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Around the closure of the set of commutators of idempotents in $\mathcal{B}(\mathcal{H})$: Biquasitriangularity and factorisation

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In this paper, we continue our study of the norm-closure of the set $\mathfrak{C}_{\mathfrak{E}}$ of bounded linear operators acting on a complex, infinite-dimensional, separable Hilbert space $\mathcal{H}$ which may be expressed as the commutator of two idempotent operators. In particular, we identify which biquasitriangular operators belong to the norm-closure $\text{CLOS}(\mathfrak{C}_{\mathfrak{E}})$ of $\mathfrak{C}_{\mathfrak{E}}$, and we exhibit an index obstruction to membership in $\text{CLOS}(\mathfrak{C}_{\mathfrak{E}})$. Finally, we consider factorisations of bounded linear operators on $\mathcal{H}$ as sums and products of elements in $\mathfrak{C}_{\mathfrak{E}}$ and related sets.

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1. Introduction

1.1. Let $\mathcal{H}$ be a complex, separable Hilbert space, and let $\mathcal{B}(\mathcal{H})$ denote the algebra of bounded linear operators acting on $\mathcal{H}$. By $\mathcal{K}(\mathcal{H})$ we denote the two-sided ideal of compact operators in $\mathcal{B}(\mathcal{H})$. An element $E \in \mathcal{B}(\mathcal{H})$ is said to be **idempotent** if $E^2 = E$, and we denote by $\mathfrak{E}$ the set of all idempotent operators on $\mathcal{H}$. A number of authors [22, 36,18] have considered the problem of determining which elements of $\mathcal{B}(\mathcal{H})$ may be written as a **commutator** $[E, F] := EF - FE$ or a difference $E - F$ of idempotents. As pointed out in the paper [28], one very good reason for being interested in this problem is that if there exists a pair $A, B \in \mathcal{B}(\mathcal{H})$ of operators each satisfying a quadratic equation (meaning that each is a translation of an idempotent operator by a scalar operator), and if the commutator of A and B is quasinilpotent but not triangularisable, then there exists an operator $T \in \mathcal{B}(\mathcal{H})$ without invariant subspaces! In the finite-dimensional setting, characterisations of the sets $\mathfrak{C}_{\mathfrak{E}} = \{[E, F] : E, F \in \mathfrak{E}\}$ and $\mathfrak{D}_{\mathfrak{E}} := \{E - F : E, F \in \mathfrak{E}\}$ have been obtained [22,18]. The corresponding characterisations in the infinite-dimensional setting are far more delicate; partial results are due to Drnovšek, Radjavi and Rosenthal [18] and Wang and Wu [36].

The fact that $E \in \mathfrak{E}$ implies that $S^{-1}ES \in \mathfrak{E}$ whenever $S \in \mathcal{B}(\mathcal{H})$ is invertible implies that both $\mathfrak{C}_{\mathfrak{E}}$ and $\mathfrak{D}_{\mathfrak{E}}$ are invariant under conjugation by invertible operators. The work of Herrero on norm-approximation of similarity-invariant sets in the 1980s led him to conjecture the existence of a metatheorem [24] which suggests that if a similarity-invariant set $\mathcal{W} \subseteq \mathcal{B}(\mathcal{H})$ has "*sufficient structure*", then the list of all spectral, semi-Fredholm and algebraic conditions that the Riesz-Dunford functional calculus and the stability properties related to the set of semi-Fredholm operators impose upon elements of the norm-closure $\text{CLOS}\,(\mathcal{W})$ of $\mathcal{W}$ are also sufficient to characterise membership in that set. (We shall not require a precise definition of "sufficient structure" here – we refer the reader to Herrero's paper.) Loosely stated, to characterise $\text{CLOS}\,(\mathcal{W})$, one writes down the list of spectral, index and algebraic conditions required to belong to that set, and then goes about proving that the necessary conditions are also sufficient. While that last step is typically quite difficult – such problems have the advantage that one always has a conjecture as to what the characterisation should be.

In this paper, we adopt Herrero's approach as we continue our work from [25] on the problem of characterising the norm-closure of $\mathfrak{C}_{\mathfrak{E}}$. In the first paper, we described $\text{CLOS}\,(\mathfrak{C}_{\mathfrak{E}})$ and $\text{CLOS}\,(\mathfrak{D}_{\mathfrak{E}})$ when $\dim \mathcal{H} < \infty$, as well as describing $\mathcal{K}(\mathcal{H}) \cap \text{CLOS}\,(\mathfrak{C}_{\mathfrak{E}})$ in the infinite-dimensional setting. Our current goal is three-fold:

- firstly, we shall characterise the set of biquasitriangular operators which lie in $\text{CLOS}\,(\mathfrak{C}_{\mathfrak{E}})$. The biquasitriangular operators are those operators T for which $T - \lambda I$ has semi-Fredholm index equal to zero whenever it is defined. We shall obtain our characterisation by first establishing that all nilpotent operators belong to $\text{CLOS}\,(\mathfrak{C}_{\mathfrak{E}})$, and using this to help us describe the set of normal operators which lie in that set. From a theorem of Apostol, Foiaş and Voiculescu [4], one deduces that the set (BQT) of biquasitriangular operators agrees with the norm-closure of the set of all operators which are similar to normal operators, and this will prove useful in extending our result for normal operators in $\text{CLOS}(\mathfrak{C}_{\mathfrak{E}})$ to biquasitriangular operators in $\text{CLOS}(\mathfrak{C}_{\mathfrak{E}})$.

- Next, we take a first step towards determining which non-biquasitriangular operators belong to $\text{CLOS}(\mathfrak{C}_{\mathfrak{E}})$ by establishing an index obstruction which allows us to deduce that for the unilateral forward shift S and $\kappa \in \mathbb{C}$, we have that $\kappa S \in \text{CLOS}(\mathfrak{C}_{\mathfrak{E}})$ if and only if $|\kappa| \leq \frac{1}{2}$.
- Finally, we consider a number of factorisation problems for operators in $\mathcal{B}(\mathcal{H})$. More specifically, we determine that every operator in $\mathcal{B}(\mathcal{H})$ may be written as a sum of at most five elements of $\mathfrak{C}_{\mathfrak{E}}$, and as a product of at most three elements of that set.

1.2 Notation. Let us now establish a number of definitions, notations and observations we shall require below.

If $\varnothing \neq K \subseteq \mathbb{C}$ and $\delta > 0$, then $(K)_\delta := \{z \in \mathbb{C} : |z - k| < \delta \text{ for some } k \in K\}$, while $[K]_\delta := \{z \in \mathbb{C} : |z - k| \leq \delta \text{ for some } k \in K\}$.

Given an operator $T \in \mathcal{B}(\mathcal{H})$, we denote by $\sigma_p^0(T)$ the set of all isolated eigenvalues of finite multiplicity of T. For a Borel subset $\Omega \subseteq \mathbb{C}$ with $\Omega \cap \sigma(T)$ being a clopen subset of $\sigma(T)$, we denote by $\dim \mathcal{H}(\Omega; T)$ the dimension of the Riesz subspace for T corresponding to the set $\Omega \cap \sigma(T)$.

Two operators $A, B \in \mathcal{B}(\mathcal{H})$ are **approximately unitarily equivalent** (and we write $A \simeq_a B$) if there exists a sequence $(U_n)_n$ of unitary operators such that $B = \lim_n U_n^* A U_n$. It is readily verified that this is an equivalence relation. For $A \in \mathcal{B}(\mathcal{H})$, we denote by $\mathcal{U}(A) := \{U^* A U : \text{unitary } U \in \mathcal{B}(\mathcal{H})\}$ the **unitary orbit** of A, while $\mathcal{S}(A) := \{S^{-1} A S : \text{invertible operator } S \in \mathcal{B}(\mathcal{H})\}$ denotes the **similarity orbit** of A.

Let $1 \leq p < \infty$ be a real number. The **Schatten p-class** $\mathcal{C}_p(\mathcal{H})$ consists of those compact operators K with the property that the eigenvalues $(s_n)_n$ of $|K| := (K^*K)^{\frac{1}{2}}$ (i.e. the **singular numbers** of K) belong to ℓ^p. Each $\mathcal{C}_p(\mathcal{H})$ is a Banach space with the norm $\|K\|_p := \|(s_n)_n\|_p$, and each $\mathcal{C}_p(\mathcal{H})$ is a two-sided ideal of $\mathcal{B}(\mathcal{H})$. We refer the reader to [13] for more information on these sets. The space $\mathcal{C}_1(\mathcal{H})$ is also known as the set of **trace class** operators, while $\mathcal{C}_2(\mathcal{H})$ is referred to as the set of **Hilbert-Schmidt** operators.

Let $\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ denote the canonical quotient map of $\mathcal{B}(\mathcal{H})$ into the Calkin algebra. The **essential spectrum** of T is the set $\sigma_e(T) = \{\lambda \in \mathbb{C} : \pi(T) \text{ is not invertible in } \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})\}$. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be **semi-Fredholm** if $\pi(T)$ is either *left*- or *right*-invertible in the Calkin algebra. We recall that the **semi-Fredholm domain** of an operator $T \in \mathcal{B}(\mathcal{H})$ is defined as

$$\rho_{\text{s-F}}(T) := \{\lambda \in \mathbb{C} : (T - \lambda I) \text{ is semi-Fredholm}\}.$$

When T is semi-Fredholm, we define the **semi-Fredholm index** of T to be:

$$\text{IND}\, T := \text{NUL}\, T - \text{NUL}\, T^*.$$

This is well-defined (as an element of $\mathbb{Z} \cup \{\pm\infty\}$, as at least one of $\text{NUL}\, T$ and $\text{NUL}\, T^*$ is necessarily finite). A standard result [10] shows that the set of semi-Fredholm operators with a fixed index $m \in \mathbb{Z} \cup \{\pm\infty\}$ is open.

In the event that T is semi-Fredholm and its semi-Fredholm index is finite, we say that T is **Fredholm**. In this case, $\pi(T)$ is invertible in $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ [10]. We shall denote the set of Fredholm operators on $\mathcal{H}$ by $\text{FRED}(\mathcal{H})$.

We shall also denote the **spectral radius** of T by $\text{SPR}(T) = \max\{|\alpha| : \alpha \in \sigma(T)\}$.

