

Degrees of P -Grothendieck polynomials and regularity of Pfaffian varieties

by

Matthew St.Denis

A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Master of Mathematics
in
Combinatorics & Optimization

Waterloo, Ontario, Canada, 2024

© Matthew St.Denis 2024

Author's Declaration

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of Contributions

This thesis is expanded from a forthcoming paper co-authored with Oliver Pechenik [PS24]. Lemma 4.2.3 and the following corollary 4.2.4 are due to Oliver Pechenik.

Abstract

We prove a formula for the degrees of Ikeda and Naruse's P -Grothendieck polynomials using combinatorics of shifted tableaux. We show this formula can be used in conjunction with results of Hamaker, Marberg, and Pawlowski to obtain an upper bound on the Castelnuovo–Mumford regularity of certain Pfaffian varieties known as vexillary skew-symmetric matrix Schubert varieties. Similar combinatorics additionally yields a new formula for the degree of Grassmannian Grothendieck polynomials and the regularity of Grassmannian matrix Schubert varieties, complementing a 2021 formula of Rajchgot, Ren, Robichaux, St. Dizier, and Weigandt.

Acknowledgments

The author was partially supported by an NSERC Canada Graduate Scholarship (CGS-M) and by NSERC Discovery Grant RGPIN-2021-02391. Thanks to Oliver Pechenik for support and feedback in the writing of this thesis. Thanks also to Zach Hamaker, Patricia Klein and Jenna Rajchgot for feedback and suggestions, and to Elana Kalashnikov and Olya Mandelshtam for their many helpful comments on this thesis.

Table of Contents

Author's Declaration	ii
Statement of Contributions	iii
Abstract	iv
Acknowledgments	v
1 Introduction	1
2 Background	5
2.1 Classical symmetric function theory and Schubert calculus on the Grassmannian	5
2.2 Geometry of the Grassmannian and Schubert calculus	18
2.3 Grothendieck polynomials and K -theory of the Grassmannian	23
2.4 Shifted Schubert calculus and P -Schur functions	28
2.5 Both ways at once: K -theory of the orthogonal Grassmannian and P -Grothendieck polynomials	34
2.6 Matrix Schubert varieties	36
2.7 Algebraic background: Castelnuovo-Mumford regularity and degrees of Grothendieck polynomials	37
2.8 Algebraic background: Gröbner bases	38
3 Degrees of P-Grothendieck polynomials and proof of Theorem 1.0.1	42
4 Regularity of Pfaffian ideals	53
4.1 Skew-symmetric matrix Schubert varieties	53
4.2 Symplectic Grothendieck polynomials and proof of Theorem 1.0.2	54

5	Degrees of symmetric Grothendieck polynomials	57
6	The case of Q -Grothendieck polynomials	63
	References	66

Chapter 1

Introduction

Schubert calculus can be broadly described as the construction of polynomial models for the cohomology rings of certain geometric spaces. The prototypical study of the cohomology ring of the *Grassmannian* via the ring of symmetric functions has expanded immensely since the late 20th century to explain broader families of varieties, such as *determinantal varieties* or the *Lagrangian* and *orthogonal* Grassmannians of subspaces isotropic with respect to certain bilinear forms, exotic cohomology theories such as *K-theory* and *equivariant cohomology*, and other homological data, such as their *Hilbert series* and commutative-algebraic statistics that can be derived from it.

For combinatorialists, the interest of Schubert calculus lies in the fact that deep bijections lie behind the positivity of the cohomological data mentioned above. Schubert calculus is intimately tied to the theory of *symmetric functions* and *Young tableaux*, and has motivated developments such as the theory of *set-valued tableaux*; while initially created to explain the *K-theory* of the Grassmannian, these tableaux have proven to have a great deal of depth as combinatorial objects independent of their geometric birthplace. The combinatorics which appears in Schubert calculus also has a habit of showing significant parallels with combinatorial representation theory; the same symmetric functions which appear in the representation theory of the symmetric group appear again in the cohomology of the Grassmannian, and the *shifted tableaux* appearing in the theory of spin representations of S_n acquire new faces in the cohomology of the orthogonal Grassmannian.

There has been much recent interest in the degrees of *Grothendieck polynomials* (e.g., [DMS22, Haf22, PSW24, PY24, RRRSW21, RRW23, Rob23]) and related polynomials derived from the combinatorics of *K-theoretic* Schubert calculus. Beyond the intrinsic combinatorial interest of these formulas, a major motivation is that, as first observed in [RRRSW21], these degrees yield the *Castelnuovo–Mumford regularity* of *matrix Schubert varieties* X_w and certain *Kazhdan–Lusztig varieties* $X_{u,w}$. These affine varieties, introduced by [Ful92] and [WY08], respectively, are important models for local properties of Schubert varieties. Moreover, X_w and $X_{u,w}$ are generalized determinantal varieties, and in this context various special cases have a long and distinguished history in commuta-

tive algebra (see, e.g., [HE71, HT92, CH97, GM00, GK15, CDG20]). In Schubert calculus, representatives for the cohomology and K -theory classes of Schubert varieties can be obtained as torus-equivariant classes of X_w (e.g., [Ful92, FR03, KM05]), and associated combinatorial formulas may then be obtained from studying Gröbner bases (e.g., [KM05, HPW22, KW21]). Castelnuovo–Mumford regularity, meanwhile, is an important algebraic invariant that measures the complexity of the syzygies of defining ideals and simplifies Gröbner basis calculations; for a survey of regularity, see [Cha07].

Previous work on the degrees of Grothendieck polynomials has been restricted to those in “type A”, related to the Schubert varieties in *Grassmannians* $\mathrm{GL}_n(\mathbb{C})/P$ and the *complete flag variety* $\mathrm{GL}_n(\mathbb{C})/B$. (Here, B denotes the *Borel subgroup* of upper triangular matrices and $P \supset B$ is a *maximal parabolic subgroup*.) In contrast, we study the degrees of *P -Grothendieck polynomials* GP_λ , which are related in one sense to “type B” Schubert calculus and in another to “type C”. (We note that the “ P ” in the name of P -Grothendieck polynomials does not refer to a parabolic subgroup, but is instead just the letter “ P ”.)

Our main combinatorial theorem is the following:

Theorem 1.0.1. *Let λ be a strict partition and let $GP_{\lambda,n}$ be the P -Grothendieck polynomial for λ in n variables. Let $\Delta = (\Delta_1, \dots, \Delta_\ell)$ be the largest partition contained in λ such that all parts of Δ differ by at least two. Then the degree of the P -Grothendieck polynomial is*

$$\deg(GP_{\lambda,n}) = \begin{cases} |\Delta| + 2n\ell - \ell^2 - \ell, & \text{if } \Delta_\ell > 1; \\ |\Delta| + 2n\ell - \ell^2 - n, & \text{if } \Delta_\ell = 1. \end{cases}$$

Our proof of Theorem 1.0.1 is based on direct combinatorial analysis of the tableau formula for $GP_{\lambda,n}$ given in [IN13]. The first step of the proof is a reduction from general $GP_{\lambda,n}$ to a special subclass $GP_{\Delta,n}$, while the second step calculates the degree of $GP_{\Delta,n}$ by explicitly identifying a tableau contributing to its highest-degree component.

P -Grothendieck polynomials were introduced by T. Ikeda and H. Naruse [INN11, IN13] as representatives of K -theoretic Schubert classes on *maximal orthogonal Grassmannians* $\mathrm{O}_{2n+1}(\mathbb{C})/P$, where P is a particular parabolic subgroup of $\mathrm{O}_{2n+1}(\mathbb{C})$. It is in this sense that GP_λ is “type B”. By specializing GP_λ , one obtains the classical P -Schur functions P_λ that were introduced by I. Schur [Sch11] to describe the projective representation theory of symmetric groups and were connected to the cohomological Schubert calculus of maximal orthogonal Grassmannians by P. Pragacz [Pra91].

Recent work of E. Marberg and B. Pawlowski [MP21] shows that P -Grothendieck polynomials coincide with the stable limits of *vequillary symplectic Grothendieck polynomials*, representatives for K -theoretic classes of $\mathrm{Sp}_n(\mathbb{C})$ -orbit-closures on the flag variety $\mathrm{GL}_n(\mathbb{C})/B$. It is in this sense that GP_λ is related to “type C”. Just as matrix Schubert varieties provide affine models of Schubert varieties, *skew-symmetric matrix Schubert varieties* X_w^{ss} provide affine models for $\mathrm{Sp}_n(\mathbb{C})$ -orbit-closures (see [WY17]). Marberg and Pawlowski [MP20] have shown that, just as ordinary Grothendieck polynomials arise

as torus-equivariant classes of ordinary matrix Schubert varieties, symplectic Grothendieck polynomials arise as torus-equivariant classes of skew-symmetric matrix Schubert varieties X_w^{ss} . Skew-symmetric matrix Schubert varieties are instances of Pfaffian varieties [MP22] and hence have a commutative algebra history in that context (see, e.g., [HT92, DG09, DG11, DS11, LRWW17]). Indeed, we suspect that vexillary skew-symmetric matrix Schubert varieties are related to the *mixed ladder Pfaffian varieties* of [DG09], for which [DG11] determine a recursive regularity formula; forthcoming work by L. Escobar, A. Fink, J. Rajchgot, and A. Woo [EFRW24] will clarify this relationship.

To obtain algebraic consequences of Theorem 1.0.1, we leverage the connection between GP_λ and *symplectic Grothendieck polynomials*. We observe that X_w^{ss} is Cohen–Macaulay and that the degrees of symplectic Grothendieck polynomials yield its regularity. Thus, after describing the relation between degrees of symplectic Grothendieck polynomials and of P -Grothendieck polynomials, Theorem 1.0.1 implies the following, which is our main algebraic result developed in chapter 4.

Theorem 1.0.2. *Let $z \in \mathbf{S}_n$ be an FPF-vexillary fixed-point-free involution with associated symplectic shape the partition $\lambda_{\text{Sp}}(z) = \lambda$. Let k be the position of the last nonzero entry of the symplectic code $\text{SpCode}(z)$ and let Δ be the largest partition contained in λ such that all parts differ by at least two.*

Then the Castelnuovo–Mumford regularity of the vexillary skew-symmetric matrix Schubert variety X_z^{ss} satisfies

$$\text{reg}(X_z^{\text{ss}}) \leq \begin{cases} 2k\ell - \ell^2 - \ell - (|\lambda_{\text{Sp}}(z)| - |\Delta|), & \text{if } \Delta_\ell > 1; \\ 2k\ell - \ell^2 - k - (|\lambda_{\text{Sp}}(z)| - |\Delta|), & \text{if } \Delta_\ell = 1. \end{cases}$$

In contrast to earlier “type A” work (e.g., [RRRSW21, PSW24]), we obtain only an upper bound on regularity instead of an exact formula. This feature arises from the fact that we currently do not have a formula for the degrees of symplectic Grothendieck polynomials, but rather only partial information extracted from such a formula for GP_λ . It would be very interesting to study the degrees of symplectic Grothendieck polynomials directly. And indeed one might hope to be able to do so, perhaps using the combinatorics developed in [MP20, HMP22, MP22].

In Section 5, we imitate the proof of Theorem 1.0.1 to prove a new formula (Theorem 5.0.6) for the degrees of *symmetric Grothendieck polynomials*. These polynomials represent K -theoretic Schubert classes on a Grassmannian $\text{GL}_n(\mathbb{C})/P$. As shown in [RRRSW21], the degree of a symmetric Grothendieck polynomial yields the Castelnuovo–Mumford regularity of a corresponding Grassmannian matrix Schubert variety; hence, our degree formula yields a new regularity formula (Corollary 5.0.7) in this context. Our formula differs from those given previously by [RRRSW21, Haf22, RRW23, PSW24] and it is unclear how our formula relates to earlier ones. Moreover, our proof is arguably somewhat easier. Except for some background on matrix Schubert varieties in Section 2.6, Section 5

can be read independently of Sections 3 and 4 by readers whose primary interest is this type A setting.

Finally, in Section 6, we discuss obstacles to proving an analogue of Theorem 1.0.1 for *Q-Grothendieck polynomials* GQ_λ . It is perhaps surprising that this analogous problem should be substantially more difficult, since the classical *Q-Schur polynomials* obtained by specializing GQ_λ barely differ from *P-Schur* polynomials. Nonetheless, there are significant technical difficulties and we provide only partial progress. A relevant fact may be that *Q-Grothendieck* polynomials are related to the *K*-theoretic Schubert calculus of *Lagrangian Grassmannians* $\mathrm{Sp}_n(\mathbb{C})/P$ (see [IN13]) and likely related to the *symmetric matrix Schubert varieties* that give affine models for $\mathrm{O}_n(\mathbb{C})$ -orbit-closures on $\mathrm{GL}_n(\mathbb{C})/B$; the geometry of all these varieties is known to be significantly harder and worse-behaved than those of the varieties associated to GP_λ (see, e.g., [IN13, PY17, WY17, MP22] for related remarks).

The earlier sections of this paper are organized as follows. Chapter 2 recalls combinatorial and algebraic background. Chapter 3 establishes Theorem 1.0.1. Chapter 4 recalls additional background related to (skew-symmetric) matrix Schubert varieties and establishes Theorem 1.0.2.

This thesis is an expanded version of the paper [PS24].

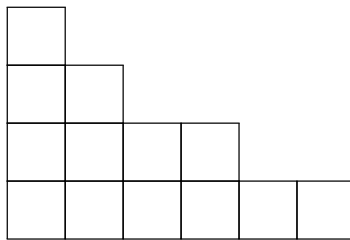
Chapter 2

Background

2.1 Classical symmetric function theory and Schubert calculus on the Grassmannian

A *partition* of a positive integer n is a weakly decreasing sequence of positive integers $\lambda = (\lambda_1, \dots, \lambda_\ell)$ such that the sum $\lambda_1 + \lambda_2 + \dots + \lambda_\ell = n$. For any partition, we can draw its *diagram*, which we construct by drawing λ_i left-justified boxes in the i th row. Diagrams of partitions are drawn in *French notation*, meaning the first row is placed at the bottom of the diagram, mimicking Cartesian coordinates. In the literature, diagrams can also be found in *English notation*, with the first row instead placed at the top, mimicking matrix coordinates. As the names may suggest, the former convention is somewhat more common in Francophone literature than it is in English, although this linguistic divide is not universal.

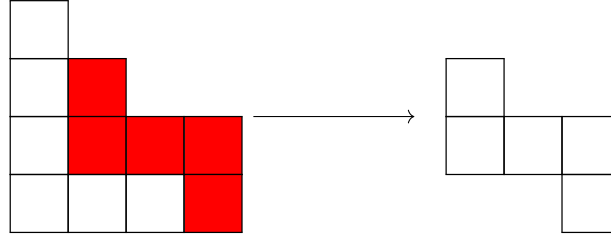
Example 2.1.1. $(6,4,2,1)$ is a partition of 13, and its diagram is



For a partition λ , we write $B \in \lambda$ to denote a specific box of the diagram. To facilitate manipulations of adjacent boxes, we use the notations $B^\uparrow, B^\rightarrow, B^\downarrow, B^\leftarrow$ to denote the boxes immediately above, to the right, below, and to the left of B , respectively. When there is no box above B in λ , it is most convenient to simply allow $B^\uparrow$ to be empty, to avoid the repetitive need to quantify statements with “provided such a box exists”. We make identical allowances in the other directions.

Definition 2.1.2. Given two partitions λ, μ with $\mu \subseteq \lambda$, the *skew diagram* λ/μ consists of the boxes B such that $B \in \lambda$ and $B \notin \mu$.

Example 2.1.3. With $\lambda = (4421)$ and $\mu = (3111)$, the skew diagram λ/μ is



The symmetric group S_n has a natural action on the polynomial ring $\mathbb{C}[x_1, \dots, x_n]$ induced from the canonical action of S_n on $[n]$ by permuting the indices of a polynomial by σ . Precisely, one defines $\sigma x_i = x_{\sigma^{-1}(i)}$ for all variables and then extends via multiplication and addition to the full polynomial ring (the inverse appears here because a monomial is, formally speaking, a function $[n] \rightarrow \mathbb{Z}_{\geq 0}^n$, and so the action induced on polynomials by an action on $[n]$ is contragredient.) We can also check this is equivalent to the left action which sends

$$\sigma(x_1^{d_1} x_2^{d_2} \dots x_n^{d_n}) = x_1^{\sigma(d_1)} x_2^{\sigma(d_2)} \dots x_n^{\sigma(d_n)}.$$

The subalgebra of $\mathbb{C}[x_1, \dots, x_n]$ invariant under the action of S_n are polynomials invariant under any permutation of their variables; this is known as the algebra of *symmetric polynomials* in n variables $\Lambda(x_1, \dots, x_n)$. The most straightforward method of producing an example of a symmetric polynomial is to begin with any monomial $x_1^{d_1} \dots x_n^{d_n}$ and force it to be symmetric by summing over its S_n -orbit. Expressions of the type $x_1^{d_1} \dots x_n^{d_n}$ will be extremely frequent, and so we will adopt the shorthand $\mathbf{x}^{\mathbf{d}}$ for a monomial to a vector exponent from now on. For example, the monomial $x_1 x_2^2 \in \mathbb{C}[x_1, x_2, x_3, x_4]$ has S_n -orbit

$$\mathcal{O}_{x_1 x_2^2} = \{x_1 x_2^2, x_1 x_3^2, x_1 x_4^2, x_2 x_3^2, x_2 x_4^2, x_3 x_4^2, x_1^2 x_2, x_1^2 x_3, x_1^2 x_4, x_2^2 x_3, x_2^2 x_4, x_3^2 x_4\},$$

and so the polynomial

$$x_1 x_2^2 + x_1 x_3^2 + x_1 x_4^2 + x_2 x_3^2 + x_2 x_4^2 + x_3 x_4^2 + x_1^2 x_2 + x_1^2 x_3 + x_1^2 x_4 + x_2^2 x_3 + x_2^2 x_4 + x_3^2 x_4$$

is symmetric.

In this example, the degree of $x_1^2 x_2$ is a partition, and in general we can reason that the S_n -orbit of any monomial contains exactly one monomial whose degree is a partition, obtained by applying the S_n -action to sort the degree in weakly decreasing order. Therefore, we can parameterize the S_n -orbits of monomials by partitions λ as

$$\mathcal{O}_\lambda := \{\mathbf{x}^{\sigma(\lambda_1, \lambda_2, \dots, \lambda_n)} : \sigma \in S_n\},$$

and we can now define the simplest family of symmetric polynomials:

Definition 2.1.4. For a partition λ , the *monomial symmetric polynomial* $m_\lambda(x_1, x_2, \dots, x_n)$ is

$$m_\lambda(x_1, x_2, \dots, x_n) := \sum_{\mathbf{x}^d \in \mathcal{O}_\lambda} \mathbf{x}^d.$$

The monomial symmetric polynomials are a linear basis of the algebra of all symmetric polynomials. They are linearly independent because by construction only m_λ contains the monomial $\mathbf{x}^\lambda$, and they span the algebra because if a symmetric polynomial contains a single monomial, then it must contain the entire orbit in order to be symmetric.

A less artificial way of producing a symmetric polynomial, and likely the first place one was encountered in the wild, is to write the coefficients of a polynomial in terms of its roots. Expanding the factored polynomial

$$(x - r_1)(x - r_2) \dots (x - r_n) = x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$$

we obtain *Vieta's formulas*, which were known, at least in a partial form, to François Viète as early as the 16th century.

$$\begin{aligned} a_0 &= (-1)^n m_{(n)}(r_1, r_2, \dots, r_n) \\ a_1 &= (-1)^{n-1} m_{(n-1)}(r_1, r_2, \dots, r_n) \\ &\vdots \\ a_{n-1} &= (-1)^1 m_{(1)}(r_1, r_2, \dots, r_n) \end{aligned} \tag{2.1.1}$$

The polynomial $m_{(k)}$ is important enough to warrant its own name; we refer to it as the *elementary symmetric polynomial* $e_k(x_1, \dots, x_n)$. Clearly the e_k do not form a linear basis of the algebra of symmetric polynomials, as there is only one of each degree. However, they do form a *multiplicative* base:

Theorem 2.1.5. For a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$, the *elementary symmetric polynomial* e_λ is defined as

$$e_\lambda(x_1, \dots, x_n) := e_{\lambda_1}(x_1, \dots, x_n) e_{\lambda_2}(x_1, \dots, x_n) \dots e_{\lambda_\ell}(x_1, \dots, x_n).$$

The polynomials e_λ form a linear basis of the algebra of symmetric polynomials.

Before we prove this, we will introduce a definition critical to the proof and which will also be useful later in this thesis, the concept of *lexicographic order* on monomials.

Definition 2.1.6. Suppose $\mathbf{x}^{d_1}$ and $\mathbf{x}^{d_2}$ are two monomials. We say $\mathbf{x}^{d_1} \prec \mathbf{x}^{d_2}$ in *lexicographic order* if the first nonzero entry of $d_1 - d_2$ is positive.

The name lexicographic refers to the similarity to the familiar dictionary order on words, which is encoded in lexicographic order if one interprets the letters $a - z$ as the integers $0 - 25$ and interprets the i th letter of a word as the degree of x_i .

Remark 2.1.7. Lexicographic order is a total order on monomials. If $\mathbf{x}^{d_1} \not\preceq \mathbf{x}^{d_2}$, then the first nonzero entry of $d_1 - d_2$ is negative, so $\mathbf{x}^{d_2} \prec \mathbf{x}^{d_1}$.

Example 2.1.8. In lexicographic order, we have the order relation

$$x_1^2 x_2 x_3 x_4 \prec x_1^2 x_3^2 x_4$$

as $(2, 1, 1, 1) - (2, 0, 2, 1) = (0, 1, -1, 0)$.

Lemma 2.1.9. *If $P(x_1, x_2, \dots, x_n) \in \Lambda(x_1, x_2, \dots, x_n)$ is a symmetric polynomial, then the maximum monomial of P (which we refer to as the leading monomial of P) has a weakly decreasing degree vector.*

Proof of Theorem 2.1.5. Let $P(x_1, \dots, x_n)$ be any symmetric polynomial in n variables. By Lemma 2.1.9, the leading monomial of P has a degree vector that is a partition. Let this partition be λ . Since $e_\lambda(x_1, \dots, x_n)$ also contains $\mathbf{x}^\lambda$, this is also the leading term of $e_\lambda(x_1, \dots, x_n)$. Therefore if $c\mathbf{x}^\lambda$ is the leading term of P , then $P - ce_\lambda(x_1, \dots, x_n)$ contains terms strictly preceding $\mathbf{x}^\lambda$. The result follows by induction on lexicographic order. $\square$

There are two other simple bases of the algebra of symmetric polynomials which, while not directly useful for the algebraic geometry we wish to discuss below, are still quite useful in other branches of mathematics, and which we will briefly mention here for completeness.

Definition 2.1.10. The *complete homogeneous symmetric polynomial* $h_k(x_1, \dots, x_n)$ is defined as

$$h_k(x_1, \dots, x_n) := \sum_{\substack{a_1 + a_2 + \dots + a_n = k \\ a_i \in \mathbb{Z}_{\geq 0}}} x_1^{a_1} x_2^{a_2} \dots x_n^{a_n}$$

In English, $h_k(x_1, \dots, x_n)$ is simply the sum of all monomials with n variables with degree k . As for the elementary symmetric polynomials, for a partition λ , we define

$$h_\lambda(x_1, \dots, x_n) := h_{\lambda_1}(x_1, \dots, x_n) h_{\lambda_2}(x_1, \dots, x_n) \dots h_{\lambda_\ell}(x_1, \dots, x_n).$$

Definition 2.1.11. The *power sum symmetric polynomial* $p_k(x_1, \dots, x_n)$ is defined as

$$p_k(x_1, \dots, x_n) := x_1^k + x_2^k + \dots + x_n^k.$$

For a partition λ , we define $p_\lambda(x_1, \dots, x_n)$ as

$$p_\lambda(x_1, \dots, x_n) := p_{\lambda_1}(x_1, \dots, x_n) p_{\lambda_2}(x_1, \dots, x_n) \dots p_{\lambda_\ell}(x_1, \dots, x_n).$$

Both the complete homogeneous symmetric polynomials and the power sum symmetric polynomials are linear bases of $\Lambda(x_1, \dots, x_n)$, and the twelve transition matrices between $m_\lambda, e_\lambda, h_\lambda$ and p_λ are quite interesting from a combinatorial perspective (see [MR15] for further discussion of these).

