Synchronous and quantum games: Graphical and algebraic methods

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

This is a mathematics thesis that contributes to an understanding of nonlocal games as formal objects. With that said, it does have connections to quantum physics and information theory.

Nonlocal games are interactive protocols modelling two players attempting to win a game, by answering a pair of questions posed by the referee, who then checks whether their answers are correct. The players may have access to a shared quantum resource state and may use a pre-arranged strategy. Upon receiving their questions, they can measure this state, subject to some separation constraints, in order to select their answers. A famous example is the CHSH game of [Cla+69], where making use of shared quantum entanglement gives the players an advantage over using classical strategies.

This thesis contributes to two separate questions arising in the study of synchronous nonlocal games: their algebraic properties, and their generalization to the quantum question-and-answer setting. Synchronous games are those in which players must respond with the same answer, given the same question.

First, we study a synchronous version of the linear constraint game, where the players must attempt to convince the referee that they share a solution to a system of linear equations over a finite field. We give a correspondence between two different algebraic objects modelling perfect strategies for this game, showing one is isomorphic to a quotient of the other. These objects are the game algebra of [OP16] and the solution group algebra of [CLS17]. We also demonstrate an equivalence of these linear system games to graph isomorphism games on graphs parameterized by the linear system.

Second, we extend nonlocal games to quantum games, in the sense that we allow the questions and answers to be quantum states of a bipartite system. We do this by quantizing the rule function, games, strategies, and correlations using a graphical calculus for symmetric monoidal categories applied to the category of finite dimensional Hilbert spaces. This approach follows the overall program of categorical quantum mechanics. To this generalized setting of quantum games, we extend definitions and results around synchronicity. We also introduce quantum versions of the classical graph homomorphism [MR16] and isomorphism [Ats+16] games, where the question and answer spaces are the algebras representing the "vertices" of quantum graphs, and we show that quantum tensor strategies realizing perfect correlations for these games correspond to morphisms between the underlying quantum graphs.

Acknowledgements

Once I thought completing a dissertation akin to reaching the end of an arduous trek, or perhaps placing the final tile of a meticulously crafted structure. I found the fact of finishing to be much more mundane – a timer going off somewhere, pulling me up, up, back to the surface from the depths. Having been interrupted on a fascinating journey, I write, at this somewhat arbitrary recess, to say thank you for helping me along thus far.

In particular, I am grateful to: Vern, my advisor at the beginning – for igniting my interest in nonlocal games, and teaching me methodically and carefully; Mike, my advisor at the end – for adopting me as a student, for your interest in this work, and for asking the right questions; Jennifer, Larissa, and Padraig – for learning with me; Hayley and Wilson – for beginning this journey together; John and Paul – for camaraderie; my friends in KW – for making it home; Meg, Allison, Leah and Maribel – for welcoming me to a new home; Alice, Melissa, Monique, and Yael – for knowing I could; Mrs. Gugoiu – for being a physics role model; Serdar and Stark – for showing me how to be a researcher; Anna, Victoria, and the folks at Owl's Nest – for giving me the space to think; Nancy, Lis, and Pavlina – for your kindness; Mom and Dad – for giving me a home away from home; Tamar and Leora – for listening; Bahir – for joining me along the way; and Max – for your steadfastness, your curiosity, and for remembering that we need to eat.

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Dedication

If you have been visited by a pandemic, a brain injury, a newborn baby, or any other complex, life-recalibrating arrival, remember that your world has suddenly changed, and you have persisted nonetheless.

If you have supported somebody to persist through such immense changes, this work is dedicated to you.

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List of Acronyms

Notation	Description
BCS	binary constraint system
CP	completely positive
CQM	categorical quantum mechanics
†-CFA	commutative dagger Frobenius algebra
†-SCFA	special commutative dagger Frobenius algebra
†-SFA	symmetric dagger Frobenius algebra
†-SSFA	special symmetric dagger Frobenius algebra
det	deterministic
LCS	linear constraint system
loc	local
NL	nonlocal
ns	non-signalling
POVM	positive operator-valued measure
q	quantum (tensor)
qa	quantum approximate
qc	quantum commuting
qns	quantum non-signalling
qs	quantum spatial
syncBCS	synchronous binary constraint system
syncLCS	synchronous linear constraint system

List of Symbols

Notation	Description
A, B, X, Y	Question and answer (quantum) sets.
$\mathcal{A}(\mathcal{G})$	The game algebra of a synchronous nonlocal game $\mathcal{G}$.
$\square \to_{\mathcal{H}} \square$	A quantum function, with resource wire $\mathcal{H}$.
C_t	The class of t-correlations for $t \in \{\mathrm{loc}, \mathrm{q}, \mathrm{qs}, \mathrm{qa}, \mathrm{qc}, \mathrm{ns}\}$.
$\dagger$	The dagger in a monoidal category (here almost always the Hilbert space adjoint).
Δ	The comultiplication of a quantum set.
ε	The counit of a quantum set.
FHilb	The category of finite dimensional Hilbert spaces.
$\mathcal{G}$	A nonlocal game.
$G_{A,b}$	The graph associated to the LCS $Ax = b$.
G, H	(Quantum) graphs.
$\mathcal{H}, \mathcal{K}$	Hilbert spaces.
$\mathcal{I}$	The set of questions for a synchronous NL game.
λ	In Chapters 2-3, the rule function for an NL game, and in Chapters 4-6, a quantum game.
$\lambda_{G \simeq H}$	The quantum graph isomorphism game.
$\lambda_{G \to H}$	The quantum graph homomorphism game.
m	The multiplication of a quantum set.
$\mathcal{O}$	The set of answers for a synchronous NL game.
p	A correlation.
P	A (quantum set) correlation.

Notation	Description
φ	A quantum function.
Q_t	The class of (quantum set) t-correlations for $t \in \{\mathrm{loc}, \mathrm{q}, \mathrm{qc}, \mathrm{ns}\}$.
σ	The swap map in **FHilb**.
$*$	The involution in a C*-algebra.
$\star$	The commuting product of games.
$\Box_*$	The categorical conjugate.
$\Box^*$	In Chapters 2-3, the involution in a C*-algebra, and in Chapters 4-6, the categorical dual/transpose.
u	The unit of a quantum set.
ω	The value of a strategy.
Ω	The value of a strategy.
Ω_t	The t-value of a game.
ω_t	The t-value of a game.
$\mathbb{Z}_p$	The integers modulo a prime p.

1 Introduction

> We should not be lulled into accepting the primacy of the category of sets and functions just because of its flexibility. The mere fact that we can use set theory as a framework for studying quantum phenomena does not imply that this is the most enlightening approach. Indeed, the famously counter-intuitive behavior of the microworld suggests that not only set theory but even classical logic is not optimized for understanding quantum systems. While there are no real paradoxes, and one can compute everything to one's heart's content, one often feels that one is grasping these systems 'indirectly', like a nuclear power plant operator handling radioactive material behind a plate glass window with robot arms. - John Baez, [Bae06]

1.1 Operational approach to nonlocal games

We begin by introducing games operationally. A game can be understood informally as players being asked questions and responding with answers, along with a rule for which answers are jointly correct. Traditionally there are two players named Alice and Bob, and there is a third character – the referee – whom we think of as posing the questions and verifying the answers. We generally require that Alice and Bob cannot communicate once they have received their questions. This can be done by physically separating the players.

It will be helpful to keep in mind the following story of a game, to accompany the mathematics to come.

1. *Questions:* The referee privately poses a separate question to each of Alice and Bob.
2. *Answers:* Each player, by some predetermined process, chooses an answer to their question, then sends their answer back to the referee.

3. Evaluation: The referee receives the two answers and checks the rules to determine whether the players have jointly won or lost the game.

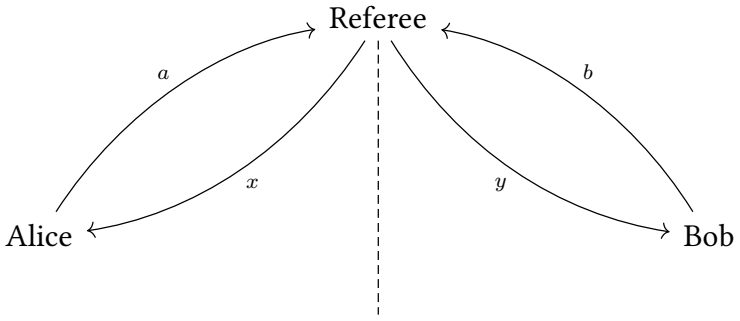


Figure 1.1: The setup for a typical nonlocal game.

The title of this thesis refers to synchronous games. When we say a game is synchronous, we describe some level of symmetry between Alice and Bob. The title of this thesis also refers to quantum games, which are a generalization of nonlocal games, where the questions and answers are now also quantum states. When we say a game is nonlocal, we imply that the players may have access to non-classical physical shared resources, like quantum entanglement.

1.2 Context for the generalization of nonlocal games to quantum games

Nonlocal games are recent objects of study in quantum information theory and complexity theory. They have been used to distinguish between various models of quantum mechanics [Cle+04; Slo19]. They have also played a role in proving that the Connes embedding problem from operator algebras has a negative answer [Ji+21]. These results and others demonstrate deep connections between quantum physical models, complexity classes, and operator theoretic constructions.

Synchronicity for nonlocal games, introduced in [Pau+16], is a well studied property in the typical setting, where questions and answers are classical. Winning a synchronous game requires players to give the same answer upon receiving the same question. In that setting,

synchronicity forces the players to have matching strategies. For those familiar with nonlocal games, we elaborate on what we mean by matching strategies: Strategies corresponding to synchronous correlations are pairs of functions or operators (depending on the strategy class) that are forced to agree on a relevant subset of their domain. Both players apply their own map to their received question to produce an answer. Furthermore, if a game is synchronous, the associated winning correlations (statistical behaviour of the players) are forced to be synchronous themselves, in that the probability of the players responding with different answers to the same question is zero.

The program of categorical quantum mechanics (CQM), initiated in [AC04], approaches quantum theory as a process theory, modelling physical systems and transformations as the objects and morphisms in a category equipped with some additional structure. Compositionality is the key feature of this approach. CQM lends itself well to the quantisation of classical protocols – once they are carefully modelled as processes in an appropriate category, one need only extend the underlying category to automatically model a quantum version of the protocol. One compelling example of this is that quantum teleportation and one-time pad encryption can be seen as quantum and classical models of the same compositional construction [HV19, Section 3.2].

We use CQM ideas to extend the theory of nonlocal games, including features of synchronicity, to *quantum games* – nonlocal games with quantum questions and answers – using string diagrams interpreted in the category of finite dimensional Hilbert spaces. We also upgrade other notions from nonlocal games to the quantum game setting. We make some initial connections of quantum games to another quantized object: quantum graphs. We hope that this notion of quantum games will be found to have deep connections to other quantized objects, such as quantum groups.

1.3 Chapter layout and summary of results

We use Chapter 2 to provide background on the typical nonlocal games setting, where the questions and answers are classical. We review correlation and strategy classes, the definition and some examples of synchronous games, and the definition of the $*$-algebra of a synchronous game.

In Chapter 3, we discuss a synchronous version of the linear constraint system game, and prove two of our major results. The first of the two results (Theorem 3.4) is the equivalence of two algebraic objects describing the game: its solution group, and its $*$-algebra. The second result (Theorem 3.6) is the equivalence of a linear system game to the graph isomorphism game for a pair of graphs based on the linear system.

Then, in Chapter 4, we use a graphical language to define quantum games: a generalisation of nonlocal games where the questions and answers can be quantum states. The key concepts we generalise are *games*, *correlations*, and *strategies*. We extend certain correlation classes to the quantum game setting by defining *non-signalling* correlations, and *quantum commuting*, *quantum*, and *deterministic* strategies, generalizing these notions as defined for typical nonlocal games. We also define some *products* of quantum games, and the *value* of games and correlations. Finally, we interpret how a quantum game may be implemented operationally, and we use this interpretation to connect quantum games to related notions in the literature. Chapter 4 is definition-heavy, and can mostly be thought of as preparation for the next two chapters.

We are then equipped to extend *synchronicity* to quantum games in Chapter 5. This definition builds off something that we call *sharing*, which we also use to graphically represent the *classical dimension* of a quantum set.

In Section 5.2, we extend properties of typical synchronicity. One of our main results is that perfect correlations/strategies for synchronous quantum games are forced to be synchronous (Theorem 5.9). We examine the structure of synchronous strategies and correlations. We show that Alice's operators determine Bob's operators when implementing a synchronous quantum-commuting correlation (Theorem 5.22) or strategy (Theorem 5.25). As a key tool, we present a diagrammatic version of the Cauchy-Schwarz inequality for quantum functions (Theorem 5.24). In Section 5.3, we discuss *bistrategies*, *bicorrelations*, and *bisynchronicity*. We show (Theorem 5.36) that bisynchronicity of increasingly structured maps preserves a growing list of quantum set properties: dimension, classical dimension, quantum isomorphism class, isomorphism class. We also describe the structure of bistrategies in different strategy classes. Then, in Section 5.4, we compare our definitions to two other notions of synchronicity in the literature [Bra+23; Bra+24; BKS23] that also aim to generalize typical synchronicity.

Finally, in Chapter 6, we come to two examples of quantum games played on quantum graphs. First we study a truly quantum, synchronous game (the quantum graph homomorphism game), generalizing the well-known graph homomorphism game of [MR16]. We connect its perfect quantum correlations to quantum homomorphisms of quantum graphs (Theorem 6.7). Second, we discuss the bisynchronous quantum graph isomorphism game, and connect its perfect bistrategies and bicorrelations to quantum graph isomorphisms (Theorem 6.16).

This thesis is an (edited) compilation of three sources, with the addition of several new sections. Chapter 2 comes mostly from the author's unpublished master's project, [Gol19]. Chapter 3 comes from the author's published article, [Gol21]. Chapters 4, 5, and 6 are from the author's preprint [Gol24], with the exception of sections 4.5, 4.6 and 4.7, which are published for the first time in this thesis. Chapter 6 has been heavily updated from the preprint version.

2 Nonlocal Games: Background

> Quantum mechanical objects can exhibit correlations with one another that are fundamentally at odds with the paradigm of classical physics; one says that the objects are 'entangled'... [E]ntanglement has come to be studied not only as a marvel of nature but as a potential resource, particularly as a resource for certain unusual kinds of communication. - [Eke+98]

This chapter is not strictly a necessary prerequisite to quantum games – after all, typical nonlocal games can be thought of as quantum games played on classical question and answer sets, so starting there is reasonable. However, most of our work is motivated by analogy to typical nonlocal games. For the intuitively curious or operationally motivated reader, this chapter is worthwhile background. The reader familiar with nonlocal games may view this chapter as a resource to access as needed, and may wish to skip ahead.

This chapter uses material from the author's unpublished Master's project, [Gol19].

In this chapter we define *nonlocal games* and *strategies* (Section 2.1) from the basics, and we then compare multiple physical models of reality under which Alice and Bob may be assumed to operate (Section 2.2). These physical models, called correlation classes, dictate what strategies are available to the players of the game, and accordingly impact the players' performance. We then describe (Section 2.3) how a synchronous game, which is a game with some symmetry between the two players, can be assigned a $*$-algebra called the *game algebra*. It encodes information about the existence of strategies that are guaranteed to win, known as *perfect* strategies. Finally (Section 2.3.4) we present some notable synchronous games.

2.1 The language of nonlocal games

In this section, we define cooperative two-player nonlocal games, and strategies for playing these games.

2.1.1 What is a nonlocal game?

Formally, we define a nonlocal game[1] as follows.

Definition 2.1 (Nonlocal game)**.** A *nonlocal game* is a tuple $\mathcal{G} = (X, Y, A, B, \lambda)$, where X, Y, A, B are finite (classical) sets and $\lambda : A \times B \times X \times Y \to \{0, 1\}$ is a function.

We call X, Y the *input* or *question* sets, and A, B the *output* or *answer* sets belonging respectively to Alice and Bob. See Figure 1.1 to understand why. We call λ the *rule function*.

Note that when Alice and Bob receive a pair of questions $(x, y) \in X \times Y$ and return a pair of answers $(a, b) \in A \times B$, then $\lambda(a, b|x, y) = 0$ means the players have lost the game, and $\lambda(a, b|x, y) = 1$ means they have won it. We will use the letters x, y to denote Alice and Bob's respective questions and a, b to denote their respective answers. The vertical bar in λ should evoke the conditional nature of the rule function.

Example 2.2. The CHSH (Clauser, Horn, Shimony, and Holt) game originating in [Cla+69] has $X = Y = A = B = \mathbb{Z}_2$, and

$$\lambda(a, b|x, y) = \begin{cases} 1, & a + b = x \cdot y, \\ 0, & \text{otherwise.} \end{cases}$$

where the operations used are addition and multiplication mod 2.

Notice that in the definition of a game, there is no information on how the referee selects which questions to ask (*question distribution*), and by what methods Alice and Bob may choose their answers (*strategies*). We think of the question distribution and the strategies as external to the game. Whenever we study a specific game, it will be done with respect to some distribution of questions and some strategy or class of strategies. We introduce these concepts in the next two sections.

[1] There are many ways to define a game. Here we are defining a specific class of games: co-operative two-player finite input/output games with classical questions and answers. In this chapter, we simply refer to them as games, but the reader should note that they are a small subclass of the quantum question-and-answer games defined in Chapter 4.

2.1.2 Strategies and correlations

Based on the kinds of behaviour that should be possible, we introduce classes of strategies or classes of correlations. Let us explain each term, and the connection between these two different terms: 'strategy' and 'correlation'. In brief, the difference is in point of view – whether one is trapped inside the machine, being a player of the game, or looks on from the outside, as an observer or referee might.

Strategies

A *class of strategies* can be thought of as an algorithm that Alice and Bob can use to decide their answers given their questions. The algorithm has modifiable parameters, that Alice and Bob can choose in order to maximize their chance of winning the game. Additionally, this algorithm does not need to be deterministic, in that it can involve evaluating random variables or doing quantum measurements. In Section 2.2, in order to describe a class of strategies, we list some mathematical objects (determined by some parameters) that the players can use, along with a way the players can extract their answers from these objects.

Given a class of strategies, in order to fix a single *strategy* within a class of strategies, the players agree in advance on a choice of the parameters of the given mathematical objects. We often identify a strategy with the choice of objects themselves, implicitly remembering the method by which Alice and Bob use the objects to produce answers. This is similar to the way a group is often referred to by its underlying set, with implicit knowledge of the associated operation.

To demonstrate the concept of a strategy, we introduce a simple class of strategies, known as the deterministic strategies.

Definition 2.3 (Deterministic strategy)**.** A *deterministic strategy* for a game is a pair of functions $f_A : X \to A$ and $f_B : Y \to B$.

These functions can be chosen together by Alice and Bob before playing, with knowledge of the rules of the game (i.e. knowledge of the function λ). When playing the game, Alice receives question x and returns answer $a = f_A(x)$. Similarly, Bob receives question y and returns answer $b = f_B(y)$. Alice and Bob do not communicate during this process.

Example 2.4. One very simple deterministic strategy for CHSH (see Example 2.2) is given by $f_A(x) = 0, f_B(y) = 0$ for $x, y \in \mathbb{Z}_2$. Note that under this strategy, when $(x, y) = (1, 1)$, the result is a loss, as

$$f_A(1) + f_B(1) = 0 + 0 = 0 \neq 1 = 1 \cdot 1 = x \cdot y.$$

However, when any other pair of questions is posed, then $x \cdot y = 0$, so the strategy results in a win.

Correlations

Instead of thinking about a game algorithmically, or from the players' point of view, in the sense of the players applying a strategy to turn questions into answers, we can think of a game probabilistically, or from an observer's point of view, in the sense of the players behaving according to some discrete conditional probability density function, referred to here as a *correlation*. A correlation models the likelihood of a certain pair of answers given a specific pair of questions. These correlations are also referred to in other work as *densities* or *behaviours*.

Definition 2.5 (Correlation)**.** A *correlation* $p(a, b|x, y)$ is a function $p : A \times B \times X \times Y \to [0, 1]$, satisfying

$$\sum_{(a,b)\in A\times B} p(a, b|x, y) = 1$$

for any $(x, y) \in X \times Y$.

If we think of Alice and Bob as random players, $p(a, b|x, y)$ represents the conditional probability of Alice giving answer $a \in A$ and Bob giving answer $b \in B$ given that Alice has received question $x \in X$ and Bob has received $y \in Y$.

To illustrate the correspondence between strategies and correlations, let us describe the class of *deterministic correlations* corresponding to the deterministic strategies. We denote this class C_{det}. For a given deterministic strategy f_A, f_B, let

$$S_A := \{(f_A(x), x)|x \in X\} \subseteq A \times X$$
$$S_B := \{(f_B(y), y)|y \in Y\} \subseteq B \times Y$$

These are the sets of answer/question pairs Alice and Bob plan to use for their chosen strategy. Let $\chi_S(\cdot)$ denote the indicator function taking value 1 on the set S and 0 otherwise. That is,

$$\chi_S(s) = \begin{cases} 1 & s \in S \\ 0 & \text{otherwise} \end{cases}$$

The correlation corresponding to the strategy where Alice uses f_A and Bob uses f_B, in the sense of the statistical pattern observed when the players follow this strategy, is

$$p(a, b|x, y) = \chi_{S_A}(a, x)\chi_{S_B}(b, y)$$

which is 1 exactly when both Alice and Bob apply their strategy functions, and 0 otherwise. The class of deterministic correlations, C_{det}, includes only correlations that can be written in this form.

Example 2.6. The correlation corresponding to the strategy of Example 2.4 is given by $p(0,0|x,y) = 1$, and for $(a,b) \neq (0,0)$, $p(a,b|x,y) = 0$.

Notice that in the case of any deterministic correlation, the only possible values for $p(a,b|x,y)$ are 0 and 1, and for each pair (x,y), there is exactly one pair (a,b) that maps to the value 1.

2.2 Classes of strategies

We will now introduce several models of physical reality in which Alice and Bob may operate. More precisely, we introduce different classes of strategies corresponding to different physical models. Different kinds of answers may be possible in different physical realities. For example, the reader hopefully agrees that in any reasonable model, given our non-communication clause, Bob's answer should not directly use knowledge of Alice's question (this property is called non-signalling and is defined in Section 2.2.3).

We first introduce strategies that reflect a classical model of reality, with no quantum entanglement. These strategies are referred to as *local strategies*, and include the deterministic strategies as a subclass. After giving a picture of the classical model, we go on to describe some classes of *nonlocal strategies*. These strategies have the power to exploit entanglement.

2.2.1 Local strategies

Local strategies are a probabilistic generalization of deterministic strategies. We introduce them for completeness, but we will eventually explain why probabilistic strategies need not be considered, as they provide no advantage in our setting over deterministic strategies. However, there are other settings in which they could be advantageous. For example, they may be useful in a multi-round setting, if the referee is an adversary attempting to spy on Alice and Bob and learn their strategy.

Definition 2.7 (Local strategy)**.** A (probabilistic or) *local strategy* for a game is given by a probability space $(\Omega, \mathcal{F}, P)$, along with two sets of random variables $\{g_A^x : \Omega \to A \mid x \in X\}$ and $\{g_B^y : \Omega \to B \mid y \in Y\}$.

As in the deterministic case, these random variables are chosen before playing, with knowledge of the rules of the game. When playing the game, Alice receives question $x \in X$ and samples the appropriate random variable g_A^x to return answer $a \in A$ with probability $P(g_A^x(\omega) = a)$, and similarly for Bob. We can imagine Alice and Bob using the price of some mutually-agreed upon stock at a given time of day to decide how to answer. The idea is that both Alice and Bob have shared access to a source of classical randomness.

Notice that a deterministic strategy f_A, f_B can be framed as a local strategy by setting $g_A^x(\omega) := f_A(x)$ for all $x \in X, \omega \in \Omega$ and $g_B^y(\omega) := f_B(y)$ for all $y \in Y, \omega \in \Omega$. So we can think of the deterministic strategies as a subset of the local strategies.

If we want to think in terms of correlations, the corresponding class of *local correlations*, denoted C_{loc} are correlations of the form

$$p(a, b|x, y) = P(g_A^x(\omega) = a \wedge g_B^y(\omega) = b).$$

The above form can always be rewritten as a convex combination of deterministic correlations. We discuss this in Section 2.2.5.

In Section 2.2.4, we will define the value of a game. It will follow that a deterministic strategy can always replace a probabilistic strategy and perform at least as well. That is to say, Alice and Bob cannot play better by (locally) randomizing their actions. Therefore, it will be enough to think of local strategies as deterministic. Local strategies are mostly introduced here to contrast the ineffectiveness of shared classical randomness with the perhaps surprising effectiveness of shared entanglement.

2.2.2 Some useful language from quantum information

Before we introduce nonlocal strategies, we need to have an idea of what a quantum experiment is. In the worldview of quantum information, we consider states (representing the physical state of a particle or a system) and transformations of those states. Pure states are represented by unit vectors in a Hilbert space, and the transformations applied are represented by a special class of operators on that space.

Formally, we will let $\mathcal{H}$ denote a Hilbert space, and $|\phi\rangle \in \mathcal{H}$ denote a *state*[2], which is merely some unit vector in $\mathcal{H}$.

To model measurement of a quantum system, we use something called a measurement system.

[2]For the reader who is not familiar with bra-ket notation: If ϕ behaves as a vector it is written as a *ket* $|\phi\rangle$ and if it behaves as a dual vector or linear functional it is written as a *bra* $\langle\phi|$.

Definition 2.8 (Measurement system)**.** Given Hilbert spaces $\mathcal{H}, \mathcal{K}$, a *measurement system* with n outcomes, denoted $\{M_i\}_{i=1}^n$, is a set of bounded linear operators $M_i \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ satisfying

$$\sum_i M_i^* M_i = I.$$

Measurement systems model physical measurements of quantum systems in that each index i in the above definition refers to a particular *outcome* of the measurement. Given that the system starts in state $|\phi\rangle$, the probability of outcome i is given by $\langle\phi| M_i^* M_i |\phi\rangle$.

One can also use M_i to compute the state after measurement, given that outcome i was observed. However, in this work, we are only interested in probabilities, and not in the state after measurement. This is due to the fact that the games we study here are static, not continuing beyond a single round of questions and answers. Because we are only concerned with probabilities, it is enough to work with the positive bounded linear operators on $\mathcal{H}$ given by $X_i := M_i^* M_i$. There is a formalization of this model of measurements, called POVM[3] measurements, defined as follows.

Definition 2.9 (POVM)**.** A *POVM measurement* with n outcomes, denoted $\{E_i\}_{i=1}^n$, is a set of n positive bounded linear operators on $\mathcal{H}$ summing to identity. That is to say, for each i, $E_i \in \mathcal{B}(\mathcal{H})$ and

$$E_i \geq 0, \qquad \sum_i E_i = I.$$

Because any positive operator E can be decomposed as $M^* M$ in some way, this definition models the same measurements as the previous definition. The main difference here is that POVMs simplify some calculations while obscuring the state after measurement. For a thorough treatment of the subject, see [NC10].

To model *entanglement* between two physical systems represented by Hilbert spaces $\mathcal{H}_A, \mathcal{H}_B$, we consider states in $\mathcal{H} := \mathcal{H}_A \otimes \mathcal{H}_B$ given by the tensor product of the two spaces. Operators on $\mathcal{H}_A$ (or $\mathcal{H}_B$) give rise to operators on all of $\mathcal{H}$ by tensoring on the left with the identity of $\mathcal{H}_B$ (respectively, on the right with the identity of $\mathcal{H}_A$).

2.2.3 Nonlocal strategies

If we update our model of physical reality to reflect the phenomenon of quantum entanglement, we can look at strategies where Alice and Bob can choose their answers after performing measurements on a (possibly entangled) bipartite state. This may make it sound like the first player

[3] POVM stands for positive operator-valued measure

to act on the system can somehow communicate to the second one, but that is not the case. This is because the operations performed by the players will always be required to commute. We will see shortly that strategies built from commuting operations always give rise to correlations with a *non-signalling* property that guarantees no useful communication happens between the players.

In fact, this non-signalling property is common to all the classes of strategies (or correlations) discussed here. The purpose of the non-signalling property is to impose a non-communication requirement on the players while the game is in progress. It means that the probability that Alice gives a specific answer cannot depend on Bob's question, and vice versa.

Definition 2.10 (Non-signalling)**.** A correlation $p(a, b|x, y)$ is *non-signalling* if the following quantities are well-defined:

$$p_A(a|x) := \sum_{b\in B} p(a, b|x, y) \qquad \text{for any fixed } y \in Y$$
$$p_B(b|y) := \sum_{a\in A} p(a, b|x, y) \qquad \text{for any fixed } x \in X$$

In the above definition, 'well-defined' means that the value of $p_A(a|x)$ is independent of the choice of $y \in Y$, and the value of $p_B(b|y)$ is independent of the choice of $x \in X$.

Standard quantum model

The first and most widely studied class of strategies we discuss are the (standard) quantum strategies or q-strategies.

Definition 2.11 (Quantum strategy)**.** A *quantum strategy* (or q-*strategy*) for a game is a state $|\phi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$, where $\mathcal{H}_A, \mathcal{H}_B$ are finite-dimensional Hilbert spaces, along with a POVM measurement on $\mathcal{H}_A$ given by $\{E_{x,a}\}_{a\in A}$ for each $x \in X$ and a POVM measurement on $\mathcal{H}_B$ given by $\{F_{y,b}\}_{b\in B}$ for each $y \in Y$.

Note that $\mathcal{H}_A$ and $\mathcal{H}_B$ represent Alice and Bob's respective physical systems or labs. In order to play the game, upon receiving a question $x \in X$, Alice hunts down the corresponding measurement $\{E_{x,a}\}_{a\in A}$ and measures her part of $|\phi\rangle$, that is, she performs the measurement locally, in her own lab. The outcome of the measurement is some $a \in A$. She gets answer a with probability $\langle\phi| E_{x,a} \otimes I |\phi\rangle$. She sends her answer to the referee. Bob follows the same procedure, receiving question y, and giving answer b with probability $\langle\phi| I \otimes F_{y,b} |\phi\rangle$. However,

since Alice's and Bob's answers are not independently generated (both depend on the state $|\phi\rangle$), we cannot multiply these two probabilities to find $p(a, b|x, y)$ as we could in the local case. Rather, in the case of a quantum correlation,

$$p(a, b|x, y) = \langle\phi| E_{x,a} \otimes F_{y,b} |\phi\rangle .$$

Alice and Bob's joint measurement of the system, given question pair (x, y), is

$$\mathbb{E}_{x,y} := \{E_{x,a} \otimes F_{y,b}\}_{(a,b)\in A\times B},$$

which is built from local operations. The class of *quantum correlations* is denoted C_{q}. Note that in general,

$$\langle\phi| E_{x,a} \otimes F_{y,b} |\phi\rangle \neq \langle\phi| E_{x,a} \otimes I |\phi\rangle \langle\phi| I \otimes F_{y,b} |\phi\rangle ,$$

but when the above holds with equality for all a, b, x, y, the strategy in question is actually a local strategy dressed up as a quantum strategy.

Remark 2.12. Note that if quantum strategies are defined to use PVMs instead of POVMs, i.e., if the operators are required to be projections, the resulting class of correlations is still the same. See [Jun+11; Fri12].

Infinite dimensional models

Definition 2.13 (Quantum spatial strategy)**.** A *quantum spatial strategy* (or qs-strategy) is a generalization of a quantum strategy (see Definition 2.11) in that the Hilbert spaces $\mathcal{H}_A, \mathcal{H}_B$ are allowed to be infinite-dimensional.

The class of associated *quantum spatial correlations* is denoted C_{qs}.

Allowing the Hilbert spaces to be infinite dimensional is only one way of removing the finiteness constraint that is present for quantum strategies. Another way of generalizing q-strategies is to take the closure of C_{q}. The topological notion of closure here is unambiguous, as the correlations live in a finite dimensional space. Specifically, they can be viewed as a subset of $[0, 1]^n \subseteq \mathbb{R}^n$ where $n = |X| \cdot |Y| \cdot |A| \cdot |B|$. This gives us a class of correlations called the *quantum approximate* (qa) correlations, denoted C_{qa}. Because of the nature of this construction, we cannot describe the corresponding strategies directly, but rather must think of them as limits of q-strategies.