An **involution** is an invertible operator $S \in \mathcal{B}(\mathcal{H})$ such that $S = S^{-1}$. We define the sets

$$\text{NIL}(\mathcal{H}) := \{T \in \mathcal{B}(\mathcal{H}) : T^k = 0 \text{ for some } k \geq 1\};$$
$$\text{NEG}_S(\mathcal{H}) := \{T \in \mathcal{B}(\mathcal{H}) : T \text{ is similar to } -T\};$$
$$\text{NEG}_{\text{INVS}}(\mathcal{H}) := \{T \in \mathcal{B}(\mathcal{H}) : T \text{ is involution-similar to } -T\}.$$

1.3 Definition. An element $T \in \mathcal{B}(\mathcal{H})$ is said to be **weakly balanced** if

(a) $\sigma(T) = \sigma(-T)$.
(b) If G_1, G_2 are disjoint open sets such that $\sigma(T) \subseteq G_1 \cup G_2$, then

$$\dim\, \mathcal{H}(G_1; T) = \dim\, \mathcal{H}(-G_1; T).$$

We say that T is **balanced** if T is weakly balanced *(*in the above sense*)* and for all $\lambda \in \mathbb{C}$, $T - \lambda I$ is semi-Fredholm if and only if $T + \lambda I$ is semi-Fredholm, in which case we also require that

$$\text{IND}\,(T - \lambda I) = \text{IND}\,(T + \lambda I).$$

We denote by $\text{WBAL}(\mathcal{H})$ (resp. $\text{BAL}(\mathcal{H})$) the set of all elements of $\mathcal{B}(\mathcal{H})$ that are weakly balanced (resp. balanced).

It is worth observing (and we shall use this fact below) that if $T \in \mathcal{B}(\mathcal{H})$ is balanced, then $\sigma_e(T) = \sigma_e(-T)$. Indeed, if T is balanced and $\lambda \notin \sigma_e(T)$, then $T - \lambda I$ is Fredholm and has finite index, whence $T + \lambda I$ is Fredholm with finite index, i.e. $-\lambda \notin \sigma_e(T)$ or equivalently, $\lambda \notin \sigma_e(-T)$. It is routine to verify that T is balanced if and only if $-T$ is balanced, from which the result easily follows.

We recall the following result [25, Proposition 2.2]:

1.4 Proposition. *For any Hilbert space $\mathcal{H}$,*

$$\mathfrak{C}_{\mathfrak{E}} \subseteq \text{NEG}_{\text{INVS}}(\mathcal{H}) \subseteq \text{NEG}_S(\mathcal{H}) \subseteq \text{BAL}(\mathcal{H}).$$

1.5. Finally, given a non-empty subset $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H})$, we denote by $\text{Lat}\,\mathcal{S}$ the **invariant subspace lattice** of $\mathcal{S}$;

$$\text{Lat}\,\mathcal{S} := \{\mathcal{M} \subseteq \mathcal{H} : \mathcal{M} \text{ a closed subspace of } \mathcal{H} \text{ and } S\mathcal{M} \subseteq \mathcal{M} \text{ for all } S \in \mathcal{S}\}.$$

When $\mathcal{S} = \{S\}$ is a singleton set, we simply write $\text{Lat}\,S$.

2. Nilpotent operators in $\mathfrak{C}_{\mathfrak{E}}$ and in clos $(\mathfrak{C}_{\mathfrak{E}})$

2.1. Recall that an element q of a Banach algebra $\mathcal{A}$ is said to be **quasinilpotent** if $\sigma(q) = \{0\}$. As we shall now see, a quasinilpotent operator $Q \in \mathcal{B}(\mathcal{H})$ belongs to $\mathfrak{C}_{\mathfrak{E}}$ if and only if there exists an involution $J \in \mathcal{B}(\mathcal{H})$ such that $J^2 = I$ and $-Q = JQJ$. Unfortunately, given $Q \in \mathcal{B}(\mathcal{H})$ quasinilpotent, it is highly non-trivial to determine whether or not such an involution exists.

We first require the following result from [18, Theorem 1].

2.2 Theorem *(Drnovsek, Radjavi and Rosenthal). Let R be a unital ring with identity 1 in which the element 2 is invertible, and $t \in R$. Then t is a commutator of a pair of idempotents in R if and only if there exist j and $s \in R$ such that $j^2 = 1$, $jt + tj = 0$, $js = sj$, $st = ts$ and $s^2 = t^2 + \frac{1}{4}$.*

In other words, t is a commutator of idempotents in R if and only if $t^2 + \frac{1}{4}$ admits a square root which not only commutes with t, but also with an involution j which implements the similarity of t and $-t$.

We say that a compact subset $K \subseteq \mathbb{C}$ **does not separate** 0 **from** ∞ if 0 lies in the unbounded component of $\mathbb{C} \setminus K$.

2.3 Proposition. *Let $T \in \mathcal{B}(\mathcal{H})$. If $T \in \mathfrak{C}_{\mathfrak{E}}$, then T is similar to $-T$ via an involution operator. Conversely, if the spectrum of $T^2 + \frac{1}{4}I$ does not separate 0 from ∞ and T is similar to $-T$ via an involution operator, then $T \in \mathfrak{C}_{\mathfrak{E}}$.*

Proof. If $T \in \mathfrak{C}_{\mathfrak{E}}$, then by Theorem 2.2, there exists $J \in \mathcal{B}(\mathcal{H})$ such that $J^2 = I$ and $JT + TJ = 0$, or equivalently, $-T = JTJ$.

To prove the converse, suppose that J is an involution operator and that $JT + TJ = 0$. Then

$$J(T^2 + \frac{1}{4}I) = -TJT + \frac{1}{4}J = (T^2 + \frac{1}{4}I)J.$$

Since $T^2 + \frac{1}{4}I$ does not separate 0 from ∞, from the Riesz functional calculus, $T^2 + \frac{1}{4}I$ admits a square root S which lies in the norm-closed algebra generated by $T^2 + \frac{1}{4}I$. Since $T^2 + \frac{1}{4}I$ commutes with both J and T, we have that $SJ = JS$ and $ST = TS$. By Theorem 2.2, $T \in \mathfrak{C}_{\mathfrak{E}}$. $\square$

2.4 Corollary. *Let $T \in \mathcal{B}(\mathcal{H})$ with* $\text{SPR}(T) < \frac{1}{2}$ *or T be a compact operator with $\pm\frac{\mathrm{i}}{2} \notin \sigma(T)$. Then $T \in \mathfrak{C}_{\mathfrak{E}}$ if and only if T is similar to $-T$ via an involution operator.*

In particular, if $Q \in \mathcal{B}(\mathcal{H})$ is quasinilpotent, then $Q \in \mathfrak{C}_{\mathfrak{E}}$ if and only if Q is similar to $-Q$ via an involution operator.

The next corollary will prove useful in Section 3.

2.5 Corollary. *Let $Q \in \mathcal{B}(\mathcal{H})$ be a quasinilpotent operator and let $\alpha \in \mathbb{C} \setminus \{\pm\frac{i}{2}\}$. Then*

$$R_\alpha := \begin{bmatrix} \alpha I + Q & 0 \\ 0 & -\alpha I - Q \end{bmatrix} \in \mathfrak{C}_{\mathfrak{E}}.$$

Proof. Note that R_α is similar to $-R_\alpha$ via the involution operator $\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$. If $\alpha \neq \pm\frac{i}{2}$, then $\sigma(R_\alpha^2 + \frac{1}{4}I) = \{\alpha^2 + \frac{1}{4}\}$ does not separate 0 from ∞. By Proposition 2.3, $R_\alpha \in \mathfrak{C}_{\mathfrak{E}}$. □

2.6. There have been multiple examples of nilpotent operators M of order three such that M is not similar to $-M$ (and consequently $M \notin \mathfrak{C}_{\mathfrak{E}}$). These examples resemble each other in the sense that they are all of the form

$$M = \begin{bmatrix} 0 & D & I \\ 0 & 0 & D \\ 0 & 0 & 0 \end{bmatrix}$$

for an appropriate operator $D \in \mathcal{K}(\mathcal{H})$. The first construction we know of is [3, Example 9.48] where $D = \text{DIAG}(\frac{1}{n!})$. There, Apostol, Fialkow, Herrero and Voiculescu show that $M \sim \alpha M$ if and only if $\alpha = 1$. The second construction of which we are aware appears in the proof of [36, Proposition 1.11] where $D = \text{DIAG}(\varepsilon^n)$ for some fixed $\varepsilon > 0$.

In fact, the family of examples of nilpotent operators M of this form which are not similar to their negatives can be extended considerably, as we shall now prove.

2.7 Lemma. *Suppose that $0 \leq P$ is of the Hilbert-Schmidt class, but that it is not a trace class operator. If $Z \in \mathcal{B}(\mathcal{H})$ is invertible, then*

$$ZPZ^{-1} + P \notin \mathcal{C}_1(\mathcal{H}).$$

Proof. We shall argue by contradiction. Suppose, to that end, that $ZPZ^{-1} + P \in \mathcal{C}_1(\mathcal{H})$. Let $Z = U|Z|$ be the polar decomposition of Z, and observe that U is unitary. Since $\mathcal{C}_1(\mathcal{H})$ is invariant under unitary conjugation (it is in fact an ideal of $\mathcal{B}(\mathcal{H})$), we see that

$$|Z|P|Z|^{-1} + U^*PU \in \mathcal{C}_1(\mathcal{H}).$$

Moreover, $|Z|$ is normal, and thus by a result of Voiculescu [35] (see also [14]), given $\varepsilon > 0$, there exists a Hilbert-Schmidt operator K with $\|K\| \leq \|K\|_2 < \varepsilon$ such that $D := |Z| - K$ is diagonalisable. Note that by choosing ε sufficiently small (for example $0 < \varepsilon < \|Z^{-1}\|^{-1}$), we may guarantee that D is invertible.

Defining $L := |Z|^{-1} - D^{-1}$, we have that

$$I = |Z|^{-1}|Z| = (D^{-1} + L)(D + K) = I + D^{-1}K + LD + LK.$$

Thus

$$LD = -(D^{-1} + L)K \in \mathcal{C}_2(\mathcal{H}).$$

Since D is invertible and $\mathcal{C}_2(\mathcal{H})$ is also an ideal of $\mathcal{B}(\mathcal{H})$, $L = (LD)D^{-1} \in \mathcal{C}_2(\mathcal{H})$.

Thus

$$\begin{aligned}|Z|P|Z|^{-1} + U^*PU &= (D + K)P(D^{-1} + L) + U^*PU\\ &= DPD^{-1} + DPL + KPD^{-1} + KPL + U^*PU.\end{aligned}$$

Since P, L and $K \in \mathcal{C}_2(\mathcal{H})$, and since the product of two Hilbert-Schmidt operators is a trace-class operator, it follows that DPL, KPD^{-1} and $KPL \in \mathcal{C}_1(\mathcal{H})$. From this we see that

$$DPD^{-1} + U^*PU \in \mathcal{C}_1(\mathcal{H}).$$

Fix an orthonormal basis $\{e_n\}_n$ relative to which D is diagonal, say $D = \textsc{diag}(d_n)_n$. Let $\alpha_n := \langle Pe_n, e_n\rangle$ and $\beta_n := \langle PUe_n, Ue_n\rangle$, $n \geq 1$. Observe that $\alpha_n, \beta_n \geq 0$ for all n.