Before introducing the basis of symmetric polynomials which turns out to be by far the most important in practice, we should begin with a remark on power series. The most frequent situation which arises in symmetric function theory is that theorems do not depend in a serious way on how large the number of variables n is, provided n is "big enough" (whatever that may mean in the context of the theorem) for the assumptions to hold. Rather than bookkeeping the number of variables, what one would really like to write is simply the symbol m_λ or e_μ , with the assumption that the number of variables is enough. To make this idea precise, we usually work with an infinite variable set, as follows. Replace a finite symmetric group S_n with the infinite symmetric group S_∞ of permutations of $\mathbb{Z}_{>0}$ with a finite number of non-fixed points, and replace the polynomial ring $\mathbb{C}[x_1, \dots, x_n]$ with the ring of formal power series in infinitely many variables $\mathbb{C}[[x_1, x_2, \dots]]$. The orbit of any monomial under the action of S_∞ is now no longer a polynomial, but a homogeneous infinite formal power series which we refer to as a *symmetric function* (despite not being, in any natural sense, a function), and so the monomial, elementary, complete homogeneous, and power sum symmetric polynomials $m_\lambda(x_1, \dots, x_n), e_\lambda(x_1, \dots, x_n), h_\lambda(x_1, \dots, x_n), p_\lambda(x_1, \dots, x_n)$ are replaced respectively by the monomial, elementary, complete homogeneous, and power sum symmetric *functions* $m_\lambda, e_\lambda, h_\lambda, p_\lambda$.

As we shall see in this section, Section 2.6, and in the main theorem of this thesis, situations occasionally do arise when we really are interested in a finite symmetric polynomial, and a power series will not do. However, this does not present a problem, as we can immediately drop back down to any finite variable set of length k by specializing all x_i with $i > k$ to 0. Conversely, Λ is the *inverse limit* $\lim_{\leftarrow} \Lambda(x_1, \dots, x_n)$ in the category of graded algebras; we will not go afield into this category-theoretic construction, except to mention that from this fact it follows that we can also prove things in the other direction; if a theorem about the structure of $\Lambda(x_1, \dots, x_n)$ holds for all $\Lambda(x_1, \dots, x_n)$ with $n > n_0$, then it holds in Λ as well. We will also see examples of this below, where some theorems may be more convenient to prove in a finite variable set.

With these technicalities dealt with, we can now introduce the family of symmetric functions which turns out to be by far the most important in practice, although not the most obvious to define.

Given a Young diagram λ and some alphabet $\mathbb{A}$ (which can be any arbitrary set, although generally taken to be totally ordered), a *filling* or *tableau* on λ is a function $T : \lambda \rightarrow \mathbb{A}$ which we interpret as "filling" the boxes of λ with elements of $\mathbb{A}$. The most classical example of a Young diagram filling is a *Semistandard Young tableau*:

Definition 2.1.12. A *Young tableau* T on a diagram λ is a filling $T : \lambda \rightarrow \mathbb{Z}_{>0}$ with the additional properties that for all boxes B :

1. $T(\mathbf{B}) \leq T(\mathbf{B}^\rightarrow)$
2. $T(\mathbf{B}) < T(\mathbf{B}^\uparrow)$

Definition 2.1.13. The *content vector* $c(T)$ of a tableau

$$c(T) = (\# \text{ of ones in } T, \# \text{ of twos in } T, \dots)$$

and we say the *degree* of T is $d(T) = \sum c(T)$ ($d(T)$ can be visually interpreted as the number of boxes of T .)

Example 2.1.14. A Young tableau of shape $(6, 4, 2, 1)$ and content $(1, 2, 3, 1, 3, 3)$.

6					
4	6				
3	3	5	5		
1	2	2	3	5	6

Definition 2.1.15. The set of Young tableaux on λ with entries in $[n]$ and the set of Young tableaux on λ with entries in $\mathbb{Z}_{>0}$ are denoted by

$$\text{SSYT}(\lambda, n) \text{ and } \text{SSYT}(\lambda),$$

respectively.

Given sets of combinatorial objects, we can pack these into a generating function, weighted by their content, to obtain what is known as a *Schur polynomial* or a *Schur function*.

Definition 2.1.16. The *Schur polynomial* $s_\lambda(x_1, \dots, x_n)$ is

$$s_\lambda(x_1, \dots, x_n) := \sum_{T \in \text{SSYT}(\lambda, n)} \mathbf{x}^{c(T)}.$$

The *Schur function* s_λ is

$$s_\lambda := \sum_{T \in \text{SSYT}(\lambda)} \mathbf{x}^{c(T)}.$$

Example 2.1.17. The family of Young tableaux $\text{SSYT}((2, 1), 3)$ is

2		2		3		3		3		3		2		3	
1	1	1	2	1	3	1	1	2	2	2	3	1	3	1	2

which leads to the Schur polynomial

$$s_{(2,1)}(x_1, x_2, x_3) = x_1^2 x_2 + x_1 x_2^2 + x_1 x_3^2 + x_1^2 x_3 + x_2^2 x_3 + x_2 x_3^2 + 2x_1 x_2 x_3.$$

This example turns out to be symmetric, but unlike the previous bases, it is not at all obvious why (or if) these polynomials should be symmetric in general. There are many paths to the symmetry of the Schur polynomials, and in fact the Schur polynomials are such fundamental objects that there are many equivalent ways to *define* the Schur polynomials. They can be defined algebraically with Vandermonde determinants, direct combinatorial proofs involving Young tableau bijections, as well as proofs which saddle the divide between algebra and combinatorics, such as the Lindström-Gessel-Viennot proof of the Jacobi-Trudi formula using lattice paths all yield proofs of symmetry of the Schur polynomials, as well as alternative definitions. We will give a bijective proof here. At least two bijective proofs exist; the historical first proof given is due to E. Bender and D. Knuth, and another can be given using crystal operators. We choose to give the proof by the so-called *Bender-Knuth involutions* here; crystal operators on Young tableaux arise very naturally from an extremely fruitful body of theory around *crystal bases* [BS17], but we will have no need to develop any crystal theory in this thesis.

Theorem 2.1.18. *The polynomial $s_\lambda(x_1, \dots, x_n)$ is symmetric.*

Proof. [BK72] For each positive integer i , we define an involution σ_i on $\text{SSYT}(\lambda)$ which reverses the number of i and $i + 1$ in the tableau, i.e. if $T(\lambda)$ contains m copies of i and n copies of $i + 1$, then $\sigma_i(T(\lambda))$ will contain n copies of i and m copies of $i + 1$. Any permutation of the content vector of $T(\lambda)$ is reachable by some sequence of swaps of adjacent contents, and will therefore have the same leading coefficient. This in turn proves that the Schur polynomials are a sum of symmetric group orbits, i.e. they are symmetric.

To define the involution σ_i , call an entry in the tableau *pinned* if $T(\mathbf{B}) = i$ and $T(\mathbf{B}^\uparrow) = i + 1$, or if $T(\mathbf{B}) = i + 1$ and $T(\mathbf{B}^\downarrow) = i$. If an i or an $i + 1$ is not pinned, we call it *free*. Note that each i is pinned by a unique $i + 1$ and vice-versa, and in particular the number of pinned i and pinned $i + 1$ is equal.

Next, we observe that the free entries form a single continuous block $i^a(i + 1)^b$ in each row, which cannot overlap with the block in another row (or else some of the entries would be paired). So we can define σ_i as replacing each free block $i^a(i + 1)^b$ with a block $i^b(i + 1)^a$, with the result that the quantities of i and $i + 1$ in the tableau have been reversed, as desired. $\square$

Example 2.1.19. To see how this works, let us apply the Bender-Knuth involution σ_4 to the tableau

8						
7	8					
5	5	5	6	8		
3	3	4	5	6		
2	3	3	4	4	5	5

which has three 4s and six 5s. We highlight the pinned entries in red ■, and the free entries in blue ■, as follows:

8						
7	8					
5	5	5	6	8		
3	3	4	5	6		
2	3	3	4	4	5	5

We then apply the involution to each block of free entries, and obtain the tableau

8						
7	8					
4	4	5	6	8		
3	3	4	5	6		
2	3	3	4	4	4	5

which now contains three 5s and six 4s.

Remark 2.1.20. Every step of the above theory can be carried out just as well if we replace λ with a skew diagram λ/μ ; by doing so one obtains *skew Young tableaux*, or simply *skew tableaux*, whose generating functions are the *skew Schur functions*. The Bender-Knuth involutions apply without modification to show that the skew Schur functions are also symmetric.

This proves the symmetry of the Schur functions in a combinatorial manner, but the Schur functions possess a great deal more structure than simply being symmetric. Seeing another family of symmetric functions indexed by partitions should arouse suspicions that these are another linear basis of Λ , and this is indeed the case:

Theorem 2.1.21. *The polynomials $\{s_\lambda\}_{\ell(\lambda) \leq n}$ are a linear basis of $\Lambda(n)$.*

Proof. We can prove this fact in a mostly combinatorial manner by showing that the transition matrix from the monomial symmetric polynomials to the Schur polynomials is triangular. We arrange the indices of our matrix in *lexicographical order*, which means $\mu < \lambda$ if the first nonzero entry of $\lambda - \mu$ is positive. Let us refer to the $m - s$ transition matrix as K . It is clear that $K_{\lambda\lambda} > 0$, since one can construct a Young tableau of shape λ and content λ by filling the i th row with λ_i copies of i (in fact, with a little more thought

one can convince oneself that this is the *only* possible tableau of shape λ and content λ , and so the matrix K is actually unitriangular). For example, the only Young tableau of shape 43321 and content 43321 is

5				
4	4			
3	3	3		
2	2	2		
1	1	1	1	

The form of the tableau of shape λ and content λ gives a suggestion as to why $K_{\lambda\mu} = 0$ whenever $\lambda < \mu$. It is an elementary but important observation that no tableau can contain an i higher than the i th row due to column-increasingness. Thus if j is the first index for which $\mu_j - \lambda_j \neq 0$ and $\mu_j - \lambda_j > 0$, the first $j - 1$ rows of the tableau can only be completely filled with μ_{j-k} copies of $j - k$ in each row. And since no rows higher than j can contain a j , we are left with only λ_j boxes in which to place μ_j entries, which is impossible. Therefore, $K_{\lambda\mu} = 0$ for all $\mu > \lambda$, and the matrix is triangular in this ordering. $\square$

The Schur basis possesses the rather remarkable property that products of Schur functions expand positively in the Schur basis, i.e. given two Schur polynomials

$$s_\lambda s_\mu = \sum_{\nu} c_{\lambda,\mu}^{\nu} s_{\nu},$$

all coefficients $c_{\lambda,\mu}^{\nu} \geq 0$. In a moment, we will prove not only the non-negativity of these coefficients but also give a combinatorial interpretation of them. First, however, will cover a very special case which is nonetheless important enough to the upcoming geometric theory to warrant its own name.

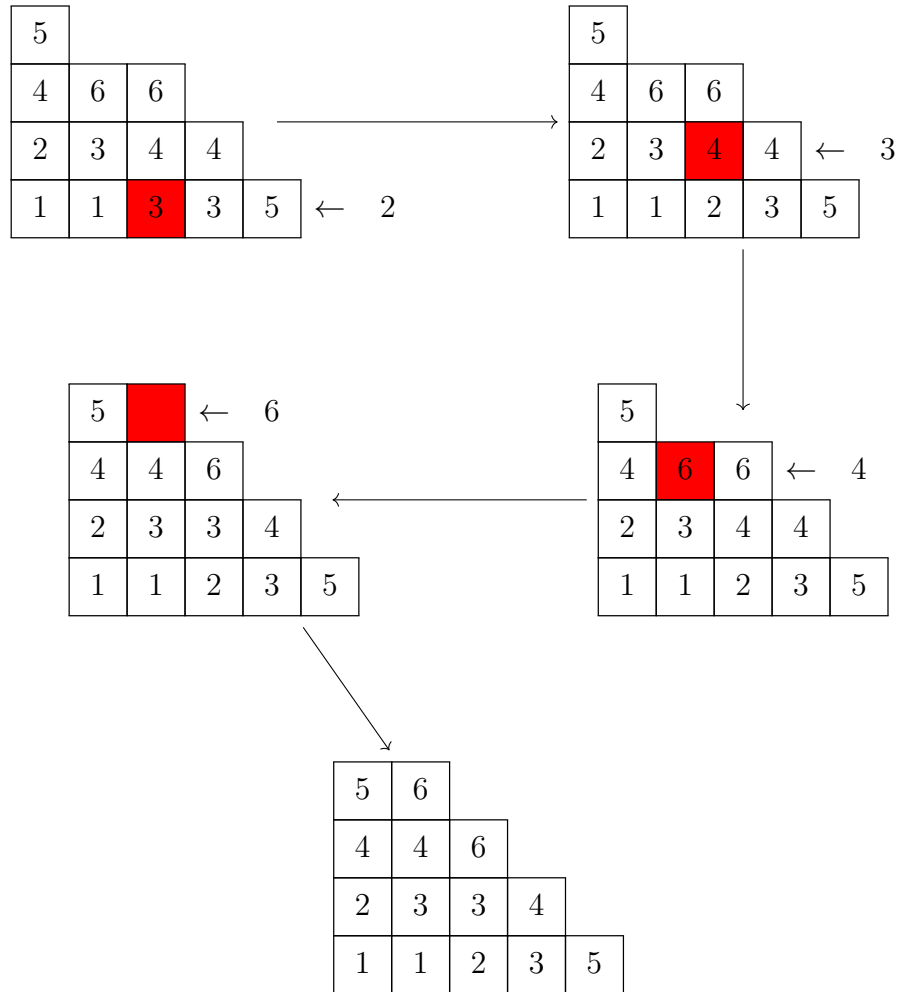
Definition 2.1.22. Given a tableau $T(\lambda)$ and a number n , we define the *row insertion* $T'(\lambda) := T(\lambda) \leftarrow n$ via the following algorithm. Let $\mathbf{B}$ be the leftmost in the first row of λ such that $T(BBB) > n$. If $\mathbf{B}$ does not exist, then add a new box containing n to the end of the row and terminate. Otherwise, let n_1 be $T(\mathbf{B})$, and insert n_1 into the final $n - 1$ rows of λ recursively. In common parlance, we say that n “bumps” n_1 .

This insertion algorithm preserves the row and column-increasingness of the tableau if it was already semistandard, and will output a tableau of shape λ' such that λ'/λ consists of a single box added to an outer corner of λ .

Example 2.1.23. To insert 2 into the tableau

5				
4	6	6		
2	3	4	4	
1	1	3	3	5

we follow the sequence of steps (with bumped entries highlighted in red ■)



Remark 2.1.24. A simple but extremely important remark is that the insertion algorithm of Definition 2.1.22 is completely reversible, *provided one knows the location of the box λ'/λ* . If we know which box was added, we can locate the box it bumped from the row below, and run the algorithm in reverse to recover the original tableau and the element which was inserted.

Theorem 2.1.25 (Pieri's rule). *Let $S(\lambda, k)$ be the set of all partitions μ such that the diagram of μ/λ consists of k boxes, with at most one box in each row. Then*

$$s_\lambda e_k = \sum_{\mu \in S(\lambda, k)} s_\mu$$

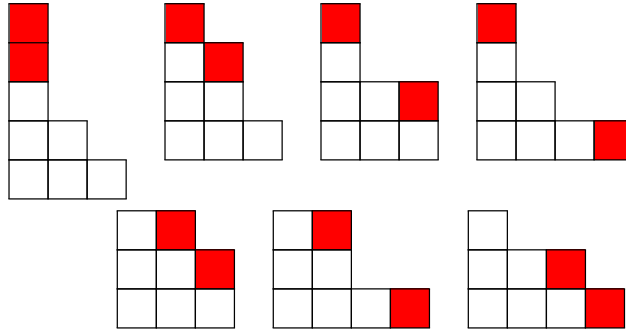
Lemma 2.1.26. *Let $n_1 < n_2$, and let $T(\lambda)$ be a tableau of shape λ . Then the new box of $(T(\lambda) \leftarrow n_2) \leftarrow n_1$ sits strictly above the new box of $T(\lambda) \leftarrow n_2$.*

Proof. From the observation that the box bumped from the first row of $T(\lambda)$ by n_2 is strictly greater than the box bumped from the first row of $T(\lambda) \leftarrow n_2$ by n_1 (provided n_2 bumped any box at all), the lemma follows by induction. $\square$

Remark 2.1.27. By repeated application, Lemma 2.1.26 applies equally well to a sequence $n_1 < n_2 < \dots < n_k$ of arbitrary length. This fact will be used in the proof of Theorem 2.1.25 below.

Proof of Theorem 2.1.25. A combinatorial proof follows quickly from Lemma 2.1.26. Given a pair of a tableau on λ and a sequence of k numbers, inserting the k numbers into λ in decreasing order produces a tableau of shape $\mu \in S(\lambda, k)$. To reverse the process and thereby prove the identity by a bijection, if $\mu \in S(\lambda, k)$, we can reverse insert the k boxes of μ/λ from top to bottom to obtain a tableau of shape λ and a sequence of k numbers, which are seen to be strictly increasing by Lemma 2.26. $\square$

Example 2.1.28. To see how this works, let us try expanding the product $s_{(3,2,1)}e_2$. The set $S((3,2,1), 2)$ consists of the diagrams (added boxes highlighted in red)



So we can read off the product as

$$s_{(3,2,1)}e_2 = s_{(3,2,1,1,1)} + s_{(3,2,2,1)} + s_{(3,3,1,1)} + s_{(4,2,1,1)} + s_{(3,3,2)} + s_{(4,2,2)} + s_{(4,3,1)}.$$

Pieri's rule is a special case of the multiplication $s_\lambda s_\mu$ when $\mu = 1^k$. Structurally, this rule completely determines multiplication in Λ , but a general rule for expanding two arbitrary Schur functions as a linear combination of Schur functions is desirable both for

geometric reasons we will soon see and because in the presence of non-negative integers, some combinatorial rule should be suspected to exist. Such a combinatorial rule is considerably more subtle; while the rule was given by Littlewood and Richardson in the 1930s, they were unable to prove it in general, and subsequent attempts at proof were notorious for subtle gaps. The first complete proof did not appear until the 1970s [Tho78]. There are nowadays a plethora of proofs of the Littlewood-Richardson rule, including some incredibly brief ones. The proof below is due to Stembridge [Ste02], and although it is inspired by fairly deep mathematics involving crystal basis theory, it requires essentially no more machinery than the Bender-Knuth involutions 2.1.18.

Definition 2.1.29. We use the notation $T_{\leq j}$ to denote the first j columns of a tableau T . We interpret $T_{< j}, T_{\geq j}, T_{> j}$ in the same manner.

Theorem 2.1.30. (*Littlewood-Richardson rule*) Let λ and μ be partitions. Then

$$s_\lambda s_\mu = \sum s_{\lambda+c(T)}$$

where the sum ranges over all $T \in \text{SSYT}(\mu)$ such that $\lambda + c(T_{\geq j})$ is a partition for all $j \geq 1$.

We will obtain this expansion as an immediate corollary of the following lemma:

Lemma 2.1.31. Fix some number of variables $n \in \mathbb{Z}_{>0}$. For a partition λ , define the alternant

$$a_\lambda := \det \left[x_i^{\lambda_j} \right] = \sum_{w \in S_n} \text{sgn}(w) \mathbf{x}^{w\lambda}.$$

Then

$$a_{\lambda+\rho} s_\mu = \sum a_{\lambda+\rho+c(T)} \tag{2.1.2}$$

where the sum ranges over all $T \in \text{SSYT}(\mu)$ such that $\lambda + c(T_{\geq j})$ is a partition for all $j \geq 1$.

Proof. We begin by rewriting the left-hand product as

$$\sum_{w \in S_n} \text{sgn}(w) \mathbf{x}^{w(\lambda+\rho)} \sum_{T \in \text{SSYT}(\mu)} \mathbf{x}^{c(T)}.$$

Since s_μ is symmetric, we can choose to rewrite

$$\sum_{T \in \text{SSYT}(\mu)} \mathbf{x}^{c(T)}$$

as

$$\sum_{T \in \text{SSYT}(\mu)} \mathbf{x}^{w(c(T))}$$

for any fixed $w \in S_n$. This simplifies the product to

$$\sum_{w \in S_n} \sum_{T \in \text{SSYT}(\mu)} \text{sgn}(w) \mathbf{x}^{w(\lambda + \rho + c(T))} = \sum_{T \in \text{SSYT}(\mu)} a_{\lambda + \rho + c(T)}.$$

To reach the result, we need only to show that all terms for which $c(T_{\geq j})$ is not a partition cancel each other. Following Stembridge, we call such a T a "bad guy", and thus we would like to produce a sign-reversing involution $T \rightarrow T^*$ on the set of bad guys. Amazingly, this is accomplished by nothing more than a clever application of a Bender-Knuth involution.

If T is bad, then there is a maximal column j for which $\lambda + c(T_{\geq j})$ fails to be a partition, and for this j , there is a minimal k such that

$$\lambda_k + c(T_{\geq j})_k < \lambda_{k+1} + c(T_{\geq j})_{k+1}. \quad (2.1.3)$$

Since this is the maximal j which is not a partition, and since the imbalance between k and $k+1$ can only change by either 1 or 0 per column, this implies there is a free (in the terminology of Theorem 2.1.18) $k+1$ in the j th column of T , and that the inequality 2.1.3 is in fact only off by one. Therefore we actually have

$$\lambda_k + c(T_{\geq j})_k + 1 = \lambda_{k+1} + c(T_{\geq j})_{k+1}. \quad (2.1.4)$$

Now to define T^* we apply the Bender-Knuth involution σ_k to $T_{< j}$ and fix $T_{\geq j}$. This is evidently an involution, since $T_{\geq j}$ does not change at all, so T^* is a bad guy for the same j, k pair as T . To finish the proof, we need to show that $a_{\lambda + \rho + c(T)} = -a_{\lambda + \rho + c(T^*)}$. This is a straightforward manipulation of the right-hand expression:

$$\begin{aligned} a_{\lambda + \rho + c(T^*)} &= \sum_{w \in S_n} \text{sgn}(w) \mathbf{x}^{w(\lambda + \rho + c(T^*))} \\ &= \sum_{w \sigma_k \in S_n} -\text{sgn}(w) \mathbf{x}^{w \sigma_k(\lambda + \rho + c(T^*))}. \end{aligned}$$

We conclude with the observation that $\sigma_k(\lambda + \rho + c(T^*)) = \lambda + \rho + c(T)$. In fact, the identity $\sigma_k(T_{< j}^*) = T_{< j}$ is direct from our definition of T^* , and the identity $\lambda_k + c(T_{\geq j})_k + 1 = \lambda_{k+1} + c(T_{\geq j})_{k+1}$ has already been established in 2.1.4.

□

As a straightforward corollary of this expansion, we can obtain the so-called *bialternant formula* for a Schur function, which is sometimes taken as the definition of the Schur functions by the more algebraically minded.

Corollary 2.1.32. $s_\mu = \frac{a_{\mu + \rho}}{a_\rho}.$

Proof. Setting $\lambda = \emptyset$ in Lemma 2.1.31, we obtain the summation

$$a_\rho s_\mu = \sum a_{\rho+c(T)}.$$

However, $c(T_{\geq j})$ is only a partition for all j for the single case of the partition constructed in Theorem 2.1.21, for which the i th row of the tableau is filled entirely by i . The right-hand sum therefore collapses into a single term, and upon dividing by a_ρ the result follows. $\square$

Proof of Theorem 2.1.30. The result follows by dividing both sides of equation 2.1.2 by a_ρ and applying corollary 2.1.32. $\square$

Remark 2.1.33. The above is a brief sketch of a few pieces of the theory of symmetric functions necessary for the following sections. For much deeper textbook coverage, the classic combinatorial treatment of symmetric functions is chapter 7 of [Sta24]; the book [Mac15], often referred to as the "bible" of symmetric functions, contains an extremely thorough development from a much more algebraic perspective. Some more modern developments in the combinatorial theory can be found in [MR15].

2.2 Geometry of the Grassmannian and Schubert calculus

Remark 2.2.1. We will not give proofs of any of the geometric theorems stated in this section. The traditional textbook source for the Schubert calculus of the Grassmannian, at a quite accessible level, is [Ful97]; unless a different source is cited, any claims made without a proof can be readily found there. The study of the Grassmannian leads naturally to the *flag variety* and its corresponding *Schubert polynomials*, which do not form the topic of this thesis; for this theory, one can consult [Man01]. At a much higher level of abstraction, one can also find coverage of Schubert calculus in sources written for an audience of algebraic geometers, such as [AF24, EH16, GH78].

Let us begin by defining our central object of study for this section.

Definition 2.2.2. The *Grassmannian* $Gr(n, k)$ is the set of all k -dimensional subspaces of $\mathbb{C}^n$.

To make points in the Grassmannian more concrete, it is quite useful to think of them as $k \times n$ full-rank matrices, with a matrix corresponding to its row space. Many bases can be chosen, corresponding to many different matrices with the same row space, but all such matrices have the same reduced row echelon form. For example, the two matrices

$$\begin{bmatrix} 0 & 1 & 1 & 2 & 3 & 1 & -2 & 0 & 1 \\ 0 & 1 & 1 & 2 & 3 & 2 & -3 & 1 & 3 \\ 0 & 1 & 1 & 2 & 3 & 0 & -1 & -1 & -1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 2 & -2 & 1 & 3 \\ 0 & -1 & -1 & -2 & -3 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 & 3 & 1 & -2 & 1 & 2 \end{bmatrix}$$

both have reduced row echelon form

$$\begin{bmatrix} 0 & 1 & 1 & 2 & 3 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \quad (2.2.1)$$

Defined from scratch this way, the Grassmannian is simply a set of subspaces with no additional structure, but we can realize it as a geometric object by embedding it as a projective variety. The $k \times k$ minors of a matrix stand out as candidates for projective coordinates, as they are k -linear and therefore any two matrices with the same rowspace should map to the same point in projective space, and they cannot all be zero since the matrix has full rank. This idea leads to the following definition:

Definition 2.2.3. The *Plücker embedding* is the map $G(n, k) \rightarrow \mathbb{P}^{\binom{n}{k}-1}$ which maps a subspace U to all $k \times k$ minors of any $k \times n$ matrix M with rowspace U .