The final class of strategies we introduce involves removing the tensor product structure requirement on Alice and Bob's physical systems. Instead, we model their joint physical system as

one large (possibly infinite dimensional) Hilbert space. Alice and Bob still choose their answers via POVM measurements on this space. However, in order to ensure that Alice and Bob cannot communicate during the game, we require that Alice's measurement operators commute with Bob's measurement operators. This works by preventing Alice from being able to perform her measurement and thereby communicate to Bob by modifying the state of the system before he chooses his answer. To see formally why the commutation condition prevents communication, see Section 2.2.5, where we discuss why these strategies are indeed non-signalling. We call these the quantum commuting strategies, and the associated class of correlations is denoted C_{qc}. Formally:

Definition 2.14 (Quantum commuting strategy)**.** A *quantum commuting strategy* (or qc-strategy) consists of a state $|\phi\rangle$ in a Hilbert space $\mathcal{H}$ along with POVM measurements on $\mathcal{H}$ given by $\{E_{x,a}\}_{a\in A}$ for each $x \in X$ and $\{F_{y,b}\}_{b\in B}$ for each $y \in Y$, subject to the requirement that for all x, y, a, b, $E_{x,a}F_{y,b} = F_{y,b}E_{x,a}$.

Note that the correlation corresponding to the above qc-strategy is given by

$$p(a, b|x, y) = \langle\phi| E_{x,a}F_{y,b} |\phi\rangle$$

and the class of *quantum commuting correlations* is denoted C_{qc}.

Similarly to Remark 2.12, if qc-strategies are defined to use PVMs instead of POVMs, the resulting class of correlations is unchanged, as shown in [Jun+11; Fri12].

2.2.4 The many values of a game

Now that we have defined games and the strategy classes, we will define the value of a game corresponding to a given class. This value measures how likely it is that the players can win, given they are using an optimal strategy from that class.

We first define the value of a specific correlation for a game $\mathcal{G}$, and then define the value of a game by taking the supremum over all correlations from that class.

The value of a correlation is contingent not only on $\mathcal{G}$ and p, but depends also on the probability of receiving each pair of questions. Recall that a (usual, not conditional) discrete probability density function on a finite set S is a function $\Pi : S \to [0, 1]$ such that $\sum_{s\in S} \Pi(s) = 1$. Π can also be thought of as a probability measure.

Definition 2.15 (Value of a correlation)**.** Given a game $\mathcal{G}$, a correlation $p(a, b|x, y)$ and a probability density function $\Pi : X \times Y \to [0, 1]$, the *value* of the correlation is

$$\omega(\mathcal{G}, \Pi, p) := \sum_{\substack{x\in X\\ y\in Y}} \Pi(x, y) \sum_{\substack{a\in A\\ b\in B}} p(a, b|x, y)\lambda(a, b|x, y)$$

Class	**t**	**Strategy**	$\mathbf{p(a,b\|x,y)} = \ldots$
Deterministic	det	f_A, f_B	$\chi_{S_A}(a,x)\chi_{S_B}(b,y)$
Local	loc	$(\Omega, \mathcal{F}, P), \{g_A^x\}_{x\in X}, \{g_B^y\}_{y\in Y}$	$P(g_A^x(\omega) = a \wedge g_B^y(\omega) = b)$
Quantum	q	$\|\phi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ finite dimensional shared state, POVM measurements $\{E_{x,a}\}_a, \{F_{y,b}\}_b$ for all x, y	$\langle\phi\| E_{x,a} \otimes F_{y,b} \|\phi\rangle$
Quantum spatial	qs	$\|\phi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ shared state, POVM measurements $\{E_{x,a}\}_a, \{F_{y,b}\}_b$ for all x, y	$\langle\phi\| E_{x,a} \otimes F_{y,b} \|\phi\rangle$
Quantum approximate	qa	-	Closure of C_q
Quantum commuting	qc	$\|\phi\rangle \in \mathcal{H}$ shared state, pairwise commuting POVM measurements $\{E_{x,a}\}_a, \{F_{y,b}\}_b$ for all x, y	$\langle\phi\| E_{x,a}F_{y,b} \|\phi\rangle$
Non-signalling	ns	-	Has non-sig. property

Table 2.1: Classes of strategies/correlations for nonlocal games

For the probabilists out there, note that the above is just the expected value of λ if we recall that the probability of the tuple (x, y, a, b) is given by $\Pi(x, y)p(a, b|x, y)$. For computation purposes, note that if p corresponds to a deterministic strategy given by f_A, f_B, and Π is uniform, then the value of the correlation is given by

$$\omega(\mathcal{G}, \Pi, p) := \frac{1}{|X||Y|} \sum_{(x,y)} \lambda(f_A(x), f_B(y)|x, y). \tag{2.1}$$

Definition 2.16 (t-Value of a game)**.** Given $t \in \{\mathrm{loc}, \mathrm{q}, \mathrm{qs}, \mathrm{qa}, \mathrm{qc}, \mathrm{ns}\}$, a game $\mathcal{G}$ and a probability density function $\Pi : X \times Y \to [0, 1]$, the *$t$-value* of the game is

$$\omega_t(\mathcal{G}, \Pi) := \sup_{p\in C_t} \omega(\mathcal{G}, \Pi, p).$$

If Π is uniform, we may omit any mention of Π and just write $\omega_t(\mathcal{G})$. In fact, if it is clear which game is under consideration, we will simply write ω_t to refer to the t-value of the game.

Example 2.17. If we consider CHSH with a uniform distribution on the questions, and the deterministic strategy of Example 2.4 we get a value of

$$\frac{1}{4}(1+1+1+0) = \frac{3}{4}.$$

This is calculated using (2.1). A short argument, given in [Cle19], demonstrates that in fact this is an optimal deterministic strategy. That is to say, $\omega_{\text{det}}(\text{CHSH}) = \frac{3}{4}$.

We are especially concerned with the question of which games can be won with certainty. In light of that, we define perfect strategies.

Definition 2.18 (Perfect strategy)**.** We say that a correlation p corresponds to a *perfect strategy* for $\mathcal{G}$ if $\lambda(a,b|x,y) = 0$ implies $p(a,b|x,y) = 0$.

We may also just say p is a perfect correlation. If $p \in C_t$ is perfect, we call the corresponding strategy a perfect t-strategy for $\mathcal{G}$. People are often interested in the question: Does $\mathcal{G}$ have a perfect strategy within a given class of strategies? This can correspond to the existence of a certain type of morphism between mathematical objects encoded by the game.

Remark 2.19. If p is a perfect correlation for $\mathcal{G}$ then for all probability density functions Π, $\omega(\mathcal{G}, \Pi, p) = 1$.

Thus, whether or not a game can be won with certainty is independent of the particular values of Π.

2.2.5 Hierarchy of strategies

We saw in Section 2.2.1 that the deterministic strategies are a subclass of local strategies. It is easy to come up with an example of a non-deterministic local strategy, for instance, take two different deterministic strategies and have each player randomly decide to use one or the other when preparing their answer. This is non-deterministic, so we are dealing with a strict subclass. In terms of correlations, we see that $C_{\text{det}} \subsetneq C_{\text{loc}}$.

In this section, we will explain the following chain of containment between the classes of correlations. We sketch some proofs and give examples to demonstrate each strict containment, when these are simple to write down. When the proofs are more involved, we give references.

$$C_{\text{det}} \subsetneq C_{\text{loc}} \subsetneq C_{\text{q}} \subsetneq C_{\text{qs}} \subsetneq C_{\text{qa}} \subsetneq C_{\text{qc}} \subsetneq C_{\text{ns}}$$

Note that when we write each C_t, we mean with respect to fixed cardinalities of X, Y, A, B. Therefore, it makes sense to talk about the chain of strategy classes for any fixed game, but it does not make sense to compare strategies for games with different sized question/answer sets.

All of the above containments are known to be strict, as we will see shortly. We now discuss the containments one by one.

Local strategies are quantum strategies

We will now explain why $C_{\text{loc}} \subseteq C_{\text{q}}$. See [Gol19, Section 4.1] for a detailed proof.

Sketch of proof. One way to see this is that $C_{\text{det}} \subseteq C_{\text{q}}$, that C_{q} is convex, and that C_{loc} is the convex hull of C_{det}.

To see that $C_{\text{det}} \subseteq C_{\text{q}}$, we need to disguise any deterministic strategy as a quantum strategy, resulting in the same correlation. This can be done by picking Hilbert spaces and state on their tensor product, and then letting Alice use the following family of POVMs. For each question x, the associated POVM is the identity on answer $a = f_A(x)$ and zero on all other answers, and similarly for Bob. One can then check that the resulting correlation is unchanged.

Next, C_{q} can be shown to be convex by building convex combinations of resident correlations from direct sums of the POVMs that realize these correlations.

Finally, let's discuss why C_{loc} is the convex hull of C_{det}. Combining deterministic strategies by weighting them according to a given convex combination produces a local strategy. Conversely, given a local strategy, one can partition the associated probability space, Ω, into finitely many sets on which the random variables defining the strategy are constant. One can then define a mixture of deterministic strategies, where the weights come from the probabilities of each set in the partition. This reproduces the original local strategy. □

We now conclude that $C_{\text{loc}} \subseteq C_{\text{q}}$. To see that the containment is strict, it is enough to observe that there are games for which the quantum value is strictly greater than the deterministic value (as deterministic correlations are the extreme points of the local correlations). Example 2.20 demonstrates this.

Example 2.20. The CHSH game introduced in Example 2.2, along with the uniform question distribution, has deterministic value $\omega_{\text{det}} = \frac{3}{4}$, as mentioned in Example 2.17. However, it is known that the quantum value is strictly greater: $\omega_{\text{q}} = \frac{2+\sqrt{2}}{4}$. For an analysis of the CHSH game, see [Cle19].

Relationship between the quantum strategy classes

We will now explore the relationship between all of the quantum strategy classes. It is fairly clear that $C_{\mathrm{q}} \subseteq C_{\mathrm{qs}}$, because the models are identical except that in the quantum spatial model, any Hilbert space $\mathcal{H}$ can be chosen, whereas in the quantum model, $\mathcal{H}$ is required to be a finite dimensional space.

From the definition of the quantum approximate strategies as the closure of C_{q}, it is clear that $C_{\mathrm{q}} \subseteq C_{\mathrm{qa}}$, but then we must ask where do the quantum spatial strategies live? It is shown in [SW08] that $C_{\mathrm{qs}} \subseteq C_{\mathrm{qa}}$, and hence C_{qa} is also the closure of C_{qs}. So we have that

$$C_{\mathrm{q}} \subseteq C_{\mathrm{qs}} \subseteq C_{\mathrm{qa}}.$$

There is an example constructed in [CS18] demonstrating that $C_{\mathrm{q}} \neq C_{\mathrm{qs}}$. It is demonstrated in [Slo19] that C_{q} and C_{qs} are both not closed, so that $C_{\mathrm{qs}} \neq C_{\mathrm{qa}}$. This proof implicitly defines a game that has no perfect quantum spatial strategy but does have a perfect quantum approximate strategy.

To see that $C_{\mathrm{qa}} \subseteq C_{\mathrm{qc}}$, we first claim that $C_{\mathrm{q}} \subseteq C_{\mathrm{qc}}$. This is achieved by disguising quantum strategies as quantum commuting strategies, using the fact that operators of the form $E \otimes I$ commute with operators that look like $I \otimes F$. Then noting that C_{qc} is closed, we get the desired containment. The proof that C_{qc} is closed is somewhat technical, and follows from the fact that C_{qc} is the image of the weak-$*$ closed set of states on a given C* algebra under a continuous mapping.

With much labour, it was shown that C_{qa} and C_{qc} are not equal. Their possible equality was known until recently as the Weak Tsirelson Conjecture, which was shown to have a negative answer in [Ji+21]. It is shown in [Jun+11; Oza12; Fri12] that the two sets are equal if and only if Connes' embedding problem has a positive solution, so [Ji+21] also provides a negative answer to this longstanding problem in operator algebras.

Finally, let us see that all of the strategies we have discussed so far are non-signalling. We need only observe that qc-strategies are non-signalling, as qc is the largest class discussed so far. To see this, note that the expecations of each player's POVM operators with respect to the shared state gives us their marginal distribution. See [Gol19, Section 4.2] for details. The containment is strict, i.e. $C_{\mathrm{qc}} \neq C_{\mathrm{ns}}$. For one example, see [Pau16].

Ordering of values

From the chain of containment of correlation classes, we get a similar chain of values. That is, for a fixed game and question distribution,

$$\omega_{\text{det}} = \omega_{\text{loc}} \leq \omega_{\text{q}} \leq \omega_{\text{qs}} \leq \omega_{\text{qa}} \leq \omega_{\text{qc}} \leq \omega_{\text{ns}} \tag{2.2}$$

Note that a strict containment of correlation classes does not imply a strict inequality here. Indeed, the first containment of classes does not allow for greater values, that is, the best local strategies can be taken to be deterministic. The local correlations are the convex hull of the deterministic correlations and thus, by linearity of the value function, the maximum value is achieved at an extreme point, i.e. at a deterministic correlation. This means that a local strategy can always be replaced with a deterministic strategy with no decrease in value.

2.3 Synchronous games and *-algebras

Definition 2.21 (Synchronous game)**.** A *synchronous game* is a game where $X = Y =: \mathcal{I}$, $A = B =: \mathcal{O}$, and for $x \in \mathcal{I}, a, b \in \mathcal{O}$, if $a \neq b$ then $\lambda(a, b|x, x) = 0$. We denote the game by the triple $(\mathcal{I}, \mathcal{O}, \lambda)$.

From the definition we can see that in a synchronous game, if Alice and Bob receive the same question, they lose unless they give the same answer. This means that there is some degree of symmetry between the players. However, for $x \neq y$, there is no guarantee that $\lambda(a, b|x, y) = \lambda(b, a|y, x)$, i.e. if Alice and Bob trade roles, the outcome of the game may change. However, even if the rule function is not symmetric, it turns out that perfect strategies are symmetric in the following sense.

In [Hel+19], it is shown that every synchronous game can be symmetrized, in the sense that perfect strategies for the symmetric version and the original version coincide. This means a perfect strategy for a synchronous game is always a symmetric strategy, in the sense that $p(a, b|x, y) = p(b, a|y, x)$. Therefore, there is no loss in assuming a synchronous game has a symmetric rule function.

We will see that synchronous games can be modelled well by *-algebras. In particular, each synchronous game can be assigned a *-algebra that encodes the existence of perfect strategies in various classes.

2.3.1 Synchronous correlations

When working with a synchronous game, we consider a subclass of strategies or correlations that are called synchronous. In this section we will see why we can abandon non-synchronous strategies.

Definition 2.22 (Synchronous correlation)**.** A correlation $p(a, b|x, y)$ is called a *synchronous correlation* when $p(a, b|x, x) = 0$ for $a \neq b \in \mathcal{O}, x \in \mathcal{I}$.

We call the corresponding strategy a *synchronous strategy*. The motivation behind the above definition comes from the definition of a perfect strategy. By combining the definitions of perfect strategies, synchronous games, and synchronous correlations, we get the following remark.

Remark 2.23. A perfect strategy for a synchronous game must be synchronous.

This motivates a refinement of the classes of correlations when studying synchronous games. We denote the synchronous subset of any class of correlations C_t by C_t^s.

2.3.2 The *-algebra of a synchronous game

To a synchronous game we can associate an algebra with an involution, $*$. We call this associated algebra the *game algebra*. It captures the structure of the game. This definition is adapted from the original definition of [Hel+19].

Definition 2.24 (Game algebra)**.** Given a synchronous game $\mathcal{G} = (\mathcal{I}, \mathcal{O}, \lambda)$, the *game algebra* $\mathcal{A}(\mathcal{G})$ is the unital $*$-algebra over the complex numbers, generated by $e_{i,x}$ for $i \in \mathcal{I}, x \in \mathcal{O}$, with the following relations:

1. $e_{i,x}^2 = e_{i,x} = e_{i,x}^*$ for all $i \in \mathcal{I}, x \in \mathcal{O}$,
2. $\sum_{x \in \mathcal{O}} e_{i,x} = 1$ for all $i \in \mathcal{I}$,
3. $\lambda(x, y|i, j) = 0 \implies e_{i,x} e_{j,y} = 0$ for all $i, j \in \mathcal{I}, x, y \in \mathcal{O}$.

Each projection $e_{i,x}$ corresponds to one of the players receiving the input i and responding with output x. Relation 2 captures the fact that a player must always respond with *some* output. Relation 3 captures the rules of the game, imposing orthogonality on input-output pairs that lose the game. In other words, the relations say that the generators should be families of projections that send losing pairs of answers to zero under multiplication.

The reader may be familiar with the related idea of the C^*-algebra of a game as in [Lup+20]. The C^*-algebra of a game can be realized as a completion of the game algebra.

The game algebra captures a number of interesting properties about the existence of perfect strategies in various physical classes (corresponding to various quantum models). We now state a result from [Hel+19] that demonstrates the utility of the game algebra. This result will be applied in Section 2.3.4.

Theorem 2.25 (Helton, Mayer, Paulsen, Satriano)**.** *Let $\mathcal{G} = (\mathcal{I}, \mathcal{O}, \lambda)$ be a synchronous game, with game algebra $\mathcal{A}(\mathcal{G})$.*

1. *$\mathcal{G}$ has a perfect deterministic strategy if and only if there exists a unital $*$-homomorphism from $\mathcal{A}(\mathcal{G})$ to $\mathbb{C}$.*
2. *$\mathcal{G}$ has a perfect q-strategy if and only if there exists a unital $*$-homomorphism from $\mathcal{A}(\mathcal{G})$ to $B(\mathcal{H})$ for some non-zero finite dimensional Hilbert space $\mathcal{H}$.*
3. *$\mathcal{G}$ has a perfect qc-strategy if and only if there exists a unital C^*-algebra $\mathcal{C}$ with a faithful trace and a unital $*$-homomorphism $\pi : \mathcal{A}(\mathcal{G}) \to \mathcal{C}$.*

The proof is quite short, but it rests heavily on a more involved result from [Pau+16] regarding synchronous correlations. The result tells us that for a synchronous qc-correlation, there is a C^*-algebra $\mathcal{C}$ with a tracial state $\tau : \mathcal{C} \to \mathbb{C}$ and families of projections $\{E_{x,a}\} \subseteq \mathcal{C}$ satisfying $\sum_a E_{x,a} = I$ for all x such that $p(a, b|x, y) = \tau(E_{x,a}E_{y,b})$. This leads to a proof of statement 3. The proof then works backwards through the statements of the theorem, adding stricter conditions on the correlation $p(a, b|x, y)$ to specialize the guaranteed homomorphism to satisfy statements 2 and 1. The specialization uses the fact that perfect strategies in the smaller classes are also perfect strategies in the larger classes.

There are also conditions on $\mathcal{A}(\mathcal{G})$ for which $\mathcal{G}$ has a perfect qa-strategy but the mathematical objects involved are beyond the scope of this thesis. Here are the conditions as stated in [Hel+19].

Theorem 2.26 (Helton, Mayer, Paulsen, Satriano)**.** *$\mathcal{G}$ has a perfect* qa*-strategy if and only if there exists a unital $*$-homomorphism of $\mathcal{A}(\mathcal{G})$ into the von Neumann algebra $\mathcal{R}^\omega$.*

2.3.3 Notable synchronous nonlocal games

In this section, we will introduce three fairly general-purpose paramaterized families of synchronous nonlocal games. These games will be used and extended in Chapters 3 and 6.

Linear constraint system games

Let $A = [a_{i,j}]$ be an $m \times n$ matrix with entries in $\mathbb{Z}_2$. Let $b \in \mathbb{Z}_2^m$. Let

$$V_i := \{j \in \{1, \ldots, n\} : a_{i,j} \neq 0\}$$

denote the support of the i^{th} row of A. Let

$$S_i := \{x \in \mathbb{Z}_2^n : \sum_{j=1}^{n} a_{i,j}x_j = b_i,\ x_j = 0\ \forall j \notin V_i\}$$

denote the set of solutions to the i^{th} equation that are 0 off the support of that row. A synchronous binary constraint system (BCS) game is a game where the players demonstrate partial solutions to the linear system $Ax = b$ as follows. A non-synchronous variant was first introduced in [CM14]. It can be shown that they are equivalent to these games in terms of existence of perfect strategies.

Definition 2.27 (SyncBCS game)**.** The *synchronous BCS game* associated with the system $Ax = b$ is given by $\mathrm{syncBCS(A, b)} := (\mathcal{I}, \mathcal{O}, \lambda)$ where $\mathcal{I} := \{1, \ldots, m\}$, $\mathcal{O} := \mathbb{Z}_2^n$, and for $i, k \in \mathcal{I}$, $x, y \in \mathcal{O}$, $\lambda(x, y|i, k) = 1$ iff

1. $x \in S_i$,
2. $y \in S_k$, and
3. $x_j = y_j \quad$ for all $j \in V_i \cap V_k$.

Less formally, Alice and Bob are each required to solve a linear equation corresponding to a specific row of the system, so that their solutions are compatible. Here, compatible means that when both equations feature the same variable (the corresponding matrix entry is nonzero), Alice and Bob must both assign it the same value. Another way to say this is that Alice and Bob's solutions must agree on the common support of their question rows.

We will generalize the above definition by allowing the entries of A, b to range over $\mathbb{Z}_p$ for some $p \in \mathbb{N}$ prime. See Definition 3.1. We call this more general class of games synchronous linear constraint system (LCS) games. Note that there are non-synchronous LCS games as well. For example, the BCS games introduced in [CM14] have different input and output sets for Alice and Bob, and are therefore not synchronous.

Linear constraint system games let us weaken the notion of solving a linear system of equations, giving rise to the idea of a quantum solution to a linear system. More precisely:

Proposition 2.28. *Perfect deterministic strategies for* $\mathrm{syncBCS}(A, b)$ *are in one-to-one correspondence with solutions to the linear system* $Ax = b$*.*

Proof. One direction of this correspondence is immediate – a shared solution x to the linear system allows Alice and Bob to play the game perfectly, by each answering with the entries of x. That is, $f(i) = g(k) = x$. Their answers will solve any row of the system and will be mutually consistent. The other direction of this correspondence can be seen as follows. Say Alice and Bob have a perfect deterministic strategy. First off, their answer functions f and g must be equal on V_i for each row i – if they receive the same row, they must always give the same answer, otherwise their strategy would not win in every scenario. The values of $g(i)_j$ for $j \notin V_i$ are irrelevant to the strategy so can be assumed equal to $f(i)_j$. So we can assume $g = f$. Now given two rows i, k with $j \in V_i \cap V_k$, if the answer function assigns a different value to the j^{th} variable, i.e. $f(i)_j \neq f(k)_j$, this means $f(i)_j \neq g(k)_j$, which says that Alice and Bob give inconsistent answers when presented with the pair of rows (i, k), resulting in a losing scenario. This is not possible given that their strategy is perfect, so the answer function must have $f(i)_j = f(k)_j$ for all rows i, k and all indices j. This means that $f_j := f(i)_j$ is well-defined and it is easy to see that the vector f_j solves the linear system. □

The above correspondence motivates the definition of a t-solution to the linear system $Ax = b$, for $t \in \{\text{loc}, \text{q}, \text{qs}, \text{qa}, \text{qc}, \text{ns}\}$, as a perfect t-strategy for $\text{syncBCS}(A, b)$.

Graph homomorphism games

The graph homomorphism game was introduced in [MR16], extending the much earlier graph coloring game [GW02; Cle+04]. Before describing the graph homomorphism game here, we quickly introduce some notation about graphs. If G is a graph, then $V(G)$ denotes the vertex set of G, and for $v, w \in V(G)$, we write $v \sim w$ if (v, w) is an edge, and $v \not\sim w$ otherwise. In this work, all graphs are undirected, and are assumed to have no loops.

Definition 2.29 (Graph homomorphism game, [MR16])**.** Let G, H be two graphs. The *graph homomorphism game* associated with the pair (G, H) is given by $\text{Hom}(G, H) := (\mathcal{I}, \mathcal{O}, \lambda)$ where $\mathcal{I} := V(G)$, $\mathcal{O} := V(H)$, and for $v, w \in V(G)$, $a, b \in V(H)$, $\lambda(a, b|v, w) = 1$ if

1. $v = w$ and $a = b$, or
2. $v \sim w$ and $a \sim b$, or
3. $v \not\sim w$

Similarly to Proposition 2.28, it is not difficult to verify that perfect deterministic strategies for $\text{Hom}(G, H)$ are in correspondence with graph homomorphisms $G \to H$. In fact, this correspondence is what motivated the definition of this game. We can thus define a t-homomorphism from G to H as a perfect t-strategy for $\text{Hom}(G, H)$, for $t \in \{\text{loc}, \text{q}, \text{qs}, \text{qa}, \text{qc}, \text{ns}\}$.

Graph isomorphism games

In order to simply present the rules of the graph isomorphism game, as introduced in [Ats+16] and studied further in [Bra+19], we first define an auxiliary 'relation' function on the vertices of a graph. For a graph G, let $\mathrm{rel} : V(G) \times V(G) \to \{-1, 0, 1\}$ be given by

$$\mathrm{rel}(v, w) := \begin{cases} 0, & v = w, \\ -1, & v \sim w, \\ 1, & \text{otherwise.} \end{cases} \tag{2.3}$$

Note that for rel to be well-defined, there must be no loops in the graph.

Definition 2.30 (Graph isomorphism game, [Ats+16])**.** Let G, H be two graphs. The *graph isomorphism game* associated with the pair (G, H) is given by $\mathrm{Iso}(G, H) := (\mathcal{I}, \mathcal{O}, \lambda)$ where $\mathcal{I}, \mathcal{O} := V(G) \sqcup V(H)$, and for $v, w, a, b \in V(G) \sqcup V(H)$, $\lambda(a, b|v, w) = 1$ if all of the following conditions are met

1. v and a belong to different graphs and w and b belong to different graphs
2. If v, w are from the same graph, $\mathrm{rel}(v, w) = \mathrm{rel}(a, b)$
3. If v, w are from different graphs, $x = w$ iff $y = v$

As was the case with $\mathrm{Hom}(G, H)$, it is not difficult to verify that perfect deterministic strategies for $\mathrm{Iso}(G, H)$ are in correspondence with graph isomorphisms $G \to H$. We can thus define a t-isomorphism from G to H as a perfect t-strategy for $\mathrm{Iso}(G, H)$, for $t \in \{\mathrm{loc}, \mathrm{q}, \mathrm{qs}, \mathrm{qa}, \mathrm{qc}, \mathrm{ns}\}$. We write $G \cong_t H$ to mean G and H are t-isomorphic.

2.3.4 Algebraic equivalence of games

In this section we introduce algebraic equivalence of games.

There are several notions of equivalence for games. Here we are interested in studying the existence of perfect strategies. We want a notion of equivalence that guarantees a perfect t-strategy in one game if and only if we have a perfect t-strategy in the other. One way to guarantee this is with algebraic equivalence (also referred to as $*$-equivalence).

Definition 2.31 (Algebraic equivalence)**.** Let $\mathcal{G}_1, \mathcal{G}_2$ be synchronous games. We say these games are *algebraically equivalent* or *$*$-equivalent* if there are unital $*$-homomorphisms $\mathcal{A}(\mathcal{G}_1) \to \mathcal{A}(\mathcal{G}_2)$ and $\mathcal{A}(\mathcal{G}_2) \to \mathcal{A}(\mathcal{G}_1)$.

Note that the homomorphisms are not required to be mutual inverses, i.e. the algebras are not necessarily isomorphic.

This notion of equivalence is notably quite a bit weaker than isomorphism of the game algebras. It is motivated by its preservation of perfect strategies in all of the relevant strategy classes, as follows.

Perfect strategies in equivalent games

Due to Theorem 2.25 (and other omitted results regarding perfect strategies in other classes), if two games are algebraically equivalent, then mutual existence or non-existence of perfect strategies is immediate, by the composition of $*$-homomorphisms. The following proposition[4] summarizes the above remark:

Proposition 2.32 ([Bra+19]). *Let $t \in \{\mathrm{loc}, \mathrm{q}, \mathrm{qa}, \mathrm{qc}\}$. If $\mathcal{G}_1, \mathcal{G}_2$ are synchronous games that are algebraically equivalent, then $\mathcal{G}_1$ has a perfect t-strategy if and only if $\mathcal{G}_2$ has a perfect t-strategy.*

The proof of this result makes use of Theorem 2.25. This proposition is practically applicable. Let's say perfect strategies are known to exist in various classes but not exist in other classes for some fixed game $\mathcal{G}$. If any other game can be shown to be algebraically equivalent to $\mathcal{G}$, then there is no further proof required in characterizing its perfect strategies. This simplification makes the game algebra an attractive tool.

[4] The original result is stated for two more classes of strategies, as well as for hereditary equivalence, but they are not defined in this work so we omit them here.

3 Synchronous linear constraint system games: Algebraic methods

> Mathematics compares the most diverse phenomena and discovers the secret analogies that unite them. - Joseph Fourier

Here we unify and expand upon results about a class of nonlocal games called syncLCS(A, b), parameterized by a system of equations $Ax = b$ over the finite field of prime order p. The game is meant to verify whether two players share a solution to the system of equations. Many tasks can be formulated in terms of solving a linear system of equations, so in a sense this is a general-purpose game. The game was introduced in [KPS18] as a synchronous version of the binary constraint system game from [CM14]. We define this game in Section 3.1.

Two distinct algebraic objects have been associated to this game, both encoding information about the existence of perfect strategies. One such object is a group called the solution group, due to [CLS17], and the other is a *-algebra called the game algebra (see Section 2.3.2), due to [OP16] and described in more detail in [Hel+19]. We demonstrate an equivalence of these objects in Section 3.2. In more detail, our first result of this chapter, Theorem 3.4, states that the game algebra is *-isomorphic to a suitable quotient of the group algebra of the solution group. In other words, both algebraic objects encode the same information about the existence of perfect strategies.

Then, in Section 3.3, we prove the second main result of this chapter. Theorem 3.6, improves upon [Bra+19, Theorem 5.6]. We show that the game syncLCS(A, b) is *-equivalent to the graph isomorphism game, introduced in [Ats+16], for a suitable pair of graphs. The notion of *-equivalence, introduced in [Hel+19], is useful for the following reason. When two games are *-equivalent, if there is a perfect strategy for one game in any of the standard correlation sets (local, quantum tensor, quantum approximate, quantum commuting, etc.) then there is a perfect strategy for the other game in the same set.

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3.1 The syncLCS(A, b) games

Let p be a prime number. Given an $m \times n$ matrix $A = (A_{i,j})$ and a vector $b = (b_i)$ over the field $\mathbb{Z}_p := \mathbb{Z}/p\mathbb{Z}$, the game syncLCS$(A, b)$ has Alice and Bob convincing a referee that they share a solution x to the equation $Ax = b$. This game generalizes the syncBCS game introduced in [KPS18], allowing more than just binary coefficients. LCS stands for 'linear constraint system' whereas BCS stands for 'binary constraint system', because in syncBCS, $p = 2$. There are also nonlinear BCS games – syncLCS generalizes linear BCS games.

For a vector $x \in \mathbb{Z}_p^n$, let $\text{supp}(x) := \{j \in \{1, \ldots, n\} : x_j \neq 0\}$ be the support of x, and let $f_i = [0 \ldots 0\ 1\ 0 \ldots 0]^T$ be the i^{th} standard basis vector in $\mathbb{Z}_p^m$. Notice that the i^{th} row of A is given by $f_i^T A$. For rows $i = 1, \ldots, m$, we let

$$V_i := \text{supp}(f_i^T A).$$

So V_i is the support of the i^{th} row of A. In other words, V_i is the set of indices where the i^{th} row of A has a nonzero coefficient. We then define

$$S_i := \{x \in \mathbb{Z}_p^n : f_i^T Ax = b_i \text{ and } \text{supp}(x) \subseteq V_i\}.$$

Thus, S_i is the set of solutions to the i^{th} equation, modulo discrepancies where coefficients of A are 0. When we are working with multiple games, there may be confusion over which linear system S_i refers to. In that case, we will write $S_i(A, b)$ instead.

We now define the synchronous game associated to the system $Ax = b$.

Definition 3.1 (SyncLCS game)**.** Let A be an $m \times n$ matrix over $\mathbb{Z}_p$ and let $b \in \mathbb{Z}_p^m$. Then the *synchronous LCS game* associated to the linear system $Ax = b$ and denoted $\text{syncLCS}(A, b)$ is the tuple $(\mathcal{I}, \mathcal{O}, \lambda)$, where

1. the input set $\mathcal{I} = \{1, \ldots, m\}$, corresponding to rows of A,
2. the output set $\mathcal{O} = \mathbb{Z}_p^n$, corresponding to potential solutions to the system, and
3. given inputs (i, j) and outputs (x, y), $\lambda(x, y|i, j) = 1$ iff $(x, y) \in S_i \times S_j$ and $x_k = y_k$ for all $k \in V_i \cap V_j$.