Since $DPD^{-1} + U^*PU \in \mathcal{C}_1(\mathcal{H})$, we obtain (keeping in mind that series of positive terms may be rearranged in any order without changing their sum):

$$\begin{aligned}\textsc{tr}(DPD^{-1} + U^*PU) &= \sum_n \langle PD^{-1}e_n, D^*e_n\rangle + \langle PUe_n, Ue_n\rangle\\ &= \sum_n \langle Pd_n^{-1}e_n, \overline{d_n}e_n\rangle + \beta_n\\ &= \sum_n (\alpha_n + \beta_n)\\ &= \sum_n \alpha_n + \sum_n \beta_n.\end{aligned}$$

But $\sum_n \alpha_n = \sum_n \beta_n = \textsc{tr}(P) = \infty$, since $P \notin \mathcal{C}_1(\mathcal{H})$, a contradiction. Thus

$$ZPZ^{-1} + P \notin \mathcal{C}_1(\mathcal{H}). \quad \square$$

2.8 Proposition. *Let $0 \leq D$ be an element of $\mathcal{C}_3(\mathcal{H})$ such that D is injective, has dense range, and D^2 is not a trace-class operator. Let*

$$M := \begin{bmatrix} 0 & D & I \\ 0 & 0 & D \\ 0 & 0 & 0 \end{bmatrix}.$$

Then M is not similar to $-M$.

Proof. Suppose that $S = [S_{ij}]$ is invertible and $SMS^{-1} = -M$. Then $SM = -MS$ and so

$$\begin{bmatrix} 0 & S_{11}D & S_{11} + S_{12}D \\ 0 & S_{21}D & S_{21} + S_{22}D \\ 0 & S_{31}D & S_{31} + S_{32}D \end{bmatrix} = - \begin{bmatrix} DS_{21} + S_{31} & DS_{22} + S_{32} & DS_{23} + S_{33} \\ DS_{31} & DS_{32} & DS_{33} \\ 0 & 0 & 0 \end{bmatrix}.$$

Since D is injective with dense range, we immediately conclude by considering the $(3,2)$-operator entry that $S_{31} = 0$. By next looking at the $(1,1)$- and the $(3,3)$-operator entries, we conclude that $S_{21} = 0 = S_{32}$, whence

$$S = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ 0 & S_{22} & S_{23} \\ 0 & 0 & S_{33} \end{bmatrix}.$$

A similar computation applied to S^{-1} shows that it too is block upper-triangular, and thus that S_{jj} is invertible, $1 \leq j \leq 3$.

The above matrix equation also implies that

(i) $S_{11}D = -DS_{22}$;
(ii) $S_{22}D = -DS_{33}$; and
(iii) $S_{11} + S_{12}D = -DS_{23} - S_{33}$.

From these we see that

(iv) $S_{11}D^2 = D^2 S_{33}$, and that
(v) $S_{11} + S_{33} = -(S_{12}D + DS_{23})$.

Hence

$$(S_{11} + S_{33})D^2 = S_{11}D^2 + S_{33}D^2 = D^2 S_{33} + S_{33}D^2 = -(S_{12}D + DS_{23})D^2.$$

Observe that $S_{12}D, DS_{23} \in \mathcal{C}_3(\mathcal{H})$ since $D \in \mathcal{C}_3(\mathcal{H})$ and the latter is an ideal, and that $D^2 \in \mathcal{C}_{3/2}(\mathcal{H})$ since $D \in \mathcal{C}_3(\mathcal{H})$. From this and Hölder's inequality with $p = 3$ and $q = 3/2$, it follows that the right-hand side of the above equation lies in $\mathcal{C}_1(\mathcal{H})$, the set of trace-class operators on $\mathcal{H}$.

As for the left-hand side of the equation, by setting $Z := S_{33}$ and $P = D^2$ in Lemma 2.7, we see that

$$S_{33}D^2S_{33}^{-1} + D^2 \notin \mathcal{C}_1(\mathcal{H}),$$

whence

$$S_{33}D^2 + D^2S_{33} \notin \mathcal{C}_1(\mathcal{H}),$$

a contradiction. Thus M is not similar to $-M$. □

2.9. As previously mentioned, it is not always easy to determine whether or not a given quasinilpotent operator M is similar to $-M$. For example, let V denote the classical Volterra operator in $\mathcal{B}(L^2([0,1],dx))$. As of yet, we have been unable to determine whether $M \sim -M$, where

$$M = \begin{bmatrix} 0 & V & I \\ 0 & 0 & V \\ 0 & 0 & 0 \end{bmatrix}.$$

More delicate still is the following question which we leave to the interested reader.

Question. Suppose that $M \in \mathcal{B}(\mathcal{H})$ is nilpotent of order $k \geq 3$ and that M is similar to $-M$. Does there exist an *involution* $J \in \mathcal{B}(\mathcal{H})$ such that $-M = JMJ$?

Fortunately, the question of determining which nilpotent operators lie in $\mathrm{CLOS}\,(\mathfrak{C}_{\mathfrak{E}})$ is much more tractable, and we now turn our attention to this problem.

2.10. If $\mathfrak{X}$ and $\mathfrak{Y}$ are Banach spaces, $R \in \mathcal{B}(\mathfrak{Y})$, $T \in \mathcal{B}(\mathfrak{X})$, we may define the corresponding **Sylvester-Rosenblum operator**

$$\begin{array}{rccc} \tau_{R,T}: & \mathcal{B}(\mathfrak{X},\mathfrak{Y}) & \to & \mathcal{B}(\mathfrak{X},\mathfrak{Y}) \\ & Z & \mapsto & RZ - ZT. \end{array}$$

It is a standard result (due independently to Sylvester in the matrix setting [32] and to a number of people (see [8]) including Krein, Dalecki and Rosenblum [29] in the operator setting) that if $\sigma(R) \cap \sigma(T) = \varnothing$, then $\tau_{R,T}$ is invertible. Indeed, the operators

$$\begin{array}{rccc} \lambda_R: & \mathcal{B}(\mathfrak{X},\mathfrak{Y}) & \to & \mathcal{B}(\mathfrak{X},\mathfrak{Y}) \\ & Z & \mapsto & RZ \end{array} \quad \text{and} \quad \begin{array}{rccc} \varrho_T: & \mathcal{B}(\mathfrak{X},\mathfrak{Y}) & \to & \mathcal{B}(\mathfrak{X},\mathfrak{Y}) \\ & Z & \mapsto & ZT \end{array}$$

are commuting elements of $\mathcal{B}(\mathcal{B}(\mathfrak{X},\mathfrak{Y}))$, and by standard Banach algebra techniques, it follows that

$$\sigma(\tau_{R,T}) = \sigma(\lambda_R - \varrho_T) \subseteq \sigma(\lambda_R) - \sigma(\varrho_T) \subseteq \sigma(R) - \sigma(T).$$

We shall use this in the following way. Suppose that $A \in \mathcal{B}(\mathcal{H})$ and that $\mathcal{M} \in \text{Lat}\, A$. Write

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & A_4 \end{bmatrix}$$

relative to the decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$. If $\sigma(A_1) \cap \sigma(A_4) = \varnothing$, then A is similar to $B := A_1 \oplus A_4$. Indeed, since $\tau_{A_1,A_4} : \mathcal{B}(\mathcal{M}^\perp, \mathcal{M}) \to \mathcal{B}(\mathcal{M}^\perp, \mathcal{M})$ is invertible, there exists $X \in \mathcal{B}(\mathcal{M}^\perp, \mathcal{M})$ such that $A_1 X - X A_4 = A_2$.

Let $S = \begin{bmatrix} I & X \\ 0 & I \end{bmatrix}$ (relative to the same decomposition of $\mathcal{H}$), and note that S is clearly invertible. Furthermore,

$$S^{-1} B S = \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix} \begin{bmatrix} A_1 & 0 \\ 0 & A_4 \end{bmatrix} \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} = \begin{bmatrix} A_1 & A_1 X - X A_4 \\ 0 & A_4 \end{bmatrix} = A.$$

2.11. Recall that if $0 \neq M \in \mathcal{B}(\mathcal{H})$ is nilpotent of order $k \geq 2$ and $\mathcal{H}_j := \ker M^j \ominus \ker M^{j-1}$, $1 \leq j \leq k$, then

$$\dim \mathcal{H}_{j-1} \geq \dim \mathcal{H}_j, \qquad 2 \leq j \leq k.$$

2.12 Lemma. *Let $0 \neq M \in \mathcal{B}(\mathcal{H})$ be nilpotent of order $k \geq 2$, and suppose that*

$$\dim(\ker M^{k-1})^\perp = \infty.$$

Then $M \in \text{clos}\,(\mathfrak{C}_{\mathfrak{E}})$.

Proof. As before, let $\mathcal{H}_j := \ker M^j \ominus \ker M^{j-1}$, $1 \leq j \leq k$. By the observation preceding the Lemma, $\dim \mathcal{H}_j = \infty$ for all $1 \leq j \leq k$. Note that relative to the decomposition $\mathcal{H} = \oplus_{j=1}^k \mathcal{H}_j$, we may write

$$M = \begin{bmatrix} 0 & M_{12} & M_{13} & \cdots & M_{1k} \\ 0 & 0 & M_{23} & \cdots & M_{2k} \\ 0 & 0 & 0 & \cdots & M_{3k} \\ 0 & 0 & 0 & \ddots & M_{k-1,k} \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Let $\varepsilon > 0$. For $1 \leq j \leq k$, let $D_j \in \mathcal{B}(\mathcal{H}_j)$ be a normal operator with $\sigma(D_j) = \sigma_e(D_j) = \{\frac{-\varepsilon}{2}, \frac{\varepsilon}{2}\}$, and let $D = \oplus_{j=1}^k D_j$. Then $\|D\| = \text{spr}(D) < \varepsilon$, and

$$M + D = \begin{bmatrix} D_1 & M_{12} & M_{13} & \cdots & M_{1k} \\ 0 & D_2 & M_{23} & \cdots & M_{2k} \\ 0 & 0 & D_3 & \cdots & M_{3k} \\ 0 & 0 & 0 & \ddots & M_{k-1,k} \\ 0 & 0 & 0 & \cdots & D_k \end{bmatrix}.$$

By repeated applications of the observation in Section 2.10, we see that $M+D$ is similar to D. But D is normal and $D \sim -D$. By [18, Proposition 3], we conclude that $D \in \mathfrak{C}_{\mathfrak{E}}$. Since $\mathfrak{C}_{\mathfrak{E}}$ is invariant under similarity, $M+D \in \mathfrak{C}_{\mathfrak{E}}$ as well.