We can quickly convince ourselves that this embedding is algebraic; consider M to be a $k \times n$ full-rank matrix in reduced row echelon form. The k pivot columns of M consist of a $k \times k$ identity matrix. If we consider A as the matrix consisting of all non-pivot columns, then the $k \times k$ identity column allows us to view *all* minors of A in bijection with $k \times k$ minors of the full matrix (except for the identity minor). In particular, the 1×1 minors of A are $k \times k$ minors of M , and so clearly every other minor is a polynomial in these.

However, such a family of equations is not particularly desirable, both because the degrees of the equations become large and because the equations are inhomogeneous. There is a family of homogeneous quadrics known as the *Plücker relations* which cut out the Grassmannian as a projective variety in a more efficient way. Constructing the Plücker relations is somewhat delicate, and we will not do it here; the details of the construction can be found in [MS05].

All reduced row echelon forms can be classified by the shape of their nonzero entries; for example, if we replace all free entries of 2.2.1 with a generic $\bullet$, we obtain

$$\begin{bmatrix} 0 & 1 & \bullet & \bullet & \bullet & 0 & \bullet & 0 & \bullet \\ 0 & 0 & 0 & 0 & 0 & 1 & \bullet & 0 & \bullet \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \bullet \end{bmatrix}$$

In this example, this partition is $(5, 2, 1)$. For geometric reasons, it is more natural to tweak this definition slightly; observe that the largest possible partition corresponding to the free entries in a $k \times n$ matrix is $(n - k)^k$, corresponding to the matrix for which the first k columns are all pivots and the trailing $k \times (n - k)$ submatrix is entirely free. Since all possible partitions live inside a $k \times (n - k)$ rectangle, we can make the following definition:

Definition 2.2.4. Let λ be a partition such that $\lambda \subseteq (n - k)^k$. The *complement* λ^c with respect to this rectangle is the partition obtained by aligning λ in the upper-right corner of a $k \times n - k$ rectangle and taking $(n - k)^k / \lambda$.

Example 2.2.5. In the above example, where $k = 3, n - k = 6$, and $\lambda = (5, 2, 1)$, we draw the following diagram

	•	•	•	•	•
				•	•
					•

so $\lambda^c = (5, 4, 1)$.

Remark 2.2.6. In the literature, especially among authors using English diagram notation, it is common to find a slightly varied definition of the reduced row echelon form. To cause the complement to correspond to an English diagram of a partition without re-aligning, one begins the reduction algorithm from the bottom-left rather than the top left, and arrives at a matrix upside-down of the form we will use here.

The condition that matrices representing a point $V \in Gr(n, k)$ correspond to the partition λ can be restated algebraically, which is useful both to be slightly more formal and because it makes geometric operations on the Schubert cell more apparent.

Definition 2.2.7. Let $\mathcal{B} = (b_1, b_2, \dots, b_n)$ be an ordered basis of $\mathbb{C}^n$. The sequence of subspaces $F_1(\mathcal{B}) = \text{span}(b_1), F_2(\mathcal{B}) = \text{span}(b_1, b_2), \dots, F_n(\mathcal{B}) = \text{span}(b_1, b_2, \dots, b_n)$ is referred to as the *complete flag* $F(\mathcal{B})$. The *standard complete flag* is the flag F obtained when $\mathcal{B}$ is the standard basis of $\mathbb{C}^n$, and the *opposite complete flag* is the flag E obtained when $\mathcal{B}$ is the standard basis in reverse order.

Definition 2.2.8. The family of all points of $Gr(n, k)$ whose reduced row echelon form has corresponding partition λ^c is the *Schubert cell* Ω_λ° . Algebraically, the Schubert cell Ω_λ° is

$$\Omega_\lambda^\circ := \{V \in Gr(n, k) : \dim(V \cap F_{n-k+i-\lambda_i}) = i\} \quad (2.2.2)$$

Definition 2.2.9. The *Schubert variety* $\Omega_\lambda(F)$ is the Zariski closure of the Schubert cell $\Omega_\lambda(F(\bullet))^\circ$. Algebraically, this means

$$\Omega_\lambda(F) := \{V \in Gr(n, k) : \dim(V \cap F_{n-k+i-\lambda_i}) \geq i\}. \quad (2.2.3)$$

Remark 2.2.10. The above definitions rely on the flag F , and identical definitions can be made of Schubert varieties $\Omega(F(\mathcal{B}))$ for any complete flag $F(\mathcal{B})$. This is a reasonable situation, as we have already implicitly introduced the standard basis by moving from an abstract vector space to a concrete matrix. By changing the basis of the matrix, we obtain a Schubert variety defined with respect to a different flag.

There is little interesting to say about the internal geometry of a Schubert cell; Ω_λ° is simply homeomorphic to an affine space of codimension $|\lambda|$ sitting in $Gr(n, k)$. However, as subspaces homeomorphic to affine spaces, we can assemble them into an *cell complex* of

$Gr(n, k)$ (explaining the choice of term "Schubert cell"). The only 0-dimensional Schubert cell is the cell $\Omega_{(n-k)^k}^\circ$, and since removing a corner box of the partition increases the (complex) dimension of the cell by one, and therefore the real dimension of the cell by two, we can inductively construct the k -skeleton from the $(k-2)$ -skeleton. As noted above, since the partition λ is invariant for any matrix spanning a point in the cell, $\Omega_\lambda^\circ \cap \Omega_\mu^\circ = \emptyset$ whenever $\lambda \neq \mu$. Furthermore, the closure map provides the necessary continuous gluing along the boundaries. For example, a cellular decomposition of $Gr(6, 3)$ is

$$\begin{array}{ccccccc}
X^{18} & & & & \Omega_\emptyset & & \\
\sqcup & & & & & & \\
X^{16} & & & & \Omega_{\square} & & \\
\sqcup & & & & & & \\
X^{14} & & \Omega_{\begin{smallmatrix} \square & \\ \square & \end{smallmatrix}} & & & \Omega_{\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}} & \\
\sqcup & & & & & & \\
X^{12} & & \Omega_{\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}} & & \Omega_{\begin{smallmatrix} \square & \square & \\ \square & \square & \end{smallmatrix}} & & \Omega_{\begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \end{smallmatrix}} \\
\sqcup & & & & & & \\
X^{10} & & \Omega_{\begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \end{smallmatrix}} & & \Omega_{\begin{smallmatrix} \square & \square & \square & \\ \square & \square & \square & \end{smallmatrix}} & & \Omega_{\begin{smallmatrix} \square & \square & \square & \square \\ \square & \square & \square & \square \end{smallmatrix}} \\
\sqcup & & & & & & \\
X^8 & & \Omega_{\begin{smallmatrix} \square & \square & \square & \\ \square & \square & \square & \end{smallmatrix}} & & \Omega_{\begin{smallmatrix} \square & \square & \square & \square \\ \square & \square & \square & \square \end{smallmatrix}} & & \Omega_{\begin{smallmatrix} \square & \square & \square & \square & \\ \square & \square & \square & \square & \end{smallmatrix}} \\
\sqcup & & & & & & \\
X^6 & & \Omega_{\begin{smallmatrix} \square & \square & \square & \square \\ \square & \square & \square & \square \end{smallmatrix}} & & \Omega_{\begin{smallmatrix} \square & \square & \square & \square & \\ \square & \square & \square & \square & \end{smallmatrix}} & & \Omega_{\begin{smallmatrix} \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square \end{smallmatrix}} \\
\sqcup & & & & & & \\
X^4 & & \Omega_{\begin{smallmatrix} \square & \square & \square & \square & \\ \square & \square & \square & \square & \end{smallmatrix}} & & & \Omega_{\begin{smallmatrix} \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square \end{smallmatrix}} & \\
\sqcup & & & & & & \\
X^2 & & & & \Omega_{\begin{smallmatrix} \square & \square & \square & \square & \square & \\ \square & \square & \square & \square & \square & \end{smallmatrix}} & & \\
\sqcup & & & & & & \\
X^0 & & & & \Omega_{\begin{smallmatrix} \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \end{smallmatrix}} & &
\end{array}$$

Due to the Grassmannian possessing a good amount of desirable geometric qualities, this cell decomposition allows us to say virtually everything we would like about its homology. Since the cell complex above has no cells in odd degree, it provides us with a full description of the homology groups of the Grassmannian:

Theorem 2.2.11. *The homology groups $H_{2j}(Gr(n, k))$ are freely generated by the set of homology classes $\{[\Omega_\lambda] : \lambda \subseteq (n-k)^k, |\lambda| = j\}$.*

By Poincaré duality, the classes $[\Omega_\lambda]$ also additively generate the cohomology ring $H^*(Gr(n, k))$. The class $[\Omega_\lambda]$ is denoted σ_λ , and referred to as a *Schubert class*, or sometimes as a *Schubert cycle*. The multiplicative structure of $H^*(Gr(n, k))$ corresponds to intersection of classes; justifying this claim is not trivial, and requires going through an algebro-geometric construction called the *Chow ring*, which is a ring of (certain equivalence classes of) algebraic subvarieties. Multiplication in the Chow ring corresponds to

intersection, and for varieties which are in some geometric way “nice enough”, such as Grassmannians, The Chow ring is isomorphic to the cohomology ring. We have in summary the following theorem:

Theorem 2.2.12. *The ring $H^{2j}(Gr(n, k))$ has a $\mathbb{Z}$ -linear basis of classes*

$$\sigma_\lambda := [\Omega_\lambda]$$

and the product $\sigma_\lambda \sigma_\mu$ is given by

$$\sigma_\lambda \sigma_\mu = [\Omega_\lambda(F) \cap \Omega_\mu(E)]$$

We therefore have obtained, at least in theory, a nearly complete picture of the cohomology ring $H^*(Gr(n, k))$. In practice, however, a multiplication rule which does not specify concretely what the result is seems less than satisfactory; it would be nicer to state explicitly in terms of the σ -basis what the result of the product $\sigma_\lambda \sigma_\mu$ is. In contrast to the theorems above, which require high-powered topological machinery, a special case of this multiplication rule below can be proven by elementary linear algebra on the matrices of Schubert cycles [Ful97]. As will be apparent immediately afterwards, the structure of this seemingly special case will mean give us enough to completely describe the multiplication.

Theorem 2.2.13 (Pieri’s rule). *Let $\sigma_\lambda, \sigma_{(k)}$ be two Schubert cycles in the cohomology ring $H^*(Gr(n, k))$. Then*

$$\sigma_\lambda \sigma_{(k)} = \sum_{\substack{\mu \in S(\lambda, k) \\ \mu \subseteq (n-k)^k}} \sigma_\mu \tag{2.2.4}$$

Recall that we have already proven a Pieri rule in Theorem 2.1.25, and, in fact, it is almost the same Pieri rule. Theorem 2.1.25 only lacks the restriction that the partitions must live inside the box $(n - k)^k$, but since the set of Schur functions $(s_\lambda : \lambda \not\subseteq (n - k)^k)$ generates an ideal, upon quotienting by this ideal, it becomes truly the same Pieri rule, and the cohomological structure of the Grassmannian is essentially exactly the same as the ring structure of Λ . Since the Pieri rule completely determines the multiplicative structure of Λ (recall that e_k is a set of algebraic generators), we have arrived at the following theorem:

Theorem 2.2.14. *The map $\Omega_\lambda \rightarrow s_\lambda$ is an isomorphism of rings*

$$H^*(Gr(n, k)) \cong \Lambda / (s_\lambda, \lambda \not\subseteq (n - k)^k).$$

Since we possess the ability to multiply arbitrary symmetric functions (by the Littlewood-Richardson rule, or quite quickly by hand or computer for small examples), this isomorphism allows us to perform any computation we like in the cohomology ring of the Grassmannian by simply multiplying symmetric polynomials.

Example 2.2.15. [GH78] In the particular case of the cohomology of the Grassmannian, products of Schubert cycles have an interesting enumerative meaning. Suppose we would like to know the number of 2-planes in $\mathbb{C}^5$ intersecting six general 2-planes. An affine 2-plane is a line in projective space, and the class of a line is the Schubert variety σ_1 . One can use the interpretation of cohomology as intersections of classes to reinterpret this as the intersection of Schubert varieties σ_1^6 . By applying the isomorphism to $\Lambda/(s_\lambda, \lambda \notin (n-k)^k)$ and carrying out this expansion in terms of Schur functions instead of cohomology classes, we find that the intersection collapses to a single term:

$$s_1^6 = s_{111111} + 5s_{21111} + 9s_{2211} + 5s_{222} + 10s_{3111} + 16s_{321} + 5s_{33} + 10s_{411} + 9s_{42} + 5s_{51} + s_6$$

Since s_{33} is the class of a point, and the coefficient is 5, we conclude there are five points living in this space, i.e. that the answer to our question of how many 2-planes live in this intersection is 5.

Remark 2.2.16. While the computation of quantities such as the above example was Schubert’s initial motivation for the construction of his calculus, it would be a restrictive viewpoint to suggest that such enumerative problems are the primary goal of Schubert calculus as a whole. In the following sections, we will review some variations on this idea that are either not motivated by enumerative questions or for which the enumerative questions that can be constructed are somewhat less natural than the example above. Nevertheless, in such contexts a polynomial model of a cohomology ring is still both extremely useful for understanding the topology of a space on a detailed level and quite interesting for its own sake, especially as beautiful combinatorial explanations have been developed for many such polynomials.

2.3 Grothendieck polynomials and K -theory of the Grassmannian

K -theory is a cohomology theory constructed by Atiyah and Hirzebruch [AH59] that, roughly speaking, constructs a cohomology ring out of vector bundles on a topological space. K -theory is capable of detecting more topological detail than ordinary cohomology, at the expense of often being quite difficult to compute in practice. Therefore, models that ease K -theoretic computations are highly valuable. In the 1980s, A. Lascoux developed a polynomial model of the K -theory of the Grassmannian [LS82], in which the K -classes are represented by *Grothendieck polynomials* (the proto-history of K -theory stems from work of Grothendieck, which many naming conventions pay respect to; the 0th K -group K_0 of a space is known as its *Grothendieck group* while the full ring K° is referred to as the *Grothendieck ring*.) However, Lascoux’s polynomials were defined via divided difference operators, and the tableau formulas we will review in this section did not arise until the work of Buch [Buc02], although an alternative combinatorial description due to Fomin and

Kirillov [FK96] in terms of binary matrices predates this. The overall structure should appear quite similar to that of Section 2.1, and this trend will be repeated in Sections 2.4 and 2.5.

Definition 2.3.1. [Buc02] A *set-valued tableau* T on a diagram λ is a filling $T : \lambda \rightarrow \mathcal{P}(\mathbb{Z}_{>0})$ with the additional properties that for all boxes $\mathbf{B}$:

1. $\max(T(\mathbf{B})) \leq \min T(\mathbf{B}^{\rightarrow})$
2. $\max(T(\mathbf{B})) < \min T(\mathbf{B}^{\uparrow})$.

Example 2.3.2. A set-valued tableau of shape $(6, 4, 2, 1)$.

56					
4	6				
3	345	5	5		
12	2	23	3	45	6

Definition 2.3.3. The set of set-valued tableaux on λ with entries in $[n]$ and the set of Young tableaux on λ with entries in $\mathbb{Z}_{>0}$ are denoted by

$$\text{SVT}(\lambda, n) \text{ and } \text{SVT}(\lambda),$$

respectively.

Definition 2.3.4 ([LS82, Buc02]). The *symmetric Grothendieck polynomial* $G_{\lambda}(x_1, \dots, x_n)$ is

$$G_{\lambda}(x_1, \dots, x_n) := \sum_{T \in \text{SVT}(\lambda, n)} \mathbf{x}^{c(T)} \beta^{d(T) - |\lambda|}$$

and the *stable symmetric Grothendieck polynomial* G_{λ} is

$$G_{\lambda} := \sum_{T \in \text{SVT}(\lambda)} \mathbf{x}^{c(T)} \beta^{d(T) - |\lambda|}$$

Remark 2.3.5. In definition 2.3.4, a dummy variable β is used to keep track of how much “extra” content a tableau has beyond the minimum one entry per box. We view ourselves as working over the polynomial (respectively power series) ring $\mathbb{Z}[\beta][x_1 \text{ dots } x_n]$, and so we treat β the degree of β as zero. From a combinatorial perspective, it might seem cleaner to simply set $\beta = 1$, so the generating series is a direct count of the tableaux, but in the K -theory literature it is often convenient to specialize $\beta = -1$, so the most useful and most common choice, which we will take here, is simply to remain agnostic about the value of β unless a choice is necessary.

Remark 2.3.6. For compactness in later sections, we will adopt the notation $G_{\lambda,n}$ for the Grothendieck polynomial $G_{\lambda}(x_1, \dots, x_n)$.

Example 2.3.7. The family of set-valued tableaux $\text{SVT}((2, 1), 3)$ is

2		2		2		2		2		2		2		2		2	
1	1	1	2	1	3	1	12	1	13	1	23	1	123	1	123	1	123
3		3		3		3		3		3		3		23		23	
1	2	1	3	1	12	1	13	1	23	1	123	1	1	1	1	1	1
23		23		23		23		3		23		23		23		23	
1	2	1	3	1	12	1	13	1	1	1	23	1	123	1	123	1	123
3		3		3		3		3		3		3		2		2	
12	2	12	23	12	3	2	2	2	23	2	3	2	3	1	123	1	123

which leads to the polynomial

$$\begin{aligned}
G_{(2,1),3} = & x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2 + x_1 x_2 x_3 + x_1 x_2 x_3 \\
& + \beta(x_1^2 x_2^2 + x_2^2 x_3^2 + x_1^2 x_3^2 + x_1^2 x_2 x_3 + x_1^2 x_2 x_3 + x_1^2 x_2 x_3 \\
& + x_1 x_2^2 x_3 + x_1 x_2^2 x_3 + x_1 x_2^2 x_3 + x_1^2 x_2^2 x_3 + x_1 x_2 x_3^2 + x_1 x_2 x_3^2 + x_1 x_2 x_3^2) \\
& + \beta^2(x_1^2 x_2^2 x_3 + x_1^2 x_2^2 x_3 + x_1^2 x_2 x_3^2 + x_1^2 x_2 x_3^2 + x_1 x_2^2 x_3^2 + x_1 x_2^2 x_3^2) \\
& + \beta^3 x_1^2 x_2^2 x_3^2
\end{aligned}$$

This polynomial is also symmetric, which should perhaps be less surprising after reading the previous section, and one can see in the lowest-degree component the Schur polynomial $s_{(12)}(x_1, x_2, x_3)$. Analogous determinantal identities to Jacobi-Trudi exist for Grothendieck polynomials [Kir16], but they are proven by purely algebraic means; finding a bijective proof using lattice path machinery would be an interesting problem. However, analogs of both involutive proofs have been worked out. The set-valued analog of Bender-Knuth involutions we give below are due to T. Ikeda and T. Shimazaki [IS14], and set-valued crystal operators have been defined in [MPS21].

Theorem 2.3.8. *The polynomial $G_{\lambda,n}$ is symmetric.*

Proof. [IS14] A similar set of involutions to those in Theorem 2.1.18 will provide an analogous involution for set-valued tableaux. We adopt identical definitions of a *pinned* and a *free* $i, i + 1$. Note that an entry can only be pinned by an entry in the box above it; this means that if $\{i, i + 1\} \subseteq T(\mathbf{B})$, both of these entries are free. We then replace each contiguous block of $i^a(i + 1)^b$ with $i^b(i + 1)^a$. Two edge cases appear in the set-valued

setting which does not appear in the ordinary setting, which we must take special care to address. In the case of three adjacent free entries of the form

$$\begin{array}{|c|c|} \hline i+1 & i+1 \\ \hline i & \\ \hline \end{array} \text{ or } \begin{array}{|c|c|} \hline i & i+1 \\ \hline & i \\ \hline \end{array},$$

the involution will attempt to turn them into

$$\begin{array}{|c|c|} \hline i & i+1 \\ \hline i & \\ \hline \end{array} \text{ or } \begin{array}{|c|c|} \hline i & i+1 \\ \hline & i+1 \\ \hline \end{array},$$

which is clearly nonsensical. This situation can be quickly remedied by requiring that in this situation, the involution shifts the doubled entry one box to the left/right respectively, leading to the following two cases instead:

$$\begin{array}{|c|c|} \hline i & i+1 \\ \hline & i \\ \hline \end{array} \text{ or } \begin{array}{|c|c|} \hline i+1 & i+1 \\ \hline i & i+1 \\ \hline \end{array}.$$

This cannot cause any increasingness issues since these entries are free, so there is no other $i, i+1$ in either column, and these two rules are involutions of each other. With this amendment, the Bender-Knuth involutions reverse the number of $i, i+1$ in any tableau, and thereby prove the Grothendieck polynomials are symmetric. $\square$

Example 2.3.9. Let us apply the involution σ_4 again, but this time to the set-valued tableau

78						
6	78					
4	5	5	56	78		
3	3	3	4	57		
1	2	2	3	4	45	56

which contains four 4s and six 5s. As in example 2.1.19, we highlight the free entries in blue, and the pinned entries in red, (not highlighting the entire box since it may contain other entries irrelevant to the involution):

78						
6	78					
4	5	5	56	78		
3	3	3	4	57		
1	2	2	3	4	45	56

Applying the involution to each block of free entries, we obtain the tableau

78						
6	78					
4	4	5	56	78		
3	3	3	4	57		
1	2	2	3	4	4	456

which contains six 4s and four 5s.

In direct analogy with

Theorem 2.3.10. [Buc02, LS82] *The Grothendieck polynomials G_λ are polynomial representatives of the K -classes of the Schubert varieties $\mathcal{O}_{\Omega_\lambda}$; that is, the map*

$$K^\circ(\text{Gr}(n, k)) \rightarrow \Lambda[\beta] / (s_\mu : \mu \not\subseteq (n - k)^k) : [\mathcal{O}_{\Omega_\lambda}] \mapsto G_\lambda$$

is a ring isomorphism.

So-called “ K -analogs” have been developed to rebuild many parts of the structure of ordinary Schubert calculus. The most foundational of these is the K -theoretic Littlewood-Richardson rule; this is due originally to Buch [Buc02], but a very short proof exists due to [IS14] using the set-valued Bender-Knuth involutions which follows Stembridge’s proof of theorem 2.1.30 very closely, in direct analogy to our proof of the symmetry of Grothendieck polynomials essentially only requiring a small tweak of the proof of symmetry of Schur polynomials.

Theorem 2.3.11. [IS14] *Let λ, μ be partitions. Then*

$$G_\lambda G_\mu = \sum \beta^{d(T) - |\mu|} G_{\lambda + c(T)}$$

where the sum ranges over all $T \in \text{SVT}(\mu)$ such that $\lambda + c(T_{\geq j})$ is a partition for all $j \geq 1$.

Remark 2.3.12. From this perspective, the ordinary cohomology of the Grassmannian can be viewed as a specialization of the K -theory, as the K -theoretic representatives contain strictly more information; by specializing $\beta = 0$, one recovers the Schur function for the same partition from the Grothendieck polynomial, and so any calculation involving Schur functions can be evaluated as a specialization of a calculation involving Grothendieck polynomials.

2.4 Shifted Schubert calculus and P -Schur functions

The *orthogonal Grassmannian* is a variety closely related to the Grassmannian with additional structure introduced by a symmetric bilinear form. From a Coxeter-theoretic perspective, the orthogonal Grassmannian is the “type B” analog of the role the Grassmannian plays in type A. Interesting Schubert calculus can be carried out in all Coxeter types; by introducing a skew-symmetric form instead of a symmetric form, we construct the type C *Lagrangian Grassmannian*, and *affine Grassmannians* also exist in affine Coxeter types [LLM⁺14]. In this thesis, we will be concerned mostly with the type B case, although we will make some exploratory remarks on the type C case and the difficulties associated with it in Chapter 6 and observe in Chapter 5 that the methods developed in this thesis can also be applied to the type A Grassmannian.

Definition 2.4.1. Consider an odd-dimensional space $\mathbb{C}^{2n+1}$, and fix a nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$. The *orthogonal Grassmannian* $OG(2n+1, k)$ is the variety of all k -dimensional subspaces of $\mathbb{C}^{2n+1}$ isotropic with respect to $\langle \cdot, \cdot \rangle$.

Remark 2.4.2. While the orthogonal Grassmannian can be defined for any k , it is most often chosen to be n , because as we will soon see, the combinatorics in this case is particularly nice. Henceforth, we will only consider the orthogonal Grassmannian $OG(2n+1, n)$, and will simply refer to this as “the” orthogonal Grassmannian.

