+1} is the block decomposition of H_k plus a single vertex.

- $|E(H_{k+1}) \setminus E(H_k)| = 2$. Suppose that v_{k+1} is adjacent to v_{j_1}, v_{j_2} in H_{k+1} and $g = (v_{j_1}, v_{j_2}) \in E(G_{k+1}), e = (v_{j_1}, v_{k+1}), f = (v_{k+1}, v_{j_2}) \in E(H_k)$. Remark 2.9.3 implies that $t(v_{k+1}) = t(v_{j_1}) + x_e$. It's not possible that the algorithm removes g, the reason is that if $\rho + m\Delta \in (t(v_{j_1}), t(v_{j_1}) + x_g)$, then either $\rho + m\Delta \in (t(v_{j_1}), t(v_{j_1}) + x_e)$, or $\rho + m\Delta \in (t(v_{k+1}), t(v_{k+1}) + x_f)$. This is because $x_e + x_f \geq x_g$ (The triangle inequality). Thus $g \in E(H_{k+1})$. Now we have 2 cases,

  - $v_{j_1}, v_{j_2} \in B$ for some block $B \in E(H_k)$. This implies that $v_{k+1} \in B'$ such that $V(B') = V(B) \cup \{v_{k+1}\}$, and $G_{k+1}[B'] = B'$.
  - $v_{j_1} \in B_1, v_{j_2} \in B_2$. This implies that g is an edge-cut in H_k. It can be derived that the triangle with vertices $v_{j_1}, v_{j_2}, v_{k+1}$ is a block in G_{k+1}.

□

Lemma 2.9.6. *Let B be a block in $\vec{H}$. Then there exists a vertex $r_B \in V(B)$ such that*

$$d_G^x(r_B, u) \leq \Delta \quad \textit{for all } u \in V(B),$$

and therefore $\operatorname{diam}_H^x(B) \leq 2\Delta$.

Proof. By Theorem 2.9.1, we have $\vec{G}[B] = B$. If $B = K_2$, the statement is immediate. Otherwise, B is a 2-vertex-connected induced subgraph of G. By Lemma 2.9.4, there exists a vertex $v_l \in V(B)$ such that for every $v_j \in V(B)$, the path P_{v_j} contains v_l, and moreover the subpath $P_{v_j}[v_l, v_j]$ is entirely contained in B. By Remark 2.9.5, we have

$$t(v_j) - t(v_l) \;=\; x\big(P_{v_j}[v_l, v_j]\big)\,.$$

If $x\big(P_{v_j}[v_l, v_j]\big) > \Delta$, then there would exist some integer $l \geq 0$ such that $\rho + l\Delta \in (t(v_l), t(v_j))$, implying that one of the edges of $P_{v_j}[v_l, v_j]$ would have been removed by the algorithm—contradicting the fact that B is preserved. Hence

$$x\big(P_{v_j}[v_l, v_j]\big) \leq \Delta.$$

Thus every vertex $v_j \in V(B)$ lies within distance Δ of v_l. Taking $r_B = v_l$, it follows that any two vertices in B are at distance at most 2Δ, and therefore $\operatorname{diam}_H^x(B) \leq 2\Delta$. □

We now disregard the orientation of edges in H (the graph obtained by Algorithm 4). The graph H is not necessarily connected, so let its connected components be $H_1, \ldots, H_k$. By Lemma 2.9.6, every block of each H_i has x-diameter at most 2Δ. We next perform

Algorithm 5 independently on each component H_i. At this stage, we do not use the fact that H_i is a subgraph of some maximal outer-planar graph; the only property we require is that every block of H_i has x-diameter at most 2Δ. Fix a component H_i for the remainder of the discussion, and note that in this stage H_i is treated as an undirected graph.

Algorithm 5 Second Stage

1: Let $r \in V(H_i)$ be an arbitrary root.
2: Choose $\rho' \in (0, \Delta]$ uniformly at random.
3: For each $j \geq 1$, define

$$U_j := \{\, v \in V(H_i) \mid \rho' + (j-1)\Delta < d^x_{H_i}(r, v) \leq \rho' + j\Delta \,\}.$$

4: Remove $\bigcup_j \delta_{H_i}(U_j)$.