We can think of Alice and Bob winning the game if they provide compatible solutions to their respective equations. In fact, the notion of compatibility is quite useful in the remainder of this work, so we formalize it as follows.

Definition 3.2 (Compatible solutions)**.** Let $i, j \in \{1, \ldots, m\}$, and $x \in S_i(A, b), y \in S_j(A, b)$. We say (x, y) are *compatible* solutions for rows (i, j) if for all $k \in V_i \cap V_j$, $x_k = y_k$. Otherwise, we call the pair (x, y) *incompatible* for rows (i, j).

Note that when $i = j$, (x, y) being compatible solutions to row i is simply equivalent to $x = y$. Therefore, $\mathrm{syncLCS}(A, b)$ is a synchronous game.

3.2 The solution group of syncLCS(A, b)

When $p = 2$ in the syncLCS game, the game is referred to as $\mathrm{syncBCS}(A, b)$, as introduced in [KPS18]. A nonsynchronous version of this game was introduced earlier in [CLS17], where properties of the game were shown to be captured by a group the authors called the solution group, denoted $\Gamma(A, b)$. The solution group was defined there for $p = 2$. It was noted in that paper that the solution group could be defined for any prime p. Following that note, we set down a fairly direct generalization to the case when $p > 2$, allowing us to associate a group to the syncLCS game. There is a similar definition in [CS19], where the authors study LCS games over the ring $\mathbb{Z}_d$ for positive integers d.

Definition 3.3 (Solution group)**.** The *solution group* $\Gamma(A, b)$ is the group generated by symbols $g_1, \ldots, g_n$ and J satisfying the following relations.

1. $g_j^p = 1$ for $j = 1, \ldots, n$.
2. $J^p = 1$.
3. $[g_j, J] = 1$ for $j = 1, \ldots, n$.
4. $[g_j, g_\ell] = 1$ if $j, \ell \in V_i$ for some $i \in \{1, \ldots, m\}$.
5. $$\prod_{j=1}^{n} g_j^{A_{i,j}} = J^{b_i} \text{ for } i = 1, \ldots, m.$$

Each g_j represents a variable x_j in the system of equations $Ax = b$. The element J is meant to represent a primitive p^{th} root of unity. (In the case that $p = 2$, the solution group can be thought of as a multiplicative representation of the system of equations, where 0s become 1s, and 1s become Js, which should be thought of as -1s.) Any two variables that appear in the same equation in the system $Ax = b$ are required to have corresponding group elements commute. Finally, the variables in a given equation should have their corresponding group elements multiply to an appropriate power of J depending on the value of b_i. Note that there is a one-to-one correspondence between powers of J and possible values for b_i.

3.2.1 Connection to the game algebra

Let $\mathbb{C}\Gamma(A, b)$ denote the group $*$-algebra of the group $\Gamma(A, b)$. Its elements can be thought of as $\mathbb{C}$-linear combinations of group elements. Multiplication extends bilinearly from the group multiplication. The $*$-operation is defined on group elements to be the group's inverse operation and then extended conjugate-linearly. This algebra is unital, with the unit given by the identity element of the group.

Let $\omega \in \mathbb{C}$ be a primitive p^{th} root of unity. In $\mathbb{C}\Gamma(A, b)$, there is an element $\omega 1$ and an element J. In keeping with the motivation for constructing the solution group, we identify J with $\omega 1$ through a quotient. (In the $p = 2$ case, this is the identification that ensures J really does represent -1.) Similarly to $\mathcal{A}(\text{syncLCS}(A, b))$, the resulting $*$-algebra encodes information about perfect strategies in the syncLCS game. In fact, the two algebras are isomorphic – the first of the main results of this chapter.

Theorem 3.4. *The following algebras are isomorphic as unital $*$-algebras.*

$$\mathcal{A}(\text{syncLCS}(A, b)) \cong \frac{\mathbb{C}\Gamma(A, b)}{\langle J - \omega 1 \rangle}$$

There is a related notion to the game algebra, called the C^*-algebra of a game, denoted $C^*(\mathcal{G})$, that also encodes information about perfect strategies. The C^*-algebra $C^*(\mathcal{G})$ is a quotient of the game algebra. Theorem 3.4 resembles [Lup+20, Corollary 4.10] but is more general in that it is a result at the level of *-algebras as opposed to difficult-to-construct quotients. This result is also more general in that we do not restrict $p = 2$.

Proof. To begin, let us briefly examine the algebra $\mathcal{A}(\text{syncLCS}(A, b))$. Following the game-algebra construction, we get a $*$-algebra over $\mathbb{C}$ generated by self-adjoint idempotents $a_{i,x}$ where $i \in \{1, \ldots, m\}$ and $x \in S_i(A, b)$, satisfying the following relations:

1. $\sum_{x \in S_i} a_{i,x} = 1$.
2. When (x, y) are incompatible solutions to rows (i, k) then $a_{i,x} a_{k,y} = 0$.

We will define two $*$-algebra homomorphisms $\tilde{\phi}$ and $\overline{\psi}$ and then show that they are mutual inverses.

Define $\phi : \mathbb{C}\Gamma(A, b) \to \mathcal{A}(\mathrm{syncLCS}(A, b))$ on the generators by:

$$\phi(g_j) = \sum_{x \in S_i} \omega^{x_j} a_{i,x},$$
$$\phi(J) = \omega 1,$$

where i is chosen so that $j \in V_i$.

We must check that ϕ is well-defined, i.e. that $\sum_{x \in S_i} \omega^{x_j} a_{i,x}$ doesn't depend on the choice of i.

For each $1 \leq i \leq m$, $1 \leq j \leq n$, and $t \in \mathbb{Z}_p$, define

$$P_{i,j}(t) := \sum_{\substack{x \in S_i \\ x_j = t}} a_{i,x},$$

with $P_{i,j}(t) = 0$ if the above sum is empty. By the relations that the generators $a_{i,x}$ satisfy, it follows that:

1. $P_{i,j}(t)^2 = P_{i,j}(t)$ for all i, j and t,
2. $P_{i,j}(t) P_{k,j}(s) = 0$ if $j \in V_i \cap V_k$ and $s \neq t$,
3. $\sum_{t \in \mathbb{Z}_p} P_{i,j}(t) = 1$ for all i, j.

Therefore, if $j \in V_i \cap V_k$, then using Properties 2 and 3, we get

$$P_{i,j}(t) = P_{i,j}(t) \left(\sum_{s \in \mathbb{Z}_p} P_{k,j}(s) \right) = P_{i,j}(t) P_{k,j}(t)$$
$$= \left(\sum_{t \in \mathbb{Z}_p} P_{i,j}(s) \right) P_{k,j}(t) = P_{k,j}(t).$$

Thus $\phi(g_j) = \sum_{t\in\mathbb{Z}_p} \omega^t P_{i,j}(t) = \sum_{t\in\mathbb{Z}_p} \omega^t P_{k,j}(t)$. That is, $\phi(g_j)$ does not depend on the choice of i.

To see that ϕ is a $*$-algebra homomorphism, one must check that the relations of $\Gamma(A,b)$ given in Definition 3.3 are satisfied in the image. These relations are easily verified, except perhaps for relation 5, so we verify it here.

Let $i \in \{1,\dots,m\}$. We must check that $\prod_{j\in V_i} \phi(g_j)^{A_{i,j}} = \phi(J)^{b_i}$.

$$\begin{aligned}\prod_{j\in V_i} \phi(g_j)^{A_{i,j}} &= \prod_{j\in V_i} \left(\sum_{x\in S_i} \omega^{A_{i,j}x_j} a_{i,x}\right) = \sum_{x\in S_i} \left(\prod_{j\in V_i} \omega^{A_{i,j}x_j}\right) a_{i,x} \\ &= \sum_{x\in S_i} \omega^{b_i} a_{i,x} = \omega^{b_i} 1 = \phi(J)^{b_i}.\end{aligned}$$

So $\phi : \mathbb{C}\Gamma(A,b) \to \mathcal{A}(\mathrm{syncLCS}(A,b))$ is a homomorphism of $*$-algebras. Because $J - \omega 1 \in \ker(\phi)$, there is an induced $*$-homomorphism $\tilde{\phi} : \frac{\mathbb{C}\Gamma(A,b)}{\langle J-\omega 1\rangle} \to \mathcal{A}(\mathrm{syncLCS}(A,b))$ where $\langle J-\omega 1\rangle$ is the ideal generated by $J - \omega 1$.

Now for the reverse direction we define

$$\begin{aligned}\psi : \mathcal{A}(\mathrm{syncLCS}(A,b)) &\to \mathbb{C}\Gamma(A,b) \\ a_{i,x} &\mapsto \prod_{j\in V_i} f_j(x_j),\end{aligned}$$

where for $j \in \{1,\dots,m\}$, we have

$$\begin{aligned}f_j : \mathbb{Z}_p &\to \mathbb{C}\Gamma(A,b) \\ s &\mapsto \frac{1}{p}\sum_{t=0}^{p-1} (\omega^{-s} g_j)^t.\end{aligned}$$

After checking ψ is a $*$-homomorphism, we can compose it with the quotient map $\mathbb{C}\Gamma(A,b) \to \frac{\mathbb{C}\Gamma(A,b)}{\langle J-\omega 1\rangle}$ to get the $*$-homomorphism we are looking for:

$$\overline{\psi} : \mathcal{A}(\mathrm{syncLCS}(A,b)) \to \frac{\mathbb{C}\Gamma(A,b)}{\langle J-\omega 1\rangle}.$$

In order to see that ψ is a $*$-homomorphism, we will use some facts about the family f_j. First and foremost, each $f_j(s)$ is a self-adjoint projection. Furthermore, each g_j decomposes as

a linear combination of the $f_j(s)$ as follows:

$$g_j = \sum_{s=0}^{p-1} \omega^s f_j(s).$$

Note also that if there is some $i \in \{1, \ldots, m\}$ with $j, \ell \in V_i$ then $f_j(s)$ commutes with $f_\ell(r)$ for all r, s, precisely because g_j commutes with g_ℓ. Finally, for each fixed j, $\{f_j(s) : s \in \mathbb{Z}_p\}$ form an orthogonal family summing to 1, where by orthogonal we mean that for $s \neq r$, $f_j(s)f_j(r) = 0$.

With these properties of $f_j(s)$, it is fairly easy to extend them and prove all but one required property of ψ.

For each $a_{i,x}$, we have $\psi(a_{i,x})$ is a self-adjoint projection, as it is a commuting product of self-adjoint projections.

Now let (x, y) be a pair of incompatible solutions to rows (i, k). Then there exists $u \in V_i \cap V_k$ with $x_u \neq y_u$. But then $f_u(x_u)f_u(y_u) = 0$. So

$$\begin{aligned}\psi(a_{i,x})\psi(a_{k,y}) &= \prod_{j \in V_i} f_j(x_j) \prod_{\ell \in V_k} f_\ell(y_\ell) \\ &= \left(\prod_{j \in V_i \setminus \{u\}} f_j(x_j) \right) f_u(x_u)f_u(y_u) \left(\prod_{\ell \in V_k \setminus \{u\}} f_\ell(y_\ell) \right) = 0.\end{aligned}$$

The last property we need to show is that $\sum_{x \in S_i} \psi(a_{i,x}) = 1$. This is the trickiest one to prove and requires the use of some involved symmetry arguments. We will first note that because $A_{i,j} \neq 0 \in \mathbb{Z}_p$ for all $j \in V_i$, we can apply a change of variables in our definition of f_j as follows, replacing t with $tA_{i,j}$, to get

$$f_j(s) = \frac{1}{p} \sum_{t=0}^{p-1} (\omega^{-s} g_j)^{tA_{i,j}}.$$

We will let the set $\mathcal{P}_i$ denote the set of partitions λ of the set V_i into exactly p blocks, where the t^{th} block is denoted λ_t. More precisely, each $\lambda = \{\lambda_t : t = 0, \ldots, p-1\}$, satisfying $V_i = \cup_t \lambda_t$ and $\lambda_t \cap \lambda_s = \emptyset$ when $s \neq t$. Note that λ_t may be empty.

We then have

$$\begin{aligned} p^{|V_i|} \sum_{x \in S_i} \psi(a_{i,x}) &= \sum_{x \in S_i} \prod_{j \in V_i} \sum_{t=0}^{p-1} (\omega^{-x_j} g_j)^{tA_{i,j}} \\ &= \sum_{x \in S_i} \sum_{\lambda \in \mathcal{P}_i} \prod_{t=0}^{p-1} \prod_{j \in \lambda_t} (\omega^{-x_j} g_j)^{tAi,j} \\ &= \sum_{\lambda \in \mathcal{P}_i} \sum_{x \in S_i} \prod_{t=0}^{p-1} \left(\omega^{-\sum_{j \in \lambda_t} A_{i,j} x_j} \prod_{j \in \lambda_t} g_j^{A_{i,j}} \right)^t. \end{aligned}$$

We will divide the outer sum into a sum where λ has exactly one non-empty block $\lambda_s = V_i$ and a sum where λ has more than one non-empty block. We will denote the second set of partitions by $\tilde{\mathcal{P}}_i$. The sum splits into the following two pieces.

$$\sum_{s=0}^{p-1} \sum_{x \in S_i} \left(\omega^{-\sum_{j \in V_i} A_{i,j} x_j} \prod_{j \in V_i} g_j^{A_{i,j}} \right)^s \tag{3.1}$$

$$\sum_{\lambda \in \tilde{\mathcal{P}}_i} \sum_{x \in S_i} \prod_{t=0}^{p-1} \left(\omega^{-\sum_{j \in \lambda_t} A_{i,j} x_j} \prod_{j \in \lambda_t} g_j^{A_{i,j}} \right)^t \tag{3.2}$$

We have that $p^{|V_i|} \sum_{x \in S_i} \psi(a_{i,x})$ is the sum of (3.1) and (3.2). We will now show that (3.1) $= p^{|V_i|}$ and (3.2) $= 0$, completing the proof of this particular property. First, we have

$$(3.1) = \sum_{s=0}^{p-1} \sum_{x \in S_i} \left(\omega^{-b_i} \omega^{b_i} \cdot 1 \right)^s = \sum_{s=0}^{p-1} \sum_{x \in S_i} 1 = p \cdot |S_i| = p^{|V_i|}.$$

Second, we have

$$\begin{aligned} (3.2) &= \sum_{\lambda \in \tilde{\mathcal{P}}_i} \sum_{x \in S_i} \prod_{t=0}^{p-1} \omega^{-t \sum_{j \in \lambda_t} A_{i,j} x_j} \prod_{j \in \lambda_t} g_j^{tA_{i,j}} \\ &= \sum_{\lambda \in \tilde{\mathcal{P}}_i} \left(\sum_{x \in S_i} \omega^{-\sum_{t=0}^{p-1} \sum_{j \in \lambda_t} tA_{i,j} x_j} \right) \prod_{t=0}^{p-1} \prod_{j \in \lambda_t} g_j^{tA_{i,j}}. \end{aligned}$$

For the following paragraph, fix $\lambda \in \tilde{\mathcal{P}}_i$. Let

$$\begin{aligned} \mu_i : \mathbb{Z}_p^n &\to \mathbb{Z}_p \\ x &\mapsto -\sum_{t=0}^{p-1} \sum_{j \in \lambda_t} tA_{i,j} x_j. \end{aligned}$$

As x ranges over S_i (a vector space over $\mathbb{Z}_p$), the linear functional $\mu_i(x)$ could either be constantly 0 or hit every value in $\mathbb{Z}_p$ equally many times. But because λ has at least two blocks, this functional is not a constant multiple of $\sum_{j \in V_i} A_{i,j} x_j$, so has a kernel defining a hyperplane that is not equal to S_i. In other words, μ_i is not constantly 0 for $x \in S_i$. Thus, μ_i must take each value equally many times, and so we get that

$$\sum_{x \in S_i} \omega^{\mu_i(x)} = 0.$$

Thus, (3.2) $= 0$.

Putting both pieces together, we get that

$$\sum_{x \in S_i} \psi(a_{i,x}) = p^{-|V_i|}(p^{|V_i|} + 0) = 1.$$

This completes the argument that $\psi : \mathcal{A}(\mathrm{syncLCS}(A, b)) \to \mathbb{C}\Gamma(A, b)$ is a $*$-homomorphism.

Finally, we must show that $\overline{\psi}$ and $\tilde{\phi}$ are mutual inverses.

Let $\ell \in \{1, \ldots, n\}$. Let i be such that $\ell \in V_i$.

$$\begin{aligned}
\psi(\phi(g_\ell)) &= \psi\left(\sum_{x \in S_i} \omega^{x_\ell} a_{i,x}\right) = \sum_{x \in S_i} \omega^{x_\ell} \prod_{j \in V_i} f_j(x_j) \\
&= \sum_{x \in S_i} \omega^{x_\ell} f_\ell(x_\ell) \prod_{j \in V_i \setminus \{\ell\}} f_j(x_j) = \sum_{x \in S_i} g_\ell f_\ell(x_\ell) \prod_{j \in V_i \setminus \{\ell\}} f_j(x_j) \\
&= g_\ell \sum_{x \in S_i} \psi(a_{i,x}) = g_\ell,
\end{aligned}$$

where the fourth equality holds because of the following:

$$\omega^s f_j(s) = \frac{1}{p} \sum_{t=0}^{p-1} \omega^{-s(t-1)} g_j^t = \frac{g_j}{p} \sum_{t=-1}^{p-2} \omega^{-st} g_j^t = g_j f_j(s).$$

If q is the canonical quotient map $\mathbb{C}\Gamma(A, b) \to \frac{\mathbb{C}\Gamma(A,b)}{\langle J - \omega 1 \rangle}$, then for all j,

$$\overline{\psi} \circ \tilde{\phi}(q(g_j)) = q(g_j).$$

Also, $\psi \circ \phi(J - \omega 1) = \psi(0) = 0$. Thus,

$$\overline{\psi} \circ \tilde{\phi}(q(J)) = q(J).$$

In summary, we get that $\overline{\psi} \circ \tilde{\phi}$ is the identity on $\frac{\mathbb{C}\Gamma(A,b)}{\langle J - \omega 1 \rangle}$.

In the other direction, let $i \in \{1, \ldots, m\}$, and $y \in S_i$. Then we have

$$\begin{aligned}
\phi(\psi(a_{i,y})) &= \phi\left(\prod_{j \in V_i} f_j(y_j)\right) = \prod_{j \in V_i} \frac{1}{p} \sum_{t=0}^{p-1} \left(\omega^{-y_j} \phi(g_j)\right)^t \\
&= \frac{1}{p^{|V_i|}} \prod_{j \in V_i} \sum_{t=0}^{p-1} \left(\omega^{-y_j} \sum_{x \in S_i} \omega^{x_j} a_{i,x}\right)^t \\
&= \frac{1}{p^{|V_i|}} \prod_{j \in V_i} \sum_{x \in S_i} \left(\sum_{t=0}^{p-1} \omega^{(x_j - y_j)t}\right) a_{i,x} \\
&= \frac{1}{p^{|V_i|}} \sum_{x \in S_i} \left(\prod_{j \in V_i} \sum_{t=0}^{p-1} \omega^{(x_j - y_j)t}\right) a_{i,x} \\
&= \frac{1}{p^{|V_i|}} \prod_{j \in V_i} \sum_{t=0}^{p-1} \omega^{(y_j - y_j)t} a_{i,y} = \frac{1}{p^{|V_i|}} \left(\prod_{j \in V_i} p\right) a_{i,y} = a_{i,y},
\end{aligned}$$

where the third-to-last equality holds because whenever there is some $x_j \neq y_j$, we get the p distinct p^{th} roots of unity which sum to 0. Thus, the only x that survives is $x = y$.

Note that $\tilde{\phi}$ is an induced homomorphism satisfying $\tilde{\phi} \circ q = \phi$. So

$$\tilde{\phi} \circ \overline{\psi} = \tilde{\phi} \circ q \circ \psi = \phi \circ \psi = \mathrm{Id}.$$

□

3.3 Equivalence to the graph isomorphism game

So far, we have seen two equivalent algebraic structures modelling the game syncLCS. Now we will see that this structure connects syncLCS to another class of games, known as graph isomorphism games.

We can associate a graph to the system $Ax = b$. This is the same $G_{A,b}$ defined in [Ats+16], but now A, b have entries in $\mathbb{Z}_p$.

Definition 3.5 (Graph of a linear system)**.** Let $G_{A,b}$ be the graph with vertex set $\{(i,x) : i \in \{1,\dots,m\}, x \in S_i(A,b)\}$ and $(i,x) \sim (j,y)$ if there exists $k \in V_i \cap V_j$ with $x_k \neq y_k$.

Recall that $S_i(A,b)$ is the set of solutions to row i of the system of equations $Ax = b$, and V_i is the support of row i of A. We can think of edges in $G_{A,b}$ as capturing incompatibility of solutions to the various rows.

We will now see that the game asking the players to construct an isomorphism from $G_{A,b}$ to $G_{A,0}$ is essentially the same as the game syncLCS(A,b), which asks players to demonstrate a shared solution to the system $Ax = b$. This graph game is known more generally as Iso(G,H) for two graphs G and H. See Definition 2.30.

The following theorem was first stated as such in an early version of [Bra+19] for the $p = 2$ case, but the proof had an error, as there was a third game included in the equivalence which is now known to only satisfy a weaker form of equivalence to these two games. That third game is the graph homomorphism game from K_m to $\overline{G_{A,b}}$. The result in [Bra+19] was then weakened to say the three games are hereditarily $*$-equivalent, meaning that the hereditary quotients of the game algebras possess a pair of $*$-homomorphisms.

The following theorem generalizes that equivalence result to the $p > 2$ setting, and drops the modifier "hereditarily" once again, but only for two of the three games originally considered.

Theorem 3.6. *Let A be an $m \times n$ matrix over $\mathbb{Z}_p$ and let $b \in \mathbb{Z}_p^m$. Then the following two synchronous games are $*$-equivalent:*

1. *syncLCS(A,b)*
2. *Iso$(G_{A,b}, G_{A,0})$*

Before seeing the proof, it will be useful to learn a little about the game algebra of Iso(G,H).

For graphs G, H, $\mathcal{A}(\text{Iso}(G,H))$ is generated by $e_{g,h}$ where $g \in V(G)$ and $h \in V(H)$. Looking at the definition of Iso(G,H) and the definition of the game-algebra, it seems we may need the other three possibilities of which graphs the pair (g,h) can live in. However, it can be shown that $e_{g,h} = 0$ whenever g, h are vertices in the same graph. With some additional cleverness, it can also be shown that $e_{h,g} = e_{g,h}$, eliminating the need for both index orderings.

This means that in our case, $\mathcal{A}(\text{Iso}(G_{A,b}, G_{A,0}))$ is generated by $e_{(i,x),(j,y)}$ where $i,j \in \{1,\dots,m\}, x \in S_i(A,b), y \in S_j(A,0)$. That is, each (i,x) is a vertex of $G_{A,b}$ and (j,y) is a vertex of $G_{A,0}$. The usual game algebra relations are in place: each generator must be a self-adjoint idempotent. Also for a fixed input, the sum over all outputs must be 1. In that case, this

gives us the following two relations:

$$\sum_{(i,x)} e_{(i,x),(j,y)} = 1, \quad \sum_{(j,y)} e_{(i,x),(j,y)} = 1.$$

Furthermore, the product of two generators must be 0 when the corresponding inputs and outputs don't satisfy the game's rules. That means $e_{(i,x),(j,y)}e_{(i',x'),(j',y')} = 0$ whenever the relationship of the vertices $(i,x),(i',x')$ in $G_{A,b}$ is *not* the same as the relationship of the vertices $(j,y),(j',y')$ in $G_{A,0}$. In more detail, $e_{(i,x),(j,y)}e_{(i',x'),(j',y')} = 0$ whenever:

- $(i,x) = (i',x')$ but $(j,y) \neq (j',y')$, or $(j,y) = (j',y')$ but $(i,x) \neq (i',x')$. That is, one pair of vertices is equal when the other pair is not equal.
- $(i,x) \sim (i',x')$ but $(j,y) \nsim (j',y')$, or $(j,y) \sim (j',y')$ but $(i,x) \nsim (i',x')$. That is, one pair of vertices is incompatible when the other pair is compatible.

Proof of Theorem 3.6. We claim the following map defines a unital $*$-homomorphism:

$$\begin{aligned} \phi : \mathcal{A}(\mathrm{Iso}(G_{A,b}, G_{A,0})) &\to \mathcal{A}(\mathrm{syncLCS}(A,b)) \\ e_{(i,x),(j,y)} &\mapsto \delta_{ij} a_{i,x+y} \end{aligned}$$

for all $i,j \in \{1,\dots,m\}$ and all $x \in S_i(A,b), y \in S_j(A,0)$. Each generator maps to a self-adjoint idempotent. We must verify that ϕ preserves the other relations of $\mathcal{A}(\mathrm{Iso}(G_{A,b}, G_{A,0}))$. Unitality will follow from these.

First we show that the "sum to one" relations are preserved by ϕ.

$$\sum_{(i,x)} \phi(e_{(i,x),(j,y)}) = \sum_{(i,x)} \delta_{ij} a_{i,x+y} = \sum_{x \in S_j(A,b)} a_{j,x+y} = \sum_{z \in S_j(A,b)} a_{j,z} = 1.$$

The second-to-last step here is due to the fact that the set of solutions to $Ax = b$ is preserved by adding a homogeneous solution. Similarly,

$$\sum_{(j,y)} \phi(e_{(i,x),(j,y)}) = \sum_{(j,y)} \delta_{ij} a_{i,x+y} = \sum_{y \in S_i(A,0)} a_{i,x+y} = \sum_{z \in S_j(A,b)} a_{j,z} = 1.$$

The second-to-last step here is due to a similar fact: The set of solutions to $Ax = b$ can be constructed by starting with a particular solution and adding to it all homogeneous solutions. From the above arguments, we get that ϕ sends 1 to 1.

Now we show that the relations corresponding to the rules of the isomorphism game are preserved under ϕ.

Assume $e_{(i,x),(j,y)}e_{(i',x'),(j',y')} = 0$. Then we would like to show that

$$\phi(e_{(i,x),(j,y)})\phi(e_{(i',x'),(j',y')}) = \delta_{ij}a_{i,x+y}\delta_{i'j'}a_{i',x'+y'}$$

evaluates to 0 as well. Whenever $i \neq j$ or $i' \neq j'$, the above expression is 0, so we may assume $i = j, i' = j'$, and we must show that $e_{(i,x),(i,y)}e_{(i',x'),(i',y')} = 0 \implies a_{i,x+y}a_{i',x'+y'} = 0$. That is, we must show that $(x + y, x' + y')$ are incompatible solutions to rows (i, i').

If $e_{(i,x),(i,y)}e_{(i',x'),(i',y')} = 0$, there are several cases to consider.

1. If $(i, x) = (i', x')$, we must have $(i, y) \neq (i', y')$. Because $i = i'$, we have that $y \neq y'$. Then, because $x = x'$, $x + y \neq x' + y'$, and so $(x + y, x' + y')$ are incompatible solutions to row i.

2. Now if $(i, y) = (i', y')$, we must have $(i, x) \neq (i', x')$. Exchanging x and y in the above argument shows that $(x + y, x' + y')$ are once again incompatible solutions to row i.

3. If $(i, x) \sim (i', x')$, then there exists $k_0 \in V_i \cap V_{i'}$ with $x_{k_0} \neq x'_{k_0}$. But also, we must have that $(i, y) \nsim (i', y')$, so for all $k \in V_i \cap V_{i'}$, $y_k = y'_k$. Thus, $(x + y)_{k_0} \neq (x' + y')_{k_0}$, so $(x + y, x' + y')$ are incompatible solutions to rows (i, i').

4. If $(i, y) \sim (i', y')$, then exchanging x and y in the above argument resolves this last case.

We have shown that in all cases, ϕ preserves the "rules" relations of the isomorphism game, allowing us to conclude that ϕ is a unital $*$-homomorphism $\mathcal{A}(\text{Iso}(G_{A,b}, G_{A,0})) \to \mathcal{A}(\text{syncLCS}(A, b))$.

In the other direction, let ψ be the linear map given by

$$\begin{aligned}\psi : \mathcal{A}(\text{syncLCS}(A, b)) &\to \mathcal{A}(\text{Iso}(G_{A,b}, G_{A,0})) \\ a_{i,x} &\mapsto \sum_{k=1}^{m} e_{(i,x),(k,0)}\end{aligned}$$

for all $i \in \{1, \ldots, m\}, x \in S_i(A, b)$. The element $\psi(a_{i,x})$ is self-adjoint and idempotent because $\{e_{(i,x),(k,0)} : k = 1, \ldots, m\}$ are a family of orthogonal projections.

When (x, y) are incompatible solutions to rows (i, j), we have $a_{i,x}a_{j,y} = 0$. But then $(i, x) \sim (j, y)$ in $G_{A,b}$, so

$$\psi(a_{i,x})\psi(a_{j,y}) = \sum_{k,\ell=1}^{m} e_{(i,x),(k,0)}e_{(j,y),(\ell,0)} = 0,$$

because each $(k,0) \nsim (\ell,0)$ in $G_{A,0}$ (they are either the same vertex, or distinct but compatible and therefore nonadjacent).

Finally, we must show that $\sum_{x \in S_i} \psi(a_{i,x}) = 1$. This piece of the proof relies on the fact that $\mathcal{A}(\mathrm{Iso}(X,Y))$ is *-isomorphic to $\mathcal{O}(G_X, G_Y)$ as defined in [Bra+19]. In that paper, Theorem 4.7 tells us that whenever $\mathcal{A}(\mathrm{Iso}(X,Y))$ is nonzero, it admits a faithful state. Therefore, $\mathcal{A}(\mathrm{Iso}(X,Y))$ is hereditary. A *-algebra $\mathcal{A}$ is hereditary whenever $\sum_{i=1}^{N} x_i^* x_i = 0$ implies each $x_i = 0$, where $x_i \in \mathcal{A}$.

We follow exactly the steps taken in [Bra+19] in the proof of Theorem 5.6. Let $p_i := \sum_{x \in S_i(A,b)} \sum_{k=1}^{m} e_{(i,x),(k,0)}$. We would like to show that $p_i = 1$. Note that p_i is self-adjoint, being a sum of projections, and idempotent, because $e_{(i,x),(k,0)} e_{(i,y),(\ell,0)} = 0$ unless $x = y$ and $k = \ell$. Then, $q_i := 1 - p_i$ is also a self-adjoint idempotent. Thus,

$$\begin{aligned}\sum_{i=1}^{m} q_i^* q_i = \sum_{i=1}^{m} 1 - p_i &= m \cdot 1 - \sum_{k=1}^{m} \sum_{i=1}^{m} \sum_{x \in S_i(A,b)} e_{(i,x),(k,0)} \\ &= m \cdot 1 - \sum_{k=1}^{m} 1 = 0,\end{aligned}$$

so because the algebra is hereditary, we have that each $q_i = 0$. Thus, each $p_i = 1$. We can also conclude from this that ψ is unital.

It follows that ψ is a unital $*$-homomorphism $\mathcal{A}(\mathrm{syncLCS}(A,b)) \to \mathcal{A}(\mathrm{Iso}(G_{A,b}, G_{A,0}))$. We have exhibited a unital $*$-homomorphism in both directions, meaning that the games $\mathrm{syncLCS}(A,b)$ and $\mathrm{Iso}(G_{A,b}, G_{A,0})$ are $*$-equivalent. □

Remark 3.7. In fact, composing $\phi \circ \psi$ gives the identity on $\mathcal{A}(\mathrm{syncLCS}(A,b))$. This means that ψ is an embedding of $\mathcal{A}(\mathrm{syncLCS}(A,b))$ into $\mathcal{A}(\mathrm{Iso}(G_{A,b}, G_{A,0}))$.

4 Quantum games: Graphical methods

Wherever there are words, let there be pictures. - Chris Riddell

In this chapter, we generalize the setting of typical nonlocal games to quantum question and answer games. To define games, strategies and correlations in the quantum question and answer setting, we follow the program of categorical quantum mechanics, using the language of monoidal dagger categories, and the accompanying graphical calculus. We apply this approach to the category of finite dimensional complex Hilbert spaces, where the dagger is the Hilbert space adjoint. There is an excellent exposition of this topic in [HV19]. We follow the conventions introduced there.