Finally, since $\|(M+D)-M\| = \|D\| < \varepsilon$, and since $\varepsilon > 0$ was arbitrary, we conclude that $M \in \textsc{clos}(\mathfrak{C}_{\mathfrak{E}})$. □

2.13 Theorem. *Let $0 \neq M \in \mathcal{B}(\mathcal{H})$ be nilpotent of order $k \geq 2$. Then $M \in \textsc{clos}(\mathfrak{C}_{\mathfrak{E}})$.*

Proof. Define $\mathcal{H}_j := \ker M^j \ominus \ker M^{j-1}$, $1 \leq j \leq k+1$, and choose $1 \leq k_0 \leq k$ such that $\dim \mathcal{H}_{k_0} = \infty > \dim \mathcal{H}_{k_0+1}$.

Relative to the decomposition $\mathcal{H} = \oplus_{j=1}^{k} \mathcal{H}_j$, we may write

$$M = \begin{bmatrix} 0 & M_{12} & M_{13} & \cdots & M_{1k} \\ 0 & 0 & M_{23} & \cdots & M_{2k} \\ 0 & 0 & 0 & \cdots & M_{3k} \\ & & & & \\ 0 & 0 & 0 & \ddots & M_{k-1,k} \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Observe that $\mathcal{H}_f := \mathcal{H}_{k_0+1} \oplus \mathcal{H}_{k_0+2} \oplus \cdots \oplus \mathcal{H}_k$ is finite-dimensional. Relative to the decomposition $\mathcal{H} = (\mathcal{H}_f)^{\perp} \oplus \mathcal{H}_f$, we may write

$$M = \begin{bmatrix} X_1 & X_2 \\ 0 & X_4 \end{bmatrix}.$$

Let $\varepsilon > 0$. Now, $X_1 \in \mathcal{B}(\mathcal{H}_f^{\perp})$ is a nilpotent which satisfies the conditions of Lemma 2.12, and as such, as in the proof of that Lemma, we can find an operator Y_1 such that

- $Y_1 = [E_1, F_1]$ for some idempotents $E_1, F_1 \in \mathcal{B}(\mathcal{H}_f^{\perp})$; and
- $\sigma(Y_1) = \sigma_e(Y_1)$ and $\sigma(Y_1) \cap \{0\} = \varnothing$.
- $\|X_1 - Y_1\| < \varepsilon$.

Note that X_4, is a finite-rank nilpotent operator, so it is a *bona fide* commutator of idempotents, by [18, Proposition 6]. Choose idempotents $E_4, F_4 \in \mathcal{B}(\mathcal{H}_f)$ such that

$$X_4 = [E_4, F_4].$$

Let $Y = \begin{bmatrix} Y_1 & X_2 \\ 0 & X_4 \end{bmatrix}$. Clearly

$$\|Y - M\| = \|X_1 - Y_1\| < \varepsilon.$$

Setting $E = E_1 \oplus E_4$ and $F = F_1 \oplus F_4$, we find that E and $F \in \mathcal{B}(\mathcal{H})$ are idempotents, and by the comments of Section 2.10, Y is similar to

$$Y_1 \oplus X_4 = [E_1, F_1] \oplus [E_4, F_4] = [E, F].$$

Since $Y_1 \oplus X_4 \in \mathfrak{C}_{\mathfrak{E}}$ and the latter is invariant under similarity, $Y \in \mathfrak{C}_{\mathfrak{E}}$. But $\|M - Y\| < \varepsilon$, and $\varepsilon > 0$ was arbitrary, whence $M \in \textsc{clos}\,(\mathfrak{C}_{\mathfrak{E}})$. □

2.14 Corollary. *Let $T \in \mathcal{B}(\mathcal{H})$ satisfy:*

(i) *$\sigma(T)$ is connected and $0 \in \sigma(T)$.*
(ii) *$\sigma_e(T)$ is connected and $0 \in \sigma_e(T)$.*
(iii) *$\textsc{ind}(T - \lambda I) = 0$ for all λ in the semi-Fredholm domain of T.*

Then $T \in \textsc{clos}\,(\mathfrak{C}_{\mathfrak{E}})$.

Proof. The set of operators $T \in \mathcal{B}(\mathcal{H})$ which satisfy the above conditions is precisely the norm-closure of the set of all nilpotent operators in $\mathcal{B}(\mathcal{H})$, by a celebrated theorem of Apostol, Foiaş and Voiculescu [5]. The result now follows immediately from Theorem 2.13. □

As we shall see below – Theorem 4.2 – a result of Apostol, Foiaş and Voiculescu shows that the set of operators in $\mathcal{B}(\mathcal{H})$ which satisfy condition (iii) is precisely the set of biquasitriangular operators.

3. Normal operators in clos ($\mathfrak{C}_{\mathfrak{E}}$)

3.1. It is an immediate consequence of Corollary 2.14 that if $N \in \mathcal{B}(\mathcal{H})$ is normal, $0 \in \sigma(N)$ and $\sigma(N)$ is connected, then $N \in \textsc{clos}\,(\mathfrak{C}_{\mathfrak{E}})$. Our next goal is to describe the set of all normal operators which lie in $\textsc{clos}\,(\mathfrak{C}_{\mathfrak{E}})$.

3.2 Lemma. *Let $\mathcal{A}$ be a unital Banach algebra. Let $a, b_n \in \mathcal{A}$, $n \geq 1$, and suppose that $a = \lim_n b_n$. If $\sigma(b_n) = \sigma(-b_n)$ for all $n \geq 1$, then every connected component of $\sigma(a)$ intersects $\sigma(-a)$.*

Proof. Suppose otherwise, and let $\Omega \subseteq \sigma(a)$ be a connected component such that

$$\Omega \cap -\sigma(a) = \Omega \cap \sigma(-a) = \varnothing.$$

Writing $\Delta := \sigma(a) \setminus \Omega$, we then have that $\Omega \cap -\Omega = \varnothing$ and $\Omega \cap -\Delta = \varnothing = -\Omega \cap \Delta$.

It follows that Ω, $-\Omega$ and $(\Delta \cup -\Delta)$ are three disjoint, compact sets. Being bounded, they are contained in the closed ball $B(0, \mu) := \{z \in \mathbb{C} : |z| \leq \mu\}$ for some $\mu > 0$. Furthermore, $B(0, \mu)$ is compact and Hausdorff, hence normal (as a topological space).

We can therefore find three disjoint open sets G_0, G_1 and G_2 such that $(\Delta \cup -\Delta) \subseteq G_0$, $\Omega \subseteq G_1$ and $-\Omega \subseteq G_2$. Let $H_1 := G_1 \cap -G_2$ and $H_2 = -H_1$, so that H_1, H_2 are open, each being the intersection of two open sets. Furthermore, $H_1 \subseteq G_1$ and $H_2 \subseteq G_2$, so that G_0, H_1 and H_2 are disjoint open sets with $\Omega \subseteq H_1$ and $-\Omega \subseteq H_2$. Clearly $\sigma(a) \subseteq G_0 \cup H_1$.

Using the upper semi-continuity of the spectrum and Newburgh's Theorem (cf. [23, Theorem 1.1]), there exists $\delta > 0$ such that $\|y - a\| < \delta$ implies that $\sigma(y) \subseteq G_0 \cup H_1$ and $\sigma(y) \cap G_0 \neq \varnothing \neq \sigma(y) \cap H_1$. If, furthermore, $\sigma(y) = \sigma(-y)$, then $\sigma(y) \cap H_1 \neq \varnothing$ implies that $\sigma(y) \cap H_2 \neq \varnothing$, a contradiction.

This completes the proof. □

3.3 Lemma. *Let $T \in \textsc{clos}\,(\mathrm{WB}\textsc{al}(\mathcal{H}))$. If $\alpha, -\alpha$ are isolated points in $\sigma(T)$, then*

$$\dim \mathcal{H}(\{\alpha\}; T) = \dim \mathcal{H}(\{-\alpha\}; T).$$

Proof. Suppose that $T \in \textsc{clos}\,(\mathrm{WB}\textsc{al}(\mathcal{H}))$, and that $\alpha, -\alpha \in \sigma(T)$ are isolated points. Then there exist disjoint open sets G_0, G_1 such that $\alpha \in G_1$, $-\alpha \in G_2 := -G_1$ and $\Delta := \sigma(T) \setminus \{\alpha, -\alpha\} \subseteq G_0$.

As in Lemma 3.2, the upper semi-continuity of the spectrum and Corollary 1.6 of [23], there exists $\delta > 0$ such that $\|Y - T\| < \delta$ implies that

- $\sigma(Y) \subseteq G_0 \cup G_1 \cup G_2$,
- $\sigma(Y) \cap G_k \neq \varnothing$, $k = 1, 2, 3$, and
- $\dim \mathcal{H}(\{\alpha\}; T) = \dim \mathcal{H}(\sigma(Y) \cap G_1; Y)$, $\dim \mathcal{H}(\{-\alpha\}; T) = \dim \mathcal{H}(\sigma(Y) \cap G_2; Y)$.

In particular, if $B \in \mathrm{WB}\textsc{al}(\mathcal{H})$ and $\|B - T\| < \delta$, then

$$\dim \mathcal{H}(\{\alpha\}; T) = \dim \mathcal{H}(\sigma(B) \cap G_1; B) = \dim \mathcal{H}(\sigma(B) \cap G_2; B) = \dim \mathcal{H}(\{-\alpha\}; T). \quad \square$$

3.4 Example. The next example will hopefully be of use in helping the reader understand the conditions of Theorem 3.5 below. Let $N \in \mathcal{B}(\mathcal{H})$ be a normal operator with $\sigma(N) = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$, where

- $\Gamma_1 = \{-1\}$;
- $\Gamma_2 = \{-\frac{9}{10} + iy : -1 \leq y \leq 1\}$;
- $\Gamma_3 = \{-\frac{1}{2} + iy : -1 \leq y \leq 1\}$; and
- $\Gamma_4 = [0, 1]$.

Suppose furthermore that -1 is an isolated eigenvalue of N of multiplicity 3. Then $N \in \textsc{clos}\,(\mathfrak{C}_{\mathfrak{C}})$.

First note that $N \simeq_a I_3 \oplus N$, so that $N \simeq_a (-I_3 \oplus I_3) \oplus N^\circ$, where N° is the compression of N to $\ker(N + I)^\perp$. Since $(-I_3 \oplus I_3) \in \mathfrak{C}_{\mathfrak{C}}$, it suffices to prove that $N^\circ \in \textsc{clos}\,(\mathfrak{C}_{\mathfrak{C}})$.