Let H' be the graph obtained after performing the second stage on H_i.

Lemma 2.9.7. *For every edge $e \in E(H_i)$, the probability that e is removed in the second stage is at most $\frac{x(e)}{\Delta}$.*

Proof. Fix an edge $e = (u, v) \in E(H_i)$. Without loss of generality, assume

$$d^x_{H_i}(r, u) \leq d^x_{H_i}(r, v).$$

Since e is the edge between u, v, we also have

$$d^x_{H_i}(r, v) \leq d^x_{H_i}(r, u) + x(e).$$

Thus the interval $(d^x_{H_i}(r, u),\, d^x_{H_i}(r, v)]$ has length at most $x(e)$. Algorithm 5 removes e if and only if, for some integer $l \geq 0$, the shifted boundary point $\rho' + l\Delta$ lies inside this interval. Since ρ' is chosen uniformly at random from $(0, \Delta]$, the probability of this event is at most

$$\frac{d^x_{H_i}(r, v) - d^x_{H_i}(r, u)}{\Delta} \;\leq\; \frac{x(e)}{\Delta}.$$

□

This implies that after the execution of both stages of the algorithm, each edge $e \in E(G)$ is removed with probability at most $\frac{2x(e)}{\Delta}$. We now prove that every connected component H' of the resulting graph has x-diameter strictly 4Δ.

Lemma 2.9.8. *Let u, v be two vertices in the same connected component C of H'. Then*

$$d^x_{H_i}(u, v) < 4\Delta.$$

Proof. Let B_u and B_v denote the blocks of H_i that contain u and v, respectively, taken before performing Algorithm 5. If $B_u = B_v$, then by Lemma 2.9.6 we have $d^x_{H_i}(u, v) \leq 2\Delta$, and the claim follows. Now assume $B_u \neq B_v$. Let B_0 be the lowest common ancestor block of B_u and B_v in the block decomposition of H_i rooted at r. By definition of the block tree, there exists a cut vertex $r_u \in V(B_0)$ such that any path from r to u in H_i must pass through r_u. Similarly, there exists $r_v \in V(B_0)$ with the analogous property for v. Moreover, any u–v path in H_i must contain both r_u and r_v. Since u and v lie in the same connected component C of H', it follows that $r_u, r_v \in V(C)$.

We now bound the distances. By construction of the sets U_j in Algorithm 5, if u and r_u remain connected in C, then

$$d^x_{H_i}(r, u) - d^x_{H_i}(r, r_u) < \Delta.$$

Otherwise u would have been separated from r_u. Note that the left-hand side above equals $d^x_{H_i}(r_u, u)$, since every r–u path must pass through r_u once r_u is identified as a cut vertex. Hence,

$$d^x_{H_i}(r_u, u) < \Delta.$$

An identical argument gives

$$d^x_{H_i}(r_v, v) < \Delta.$$

Finally, since $r_u, r_v \in V(B_0)$, Lemma 2.9.6 implies

$$d^x_{H_i}(r_u, r_v) \leq 2\Delta.$$

Applying the triangle inequality, we obtain

$$d^x_{H_i}(u, v) \;\leq\; d^x_{H_i}(u, r_u) + d^x_{H_i}(r_u, r_v) + d^x_{H_i}(r_v, v) \;<\; \Delta + 2\Delta + \Delta \;=\; 4\Delta.$$

This completes the proof. □

Theorem 2.9.2. *$\alpha_{\mathcal{M}} \leq 8$, where $\mathcal{M}$ is the family of all instances of the minimum multicut problem in outer-planar graphs.*

Proof. We choose $\Delta = \frac{1}{4}$ and run the randomized rounding algorithm using the optimal solution x of the LP relaxation. Each edge e is removed with probability at most

$$\frac{2x_e}{\Delta} \;=\; \frac{2x_e}{1/4} \;=\; 8x_e.$$

Furthermore, the diameter of every connected component is strictly less than $4\Delta = 1$, which ensures that all source–sink terminals are separated from each other. Thus the expected cost of the multicut produced is at most $8 \cdot \mathrm{OPT}_{LP}$. □

Since the family of cactus graphs is a subfamily of outer-planar graphs, the lower bound from Section 2.5 applies in this setting as well. Hence we have

$$\tfrac{20}{9} \;\leq\; \alpha_{\text{OUTER-PLANAR}} \;\leq\; 8.$$

Closing this gap remains an open question.