First, in Section 4.1, we define quantum sets and quantum functions, and make our first use of the graphical calculus. Then, building on this, in Section 4.2, we make the basic graphical definitions for the quantum question-and-answer game setting, including quantized notions of games, strategies, and correlations. In Section 4.3, we define quantum question-and-answer analogues to some of the strategy classes introduced in Section 2.2. We go on to define various products of games (Section 4.4) and then using an operational interpretation of quantum games (Section 4.5), we define values of games and correlations (Section 4.6). Finally, in Section 4.7, we discuss other definitions for quantum question-and-answer games, and compare them to our approach.

4.1 Quantum sets and functions

Instead of question and answer spaces being finite sets, they will now be some kind of "quantum sets". Quantum sets are introduced in [MRV18] (and are equivalent to the finite quantum sets of [Kor20]). We include the definition here, but first, we must define dagger Frobenius algebras.

In the following, let **FHilb** be the dagger compact category of finite dimensional complex Hilbert spaces with linear maps as morphisms. Notably, it has a monoidal structure, given by the Hilbert space tensor product, $\otimes$, and a biproduct structure, given by the direct sum of Hilbert spaces, $\oplus$. In this category, the dagger ($\dagger$) is the Hilbert space adjoint.

Definition 4.1 (Dagger Frobenius algebra)**.** A *dagger Frobenius algebra* in **FHilb** is an object X of **FHilb** that has an associative unital multiplication $m : X \otimes X \to X$ with unit $u : \mathbb{C} \to X$, and a coassociative counital comultiplication $\Delta : X \to X \otimes X$ with counit $\varepsilon : X \to \mathbb{C}$. These maps must satisfy $\Delta = m^\dagger, \varepsilon = u^\dagger$, and

$$(1 \otimes m)(\Delta \otimes 1) = \Delta m = (m \otimes 1)(1 \otimes \Delta) \tag{4.1}$$

Here, we use 1 to refer to the identity map on X. By *associative*, we mean

$$m(m \otimes 1) = m(1 \otimes m) \tag{4.2}$$

and by *unital*, we mean

$$m(u \otimes 1) = 1 = m(1 \otimes u) \tag{4.3}$$

By *coassociative* and *counital*, we mean that Δ and ε satisfy the daggers of the above equations. These properties are automatic given that $\Delta = m^\dagger$ and $\varepsilon = u^\dagger$, but we include them in the definition to emphasize that we have both an algebra and coalgebra structure on the same vector space.

Equation (4.1) is called the Frobenius equation.

Dagger Frobenius algebras are related to Hopf algebras[1], but should not be confused for each other, despite the similar notation used here.

Definition 4.2 (Quantum set, [MRV18])**.** A dagger Frobenius algebra X in **FHilb** is called a *quantum set* if it is *special*, that is

$$m\Delta = 1, \tag{4.4}$$

and *symmetric*, meaning

$$\varepsilon m \sigma = \varepsilon m, \tag{4.5}$$

where σ is the canonical isomorphism that swaps tensor factors.

These dagger Frobenius algebras are an alternate presentation of finite-dimensional unital C*-algebras [Vic10, Theorem 4.6]. Their underlying algebras are direct sums of finite-dimensional complex matrix algebras. The inner product coming from the Hilbert space structure determines ε and Δ.

[1] Both have algebra and coalgebra structures, but the ways these structures interact are different.

From an operator algebraic perspective, we can think of (4.4) as requiring that Δ is an isometry, and of (4.5) as requiring that ε is tracial.

Quantum sets are referred to most everywhere else as †-SSFAs (for **s**pecial **s**ymmetric dagger **F**robenius **a**lgebras) in **FHilb**, and when they are classical, as †-SCFAs (where the **c** stands for **c**ommutative). We use the terminology quantum set here because it is more evocative, given that we are generalizing nonlocal games with classical question and answer sets.

If a dagger Frobenius algebra is only symmetric, but not special, we will refer to it as a †-SFA (and if commutative, as a †-CFA).

Remark 4.3 (Structure of quantum sets)**.** As laid out in [HV19, Corollary 5.34], all quantum sets are specially weighted direct sums of complex matrix algebras. More precisely, we mean that the underlying set of a quantum set A is a finite direct sum of full matrix algebras $\bigoplus_k M_{n_k}(\mathbb{C})$. The multiplication m is given by the usual matrix multiplication, but divided by $\sqrt{n_k}$ on each block, so that given $a, b \in A$ with $a = \bigoplus_k a_k, b = \bigoplus_k b_k$,

$$m(a \otimes b) = \bigoplus_k \frac{1}{\sqrt{n_k}} a_k b_k.$$

The unit u is given by the usual identity matrix, but scaled in each block by $\sqrt{n_k}$, so that

$$u(1) = \bigoplus_k \sqrt{n_k} I_{n_k}.$$

Taking the dagger to be the Hermitian adjoint (conjugate-transpose) operator, the compatible Hilbert space inner product that arises is $\langle a, b \rangle = \mathrm{Tr}(ab^\dagger)$. Under this inner product, the standard matrix-unit basis $\{E_{ij}^k : 1 \leq i, j \leq n_k\}$ is orthonormal, where $E_{ij}^k \in A$ has a 1 in the (i, j) entry of block k and 0s elsewhere. The comultiplication is then given on this basis by

$$\Delta(E_{ij}^k) = \frac{1}{\sqrt{n_k}} \sum_t E_{it}^k \otimes E_{tj}^k$$

and the counit is given by

$$\varepsilon(a) = \sum_k \sqrt{n_k} \mathrm{Tr}(a_k).$$

Remark 4.4 (Normalization choices)**.** If specialness is not required, we are free to adjust the normalization coefficients α_k of the unit $u(1) = \bigoplus_k \alpha_k I_{n_k}$. (Normalization coefficients of the multiplication follow from unitality.) Different choices of α_k affect the scalar that pops out most naturally in graphical calculations, given by εu.

$\alpha_k = \sqrt{n_k}$: This choice of normalization ensures specialness. This can be interpreted as a counting normalization, in the sense that εu evaluates to the dimension of the underlying vector space. (In the commutative case of the algebra of functions on a finite set S, we have $\varepsilon u = |S|$.)

$\alpha_k = \frac{1}{\sqrt{n_k}}$: In the commutative case, this is the same as above, because each $n_k = 1$, but generally this violates the specialness condition so does not result in a quantum set. Under this choice of normalization, εu counts the number of matrix blocks. This is a second type of counting normalization.

$\alpha_k = 1$: This is the naive normalization condition, and also does not generally result in a quantum set. In this case, εu evaluates to $\sum_k n_k$, which gives the dimension of the Hilbert space naturally acted on by the algebra.

Dividing by $\sqrt{K}$: Three more natural choices come by dividing the above α_k by a factor of $\sqrt{K}$, where K is the number of matrix blocks in the decomposition. The first results in εu giving the average dimension of a matrix block, and the second results in $\varepsilon u = 1$. This second choice, with $\alpha_k = \frac{1}{\sqrt{K n_k}}$, is the most suitable for a quantisation of a discrete probability space. The third results in εu giving the average dimension of the Hilbert space acted on by a block.

The structure result of Remark 4.3 illustrates a sliding-scale of "quantumness". When all matrix algebras are size $n_k = 1$, a quantum set is a finite direct sum of copies of $\mathbb{C}$, and the above multiplication becomes commutative. We call the resulting quantum set *classical*. On the other end of the spectrum, when there is only one direct summand, so the quantum set is a full matrix algebra, we call it *maximally quantum*.

Example 4.5 (Classical set)**.** Because we will refer to classical sets often, let's specify that we will often implicitly identify the set element $i \in [n]$ with the standard basis vector $|i\rangle \in \mathbb{C}^n$. The operations in Remark 4.3 become

$$\begin{aligned} m(|i\rangle \otimes |j\rangle) &= \delta_{ij} |i\rangle , \\ u(1) &= \sum_i |i\rangle , \\ \Delta(|i\rangle) &= |i\rangle \otimes |i\rangle , \\ \varepsilon(|i\rangle) &= 1. \end{aligned}$$

Here multiplication can be thought of as *merging* of classical elements, the unit is (an unnormalized copy of) the maximally mixed state, comultiplication can be thought of as *copying* classical elements, and the counit has a *deletion* effect on classical elements.

Classical sets are also commonly identified with diagonal matrices under $|i\rangle \mapsto E_{ii}$. In that case the above operations give us back matrix multiplication (or the Schur product, as they coincide here).

Example 4.6 (Maximally quantum set)**.** In the case that the quantum set is a full matrix algebra M_n, the operations of Remark 4.3 become

$$\begin{aligned} m(a \otimes b) &= \frac{1}{\sqrt{n}} ab, \\ u(1) &= \sqrt{n} I_n, \\ \Delta(E_{ij}) &= \frac{1}{\sqrt{n}} \sum_t E_{it} \otimes E_{tj}, \\ \varepsilon(a) &= \sqrt{n} \mathrm{Tr}(a). \end{aligned}$$

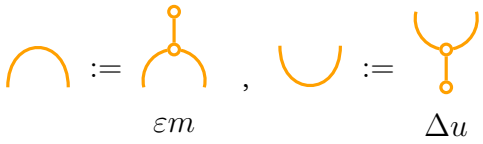
It is convenient and elegant to use a graphical calculus to work with Frobenius algebras. Symbolic equations only possess one dimension of latitude in which to perform two types of composition: tensor products, which in physics represent composing across space, and composition of maps, physically representing composing in time. This results in the eyes jumping back and forth matching n^{th} tensor factors when performing compositions. Using a two dimensional graphical calculus allows us to depict tensor products horizontally and composition of maps vertically, freeing the brain from this extra matching step.

From this point onwards, equations will be presented diagrammatically. Using the graphical calculus given in [HV19], we encode Frobenius algebras as follows. The diagrams are read from bottom to top[2], as processes evolving in time, and left to right for tensor products. Each string or wire represents the Frobenius structure X, and the structural maps are represented by nodes with the appropriate number of X inputs and outputs. Empty space represents $\mathbb{C}$ (the tensor unit of **FHilb**).

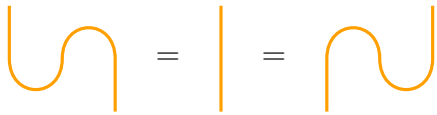


We will often omit the small circles for m and Δ, as the multiplication or comultiplication can be inferred just from the joining or splitting of strings. For convenience, we will also draw caps and cups to represent the common compositions εm and Δu.

[2] I was recently told by Andre Kornell that some call this the optimistic convention.



These cups and caps define a duality (in the sense of monoidal categories) of X to itself, resulting in the "snake equations". This is the chosen duality for quantum sets in our work.



Given quantum sets X, A if $f : X \to A$ is a linear map, then we represent f as a node transforming an X-wire to an A-wire.



We now explain the canonical ways a linear map f can be flipped and rotated. These correspond to the duals and conjugates that arise in dagger symmetric monoidal categories. For an explanation of duals and conjugates at the categorical level, see [HV19]. In the following it suffices to understand the relationship between duals and conjugates and how to manipulate them graphically. For the interested reader, we also provide some computational details.

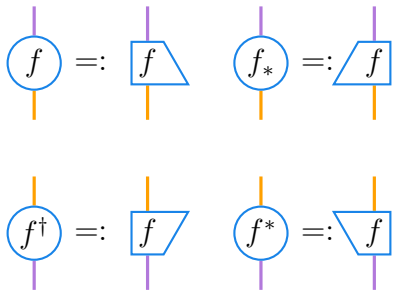
The dagger of the map f is denoted $f^\dagger$. The categorical dual (transpose) of the map f is denoted f^*. The categorical conjugate of f, denoted f_*, is defined as the composition of dagger and transpose (which can be shown to commute).

Taking the dagger of a diagram is done by flipping the whole diagram upside down, and applying daggers to any maps, which corresponds in **FHilb** to the Hilbert space adjoint operation.

When working with an entire diagram and taking conjugates and transposes, a horizontal flip of each element of a diagram corresponds to its categorical conjugate, and a half-turn (rotation by π or[3] $-\pi$) of the diagram while keeping inputs and outputs fixed corresponds to its transpose.

In accordance with the above graphical rule, it is sometimes convenient, when working with daggers, transposes, and conjugates, to use the following trapezium notation for maps.

[3]They are equivalent by symmetry of **FHilb** as a monoidal category.



For the reader interested in doing calculations, consider $f : X \to A$ where X and A are quantum sets that decompose into blocks with matrix unit bases $\{x_{ij}^s\}$ and $\{a_{kl}^t\}$. From the fact that the dagger is the Hilbert space adjoint and then from the way self-duality interacts with matrix units, we can deduce that

$$\langle f x_{ij}^s, a_{kl}^t \rangle_A = \langle x_{ij}^s, f^\dagger a_{kl}^t \rangle_X = \langle a_{lk}^t, f_* x_{ji}^s \rangle_A = \langle f^* a_{lk}^t, x_{ji}^s \rangle_X$$

In particular, for classical quantum sets, these equations tell us that when f is represented as a matrix in the distinguished orthonormal bases, the dagger is the Hermitian adjoint, the conjugate is the entrywise complex conjugate, and the dual is the matrix transpose.

So far we have only discussed the (self-dual) quantum set setting. In general, if there is no quantum set structure present, i.e., if $f : \mathcal{H} \to \mathcal{K}$, then letting $\mathcal{H}^*$ be the Hilbert space dual of $\mathcal{H}$,

$$f^\dagger : \mathcal{K} \to \mathcal{H}, \qquad f_* : \mathcal{H}^* \to \mathcal{K}^*, \qquad f^* : \mathcal{K}^* \to \mathcal{H}^*$$

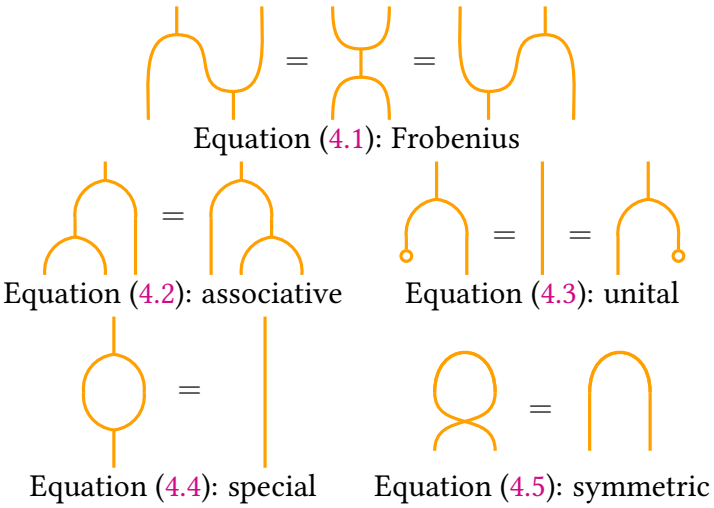
In a Hilbert space without a designated quantum set structure, these transformations, although basis independent, can be computed in any basis $\{|j\rangle\}$ of $\mathcal{H}$ along with its canonical dual basis $\{\langle j|\}$ of $\mathcal{H}^*$ (and similarly $\{|i\rangle\}$ of $\mathcal{K}$ and $\{\langle i|\}$ of $\mathcal{K}^*$). Here,

$$\langle i| f |j\rangle = \overline{\langle j| f^\dagger |i\rangle} = \overline{(f_* \langle j|) |i\rangle} = (f^* \langle i|) |j\rangle .$$

Under the non-canonical linear identification $\mathcal{H}^* \to \mathcal{H}$ extending $\langle i| \mapsto |i\rangle$ (and similarly for $\mathcal{K}$), if we let $[f]_{ij} := \langle i| f |j\rangle$ be the matrix of f in the given bases, we recover the simple computational form that we saw for classical quantum sets, with

$$[f^\dagger] = \overline{[f]}^T, \qquad [f_*] = \overline{[f]}, \qquad [f^*] = [f]^T$$

We can now graphically represent the equations defining quantum sets.



Example 4.7 (Example 4.5 continued)**.** When X is classical, the Frobenius equation shows us three equivalent ways of projecting onto pairs of equal elements. Given a pair of elements (x, y), either copy y and then check one output against x, or check x against y and then copy the output, or copy x and check one output against y. The specialness equation says that copying an element x and then checking it against itself does nothing. The symmetry equation says that checking (x, y) against each other and deleting the result is the same as if the input was (y, x). In fact, in the classical case, we have the stronger property – commutativity. Checking (x, y) against each other is the same as checking (y, x) against each other.

Example 4.8 (Example 4.6 continued)**.** When X is maximally quantum, X can be thought of as $\mathbb{C}^n \otimes (\mathbb{C}^n)^*$, and we can prove the above properties by splitting the X wire into two oriented Hilbert space wires (where the upward arrow depicts a Hilbert space and the downward arrow its canonical dual space), and depicting multiplication and unit as follows:



This is a weighted instance of the infamous graphical *pair of pants algebra*, named for the graphical similarity of the multiplication to the article of clothing. See, for instance, [HV19] for details.

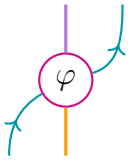
In the general quantum set case, X can be graphically broken down as a direct sum of normalized pair of pants algebras. We will use this in several proofs.

Remark 4.9 (Graphical proofs)**.** The result of all of these diagrammatic definitions is that the algebraic rules for quantum sets allow for somewhat freely manipulating their diagrams in the

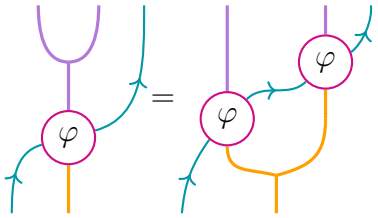
plane. Wires can be bent and passed over nodes. Nodes can slide along wires. The quantum set operations can be rotated inside the plane, keeping their input and output wires in the same relative positions. These are all correct manipulations up to *planar oriented isotopy*, as stated to be valid for †-SSFAs in a pivotal category (which **FHilb** is) in [HV19, Theorem 3.28]. Referencing each algebraic property would make some of our proofs quite arduous, so we will proceed graphically at times.

Now that we have introduced our quantum objects, let us speak of the morphisms between them.

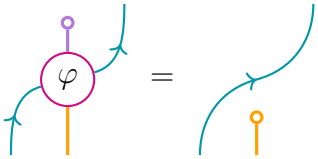
Definition 4.10 (Quantum function, [MRV18])**.** Given quantum sets X, A and a Hilbert space $\mathcal{H}$, a *quantum function*



from X to A over $\mathcal{H}$, denoted $\varphi : X \to_{\mathcal{H}} A$, is a linear map $\mathcal{H} \otimes X \to A \otimes \mathcal{H}$ satisfying



(4.6)



(4.7)



(4.8)

Letting $\mathcal{H} = \mathbb{C}$ in the above definition recovers the definition of a $*$-cohomomorphism from X to A, thought of as tracial C*-algebras. In that case, (4.6) becomes comultiplicativity,

(4.7) becomes counitality or trace preservation, and (4.8) becomes $*$-preservation. Note too that this is equivalent to $\varphi^\dagger$ being a $*$-homomorphism $A \to X$ – a more commonly encountered definition in operator algebras.

Quantum sets as objects with quantum functions as morphisms form a category, as shown in [MRV18]. The Hilbert space wire in the definition of a quantum function can be thought of as a resource space. When X, A are classical, a quantum function is a family of PVMs indexed by X with outcomes in A.

The wire $\mathcal{H}$ could be infinite-dimensional in general, but because several results in this thesis rely on traces, we will stick to finite-dimensional spaces here. We expect our results generalize to $\mathcal{H}$ infinite dimensional at least whenever the dual space $\mathcal{H}^*$ is not involved in the proof. In fact, infinite dimensional Hilbert spaces do not have duals in the dagger categorical sense [HV19]. The Hilbert space dual is only a dagger categorical dual for finite dimensional Hilbert spaces.

4.2 Games, strategies, and correlations

In the following, let X, Y, A, B be quantum sets. We will sometimes use a single wire instead of two to graphically represent the quantum set $X \otimes Y$ and similarly $A \otimes B$. In order to make sense of this, note that $X \otimes Y$ is itself a quantum set when equipped with operations

$$m := (m_X \otimes m_Y)(\mathrm{Id}_X \otimes \sigma \otimes \mathrm{Id}_Y), \tag{4.9}$$

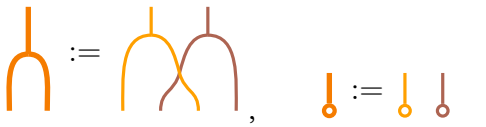
$$u := u_X \otimes u_Y \tag{4.10}$$

and $\Delta := m^\dagger, \varepsilon := u^\dagger$.

Graphically, this looks like using a thick wire to replace the pair of wires $X \otimes Y$



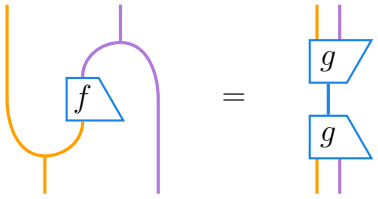
and then defining a new multiplication and unit



Equation (4.9) Equation (4.10)

In order to quantize games and correlations, we will need a diagrammatic definition of complete positivity.

Definition 4.11 (Complete positivity)**.** A linear map $f : X \to A$ for quantum sets X, A is *completely positive* (CP) if there exists a linear map $g : X \otimes A \to Q$ for some other quantum set Q satisfying

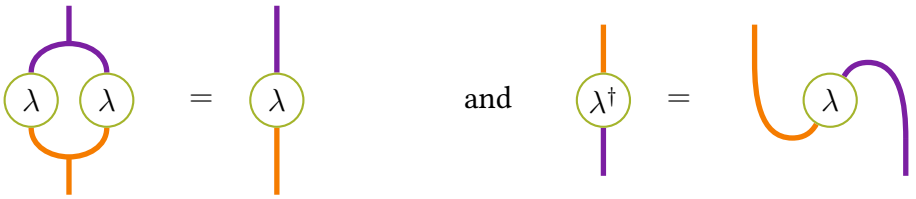


In **FHilb**, this diagrammatic notion agrees with the usual definition of complete positivity. See [HV19, Theorem 7.18] for details.

4.2.1 Games and strategies

We now generalize nonlocal games to allow quantum questions and answers.

Definition 4.12 (Game)**.** A *(quantum) game* is a linear map $\lambda : X \otimes Y \to A \otimes B$ satisfying

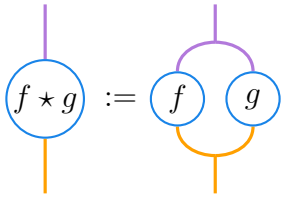


In the remainder of this section, when we refer to the game λ, we mean $\lambda : X \otimes Y \to A \otimes B$.

Remark 4.13 (Games with classical questions and answers)**.** In case X, Y, A, B are commutative, they can be thought of as classical sets[4], and then λ is a linear map $C(X \times Y) \to C(A \times B)$ with idempotent coefficients. This can be thought of as the usual rule function $X \times Y \times A \times B \to \{0, 1\}$, for nonlocal games with classical questions and answers.

In general, given two linear maps between quantum sets, $f, g : X \to A$, we can form their $\star$-product $f \star g$, quantizing the Schur product of matrices, and given by

[4]This is via Gelfand duality, and is discussed in greater detail in [MRV18].

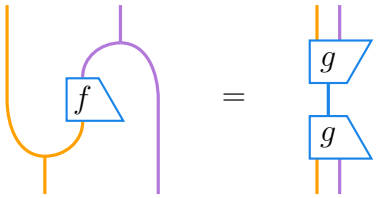


With this in mind, the first condition on games is $\lambda \star \lambda = \lambda$, quantizing the Schur-idempotency condition of a nonlocal game's rule matrix.

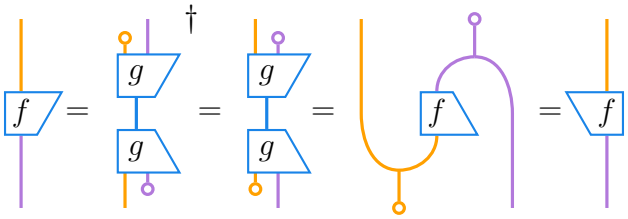
Remark 4.14 (Equivalent characterization of games)**.** The second (self-conjugacy) condition for games is not well-motivated by analogy to the classical setting. In order to motivate it, let us mention an equivalent characterization of games, as $\star$-idempotent CP maps $X \otimes Y \to A \otimes B$. This characterization is equivalent to Definition 4.12, given the following two propositions.

Proposition 4.15. *CP maps between quantum sets are self-conjugate.*

Proof. Let X, A be quantum sets and let $f : X \to A$ be completely positive. By definition, there is some $g : X \otimes A \to Q$ for some other quantum set Q so that f satisfies



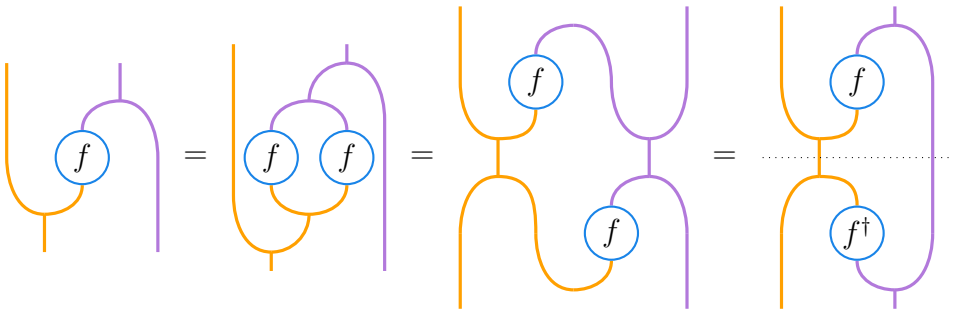
But then we have



□

Proposition 4.16. *$\star$-idempotent, self-conjugate maps between quantum sets are completely positive.*

Proof. Let X, A be quantum sets, and $f : X \to A$ be a linear map that is $\star$-idempotent, in that $f \star f = f$, and be self-conjugate, in that $f^\dagger = f^*$. Then,



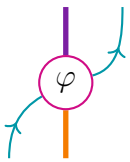
□

An earlier version of the above results, for the case that $X = A$, can be found in [Mat22, Lemmas 2.22, 2.23], in a discussion of the (graphical) definition of a quantum graph.

Remark 4.17 (Games are bipartite quantum relations)**.** Consider linear maps $f : X \otimes Y \to A \otimes B$. Given a game λ, the space of all linear maps f such that $f \star \lambda = f$ is a quantum relation as defined in [Kor20], which are known to be equivalent to the quantum relations of [Wea10]. In this sense, a quantum game is a quantum relation from the question space to the answer space. There are connections to be explored in future work between the following presentation of quantum games, and the point of view of quantum relations.

We now introduce a "container" object that can hold any type of strategy.

Definition 4.18 (Combined strategy)**.** A *combined strategy* for the game λ over a Hilbert space $\mathcal{H}$ is a quantum function $\varphi : X \otimes Y \to_{\mathcal{H}} A \otimes B$, depicted by



The main takeaway here is that all strategies will be thought of as maps from the space of question pairs to the space of answer pairs with a Hilbert space resource.

Example 4.19. With classical questions and answers, a quantum tensor strategy given by PVMs $\{E_{xa}\}_a, \{F_{yb}\}_b$ can be encoded as a linear map $|\psi\rangle \otimes x \otimes y \mapsto \sum_{a,b} a \otimes b \otimes (E_{xa} \otimes F_{yb}) |\psi\rangle$.

The word *combined* is used because it rolls both players' strategies together into a single map. Thus far, we have made no guarantees that a given combined strategy can be implemented

under non-communication requirements on the players. That is because there is no separation of Alice and Bob baked into the definition. Later we will define various types of *valid strategies*, meaning strategies that can be understood to physically model spacetime-separated players with a non-communication requirement. The definition of combined strategy is useful merely because it allows us to talk about all types of valid strategies at once.

4.2.2 States and correlations from channels

When we aren't interested in varying the state of the resource system $\mathcal{H}$ used by the players, we use a correlation instead of a strategy. To define correlations, we first diagrammatically define channels between quantum sets, and to do this we make use of the diagrammatic definition of complete positivity, as well as another diagrammatic condition, counitality.

Definition 4.20 (Counital)**.** A linear map $f : X \to A$ for quantum sets X, A is *counital* if



Counitality implements the physical property causality when the counit is thought of as a discarding map. See [CK17] for details. This condition is called trace-preserving when thinking of quantum sets as C^*-algebras or matrix algebras, although note that the counit will not generally be given by the canonical trace, but rather by a reweighted version. See Remark 4.3.

Definition 4.21 (Channel)**.** Given quantum sets X, A, a *channel* from X to A is a completely positive, counital linear map $f : X \to A$.

Channels here are just the channels in [CHK13], given as normalized morphisms in $\mathbf{CP}^*[\mathbf{FHilb}]$.

Remark 4.22 (Kraus operators)**.** Given a channel f, decompose the map g of Definition 4.11, coming from complete positivity of f, as

$$g = \sum_i |i\rangle \langle i| \, g$$

over a basis $\{i\}$ of Q. Each $K_i := \langle i| \, g$ corresponds to a Kraus operator.

Example 4.23 (Maximally quantum and classical channels)**.** In the maximally quantum case, this notion of channel recovers the usual definition of a quantum channel as a CPTP map between matrix algebras, but with scaled traces.

In the classical case, this definition picks out exactly classical channels, which are probability distributions $p(a|x)$ on the set A conditioned on the set X. In that setting, complete positivity is equivalent to non-negativity of p and counitality is equivalent to $\sum_a p(a|x) = 1$.

Definition 4.24 (Correlation)**.** A (bipartite) *quantum set correlation* is a channel P from $X \otimes Y$ to $A \otimes B$.

Note, that this should not be confused with the typical notion of correlations induced by the quantum strategies of Definition 2.11, but rather is a generalization of correlations (Definition 2.5) to when the question and answer sets are quantum. With that said, in the following we will drop the words *quantum set* and just write *correlation.* Our definition of correlation is the dagger dual of [BKS23, Definition 5.3].

Now that we have channels, we can use them to define states of quantum sets.

Definition 4.25 (State of a quantum set)**.** A *state* of a quantum set X is a channel $\mathbb{C} \to X$.

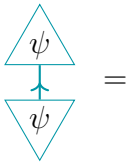
When the quantum set is maximally quantum, so the underlying vector space is some $B(\mathcal{H})$, this definition of state corresponds to the density matrices of standard quantum mechanics.

To relate strategies and correlations, we also need a notion of pure states. We use the normalized version of [HV19, Definition 1.10], as follows.

Definition 4.26 (Pure state)**.** Let $\mathcal{H}$ be a finite-dimensional Hilbert space. A *pure state* of $\mathcal{H}$ is a linear map $|\psi\rangle : \mathbb{C} \to \mathcal{H}$, depicted



that is *normalized* in the sense that



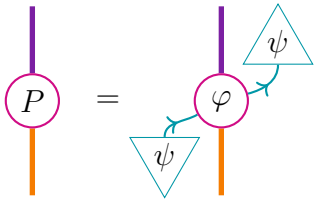
(Recall that the empty diagram above represents $\mathrm{Id}_{\mathbb{C}}$, or scalar multiplication by 1.)

We denote the corresponding *effect* by $\langle\psi| := |\psi\rangle^\dagger$, and depict it



Remark 4.27 (State disambiguation)**.** When we say a state of a quantum set X, we are always referring to Definition 4.25, which corresponds to a mixed state of the Hilbert space that the quantum set naturally acts on, rather than to a pure state of the underlying Hilbert space. We denote both of these types of states by $|\cdot\rangle$, or by triangles pointing downwards in the graphical notation. The way to tell them apart is to determine whether they map to a Hilbert space with no additional structure (an oriented wire) or to a quantum set (an unoriented wire).

Definition 4.28 (Realized by)**.** A correlation P is said to be *realized by* a combined strategy φ on $\mathcal{H}$ and a normalized state $|\psi\rangle \in \mathcal{H}$ when

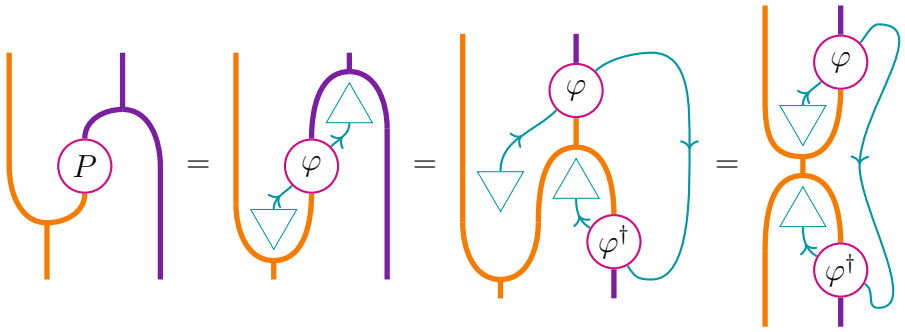


The physical interpretation of a correlation as a channel from question pairs to answer pairs emphasizes the observable “signature" of a strategy, from the referee’s perspective, not having access to the players’ resource state.