Remark 2.4.3. The particular bilinear form chosen does not have any effect on the geometry of the orthogonal Grassmannian; all choices lead to isomorphic varieties. Therefore, it is often assumed (and there is no harm in assuming) some particular form. Taking the form

$$\langle a, b \rangle := \sum_{i=1}^{2n+1} a_i b_{2n+1-i+1}$$

yields the convenient property that the usual standard and opposite flags F and E are flags of orthogonal subspaces.

The reasoning behind remark 2.4.2 is that it allows us to perform the following combinatorial trick, which brings a new form of tableau combinatorics into the picture.

Example 2.4.4. Let us consider an n -dimensional subspace of $\mathbb{C}^{2n+1}$; for concreteness, let $n = 3$. Since this is simply a point of $Gr(7, 3)$, the analysis of Section 2.1 allows us to represent this subspace in a reduced row echelon form corresponding to a partition. For example, if the partition is $(4, 1, 1)$, then from the diagram

	•	•	•
	•	•	•

we can compute its complement to be $(3, 3)$, and so it is represented by matrices with reduced row echelon form of shape

$$X = \begin{bmatrix} 0 & 1 & 0 & \bullet & \bullet & \bullet & 0 \\ 0 & 0 & 1 & \bullet & \bullet & \bullet & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

For a point in $Gr(2n+1, n)$, all $\bullet$ are free, but if we request that this subspace be isotropic, then there are relations imposed upon the $\bullet$ entries by the orthogonality relations. By taking the product of the first row with itself, the second row with itself, and the first row with the second, we obtain the relations

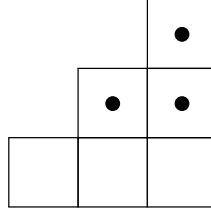
$$\begin{aligned} X_{1,4}^2 + 2X_{1,6} &= 0, \\ X_{2,4}^2 + 2X_{2,6} &= 0, \\ X_{1,4}X_{2,4} + X_{1,5} + X_{2,6} &= 0. \end{aligned}$$

From these relations it follows that the values of $X_{1,4}$, $X_{1,5}$ and $X_{2,4}$ are dependent on the values of $X_{1,6}$, $X_{2,5}$, and $X_{2,6}$; if the latter three are fixed, the former three are determined uniquely. These conditions correspond visually to a sort of “reflection” condition on the diagram, that is, the values below the **red** line

	•	•	•
	•	•	•

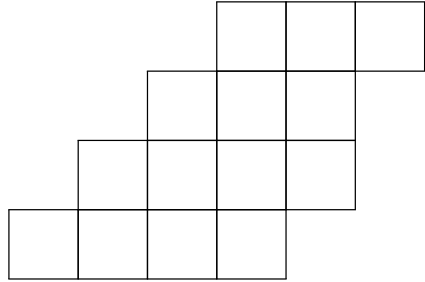
determine the above values. In particular, the partition itself must be symmetric with respect to the line, or there will be conditions imposed that are impossible to satisfy.

So in some sense, isotropic subspaces are indexed by half a partition that is symmetric with respect to this line. By visually “folding” the rectangle over the line, we obtain a Young diagram-like object that looks like



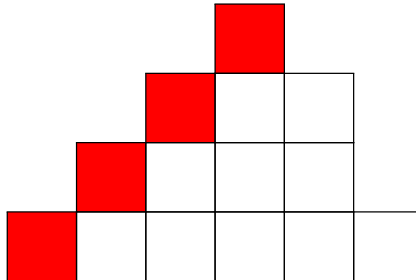
This is the foundational combinatorial object of orthogonal Schubert calculus; it is called a *shifted diagram*, and for this reason the Schubert calculus of the orthogonal Grassmannian is often referred to as *shifted Schubert calculus*.

A partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ is called *strict* if $\lambda_1 > \lambda_2 > \dots > \lambda_\ell$. For a strict partition λ , it makes graphical sense to slide the i th row of the Young diagram to the right by i boxes without creating overhanging boxes; for example, if we tried to shift the diagram of the partition $(4, 4, 3, 3)$, we would end up with the visual



The purpose of restricting to strict partitions is to forbid such cases, ensuring that every box of the tableau has a box underneath it (except the bottom row). This draws what is referred to as the *shifted diagram* of λ .

Example 2.4.5. The partition $(6, 4, 3, 1)$ is strict, and its shifted diagram is



A notion that exists for shifted diagrams that does not exist in the unshifted setting is that of the *main diagonal* of the diagram, which is the sequence of leftmost boxes in each row. The main diagonal in example 2.4.5 is highlighted in red.

The procedure of the above example can be carried out for any subspace, and the Schubert varieties of $OG(2n+1, n)$ can be indexed by the *shifted partitions* which live inside half of the diagram of $(n+1)^n$:

Definition 2.4.6. Let F be a flag of orthogonal subspaces, and let μ be a shifted partition contained inside half of the diagram of $(n+1)^n$. Write λ for the shifted partition obtained by folding μ over the line of symmetry. The *Schubert variety* X_λ is

$$X_\lambda := \{V \in OG(2n+1, n) : \dim(V \cap F_{n+i+1-\mu_i}) \geq i\}$$

The Schubert classes $[X_\lambda]$ form a basis of the cohomology ring $H^*(OG(2n+1, n))$, so to complete a Schubert calculus program for the orthogonal Grassmannian, we would like to find polynomial representatives that multiply like these classes, preferably with a combinatorial definition. This is achieved by the so-called *P-Schur functions*, which we will now define.

We will begin by defining rules for tableaux on shifted diagrams. Shifted tableaux are filled according to different rules than classical tableaux. The filling alphabet is not $\mathbb{Z}_{>0}$ but the doubled alphabet

$$\mathbb{S} = 1' < 1 < 2' < 2 < 3' < 3 < \dots$$

We refer to letters of the form n' as *primed* letters. Primed letters obey opposite rules to unprimed letters; while unprimed letters increase weakly along rows and strictly up columns, primed letters increase weakly up columns and strictly along rows.

Definition 2.4.7. A *P-shifted tableau* on a shifted diagram λ is a filling $T : \lambda \rightarrow \mathbb{S}$ with the additional properties that for all boxes B :

1. If $T(B)$ is primed, then

- $T(B) \leq T(B^\uparrow)$.
- $T(B) < T(B^\rightarrow)$.

2. If $T(B)$ is unprimed, then

- $T(B) < T(B^\uparrow)$.
- $T(B) \leq T(B^\rightarrow)$.

3. No boxes on the main diagonal contain any primed entries.

Example 2.4.8. A *P-shifted tableau* on $(6, 4, 2, 1)$.

						5					
						3	5'				
					2	2	3'	4'			
		1	1	2'	3'	4'	4				

Definition 2.4.9. For a shifted tableau T , the *content* of T is the vector

$$c(T) := (\# \text{ of ones in } T, \# \text{ of twos in } T, \dots).$$

Note that we do *not* distinguish between a and a' when counting content of a shifted tableau. Both letters are treated as content a .

Definition 2.4.10. To denote intervals in the shifted alphabet $\mathbb{S}$, we write

$$[a, b]_{\mathbb{S}} := \{c \in \mathbb{S} : a \leq c \leq b\}.$$

Definition 2.4.11. The set of P -shifted tableaux on λ with entries in $[1, n]_{\mathbb{S}}$ and the set of P -shifted tableaux on λ with entries in $\mathbb{Z}_{>0}$ are denoted by

$$\text{PT}(\lambda, n) \text{ and } \text{PT}(\lambda),$$

respectively.

Remark 2.4.12. The naming choice of P -shifted tableau, instead of simply shifted tableau, may seem obscure; the reason for this is to distinguish them from the dual notion of a Q -shifted tableau, which follows definition 2.4.7 exactly except for the deletion of condition (3), allowing for primed entries on the main diagonal. Despite this minor difference in form, there are substantial differences in both the combinatorial behaviour of these tableaux and in the geometric behaviour of objects associated with them. In particular, the theorems of this thesis and the papers cited by this thesis apply only to the P -shifted setting and fail to be true (or to exist at the time of writing) for Q -shifted tableaux, and so we will deal only with P -shifted tableaux except for a brief summary of the situation in chapter 6.

Definition 2.4.13. Given a strict partition λ , the P -Schur polynomial $P_{\lambda}(x_1, \dots, x_n)$ is the polynomial

$$P_{\lambda}(x_1, \dots, x_n) := \sum_{T \in \text{PT}(\lambda, n)} \mathbf{x}^{c(T)}$$

and the P -Schur function P_{λ} is

$$P_{\lambda} := \sum_{T \in \text{PT}(\lambda)} \mathbf{x}^{c(T)}.$$

Example 2.4.14. For the strict partition $\lambda = (3, 1)$, the family of shifted tableaux

$\text{PT}((3, 1), 3)$ is

<table><tr><td></td><td>2</td><td></td></tr><tr><td>1</td><td>1</td><td>1</td></tr></table>		2		1	1	1	<table><tr><td></td><td>2</td><td></td></tr><tr><td>1</td><td>1</td><td>2</td></tr></table>		2		1	1	2	<table><tr><td></td><td>2</td><td></td></tr><tr><td>1</td><td>1</td><td>3</td></tr></table>		2		1	1	3	<table><tr><td></td><td>3</td><td></td></tr><tr><td>1</td><td>1</td><td>1</td></tr></table>		3		1	1	1	<table><tr><td></td><td>3</td><td></td></tr><tr><td>1</td><td>1</td><td>2</td></tr></table>		3		1	1	2
	2																																	
1	1	1																																
	2																																	
1	1	2																																
	2																																	
1	1	3																																
	3																																	
1	1	1																																
	3																																	
1	1	2																																
<table><tr><td></td><td>3</td><td></td></tr><tr><td>1</td><td>1</td><td>3</td></tr></table>		3		1	1	3	<table><tr><td></td><td>3</td><td></td></tr><tr><td>1</td><td>2</td><td>2</td></tr></table>		3		1	2	2	<table><tr><td></td><td>3</td><td></td></tr><tr><td>1</td><td>2</td><td>3</td></tr></table>		3		1	2	3	<table><tr><td></td><td>3</td><td></td></tr><tr><td>2</td><td>2</td><td>2</td></tr></table>		3		2	2	2	<table><tr><td></td><td>3</td><td></td></tr><tr><td>2</td><td>2</td><td>3</td></tr></table>		3		2	2	3
	3																																	
1	1	3																																
	3																																	
1	2	2																																
	3																																	
1	2	3																																
	3																																	
2	2	2																																
	3																																	
2	2	3																																
<table><tr><td></td><td>2</td><td></td></tr><tr><td>1</td><td>1</td><td>2'</td></tr></table>		2		1	1	2'	<table><tr><td></td><td>3</td><td></td></tr><tr><td>1</td><td>1</td><td>2'</td></tr></table>		3		1	1	2'	<table><tr><td></td><td>2</td><td></td></tr><tr><td>1</td><td>1</td><td>3'</td></tr></table>		2		1	1	3'	<table><tr><td></td><td>3</td><td></td></tr><tr><td>1</td><td>1</td><td>3'</td></tr></table>		3		1	1	3'	<table><tr><td></td><td>3</td><td></td></tr><tr><td>1</td><td>2</td><td>3'</td></tr></table>		3		1	2	3'
	2																																	
1	1	2'																																
	3																																	
1	1	2'																																
	2																																	
1	1	3'																																
	3																																	
1	1	3'																																
	3																																	
1	2	3'																																
<table><tr><td></td><td>2</td><td></td></tr><tr><td>1</td><td>2'</td><td>2</td></tr></table>		2		1	2'	2	<table><tr><td></td><td>2</td><td></td></tr><tr><td>1</td><td>2'</td><td>3'</td></tr></table>		2		1	2'	3'	<table><tr><td></td><td>2</td><td></td></tr><tr><td>1</td><td>2'</td><td>3</td></tr></table>		2		1	2'	3	<table><tr><td></td><td>3</td><td></td></tr><tr><td>1</td><td>2'</td><td>2</td></tr></table>		3		1	2'	2	<table><tr><td></td><td>3</td><td></td></tr><tr><td>1</td><td>2'</td><td>3'</td></tr></table>		3		1	2'	3'
	2																																	
1	2'	2																																
	2																																	
1	2'	3'																																
	2																																	
1	2'	3																																
	3																																	
1	2'	2																																
	3																																	
1	2'	3'																																
	<table><tr><td></td><td>3</td><td></td></tr><tr><td>1</td><td>2'</td><td>3</td></tr></table>		3		1	2'	3	<table><tr><td></td><td>3</td><td></td></tr><tr><td>1</td><td>3'</td><td>3</td></tr></table>		3		1	3'	3	<table><tr><td></td><td>3</td><td></td></tr><tr><td>2</td><td>3'</td><td>3</td></tr></table>		3		2	3'	3													
	3																																	
1	2'	3																																
	3																																	
1	3'	3																																
	3																																	
2	3'	3																																

and the associated P -Schur polynomial is

$$\begin{aligned}
P_{(3,1)}(x_1, x_2, x_3) = & x_1^3 x_2 + x_1^2 x_2^2 + x_1^2 x_2 x_3 + x_1^3 x_3 + x_1^3 x_2 x_3 \\
& + x_1^2 x_3^2 + x_1 x_2^2 x_3 + x_1 x_2 x_3^2 + x_2^3 x_3 + x_2^2 x_3^2 \\
& + x_1^2 x_2^2 + x_1^2 x_2 x_3 + x_1^2 x_2 x_3 + x_1^2 x_3^2 + x_1 x_2 x_3^2 \\
& + x_1 x_2^3 + x_1 x_2^2 x_3 + x_1 x_2^2 x_3 + x_1 x_2^2 x_3 + x_1 x_2 x_3^2 \\
& + x_1 x_2 x_3^2 + x_1 x_3^3 + x_2 x_3^3.
\end{aligned}$$

As should be expected by now, the P -Schur polynomials are again symmetric, although they span only a subspace of Λ since their indexing set is smaller.

Theorem 2.4.15. *The polynomial P_λ is symmetric.*

Several combinatorial proofs of theorem 2.4.15 exist: there is a Bender-Knuth-like proof due also to Stembridge [Ste89], as well as a proof by Cho [Cho13] using combinatorial objects called *decomposition tableaux*.

While the P -Schur functions are quite old, having been introduced by Schur in 1911 [Sch11], they were originally used for representation-theoretic applications, and the development of a geometric theory, or of a combinatorial model, took many more decades to come to the surface. The combinatorial theory of the P -Schur functions and shifted

tableaux was developed in the 1980s independently by Sagan [Sag87] and Worley [Wor84], and the Schubert-theoretic realization of these polynomials as representatives of cohomology classes was accomplished a few years later:

Theorem 2.4.16. [Pra91] *The P -Schur function P_λ is a representative of the class $[X_\lambda]$ in the cohomology of the orthogonal Grassmannian $OG(2n+1, n)$.*

2.5 Both ways at once: K -theory of the orthogonal Grassmannian and P -Grothendieck polynomials

After reading the previous two sections, the idea to combine them seems quite natural. If allowing sets in our tableaux turns cohomology into K -theory, and shifting the tableaux gives us the cohomology of the orthogonal Grassmannian, we ought to immediately wonder if we could get the K -theory of the orthogonal Grassmannian by considering set-valued fillings of shifted tableaux. This guess turns out to be exactly correct.

Definition 2.5.1. A P -shifted set-valued tableau on a shifted diagram λ is a filling $T : \lambda \rightarrow \mathcal{P}(\mathbb{S})$ with the additional properties that for all boxes B :

1. If $\max(T(B))$ is primed, then
 - $\max(T(B)) \leq \min(T(B^\uparrow))$.
 - $\max(T(B)) < \min(T(B^\rightarrow))$.
2. If $T(B)$ is unprimed, then
 - $\max(T(B)) < \min(T(B^\uparrow))$.
 - $\max(T(B)) \leq \min(T(B^\rightarrow))$.

3. No boxes on the main diagonal contain any primed entries.

Definition 2.5.2. [INN11] Given a strict partition λ , the P -Grothendieck polynomial $GP_\lambda(x_1, \dots, x_n)$ is the polynomial

$$GP_\lambda(x_1, \dots, x_n) = GP_{\lambda, n} := \sum_{T \in \text{PSVT}(\lambda), n} \mathbf{x}^T \beta^{d(T) - |\lambda|}$$

and the *stable P -Grothendieck polynomial* GP_λ is

$$GP_\lambda := \sum_{T \in \text{PSVT}(\lambda)} \mathbf{x}^T \beta^{d(T) - |\lambda|}.$$

Remark 2.5.3. We will use a similar convention to remark 2.3.6 and write $GP_{\lambda, n}$ to refer to the P -Grothendieck polynomial $GP_\lambda(x_1, \dots, x_n)$.

Example 2.5.4. For the strict partition $\lambda = (2, 1)$, the set of shifted set-valued tableaux $\text{PSVT}((2, 1), 3)$ is

	2		2		2		3		23		3
1	1	1	2'	1	12'	1	1	1	1	1	2'
	23		3		3		3		3		3
1	2'	1	3'	1	2'2	1	12'2	1	$\frac{23'}{12'}$	1	2'3'
	3		3		3		3		3		3
1	23'	1	12'	1	12	1	13'	1	123'	1	12'3'
	3		3		3		3		3		3
12	2	12	3'	12	23'	2	2	2	23'	2	3'
									3		3
								1	12'2	1	2'23'

which leads to the P -Grothendieck polynomial

$$\begin{aligned}
GP_{(2,1),3} = & x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2 + x_1 x_2 x_3 \\
& + \beta(x_1^2 x_2 x_3 + x_1^2 x_2 x_3 + x_1^2 x_2 x_3 + x_1 x_2^2 x_3 + x_1 x_2^2 x_3 + x_1 x_2^2 x_3 \\
& + x_1 x_2 x_3^2 + x_1 x_2 x_3^2 + x_1 x_2 x_3^2 + x_1^2 x_3^2 + x_1^2 x_2^2 + x_2^2 x_3^2) \\
& + \beta^2(x_1^2 x_2^2 x_3 + x_1^2 x_2^2 x_3 + x_1^2 x_2^2 x_3 + x_1^2 x_2^2 x_3 + x_1 x_2^2 x_3^2 + x_1 x_2^2 x_3^2) \\
& + \beta^3(x_1^2 x_2^2 x_3^2)
\end{aligned}$$

As should be expected by now, we have the following symmetry theorem:

Theorem 2.5.5. [INN11, IN13] *The P -Grothendieck polynomials are symmetric.*

The combinatorial theory of shifted set-valued tableaux is not quite as mature as that seen in previous three sections. As of the time of writing, there is no known analog of the Bender-Knuth involutions for shifted set-valued tableaux. Theorem 2.5.5 above is proved by an insertion algorithm in [INN11], similar to the work of [Sag87, Wor84] on shifted tableaux.

As mentioned at the beginning of this chapter, these polynomials provide K -theory representatives for the orthogonal Grassmannian:

Theorem 2.5.6. [INN11, IN13] *The symmetric function GP_λ is a representative of the K -theory class $\mathcal{O}_{X_\lambda}$ in the ring $K^\circ(OG(2n+1, n))$.*

2.6 Matrix Schubert varieties

In this section we review some necessary definitions for Chapters 4 and 5; for a textbook treatment of the material in this section, see [MS05].

For a permutation $w \in \mathbf{S}_n$, the *permutation matrix* $P^w \in \text{Mat}_n$ associated to w is defined as the $n \times n$ matrix

$$P_{i,j}^w = \begin{cases} 1, & \text{if } j = w(i); \\ 0, & \text{otherwise.} \end{cases}$$

For example, the permutation matrix P^{52134} is

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

For any matrix $A \in \text{Mat}_n$ and any subsets $I, J \subseteq [n]$, we define A_{IJ} to be the submatrix

$$\{A_{i,j} : (i, j) \in I \times J\}.$$

In particular, the matrix $A_{[i][j]}$ is the principal $i \times j$ minor of A .

The *rank matrix* R^w of w is the matrix defined by the condition that $R_{i,j}^w$ is the rank of the $i \times j$ principal minor $P_{[i][j]}^w$. For example, the rank matrix R^{52134} is

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 2 \\ 1 & 2 & 2 & 2 & 3 \\ 1 & 2 & 3 & 3 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}.$$

The *matrix Schubert variety* X_w is the set of matrices

$$X_w := \{A \in \text{Mat}_n : \text{rank } A_{[i][j]} \leq R_{i,j}^w\},$$

which is indeed an affine algebraic variety since each inequality is equivalent to the polynomial condition that all $(R_{i,j}^w + 1) \times (R_{i,j}^w + 1)$ minors of $A_{[i][j]}$ vanish. These varieties, first introduced by W. Fulton [Ful92], provide an affine model of Schubert varieties, and have been of significant interest for many years (see, e.g., [EM16, FRS16, HPW22, Hsi13,

[KW21, KM05, KMY09, PSW24]). Fulton [Ful92] shows that the defining determinantal conditions generate a prime ideal I_w of the ring $S := \mathbb{C}[x_{i,j} : 1 \leq i, j \leq n]$. We are interested in the coordinate ring S/I_w .

A permutation $w \in \mathbf{S}_n$ that fixes n determines a permutation $w' \in \mathbf{S}_{n-1}$. We routinely identify w with w' or write $w = w' \times 1$ for clarity. Everything we study is invariant under the transformation $w \mapsto w \times 1$; in particular, $S/I_w \cong S/I_{w'}$.

A permutation $w \in \mathbf{S}_n$ is *Grassmannian* if there is at most one value $1 \leq i < n$ such that $w(i) > w(i+1)$. In Chapter 5, we will be interested in matrix Schubert varieties X_w with w Grassmannian. The *code* of a permutation $w \in \mathbf{S}_n$ is $\text{BCode}(w) = (c_1, \dots, c_n)$, where c_i is the number of integers j with $j > i$ and $w(j) < w(i)$. Sorting the entries of $\text{BCode}(w)$ into a partition yields the shape $\lambda_{\mathbf{B}}(w)$ of w .

Permutations are uniquely determined by their codes. Given a nonempty partition λ and a positive integer $n \geq \ell(\lambda)$, there is a unique Grassmannian permutation w_λ with $w(n) > w(n+1)$ and $\lambda_{\mathbf{B}}(w) = \lambda$. To find this permutation, extend λ to have length n by appending the needed number of terminal 0s and then reverse λ to obtain a weakly increasing sequence. This sequence is uniquely the code of the desired permutation w_λ .

2.7 Algebraic background: Castelnuovo-Mumford regularity and degrees of Grothendieck polynomials

Most of the background summarized in this section consists of standard commutative algebra, and proofs of all stated facts without their own citations can be found, for example, in [Eis95] or [MS05].

Take $S = \mathbb{C}[x_1, \dots, x_n]$ a polynomial ring with the standard grading $\deg(x_i) = 1$, and take $I \subseteq S$ a homogeneous ideal. For any finitely generated graded S -module M , we denote the $\mathbb{C}$ -vector space of all homogeneous degree a elements of M by M_a . Since M is finitely generated, the dimension of M_a is finite for all a , and so we can define the *Hilbert series* of M as the formal power series

$$H(M; t) := \sum_{a \in \mathbb{Z}_{\geq 0}} \dim_{\mathbb{C}}(M_a) t^a.$$

It is often useful to write the Hilbert series as a ratio of two polynomials

$$H(M; t) = \frac{\mathcal{K}(M; t)}{(1-t)^n},$$

in which case the polynomial $\mathcal{K}(M; t)$ is referred to as the *$\mathcal{K}$ -polynomial* of the module M ; see [MS05] for further discussion.

We define the degree-shifted module $M(-j)$ via the condition $M(-j)_a = M_{a-j}$ for all a . With this notation, a *free resolution* of M is an exact sequence of graded S -modules

$$\cdots \rightarrow \bigoplus_{i \in \mathbb{Z}} S(-i)^{b_i^k} \rightarrow \cdots \rightarrow \bigoplus_{i \in \mathbb{Z}} S(-i)^{b_i^1} \rightarrow \bigoplus_{i \in \mathbb{Z}} S(-i)^{b_i^0} \rightarrow M \rightarrow 0.$$

A free resolution is called *minimal* if the value of b_i^j is minimized simultaneously for all indices i, j . In our situation, there is a unique finite minimal free resolution for any finitely generated graded M by Hilbert's Syzygy Theorem.

Since minimal free resolutions are unique, we can define the (*Castelnuovo–Mumford*) *regularity* of M as

$$\text{reg}(M) := \max\{i - j : b_i^j \neq 0\}.$$

For additional background on regularity, we refer the reader to the survey [Cha07]. The definition of regularity directly offers a kind of bound on the complexity of the free resolution of M . We will only be concerned in this paper with the regularity of ideals I and their quotient modules S/I , which are essentially the same information, since $\text{reg } I = 1 + \text{reg}(S/I)$.

Computing the regularity of an arbitrary module often requires technical work with free resolutions or local cohomology. However, as noted in [RRRSW21], when S/I is a Cohen–Macaulay ring, the following lemma is known (see [BV15, Lemma 2.5] for explanation) and allows one to compute the regularity from the ideal by combinatorial methods.