Chapter 3

Conclusions

This thesis explored two directions under the unifying theme of cuts in optimization and approximation.

In the first part, we studied the complexity of split closures in mixed-integer sets with two integral and one continuous variable in $\mathbb{R}^3$. While a full extension of known results from $\mathbb{R}^2$ remains open, we made partial progress by constructing a polyhedron Q such that $P_I \subseteq Q \subseteq P_I^{split}$, where Q is describable by polynomially many inequalities in the input size of P. Determining whether P_I^{split} itself admits a polynomial-size description remains an open problem (see section 1.3).

In addition, in section 1.4, we introduced a generalized mixed-integer rounding (MIR) procedure derived from two equations, producing a new family of valid inequalities. These inequalities are rooted in the geometry of split closures in $\mathbb{R}^3$. Although we presented only a brief version of the derivation, the procedure shows promise for further exploration and could inspire future generalizations to higher dimensions or more complex systems.

In the second part, we analyzed the integrality gap of the natural LP relaxation of the minimum multicut problem in planar graphs and sub-classes of planar graphs. Using small-diameter decompositions and novel tools, we proved in section 2.5 that the integrality gap in cactus graphs is at least $\frac{20}{9}$, strictly improving the previous best lower bound of 2. Moreover, we established an upper bound of 3.44 for cactus graphs in section 2.6.

In fact, through a more delicate application of Theorem 2.4.2, we improved the lower bound further to $\frac{16}{7}$, although this stronger result is not included in the main body of the thesis. Specifically, we show that when $w = 2$, there always exists

$$SDD\big(G,\, 2w = 4,\, \tfrac{16}{7} \cdot \tfrac{1}{2w}\big) \;=\; SDD\big(G,\, 4,\, \tfrac{4}{7}\big)\,,$$

for every $G \in \text{CACTUS}$, i.e., every cactus graph. This provides an SDD with an optimal probability parameter when $w = 2$. Motivated by this observation, we propose the following Conjecture:

Conjecture 3.0.1. *Let* CACTUS *denote the family of cactus graphs. Then for any* $w \in \mathbb{N}$ *and any* $G \in \text{CACTUS}$*, there exists an*

$$SDD\big(G,\, 2w,\, \tfrac{4}{3w+1}\big)\,,$$

which implies that $\alpha_{\text{CACTUS}} \leq \frac{8}{3}$.

We also obtained tight bounds of 2 for unicyclic graphs in section 2.7, and for path cactus graphs in section 2.8. Moreover, Algorithm 3 can be adapted with minor modifications to handle series path graphs, as illustrated in Figure 3.1, yielding a tight integrality gap of 2 for this class as well.

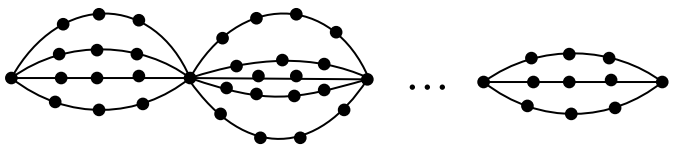


Figure 3.1: A Series Path Graph.

In section 2.9, for maximal outer-planar graphs we proved an upper bound of 8 via a randomized rounding algorithm. With minor modifications, the same argument extends to general outer-planar graphs and, in fact, to all graphs of tree-width 2.

Taken together, these results significantly refine the landscape of known bounds for the integrality gap of the minimum multicut problem in planar graphs. Table 3.1 summarizes the results on the integrality gap of the minimum multicut problem in planar graphs:

Graph Class	**Lower Bound**	**Upper Bound**
Trees	2	2
Unicyclic graphs	2	2
Series path graphs	2	2
Cactus graphs	$\frac{16}{7}$	$2 + \frac{1}{\ln 2} \approx 3.44$
Outer-planar graphs	$\frac{16}{7}$	8
Planar graphs	$\frac{16}{7}$	$O(1)$

Table 3.1: Known bounds on the integrality gap of the minimum multicut problem in various graph classes.

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