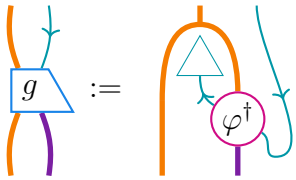
Remark 4.29. A combined strategy φ over a finite-dimensional $\mathcal{H}$ and a normalized state $|\psi\rangle \in \mathcal{H}$ always give rise to some correlation P, as in Definition 4.28.

Proof. Let φ be a combined strategy over a finite-dimensional $\mathcal{H}$ and let $|\psi\rangle \in \mathcal{H}$ be a normalized state. Let P be the expectation of φ with respect to $|\psi\rangle$, as in Definition 4.28. We show that P is a correlation.

P inherits counitality directly from (4.7) for φ, using the fact that $|\psi\rangle$ is normalized. To see that P is completely positive, we proceed diagrammatically, applying (4.6) and (4.8) to φ and bending some wires, to get



Letting



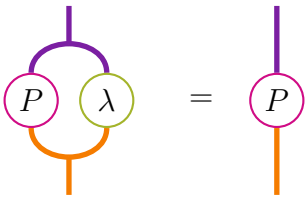
we can see that P satisfies Definition 4.11.

□

4.2.3 Perfect behaviour

In the study of nonlocal games, we are especially interested in correlations that are guaranteed to win a game, also known as perfect correlations. Let us define a perfect correlation in this general setting.

Definition 4.30 (Perfect correlation)**.** A correlation P for the game λ is called *perfect* if



Note that swapping P and λ in the definition is equivalent due to both maps being self-conjugate.

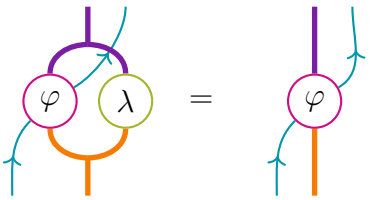
Remark 4.31 (Typical perfect correlations)**.** In the case that questions and answers are classical, this says that the question-answer pairs that occur with nonzero probability according to P are winning question-answer pairs according to λ, which is exactly the typical definition of perfect. Indeed, the above equation becomes

$$p(a,b|x,y)\lambda(a,b|x,y) = p(a,b|x,y)$$

for all $a \in A, b \in B, x \in X, y \in Y$. This is equivalent to saying $\lambda(a,b|x,y) = 0$ implies $p(a,b|x,y) = 0$.

By adding a resource wire to the definition of a perfect correlation, we can define perfect strategies.

Definition 4.32 (Perfect strategy)**.** A (combined) strategy φ for the game λ is called *perfect* if

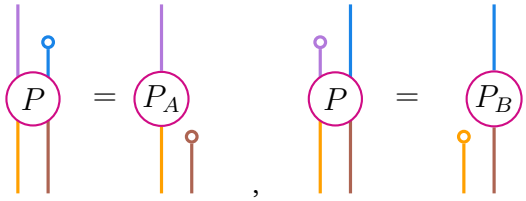


Remark 4.33 (Perfect strategies versus perfect correlations)**.** If a strategy over resource space $\mathcal{H}$ is perfect for λ and realizes a correlation P together with a normalized state in $\mathcal{H}$, then P is perfect for λ. However, the converse implication does not hold. Perfect correlations need not come from perfect strategies. Indeed, being a perfect strategy has much stronger consequences than realizing a perfect correlation. In light of this, it may be more appropriate to think of perfect strategies as *absolutely perfect*.

4.3 Strategy classes

We have the equipment to describe some of the typical strategy classes for nonlocal games (specifically – deterministic, quantum tensor, and quantum commuting strategies), and easily extend them to quantum games. All of these classes fall under the umbrella of *non-signalling strategies* (strategies realizing non-signalling correlations). We will begin with the diagrammatic definition of non-signalling, and work towards the stricter strategy classes.

Definition 4.34 (Non-signalling, marginals)**.** A quantum-set correlation $P : X \otimes Y \to A \otimes B$ is called *non-signalling* if there are linear maps $P_A : X \to A$ and $P_B : Y \to B$ such that

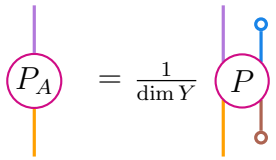


The maps P_A, P_B are called the *marginals* of P.

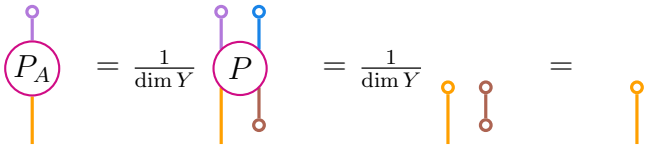
This reduces to the typical definition of non-signalling for classical question and answer sets, as given in Definition 2.10. Non-signalling quantum-set correlations are also referred to as QNS (quantum non-signalling) correlations, and were introduced in [DW16].

Corollary 4.35. *The marginals of a non-signalling correlation P are channels.*

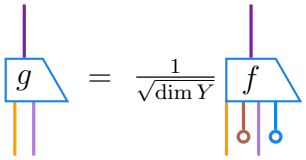
Proof. By definition of P being nonsignalling, we have



We can see that P_A is counital as follows:



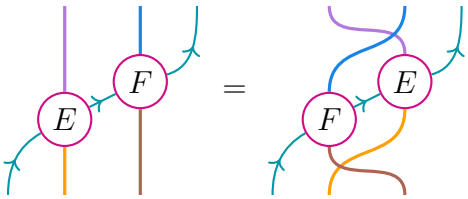
To see complete positivity, let f be a square root of P as given in the definition of complete positivity for P. Then g given as follows is a square root of P_A.



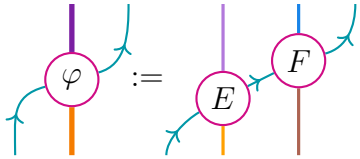
The approach is the same for P_B. □

If we are interested not just in marginals, but in Alice and Bob's implementation of their strategies, we need to look at valid strategy classes. The most general valid strategy class we will deal with here is quantum commuting.

Definition 4.36 (Quantum commuting strategy)**.** A *quantum commuting strategy* for the game λ is a pair of quantum functions (E, F) over a Hilbert space $\mathcal{H}$, given by $E : X \to_{\mathcal{H}} A, F : Y \to_{\mathcal{H}} B$, that commute on $\mathcal{H}$. In diagrams,

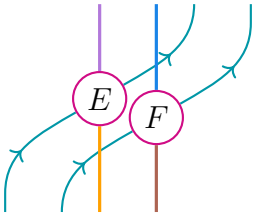


A quantum commuting strategy gives rise to a combined strategy on $\mathcal{H}$ as follows:

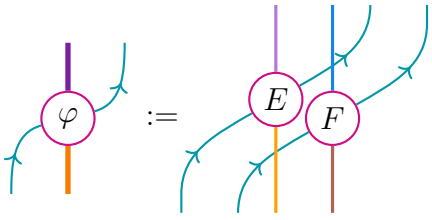


If $\mathcal{H}$ is bipartite with Alice and Bob each measuring their own part of $\mathcal{H}$, we get quantum tensor strategies.

Definition 4.37 (Quantum tensor strategy)**.** A *quantum tensor strategy* for the game λ is a pair of quantum functions (E, F), given by $E : X \to_{\mathcal{H}_A} A, F : Y \to_{\mathcal{H}_B} B$. In diagrams,



A quantum tensor strategy gives rise to a combined strategy on $\mathcal{H} := \mathcal{H}_A \otimes \mathcal{H}_B$ as follows:



Note that a quantum tensor strategy is the composition defined in [MRV18] of the quantum functions $(E \otimes I_B) \circ (I_A \otimes F)$.

Definition 4.38 (Deterministic strategy)**.** A *deterministic strategy* for the game λ is a quantum tensor strategy with classical resource spaces, i.e. $\mathcal{H}_A = \mathcal{H}_B = \mathbb{C}$.

Proposition 4.39. *Deterministic strategies are quantum tensor. Quantum tensor strategies are quantum commuting. Quantum commuting strategies realize non-signalling correlations.*

Proof. Deterministic strategies are quantum tensor by definition.

A quantum tensor strategy can be thought of as quantum commuting by setting $\mathcal{H} := \mathcal{H}_A \otimes \mathcal{H}_B$ and extending the resource space of the given E, F to act trivially on the newly accessible half of $\mathcal{H}$.

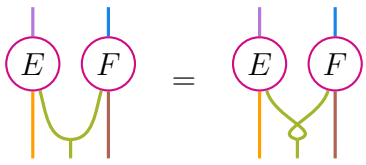
A quantum commuting strategy realizes a non-signalling correlation due to counitality of E, F. □

Remark 4.40 (Beyond quantum functions)**.** Quantum functions quantize the classical question-and-answer game strategies given by families of PVMs. Seemingly more generally, Alice and Bob may use families of POVMs, but this is made unnecessary, via Naimark's dilation theorem (a consequence of Stinespring's theorem), by working in a larger resource space. For reference, see [Pau16, Proposition 9.6 and Theorem 9.8].

In the quantum question-and-answer setting, there is a similar result to Naimark's dilation thorem presented in [BKS23].

There is a graphically natural approach to generalizing strategies in the quantum question-and-answer case as well, replacing quantum functions with channels, which we describe below for the quantum commuting model, but do not refer to in the rest of this work. It remains to be shown if these realize a larger class of correlations, or if they can always be achieved by quantum functions.

We can extend the class of quantum commuting strategies by generalizing the shared resource space $\mathcal{H}$ to be a quantum set M, and by allowing the players' quantum functions to be replaced by channels. Specifically, we require that Alice uses a channel $E : X \otimes M \to A$ and Bob uses a channel $F : M \otimes Y \to B$. Together, (E, F) is an *extended quantum commuting strategy* provided that



Our existing notion of quantum commuting strategies fit into this extended model as follows. Take a quantum commuting strategy, (E, F) over some Hilbert space $\mathcal{H}$. Let $M := B(\mathcal{H})$ as a Hilbert space, equipped with the maximal quantum set structure, and define the corresponding extended strategy (E', F') by

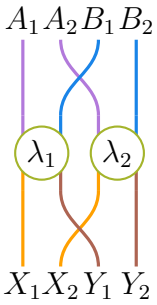


Then, the condition that E and F commute on $\mathcal{H}$ is equivalent to the above equation describing that (E', F') form an extended quantum commuting strategy.

4.4 Products of games

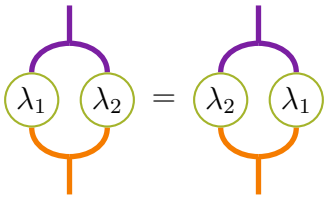
We mention two products of games – their tensor product, and their commuting product, and we relate perfect strategies for products of games to perfect strategies for the individual games.

Definition 4.41 (Tensor product of games)**.** The *tensor product* of games λ_1, λ_2 is denoted $\lambda_1 \otimes \lambda_2$ and given by



Some wires are crossed in the above definition so that the new game keeps Alice's questions and answers separate from Bob's. Note that the tensor product of games is a game. The tensor product (with the same wires crossed) of perfect strategies or correlations for each game is perfect for the product.

Definition 4.42 (Commuting product of games)**.** Given games λ_1, λ_2 with the same question and answer sets satisfying



we call the above product the *commuting product* of games. We denote this product $\lambda_1 \star \lambda_2$.

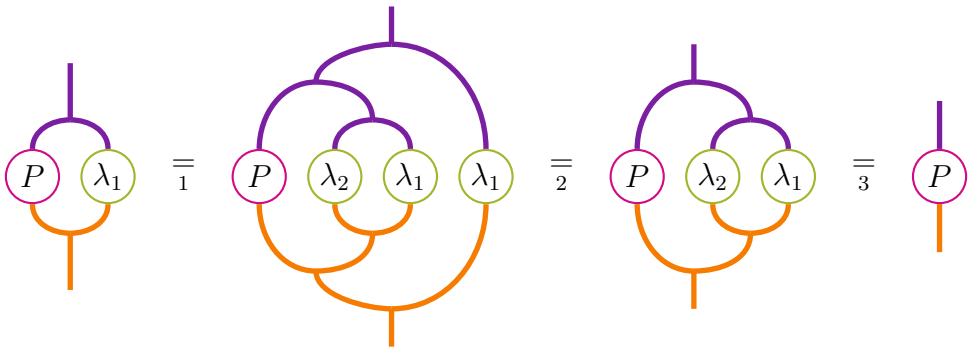
Note that the commuting product of games is a game. Note also that in the classical question and answer case, this is just the Schur product of the rule matrices.

If the commuting product of games is well-defined, we say we have *commuting games*.

Proposition 4.43. *Let λ_1, λ_2 be commuting games $X \otimes Y \to A \otimes B$. Then φ be a perfect strategy for both λ_1, λ_2 if and only if φ is a perfect strategy for $\lambda_1 \star \lambda_2$. Analogously, P is a perfect correlation for both λ_1, λ_2 if and only if P is a perfect correlation for $\lambda_1 \star \lambda_2$.*

Proof. We present the proof for a correlation P. To extend it to strategies, add a Hilbert space wire to P to get a strategy φ, and use the same graphical manipulations.

Assume P is a perfect correlation for $\lambda_1 \star \lambda_2$. We proceed graphically to show P is perfect for λ_1.



The equations are justified as follows:

1. P is perfect for $\lambda_1 \star \lambda_2$,
2. associativity of $A \otimes B$, coassociativity of $X \otimes Y$, and Schur-idempotency of λ_1, and
3. P is perfect for $\lambda_2 \star \lambda_1$.

By symmetry, P is also perfect for λ_2.

Conversely, if P is perfect for each game, then iteratively apply the definition of a perfect correlation to immediately see that P is perfect for their $\star$-product. □

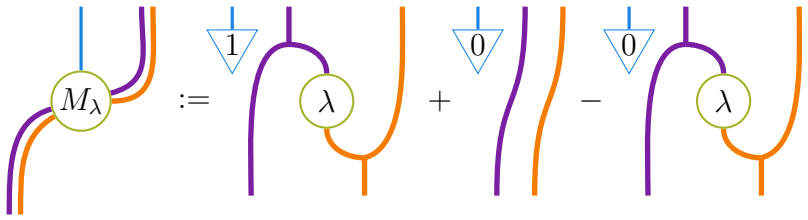
The above proposition supports viewing the commuting product of games as a logical AND operation.

4.5 Towards playing games

In this section, we attempt to connect our formalism of a quantum game λ to a physical protocol representing playing the game with correlation P and question state $|\gamma\rangle$ of $X \otimes Y$. With that said, we are missing a physical interpretation of the comultiplication.

Remark 4.44 (Games as binary measurements)**.** We use λ to define a quantum function, M_λ, capturing the rules of the game. In fact, this is a quantum function from $\mathbb{C}$ to a classical quantum set, making it just a PVM with two outcomes. This PVM models a binary measurement of the question and answer spaces, measuring 1 for correct answers and 0 for incorrect answers. This can be defined graphically by assigning to outcome 1 the projection onto the winning subspace of the question and answer space, and by assigning to outcome 0 the complementary projection.

Let $\mathbf{2}$ be the classical 2-element quantum set with distinguished orthonormal basis $\{|0\rangle, |1\rangle\}$. Define $M_\lambda : \mathbb{C} \to_{A\otimes B\otimes X\otimes Y} \mathbf{2}$ by



Using the fact that λ is a self-conjugate $\star$-idempotent, M_λ can be shown graphically to be a quantum function by some straightforward computations. In keeping with [MRV18, Remark 3.5], M_λ can be described as a quantum element of a bit.

Note that we could have alternatively defined games as binary projective measurements, given that λ can be recovered from M_λ by applying a unit to the input answer wire, a counit to the output question wire, and the effect $\langle 1|$ to the $\mathbf{2}$ wire.

We now describe the proposed protocol followed by a referee and two players Alice and Bob, who play the quantum game λ with correlation P and question state $|\gamma\rangle$ of $X \otimes Y$. Keep in mind that the comultiplication is not, a priori, physical.

1. The referee prepares the question state $|\gamma\rangle$ of $X \otimes Y$, and then applies the comultiplication to prepare the state $\Delta_{X\otimes Y} |\gamma\rangle$ of $X_1 \otimes Y_1 \otimes X_2 \otimes Y_2$ (where the subscripts are just used to distinguish instances of each quantum set).

2. The referee sends X_1 to Alice and sends Y_1 to Bob, retaining $X_2 \otimes Y_2$.

3. Alice and Bob apply the quantum channel P to $X_1 \otimes Y_1$ to get a state of $A \otimes B$, which they send back to the referee.

4. The referee then performs the binary measurement described by M_λ on the system $A \otimes B \otimes X_2 \otimes Y_2$, resulting in a bit. If the resulting bit is $|0\rangle$, this corresponds to a loss, while $|1\rangle$ corresponds to a win.

5. Optionally, the referee discards the $A \otimes B \otimes X \otimes Y$ system.

We can draw this protocol graphically as in Figure 4.1, splitting the diagram vertically into the steps above performed from bottom to top, and horizontally into the players' side on the left and the referee's side on the right.

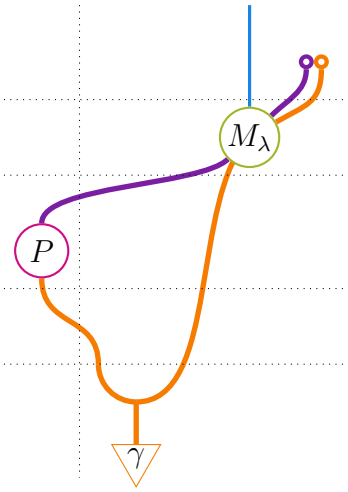


Figure 4.1: Spacetime demarcated protocol for playing a quantum game.

In other words, given a quantum game $\lambda : X \otimes Y \to A \otimes B$, a correlation $P : X \otimes Y \to A \otimes B$, and a question state $|\gamma\rangle$ of $X \otimes Y$, Figure 4.1 shows how to construct a bit whose probability of evaluating to 1 represents the probability of Alice and Bob winning the game as played.

Remark 4.45 (State preparation)**.** Other than the preparation of $\Delta_{X\otimes Y} |\gamma\rangle$, the remainder of the protocol has a physical interpretation (a channel, a binary projective measurement, and sending systems). The comultiplication can be considered as a quantum element of X over itself (that is, a quantum function $\mathbb{C} \to_X X$), but this is not enough. We leave a physical interpretation of quantum functions for future discussion.

Other approaches to quantum question-and-answer games avoid this issue. In Definition 4.52, we will see that the authors of [LTW08] allow the referee to prepare any tripartite state between the referees and the players, perhaps overgeneralizing the classical question scenario, where the question the referee tests the answers against is related to (in fact, is a copy of) the question the players use to determine their answers.

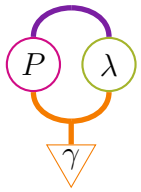
Remark 4.46. This (mostly) operational interpretation of quantum games may shed some light on the correct approach to a formal composition of quantum games. Note that the question and answer spaces, once measured, need not be discarded. This is a direction for future work.

4.6 Values of quantum correlations and quantum games

We now define the value of a correlation and of a game.

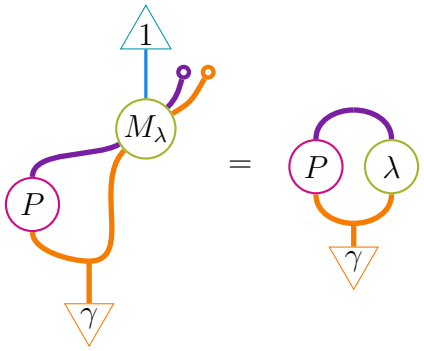
Definition 4.47 (Value of a correlation)**.** The *value* $\Omega(\lambda, P, \gamma)$ of a correlation P for the game λ played with question state $|\gamma\rangle$ is given by

$\Omega(\lambda, P, \gamma) \;=$

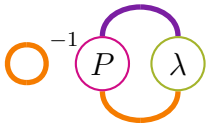


If the question and answer sets are classical, then γ defines a probability distribution on question pairs, and this definition recovers Definition 2.15.

Remark 4.48 (Value as winning probability)**.** Operationally, the value can be interpreted as the probability of the players winning the game λ with correlation P and question state γ. This probability can be computed by pairing the game protocol bit state with the effect $\langle 1|$ (measuring a 1).



Example 4.49. A natural choice for $|\gamma\rangle$ is the normalized unit of $X \otimes Y$ (which gives a uniform question distribution in the classical case), simplifying the value diagram even further to be:



Similarly to the case of nonlocal games as in Definition 2.16, we can define the t-value of a game for a strategy class t by taking the supremum over correlations from that class.

Recall the quantum set correlation classes defined in Section 4.3. Let Q_t denote the class of quantum set correlations for $t \in \{\mathrm{det}, \mathrm{q}, \mathrm{qc}, \mathrm{ns}\}$.

Definition 4.50 (t-Value of a game)**.** The *t-value* of a game λ for $t \in \{\mathrm{loc}, \mathrm{q}, \mathrm{qc}, \mathrm{ns}\}$ with question state $|\gamma\rangle$ is

$$\Omega_t(\lambda, \gamma) := \sup_{P \in Q_t} \Omega(\lambda, P, \gamma).$$

We can immediately conclude the following.

Corollary 4.51 (Corollary of Proposition 4.39)**.** $Q_{\mathrm{det}} \subseteq Q_{\mathrm{q}} \subseteq Q_{\mathrm{qc}} \subseteq Q_{\mathrm{ns}}$ *and therefore* $\Omega_{\mathrm{det}} \leq \Omega_{\mathrm{q}} \leq \Omega_{\mathrm{qc}} \leq \Omega_{\mathrm{ns}}$.

4.7 Other quantum question-and-answer games

In a somewhat speculative digression, we now discuss other authors' formulations of extensions of nonlocal games with quantum questions and answers. We begin with general approaches to quantum question-and-answer games. There is still some work to be done to explicitly connect all of these constructions to the quantum games defined above, but we give a few hints at how this may work. We then mention several papers where the authors extended specific nonlocal games to allow the questions to be quantum states.

General approaches to quantum question-and-answer games

In [LTW08, Definition 3.1], the authors present an operational definition of a quantum question-and-answer game as a protocol. We restate their definition here to compare it to ours.

Definition 4.52 (Two-player one-round quantum game)**.** Let $\mathsf{R}, \mathsf{S}, \mathsf{T}, \mathsf{A}, \mathsf{B}$ be finite-dimensional quantum registers, let $|\rho\rangle$ be a quantum state of the registers $(\mathsf{R}, \mathsf{S}, \mathsf{T})$, and let $\mathcal{M}$ be a binary measurement on the registers $(\mathsf{R}, \mathsf{A}, \mathsf{B})$.

1. The referee prepares $(\mathsf{R}, \mathsf{S}, \mathsf{T})$ in the state $|\rho\rangle$, and then sends S to Alice and T to Bob.
2. Alice and Bob transform the registers S and T sent to them however they choose, resulting in registers A and B that are sent back to the referee.
3. The referee applies the binary measurement $\mathcal{M}$ on the registers $(\mathsf{R}, \mathsf{A}, \mathsf{B})$. The outcome 1 means that Alice and Bob win, while the outcome 0 means that they lose.

Usually quantum registers are understood to hold only pure states. In that case, this is a somewhat restrictive definition of a quantum game. If the registers are understood to hold mixed states, however, we have the following result.

Proposition 4.53. *Quantum games (in our sense) can be realized as a mixed-states generalization of two-player one-round quantum games, in the sense of [LTW08].*

Proof. Beginning with a quantum game $\lambda : X \otimes Y \to A \otimes B$, define a two-player one-round quantum game as follows. Forgetting the quantum set structure and treating quantum sets as finite dimensional Hilbert spaces, take $\mathsf{S} := X$, $\mathsf{T} := Y$, and $\mathsf{R} := X \otimes Y$. To evaluate the game λ, the players must apply their strategies to a question state $|\gamma\rangle$ of $X \otimes Y$. Have the referee prepare $|\rho\rangle := \Delta_{X\otimes Y} |\gamma\rangle$. Then, potentially using another shared resource state $|\psi\rangle$ as well as their parts of $|\rho\rangle$, Alice and Bob prepare an answer state of $A \otimes B$. Finally, the referee applies the binary measurement $\mathcal{M} := M_\lambda$, as defined in Section 4.5.

One can check graphically that the value of a correlation P for the quantum game λ with question state $|\gamma\rangle$ is equal to the probability of measuring a win (the outcome of $\mathcal{M}$ labelled 1) when Alice and Bob use a strategy corresponding to P in the described two-player one-round game. □

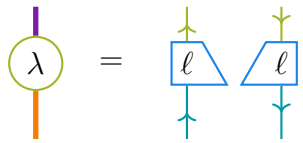
Remark 4.54. There are a couple ways in which [LTW08] have a more general definition (when allowing for mixed states) of two player one round quantum question-and-answer games than given here.

1. Their referee's register R is not required to be a second copy of registers (X, Y).

2. The tripartite question state $|\rho\rangle$ is not required to have any particular structure, whereas here it must be in the image of $\Delta_{X\otimes Y}$.

It may, however, be the case that a referee can essentially be no more powerful with an arbitrary register, or with an arbitrary state of that register. This is a question for future work.

Another general class of quantum question-and-answer games are *rank-one quantum games*, introduced in [Coo+11]. The definition is very similar to the two-player one-round quantum games of Definition 4.52. The term *rank-one* refers to the rank of the projection associated to the winning outcome in the binary measurement applied by the referee for rule-checking. Projections of rank one arise naturally when working only with pure states. A subclass of our quantum games likely connect to rank-one games by restricting to a pure question state $|\gamma\rangle$, to maximally quantum question and answer sets, and to λ that is rank one in the sense that it can be written as



for some ℓ in $B(\mathcal{H}, \mathcal{K})$ with $X \otimes Y \simeq B(\mathcal{H})$ and $A \otimes B \simeq B(\mathcal{K})$. In order to define a game, ℓ must be a pure state of $\mathcal{K} \otimes \mathcal{H}^*$ under the canonical identification.

Removing this constraint on λ gives the games introduced later in [Cra+23] called *finite-rank projection games*.

Yet another approach to quantum question-and-answer games, also given in [Cra+23], are *hypergraph quantum games*. These games are defined as probabilistic quantum hypergraphs, which encode a probability measure on (pure) question states along with a map from question states (so pure states of the Hilbert space acted on by our quantum question set $X \otimes Y$) to the projection lattice of $A \otimes B$. Hypergraph games are also discussed in [HT22; HT25]. This approach seems likely to connect well to our notion of quantum games.

A somewhat distinct approach to quantum games is taken in [TT24] where the authors define quantum non-local games to be join continuous, 0-preserving maps between projection lattices of matrix algebras corresponding to $X \otimes Y$ and $A \otimes B$ in the maximally quantum case. These games are further studied in [Bra+23; Bra+24]. There are likely connections of this approach to our quantum games. This is a direction for future work.

Partial extensions of nonlocal games to quantum question-and-answer games

An extension of nonlocal games to only allow quantum questions are the *semi-quantum games* of [Bus11]. They can be thought of as two-player one-round quantum games where the referee's register R and the answer registers A, B are classical, and Alice and Bob's question state $\mathrm{Tr}_{\mathsf{R}} |\rho\rangle$ is a separable state of (S, T).

Another partial extension are the *extended nonlocal games* of [Joh+16], generalizing two different nonlocal game extensions introduced in [Fri12; Tom+13]. These games have classical questions and answers allow the players to send a quantum system to the referee (that may be correlated to their resource state) and the value of the game is then determined by an inner product of the final state on the referee's system (after the players have done their measurements) with an observable chosen depending on the questions and answers.

Specific nonlocal games extended to have quantum questions

Classical XOR games are nonlocal games where the answer sets are binary, that is, $A = B = \{0, 1\}$, and the rule function has a particular format: The players win exactly when the binary XOR of their answers ($a \oplus b$) is equal to some binary function of their questions. Note that the CHSH game is an XOR game. In [RV15], the authors extend these games to quantum XOR games, allowing the question pair to be replaced with one of finitely many bipartite quantum states, selected at random.

The classical graph homomorphism game, described previously in 2.3.3, was extended by [BGH22] to a game requiring homomorphisms from a quantum graph to a classical graph, with the quantum question pairs corresponding to states in the Hilbert space corresponding to the operator system of a quantum graph.

To conclude this section, there is further work to be done to explicitly connect our quantum games to the other definitions discussed here, and to carefully consider the generalisation from pure states to mixed states. It seems possible that, with some further work and a possible extension to an arbitrary quantum set belonging to the referee (rather than a copy of $X \otimes Y$), the diagrammatic setting may be powerful enough to unify existing approaches to general quantum question-and-answer games, at least in the finite dimensional setting.

5 Synchronicity for quantum games

> Enthusiastic about Pauli's suggestions, Jung responded with the bold proposal of generalizing synchronicity to include acausal relationships without a mental component - that is, purely physical interactions. He did not specify quantum entanglement, but surely that fell into Jung's expanded definition. - Paul Halpern

This chapter focuses on synchronicity for quantum games, one of the central definitions of this thesis, and lays out its properties and consequences. Synchronicity for quantum games generalizes synchronicity for nonlocal games (Section 2.3). In Section 5.1, we define synchronicity for quantum games using a graphical approach. In Section 5.2, we study perfect strategies for synchronous quantum games and see that they are forced to be synchronous, then we see our first examples of synchronous quantum games, and we study the symmetric structure of synchronous correlations and strategies. As an aside, we also introduce the notion of *classical dimension* of a quantum set, which is a graphical definition that is closely linked to synchronicity. Then, in Section 5.3, we study the additional structure entailed by *bisynchronicity*. Finally, in Section 5.4, we discuss other existing generalizations of synchronicity from nonlocal to quantum games.

Remark 5.1 (The naming of things)**.** Upon finalizing this work, it is apparent that what we call here *perfect strategies* should better have been called something like *universally perfect* or *absolutely perfect* strategies, as they realize perfect correlations with respect to *any* state. This would allow the term *perfect strategies* to be used for strategies that simply realize a perfect correlation with respect to *some* state. The same applies to what we call *synchronous* strategies, which would likely better be called something like *universally* or *absolutely synchronous*. With this change in nomenclature, some of the following results for synchronous quantum games would connect more naturally to the nonlocal games literature. See Remark 4.33.

5.1 Diagrammatic approach to synchronicity

To extend synchronicity for nonlocal games to quantum games, we would like the players to use the same question and answer spaces as each other. It turns out that the classical question-and-answer results extend more elegantly and directly if the multiplication structure on the question and answer spaces is mirrored. This means that if Alice has questions in the quantum set X, then Bob's question set should be X but with the opposite algebra structure.

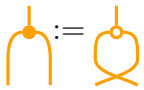
We start by defining the opposite of a quantum set, and then studying the Frobenius map, which we will also refer to as *sharing*. We then use sharing to define synchronicity, the central property of this chapter. We justify our terminology by considering the classical question and answer case.

5.1.1 Opposite of a quantum set

Definition 5.2 (Opposite of a quantum set)**.** Given a quantum set X, its *opposite*, denoted X^{op}, is another quantum set with the same underlying set X, the same unit as X,



and multiplication given by $m\sigma$,



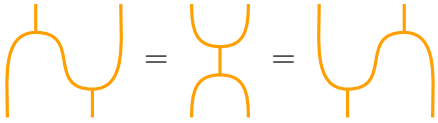
In the following, we always take $Y = X^{\mathrm{op}}$, and $B = A^{\mathrm{op}}$, with filled nodes corresponding to the opposite multiplication.

There is no need to distinguish the cups and caps that come from the original quantum set or its opposite, as (4.5) shows us that they are equal. Under the duality coming from the quantum set structure of $X \otimes X^{\mathrm{op}}$, we can check that the transpose of the filled multiplication is the empty comultiplication and vice versa. Equivalently, the filled nodes become the empty nodes under conjugation and vice versa.

5.1.2 Sharing

We now consider a *sharing* map, defined as follows. This map features in the definition of synchronicity, and will also allow us to introduce the notion of classical dimension.