Let $Q \in \mathcal{B}(\mathcal{H})$ be a quasinilpotent operator such that $q := \pi(Q) \in \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is not nilpotent. (For example, $Q := V^{(\infty)}$, where V denotes the Volterra operator.) Then by the results of Herrero [23, Proposition 8.3], Q is a *universal quasinilpotent operator* in the sense that $\text{clos}\,(\mathcal{S}(Q)) = \text{clos}\,(\text{Nil}\,(\mathcal{H}))$. Of course, $-Q$ is also a universal quasinilpotent by Herrero's result. In particular, any normal operator M whose spectrum is connected and contains 0 lies in $\text{clos}\,(\mathcal{S}(Q)) \cap \text{clos}\,(\mathcal{S}(-Q))$ [23, Proposition 5.6].

Now for any $X \in \mathcal{B}(\mathcal{H})$, $T := X \oplus -X \in \text{Bal}\,(\mathcal{H})$, and so for any $\alpha \in \mathbb{C}$,

$$R_\alpha := (\alpha I + Q) \oplus (-\alpha I - Q) \in \text{Bal}\,(\mathcal{H}).$$

In particular, by Corollary 2.5,

$$R_{\frac{9}{10}} \oplus R_{1/2} \simeq \begin{bmatrix} -\frac{9}{10}I - Q & 0 & 0 & 0 \\ 0 & -\frac{1}{2}I - Q & 0 & 0 \\ 0 & 0 & \frac{1}{2} + Q & 0 \\ 0 & 0 & 0 & \frac{9}{10}I + Q \end{bmatrix} \in \mathfrak{C}_{\mathfrak{E}}.$$

Let N_2 be a normal operator with $\sigma(N_2) = \Gamma_2$. Then $N_2 \in \text{clos}\,(\mathcal{S}(-\frac{9}{10}I - Q))$. Similarly, if N_3 is a normal operator with $\sigma(N_3) = \Gamma_3$, then $N_3 \in \text{clos}\,(\mathcal{S}(-\frac{1}{2}I - Q))$. Finally, if N_4 is a normal operator with $\sigma(N_4) = \Gamma_4 = [0, 1]$, then $N_4 \in \text{clos}\,(\mathcal{S}(\frac{1}{2}I + Q))$, and $N_4 \in \text{clos}\,(\mathcal{S}(\frac{9}{10}I + Q))$.

Since N° and $N_2 \oplus N_3 \oplus N_4$ are normal operators and $\sigma(N^\circ) = \sigma(N_2 \oplus N_3 \oplus N_4)$ has no isolated points, it follows that $\sigma_e(N^\circ) = \sigma(N^\circ) = \sigma(N_2 \oplus N_3 \oplus N_4) = \sigma_e(N_2 \oplus N_3 \oplus N_4)$. By the Weyl-von Neumann-Berg Theorem [7] (see also [15, Corollary II.4.2]),

$$\begin{aligned} N^\circ &\simeq_a N_2 \oplus N_3 \oplus N_4 \\ &\simeq_a N_2 \oplus N_3 \oplus N_4 \oplus N_4 \\ &\in \text{clos}\,(\mathcal{S}(R_{\frac{9}{10}} \oplus R_{\frac{1}{2}})) \\ &\subseteq \text{clos}\,(\mathfrak{C}_{\mathfrak{E}}). \end{aligned}$$

3.5 Theorem. *Let $N \in \mathcal{B}(\mathcal{H})$ be a normal operator. The following conditions are equivalent:*

(a) $N \in \text{clos}\,(\mathfrak{C}_{\mathfrak{E}})$.
(b) $N \in \text{clos}\,(\text{Bal}(\mathcal{H}))$.
(c) (i) *Every connected component of $\sigma(N)$ intersects $\sigma(-N)$.*
 (ii) *Every connected component of $\sigma_e(N)$ intersects $\sigma_e(-N)$.*
 (iii) *If $\alpha, -\alpha \in \sigma(N)$ are isolated points, then*

$$\dim\, \mathcal{H}(\{\alpha\}, N) = \dim\, \mathcal{H}(\{-\alpha\}, N).$$

Proof. (a) implies (b). For $N \in \mathfrak{C}_{\mathfrak{E}}$, N is similar to $-N$, therefore, $N \in \text{Bal}(\mathcal{H})$.
(b) implies (c).

Write $N = \lim_n M_n$, where each $M_n \in \mathrm{B{\scriptstyle AL}}(\mathcal{H})$. By Definition 1.3 and the comments which follow it, $\sigma(M_n) = -\sigma(M_n)$ and $\sigma_e(M_n) = -\sigma_e(M_n)$ for all $n \geq 1$. By Lemma 3.2, conditions (i) and (ii) of (c) hold. From Lemma 3.3, we see that whenever $\alpha, -\alpha \in \sigma(N)$ are isolated points,

$$\dim\, \mathcal{H}(\{\alpha\}; N) = \dim\, \mathcal{H}(\{-\alpha\}; N).$$

Thus condition (iii) holds as well.

(c) implies (a).

Step 1. We begin by reducing the problem to the case where $\sigma(N)$ has at most finitely many connected components, and any connected component of the essential spectrum contains infinitely many points.

Suppose that N satisfies conditions (i), (ii) and (iii) of (c). Let N_e be a normal operator with $\sigma(N_e) = \sigma_e(N_e) = \sigma_e(N)$, and observe that by the Weyl-von Neumann-Berg Theorem [7] (see also [15, Corollary II.4.2]), $N \simeq_a N \oplus N_e$. Let $\varepsilon > 0$, and let M_ε be a normal operator with

$$\sigma(M_\varepsilon) = \sigma_e(M_\varepsilon) = \{\lambda \in \mathbb{C} : \mathrm{dist}(\lambda, \sigma_e(N)) \leq \varepsilon\}.$$

As noted in Remark 1.4 of Davidson [12] (as applied to the case of an infinite-dimensional, separable Hilbert space), since M_ε and N_e are normal operators whose spectra coincide with their essential spectra, $d(M_\varepsilon, \mathcal{U}(N_e)) \leq \varepsilon$. By setting $N_\varepsilon = N \oplus M_\varepsilon$, we find that

$$\mathrm{dist}\,(N_\varepsilon, \mathcal{U}(N)) \leq \varepsilon.$$

If we can show that $N_\varepsilon \in \mathrm{clos}\,(\mathfrak{C}_{\mathfrak{C}})$ for each $\varepsilon > 0$, then clearly $N \in \mathrm{clos}\,(\mathfrak{C}_{\mathfrak{C}})$. The advantage of N_ε over N is that not only does N_ε satisfy conditions (i), (ii) and (iii) of (c), but by Putnam's Theorem [27], $\sigma(N_\varepsilon)$ has at most finitely many connected components. Since the problem reduces to considering N_ε, we may assume without loss of generality that $\sigma(N)$ had at most finitely many connected components to begin with, and that any connected component of the essential spectrum of N has infinitely many points.

Step 2. Next, we shall reduce to the case where item (iii) is vacuously satisfied.

If $\alpha, -\alpha \in \sigma(N)$ are isolated points, then by Step 1, they correspond to eigenvalues of finite multiplicity for N, and by condition (iii), $\mu_k := \dim \mathcal{H}(\{\alpha\}, N) = \dim\, \mathcal{H}(\{-\alpha\}, N)$. Thus

$$N \simeq \begin{bmatrix} \alpha_k I_{\mu_k} & 0 & 0 \\ 0 & -\alpha_k I_{\mu_k} & 0 \\ 0 & 0 & N^\circ \end{bmatrix},$$

where N° is the compression of N to $(\mathrm{span}\{\ker(N - \alpha_k I), \ker(N + \alpha_k I)\})^\perp$. It is not hard to see that N° still satisfies conditions (i), (ii) and (iii) of (c), and

$\sigma(N^\circ) = \sigma(N) \setminus \{\alpha_k, -\alpha_k\}$. In particular, $\sigma(N^\circ)$ has two fewer components than $\sigma(N)$.

Moreover, by Corollary 2.5, $\begin{bmatrix} \alpha_k I_{\mu_k} & 0 \\ 0 & -\alpha_k I_{\mu_k} \end{bmatrix} \in \text{CLOS}\,(\mathfrak{C}_{\mathfrak{C}})$, and so it suffices to prove that $N^\circ \in \text{CLOS}\,(\mathfrak{C}_{\mathfrak{C}})$. By repeating this argument finitely many times (keeping in mind that $\sigma(N)$ had at most finitely many connected components to begin with), we may reduce to the case where $\sigma(N)$ has at most finitely many components, N satisfies (i) and (ii) of (c), and there does not exist $\alpha \in \mathbb{C}$ such that $\alpha, -\alpha$ are isolated in $\sigma(N)$.

STEP 3. Our next objective is to remove isolated eigenvalues of finite multiplicity.

After the above reductions, we now have that if $\alpha \in \sigma(N)$ is an isolated eigenvalue of finite multiplicity, then $-\alpha \in \sigma_e(N)$, and the connected component $\Gamma_{-\alpha}$ of $\sigma(N)$ which contains $-\alpha$ has infinitely many points in it. Let $N_{-\alpha}$ denote the compression of N to the spectral subspace $\mathcal{H}(\Gamma_{-\alpha}; N)$. Since $-\alpha \in \sigma(N_{-\alpha})$, and since the latter set is connected and has infinitely many points in it, by the Weyl-von Neumann-Berg Theorem, $N_{-\alpha} \simeq_a -\alpha I_\mu \oplus N_{-\alpha}$, where $\mu := \dim \mathcal{H}(\{\alpha\}; N)$. Thus $N \simeq_a -\alpha I_\mu \oplus N$, and since $N \simeq \alpha I_\mu \oplus N^\circ$, where N° is the compression of N to the orthogonal complement of $\ker(N - \alpha I)$, we see that

$$N \simeq_a \begin{bmatrix} \alpha I_\mu & 0 & 0 \\ 0 & -\alpha I_\mu & 0 \\ 0 & 0 & N^\circ \end{bmatrix}.$$

Again, $\begin{bmatrix} \alpha I_\mu & 0 \\ 0 & -\alpha I_\mu \end{bmatrix} \in \text{CLOS}\,(\mathfrak{C}_{\mathfrak{C}})$, and it suffices to prove that N° lies in $\text{CLOS}\,(\mathfrak{C}_{\mathfrak{C}})$. Note that $\sigma(N^\circ) = \sigma(N) \setminus \{\alpha\}$. By repeating this argument finitely often (once for each isolated eigenvalue of finite multiplicity of N), we reduce to the case where $\sigma(N) = \sigma_e(N)$ has no isolated points.

STEP 4. We now deal with the remaining case.