Lemma 2.7.1. *Let $I \subseteq S$ and let S/I be a Cohen–Macaulay ring. Then*

$$\text{reg}(S/I) = \deg(\mathcal{K}(S/I; t)) - \text{ht}(I).$$

For ideals pertaining to Schubert calculus, both terms on the right side of Lemma 2.7.1 can be more approachable than the regularity itself. The height of I can often be computed from the indexing combinatorics for I , and the polynomial $\mathcal{K}(S/I; t)$ often has some known combinatorial description as a generating polynomial, such as a Grothendieck polynomial. Thus, the problem of computing the regularity of I can be solved if one is able to compute the degree of the $\mathcal{K}$ -polynomial through combinatorial means. The approach provided by this lemma is the underpinning of all combinatorial computations of regularity surveyed in Chapter 1, and will be followed in this paper as well.

2.8 Algebraic background: Gröbner bases

Definition 2.8.1. A *term order* on the monomials of $\mathbb{C}[x_1, \dots, x_n]$ is a total ordering $\prec$ on the set of monomials $\mathbf{x}^a$ with the two properties

1. $\preceq$ is stable under multiplication; that is, if $\mathbf{x}^a \prec \mathbf{x}^b$, then $\mathbf{x}^a \mathbf{x}^c \prec \mathbf{x}^b \mathbf{x}^c$.

2. The constant monomial $\mathbf{x}^0 = 1$ is the unique minimum element of $\prec$.

Example 2.8.2. We have already defined one term order, the *lexicographic* order in Definition 2.1.6. This is often referred to simply as *Lex*, for short.

Definition 2.8.3. If $\prec$ is a term order, the *leading term* of a polynomial f with respect to $\prec$ is the maximum monomial of f . We denote the leading term of f with respect to $\prec$ by $L_\prec(f)$, or just by $L(f)$ if $\prec$ is clear from context.

Example 2.8.4. We can also define the *reverse lexicographic* term order *RevLex*, where $\mathbf{x}^a \prec \mathbf{x}^b$ if the *last* nonzero entry of $b - a$ is negative. For example, $x_1^3 x_4^3 \prec x_1^2 x_2 x_3 x_4^2$ in RevLex (observe that in this example $x_1^3 x_4^3 \prec x_1^2 x_2 x_3 x_4^2$ in lexicographic order as well, so RevLex is not simply the opposite of Lex order. To obtain RevLex from Lex, we have to both read the degree backward *and* reverse the natural ordering of the variables.)

Example 2.8.5. Any term order also furnishes a graded version; given a term order $\prec$, we can define a term order $\prec_{gr}$ so that $\mathbf{x}^a \prec_{gr} \mathbf{x}^b$ if $|a| < |b|$ or $|a| = |b|$ and $\mathbf{x}^a \prec \mathbf{x}^b$. This is particularly often applied to RevLex, yielding the *GrevLex* term order.

Example 2.8.6. For this example, suppose the ambient polynomial ring is $\mathbb{C}[x_{11}, x_{12}, \dots, x_{nn}]$. A term order is referred to as *diagonal* if the leading term of any minor of the generic matrix

$$\begin{bmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ x_{n1} & x_{n2} & \dots & x_{nn} \end{bmatrix}$$

is the product of the main diagonal. Conversely, a term order is *antidiagonal* if the leading term is the product of the main antidiagonal. Diagonal and antidiagonal term orders appear naturally in the theory of determinantal ideals such as those considered in Section 2.6 and again later on in this paper in Chapter 4 due to their relationship with the minors generating them.

Definition 2.8.7. For an ideal $I \in \mathbb{C}[x_1, \dots, x_n]$ and a term order $\prec$, the *initial ideal* $\text{in}_\prec(I)$ is the ideal generated by the leading terms of all polynomials in I . If the order $\prec$ is clear from context, we will occasionally simply write $\text{in}(I)$.

The initial ideal is usually much simpler than the original ideal, as monomial ideals are generally very approachable and many tools exist for working with them. Geometrically, the space cut out by the initial ideal is simply a union of coordinate subspaces corresponding to each monomial in the span (applying this approach to a geometric space is called *Gröbner degeneration*, and has been used to great effect with various matrix Schubert varieties). Nevertheless, much desirable information about I is still remembered by passing to $\text{in}(I)$;

for example, $R/\text{in}(I)$ and R/I have the same Hilbert series. Other information is also remembered by the initial ideal (although not necessarily *carried over to* the initial ideal), as $\text{in}(I)$ tends to be strictly “worse” in most ways than I . For example, if $R/\text{in}(I)$ is Cohen-Macaulay, then R/I must have been Cohen-Macaulay as well. Of most interest to this paper is the fact that regularity can only get worse upon passing to the initial ideal; $\text{reg}(R/I) \leq \text{reg}(R/\text{in}(I))$. Proofs of these claims can be found in most books on commutative algebra or Gröbner bases, for example [Eis95].

However, the definition of the initial ideal is somewhat abstract and not particularly friendly to compute. Hilbert’s basis theorem guarantees the existence of some finite generating set of $\text{in}(I)$, so it is a reasonable goal to look for a finite set of generators. Such a question leads to the notion of a *Gröbner basis*.

Definition 2.8.8. Let $\prec$ be a term order, and $I \leq \mathbb{C}[x_1, \dots, x_n]$ be an ideal. A *Gröbner basis* $\mathcal{G}$ of I is a finite set such that

$$\text{in}(I) = (L_{\prec}(g) : g \in \mathcal{G})$$

Remark 2.8.9. There is no minimality requirement baked into the definition of a Gröbner basis: one can be arbitrarily large, of arbitrarily high degree, and contain arbitrarily much excess information. This lack of restrictions has the implication that any nontrivial ideal has infinitely many Gröbner bases. With these considerations, it seems necessary to look for a “best” Gröbner basis, which we would like to be uniquely defined and minimal in some sense.

The first natural place to start looking for uniqueness is to clear leading coefficients of elements in $\mathcal{G}$: a basis is nicest when all leading coefficients are one. A second natural trimming is to remove all terms $g \in \mathcal{G}$ such that $h|g$ for some $h \in \mathcal{G}$; such terms are clearly extraneous. These two obvious reductions define what is called a *minimal* Gröbner basis:

Definition 2.8.10. A Gröbner basis $\mathcal{G}$ of I is *minimal* if it satisfies the conditions

- All $g \in \mathcal{G}$ are monic.
- For all g_i, g_j in $\mathcal{G}$, $g_i \nmid g_j$.

Example 2.8.11. Minimal Gröbner bases satisfy some natural irredundancy requirements, but these assumptions are still not strong enough to guarantee uniqueness. For example, consider the ideal

$$I = (x_1, x_2, x_3)$$

in Lex order. Both $\mathcal{G}_1 = \{x_1, x_2, x_3\}$ and $\mathcal{G}_2 = \{x_1, x_2, x_1 + x_2 + x_3\}$ are minimal Gröbner bases of I .

From inspecting the example above, it seems the problem we have forgotten to account for is that if $g, h \in \mathcal{G}$ and $h \prec g$, then we can replace g in the Gröbner basis by $g + h$. Therefore, to ensure minimality, we should add an additional restriction that this cannot be done, that is, $g - L(g) \notin \text{in}(I)$ for all g . This is called a *reduced* Gröbner basis.

Definition 2.8.12. Let $\prec$ be a term order and let $I \leq \mathbb{C}[x_1, \dots, x_n]$ be an ideal. A Gröbner basis $\mathcal{G}$ of I is called *reduced* if it is minimal and satisfies the additional condition that for all $g \in \mathcal{G}$, $g - L(g) \notin \text{in}(I)$.

Example 2.8.13. In Example 2.8.11, $\mathcal{G}_1$ is reduced, but $\mathcal{G}_2$ is not, because $x_1 + x_2 + x_3 - L(x_1 + x_2 + x_3) = x_1 + x_2$ lives in the initial ideal.

In fact, this definition is all that is necessary:

Theorem 2.8.14. *Reduced Gröbner bases are unique.*

We remarked that the regularity of the initial ideal provides a bound on the regularity of R/I . In the special case of a GrevLex order, we can tighten this to an exact result using Gröbner bases, due to a theorem of Bayer and Stillman:

Theorem 2.8.15 ([BS87, Corollary 2.5 & Proposition 2.11]). *Fix a GrevLex term order. If $I \subset S$ is a homogeneous ideal in generic coordinates with Castelnuovo–Mumford regularity m , then the highest degree element of the reduced Gröbner basis for I has degree exactly m .*

Gröbner theory has been applied to great effect in Schubert calculus; there is a beautiful combinatorial Gröbner theory for matrix Schubert varieties, which is quite well developed (e.g., [KM05, KMY09, HPW22, CDG22, Kle23, KW21]) and explains algebraically many of the combinatorial formulas for Schubert, Schur, and Grothendieck polynomials. In contrast, the analogous theory for skew-symmetric matrix Schubert varieties is understood for only a single special term order [MP22]. Through Theorem 2.8.15, our Theorem 1.0.2 may provide a hint toward a broader theory.

Chapter 3

Degrees of P -Grothendieck polynomials and proof of Theorem 1.0.1

By Definition 2.5.2, the degree of the P -Grothendieck polynomial $GP_{\lambda,n}$ is the maximum degree amongst all tableaux $T \in \text{PSVT}(\lambda, n)$. We compute this number in two steps. First, we establish that for all partitions λ , the degree of $GP_{\lambda,n}$ is equal to the degree of $GP_{\Delta,n}$, where Δ is the largest D-partition contained in λ . This reduces the problem to computing the maximum degree of a tableau in $\text{PSVT}(\Delta, n)$, in which case we explicitly construct a tableau of maximum degree, with a sufficiently simple form that the degree can be calculated directly by elementary counting.

Lemma 3.0.1. *Suppose $\mu \subseteq \lambda$ are strict partitions and let $n \geq \ell(\lambda)$. If $T \in \text{PSVT}(\mu, n)$ is a P -shifted set-valued tableau, then there exists a $T' \in \text{PSVT}(\lambda, n)$ with $d(T') \geq d(T)$.*

Proof. By induction, it suffices to assume that $|\lambda| - |\mu| = 1$. Let the unique box of λ/μ be B_0 . We will construct T' from T by filling the box B_0 with the value n and then possibly making adjustments to other boxes to maintain a valid tableau. For an illustration of the algorithm, see Example 3.0.2.

Extending the definition from [TY09], we say a *short ribbon* is an edge-connected set of boxes that does not contain a 2×2 subshape and where each row and column contains at most two boxes. Recursively define the boxes of a short ribbon R as follows. For k odd, let $B_k = B_{k-1}^\downarrow$, while for k even, let $B_k = B_{k-1}^\leftarrow$. The short ribbon R consists of all of these boxes B_k for $k \geq 0$. We consider two cases according to whether or not B_0 lies on the main diagonal.

(Case 1: B_0 is not on the main diagonal.) Add the value n to the box B_0 . We truncate R to another short ribbon S as follows. First, say $B_1 \in S$ if $T(B_1) \ni n$; otherwise, S is the empty

ribbon. For each even $j \geq 2$, if $\mathbf{B}_{j-1} \in S$ and $|T(\mathbf{B}_{j-1})| = 1$ and $T(\mathbf{B}_j) \ni (n - \lceil \frac{j-1}{2} \rceil)'$, include $\mathbf{B}_j \in S$. For each odd $j \geq 3$, if $\mathbf{B}_{j-1} \in S$ and $|T(\mathbf{B}_{j-1})| = 1$ and $T(\mathbf{B}_j) \ni n - \lceil \frac{j-1}{2} \rceil$, include $\mathbf{B}_j \in S$.

For each box $\mathbf{B}_j \in S$, modify its filling as follows:

$$T'(\mathbf{B}_j) = \begin{cases} T(\mathbf{B}_j) \setminus \max T(\mathbf{B}_j), & \text{if } |T(\mathbf{B}_j)| > 1; \\ k - 1, & \text{if } T(\mathbf{B}_j) = k'; \\ k', & \text{if } T(\mathbf{B}_j) = k. \end{cases} \quad (3.0.1)$$

This operation is well-defined because $n \geq \ell(\lambda)$ by assumption, so the height of the ribbon S is at most n , and k cannot become non-positive before the ribbon terminates. Since the ribbon S terminates as soon as it reaches a box with more than one entry, the first case in the definition (3.0.1) can only occur at most once; therefore T' deletes at most one entry from T , and since we have added one n in the newly added box, it follows that $d(T') \geq d(T)$, as desired. It remains to check that increasingness conditions are satisfied.

Suppose there is an increasingness violation in T' ; since only the content of the ribbon S was altered, the violation must involve some box of S . First, observe that the increasingness violation cannot be between two boxes of S and that it cannot involve a box of S with more than one entry. Let j be the smallest index such that $\mathbf{B}_j$ is involved in an increasingness violation. If j is odd, then the violation must be between $\mathbf{B}_j$ and $\mathbf{B}_j^\downarrow$ or between $\mathbf{B}_j$ and $\mathbf{B}_j^\rightarrow$. The latter is clearly not possible, since the content of $\mathbf{B}_j$ has been decreased from T and these two boxes were increasing in T . A violation between $\mathbf{B}_j$ and $\mathbf{B}_j^\downarrow$ is also not possible, since if j is odd, then $T(\mathbf{B}_j)$ is, by assumption, unprimed, and so $\max(T(\mathbf{B}_j^\downarrow)) < T(\mathbf{B}_j)$, which implies $\max(T(\mathbf{B}_j^\downarrow)) \leq T'(\mathbf{B}_j)$.

Suppose instead then that j is even. Then the only two violations could be between $\mathbf{B}_j$ and $\mathbf{B}_j^\leftarrow$ or between $\mathbf{B}_j$ and $\mathbf{B}_j^\uparrow$. The latter is impossible because we have reduced the content of $\mathbf{B}_j$ while keeping $\mathbf{B}_j^\uparrow$ fixed. A violation between $\mathbf{B}_j$ and $\mathbf{B}_j^\leftarrow$ is also impossible, because $T(\mathbf{B}_j)$ is primed by assumption, so $T(\mathbf{B}_j^\leftarrow) < T(\mathbf{B}_j)$, which implies $T(\mathbf{B}_j^\leftarrow) \leq T'(\mathbf{B}_j)$.

(Case 2: $\mathbf{B}_0$ is on the main diagonal.) Add the value n to the box $\mathbf{B}_0$. Again, we truncate R to another short ribbon S , but in a slightly different way. First, say $\mathbf{B}_1 \in S$ if $T(\mathbf{B}_1) \ni n$; otherwise, S is the empty ribbon. For each even $j \geq 2$, if $\mathbf{B}_{j-1} \in S$ and $|T(\mathbf{B}_{j-1})| = 1$ and $T(\mathbf{B}_j) \ni n - \frac{j}{2} + 1$, include $\mathbf{B}_j \in S$. For each odd $j \geq 3$, if $\mathbf{B}_{j-1} \in S$ and $|T(\mathbf{B}_{j-1})| = 1$ and $T(\mathbf{B}_j) \ni n - \frac{(j-1)}{2}$, include $\mathbf{B}_j \in S$.

For each box $\mathbf{B}_j \in S$, modify its filling as follows:

$$T'(\mathbf{B}_j) = \begin{cases} T(\mathbf{B}_j) \setminus \max T(\mathbf{B}_j), & \text{if } |T(\mathbf{B}_j)| > 1; \\ T(\mathbf{B}_j) - 1, & \text{if } j \text{ is even;} \\ T(\mathbf{B}_j)', & \text{if } j \text{ is odd.} \end{cases} \quad (3.0.2)$$

Exactly as above, this operation is well-defined and the tableau T' has degree $d(T') \geq d(T)$. We now consider increasingness conditions.

Again, as mentioned above, any possible increasingness violation must involve exactly one box of the ribbon S . Suppose j is the smallest index on such a box B_j . If j is odd, then the only violations can be between B_j and $B_j^\downarrow$ or B_j and $B_j^\rightarrow$. Again, the latter is impossible because we have only decreased the content of B_j , and a violation between B_j and $B_j^\downarrow$ is also not possible, since if j is odd, then $T(B_j)$ is, by assumption, unprimed, and so $\max(T(B_j^\downarrow)) < T(B_j)$, which implies $\max(T(B_j^\downarrow)) \leq T'(B_j)$.

If, on the other hand, j is even, then there cannot possibly be any increasingness violation, since B_j is on the main diagonal for even j . Any violation would have to involve either $B_j^\leftarrow$ or $B_j^\uparrow$, but neither of these boxes exists for a box on the main diagonal.

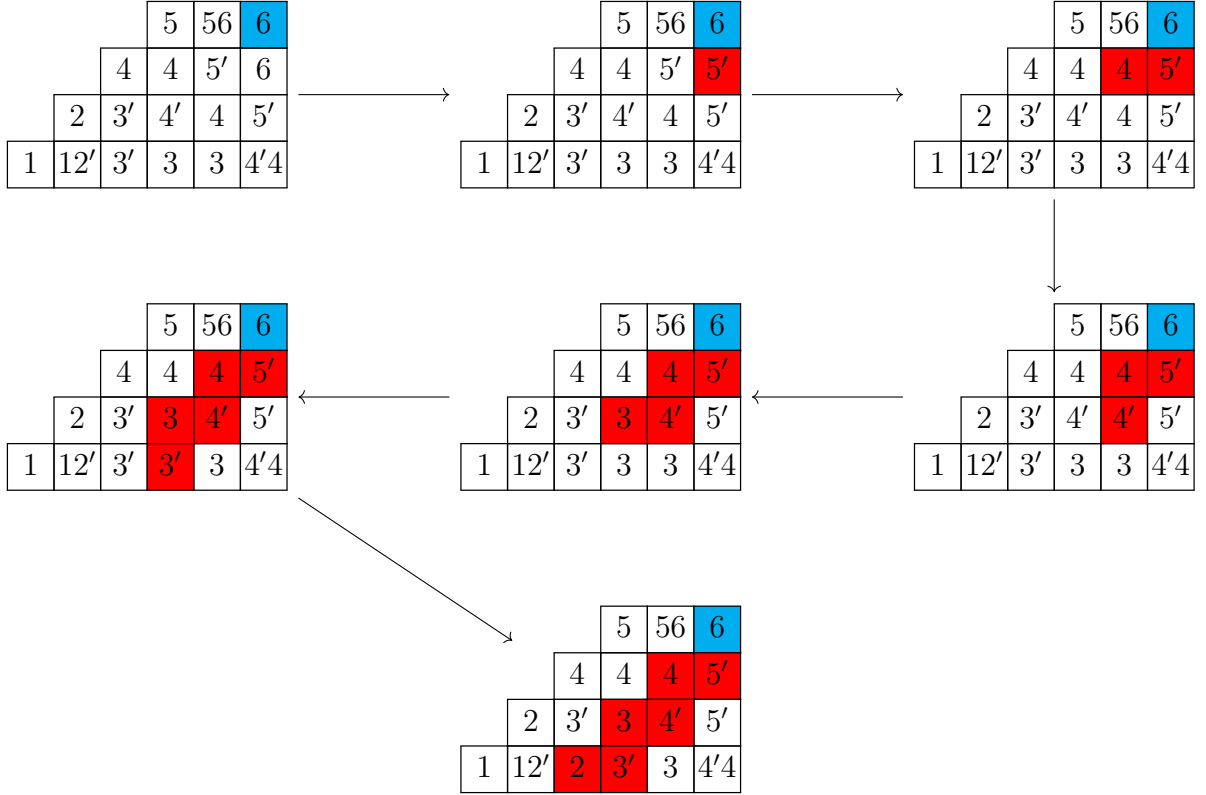
Therefore, in either case the tableau constructed by modifying the ribbon S as above and leaving all other boxes unaltered is a P -shifted set-valued tableau T' on λ with degree greater than or equal to that of T , as desired. $\square$

Example 3.0.2. Let $n = 6$, and consider the P -shifted set-valued tableau

			5	56	
		4	4	5'	6
	2	3'	4'	4	5'
1	12'	3'	3	3	4'4

of shape $\mu = (6, 5, 4, 2)$.

Suppose we wish to construct a P -shifted set-valued tableau with greater or equal weight on $\lambda = (6, 5, 4, 3) \supset \mu$. This is performed by the following sequence of steps:



Here, the unique box $B_0 \in \lambda/\mu$ is shaded in blue ■, while the boxes of the short ribbon S are shaded in red ■. In this case, the short ribbon R consists of all the shaded boxes, either blue or red. In general, R could contain further boxes below and left of the bottom of S .

Definition 3.0.3. If a partition λ satisfies $\lambda_{i-1} - \lambda_i \geq 2$ for all i , we call λ a *D-partition* (from the French, *Différence*-partition, cf. [Vie07]).

Visually, a *D-partition* satisfies the property that its shifted diagram has no two rows which end in the same column. For example, the partition in example 2.4.5 is not a *D-partition*, as rows 2 and 3 end in the same column. This property is critical to the proofs of Lemmas 3.0.4 and 3.0.8.

Lemma 3.0.4. Let $T \in \text{PSVT}(\lambda, n)$ be a *P-shifted set-valued tableau* and let Δ be the largest *D-partition* contained in λ . Then there exists a tableau $S \in \text{PSVT}(\Delta, n)$ such that $d(T) = d(S)$.

Proof. If $\lambda = \Delta$, then there is nothing to show. Otherwise, λ is not a *D-partition*, so there exists at least one row k for which $\lambda_k - 1 = \lambda_{k+1}$; choose the largest such k , and let R_0 be

the rightmost box in row k . By the choice of k , $R_0^\uparrow$ exists. Consider the longest possible sequence of boxes

$$R_0, U_1, R_1, U_2, R_2, \dots, U_j, (R_j)$$

such that $U_i = R_{i-1}^\uparrow$ and $R_i = U_i^\leftarrow$ (where the final box R_j may or may not exist). This sequence of boxes forms a *short ribbon* in the sense of [TY09] and contains at least two boxes. Two nominally distinct cases can occur, based on whether the final box in the ribbon is an R or a U:

$$(A) \quad \begin{array}{|c|c|} \hline R_j & U_j \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline R_{j-1} & U_{j-1} \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline \ddots & \ddots \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline \ddots & U_2 \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline R_1 & U_1 \\ \hline \end{array} \quad \begin{array}{|c|} \hline R_0 \\ \hline \end{array} \quad \text{or} \quad (B) \quad \begin{array}{|c|} \hline U_j \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline R_{j-1} & U_{j-1} \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline \ddots & \ddots \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline \ddots & U_2 \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline R_1 & U_1 \\ \hline \end{array} \quad \begin{array}{|c|} \hline R_0 \\ \hline \end{array} \quad (3.0.3)$$

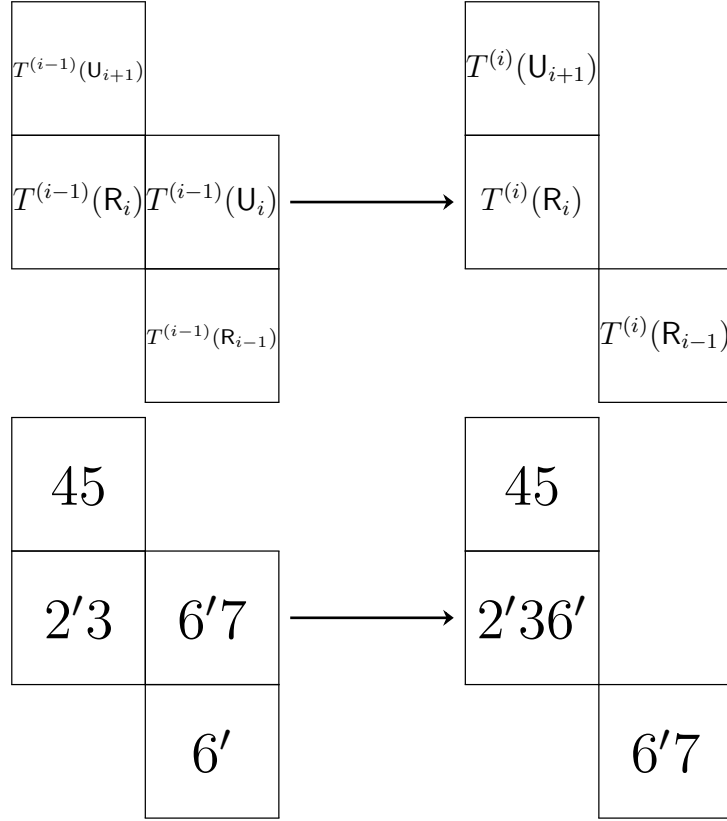
Observe that, by choice of k , the boxes $U_i^\uparrow$ and $U_i^\rightarrow$ do not exist for any i . In case (A), where R_j exists, the box $R_j^\uparrow$ does not exist. In case (B), the box $U_j^\leftarrow$ does not exist.