Definition 5.3 (Sharing)**.** The *sharing map* of a quantum set X is the Frobenius map of (4.1) given by



It may seem unnecessary to give a new name to the Frobenius map, but it makes some of our statements more operationally accessible. It also leaves room for different sharing maps to be used in the definition of synchronicity – for instance, we could introduce a twist into the sharing map, and define synchronicity that way, with many of the same properties. We will elaborate on this in future work.

In the classical case, the sharing map is the projection onto the diagonal (or "shared") questions $|x\rangle \otimes |x\rangle$, which is the reason for calling it sharing.

We will often work with the thick wire $X \otimes X^{\mathrm{op}}$ (and $A \otimes A^{\mathrm{op}}$). It will be convenient to have a shorthand notation for sharing, using thick wires.



We take note of some straightforward but useful properties of the sharing map.

Lemma 5.4 (Properties of sharing)**.** *Consider the sharing map of X.*

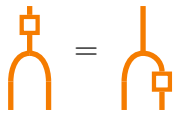
(a) Sharing is self-adjoint.

(b) Sharing is a projection.

(c) Sharing the unit produces the cup state.



(d) Sharing commutes with the right leg of multiplication.



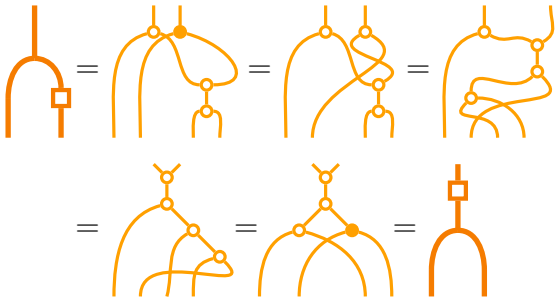
Proof. (a) Self-adjointness follows from applying a dagger to the Frobenius equations (4.1), or graphically by noticing their vertical symmetry.

(b) Composing the sharing map with itself and then applying specialness of the quantum set X returns the sharing map.

(c) Let's see that sharing the unit produces the cup state.

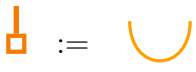


(d) Using thin wires, and repeated applications of the Frobenius equation and associativity, we have



□

Property (c) of sharing motivates one more piece of shorthand notation, allowing us to represent the cups and caps of X using only thick wires. We visually retract the unit in property (c) to define

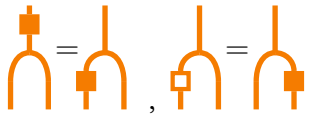


We denote the conjugate of sharing, with respect to the duality given by the thick-wire cups and caps of $X \otimes X^{\mathrm{op}}$, by the filled square,



This is also the transpose of sharing, due to self-adjointness, and is also sharing in the opposite quantum set X^{op}.

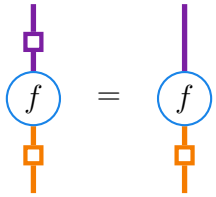
Property (d) of sharing is equivalent to two other equations relating sharing, the conjugate of sharing, and multiplication. The equivalence comes from bending wires.



Another way to approach the relationship between sharing and multiplication, is that if we think of $X \otimes X^{\mathrm{op}}$ as a left-module over itself, then sharing is a left-module endomorphism. Similarly, opposite sharing is a right-module endomorphism.

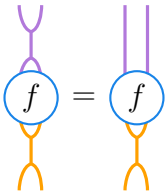
5.1.3 Synchronicity

Definition 5.5 (Synchronous map)**.** We say a linear map $f : X \otimes X^{\mathrm{op}} \to A \otimes A^{\mathrm{op}}$ is *synchronous* if



The above definition, in casual terms, says a map is synchronous if "shared inputs lead to shared outputs". If the linear map is a game or correlation, we get the notion *synchronous game*, or *synchronous correlation*.

It will sometimes be useful to use the equivalent thin-wire version of the above definition, given by



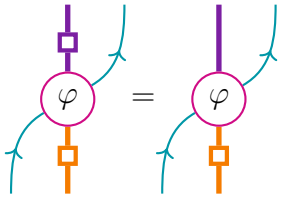
Synchronous games and correlations with classical questions and answers are already an established notion, defined in [Pau+16]. Let's briefly call that property *typically synchronous*. A correlation or game $f(a, b|x, y)$ is typically synchronous if for all $a \neq b$, $f(a, b|x, x) = 0$ for all x. It may be momentarily confusing to not give our more general property a unique name, but Proposition 5.6 resolves this confusion.

Proposition 5.6. *A classical correlation (game) is synchronous if and only if it is typically synchronous.*

Proof. Let X, A be classical. Let P be a correlation $X \otimes X \to A \otimes A$. Denote the coefficients $p(a, b|x, y) := \langle a| \langle b| P |x\rangle |y\rangle$. Being synchronous says that $P |x\rangle |x\rangle$ is in the diagonal subspace of $A \otimes A$. This means that the coefficients, $p(a, b|x, x)$ are zero unless $a = b$. The same proof applies when replacing the correlation P by a game λ. □

Remark 5.7 (A quantum processes interpretation of synchronicity for maps on classical sets)**.** In the case that X, A are classical, this equation can be understood as saying that after one instance of decoherence (the map Δm), the map f commutes with any further decoherence. See [CK17] for a diagrammatic definition of decoherence.

Definition 5.8 (Synchronous quantum function)**.** We say a quantum function $\varphi : X \otimes X^{\mathrm{op}} \to_{\mathcal{H}} A \otimes A^{\mathrm{op}}$ is *synchronous* if



The above definition gives us the notion of a *synchronous strategy*. We will see that this is a much stronger condition than realizing a synchronous correlation. Note that if a strategy on $\mathcal{H}$ is synchronous, then together with any normalized state $|\psi\rangle \in \mathcal{H}$, it realizes a synchronous correlation.

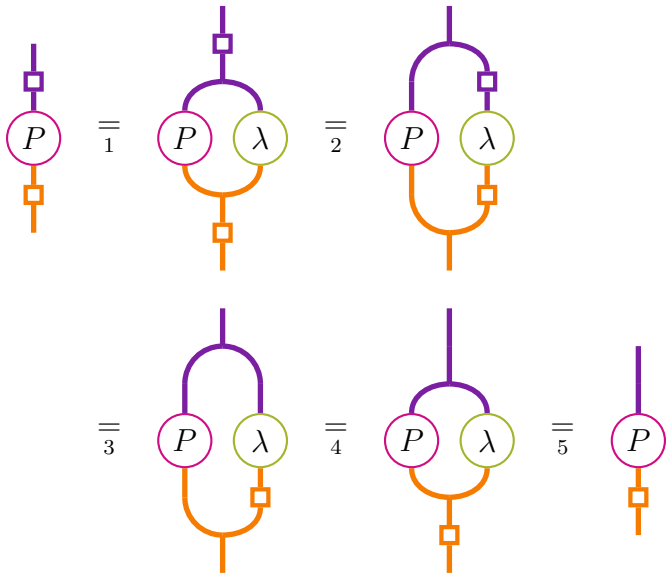
5.2 Consequences of synchronicity

5.2.1 Impact on perfect strategies

One of the most useful properties of (typical) synchronous games is that they force associated perfect strategies to be synchronous. We generalize this result to quantum question and answer games.

Theorem 5.9. *Perfect correlations for synchronous games are synchronous.*

Proof. Let P be a perfect correlation for a synchronous game λ. We have that



where

1. uses the definition of a perfect correlation,
2. uses property (d) of sharing,
3. uses synchronicity of λ,
4. again uses property (d) of sharing, and
5. again uses the definition of a perfect correlation.

□

Corollary 5.10. *Perfect strategies for synchronous games are synchronous.*

Proof. While not truly a corollary, the steps of the proof are identical to the above, with P replaced by a strategy φ, and Hilbert space wires tacked on to φ in each diagram. □

We have presented the results in this order as the proof is cleaner and easier to read without the additional Hilbert space wire.

5.2.2 First examples of synchronous games

Let us now make some guesses as to what may constitute a synchronous quantum game. Our candidates are the identity map and the sharing map. In the classical case, the identity map can be thought of as the game *Simon Says*, where Alice and Bob must each repeat their question as their answer, and the sharing map can be thought of as an unfair *Simon Says*, where they can only win if their questions also agree with each other.

These are both synchronous maps, due to the fact that sharing is a projection. The identity is always a game, by specialness of a quantum set. However, this is not true of sharing.

Proposition 5.11. *X is classical if and only if sharing is a game.*

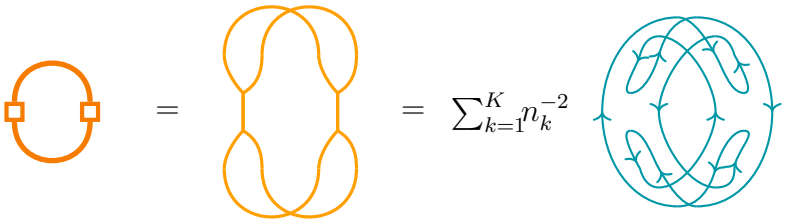
Proof. Let X be classical. Then sharing is self-conjugate because $X = X^{\mathrm{op}}$.

For classical sets, we have access to the spider theorem for †-SCFAs, that asserts that all connected diagrams from n input wires to k output wires using only the quantum set operations are equal. For a full treatment of the spider theorem, see [HV19, Section 5.2]. We can use the spider theorem to show sharing is a game. The left-hand-side of the idempotency equation for games is a connected 2-input, 2-output classical spider. By the spider theorem, this is equal to any other connected 2-input, 2-output spider, and so the idempotency equation defining a game holds.

Conversely, assume sharing is a game. Recall that $X = \bigoplus_{k=1}^{K} M_{n_k}$, as in Remark 4.3. We first show graphically that

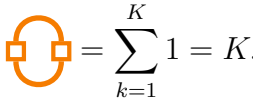


as follows.

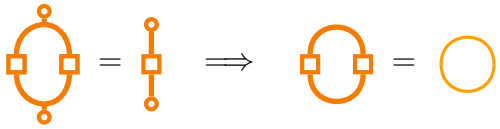


with the coefficients introduced as given in Remark 4.3, due to four instances of multiplication or comultiplication. But the complicated looking knot is just two closed loops in $\mathbb{C}^{n_k}$, so is equal

to $\dim^2 \mathbb{C}^{n_k} = n_k^2$, so we get



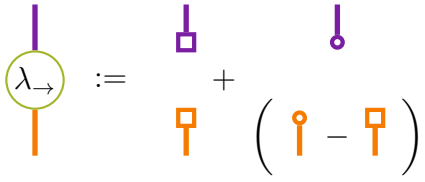
Then applying units and counits to one of the defining equations for sharing being a game, we get



The loop evaluates to $\dim X$. We conclude that $K = \dim X$, so we must have $n_k = 1$ for all k. Therefore, X is classical. □

So in general, sharing is synchronous but it is not a game outside of the classical setting. The identity map is our only example so far of a synchronous game with non-classical questions and answers.

We can also look at the *function* game $\lambda_{\to}$, given by



This is a game, and is synchronous. In fact, it is the "easiest" synchronous game, in the following sense.

Proposition 5.12 (Function game is the easiest synchronous game)**.** *Given any synchronous game, $\lambda : X \otimes X^{\mathrm{op}} \to A \otimes A^{\mathrm{op}}$, the function game on the same sets commutes with λ, and satisfies*

$$\lambda \star \lambda_{\to} = \lambda.$$

The proof follows directly from computing both products, and using synchronicity of λ (with respect to both sharing and conjugate sharing) to cancel two of the three terms.

We will construct examples, in Chapter 6, of more interesting synchronous games with truly quantum questions and answers.

We can also always construct more synchronous games by taking the commuting product of any game that commutes with a synchronous game, because of the following fact.

Proposition 5.13. *If λ is synchronous, then $\lambda' \star \lambda$ is synchronous.*

Proof. The proof is identical to steps 2-4 of the proof of Theorem 5.9, replacing P with λ'. □

Some of the games in this section motivate one more definition.

Definition 5.14 (Fair game)**.** We say a game is *fair* if there is no (self-conjugate) state $|\gamma\rangle$ of $X \otimes Y$ such that $\lambda(|\gamma\rangle) = 0$.

The identity map and the function game are fair games. The graph games in Chapter 6 are also fair.

5.2.3 Classical dimension

The proof of Proposition 5.11 motivates us to define the classical dimension of X as the number K of direct summands when X is decomposed as a direct sum of matrix algebras. In diagrams, we can use the sharing map to make our definition without appealing to a structure theorem.

Definition 5.15 (Classical dimension)**.** The *classical dimension* of a quantum set X (seen from the perspective of $X \otimes X^{\mathrm{op}}$) is

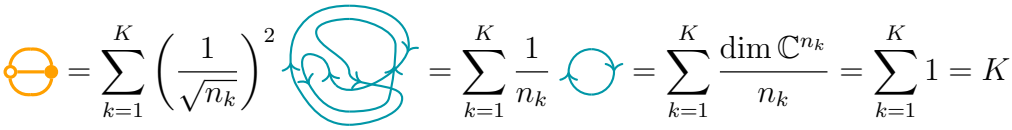


We call this notion "classical dimension" because it counts classically distinct subspaces. At the level of Abelian categories, this notion is called the *length* of X as an object of **FHilb**. It should not be confused with other notions of classical dimension that may be in the literature.

We have defined classical dimension of X using the wires of $X \otimes X^{\mathrm{op}}$. Using the notation for the opposite Frobenius algebra, we can give a simpler, equivalent definition.

Proposition 5.16. *The classical dimension of a quantum set X is also given by* .

Proof. This diagram also counts the number of matrix blocks (denoted K) in the quantum set X, as follows (see Remark 4.3 for block decomposition of X):



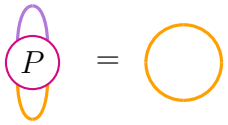
□

We can now restate the key takeaway of Proposition 5.11.

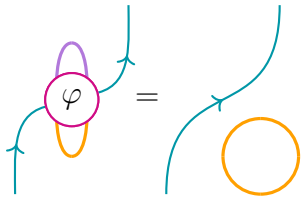
Corollary 5.17. *A quantum set X is classical if and only if its (quantum) dimension is equal to its classical dimension.*

5.2.4 Structure of synchronous correlations

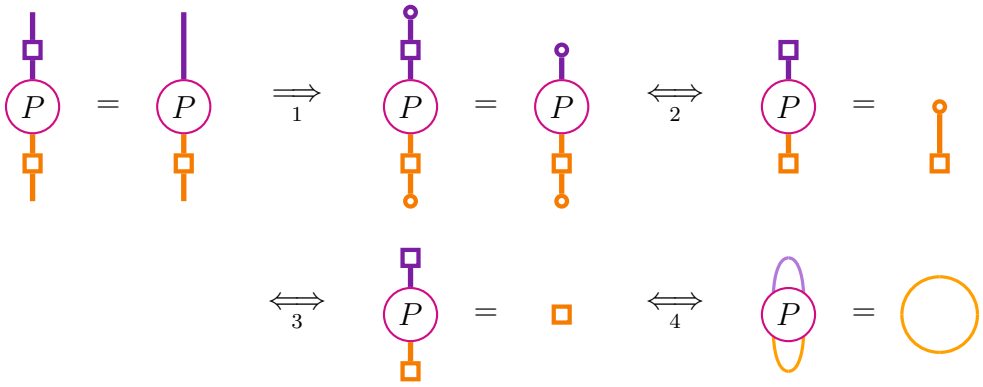
Lemma 5.18. *A synchronous correlation $P : X \otimes X^{\mathrm{op}} \to A \otimes A^{\mathrm{op}}$ can "slide upwards" off a loop:*



A synchronous (combined) strategy $\varphi : X \otimes X^{\mathrm{op}} \to_{\mathcal{H}} A \otimes A^{\mathrm{op}}$ can "slide upwards" off a loop, leaving behind its resource wire:



Proof. Beginning with the definition of a synchronous correlation, we make a series of graphical derivations.



The steps are justified as follows.

1. Apply units and counits.

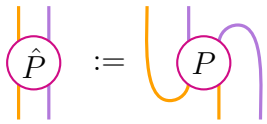
2. Contract the units and counits using Lemma 5.4, and use the fact that P is counital.

3. Contract the remaining counit.

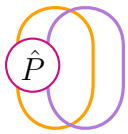
4. Rewrite using single wire notation, and use specialness of X.

All of the above steps can be applied to a synchronous strategy φ instead of a correlation P, by adding a resource Hilbert space wire through the map. Then, in step (2), on the right hand side, we apply (4.7), and the resource wire remains in the diagram. □

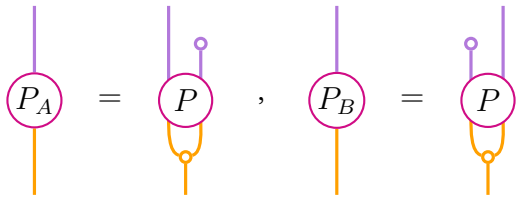
Remark 5.19. Notice that by defining $\hat{P}$ to be an operator on $X \otimes A$, given by the partial transpose of P



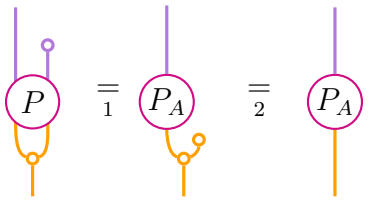
Lemma 5.18 says that for P synchronous, $\mathrm{tr}\hat{P} = \dim X$ where the trace of $\hat{P}$ is given by



Proposition 5.20. *A non-signalling correlation $P : X \otimes X^{\mathrm{op}} \to A \otimes A^{\mathrm{op}}$ has marginals given by*



Proof. We give a diagrammatic proof for P_A



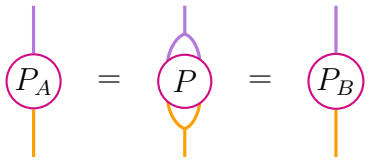
with the steps justified as follows:

1. P is nonsignalling, and
2. comultiplication is counital.

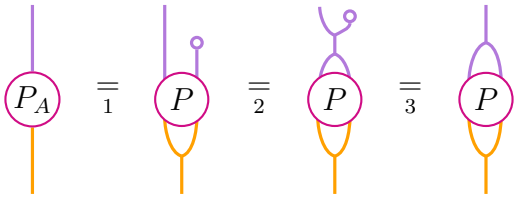
Flip the proof horizontally for P_B. □

Note that conjugating the above statement allows us to use the filled quantum set instead, as P_A and P_B are channels and so are self-conjugate. When Alice and Bob have the same sets, we define the swapped correlation $P^\sigma := \sigma P \sigma$, corresponding to Alice and Bob switching places. When P is non-signalling, P^σ is also non-signalling, and $P_B = P_A^\sigma$. If additionally, P is synchronous, we get $P_A = P_B$, as follows.

Proposition 5.21. *A non-signalling synchronous correlation P has equal marginals given by*



Proof. We give a diagrammatic proof

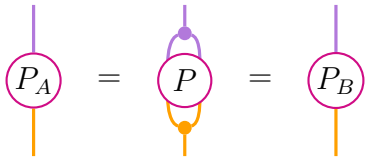


with the steps justified as follows:

1. Apply Proposition 5.20.
2. P is synchronous.
3. Contract loose counit.

□

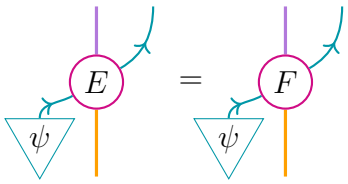
Note that conjugating the statement of the above proposition replaces all multiplication and comultiplication with that of the opposite algebra. So we also have



In the classical question and answer case, consider the quantum processes interpretation (see [CK17]) of the above proposition: Each of Alice's and Bob's marginals looks like encoding their own question as a quantum state, applying P, and then measuring to get back classical data.

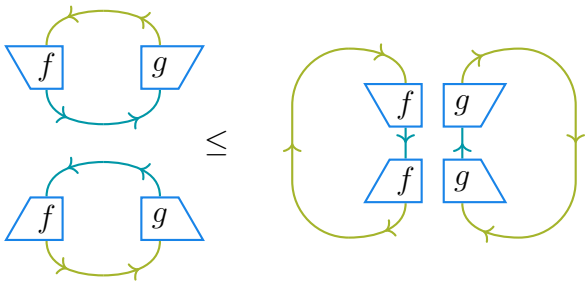
If the non-signalling correlation P is realized by some quantum commuting strategy and a shared state, we can say even more about the strategy. This is a diagrammatic alternative to the proof of [BKS23, Theorem 6.6, (1)].

Theorem 5.22. *If a synchronous correlation is realized by a quantum commuting strategy (E, F) on a finite-dimensional resource space $\mathcal{H}$, and a normalized state $|\psi\rangle \in \mathcal{H}$, then*



Before proving this result, we will present a diagrammatic version of the Cauchy-Schwarz inequality for **FHilb**. (Note that the diagrams being compared in the following equation are both non-negative scalars.)

Lemma 5.23 (Cauchy-Schwarz)**.** *Let $f, g : \mathcal{K} \to \mathcal{H}$ in **FHilb**. Then*



(5.1)

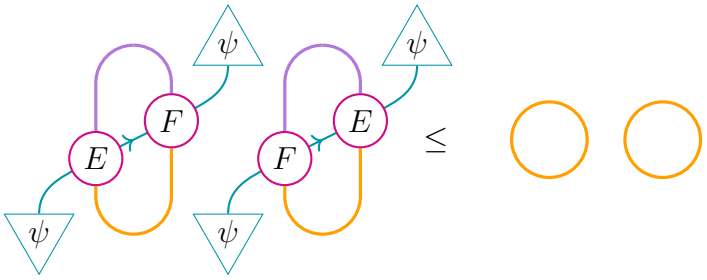
with equality if and only if $f = \alpha g$ for some $\alpha \in \mathbb{C}$.

Proof. By sliding the left-side fs along the $\mathcal{H}$ wire to the other side of their loops, this just says $\langle f, g\rangle\langle g, f\rangle \leq \langle f, f\rangle\langle g, g\rangle$, which is the Cauchy-Schwarz inequality, and it holds with equality exactly when f is a scalar multiple of g. □

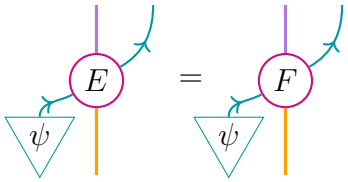
The Cauchy-Schwarz inequality in this format leads to two inequalities for pairs of quantum functions.

Theorem 5.24. *Let $E : X \to_{\mathcal{H}} A, F : X^{\mathrm{op}} \to_{\mathcal{H}} A^{\mathrm{op}}$ be quantum functions, with $\mathcal{H}$ finite-dimensional.*

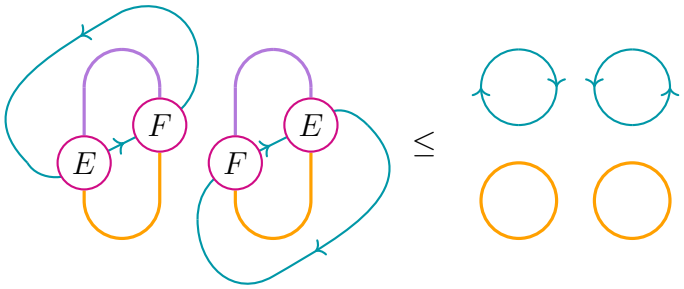
1. *Let $|\psi\rangle \in \mathcal{H}$ be a normalized state.*



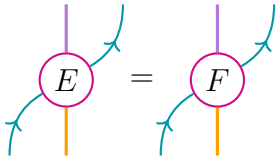
   with equality if and only if



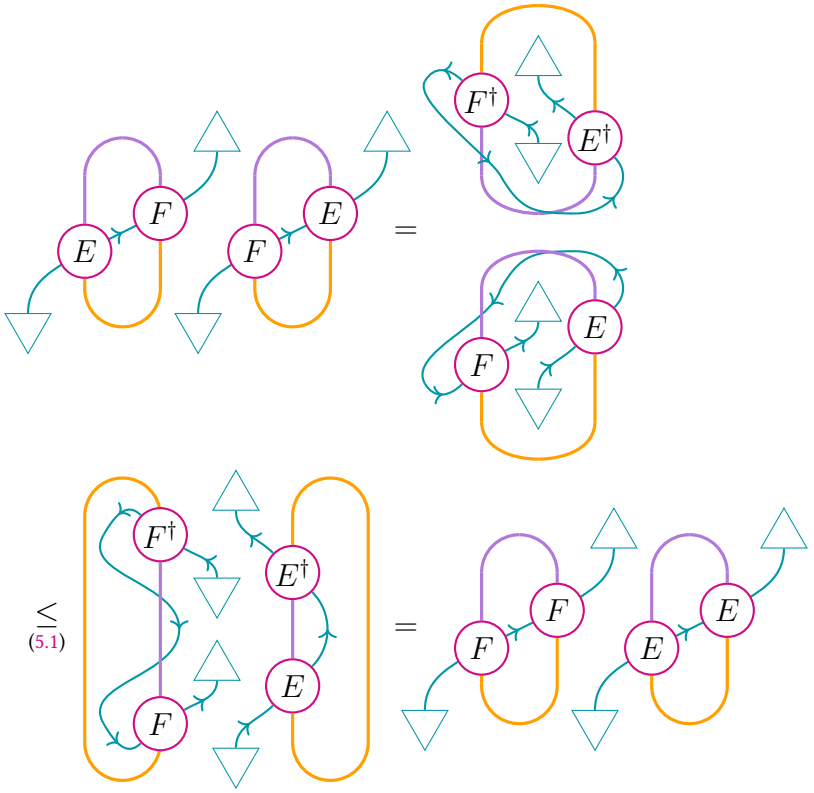
2. *We have*



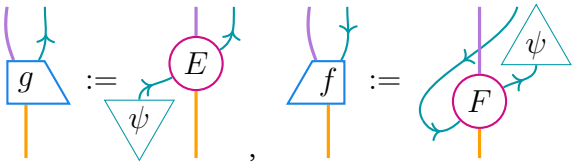
   with equality if and only if



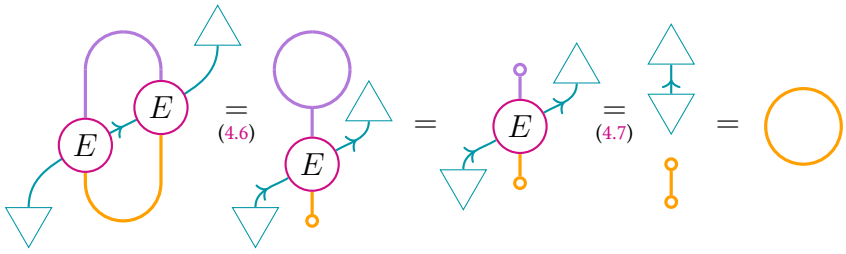
Proof of Theorem 5.24. 1. We have



where the inequality is the Cauchy-Schwarz inequality 5.23 with

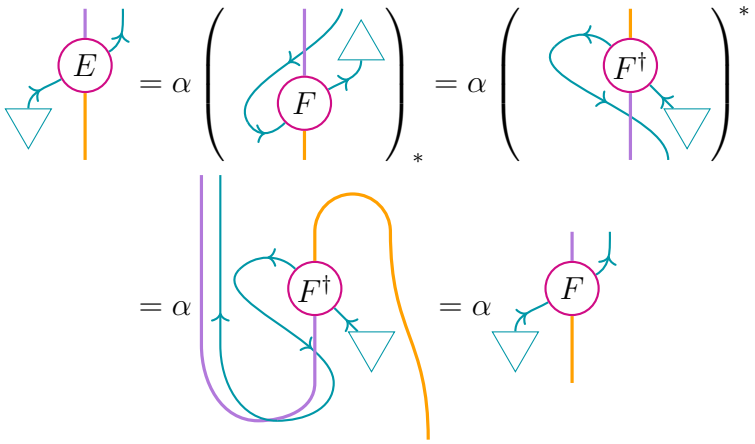


But, given that E and F are quantum functions, we can further simplify this by computing

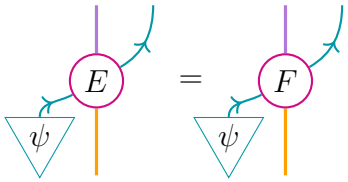


using the properties of E as a quantum function, the definition of the cup, and the fact that X, A are special and $|\psi\rangle$ is normal. The same holds for F, using the symmetry property of the quantum set to postcompose a swap with the cup of X, using the opposite comultiplication in the first equation. Substituting these loops into the above inequality, we have our desired result.

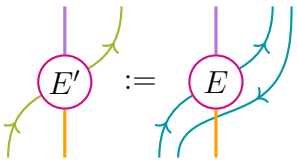
Now, let us examine the case of equality. Equality holds if and only if $g := \alpha f$ for some α in $\mathbb{C}$. This is exactly the condition that



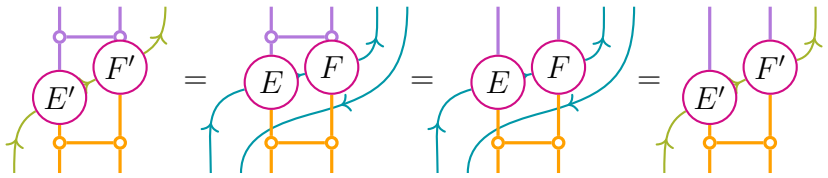
Because E is a quantum function, due to (4.7) we must have $\alpha = 1$, resulting in equality if and only if



2. Let $\mathcal{H}^*$ denote the linear dual of $\mathcal{H}$. Consider the map $E' : \mathcal{H} \otimes \mathcal{H}^* \otimes X \to A \otimes \mathcal{H} \otimes \mathcal{H}^*$, given by

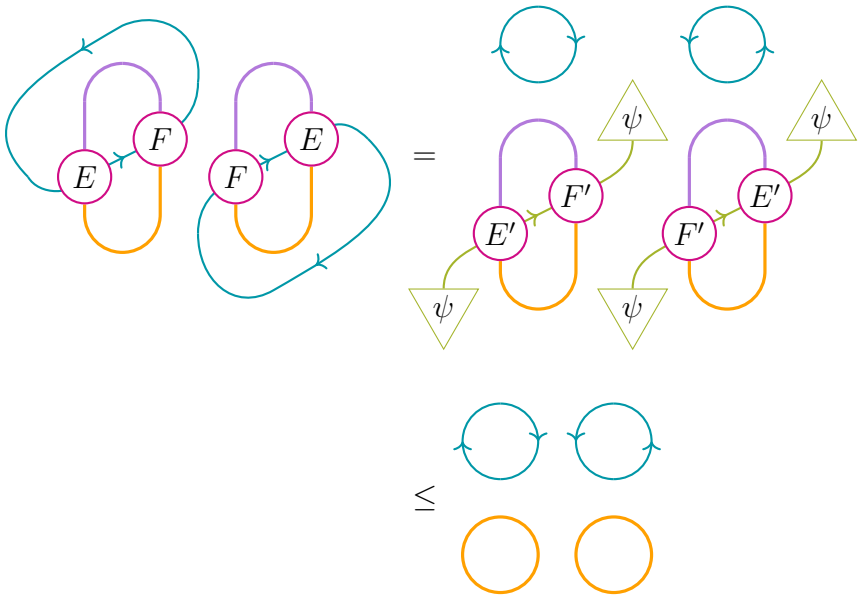


Observe that E' is again a quantum function $X \to_{\mathcal{H}\otimes\mathcal{H}^*} A$, by noting that the added $\mathcal{H}^*$ wire does not interfere with the diagrammatic properties of E. Construct F' in the same way from F. Observe that (E', F') is again a synchronous strategy, by adding an $\mathcal{H}^*$ wire to the synchronicity equation for (E, F), as pictured.

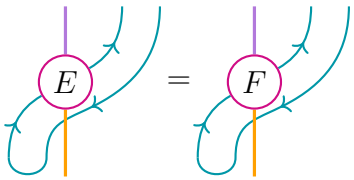


Let $|\psi\rangle$ be the normalized cup state of $\mathcal{H} \otimes \mathcal{H}^*$ coming from the duality of $\mathcal{H}^*$ to $\mathcal{H}$. Then (E', F') together with $|\psi\rangle$ generates a synchronous correlation, P.

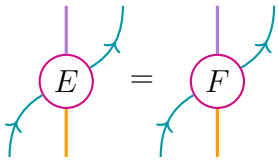
We can now apply part (1) to get



with equality if and only if



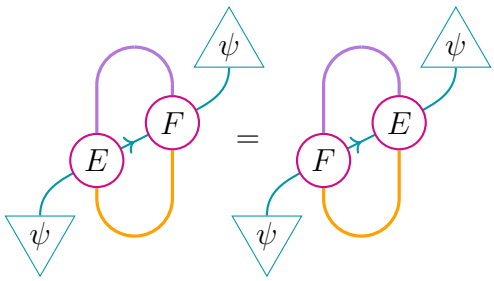
or equivalently, by the snake equations,



□

With these tools in hand, we are equipped to prove the previous theorem.