At this stage, we have that $\sigma(N) = \sigma_e(N)$ has finitely many connected components, each of which contains infinitely many points and intersects $\sigma(-N)$ non-trivially. Our approach is similar to that used in Step 1. Let $\varepsilon > 0$, and let M_ε be a normal operator with

$$\sigma(M_\varepsilon) = \sigma_e(M_\varepsilon) = \{\lambda \in \mathbb{C} : \text{dist}(\lambda, \sigma_e(N)) \le \varepsilon\}.$$

From Davidson's result, $\text{dist}(M_\varepsilon, \mathcal{U}(N_e)) = \varepsilon$, so that by setting $N_\varepsilon = N \oplus M_\varepsilon$, we find that

$$\text{dist}\,(N_\varepsilon, \mathcal{U}(N)) \le \varepsilon.$$

If we can show that $N_\varepsilon \in \text{CLOS}\,(\mathfrak{C}_{\mathfrak{C}})$ for each $\varepsilon > 0$, then clearly $N \in \text{CLOS}\,(\mathfrak{C}_{\mathfrak{C}})$. The advantage of N_ε over N is that not only does $\sigma(N_\varepsilon) = \sigma_e(N_\varepsilon)$ has finitely many connected components, but the intersection of each component with $\sigma(-N)$ contains a open disc. Since the problem reduces to considering N_ε, we may assume without loss of generality that $\sigma(N) = \sigma_e(N)$ had at most finitely many connected components, say $\Gamma_1, \Gamma_2, \ldots, \Gamma_m$, to begin with, and $\Gamma_k \cap \sigma(-N)$ contains an open disc, $1 \le k \le m$.

For each $1 \le k \le m$, choose a point $\alpha_k \in \Gamma_k$ such that $-\alpha_k \in \sigma(N)$, and $\alpha_k \ne \pm\frac{i}{2}$. Let $\tau(k) \in \{1, 2, \ldots, m\}$ be the unique element such that $-\alpha_k \in \Gamma_{\tau(k)}$. Let $Q \in \mathcal{B}(\mathcal{H})$ be a universal quasinilpotent in the sense of Herrero [23], in that $\pi(Q) \in \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is quasinilpotent but not nilpotent, and thus $\text{CLOS}\,(\mathcal{S}(Q)) = \text{CLOS}\,(\text{NIL}(\mathcal{H}))$. By the Apostol-Foiaş-Voiculescu characterisation of $\text{CLOS}\,(\text{NIL}(\mathcal{H}))$, we see that $N_k \in \text{CLOS}\,(\mathcal{S}(\alpha_k I + Q))$, $1 \le k \le m$, where N_k is the compression of N to the spectral subspace determined by Γ_k.

For each $1 \le k \le m$, by Corollary 2.5, the operator $R_k := \begin{bmatrix} \alpha_k I + Q & 0 \\ 0 & -\alpha_k I - Q \end{bmatrix} \in \mathfrak{C}_{\mathfrak{E}}$, and from above,

$$N_k \oplus N_{\tau(k)} \in \text{CLOS}\,(\mathfrak{C}_{\mathfrak{E}}).$$

Thus $\oplus_{k=1}^m (N_k \oplus N_{\tau(k)}) \in \text{CLOS}\,(\mathfrak{C}_{\mathfrak{E}})$, and since $N \simeq_a \oplus_{k=1}^m (N_k \oplus N_{\tau(k)})$ by the Weyl-von Neumann-Berg Theorem (here we use the fact that $\sigma(N) = \sigma_e(N)$ so that we may repeat terms N_k or $N_{\tau(k)}$ as often as we wish), we see that $N \in \text{CLOS}\,(\mathfrak{C}_{\mathfrak{E}})$. This completes the proof. □

We remark that the normality of N was not required to prove that (a) implies (b) nor that (b) implies (c). These conditions will reappear in Theorem 4.11 below, applied to biquasitriangular operators.

4. Biquastriangular operators in clos $(\mathfrak{C}_{\mathfrak{E}})$

4.1. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be **triangular** if there exists an ONB $\{e_n\}_{n=1}^\infty$ for $\mathcal{H}$ relative to which the operator matrix $[T] := [\langle Te_j, e_i\rangle]$ is upper-triangular; that is, $\langle Te_j, e_i\rangle = 0$ for all $i > j \ge 1$. We say that T is **quasitriangular** (and we write $T \in (\text{QT})$) if there exists $K \in \mathcal{K}(\mathcal{H})$ such that $T - K$ is triangular. It is known [23, Theorem 6.4] that if $T \in (\text{QT})$, then for all $\varepsilon > 0$ there exists $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \varepsilon$ such that $T - K$ is triangular, and thus (QT) is the norm-closure of the set of triangular operators. Finally, we say that T is **biquasitriangular** if both T and T^* are quasitriangular, in which case we write $T \in (\text{BQT})$.

The following deep theorem of Apostol, Foiaş and Voiculescu [4] yields a very useful and practical way of determining whether or not a given operator is quasitriangular.

4.2 Theorem *(Apostol, Foiaş, and Voiculescu). An operator $T \in \mathcal{B}(\mathcal{H})$ is quasitriangular if and only if*

$$\text{IND}\,(T-\lambda I) := \text{NUL}\,(T-\lambda I) - \text{NUL}\,(T-\lambda I)^* \geq 0 \textit{ for all } \lambda \in \rho_{\text{s-F}}(T).$$

Thus $T \in (\text{BQT})$ *if and only if* $\text{IND}\,(T-\lambda I) = 0$ *for all* $\lambda \in \rho_{\text{s-F}}(T)$.

Theorem 6.15 of Herrero's book [23] establishes no fewer than seventeen equivalent formulations of biquasitriangularity of Hilbert space operators. The formulation which we shall require below is the following, which – as observed by Herrero – is a relatively straightforward corollary of the characterisation by Voiculescu [33] of (BQT) as the closure of the set of **algebraic operators** in $\mathcal{B}(\mathcal{H})$ – i.e. those operators $T \in \mathcal{B}(\mathcal{H})$ for which there exists a non-zero polynomial p such that $p(T) = 0$.

4.3 Theorem. *An operator* $T \in \mathcal{B}(\mathcal{H})$ *is biquasitriangular if and only if* $T = \lim_n T_n$*, where each* T_n *is similar to a normal operator.*

The fact that $\mathfrak{C}_{\mathfrak{E}}$ is invariant under conjugation by invertible operators, combined with the above theorem and the results of the last section will allow us to determine precisely which biquasitriangular operators lie in $\text{CLOS}\,(\mathfrak{C}_{\mathfrak{E}})$, and this is the main goal of this section. Our approach will be similar to that taken in the previous section: this time, given a biquasitriangular operator T, we first wish to "fatten up" the spectrum of T to eliminate all but finitely many isolated eigenvalues of finite multiplicity. That we can do so without sacrificing biquasitriangularity (Proposition 4.8 below) is a nice surprise.

We shall also require the following, which is an immediate consequence of Proposition 1.4 and Lemma 3.2.

4.4 Lemma. *Let* $T \in \text{CLOS}\,(\mathfrak{C}_{\mathfrak{E}})$*. Let* $\lambda \in \sigma(T)$*, and let* Γ_λ *denote the connected component of* λ *in* $\sigma(T)$*. Then* $\Gamma_\lambda \cap \sigma(-T) \neq \varnothing$.

Given $Y \in \mathcal{B}(\mathcal{H})$, we denote by $\rho_{\ell e}(Y)$ the **left essential resolvent of** Y; i.e. the set of all $\alpha \in \mathbb{C}$ for which there exist $X \in \mathcal{B}(\mathcal{H})$ and $K \in \mathcal{K}(\mathcal{H})$ such that

$$X(\alpha I - Y) = I + K.$$

Equivalently, $\rho_{\ell e}(Y)$ may be thought of as the set of all $\alpha \in \mathbb{C}$ for which $\pi(Y - \alpha I)$ is left-invertible in the Calkin algebra. Analogously, there is an obvious corresponding notion of a right essential resolvent $\rho_{re}(Y)$ for Y.

We also define the **left essential spectrum** $\sigma_{\ell e}(Y) := \mathbb{C} \setminus \rho_{\ell e}(Y)$, the **right essential spectrum** $\sigma_{re}(Y) := \mathbb{C} \setminus \rho_{re}(Y)$, and the **left-right essential spectrum of** Y as $\sigma_{\ell re}(Y) := \sigma_{\ell e}(Y) \cap \sigma_{re}(Y)$.

4.5 Lemma. *Let* $T \in \mathcal{B}(\mathcal{H})$ *and* $t := \pi(T) \in \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$*. Let* $\mathcal{H}_\varphi$ *be a separable Hilbert space,* $\varphi : C^*(t) \to \mathcal{B}(\mathcal{H}_\varphi)$ *be a faithful, unital* **-representation, and set* $X := \varphi(t)$*. Then* $\rho_{\ell e}(T) = \rho_{\ell e}(X^{(\infty)})$.

Proof. By Voiculescu's non-commutative Weyl-von Neumann Theorem [34] (see also [15, Corollary II.5.5]), $T \simeq_a T \oplus X^{(\infty)}$.

- Suppose that $\alpha \in \rho_{\ell e}(T)$. Then clearly $\alpha \in \rho_{\ell e}(T \oplus X^{(\infty)})$ and so the operator $T \oplus X^{(\infty)} - \alpha(I \oplus I^{(\infty)})$ is left-invertible modulo the compacts, say

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot \begin{bmatrix} T - \alpha I & 0 \\ 0 & X^{(\infty)} - \alpha I^{(\infty)} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I^{(\infty)} \end{bmatrix} + K$$

for some compact operator $K \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H}_\varphi^{(\infty)})$. By considering the $(2, 2)$ entry of this operator, we see that

$$D(X^{(\infty)} - \alpha I^{(\infty)}) = I^{(\infty)} + L$$

for some compact operator $L \in \mathcal{B}(\mathcal{H}_\varphi^{(\infty)})$ (namely the compression of K to this corner), and thus $\alpha \in \rho_{\ell e}(X^{(\infty)})$. Hence, $\rho_{\ell e}(T) \subseteq \rho_{\ell e}(X^{(\infty)})$.
- Now suppose that $\alpha \in \rho_{\ell e}(X^{(\infty)})$. Then there exists $Y \in \mathcal{B}(\mathcal{H}_\varphi^{(\infty)})$ such that

$$Y(X^{(\infty)} - \alpha I^{(\infty)}) = Y(X - \alpha I)^{(\infty)} = I + K$$

for some $K \in \mathcal{K}(\mathcal{H}_\varphi^{(\infty)})$. Writing $Y = [Y_{ij}] \in \mathcal{B}(\mathcal{H}_\varphi^{(\infty)})$, we obtain

$$\lim_n Y_{nn}(X - \alpha I) = I,$$

whence $X - \alpha I$ is left-invertible. From this it easily follows that $|X - \alpha I|$ is bounded below, say by $\delta > 0$; in other words, $\sigma(|X - \alpha I|) \subseteq [\delta, \infty)$.
Since φ is injective (and therefore induces a *-isomorphism between $C^*(t)$ and $C^*(X)$ that sends $|t - \alpha 1|$ to $|X - \alpha I|$), we find that

$$\sigma(|t - \alpha 1|) \subseteq [\delta, \infty).$$

In $\mathcal{B}(\mathcal{H})$, we consider the polar decomposition $T - \alpha I = V_\alpha |T - \alpha I|$ and set $v_\alpha := \pi(V_\alpha)$. Since π is a *-homomorphism we have:

$$t - \alpha 1 = v_\alpha |t - \alpha 1|,$$

and so $|t - \alpha 1|^{-1} v_\alpha^*$ is a left-inverse for $t - \alpha 1$. In other words, $\alpha \in \rho_{\ell e}(T)$. Hence, $\rho_{\ell e}(X^{(\infty)}) \subseteq \rho_{\ell e}(T)$. □

By taking adjoints, we see that $\rho_{re}(T) = \rho_{re}(X^{(\infty)})$.