We define a sequence of fillings (not necessarily P -shifted set-valued tableaux) recursively as follows. Let $\lambda^{(0)} = \lambda$ and $T^{(0)} = T$. For $i > 0$, let $\lambda^{(i)} = \lambda^{(i-1)}/U_i$. Take $T^{(i)}$ to be the filling of $\lambda^{(i)}$ defined by:

- $T^{(i)}(R_{i-1}) = T^{(i-1)}(R_{i-1}) \cup T^{(i-1)}(U_i)$,
- $T^{(i)}(R_i) = T^{(i-1)}(R_i) \cup \left(T^{(i-1)}(U_i) \cap T^{(i-1)}(R_{i-1}) \right)$,
- $T^{(i)}(B) = T^{(i-1)}(B)$, for all other boxes B .

That is to say, at the i th step of the recursion, we delete the box U_i from the diagram, slide the contents of U_i down into R_{i-1} , and to preserve degree, place any intersection between $T^{(i-1)}(U_i)$ and $T^{(i-1)}(R_{i-1})$ into R_i . (One might be concerned that this description requires, in case (A), placing labels in R_j when that box does not exist; however, we will show that this does not occur.) For example, the local configuration below changes in the following

way:



We make the following claims about the sequence $(\lambda^{(i)}, T^{(i)})_{i=0}^j$.

Claim 3.0.5. *The sequence $(\lambda^{(i)}, T^{(i)})$ satisfies*

1. $\Delta \subseteq \lambda^{(j)} \subset \lambda^{(j-1)} \subset \dots \subset \lambda^{(1)} \subset \lambda^{(0)}$;
2. if $T^{(i)}$ is not a P -shifted set-valued tableau, then the only violation is an increasingness violation that occurs between the boxes R_i and U_{i+1} ;
3. $T^{(i-1)}(U_i) \cap T^{(i-1)}(R_{i-1})$ consists only of primed letters; and
4. $d(T^{(i)}) = d(T^{(i-1)})$.

Proof of the claim. We prove the four statements in turn.

1. Each step from $\lambda^{(i)}$ to $\lambda^{(i+1)}$ replaces two consecutive rows of the partition $(r, r-1)$ with $(r, r-2)$; this cannot change the largest D -partition contained within the partition.

2. This follows inductively, the base case $i = 0$ being true by the assumption that $T^{(0)} = T$ is a P -shifted set-valued tableau. If $T^{(i-1)}$ is a valid tableau except for U_i , then by removing the box U_i and only changing the filling by adding elements to the boxes R_i and R_{i-1} which are larger than all of the current contents, we can only introduce an increasingness violation involving these two boxes and a box above or to the right. R_{i-1} no longer has any boxes above or to the right in $\lambda^{(i)}$, and R_i has nothing to its right, so the only possible violation is between R_i and U_{i+1} , as claimed.
3. This also follows by induction, the base case $i = 1$ following from the fact that T is a valid tableau so the intersection in question is either empty or a single primed letter. By definition, $T^{(i-1)}(R_{i-1})$ is $T(R_{i-1}) \cup \left(T^{(i-2)}(U_{i-1}) \cap T^{(i-2)}(R_{i-2}) \right)$, which is by induction $T(R_{i-1})$ together with (perhaps) some primed letters. We have $T^{i-1}(U_i) = T(U_i)$, and since T is a P -shifted set-valued tableau, the intersection $T(U_i) \cap T(R_{i-1})$ is either empty or a single primed letter. This proves this part of the claim.
4. If $T^{(i-1)}(U_i) \cap T^{(i-1)}(R_{i-1}) = \emptyset$, then this is trivial. Otherwise, $T^{(i-1)}(U_i) \cap T^{(i-1)}(R_{i-1})$ is a set of primed letters by Claim (3), and because the original filling T is row-increasing, none of these primed letters are contained in $T^{(i-1)}(R_i)$, so

$$|T^{(i)}(R_i)| = |T^{(i-1)}(R_i)| + |T^{(i-1)}(U_i) \cap T^{(i-1)}(R_{i-1})|$$

to compensate for the fact that

$$|T^{(i)}(R_{i-1})| = |T^{(i-1)}(R_{i-1})| + |T^{(i-1)}(U_i)| - |T^{(i-1)}(U_i) \cap T^{(i-1)}(R_{i-1})|,$$

and the degrees balance.

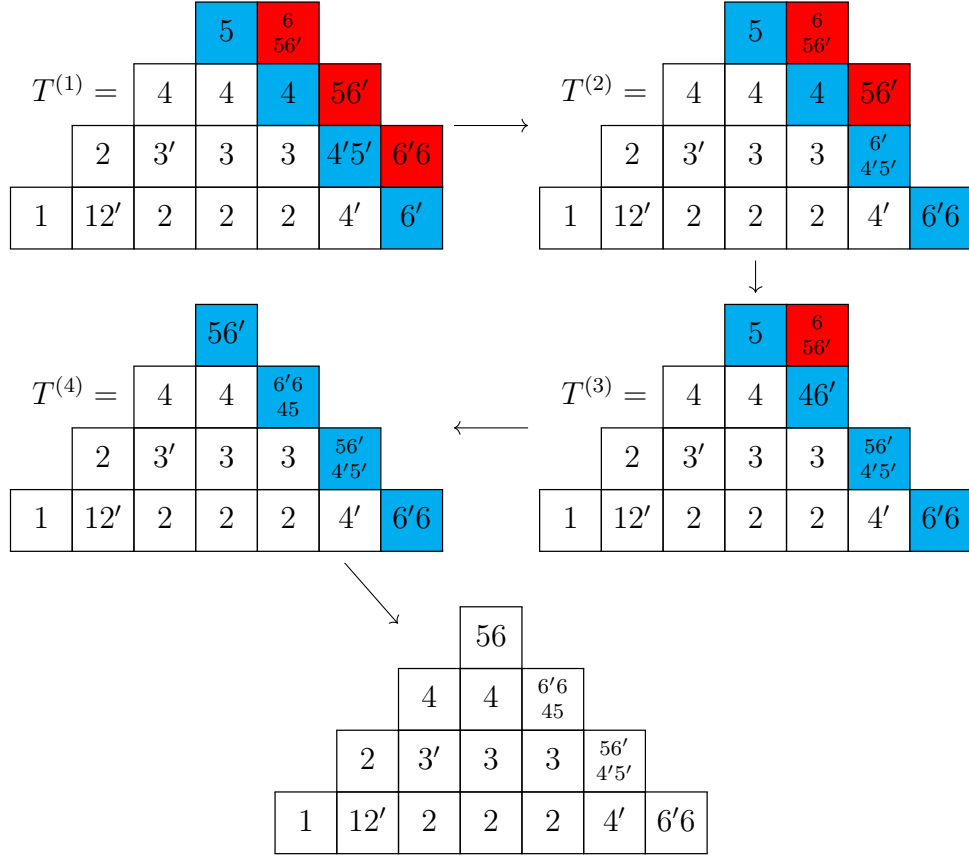
This completes the proof of the claim. □

Now, we deal with two edge case concerns, one for each of cases (A) and (B). first, the concern that case (B) appears to define a filling for a box R_j that does not exist. However, it can be seen that, in case (B), the definition of $T^{(j)}(R_j)$ will always be empty; since U_j is on the main diagonal, $T^{(j-1)}(U_j) = T(U_j)$ contains no primed entries, and so the intersection $T^{(j-1)}(U_j) \cap T^{(j-1)}(R_{j-1})$ is empty. Since $T(R_j)$ is also originally empty, the $T^{(j)}(R_j)$ prescribed by the algorithm is empty, and this procedure is in fact well-defined.

Second, if we are in case (A) and the final box R_j lies on the main diagonal, then it will be necessary to unprime all entries added to R_j . There is no possible issue introduced by doing this, since by claim 3 everything added to R_j by the algorithm was a primed entry of U_j , and if $x' \in U_j$ then by definition $\max R_j < x'$, so $\max R_j < x$.

It then follows from Claim (2) that $T^{(j)}$ is a P -shifted set-valued tableau on $\lambda^{(j)}$ (since U_{j+1} does not exist), from Claim (1) that $\Delta \subseteq \lambda^{(j)} \subset \lambda$, and from Claim (4) that $d(T^{(j)}) = d(T)$, and so the P -shifted set-valued tableau $T^{(j)}$ on the strict partition $\lambda^{(j)}$ suffices to establish the lemma. □

Example 3.0.6. Below, we trace through the algorithm of Lemma 3.0.4 applied to a tableau of shape 7642 to reduce to a tableau of D-partition shape 7531. Boxes denoted U_i are shaded in red ■, while boxes denoted R_i are shaded in blue ■.



Definition 3.0.7. Let Δ be a D-partition of length ℓ and fix a natural number $n \geq \ell$. If $\Delta_\ell > 1$, we define the P -shifted set-valued tableau $\mathcal{M}_{\Delta,n}$ to be

$$\begin{array}{cccccccccccc}
 & & & & \ell & \ell & \dots & [\ell, n]_{\mathbb{S}} & & & & \\
 & & & & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & & \\
 & & & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \\
 & & & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \\
 & & 2 & 2 & 2 & 2 & 2 & 2 & 2 & \dots & [2, n]_{\mathbb{S}} & \\
 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \dots & 1 & [1, n]_{\mathbb{S}}
 \end{array} \quad (3.0.4)$$

That is, each box in row i is filled with the value i , except the rightmost box in row i which receives the interval $[i, n]_{\mathbb{S}}$. If instead $\Delta_\ell = 1$, we define $\mathcal{M}_{\Delta, n}$ to be

$$\begin{array}{cccccccccccc}
& & & & & [\ell, n] & & & & & & \\
& & & & & k & \cdots & k & [k, n]_{\mathbb{S}} & & & \\
& & & & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & & \\
& & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & & \\
& 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & \cdots & [2, n]_{\mathbb{S}} & \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots & 1 & [1, n]_{\mathbb{S}}
\end{array}, \quad (3.0.5)$$

where $k = \ell - 1$. That is, each box outside the top row is filled as before, while the unique box of the top row receives the integer interval $[\ell, n]$.

Lemma 3.0.8. *For any D -partition Δ and any $n \in \mathbb{N}$, the tableau $\mathcal{M}_{\Delta,n}$ has maximum degree among all tableaux in $\text{PSVT}(\Delta, n)$.*

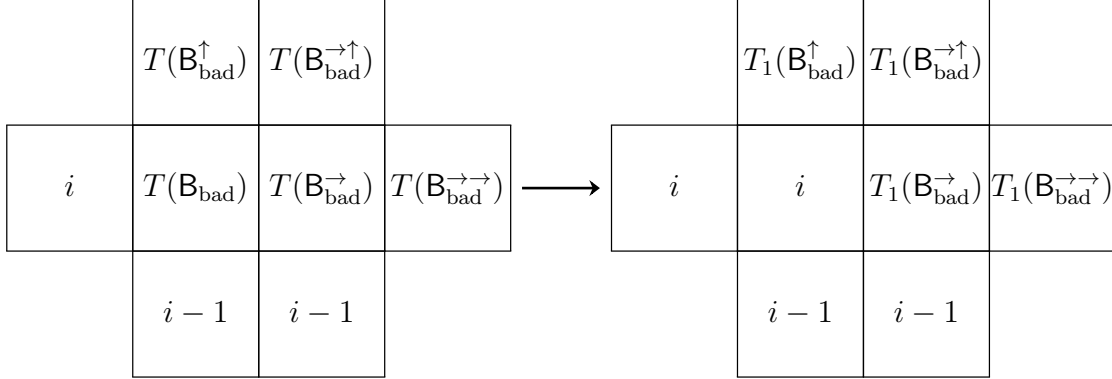
Proof. For an arbitrary tableau $T \in \text{PSVT}(\Delta, n)$, we will construct a finite sequence of tableaux $T = T_0, T_1, \dots, T_{j-1}, T_j = \mathcal{M}_{\Delta, n}$ such that $d(T_0) \leq d(T_1) \leq \dots \leq d(T_j)$. If $T = \mathcal{M}_{\Delta, n}$, there is nothing to show, so we assume there exists at least one box $\mathbf{B} \in \Delta$ for which $T(\mathbf{B}) \neq \mathcal{M}_{\Delta, n}(\mathbf{B})$. Find the smallest index i for which a box in row i of Δ differs between T and $\mathcal{M}_{\Delta, n}$, and find the leftmost box $\mathbf{B}_{\text{bad}}$ in this row such that $T(\mathbf{B}_{\text{bad}}) \neq \mathcal{M}_{\Delta, n}(\mathbf{B}_{\text{bad}})$. There are three nominally distinct cases to treat:

(Case 1: B_{bad} is not the rightmost box in row i .) In this case, we define the tableau T_1 as follows:

- $T_1(\mathbf{B}_{\text{bad}}) = \{i\}$,
- $T_1(\mathbf{B}_{\text{bad}}^{\rightarrow}) = T(\mathbf{B}_{\text{bad}}) \cup T(\mathbf{B}_{\text{bad}}^{\rightarrow})$,
- $T_1(\mathbf{B}) = T(\mathbf{B})$, for all other boxes $\mathbf{B}$.

The only two boxes which are different in T_1 and T are B_{bad} and $B_{\text{bad}}^{\rightarrow}$, so to confirm that T_1

remains a valid tableau we only need to check the following local region of rows $i-1, i, i+1$.



Most increasingness conditions follow immediately from inspection and $T \in \text{PSVT}(\Delta, n)$; we only need to remark that since we only add elements (weakly) less than the minimum of $T(\mathbf{B}_{\text{bad}}^{\rightarrow})$ to $\mathbf{B}_{\text{bad}}^{\rightarrow}$, we cannot introduce a violation with the boxes $\mathbf{B}_{\text{bad}}^{\rightarrow\uparrow}$ or $\mathbf{B}_{\text{bad}}^{\rightarrow\rightarrow}$. To conclude that $d(T) \leq d(T_1)$, observe that $|T(\mathbf{B}_{\text{bad}}) \cap T(\mathbf{B}_{\text{bad}}^{\rightarrow})| \leq 1$, and so

$$|\{i\}| + |T(\mathbf{B}_{\text{bad}}) \cup T(\mathbf{B}_{\text{bad}}^{\rightarrow})| \geq 1 + (|T(\mathbf{B}_{\text{bad}})| + |T(\mathbf{B}_{\text{bad}}^{\rightarrow})| - 1).$$

Thus, the total content of these two boxes in T_1 is weakly larger than in T .

(Case 2: $\mathbf{B}_{\text{bad}}$ is the rightmost box in row i and does not lie on the main diagonal): In this case, we set $T_1(\mathbf{B}_{\text{bad}}) = [i, n]_{\mathbb{S}}$ and $T_1(\mathbf{B}) = T(\mathbf{B})$ for all other boxes $\mathbf{B}$. Since $\mathbf{B}_{\text{bad}}$ is in row i , $\min(T(\mathbf{B}_{\text{bad}})) \geq i$, so $T(\mathbf{B}_{\text{bad}}) \subseteq [i, n]_{\mathbb{S}}$, and therefore $d(T) \leq d(T_1)$. The tableau T_1 is still a valid tableau, because by assumption $T_1(\mathbf{B}_{\text{bad}}^{\leftarrow}) = \{i\}$ and $T_1(\mathbf{B}_{\text{bad}}^{\downarrow}) = \{i-1\}$ (or is empty, if $i = 1$), and boxes in the other two directions do not exist. Since $T(\mathbf{B}_{\text{bad}}) \subseteq [i, n]_{\mathbb{S}}$, it follows that $|T(\mathbf{B}_{\text{bad}})| \leq |[i, n]_{\mathbb{S}}|$, and since this is the only box which changes between T and T_1 , we conclude $d(T) \leq d(T_1)$.

(Case 3: $\mathbf{B}_{\text{bad}}$ is the rightmost box in row i and lies on the main diagonal): This can only happen if row i consists of only the single box $\mathbf{B}_{\text{bad}}$. In this case, we proceed identically as Case 2, except that we are prohibited from having primed entries in $\mathbf{B}_{\text{bad}}$, so we set $T_1(\mathbf{B}_{\text{bad}}) = [i, n]$ instead, and as in Case 2, we see that T_1 remains a valid tableau with $d(T) \leq d(T_1)$.

If $T_1 \neq \mathcal{M}_{\Delta, n}$, then we can produce another tableau T_2 by applying the same construction to T_1 , and continue to produce a sequence of P -shifted set-valued tableaux $T_0, T_1, T_2, \dots$ with the property that $d(T_i) \leq d(T_{i+1})$ for all i . This sequence must be finite because, by construction, the bad box of T_{i+1} is either in the same row as the bad box of T_i but strictly further right or in a strictly higher row. The sequence will stop when we are unable to find any box of T_j different from $\mathcal{M}_{\Delta, n}$, that is, when $T_j = \mathcal{M}_{\Delta, n}$, which completes the proof of the lemma. $\square$

Applying Lemmas 3.0.4 and 3.0.8 reduces the calculation of the degree of $GP_{\lambda,n}$ to directly counting how many entries are in $\mathcal{M}_{\Delta,n}$, so we can now finish the proof.

Proof of Theorem 1.1. Now, we complete the proof of Theorem 1.0.1 by combining the previous lemmas. Let Δ be the largest D-partition contained in λ and let the last nonzero part of Δ be Δ_ℓ . By Lemma 3.0.1, $d(GP_{\Delta,n}) \leq d(GP_{\lambda,n})$. By Lemma 3.0.4, we have $d(GP_{\Delta,n}) \geq d(GP_{\lambda,n})$. Therefore, $d(GP_{\Delta,n}) = d(GP_{\lambda,n})$. By Lemma 3.0.8, $d(GP_{\Delta,n}) = d(\mathcal{M}_{\Delta,n})$. If $\Delta_\ell \neq 1$, then by direct inspection of the tableau $\mathcal{M}_{\Delta,n}$, we have

$$c(\mathcal{M}_{\Delta,n}) = (\Delta_1, \Delta_2 + 2, \dots, \Delta_\ell + 2\ell, 2\ell, 2\ell, \dots, 2\ell),$$

so that

$$d(\mathcal{M}_{\Delta,n}) = |\Delta| + \sum_{i=1}^{\ell} 2i + 2\ell(n - \ell) = |\Delta| + 2n\ell - \ell^2 - \ell.$$

Otherwise, $\Delta_\ell = 1$ and again by inspection we find

$$c(\mathcal{M}_{\Delta,n}) = (\Delta_1, \Delta_2 + 2, \dots, \Delta_{\ell-1} + 2(\ell - 1), \Delta_\ell + 2\ell - 2, 2\ell - 1, 2\ell - 1, \dots, 2\ell - 1),$$

so that

$$d(\mathcal{M}_{\Delta,n}) = |\Delta| + 2n\ell - \ell^2 - n. \quad \square$$

Remark 3.0.9. Lemma 3.0.8 explicitly identifies an element of $\text{PSVT}(\lambda, n)$ of maximum degree, when λ is a D-partition. Through Lemmas 3.0.1 and 3.0.4, this is sufficient to allow us to determine the degree of $GP_{\lambda,n}$ for general λ , but without identifying explicit elements of top degree outside of the D-partition case. While one can be constructed via the algorithm described in Lemma 3.0.1, the algorithm does a large amount of damage to the structure of the tableau, especially when iterated, and it is not clear what the explicit form of the final result is. It would be very interesting to find and describe such representatives.

Chapter 4

Regularity of Pfaffian ideals

In this section, we relate the combinatorics of the previous section to commutative algebra. First, in Section 4.1, we define skew-symmetric matrix Schubert varieties as studied in [MP22], and observe that they are Cohen–Macaulay. Then, in Section 4.2, we recall the symplectic Grothendieck polynomials defined in [WY17, MP21] and use them in combination with Theorem 1.0.1 to establish our main algebraic result, Theorem 1.0.2.

4.1 Skew-symmetric matrix Schubert varieties

Let Mat_n^{ss} denote the variety of $n \times n$ skew-symmetric matrices, a linear subspace of Mat_n . Our focus is on the *skew-symmetric matrix Schubert varieties* $X_w^{\text{ss}} := X_w \cap \text{Mat}_n^{\text{ss}}$ as studied by E. Marberg and B. Pawlowski in [MP22]. These varieties provide an affine model of orbit-closures for the action of the symplectic group $\text{Sp}_n(\mathbb{C})$ on the flag variety Flag_n .

While X_w^{ss} is defined for arbitrary permutations, this is not the appropriate generating set, as for example, $X_{21}^{\text{ss}} = X_{12}^{\text{ss}}$, since the diagonal entries of a skew-symmetric matrix are necessarily zero. Instead, in the symplectic context the relevant indexing family for cohomology classes is the set of *fixed-point-free involutions*, which is the set

$$\mathbf{FPF}_n := \{z \in \mathbf{S}_n : z^2 = 1 \text{ and } z(i) \neq i \text{ for } 1 \leq i \leq n\}.$$

Note this set is empty if n is odd, so from this point forward we assume we are working with $2n \times 2n$ matrices and the symmetric group $\mathbf{S}_{2n}$.

A second, subtler issue which arises is that the ideal of minors defining X_z^{ss} is not generally prime. This differs in a significant way from the generic matrix setting, where it is established by [Ful92] that the ideal of minors defining X_w is always prime. Constructing the ideal $I(X_z^{\text{ss}})$ requires some intricacies with Pfaffian polynomials which we cover below, closely following [MP22], where one can look for full proofs and additional information.

Recall that a skew-symmetric matrix $A \in \text{Mat}_{2n}^{\text{ss}}$ carries an invariant polynomial known as the *Pfaffian* $\text{pf}(A)$ with the property that $\text{pf}(A)^2 = \det(A)$. Formally, we define it as

$$\text{pf}(A) := \frac{1}{2^n n!} \sum_{\sigma \in \mathbf{S}_{2n}} \text{sgn}(\sigma) \prod_{i=1}^n a_{\sigma(2i-1), \sigma(2i)}.$$

For background on combinatorial appearances of Pfaffians, see [God93].

Fix n and let $S = \mathbb{C}[x_{i,j} : 1 \leq j < i \leq 2n]$ be a polynomial ring in $\binom{2n}{2}$ independent indeterminates.

Theorem 4.1.1. [MP22] *Let $z \in \mathbf{FPF}_{2n}$, and let $\mathcal{X}^{\text{ss}}$ be the $2n \times 2n$ skew-symmetric matrix*

$$\begin{pmatrix} 0 & -x_{2,1} & \cdots & -x_{2n,1} \\ x_{2,1} & 0 & \cdots & -x_{2n,2} \\ \vdots & \ddots & \ddots & \vdots \\ x_{2n,1} & \cdots & \cdots & 0 \end{pmatrix}.$$

Then the radical ideal $I(X_z^{\text{ss}}) \subset S$ is

$$I(X_z^{\text{ss}}) = \langle \text{pf}(\mathcal{X}_{UU}^{\text{ss}}) : \exists(i, j) \in (2n \times 2n), i \geq j, U \subseteq [i], |U \cap [j]| > R_{i,j}^z \rangle.$$

Theorem 4.1.2. *The coordinate ring $S/I(X_z^{\text{ss}})$ of the variety X_z^{ss} is Cohen–Macaulay.*

Proof. Marberg–Pawlowski [MP22] show that there is a term order such that an initial ideal of $I(X_z^{\text{ss}})$ is squarefree and that the corresponding Stanley–Reisner simplicial complex is shellable, so that for this term order $\text{in}(I(X_z^{\text{ss}}))$ is Cohen–Macaulay (see, e.g., [EH12, Theorem 5.13]). Thus it follows (see, e.g., [EH12, Corollary 6.9]) that $S/I(X_z^{\text{ss}})$ is Cohen–Macaulay. $\square$

Remark 4.1.3. E. De Negri and E. Gorla [DG09] show that *mixed ladder Pfaffian varieties* are arithmetically Cohen–Macaulay. We believe this is related to the vexillary case of Theorem 4.1.2, but do not currently understand the precise connection; forthcoming work of L. Escobar, A. Fink, J. Rajchgot, and A. Woo [EFRW24] is expected to shed more light on this.

4.2 Symplectic Grothendieck polynomials and proof of Theorem 1.0.2

Let ∂_i be the divided difference operator $\partial_i f := \frac{f - s_i f}{x_i - x_{i+1}}$, and let $\bar{\partial}_i := \partial_i((1 - x_{i+1})f)$.

The symplectic Grothendieck polynomials are, like ordinary Schubert polynomials, constructed from a top element by sequences of divided difference operators. They are defined as follows.

Definition-Theorem 4.2.1 ([WY17, MP21]). Let $w_0 := 2n(2n-1) \dots 321$ be the longest permutation of S_{2n} . The *symplectic Grothendieck polynomials* in $2n$ variables are the unique family of polynomials $\{\mathfrak{G}_z^{\text{Sp}}\}_{z \in \mathbf{FPF}_{2n}}$ satisfying

1. $\mathfrak{G}_{w_0}^{\text{Sp}} = \prod_{1 \leq i < j \leq 2n-i} (x_i + x_j - x_i x_j)$; and
2. if $i+1 \neq z(i)$ and $i \neq z(i+1)$ and $z(i) > z(i+1)$, then $\mathfrak{G}_{s_i z s_i}^{\text{Sp}} = \bar{\partial}_i \mathfrak{G}_z^{\text{Sp}}$.