Proof of Theorem 5.22. Because (E, F) is quantum commuting, we have that

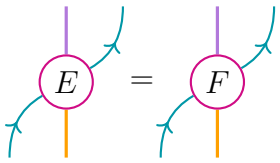


By the loop-sliding lemma 5.18, given that the realized correlation is synchronous, these are also equal to

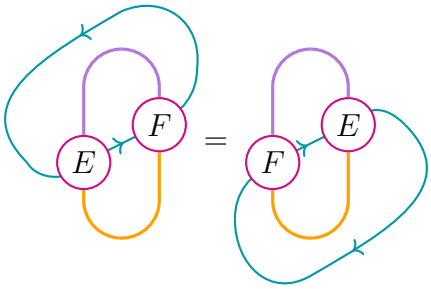


Then, we have equality in Theorem 5.24 (1), implying the desired equality. □

Theorem 5.25. *If a quantum commuting strategy (E, F) on a finite-dimensional resource space $\mathcal{H}$ is synchronous, then*



Proof. Because (E, F) is quantum commuting, we have that



By the loop-sliding lemma 5.18, given that the strategy is synchronous, these are also equal to



Then, we have equality in Theorem 5.24, (2), implying the desired equality. □

The consequence $E = F$ of Theorem 5.25 may be surprising. It shows us that synchronicity at the level of strategies is an extremely strong property – much stronger than synchronicity at the level of correlations. In a sense, it results in an n dimensional restriction on the players' behaviour instead of a 1 dimensional restriction.

As a corollary, we will show that synchronous qc strategies are essentially families of *local* strategies. Over an n-dimensional Hilbert space, these strategies are merely recipes for convex interpolation between n deterministic strategies. The specific local strategy implemented varies depending on the selected resource state. A more precise statement follows.

Corollary 5.26 (Corollary of Theorem 5.25)**.** *Let (E, F) be a synchronous quantum commuting strategy on an n-dimensional resource space $\mathcal{H}$. Then (E, F) together with any state ψ of $\mathcal{H}$ can be realized by a convex combination of n synchronous deterministic strategies.*

Proof. By Theorem 5.25, we have $E = F$. Fix a self-conjugate basis of each of X and A (this translates to being self-adjoint when thought of as operators). For each choice of basis elements x, y of X and a, b of A, composing these with the quantum commuting condition on (E, E) gives a condition

$$E_{y,b}E_{x,a} = E_{x,a}E_{y,b},$$

where $E_{x,a} := a^\dagger Ex$. Note also that each $E_{x,a}$ satisfies

$$E_{x,a}^\dagger = (a^\dagger Ex)^\dagger = x^\dagger E^\dagger a \underset{1}{=} a^* Ex_* \underset{2}{=} a^\dagger Ex = E_{x,a}$$

where step 1 is due to the realness property of the quantum function E and step 2 is due to self-conjugacy of the basis elements x, a.

So we have a family of pairwise commuting self-adjoint operators, $\{E_{x,a}\}$. Fix an orthonormal basis $\{|i\rangle : i \in [n]\}$ of $\mathcal{H}$ in which this family of operators is diagonal. Let $f_i := \langle i| E |i\rangle$. It follows that each f_i is a (classical) quantum function both $X \to_{\mathbb{C}} A$ and $X^{\text{op}} \to_{\mathbb{C}} A^{\text{op}}$ so (f_i, f_i) is a deterministic strategy for the same game.

Now let Q be the classical quantum set structure on $\mathcal{H}$ coming from the specified orthonormal basis. Let

$$|\alpha\rangle := \sum_{i=1}^{n} |\langle\psi|i\rangle|^2 |i\rangle .$$

Then the local strategy given by $\sum_{i=1}^n \langle\alpha|i\rangle f_i \otimes f_i$ realizes the same correlation as (E, E) with $|\psi\rangle$. □

Remark 5.27 (Synchronous correlations versus synchronous strategies)**.** This result does not conflict with the existence of synchronous correlations in $\mathcal{C}_{\text{qc}} \backslash \mathcal{C}_{\text{loc}}$. The key difference here is that being a synchronous *strategy* is much stronger than being a synchronous *correlation*. That is, to make the proper analogies to the classical question-and-answer literature, one must talk about synchronous correlations implemented by quantum commuting strategies, as opposed to synchronous quantum commuting strategies.

5.3 Bisynchronicity

Bisynchronous nonlocal games and correlations are introduced in [PR21]. We extend these definitions to the quantum question and answer setting.

In the classical question and answer setting, the graph isomorphism game is an example of a bisynchronous game. In Chapter 6, we discuss a quantum generalization of the graph isomorphism game, and use the fact that it is bisynchronous to relate its perfect strategies and correlations to quantum graph isomorphisms.

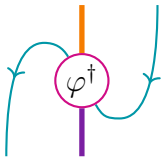
5.3.1 Bicorrelations and bistrategies

Definition 5.28 (Bicorrelation)**.** We say a correlation P is a *bicorrelation* if $P^\dagger$ is also a correlation.

A bicorrelation is simply a unital correlation.

Proposition 5.29. *Let P be a nonsignalling bicorrelation. Then $P^\dagger$ is also nonsignalling.*

Definition 5.30 (Bistrategy)**.** We say a combined strategy $\varphi : X \otimes Y \to_{\mathcal{H}} A \otimes B$ is a *bistrategy* if



is also a combined strategy $\overline{\varphi} : A \otimes B \to_{\mathcal{H}^*} X \otimes Y$.

This is equivalent to defining a bistrategy φ to be a quantum bijection as given in [MRV18, Definition 4.3]. From the perspective of C*-algebras, this can be thought of as an entanglement assisted $*$-isomorphism.

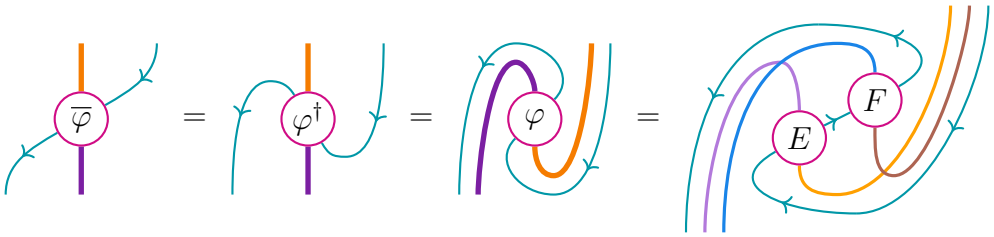
Note that correlations realized by bistrategies are automatically bicorrelations.

It turns out bistrategies have the desirable property of $\overline{\varphi}$ inheriting the strategy class of φ, as follows.

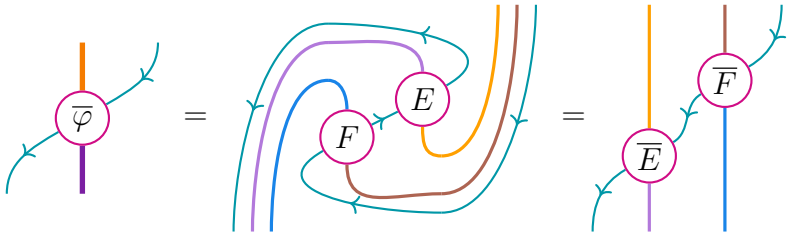
Proposition 5.31. *Let $\varphi : X \otimes Y \to A \otimes B$ be a bistrategy.*

1. *If φ is quantum commuting, then so is $\overline{\varphi}$.*
2. *If φ is quantum tensor, then so is $\overline{\varphi}$.*
3. *If φ is deterministic, then so is $\overline{\varphi}$.*

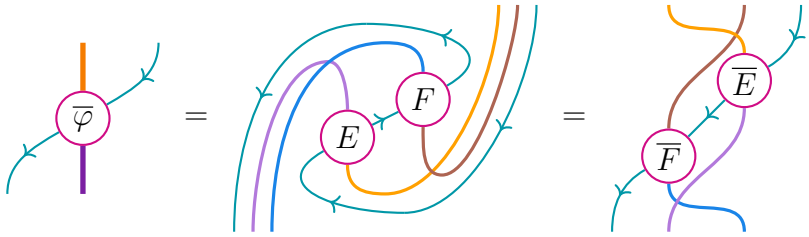
Proof. 1. If φ is quantum commuting then we can write $\overline{\varphi}$ as follows:



But then using the fact that (E, F) is quantum commuting, we have

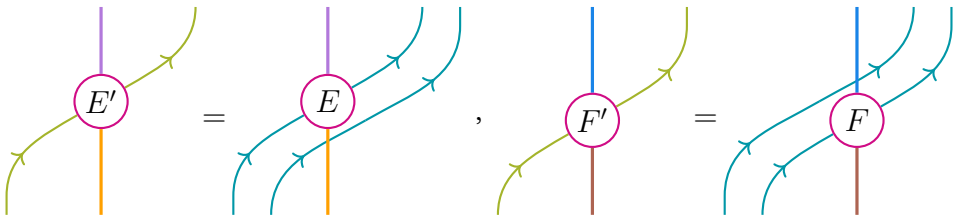


so it looks like $\overline{\varphi}$ is a quantum commuting strategy for $\lambda^\dagger$ implemented by $(\overline{E}, \overline{F})$. We will just check if the operators commute. Indeed,

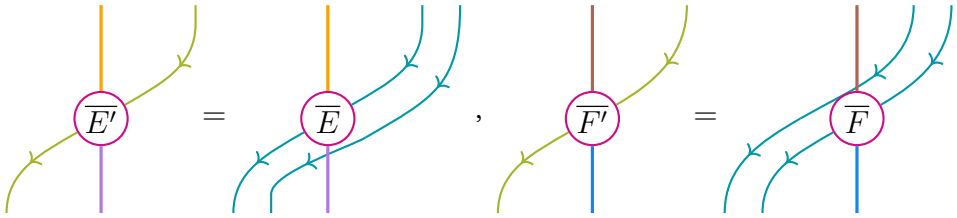


So $\overline{\varphi}$ is quantum commuting.

2. If φ is quantum tensor, implememented by some (E, F), then it is quantum commuting with operators



But then $\overline{\varphi}$ is quantum commuting and implemented by $(\overline{E'}, \overline{F'})$. These are given by



and so $\overline{\varphi}$ is a quantum tensor strategy implemented by $(\overline{E}, \overline{F})$.

3. If φ is deterministic, then $\mathcal{H} = \mathbb{C}$ but also by the above, $\overline{\varphi}$ is quantum tensor and implemented by †-cohomomorphisms $(E^\dagger, F^\dagger)$. So $\overline{\varphi} : A \otimes B \to X \otimes Y$ is deterministic.

□

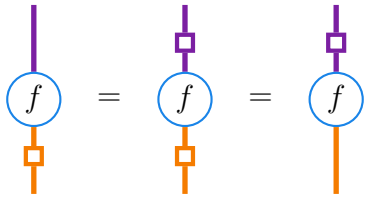
5.3.2 Bisynchronous maps

Definition 5.32 (Bisynchronous map)**.** We say a linear map $f : X \otimes X^{\mathrm{op}} \to A \otimes A^{\mathrm{op}}$ is *cosynchronous* if $f^\dagger$ is synchronous. We say f is *bisynchronous* if f is both synchronous and cosynchronous.

Definition 5.33 (Bisynchronous quantum function)**.** We say a quantum function $\varphi : X \otimes X^{\mathrm{op}} \to_{\mathcal{H}} A \otimes A^{\mathrm{op}}$ is *bisynchronous* if both φ and $\overline{\varphi}$ are synchronous.

Proposition 5.34. *A map or quantum function is bisynchronous if and only if it intertwines sharing.*

Proof. Let $f : X \otimes X^{\mathrm{op}} \to A \otimes A^{\mathrm{op}}$ be a bisynchronous linear map. If f is bisynchronous,



Let $\varphi : X \otimes X^{\mathrm{op}} \to_{\mathcal{H}} A \otimes A^{\mathrm{op}}$ be a bisynchronous quantum function. Then the same proof holds with an $\mathcal{H}$ wire tacked on.

Conversely, if f intertwines sharing, then f and $f^\dagger$ are synchronous because sharing is a projection. The same goes for φ and $\overline{\varphi}$. □

In the quantum commuting case, there is something redundant about the definition of a bisynchronous bistrategy, as follows.

Proposition 5.35. *Let φ be a synchronous quantum commuting strategy.*

1. *If φ is a bistrategy, then φ is bisynchronous.*
2. *If φ is bisynchronous and unital – in the sense that $\overline{\varphi}$ satisfies (4.7) – then φ is a bistrategy.*

Proof. By Theorem 5.25, $\varphi : X \otimes X^{\mathrm{op}} \to_{\mathcal{H}} A \otimes A^{\mathrm{op}}$ is given by some quantum function $E : X \to_{\mathcal{H}} A$ applied twice.

1. If φ is a bistrategy then applying (4.6) to E and $\overline{E}$, we see that φ commutes with sharing and is therefore bisynchronous.

2. By assumption, we have that $\overline{\varphi}$ satisfies (4.7). We know that $\overline{\varphi}$ satisfies (4.8) because φ does and the condition is the same. To see that $\overline{\varphi}$ satisfies (4.6), given that φ is bisynchronous, start with the result of Proposition 5.34, and apply a counit to the left A wire of the equation. By counitality of E, we get that (4.6) holds for E, and then it follows for φ as well. Therefore, φ is a bistrategy.

□

Theorem 5.36. *Let X, A be quantum sets.*

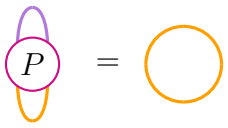
1. *If $P : X \otimes X^{\mathrm{op}} \to A \otimes A^{\mathrm{op}}$ is a bisynchronous bicorrelation, then $\dim X = \dim A$.*

2. *If additionally, $\varphi : X \otimes X^{\mathrm{op}} \to_{\mathcal{H}} A \otimes A^{\mathrm{op}}$ is a bisynchronous bistrategy, then X and A have the same classical dimension.*

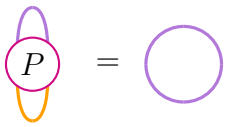
3. *If furthermore, φ is quantum commuting, then X, A are quantum isomorphic as quantum sets.*

4. *If furthermore, φ is deterministic, then X, A are isomorphic as quantum sets.*

Proof. Because P is a synchronous correlation, the loop-sliding lemma 5.18 gives us that

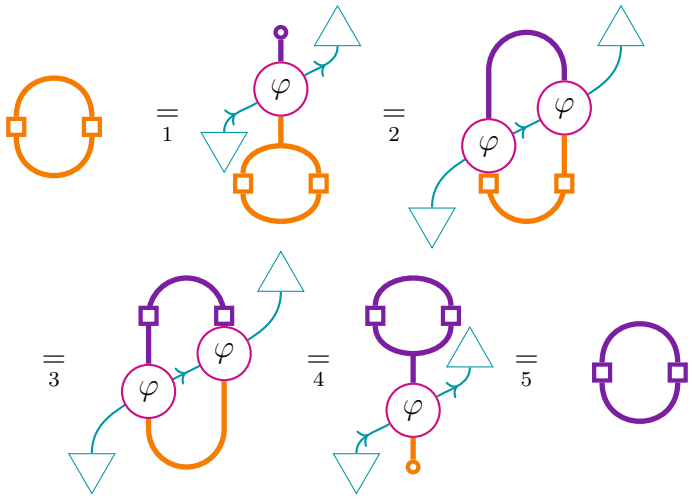


Because $P^\dagger$ is a synchronous correlation, by applying the loop-sliding lemma to $P^\dagger$ and then taking daggers (turning diagrams upside down), we have that



Therefore, the two circles are equal. In a quantum set, the circle (given by $\varepsilon m \Delta u$) computes the dimension of the underlying Hilbert space.

In the case that φ is a bisynchronous bistrategy, then using the fact that φ and $\overline{\varphi}$ are both quantum functions, and fixing any normalized state of $\mathcal{H}$,



In the above, the steps are justified as follows:

1. Apply the counitality equation (4.7) for φ.
2. Apply the comultiplicativity equation (4.6) for $\overline{\varphi}$.
3. Apply Proposition 5.34 to interchange φ with the sharing map on both sides.
4. Apply the comultiplicativity equation (4.6) for φ.
5. Apply the counitality equation (4.7) for $\overline{\varphi}$.

If, furthermore, φ is quantum commuting, then because it is synchronous, it is realized by two copies of E which is a quantum function both $X \to_{\mathcal{H}} A$ and $X^{\mathrm{op}} \to_{\mathcal{H}} A^{\mathrm{op}}$, as shown in Theorem 5.25. Because φ is a bistrategy, $\overline{E}$ is also a quantum function, and so E is a quantum bijection $X \to_{\mathcal{H}} A$.

In the case that φ is deterministic, $\mathcal{H} = \mathbb{C}$ and the quantum bijection E just becomes a quantum-set isomorphism.

□

Corollary 5.37 (of Theorem 5.36)**.** *Consider a bisynchronous game λ.*

1. *If λ has a perfect bicorrelation, then the question and answer sets have the same dimension.*
2. *If λ has a perfect bistrategy, then the question and answer sets have the same classical dimension.*
3. *If λ has a perfect quantum commuting bistrategy, then the question and answer sets are quantum isomorphic (in the sense that there is a quantum bijection from one to the other).*
4. *If λ has a perfect deterministic bistrategy, then the question and answer sets are isomorphic as quantum sets.*

Proof. A perfect correlation P for a synchronous game λ is synchronous by Theorem 5.9. We also know that $P^\dagger$ is perfect for the game $\lambda^\dagger$ by taking daggers of Definition 4.30. Because $\lambda^\dagger$ is also synchronous, we can apply Theorem 5.9 again to see that $P^\dagger$ is synchronous. Therefore, P is bisynchronous, and so we can conclude that $\dim X = \dim A$. Making the same argument to show that a perfect strategy is also forced to be bisynchronous, we can apply the rest of Theorem 5.36 to show the remaining items. □

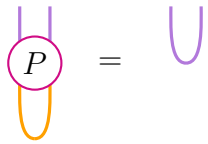
5.4 Other quantum approaches to synchronicity

We now consider two other recent generalizations of synchronous games and correlations. We relate these to our definition of synchronicity.

5.4.1 Concurrent correlations

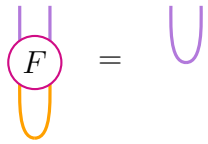
In [Bra+23; Bra+24], there is a related attempt to generalize synchronicity to quantum question and answer games. The authors call these correlations *concurrent* instead of synchronous, and their definition is specific to the case where X, A are full matrix algebras. Concurrency says a correlation takes the cup state (which is maximally entangled) to the cup state.

The question and answer sets of [Bra+23], thought of as quantum sets, are maximally quantum, as in Example 4.6. With this in mind, their definition of the property concurrent for a quantum correlation P is represented by the graphical equation



Using the above as a template, we can extend concurrency to when the question and answer sets are general quantum sets.

Definition 5.38 (Concurrent)**.** A linear map $F : X \otimes X^{\mathrm{op}} \to A \otimes A^{\mathrm{op}}$ is called *concurrent* if it satisfies



We can think of concurrency as preserving the cup state.

Note that replacing P with $P^\dagger$ gives "co-concurrency" of P. In the case that X, A are classical, this is causality of the quantum map P, as given in [CK17].

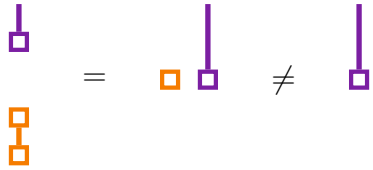
In the case that $X = A$, and F is a quantum relation on X as defined in [MRV18], concurrency is the reflexivity condition of [MRV18, Definition 7.6].

It turns out that synchronous games and correlations need not be concurrent.

Example 5.39 (Unfair function game)**.** The synchronous game



is not concurrent, because



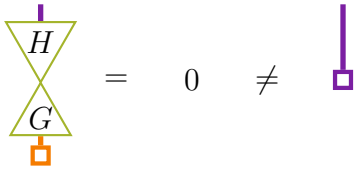
This is an unfair version of the function game, where the players cannot win if they receive different answers.

The concurrency equation fails by a scaling factor. There are also synchronous games that fail more seriously to be concurrent.

Example 5.40 (Unfair graph game)**.** Given irreflexive graphs[1] G, H, the synchronous game



is not concurrent, because

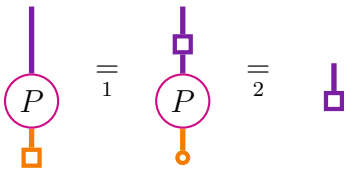


Note that these are both unfair games. It would be interesting to see a fair synchronous game that is not concurrent.

If P is a bicorrelation, then we can relate concurrency to bisynchronicity, as follows.

Proposition 5.41. *Bisynchronous bicorrelations are concurrent.*

Proof. Let $P : X \otimes X^{\mathrm{op}} \to A \otimes A^{\mathrm{op}}$ be a bisynchronous bicorrelation. Then



with steps justified as follows:

1. P preserves sharing by 5.34.
2. P is unital.

□

Note that the proof did not use the property that a bicorrelation is CP, nor that a bicorrelation is counital. This means there is a somewhat more general result here, that bisynchronous unital maps are concurrent.

[1] See Chapter 6 for a definition of quantum graphs.

5.4.2 Synchronicity in the quantum spaces of maps setting

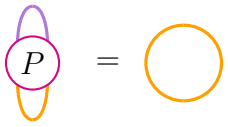
In [BKS23], there is also a definition for synchronicity of quantum question and answer correlations, which the authors refer to as quantum correlations of quantum spaces. Their input space denoted $C(\mathbb{P})$ corresponds to our question space X while their output space $C(\mathbb{O})$ corresponds to our answer space A. For the purposes of this section, we denote $X = \bigoplus_{\ell=1}^{N_X} X_\ell$ where each X_ℓ is the full matrix algebra $M_{m_\ell}(\mathbb{C})$, and $A = \bigoplus_{k=1}^{N_A} A_k$ where each A_k is the full matrix algebra $M_{n_k}(\mathbb{C})$.

Their definition of synchronous is given by the property

$$\sum_{s,t,i,j,k,\ell} \frac{1}{n_k m_\ell} {}^{kk}_{\ell\ell}P^{(st),(st)}_{(ij),(ij)} = N_X \tag{5.2}$$

where the coefficients of P come from writing the correlation P in a matrix unit basis for X and A. In [BKS23], these are denoted with an X instead of a P. See their paper for the details of index placement.

When matching summed indices using strings, this looks a lot like a consequence of our definition, given by the loop-sliding equation of Lemma 5.18,

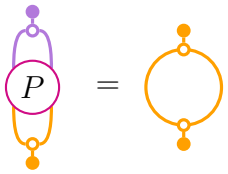


however, neither the weighting in the sum, nor the number N_X can arise when interpreting each wire as a quantum set, or even as an (unnormalized) †-SFA. (This last claim is based on the †-SFA classification result [HV19, Proposition 5.33], which is slightly more general than what's described in Remark 4.3.)

Instead, we show a graphical way to generate (5.2) using a similar loop-sliding diagram, but allowing for two Frobenius structures on the same set. The approach uses only classical structures, which means that [BKS23]'s definition of synchronicity of QNS correlations is a classical property. We compare the graphical translation of (5.2) to our definition of synchronous, and show that in the classical question and answer case, it is equivalent to our definition.

Using classical structures

Using two commutative Frobenius structures (not both normalized to be quantum sets) on X and doing the same thing for A, we can draw (5.2) like this:



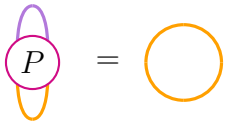
To read (5.2) from the above diagram, we must specify the †-CFAs. Each †-CFA on X uses a differently weighted Schur product and the appropriate unit. The †-CFA represented by the filled node ● has its multiplication given by the Schur product of matrices, $\circ$, with entrywise multiplication in the ℓ^{th} block weighted by the square root of its size $\sqrt{m_\ell}$ and the Schur unit (having a 1 in each nonzero entry) weighted by $\sqrt{m_\ell}^{-1}$. In other words, if $a, b \in X_\ell$, then the filled multiplication takes $a \otimes b$ to $\sqrt{m_\ell}\, a \circ b$, and the unit is given by $\bigoplus_\ell \sqrt{m_\ell}^{-1} J_\ell$ where $J_\ell \in X_\ell$ is the all-ones matrix.

The †-CFA on X represented by the empty node ○ is the same as above but the weights are replaced with their multiplicative inverses. This second †-CFA is actually special and so is indeed a quantum set, matching our empty node notation from the rest of the paper.

For the †-CFAs on A, the structures are the same as above, replacing X_ℓ with A_k and $\sqrt{m_\ell}$ with $\sqrt{n_k}$.

When the questions and answers are classical

In the case of classical questions and answers, $m_\ell = 1 = n_k$ for all k, ℓ. This means that the matrix blocks are all size 1, and the coefficients on the operations are all 1, and so the two Frobenius structures (filled and empty) coincide to be the classical quantum set that copies the diagonal matrix units. In this case, (5.2) actually *is* the loop-sliding equation



of Lemma 5.18.

In general, the loop-sliding equation is a consequence of our definition of synchronicity. In the classical question and answer setting, they are equivalent, as loop-sliding says

$$\sum_{x \in X, a \in A} p(a, a|x, x) = |X|$$

or equivalently, $p(a, b|x, x) = 0$ for $a \neq b$. This is equivalent to

$$\delta_{ab} p(a, b|x, x) = p(a, b|x, x)$$

for all a, b, x, which is the commutative interpretation of our synchronicity definition.

This equivalence in the classical setting is to be expected, as both our definition and that of [BKS23] have been shown to generalize typical synchronicity. However, the generalizations lead in different directions for quantum sets. Our definition generalizes the measure-and-encode step of checking synchronicity, replacing that map by sharing, its noncommutative counterpart. This leads to a synchronicity that observes quantum structure. The [BKS23] definition appears to be equivalent to treating the quantum set as a large classical set, in the sense of superdense coding, and then applying the typical definition of synchronicity.

6 Games on quantum graphs

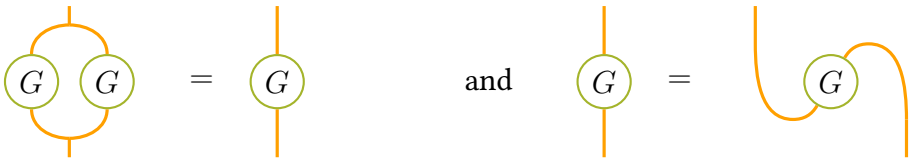
> Meaning is an automatic by-product of our recognition of any isomorphism. - Douglas R. Hofstadter

As an example of how to apply our graphical theory of games, we construct the quantum graph homomorphism game in Section 6.1, generalizing the (classical) graph homomorphism game of [MR16] to allow the question and answer spaces to be quantum graphs. We prove that it is a synchronous game, and we relate its perfect correlations and strategies to quantum graph homomorphisms.

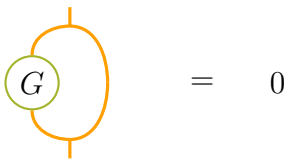
We also define the very similar quantum graph isomorphism game in Section 6.2, generalizing the (classical) graph iosomorphism game of [Ats+16], later studied by [Bra+19]. We prove that it is a bisynchronous game and relate its perfect bicorrelations and bistrategies to quantum graph isomorphisms.

For background on quantum graphs that informs our approach, see [MRV18]. We include the definition here, as well as some tools for working graphically with quantum graphs.

Definition 6.1 (Quantum graph)**.** Given a quantum set X, a *quantum graph* is a self-adjoint linear map $G : X \to X$ satisfying



We say a quantum graph X is *irreflexive* if

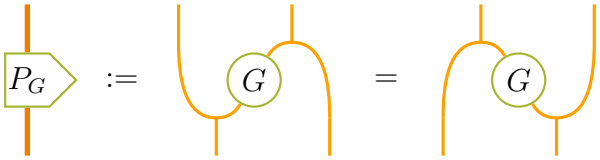


The quantum set X can be thought of as a quantum vertex set. Notice that the definition of a quantum graph looks similar to the definition of a game, plus self-adjointness. In fact, we can equivalently define a quantum graph as a self-adjoint one-player game on V, by treating Bob as a trivial player having questions and answers given by the classical one-element set, and using the fact that for a quantum set X, $X \cong X \otimes \mathbb{C}$.

In order to work more easily with quantum graphs, we introduce thick wire notation for a quantum graph as a state (or effect) of $X \otimes X^{\mathrm{op}}$, as well as notation for an equivalent formulation as a projection onto the "edges". We denote



and



which are both well-defined due to self-conjugacy of G. The notation for P_G should evoke its self-adjointness but non-self-conjugacy.

Another way of thinking about P_G is as the right-multiplication by the graph state $|G\rangle$. Indeed,

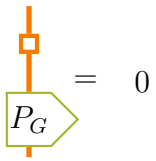


As a first example, consider the *reflexive* quantum graph given by $G = \mathrm{Id}_X$ with P_G given by sharing.

Note that the idempotency condition in the definition of a quantum graph is equivalent to

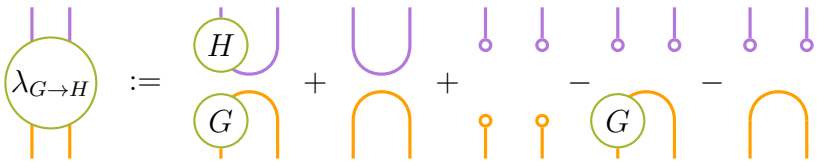


Also note that irreflexivity of G can be equivalently defined using sharing by

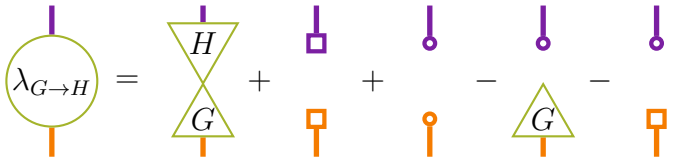


6.1 The quantum graph homomorphism game

Definition 6.2 (Quantum graph homomorphism game)**.** Given irreflexive quantum graphs $G : X \to X$ and $H : A \to A$, the *quantum graph homomorphism game* is denoted $\lambda_{G\to H} : X \otimes X^{\mathrm{op}} \to A \otimes A^{\mathrm{op}}$ and given by



or equivalently, using thick wires,



In order for the above to be a reasonable definition, we should check that it is actually a game.

Proposition 6.3. *The quantum graph homomorphism game is a game.*

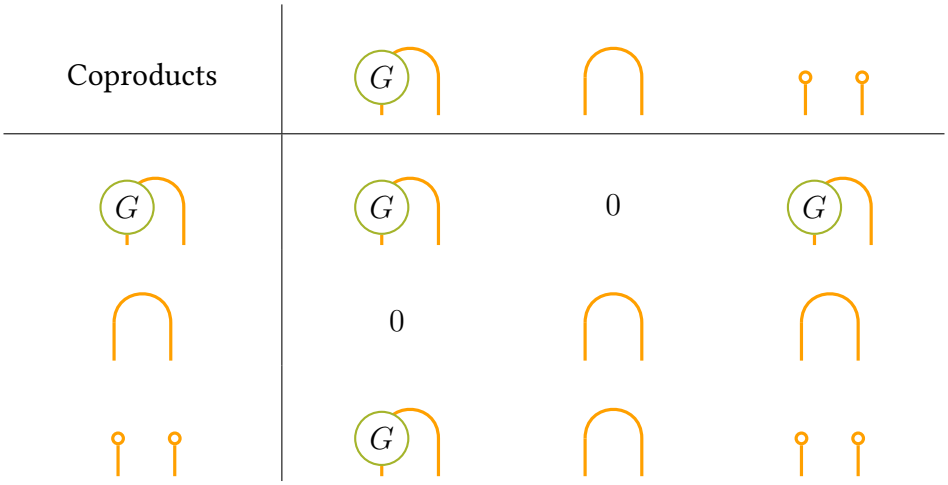
Proof. To see that $\lambda_{G\to H}$ is self-conjugate, note that it is defined by composing only self-conjugate maps.

We will now see that $\lambda_{G\to H}$ satisfies the idempotency condition of Definition 4.12. We have a sum of five terms, leading to twenty-five terms to compute. In order to make this easier, note that all of the terms are disconnected diagrams, so we may compute top and bottom halves separately. We may also take daggers of our computations.