4.6 Theorem. *Let $T \in$ (BQT) and $t := \pi(T) \in \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$. Let $\varphi : C^*(t) \to \mathcal{B}(\mathcal{H}_\varphi)$ be a faithful, unital *-representation, where $\mathcal{H}_\varphi$ is a separable Hilbert space. If $X := \varphi(t)$, then $X^{(\infty)}$ is biquasitriangular.*

Proof. Since $T \in$ (BQT), we have that $\sigma_{\ell e}(T) = \sigma_{re}(T)$. Combining this with Lemma 4.5 above, we get

$$\sigma_{\ell e}(X^{(\infty)}) = \sigma_{re}(X^{(\infty)}) = \sigma_{\ell re}(X^{(\infty)}).$$

Hence

$$\sigma_e(X^{(\infty)}) = \sigma_{\ell e}(X^{(\infty)}) \cup \sigma_{re}(X^{(\infty)}) = \sigma_{\ell e}(X^{(\infty)}) \cap \sigma_{re}(X^{(\infty)}) = \sigma_{\ell re}(X^{(\infty)}),$$

which implies that if $\alpha \notin \sigma_{\ell re}(X^{(\infty)})$, then $X^{(\infty)} - \alpha I^{(\infty)}$ is Fredholm and thus has finite index. But $\alpha \notin \sigma_e(X^{(\infty)})$ implies that

$$\text{IND}\,(X^{(\infty)} - \alpha I^{(\infty)}) = \sum_n \text{IND}\,(X - \alpha I),$$

which can only be finite if it is equal to 0. By Theorem 4.2, we conclude that $X^{(\infty)} \in$ (BQT). □

4.7 Example. Maintaining the notation and conditions of Theorem 4.6, it is natural to wonder whether or not $X = \varphi(t)$ itself must lie in (BQT). The next example shows that this need not be the case.

Let S denote the unilateral forward shift, and let

$$T = (S \oplus S^*)^{(\infty)} = S \oplus S^* \oplus S \oplus S^* \oplus \cdots \in \mathcal{B}(\oplus_n \mathcal{H}_n).$$

Then $\sigma(T) = \sigma_e(T) = \sigma_{\ell e}(T) = \sigma_{re}(T) = \sigma_{\ell re}(T) = \overline{\mathbb{D}}$. In particular, $T \in$ (BQT).

The space $\mathcal{K} := \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$ is orthogonally reducing for T, and so if we define P to be the orthogonal projection of $\oplus \mathcal{H}_n$ onto $\mathcal{K}$, then the map

$$\begin{array}{cccc} \varphi: & C^*(T) & \to & \mathcal{B}(\mathcal{K}) \\ & Y & \mapsto & PY|_{\operatorname{ran} P} \end{array}$$

is a *-homomorphism.

Note that $C^*(T) \cap \mathcal{K}(\mathcal{H}) = \{0\}$, and so φ annihilates $C^*(T) \cap \mathcal{K}(\mathcal{H})$. In particular, φ factors trivially through the Calkin algebra, and yet $X := \varphi(T) = S \oplus S^* \oplus S \notin$ (BQT) according to Theorem 4.2. As we have just seen, however, $X^{(\infty)} \simeq T \in$ (BQT).

4.8 Proposition. *Let $T \in$ (BQT) be a norm-one operator and $\delta > 0$. Then there exists a biquasitriangular operator T_δ and a unitary operator V_δ satisfying*

(i) $\|T - V_\delta^* T_\delta V_\delta\| < 2\delta$;
(ii) $\sigma_e(T_\delta) = [\sigma_e(T)]_\delta := \{z \in \mathbb{C} : |z - \alpha| \leq \delta \text{ for some } \alpha \in \sigma_e(T)\}$; *and*

(iii) *if* $\lambda \in \sigma_p^0(T)$ *and* $\text{DIST}(\lambda, \sigma_e(T)) > \delta$, *then* $\lambda \in \sigma_p^0(T_\delta)$ *and*

$$\dim \mathcal{H}(\{\lambda\}; T_\delta) = \dim \mathcal{H}(\{\lambda\}; T).$$

Proof. Let $t := \pi(T)$ be the image of T in the Calkin algebra. Without loss of generality, we may assume that $\delta < 1$. Let $\varrho : C^*(t) \to \mathcal{B}(\mathcal{H}_\varrho)$ be a faithful, unital *-representation, where $\mathcal{H}_\varrho$ is a separable Hilbert space. That such a representation exists is clear from the fact that $C^*(t)$ is separable [15, Theorem I.9.12]. By Theorem 4.6, if $X := \varrho(t)$, then $X^{(\infty)}$ is biquasitriangular, and

$$\|X^{(\infty)}\| = \|X\| = \|t\| \leq \|T\| = 1.$$

Let $\{\gamma_n\}_n$ be a countable dense subset of $\{z \in \mathbb{C} : |z| \leq \delta\}$. Then $(1 + \gamma_n)X^{(\infty)}$ is biquasitriangular for each $n \geq 1$, and since the direct sum of a countable family of biquasitriangular operators is biquasitriangular (an immediate consequence of [20, Theorem 4]),

$$T_\delta := T \oplus (\oplus_n (1 + \gamma_n)X^{(\infty)}) \in (\text{BQT}).$$

By Voiculescu's Theorem, however, $T \simeq_a T \oplus (\oplus_n X^{(\infty)})$, and

$$\|T \oplus (\oplus_n X^{(\infty)}) - T \oplus (\oplus_n (1 + \gamma_n)X^{(\infty)})\| = \sup_n \|\gamma_n X^{(\infty)}\| \leq \delta.$$

Thus there exists V_δ unitary such that

$$\|T - V_\delta^* T_\delta V_\delta\| < 2\delta.$$

Notice that

$$\sigma_e(T_\delta) = \sigma_e(T) \cup \ \cup_n \sigma((1 + \gamma_n)X^{(\infty)}) = \cup_n (1 + \gamma_n)\sigma_e(T) = [\sigma_e(T)]_\delta.$$

Furthermore, if $\lambda \in \sigma_p^0(T)$ and $\text{DIST}(\lambda, \sigma_e(T)) > \delta$, then $\lambda \notin \sigma(\oplus_n (1 + \gamma_n)X^{(\infty)})$. But T is a direct summand of T_δ, and so it follows that $\lambda \in \sigma_p^0(T_\delta)$ and

$$\dim \mathcal{H}(\{\lambda\}; T_\delta) = \dim \mathcal{H}(\{\lambda\}; T). \quad \square$$

The following result of Barría and Herrero has an extension to all normal operators with $\sigma(N) = \sigma_e(N)$; however, it simplifies nicely when $\sigma(N)$ has no isolated points. In that case, $T \in \text{CLOS}(\mathcal{S}(N))$ implies that neither the spectrum $\sigma(T)$ nor the essential spectrum $\sigma_e(T)$ admits isolated points.

4.9 Theorem. *[6, Theorem 1] Let N be a normal operator and suppose that $\sigma(N)$ has no isolated points (in which case $\sigma(N) = \sigma_e(N)$ automatically). The following are equivalent.*

(a) $T \in \textsc{clos}\,(\mathcal{S}(N))$*; i.e. T is a limit of operators similar to N.*
(b) *T satisfies the following conditions:*
  (i) *T is biquasitriangular.*
  (ii) *Neither $\sigma(T)$ nor $\sigma_e(T)$ has isolated points.*
  (iii) *$\sigma(N) \subseteq \sigma(T)$ and $\sigma_e(N) \subseteq \sigma_e(T)$.*
  (iv) *If $\lambda \in \sigma_e(T)$, then the component Γ_λ of λ in $\sigma_e(T)$ intersects $\sigma_e(N)$.*

4.10 Corollary. *Suppose that $T \in$ (BQT) is such that $\sigma_p^0(T) = \varnothing$ and $\sigma_e(T)$ has no isolated points. If $N \in \mathcal{B}(\mathcal{H})$ is normal and $\sigma(N) = \sigma_e(N) = \sigma_e(T)$, then $T \in \textsc{clos}\,(\mathcal{S}(N))$.*

4.11 Theorem. *Let $T \in \mathcal{B}(\mathcal{H})$ be a biquasitriangular operator. The following conditions are equivalent:*

(a) $T \in \textsc{clos}\,(\mathfrak{C}_{\mathfrak{E}})$*;*
(b) $T \in \textsc{clos}\,(\textsc{Bal}(\mathcal{H}))$*;*
(c) (i) *every connected component of $\sigma(T)$ intersects $\sigma(-T)$;*
  (ii) *every connected component of $\sigma_e(T)$ intersects $\sigma_e(-T)$; and*
  (iii) *If $\alpha, -\alpha \in \sigma(T)$ are isolated points, then*

$$\dim\, \mathcal{H}(\{\alpha\}; T) = \dim\, \mathcal{H}(\{-\alpha\}; T).$$

Proof. (a) implies (b) (resp. (b) implies (c)).
These follow as in the proof of Theorem 3.5 (a) implies (b) (resp. (b) implies (c)), since the normality of the operator N was not used at that point.