An involution $z \in \mathbf{FPF}_n$ has a *symplectic code* [MP21, §4.4] $\text{SpCode}(z) = (c_1, \dots, c_n)$ where c_i is the number of integers j with

$$z(i) > z(j) < i < j.$$

Sorting the entries of $\text{SpCode}(z)$ into a partition and then taking the transpose yields the *symplectic shape* $\lambda_{\text{Sp}}(z)$ of z [MP21, §4.4].

Definition 4.2.2. Let $s \in S_n, t \in S_m$ be permutations. Then the permutation $s \times t \in S_{n+m}$ is

$$(s \times t)(a) := \begin{cases} s(a), & a \leq n \\ n + t(a - n), & n < a \end{cases}.$$

It is shown in [MP20, §4.2] and [Mar20, Theorem 1.9], that the stable limit

$$\lim_{n \rightarrow \infty} \mathfrak{G}_{(21)^n \times z}^{\text{Sp}}$$

exists (in the ring of power series) and is a finite sum of GP_λ . We say that $z \in \mathbf{FPF}_n$ is *FPF-vexillary* if this stable limit is a single GP_λ for some λ . (This differs slightly from the more usual definition, but is equivalent by [MS23, Theorem 3.23].) An explicit characterization of FPF-vexillary involutions in terms of pattern avoidance appears in [HMP20, Corollary 7.9]; we do not recall it here as it is somewhat complicated and the details will not play a role in this paper.

If z is FPF-vexillary with symplectic shape $\lambda = \lambda_{\text{Sp}}(z)$, we have by [MS23, Theorem 3.23]

$$\lim_{n \rightarrow \infty} \mathfrak{G}_{(21)^n \times z}^{\text{Sp}} = GP_\lambda = \lim_{k \rightarrow \infty} GP_\lambda(x_1, \dots, x_k). \quad (4.2.1)$$

We need to understand the nature of this limit in slightly more detail. Given two polynomials $f, g \in S$ with real coefficients, we say $f \preceq g$ if, for each monomial m , we have

$$\begin{aligned} [m]f &\geq 0 \Leftrightarrow [m]g \geq 0 \quad \text{and} \\ |[m]f| &\leq |[m]g|, \end{aligned}$$

where $[m]h$ denotes the coefficient on the monomial m in h . Note that symplectic Grothendieck polynomials have real (and indeed integral) coefficients.

Lemma 4.2.3. *For $z \in \mathbf{FPF}_{2n}$, we have $\mathfrak{G}_z^{\text{Sp}} \preceq \mathfrak{G}_{21 \times z}^{\text{Sp}}$.*

Proof. As defined in [MP20, §3.3], a *Hecke atom* for z is a permutation w such that w acts on the fixed-point-free involution $2143 \dots (2n)(2n-1)$ according to an action studied in [RV13] to produce z . This action is somewhat complicated, so we omit the details; however, it is easy to see from the definition that if w is a Hecke atom for z , then $1^2 \times w$ is a Hecke atom for $21 \times z$. The result then follows from [MP20, Theorem 3.12], which gives a formula for the symplectic Grothendieck polynomial $\mathfrak{G}_u^{\text{Sp}}$ in terms of Hecke atoms of u . $\square$

Corollary 4.2.4. *If $z \in \mathbf{FPF}_{2n}$ is FPF-*vxillary* and the last nonzero entry of $\text{SpCode}(z)$ is in position k , then*

$$\mathfrak{G}_z^{\text{Sp}} \preceq GP_{\lambda_{\text{Sp}}(z), k}.$$

In particular, $\deg \mathfrak{G}_z^{\text{Sp}} \leq \deg GP_{\lambda_{\text{Sp}}(z), k}$.

Proof. Since the last nonzero entry of $\text{SpCode}(z)$ is in position k , we have $\mathfrak{G}_z^{\text{Sp}} \in \mathbb{C}[x_1, \dots, x_k]$ by the involution pipe dream formulas of [HMP22]. The corollary is then immediate from Lemma 4.2.3 combined with (4.2.1). $\square$

Finally, let us recall the theorem connecting the ideals $I(X_z^{\text{ss}})$ to our degree calculations:

Theorem 4.2.5 ([MP22]). *Let $z \in \mathbf{FPF}_{2n}$. Then*

$$\mathfrak{G}_z^{\text{Sp}}(1-t, 1-t, \dots, 1-t) = \mathcal{K}(I(X_z^{\text{ss}}); t).$$

We may now conclude the main theorem of this section, which is the precise version of Theorem 1.0.2.

Theorem 4.2.6. *Let $z \in \mathbf{FPF}_{2n}$ be an FPF-*vxillary* fixed-point-free involution with last nonzero entry of $\text{SpCode}(z)$ in position k . Further let $\Delta \subseteq \lambda_{\text{Sp}}(z)$ be the largest D -partition contained in $\lambda_{\text{Sp}}(z)$. Then*

$$\text{reg } S/I(X_z^{\text{ss}}) \leq \begin{cases} 2k\ell - \ell^2 - \ell - (|\lambda_{\text{Sp}}(z)| - |\Delta|), & \text{if } \Delta_\ell > 1; \\ 2k\ell - \ell^2 - k - (|\lambda_{\text{Sp}}(z)| - |\Delta|), & \text{if } \Delta_\ell = 1. \end{cases}$$

Proof. Since X_z^{ss} is Cohen–Macaulay by Theorem 4.1.2, we have that Theorem 2.7.1 applies. Hence $\text{reg } S/I(X_z^{\text{ss}}) = \deg \mathcal{K}(I(X_z^{\text{ss}}); t) - \text{ht } I(X_z^{\text{ss}})$. By Theorem 4.2.5, $\deg \mathfrak{G}_z^{\text{Sp}} = \deg \mathcal{K}(I(X_z^{\text{ss}}); t)$. We have $\text{ht } I(X_z^{\text{ss}}) = |\lambda_{\text{Sp}}(z)|$ (this follows from example from combining [HMP20, Theorem 1.2] with [HMP22, Theorem 1.2]). By Corollary 4.2.4, $\deg \mathfrak{G}_z^{\text{Sp}} \leq \deg GP_{\lambda_{\text{Sp}}(z), k}$. From Theorem 1.0.1 we have that

$$\deg(GP_{\lambda, k}) = \begin{cases} |\Delta| + 2k\ell - \ell^2 - \ell, & \text{if } \Delta_\ell > 1; \\ |\Delta| + 2k\ell - \ell^2 - k, & \text{if } \Delta_\ell = 1. \end{cases}$$

Thus the theorem follows. $\square$

Chapter 5

Degrees of symmetric Grothendieck polynomials

In this section, we obtain a formula for the degree of (type A) symmetric Grothendieck polynomials, complementary to that of [RRRSW21]. Our formula differs in appearance from the formula of [RRRSW21], and neither formula appears to follow directly from the other. Our approach parallels the shifted arguments of the previous sections, with an analogous collection of lemmas. The proofs of the analogues of Lemmas 3.0.1 and 3.0.4 are somewhat easier than in the shifted setting because two vertically adjacent boxes in an ordinary set-valued tableau always have disjoint content. The proof of Lemma 5.0.4 is almost verbatim that of Lemma 3.0.8.

Lemma 5.0.1. *Suppose $\mu \subseteq \lambda$ are partitions and $n \geq \ell(\lambda)$. If $T \in \text{SVT}(\mu, n)$ is a set-valued tableau, then there exists some $T' \in \text{SVT}(\lambda, n)$ with $d(T') \geq d(T)$.*

Proof. It suffices to assume that $|\lambda| - |\mu| = 1$. Let the unique box of λ/μ be B_0 . To define the filling T' , we first set $T'(B_0) = n$.

Let the boxes of the column of B_0 be $B_0, B_1, \dots$ from top to bottom. Next, we define a partial column C as follows. If $n \notin B_1$, then C is empty. Otherwise, recursively include B_i in C if

- $B_{i-1} \in C$,
- $|T(B_{i-1})| = 1$,
- and $n - i + 1 \in T(B_i)$.

If $B_i \in C$, we also refer to it as C_i .

We then define the filling T' on the partial column C as

$$T'(C_j) = \begin{cases} T(C_j) \setminus \max T(C_j), & \text{if } |T(C_j)| > 1; \\ T(C_j) - 1, & \text{if } |T(C_j)| = 1. \end{cases} \quad (5.0.1)$$

Since $n \geq \ell(\lambda)$, Equation (5.0.1) is well-defined. Finally, if $A \neq B_0$ and $A \notin C$, then we define $T'(A) = T(A)$. We claim that this tableau T' has the desired properties.

Since we have added an entry to the new box B_0 and the filling T' deletes at most one entry from T (the top case of Equation (5.0.1) can occur at most once), it follows that either $d(T') = d(T)$ or $d(T') = d(T) + 1$. In particular, $d(T') \geq d(T)$.

It only remains to check that T' satisfies the necessary increasingness conditions. First, consider B_0 . Since B_0 is the rightmost box in its row and n is the largest letter of the alphabet, there can be no row increasingness violations with B_0 . Furthermore, if $n \in T(C_1)$, then by the construction of the filling, $n \notin T'(C_1)$, so T' has no increasingness violation between boxes B_0 and C_1 .

Since the partial column C is column-increasing in T , by construction it is also column-increasing in T' . Similarly, there can be no column-increasingness violation in T' between the bottom box of C and the box directly below it.

Since we have only decreased the values in the boxes in C , we cannot have introduced any increasingness violations between a box in C and the box to its right. We then only need to check if we introduced a violation between C and the boxes to its left. We show by contradiction that this is impossible. Suppose C_k is the smallest k such that $\max(C_k^-) > \min(C_k)$. The box C_k must have had only a single entry, since otherwise its content only weakly decreases. Therefore, letting $a = T(C_k) = T'(C_k) + 1$, this implies $a \in C_k^-$. Since T was supposed to be column-increasing, this forces $\min(T(C_k^{\uparrow\leftarrow})) > a$. This, however, cannot occur, since $T'(C_{k-1}) = a$, and as C_{k-1} is the box directly right of $C_k^{\uparrow\leftarrow}$, this implies a row-increasingness violation for a smaller k than our chosen minimum, which is a contradiction (if $k = 1$, this argument implies there is a row-increasingness violation in the box $T'(B_0)$, which we already know to be impossible). $\square$

Lemma 5.0.2. *Let $T \in \text{SVT}(\lambda, n)$ be a set-valued tableau, and let μ be the largest strict partition contained in λ . Then there exists a tableau $S \in \text{SVT}(\mu, n)$ such that $d(T) = d(S)$.*

Proof. If $\lambda = \mu$, we are done. Otherwise λ is not strict, so there exists at least one row k for which $\lambda_k = \lambda_{k+1}$; choose the largest such k , and let R be the rightmost box in row k . Set $U := R^\uparrow$. We define the filling T^* on λ/U (which is a valid tableau because U is a corner box, otherwise it would be a higher row k such that $\lambda_k = \lambda_{k+1}$) as:

- $T^*(R) = T(R) \cup T(U)$,
- $T^*(B) = T(B)$, for all other boxes B .

This is simpler than the shifted case where one must track a short ribbon up the tableau, as the tableau obtained by deleting just this single box already works. The critical difference in this case is that $T(\mathbf{R}) \cap T(\mathbf{U}) = \emptyset$ by the definition of a set-valued tableau, so it is clear that $d(T^*) = d(T)$. Since R was the rightmost box in its row and we have only added strictly larger numbers to it, we have not introduced any increasingness violations with $\mathbf{R}^{\leftarrow}$ or $\mathbf{R}^{\downarrow}$. Therefore $\mu \subseteq \lambda/\mathbf{U} \subset \lambda$ and T^* is a set-valued tableau on $\lambda/\mathbf{U}$ with $d(T^*) = d(T)$. By iterating this construction until we eventually remove enough corner boxes to reach μ , we establish the lemma by induction. $\square$

Lemma 5.0.3. *If μ is the largest strict partition contained in λ and $n \geq \ell(\lambda)$, then*

$$\deg GP_{\mu,n} = \deg GP_{\lambda,n}.$$

Proof. This follows from Lemmas 5.0.1, 5.0.2, and the tableau formula for P -Grothendieck polynomials. $\square$

Lemma 5.0.4. *For any strict partition μ with length ℓ and any $n \in \mathbb{N}$, the tableau*

$$\begin{array}{|c|c|c|c|c|c|c|} \hline \ell & \dots & [\ell, n] & & & & \\ \hline \vdots & \vdots & \vdots & \vdots & & & \\ \hline 2 & 2 & 2 & \dots & [2, n] & & \\ \hline 1 & 1 & 1 & \dots & \dots & \dots & [1, n] \\ \hline \end{array} \tag{5.0.2}$$

has maximum degree among all tableaux in $\text{SVT}(\mu, n)$.

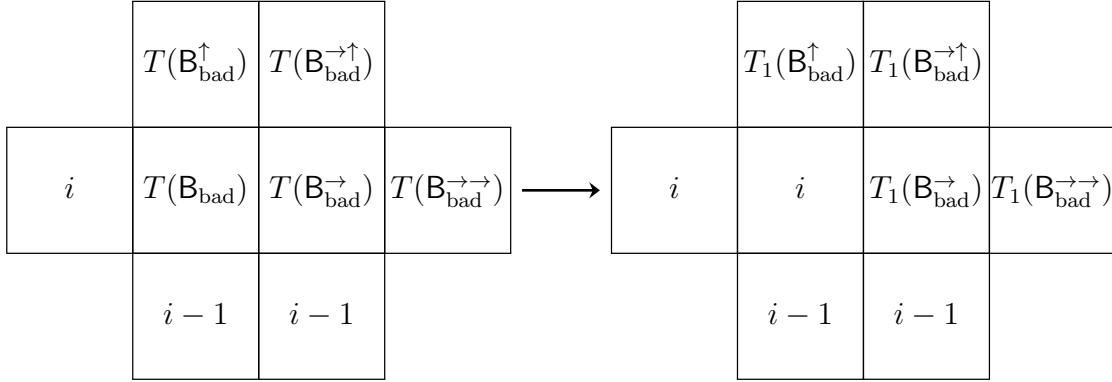
Proof. Here we can proceed very similarly to the proof of Lemma 3.0.8, with minor modifications. To further mimic the situation already proved, we call the tableau (5.0.2) by the name $\mathcal{N}_{\mu,n}$.

If $T \neq \mathcal{N}_{\mu,n}$, we find the smallest index i for which there exists a box $\mathbf{B}$ in row i such that $T(\mathbf{B}) \neq \mathcal{N}_{\mu,n}(\mathbf{B})$, denote the leftmost such box in the row $\mathbf{B}_{\text{bad}}$, and define a new tableau $T_1 \in \text{SVT}(\mu, n)$ satisfying $d(T) \leq d(T_1)$. We split into cases analogous to cases 1 and 2 in the proof of Lemma 3.0.8:

(Case 1: $\mathbf{B}_{\text{bad}}$ is not the rightmost box in row i .) In this case, we define the tableau T_1 as follows:

- $T_1(\mathbf{B}_{\text{bad}}) = \{i\}$,
- $T_1(\mathbf{B}_{\text{bad}}^{\rightarrow}) = T(\mathbf{B}_{\text{bad}}) \cup T(\mathbf{B}_{\text{bad}}^{\rightarrow})$,
- $T_1(\mathbf{B}) = T(\mathbf{B})$, for all other boxes $\mathbf{B}$.

The only two boxes which are different in T_1 and T are $\mathbf{B}_{\text{bad}}$ and $\mathbf{B}_{\text{bad}}^{\rightarrow}$, so to confirm that T_1 remains a valid tableau we only need to check the following local region in rows $i-1, i, i+1$.



Most increasingness conditions follow immediately from inspection and $T \in \text{SVT}(\mu, n)$; we only need to remark that since we only add elements (weakly) less than the minimum of $T(\mathbf{B}_{\text{bad}}^{\rightarrow})$ to $\mathbf{B}_{\text{bad}}^{\rightarrow}$, we cannot introduce a violation with the boxes $\mathbf{B}_{\text{bad}}^{\rightarrow\uparrow}$ or $\mathbf{B}_{\text{bad}}^{\rightarrow\rightarrow}$. To conclude that $d(T) \leq d(T_1)$, observe that $|T(\mathbf{B}_{\text{bad}}) \cap T(\mathbf{B}_{\text{bad}}^{\rightarrow})| \leq 1$, and so

$$|\{i\}| + |T(\mathbf{B}_{\text{bad}}) \cup T(\mathbf{B}_{\text{bad}}^{\rightarrow})| \geq 1 + (|T(\mathbf{B}_{\text{bad}})| + |T(\mathbf{B}_{\text{bad}}^{\rightarrow})| - 1).$$

Thus, the total content of these two boxes in T_1 is weakly larger than in T .

(Case 2: $\mathbf{B}_{\text{bad}}$ is the rightmost box in row i .) In this case, we set $T_1(\mathbf{B}_{\text{bad}}) = [i, n]$ and $T_1(\mathbf{B}) = T(\mathbf{B})$ for all other boxes $\mathbf{B}$. Since $\mathbf{B}_{\text{bad}}$ is in row i , a $\min(T(\mathbf{B}_{\text{bad}})) \geq i$, so $T(\mathbf{B}_{\text{bad}}) \subseteq [i, n]$, and therefore $d(T) \leq d(T_1)$. The tableau T_1 is still a valid set-valued tableau, because by assumption $T_1(\mathbf{B}_{\text{bad}}^{\leftarrow}) = \{i\}$ and $T_1(\mathbf{B}_{\text{bad}}^{\downarrow}) = \{i-1\}$ (or is empty, if $i=1$), and boxes in the other two directions do not exist. Since $T(\mathbf{B}_{\text{bad}}) \subseteq [i, n]$, it clearly follows that $|T(\mathbf{B}_{\text{bad}})| \leq |[i, n]|$, and since this is the only box whose content changes between T and T_1 , we conclude $d(T) \leq d(T_1)$.

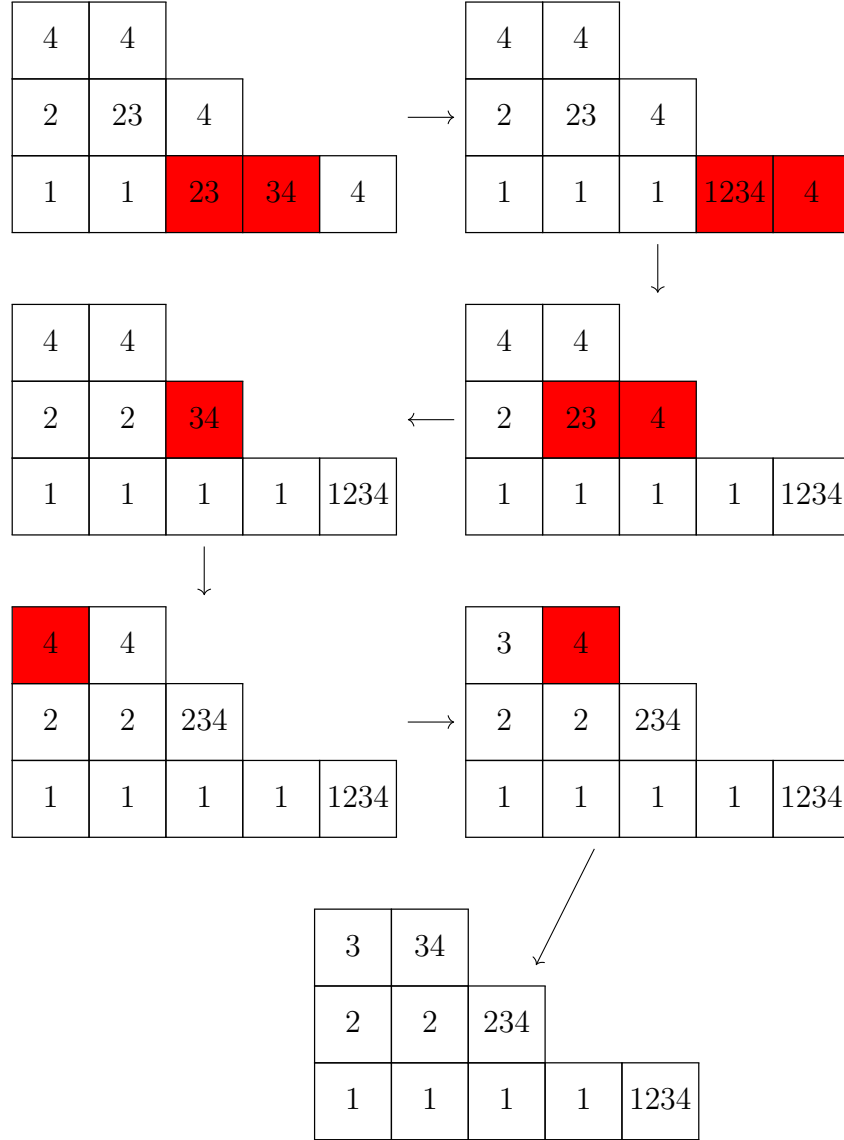
There is no need for an analogous case to case 3 of the proof of Lemma 3.0.8, since in the symmetric case there is no main diagonal and thus no varying behaviour if a box is the only box in its row or not.

Exactly as in Lemma 3.0.8, repeating this construction yields a finite sequence of tableaux $T, T_1, \dots, T_j = \mathcal{N}_{\mu, n}$ such that $d(T) \leq d(T_1) \leq \dots \leq d(T_j)$, completing the proof. $\square$

Example 5.0.5. As an example, we can transform the tableau

4	4				
2	23	4			
1	1	23	34	4	

into maximal form as follows (in each step, the boxes being altered are highlighted in red ■):



We may now prove the main results of this section, complementing the formulas of [RRRSW21] and paralleling the formulas of Theorems 1.0.1 and 1.0.2.

Theorem 5.0.6. *Let λ be a partition, and let $\mu = (\mu_1, \dots, \mu_\ell)$ be the largest strict partition contained in λ . Then*

$$\deg(G_{\lambda,n}) = |\mu| + \ell n - \frac{\ell(\ell+1)}{2}. \quad (5.0.3)$$

Proof. By Lemma 5.0.3, $\deg(G_{\lambda,n}) = \deg(G_{\mu,n})$. By Lemma 5.0.4, $\deg(G_{\mu,n})$ equals the number of entries in the tableau (5.0.2). Elementary counting of entries then yields the theorem. □

Corollary 5.0.7. *Let w be a Grassmannian permutation with $w(n) > w(n+1)$ and $\lambda_{\mathbb{B}}(w) = \lambda$. Then the matrix Schubert variety X_w has Castelnuovo–Mumford regularity*

$$\operatorname{reg}(S/I_w) = \ell n - \frac{\ell(\ell+1)}{2} - (|\lambda| - |\mu|).$$

Proof. This follows from Theorem 5.0.6 exactly as in the analogous calculations of [RRRSW21, §4.2] □

Chapter 6

The case of Q -Grothendieck polynomials

If we remove the final assumption in Definition 2.4.7 and allow primed entries on the main diagonal, then we obtain what are referred to as Q -shifted set-valued tableaux [INN11, IN13]. In particular, every P -shifted set-valued tableau is also Q -shifted. The generating function (analogous to Definition 2.5.2) for Q -shifted set-valued tableaux is the Q -Grothendieck polynomial $GQ_{\lambda,n}$. In the following, we use the notation $\text{QSVT}(\lambda, n)$ to refer to the set of all Q -shifted set-valued tableaux of shape λ with entries from $[n]_{\mathbb{S}}$.