There are three different bottom-halves, so we begin by computing their coproducts, given in Table 6.1. The same holds by taking daggers for products of the top-halves of $\lambda_{G\to H}$.

We use Table 6.2 to compute the twenty-five terms in the sum. Adding up the terms in the table, we get $\lambda_{G\to H}$, concluding the proof. □

Table 6.1: Coproducts of the bottom halves of $\lambda_{G\to H}$.



In the case of classical questions and answers, $\lambda_{G\to H}$ reduces to the graph homomorphism game between classical graphs, as introduced in [MR16]. The rule function in the classical case becomes

$$\lambda_{G\to H}(a,b|x,y) = \langle a|\, H\, |b\rangle\, \langle x|\, G\, |y\rangle + \langle a|b\rangle\langle x|y\rangle + 1 - \langle x|\, G\, |y\rangle - \langle x|y\rangle$$

With $\mathbb{1}_S$ denoting the indicator function for a subset S of all possible variable assignments, we can break the above rule function into three cases.

If $x = y$, then because G is irreflexive, we get

$$\lambda_{G\to H}(a,b|x,y) = \langle a|b\rangle = \mathbb{1}_{a=b}.$$

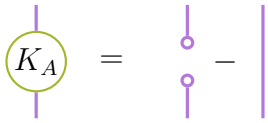
If $x \sim_G y$, we have

$$\lambda_{G\to H}(a,b|x,y) = \langle a|\, H\, |b\rangle = \mathbb{1}_{a\sim_H b}.$$

Finally, if x, y are unrelated by G,

$$\lambda_{G\to H}(a,b|x,y) = 1.$$

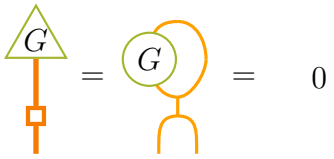
Example 6.4 (Quantum colouring a quantum graph)**.** Let G be any irreflexive quantum graph on X. Let K_A be the irreflexive quantum graph on A given by the diagram



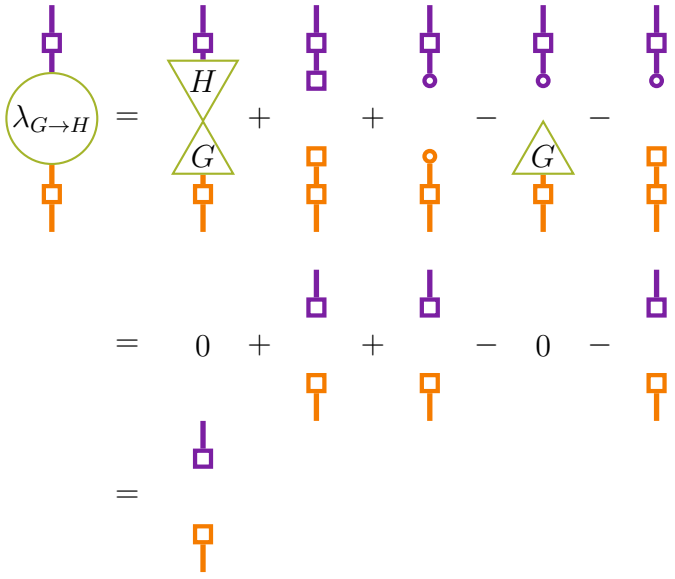
Then the game $\lambda_{G\to K_A}$ can be thought of as a quantum colouring game for G, and if the quantum vertex set A is classical of dimension n, then this simplifies to the classical n-colouring game for G.

Proposition 6.5. *The quantum graph homomorphism game is synchronous.*

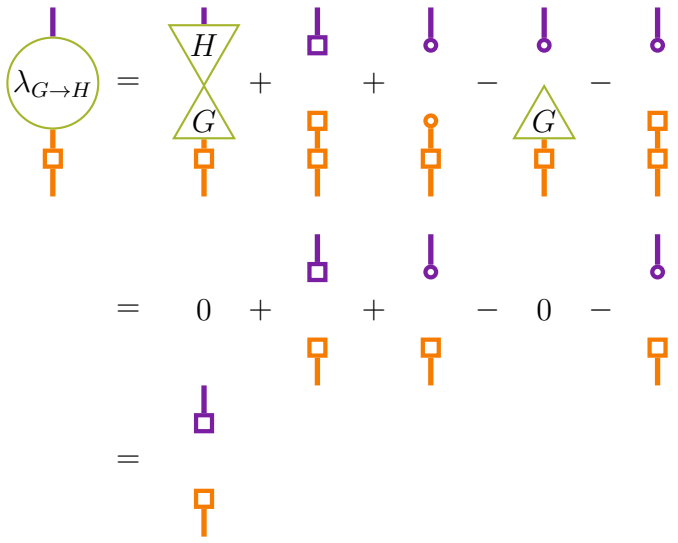
Proof. We evaluate both sides of the synchronicity equation. Let $\lambda := \lambda_{G\to H}$. By irreflexivity of G,



We have



and we also have



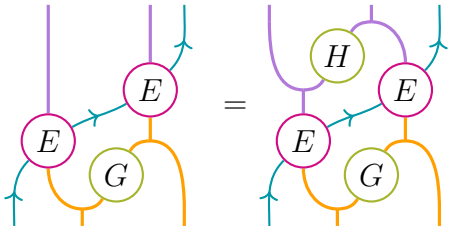
so λ is synchronous. □

One would hope that the quantum graph homomorphism game is connected to some notion of homomorphisms between quantum graphs, as is the case for classical graphs.

In the case of classical graphs, the graph homomorphism game was defined first, and then used to define quantum homomorphisms in [MR16]. Quantum homomorphisms between classical graphs are given by quantum tensor strategies for the graph homomorphism game that realize perfect correlations.

We don't need to take an analogous approach here, as there are already notions in the literature of a quantum homomorphism between quantum graphs, but there is no associated game. We restate here the quantum homomorphism definition of [MRV18].

Definition 6.6 (Quantum homomorphism of quantum graphs, [MRV18])**.** Given quantum graphs $G : X \to X, H : A \to A$, a quantum function $E : X \to_{\mathcal{K}} A$ is said to be a *quantum homomorphism* from G to H if



In [MRV18], the authors show that when the underlying graphs are classical, this notion coincides with the notion of quantum homomorphism coming from the game of [MR16].

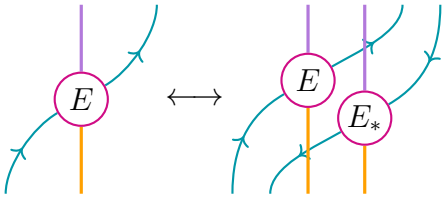
We can now relate quantum homomorphisms to the quantum graph homomorphism game, extending the results mentioned above.

In the remainder of this section, take $G : X \to X$, $H : A \to A$ to be quantum graphs.

Theorem 6.7. *There is a bijective correspondence between*

- *quantum homomorphisms from G to H, and*
- *quantum tensor strategies realizing perfect correlations (with respect to a cup state) for the quantum graph homomorphism game $\lambda_{G\to H}$.*

In order to prove the theorem, we split it into several smaller results, in the sections to come. The correspondence we set out to prove, at the level of the homomorphisms and strategies, is given by



with a quantum homomorphism on the left and a quantum tensor strategy realizing a perfect correlation on the right. We denote the strategy employed on the right by $E \otimes E_*$.

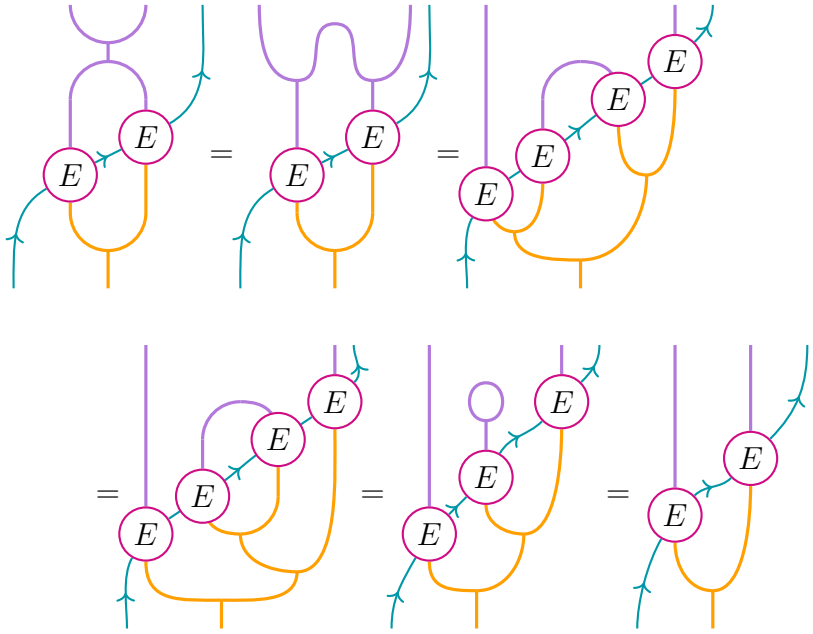
Remark 6.8. The above correspondence is bijective in the sense that there is a bijection between quantum graph homomorphisms and quantum tensor strategies with the given properties. However, note that different quantum tensor strategies may give rise to a single correlation.

6.1.1 Quantum graph homomorphisms produce perfect q-correlations

We begin with the following lemma, allowing us to build a synchronous correlation from two copies of the same quantum function.

Lemma 6.9. *Let $E : X \to_{\mathcal{H}} A$ be a quantum function. Then the quantum tensor strategy $E \otimes E_*$ realizes a synchronous correlation along with the cup state of $\mathcal{H} \otimes \mathcal{H}^*$.*

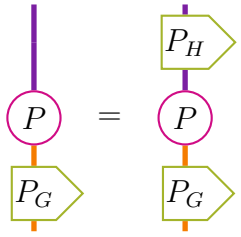
Proof. We first show that the linear map given by composing E with itself on the $\mathcal{H}$ wire is synchronous. We proceed diagramatically.



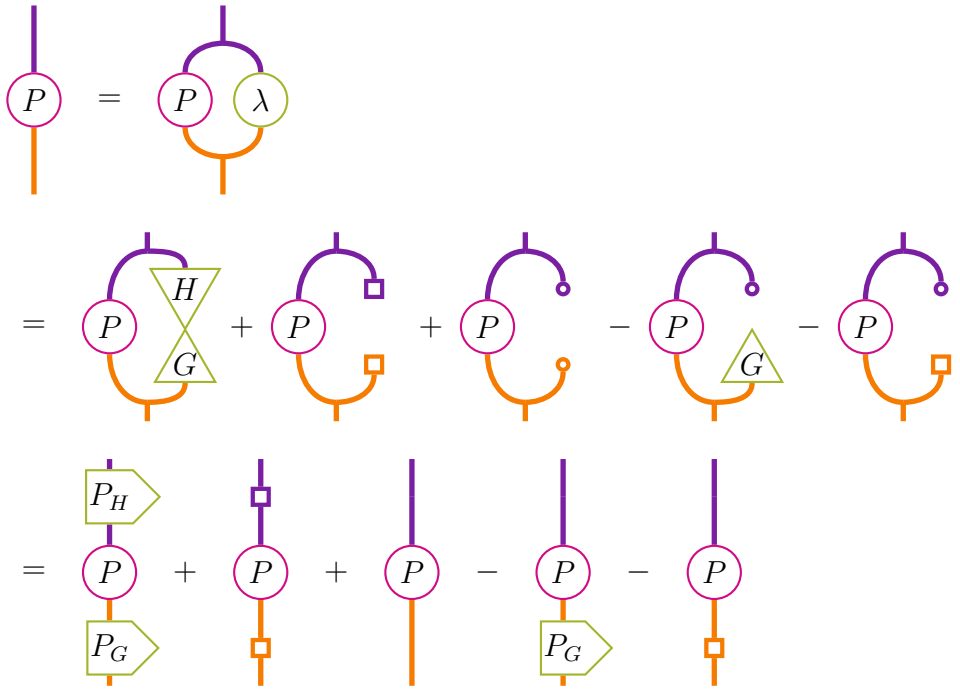
Tracing out the Hilbert space wire on the above equation and then sliding the right copy of E around the cap to get E_* gives the synchronicity condition for the correlation realized by $E \otimes E_*$ with the cup state. □

Next, we demonstrate the structure of perfect correlations for the quantum graph homomorphism game.

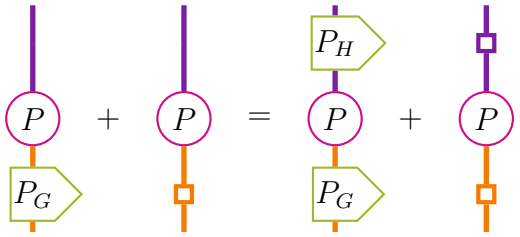
Proposition 6.10. *Perfect correlations P for $\lambda_{G\to H}$ are exactly the synchronous correlations satisfying*



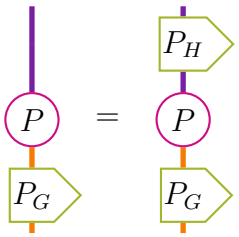
Proof. Let $P : X \otimes X^{\mathrm{op}} \to A \otimes A^{\mathrm{op}}$ be a perfect correlation for $\lambda_{G\to H}$. Then we have that



Cancelling out P from each side and rearranging, we have



By Proposition 6.5, $\lambda_{G \to H}$ is synchronous. Then, by Theorem 5.9, P is also synchronous. Cancelling the terms with sharing,



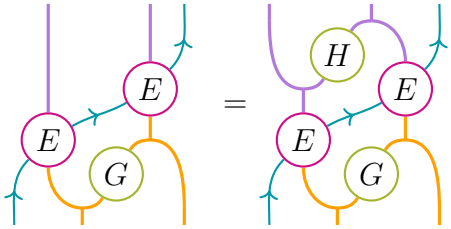
Conversely, if P is synchronous and satisfies the above equation, then we can reverse all of the steps to show that P is perfect for $\lambda_{G \to H}$. □

We now have all the tools we need to prove one direction of the correspondence of Theorem 6.7.

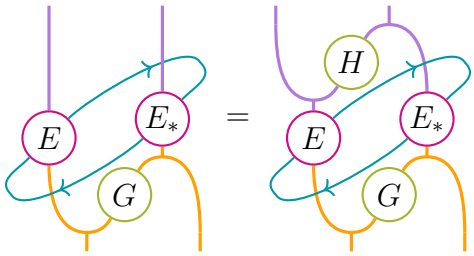
Theorem 6.11. *Given a quantum graph homomorphism E from G to H over $\mathcal{K}$, the quantum tensor strategy $E \otimes E_*$ realizes a perfect correlation for $\lambda_{G\to H}$ with respect to the cup state of $\mathcal{K} \otimes \mathcal{K}^*$.*

Proof. If $E : X \to_{\mathcal{K}} A$ is a quantum homomorphism from G to H, define φ to be the quantum tensor strategy where Alice uses E and Bob uses E_*. Along with the cup state of $\mathcal{K} \otimes \mathcal{K}^*$, φ realizes a synchronous correlation P, as shown in Lemma 6.9.

Because E is a quantum graph homomorphism, we have



Tracing out the $\mathcal{K}$ wire and sliding the righthand E around the cap, we get



which is exactly the equation condition of Proposition 6.10.

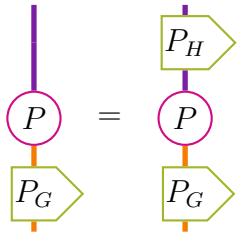
P satisfies the conditions of Proposition 6.10, so P is a perfect correlation for $\lambda_{G\to H}$. □

6.1.2 Perfect q-correlations come from quantum graph homomorphisms

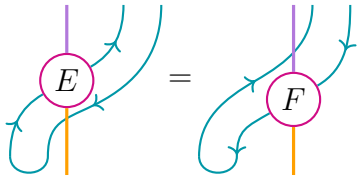
In this section we present the second direction of the proof of the correspondence of Theorem 6.7. Let $\mathcal{K}$ be a Hilbert space, as usual.

Theorem 6.12. *Let P be a perfect correlation for $\lambda_{G\to H}$ arising from a quantum tensor strategy applied to the normalized cup state of $\mathcal{K} \otimes \mathcal{K}^*$. Then the quantum tensor strategy is of the form $E \otimes E_*$ where $E : X \to_{\mathcal{K}} A$ is a quantum graph homomorphism from G to H.*

Proof. By Proposition 6.10, we have

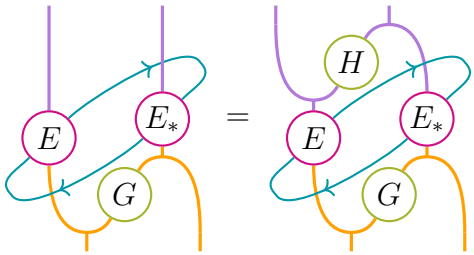


Denote the quantum tensor strategy realizing P by $E \otimes F$, where $\mathcal{H}_A = \mathcal{K}$ and $\mathcal{H}_B = \mathcal{K}^*$, with the resource state being the (normalized) cup state of $\mathcal{K} \otimes \mathcal{K}^*$. Then by Theorem 5.22,



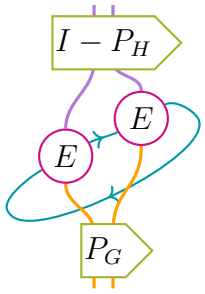
which is equivalent[1] (by the snake equations for $\mathcal{K}$) to $F = E_*$.

At the level of the quantum function E, the fact that P is perfect implies



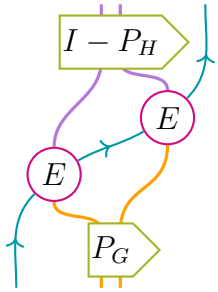
[1]We have not worked explicitly with the conjugates of quantum functions. Note that E_* is also equal to E with the transpose operation applied only to the Hilbert space wires.

or, subtracting the right-hand side from the left and then sliding E_* up around the $\mathcal{K}$ wire to get E, we have



$$= \quad 0 \tag{6.1}$$

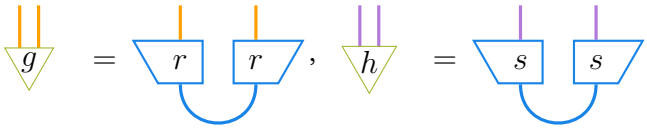
and we need to show that



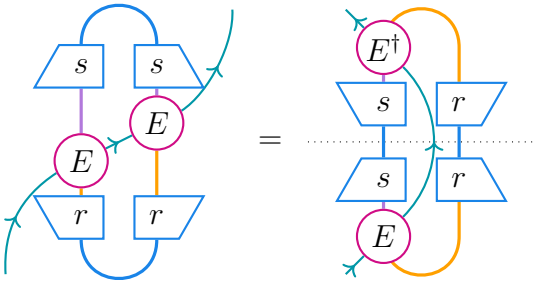
$$= \quad 0$$

In order to do this, let $\{|g\rangle\}, \{|h\rangle\}$ respectively be positive bases for $\mathrm{Im}(P_G) \subseteq \mathrm{End}(X)$, $\mathrm{Im}(I - P_H) \subseteq \mathrm{End}(Y)$ (the ranges of the two projections). We can choose positive bases because we are working with the images of (self-adjoint) projection operators.

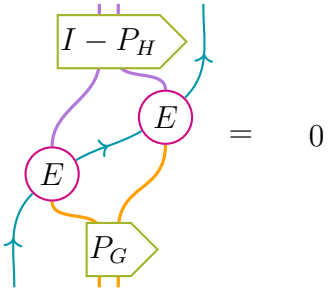
For any fixed choice of $|g\rangle, |h\rangle$, due to positivity, there are maps r, s so that they can be written



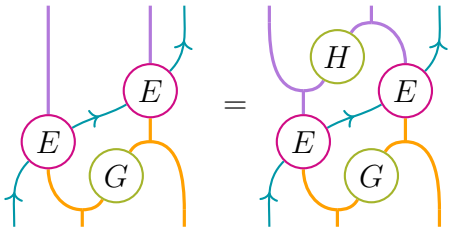
But then, by sliding maps around a loop, the following is a positive element of $B(\mathcal{K})$, as shown by vertical symmetry.



But this has trace zero, by (6.1), and so by positivity, it must be the zero operator. Because this holds in a basis, we can conclude that



which is equivalent to the condition that $E : X \to_{\mathcal{K}} A$ is a quantum graph homomorphism:



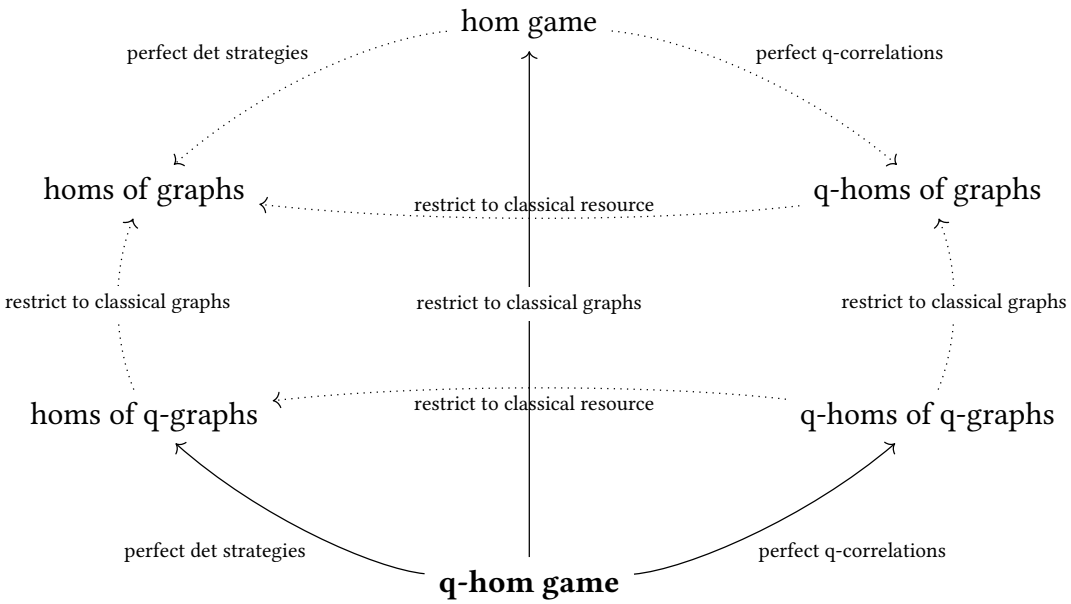
□

6.1.3 Graph homomorphisms and associated games

The preceding results allow us to connect the quantum graph homomorphism game to other notions of graph homomorphism.

We present the relationships, in Figure 6.1, between the quantum and classical graph homomorphism games, and quantum and classical homomorphisms of quantum and classical graphs. Our quantum graph homomorphism game adds a previously missing vertex to this diagram. The solid arrows are due to the work presented here.

Figure 6.1: Relationships between the classical and quantum graph homomorphism games and quantum/classical homomorphisms between quantum/classical graphs.



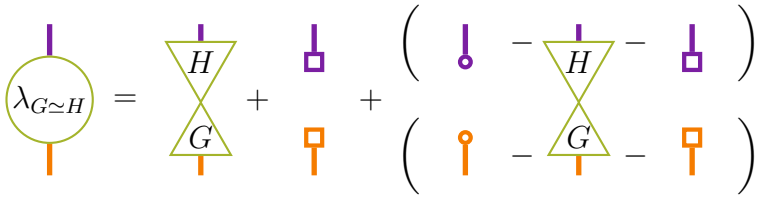
6.2 The quantum graph isomorphism game

It is also possible to similarly construct a quantum graph isomorphism game, but having more terms in the rule function. This can be constructed directly by inclusion-exclusion reasoning, in analogy to how the graph homomorphism game was defined, but can also be constructed from graph homomorphism games as a commuting product.

Definition 6.13 (Quantum graph isomorphism game)**.** Given irreflexive quantum graphs $G : X \to X$ and $H : A \to A$, the *quantum graph isomorphism game* is denoted $\lambda_{G \simeq H} : X \otimes X^{\mathrm{op}} \to A \otimes A^{\mathrm{op}}$ and given by

$$\lambda_{G \simeq H} := \lambda_{G \to H} \star \lambda^{\dagger}_{H \to G}.$$

We quickly check that the commuting product is well defined here. We just need to see that $\lambda_{G \to H}$ and $\lambda^{\dagger}_{H \to G}$ commute. This follows from the fact that Table 6.2 is symmetric, as these are the building blocks of $\lambda_{G \to H}$ and $\lambda^{\dagger}_{H \to G}$. The commuting product $\lambda_{G \to H} \star \lambda^{\dagger}_{H \to G}$ can be computed in either order to give



Proposition 6.14. *The quantum graph isomorphism game is bisynchronous.*

Proof. Because $\lambda_{G\to H}$ is synchronous, then by Proposition 5.13, $\lambda_{G\simeq H}$ is also synchronous. Note further that $\lambda_{G\simeq H}^{\dagger} = \lambda_{H\simeq G}$ is also synchronous so $\lambda_{G\simeq H}$ is bisynchronous. □

Remark 6.15 (Classical graph isomorphism game)**.** In the case of classical graphs, $\lambda_{G\simeq H}$ is a 0-1 matrix with entries $\langle a|\langle b|\lambda_{G\simeq H}|x\rangle|y\rangle$ that are 1 whenever both

$$\langle a|\langle b|\lambda_{G\to H}|x\rangle|y\rangle = 1,$$
$$\langle x|\langle y|\lambda_{H\to G}|a\rangle|b\rangle = 1.$$

Perfect deterministic bistrategies in the classical case are bijections $E : X \to A$ mapping x to a and y to b only when the above conditions are satisfied. But this is exactly the condition that E is a valid graph isomorphism from G to H. This and related results are considered in [LMR20].

This game is a simplified version of the graph isomorphism game of [Ats+16], without some of the rules that force perfect strategies to come from bijections. Instead, we look to *bistrategies* to get bijections, and only force perfect strategies to intertwine the adjacency matrices.

We would like to extend this remark to the quantum game setting, using the definition of quantum isomorphism of quantum graphs, given in [MRV18]. The following bijective correspondence is analogous to Theorem 6.7 from the homomorphism case.

Theorem 6.16. *There is a bijective correspondence between*

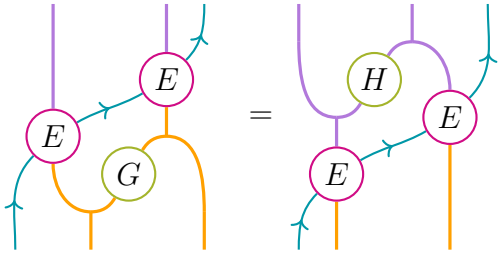
- *quantum isomorphisms from G to H, and*
- *quantum tensor bistrategies realizing perfect bicorrelations (with respect to a cup state) for the quantum graph isomorphism game $\lambda_{G\simeq H}$.*

The proof builds on results of Section 6.1.

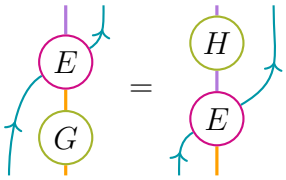
Proof of Theorem 6.16. In the forward direction, given a quantum graph isomorphism $E : G \to H$, we know by Theorem 6.11 $E \otimes E_*$ is a quantum bistrategy for $\lambda_{G\to H}$ and for $\lambda^\dagger_{H\to G}$ that realises a perfect bicorrelation for each, so by Proposition 4.43, $E \otimes E_*$ realises a perfect bicorrelation for the commuting product.

Conversely, given a quantum bistrategy $E \otimes F$ realizing a perfect bicorrelation P for the quantum graph isomorphism game with respect to a cup state, we can apply Theorem 6.12 in both directions, as follows. We know, by Proposition 4.43, that the bicorrelation P is perfect for both $\lambda_{G\to H}$ and $\lambda^\dagger_{H\to G}$. By Theorem 6.12, $F = E_*$ and E is a quantum homomorphism $G \to H$. We can also conclude that $P^\dagger$ is a perfect correlation for $\lambda_{H\to G}$, so again by Theorem 6.12, $\overline{E}$ is a quantum homomorphism $H \to G$.

But then, by combining these two quantum homomorphism equations (and taking daggers in the case of $\overline{E}$), we get that E is a quantum bijection satisfying



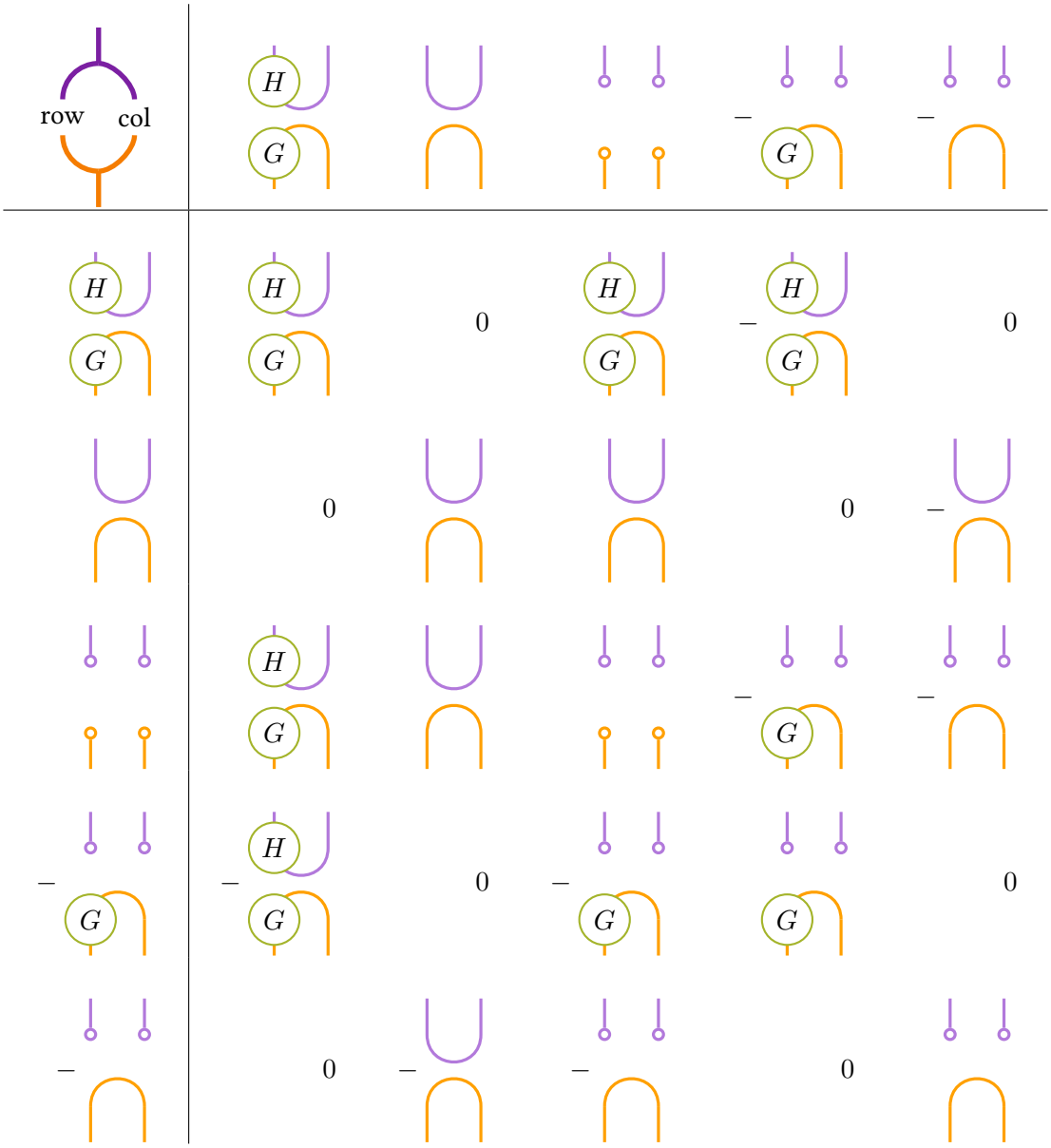
Applying a counit on the top left and a unit on the bottom right, we get



so E is a quantum isomorphism from G to H, as per [MRV18]. □

Corollary 6.17 (of Theorem 6.16)**.** *Perfect deterministic bistrategies for $\lambda_{G\simeq H}$ are in one-to-one correspondence with graph isomorphisms from G to H.*

Table 6.2: Commuting products of the terms of $\lambda_{G\to H}$.



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