Example 6.0.1. $\text{QSVT}((2, 1), 2)$ is the set

$\begin{array}{ c c } \hline & 2' \\ \hline 1' & 1 \\ \hline \end{array}$	$\begin{array}{ c c } \hline & 2' \\ \hline 1' & 2' \\ \hline \end{array}$	$\begin{array}{ c c } \hline & 2 \\ \hline 1' & 1 \\ \hline \end{array}$	$\begin{array}{ c c } \hline & 2 \\ \hline 1' & 2' \\ \hline \end{array}$	$\begin{array}{ c c } \hline & 2'2 \\ \hline 1' & 1 \\ \hline \end{array}$	$\begin{array}{ c c } \hline & 2'2 \\ \hline 1' & 2' \\ \hline \end{array}$	$\begin{array}{ c c } \hline & 2' \\ \hline 1' & 12' \\ \hline \end{array}$	$\begin{array}{ c c } \hline & 2 \\ \hline 1' & 12' \\ \hline \end{array}$	$\begin{array}{ c c } \hline & 2'2 \\ \hline 1' & 12' \\ \hline \end{array}$
$\begin{array}{ c c } \hline & 2' \\ \hline 1 & 1 \\ \hline \end{array}$	$\begin{array}{ c c } \hline & 2' \\ \hline 1 & 2' \\ \hline \end{array}$	$\begin{array}{ c c } \hline & 2 \\ \hline 1 & 1 \\ \hline \end{array}$	$\begin{array}{ c c } \hline & 2 \\ \hline 1 & 2' \\ \hline \end{array}$	$\begin{array}{ c c } \hline & 2'2 \\ \hline 1 & 1 \\ \hline \end{array}$	$\begin{array}{ c c } \hline & 2'2 \\ \hline 1 & 2' \\ \hline \end{array}$	$\begin{array}{ c c } \hline & 2' \\ \hline 1 & 12' \\ \hline \end{array}$	$\begin{array}{ c c } \hline & 2 \\ \hline 1 & 12' \\ \hline \end{array}$	$\begin{array}{ c c } \hline & 2'2 \\ \hline 1 & 12' \\ \hline \end{array}$
$\begin{array}{ c c } \hline & 2' \\ \hline 1'1 & 1 \\ \hline \end{array}$	$\begin{array}{ c c } \hline & 2' \\ \hline 1'1 & 2' \\ \hline \end{array}$	$\begin{array}{ c c } \hline & 2 \\ \hline 1'1 & 1 \\ \hline \end{array}$	$\begin{array}{ c c } \hline & 2 \\ \hline 1'1 & 2' \\ \hline \end{array}$	$\begin{array}{ c c } \hline & 2'2 \\ \hline 1'1 & 1 \\ \hline \end{array}$	$\begin{array}{ c c } \hline & 2'2 \\ \hline 1'1 & 2' \\ \hline \end{array}$	$\begin{array}{ c c } \hline & 2' \\ \hline 1'1 & 12' \\ \hline \end{array}$	$\begin{array}{ c c } \hline & 2 \\ \hline 1'1 & 12' \\ \hline \end{array}$	$\begin{array}{ c c } \hline & 2'2 \\ \hline 1'1 & 12' \\ \hline \end{array}$

(Note the presence of $1'$; although not explicitly disallowed in the definition of P -shifted tableaux, a $1'$ can only appear in the box of the main diagonal in the first row, and therefore is impossible in a P -shifted tableau.) This gives the Q -Grothendieck polynomial

$$\begin{aligned}
 GQ_{((2,1),2)} = & x_1^2 x_2 + x_1 x_2^2 + x_2^2 x_2 + x_1 x_2^2 + x_1^2 x_2^2 + x_1 x_2^3 + x_1^2 x_2^2 + x_2^2 x_2^2 + x_1^2 x_2^3 \\
 & + x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_2^2 + x_1 x_2^3 + x_1^2 x_2^2 + x_1^2 x_2^2 + x_1^2 x_2^3 \\
 & + x_1^3 x_2 + x_1^2 x_2^2 + x_1^3 x_2 + x_1^2 x_2^2 + x_1^3 x_2^2 + x_1^2 x_2^3 + x_1^3 x_2^2 + x_1^3 x_2^2 + x_1^3 x_2^3.
 \end{aligned}$$

The Q -Grothendieck polynomials have geometric significance similar to other families of Grothendieck polynomials; they are representatives of K -theoretic Schubert classes in the *Lagrangian Grassmannian* parameterizing isotropic n -planes in $\mathbb{C}^{2n}$ with respect to a nondegenerate skew-symmetric bilinear form. (For more background on the K -theory of Lagrangian Grassmannians, see [BR12, IN13, PY17].) Given the significance of the degrees of other families of Grothendieck polynomials to regularity questions, it is natural to ask if our proof can be adapted to obtain the Q -Grothendieck degrees as well. We will provide a simple counterexample below which demonstrates that this result does not work in the Q -shifted setting, and give a combinatorial explanation for why Lemma 3.0.4 fails for Q -shifted tableaux. However, due to the differences in structure between P -shifted and Q -shifted tableaux being restricted to only the diagonal, it will be possible to make a few very simple but nontrivial bounding remarks.

Remark 6.0.2. Even if one succeeds in characterizing the degrees of Q -Grothendieck polynomials, an analogous Gröbner theory to that applied in Section 4 has not yet been developed, so it is unclear what, if any, regularities these degrees track. However, see the discussion at the end of [MP22, §1] for some potentially related ideas about the geometry of *symmetric matrix Schubert varieties*. The geometry in this setting appears to be much more difficult; see [WY17] for further discussion of the challenges in this context.

The proof of Lemma 3.0.8 essentially does not depend on the fact that the tableau in question is P -shifted, rather than Q -shifted; the same proof, *mutatis mutandis*, yields that a maximum degree Q -shifted tableau for a D-partition Δ is the same as $\mathcal{M}_{\Delta,n}$, except for containing the newly allowed k' on the k th row of the main diagonal. Thus, we can still determine a tableau of maximum degree in the case that λ is a D-partition.

However, the proof of Lemma 3.0.4 *does* rely on the tableau being P -shifted, as the analysis applied to ribbons of type (B) can fail when there are primes in the top box on the main diagonal. In fact, the conclusion of Lemma 3.0.4 does not hold in the Q -shifted case; there are examples where $\deg(GQ_{\Delta,n}) < \deg(GQ_{\lambda,n})$ with Δ the largest D-partition inside λ . One such example is the partition $\lambda = 421$. The largest D-partition contained in 421 is $\Delta = 42$. However, the tableau $T \in \text{QSVT}(42, 3)$ shown below is of maximum degree 14 in $\text{QSVT}(42, 3)$, but has lower degree than the tableau $T' \in \text{QSVT}(421, 3)$, which has degree 15:

Proposition 6.0.3. *If Δ is the largest D-partition in λ and $\ell(\Delta) = \ell(\lambda)$, then*

$$\deg(GQ_{\Delta,n}) = \deg(GQ_{\lambda,n}).$$

Proof. In this case, the proof of Lemma 3.0.8 extends directly, since ribbons of type (B) (as defined in that proof) only occur when an entire row of the tableau is deleted. $\square$

Note that Proposition 6.0.3 gives only a sufficient condition for equality and is not a characterization. For example, one can compute that $\deg(GQ_{321,3}) = \deg(GQ_{31,3}) = 12$, while 31 is the largest D-partition inside 321.

It is tempting to attempt to describe the degrees of Q -Grothendieck polynomials in terms of the degrees of P -Grothendieck polynomials. We observe some bounds on the difference between these degrees.

Proposition 6.0.4. *For any strict partition λ , we have*

$$\deg(GQ_{\lambda,n}) - \deg(GP_{\lambda,n}) \in [\ell(\lambda), n].$$

In particular, if $n = \ell(\lambda)$, then $\deg(GQ_{\lambda,n}) = \deg(GP_{\lambda,n}) + n$.

Proof. To establish the lower bound, note that for any P -shifted set-valued tableau T we can produce a Q -shifted tableau T' with $d(T') = d(T) + \ell(\lambda)$ by adding to every box on the main diagonal a primed copy of the minimum element it contains.

For the upper bound, consider a tableau $S \in \text{QSVT}(\lambda, n)$. One can see that if S is of maximal degree, then every box on the main diagonal must contain both a primed and an unprimed entry. In particular, we can produce a P -shifted set-valued tableau $S^\dagger$ by deleting all primed entries from the main diagonal of S . It therefore suffices to observe that no tableau in $\text{QSVT}(\lambda, n)$ can have any particular primed value i' in more than one box on the main diagonal. This is immediate because if a box B is northeast of a box A in a shifted set-valued tableau, then $\min(T(B)) > \max(T(A))$. Thus, $\deg(GQ_{\lambda,n}) \leq \deg(GP_{\lambda,n}) + n$. $\square$

Recently, Y. Chiu and E. Marberg [CM23] proved a result that gives a signed and cancellative expansion of a Q -Grothendieck polynomial in terms of P -Grothendieck polynomials. As the formula is signed, it cannot be used directly to compute the degree of a Q -Grothendieck polynomial in n variables by specialization; in fact, the examples discussed above show that some highly coordinated cancellations can sometimes occur when specializing this formula to particular numbers of variables, causing entire leading terms of the sum to vanish. It appears difficult to understand when these cancellations occur.

References

- [AF24] David Anderson and William Fulton, *Equivariant cohomology in algebraic geometry*, Cambridge Studies in Advanced Mathematics, vol. 210, Cambridge University Press, Cambridge, 2024. MR 4655919
- [AH59] M. F. Atiyah and F. Hirzebruch, *Riemann-Roch theorems for differentiable manifolds*, Bull. Amer. Math. Soc. **65** (1959), 276–281. MR 110106
- [BK72] Edward A. Bender and Donald E. Knuth, *Enumeration of plane partitions*, J. Combinatorial Theory Ser. A **13** (1972), 40–54. MR 299574
- [BR12] Anders Skovsted Buch and Vijay Ravikumar, *Pieri rules for the K -theory of cominuscule Grassmannians*, J. Reine Angew. Math. **668** (2012), 109–132. MR 2948873
- [BS87] David Bayer and Michael Stillman, *A criterion for detecting m -regularity*, Invent. Math. **87** (1987), no. 1, 1–11. MR 862710
- [BS17] Daniel Bump and Anne Schilling, *Crystal bases*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2017, Representations and combinatorics. MR 3642318
- [Buc02] Anders Skovsted Buch, *A Littlewood-Richardson rule for the K -theory of Grassmannians*, Acta Math. **189** (2002), no. 1, 37–78. MR 1946917
- [BV15] Bruno Benedetti and Matteo Varbaro, *On the dual graphs of Cohen-Macaulay algebras*, Int. Math. Res. Not. IMRN (2015), no. 17, 8085–8115. MR 3404010
- [CDG20] A. Conca, E. De Negri, and E. Gorla, *Universal Gröbner bases and Cartwright-Sturmfels ideals*, Int. Math. Res. Not. IMRN (2020), no. 7, 1979–1991. MR 4089441
- [CDG22] Aldo Conca, Emanuela De Negri, and Elisa Gorla, *Radical generic initial ideals*, Vietnam J. Math. **50** (2022), no. 3, 807–827. MR 4447407
- [CH97] Aldo Conca and Jürgen Herzog, *Ladder determinantal rings have rational singularities*, Adv. Math. **132** (1997), no. 1, 120–147. MR 1488240

- [Cha07] Marc Chardin, *Some results and questions on Castelnuovo-Mumford regularity*, Syzygies and Hilbert functions, Lect. Notes Pure Appl. Math., vol. 254, Chapman & Hall/CRC, Boca Raton, FL, 2007, pp. 1–40.
- [Cho13] Soojin Cho, *A new Littlewood-Richardson rule for Schur P -functions*, Trans. Amer. Math. Soc. **365** (2013), no. 2, 939–972. MR 2995379
- [CM23] Yu-Cheng Chiu and Eric Marberg, *Expanding K -theoretic Schur Q -functions*, Algebr. Comb. **6** (2023), no. 6, 1419–1445. MR 4686121
- [DG09] E. De Negri and E. Gorla, *G -biliaison of ladder Pfaffian varieties*, J. Algebra **321** (2009), no. 9, 2637–2649. MR 2504494
- [DG11] Emanuela De Negri and Elisa Gorla, *Invariants of ideals generated by Pfaffians*, Commutative algebra and its connections to geometry, Contemp. Math., vol. 555, Amer. Math. Soc., Providence, RI, 2011, pp. 47–62. MR 2882673
- [DMS22] Matt Dreyer, Karola Mészáros, and Avery St. Dizier, *On the degree of Grothendieck polynomials*, preprint (2022), 29 pages, [arXiv:2209.00687](https://arxiv.org/abs/2209.00687).
- [DS11] Emanuela De Negri and Enrico Sbarra, *Gröbner bases of ideals cogenerated by Pfaffians*, J. Pure Appl. Algebra **215** (2011), no. 5, 812–821. MR 2747221
- [EFRW24] Laura Escobar, Alex Fink, Jenna Rajchgot, and Alexander Woo, *Gröbner bases, symmetric matrices, and type C Kazhdan-Lusztig varieties*, J. Lond. Math. Soc. (2) **109** (2024), no. 2, Paper No. e12856, 49. MR 4704154
- [EH12] Viviana Ene and Jürgen Herzog, *Gröbner bases in commutative algebra*, Graduate Studies in Mathematics, vol. 130, American Mathematical Society, Providence, RI, 2012.
- [EH16] David Eisenbud and Joe Harris, *3264 and all that—a second course in algebraic geometry*, Cambridge University Press, Cambridge, 2016. MR 3617981
- [Eis95] David Eisenbud, *Commutative algebra. With a view toward algebraic geometry*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995. MR 1322960
- [EM16] Laura Escobar and Karola Mészáros, *Toric matrix Schubert varieties and their polytopes*, Proc. Amer. Math. Soc. **144** (2016), no. 12, 5081–5096. MR 3556254
- [FK96] Sergey Fomin and Anatol N. Kirillov, *The yang-baxter equation, symmetric functions, and schubert polynomials*, Discrete Mathematics **153** (1996), no. 1, 123–143, Proceedings of the 5th Conference on Formal Power Series and Algebraic Combinatorics.

- [FR03] László M. Fehér and Richárd Rimányi, *Schur and Schubert polynomials as Thom polynomials—cohomology of moduli spaces*, Cent. Eur. J. Math. **1** (2003), no. 4, 418–434. MR 2040647
- [FRS16] Alex Fink, Jenna Rajchgot, and Seth Sullivan, *Matrix Schubert varieties and Gaussian conditional independence models*, J. Algebraic Combin. **44** (2016), no. 4, 1009–1046. MR 3566228
- [Ful92] William Fulton, *Flags, Schubert polynomials, degeneracy loci, and determinantal formulas*, Duke Math. J. **65** (1992), no. 3, 381–420. MR 1154177
- [Ful97] William Fulton, *Young tableaux: With applications to representation theory and geometry*, London Mathematical Society Student Texts, vol. 35, Cambridge University Press, Cambridge, 1997. MR 1464693
- [GH78] Phillip Griffiths and Joseph Harris, *Principles of algebraic geometry*, Pure and Applied Mathematics, Wiley-Interscience [John Wiley & Sons], New York, 1978. MR 507725
- [GK15] Sudhir R. Ghorpade and Christian Krattenthaler, *Computation of the a -invariant of ladder determinantal rings*, J. Algebra Appl. **14** (2015), no. 9, Paper No. 1540014, 24 pages. MR 3368266
- [GM00] Nicolae Gonciulea and Claudia Miller, *Mixed ladder determinantal varieties*, J. Algebra **231** (2000), no. 1, 104–137. MR 1779595
- [God93] C. D. Godsil, *Algebraic combinatorics*, Chapman and Hall Mathematics Series, Chapman & Hall, New York, 1993. MR 1220704
- [Haf22] Elena S. Hafner, *Vexillary Grothendieck polynomials via bumpless pipe dreams*, preprint (2022), 17 pages, [arXiv:2201.12432](https://arxiv.org/abs/2201.12432).
- [HE71] M. Hochster and John A. Eagon, *Cohen-Macaulay rings, invariant theory, and the generic perfection of determinantal loci*, Amer. J. Math. **93** (1971), 1020–1058. MR 302643
- [HMP20] Zachary Hamaker, Eric Marberg, and Brendan Pawlowski, *Fixed-point-free involutions and Schur P -positivity*, J. Comb. **11** (2020), no. 1, 65–110. MR 4015854
- [HMP22] Zachary Hamaker, Eric Marberg, and Brendan Pawlowski, *Involution pipe dreams*, Canad. J. Math. **74** (2022), no. 5, 1310–1346. MR 4504665
- [HPW22] Zachary Hamaker, Oliver Pechenik, and Anna Weigandt, *Gröbner geometry of Schubert polynomials through ice*, Adv. Math. **398** (2022), Paper No. 108228, 29 pages. MR 4379747

- [Hsi13] Jen-Chieh Hsiao, *On the F -rationality and cohomological properties of matrix Schubert varieties*, Illinois J. Math. **57** (2013), no. 1, 1–15. MR 3224557
- [HT92] Jürgen Herzog and Ngô Việt Trung, *Gröbner bases and multiplicity of determinantal and Pfaffian ideals*, Adv. Math. **96** (1992), no. 1, 1–37. MR 1185786
- [IN13] Takeshi Ikeda and Hiroshi Naruse, *K -theoretic analogues of factorial Schur P - and Q -functions*, Adv. Math. **243** (2013), 22–66. MR 3062739
- [INN11] Takeshi Ikeda, Hiroshi Naruse, and Yasuhide Numata, *Bumping algorithm for set-valued shifted tableaux*, 23rd International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2011), Discrete Math. Theor. Comput. Sci. Proc., AO, Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2011, pp. 527–538. MR 2820738
- [IS14] Takeshi Ikeda and Tatsushi Shimazaki, *A proof of K -theoretic Littlewood-Richardson rules by Bender-Knuth-type involutions*, Math. Res. Lett. **21** (2014), no. 2, 333–339. MR 3247060
- [Kir16] Anatol N. Kirillov, *On some quadratic algebras $I_{\frac{1}{2}}$: combinatorics of Dunkl and Gaudin elements, Schubert, Grothendieck, Fuss-Catalan, universal Tutte and reduced polynomials*, SIGMA Symmetry Integrability Geom. Methods Appl. **12** (2016), Paper No. 002, 172. MR 3439199
- [Kle23] Patricia Klein, *Diagonal degenerations of matrix Schubert varieties*, Algebr. Comb. **6** (2023), no. 4, 1073–1094. MR 4635090
- [KM05] Allen Knutson and Ezra Miller, *Gröbner geometry of Schubert polynomials*, Ann. of Math. (2) **161** (2005), no. 3, 1245–1318. MR 2180402
- [KMY09] Allen Knutson, Ezra Miller, and Alexander Yong, *Gröbner geometry of vertex decompositions and of flagged tableaux*, J. Reine Angew. Math. **630** (2009), 1–31. MR 2526784
- [KW21] Patricia Klein and Anna Weigandt, *Bumpless pipe dreams encode Gröbner geometry of Schubert polynomials*, preprint (2021), 50 pages, [arXiv:2108.08370](https://arxiv.org/abs/2108.08370).
- [LLM⁺14] Thomas Lam, Luc Lapointe, Jennifer Morse, Anne Schilling, Mark Shimozono, and Mike Zabrocki, *k -Schur functions and affine Schubert calculus*, Fields Institute Monographs, vol. 33, Springer, New York; Fields Institute for Research in Mathematical Sciences, Toronto, ON, 2014. MR 3379711
- [LRWW17] András C. Lőrincz, Claudiu Raicu, Uli Walther, and Jerzy Weyman, *Bernstein-Sato polynomials for maximal minors and sub-maximal Pfaffians*, Adv. Math. **307** (2017), 224–252. MR 3590517

- [LS82] Alain Lascoux and Marcel-Paul Schützenberger, *Structure de Hopf de l’anneau de cohomologie et de l’anneau de Grothendieck d’une variété de drapeaux*, C. R. Acad. Sci. Paris Sér. I Math. **295** (1982), no. 11, 629–633. MR 686357
- [Mac15] I. G. Macdonald, *Symmetric functions and Hall polynomials*, second ed., Oxford Classic Texts in the Physical Sciences, The Clarendon Press, Oxford University Press, New York, 2015, With contribution by A. V. Zelevinsky and a foreword by Richard Stanley. MR 3443860
- [Man01] Laurent Manivel, *Symmetric functions, Schubert polynomials and degeneracy loci*, SMF/AMS Texts and Monographs, vol. 6, American Mathematical Society, Providence, RI; Société Mathématique de France, Paris, 2001, Translated from the 1998 French original by John R. Swallow, Cours Spécialisés, 3. [Specialized Courses]. MR 1852463
- [Mar20] Eric Marberg, *A symplectic refinement of shifted Hecke insertion*, J. Combin. Theory Ser. A **173** (2020), 105216, 50. MR 4067984
- [MP20] Eric Marberg and Brendan Pawlowski, *K-theory formulas for orthogonal and symplectic orbit closures*, Adv. Math. **372** (2020), 107299, 43 pages. MR 4126717
- [MP21] Eric Marberg and Brendan Pawlowski, *On some properties of symplectic Grothendieck polynomials*, J. Pure Appl. Algebra **225** (2021), no. 1, Paper No. 106463, 22 pages. MR 4114972
- [MP22] Eric Marberg and Brendan Pawlowski, *Gröbner geometry for skew-symmetric matrix Schubert varieties*, Adv. Math. **405** (2022), Paper No. 108488, 56 pages. MR 4437606
- [MPS21] Cara Monical, Oliver Pechenik, and Travis Scrimshaw, *Crystal structures for symmetric Grothendieck polynomials*, Transform. Groups **26** (2021), no. 3, 1025–1075. MR 4309558
- [MR15] Anthony Mendes and Jeffrey Remmel, *Counting with symmetric functions*, Developments in Mathematics, vol. 43, Springer, Cham, 2015. MR 3410908
- [MS05] Ezra Miller and Bernd Sturmfels, *Combinatorial commutative algebra*, Graduate Texts in Mathematics, vol. 227, Springer-Verlag, New York, 2005. MR 2110098
- [MS23] Eric Marberg and Travis Scrimshaw, *Key and Lascoux polynomials for symmetric orbit closures*, preprint (2023), 44 pages, [arXiv:2302.04226](https://arxiv.org/abs/2302.04226).

- [Pra91] Piotr Pragacz, *Algebro-geometric applications of Schur S - and Q -polynomials*, Topics in invariant theory (Paris, 1989/1990), Lecture Notes in Math., vol. 1478, Springer, Berlin, 1991, pp. 130–191.
- [PS24] Oliver Pechenik and Matthew St.Denis, *Degrees of P -grothendieck polynomials and regularity of Pfaffian varieties*, preprint (2024), 25 pages.
- [PSW24] Oliver Pechenik, David E Speyer, and Anna Weigandt, *Castelnuovo–Mumford regularity of matrix Schubert varieties*, Selecta Math. **30** (2024), no. 4, Paper No. 66, 44 pages.
- [PY17] Oliver Pechenik and Alexander Yong, *Genomic tableaux*, J. Algebraic Combin. **45** (2017), no. 3, 649–685. MR 3627499
- [PY24] Jianping Pan and Tianyi Yu, *Top-degree components of Grothendieck and Lascoux polynomials*, Algebr. Comb. **7** (2024), no. 1, 109–135. MR 4715534
- [Rob23] Colleen Robichaux, *Castelnuovo–Mumford regularity for 321-avoiding Kazhdan–Lusztig varieties*, preprint (2023), 23 pages, [arXiv:2308.14208](https://arxiv.org/abs/2308.14208).
- [RRRSW21] Jenna Rajchgot, Yi Ren, Colleen Robichaux, Avery St. Dizier, and Anna Weigandt, *Degrees of symmetric Grothendieck polynomials and Castelnuovo–Mumford regularity*, Proc. Amer. Math. Soc. **149** (2021), no. 4, 1405–1416. MR 4242300
- [RRW23] Jenna Rajchgot, Colleen Robichaux, and Anna Weigandt, *Castelnuovo–Mumford regularity of ladder determinantal varieties and patches of Grassmannian Schubert varieties*, J. Algebra **617** (2023), 160–191. MR 4513784
- [RV13] Eric M. Rains and Monica J. Vazirani, *Deformations of permutation representations of Coxeter groups*, J. Algebraic Combin. **37** (2013), no. 3, 455–502. MR 3035513
- [Sag87] Bruce E. Sagan, *Shifted tableaux, Schur Q -functions, and a conjecture of R. Stanley*, J. Combin. Theory Ser. A **45** (1987), no. 1, 62–103. MR 883894
- [Sch11] J. Schur, *Über die Darstellung der symmetrischen und der alternierenden Gruppe durch gebrochene lineare Substitutionen*, J. Reine Angew. Math. **139** (1911), 155–250. MR 1580818
- [Sta24] Richard P. Stanley, *Enumerative combinatorics. Vol. 2*, second ed., Cambridge Studies in Advanced Mathematics, vol. 208, Cambridge University Press, Cambridge, [2024] ©2024, With an appendix by Sergey Fomin. MR 4621625
- [Ste89] John R. Stembridge, *Shifted tableaux and the projective representations of symmetric groups*, Adv. Math. **74** (1989), no. 1, 87–134. MR 991411

- [Ste02] John R. Stembridge, *A concise proof of the Littlewood-Richardson rule*, Electron. J. Combin. **9** (2002), no. 1, Note 5, 4. MR 1912814
- [Tho78] Glânffrwd P. Thomas, *On Schensted's construction and the multiplication of Schur functions*, Adv. in Math. **30** (1978), no. 1, 8–32. MR 511739
- [TY09] Hugh Thomas and Alexander Yong, *A jeu de taquin theory for increasing tableaux, with applications to K -theoretic Schubert calculus*, Algebra Number Theory **3** (2009), no. 2, 121–148. MR 2491941
- [Vie07] Xavier Viennot, *Énumérons! de la combinatoire énumérative classique aux nouvelles combinatoires: bijective, algébrique, expérimentale, quantique et...magique!*, Leçons de mathématiques d'aujourd'hui. Vol. 3, Le Sel et le Fer, vol. 17, Cassini, Paris, 2007, With a preface by Éric Charpentier and Nikolai Nikolski, pp. 165–238. MR 2572391
- [Wor84] Dale Raymond Worley, *A theory of shifted Young tableaux*, ProQuest LLC, Ann Arbor, MI, 1984, Thesis (Ph.D.)—Massachusetts Institute of Technology. MR 2941073
- [WY08] Alexander Woo and Alexander Yong, *Governing singularities of Schubert varieties*, J. Algebra **320** (2008), no. 2, 495–520. MR 2422304
- [WY17] B. Wyser and A. Yong, *Polynomials for symmetric orbit closures in the flag variety*, Transform. Groups **22** (2017), no. 1, 267–290. MR 3620774