(c) implies (a).
Step 1. Let $\delta > 0$ and $t = \pi(T)$ be the image of T in the Calkin algebra. Let $\varrho : C^*(t) \to \mathcal{B}(\mathcal{H}_\varrho)$ be a faithful, unital *-representation, where $\mathcal{H}_\varrho$ is separable. As in Proposition 4.8, we define the operator

$$T_\delta := T \oplus (\oplus_n (1 + \gamma_n) X^{(\infty)}) \in (\mathrm{BQT}),$$

where $X = \varrho(t)$ and $\{\gamma_n\}_n$ is a countable dense subset of $\{z \in \mathbb{C} : |z| \le \delta\}$. Then, by that Proposition,

- there exists a unitary V_δ such that $\|T - V_\delta^* T_\delta V_\delta\| < 2\delta$;
- $\sigma_e(T_\delta) = [\sigma_e(T)]_\delta$;
- if $\lambda \in \sigma_p^0(T_\delta)$, then $\dim\, \mathcal{H}(\{\lambda\}; T_\delta) = \dim\, \mathcal{H}(\{\lambda\}; T)$.

It is routine to verify that $\sigma_p^0(T_\delta) = \{\lambda \in \sigma_p^0(T) : \textsc{dist}(\lambda, \sigma_e(T)) > \delta\}$. Moreover, as a consequence of Putnam's Theorem [27], this is a finite set.
Clearly it suffices to prove that $T_\delta \in \textsc{clos}(\mathfrak{C}_{\mathfrak{E}})$ for all $\delta > 0$.

Step 2. The hypothesis that every connected component of $\sigma(T)$ intersects $\sigma(-T)$ implies that we may partition $\sigma_p^0(T_\delta)$ into two sets, say $A := \{\alpha_1, -\alpha_1, \alpha_2, -\alpha_2, \ldots,$

$\alpha_m, -\alpha_m\}$ and $B := \{\beta_1, \beta_2, \ldots, \beta_q\}$ where $B \cap -B = \varnothing$. (It is possible that $m = 0$, in which case $A = \varnothing$.) Furthermore, for each $1 \leq k \leq q$, $\{-\beta_k\}$ intersects a component of $\sigma(T)$, and since that component is not $\{-\beta_k\}$ itself, it must lie in the interior of $[\sigma_e(T)]_\delta$.

Let $\mathcal{M} = \mathcal{H}(A; T_\delta)$, and write $T_\delta = \begin{bmatrix} X_1 & X_2 \\ 0 & X_4 \end{bmatrix}$ relative to the decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$. Since $A \cap (B \cup [\sigma_e(T)]_\delta) = \varnothing$, as noted in Section 2.10, we see that T_δ is similar to $X_1 \oplus X_4$. It therefore suffices to show that $X_1 \oplus X_4 \in \text{CLOS}\,(\mathfrak{C}_{\mathfrak{C}})$.

Keeping in mind that $\mathcal{M}$ is finite-dimensional, the hypothesis that $-\alpha, \alpha \in \sigma(T)$ are isolated implies that

$$\dim \mathcal{H}(\{\alpha\}; T_\delta) = \dim \mathcal{H}(\{\alpha\}; T) = \dim \mathcal{H}(\{-\alpha\}; T) = \dim \mathcal{H}(\{-\alpha\}; T_\delta),$$

and thus by [25, Theorem 3.5], X_1 lies in $\text{CLOS}\,(\mathfrak{C}_{\mathfrak{C}})$.

We have therefore reduced the problem to showing that X_4 lies in $\text{CLOS}\,(\mathfrak{C}_{\mathfrak{C}})$.

Step 3. Note that $\sigma(X_4) = B \cup [\sigma_e(T)]_\delta$. Letting $\mathcal{N} := \mathcal{H}(B; X_4)$, we see that relative to $\mathcal{N} \oplus (\mathcal{N}^\perp \cap \mathcal{M}^\perp)$, we may write

$$X_4 = \begin{bmatrix} Y_1 & Y_2 \\ 0 & Y_4 \end{bmatrix},$$

where $\sigma(Y_1) = B$ and $\sigma(Y_4) = \sigma_e(Y_4) = [\sigma_e(T)]_\delta$. The fact that B and $[\sigma_e(T)]_\delta$ are disjoint implies that X_4 is similar to $Y_1 \oplus Y_4$, and thus it suffices to prove that $Y_1 \oplus Y_4$ lies in $\text{CLOS}\,(\mathfrak{C}_{\mathfrak{C}})$.

Let $N \in \mathcal{B}(\mathcal{H})$ be a normal operator with $\sigma(N) = [\sigma_e(T)]_\delta$, so that $\sigma(N) = \sigma_e(N)$. Recall that each $-\beta_k$ is in the interior of $\sigma_e(N)$. By the Weyl-von Neumann-Berg Theorem [7], $N \simeq_a N \oplus M$, where $M = \oplus_{k=1}^q M_k$, and $M_k = -\beta_k I_{\mu_k}$, where $\mu_k = \dim \mathcal{H}(\{\beta_k\}; T_\delta)$.

Now $Y_1 \oplus M \in \text{CLOS}\,(\mathfrak{C}_{\mathfrak{C}})$ since it is balanced and acts on a finite-dimensional space. Furthermore, the condition that each connected component of $\sigma_e(T)$ intersects $\sigma_e(-T)$ implies that each component of $\sigma_e(N) = \sigma(N) = [\sigma_e(T)]_\delta$ intersects $\sigma_e(-T) \subseteq \sigma_e(-N) = \sigma(-N)$. By Theorem 3.5, $N \in \text{CLOS}\,(\mathfrak{C}_{\mathfrak{C}})$.

By Corollary 4.10, $Y_4 \in \text{CLOS}\,(\mathcal{S}(N)) \subseteq \text{CLOS}\,(\mathfrak{C}_{\mathfrak{C}})$. Thus

$$Y_1 \oplus M \oplus Y_4 \in \text{CLOS}\,(\mathfrak{C}_{\mathfrak{C}}).$$

By Proposition 5.13 of [23], $N \simeq_a M \oplus N \in \text{CLOS}\,(\mathcal{S}(M \oplus Y_4))$, and so

$$Y_1 \oplus N \in \text{CLOS}\,(\mathfrak{C}_{\mathfrak{C}}).$$

Finally, $Y_4 \in \text{CLOS}\,(\mathcal{S}(N))$ implies that

$$Y_1 \oplus Y_4 \in \textsc{clos}\,(\mathcal{S}(Y_1 \oplus N)) \subseteq \textsc{clos}\,(\mathfrak{C}_{\mathfrak{E}}).$$

As we have seen, this is sufficient to complete the proof. □

4.12 Remark. Since $\mathfrak{C}_{\mathfrak{E}} \subseteq \textsc{Neg}_{\textsc{invs}}(\mathcal{H}) \subseteq \textsc{Neg}_{\textsc{s}}(\mathcal{H}) \subseteq \textsc{Bal}(\mathcal{H})$, by Proposition 1.4, we see that for a *biquasitriangular operator* T, the following are equivalent:

- $T \in \textsc{clos}\,(\textsc{Bal}(\mathcal{H}))$;
- $T \in \textsc{clos}\,(\textsc{Neg}_{\textsc{s}}(\mathcal{H}))$;
- $T \in \textsc{clos}\,(\textsc{Neg}_{\textsc{invs}}(\mathcal{H}))$; and
- $T \in \textsc{clos}\,(\mathfrak{C}_{\mathfrak{E}})$.

5. Non-biquastriangular operators in clos $(\mathfrak{C}_{\mathfrak{E}})$

5.1. In order to characterise $\textsc{clos}\,(\mathfrak{C}_{\mathfrak{E}})$, there remains to determine which non-biquasitriangular operators lie in that set. Unfortunately, this is currently beyond our reach. It is tempting to believe that, as in the case of operators acting on a finite-dimensional Hilbert space [25], or the case of biquasitriangular operators as in the previous section, an operator will lie in $\textsc{clos}\,(\mathfrak{C}_{\mathfrak{E}})$ if and only if its spectrum is balanced in the sense described in Theorem 4.11. The unilateral forward shift S has this property, and *a fortiori*, S is similar (in fact unitarily equivalent via an involution) to $-S$. We were therefore surprised to discover that $\mu S \in \textsc{clos}\,(\mathfrak{C}_{\mathfrak{E}})$ if and only if $|\mu| \leq \frac{1}{2}$ (Corollary 5.9). We will soon identify an obstruction to membership in $\textsc{clos}\,(\mathfrak{C}_{\mathfrak{E}})$ which hearkens back to Proposition 4 of [18].

5.2 Lemma. *Let $X, Y \in \textsc{Fred}(\mathcal{H})$. Then $\sigma_e(XY) = \sigma_e(YX)$, and*

$$\textsc{ind}\,(\lambda I - XY) = \textsc{ind}(\lambda I - YX) \qquad \textit{for all} \qquad \lambda \in \mathbb{C} \setminus \sigma_e(XY).$$

Proof. Since $X \in \textsc{Fred}(\mathcal{H})$, by Atkinson's theorem, there exists $Z \in \textsc{Fred}(\mathcal{H})$, and $K_1, K_2 \in \mathcal{K}(\mathcal{H})$, such that $XZ = I + K_1$ and $ZX = I + K_2$. Therefore,

$$Z(\lambda I - XY) = \lambda Z - Y - K_2 Y \qquad \text{and} \qquad (\lambda I - YX)Z = \lambda Z - Y - YK_1.$$

Since $Z(\lambda I - XY) - (\lambda I - XY)Z \in \mathcal{K}(\mathcal{H})$, we observe that $Z(\lambda I - XY) \in \textsc{Fred}(\mathcal{H})$ if and only if $(\lambda I - YX)Z \in \textsc{Fred}(\mathcal{H})$, in which case their Fredholm indices coincide.

Moreover, since an operator is Fredholm if and only if its image in the Calkin algebra is invertible, and since the invertible elements of any unital Banach algebra form a group, we obviously have that $\lambda I - XY \in \textsc{Fred}(\mathcal{H})$ if and only if $Z(\lambda I - XY) \in \textsc{Fred}(\mathcal{H})$, and $(\lambda I - YX)Z \in \textsc{Fred}(\mathcal{H})$ if and only if $\lambda I - YX \in \textsc{Fred}(\mathcal{H})$.

Combining this with the previous observation yields:

$$\sigma_e(XY) = \sigma_e(YX).$$

Let $\lambda \in \mathbb{C} \setminus \sigma_e(XY)$. Since $\text{IND}\, Z(\lambda I - XY) = \text{IND}(\lambda I - YX)Z$, it follows that

$$\text{IND}\ (\lambda I - XY) = \text{IND}(\lambda I - YX). \quad \square$$

The next result [11, Theorem 3.1] will prove useful below. Given an integer $p \geq 2$, we define the set $\mathcal{R}_p := \{T^p : T \in \mathcal{B}(\mathcal{H})\}$. It is clear that $\mathcal{R}_p$ is invariant under conjugation by invertible operators.

5.3 Theorem *(Conway and Morrel). Let $T \in \mathcal{B}(\mathcal{H})$ and $2 \leq p \in \mathbb{N}$. Then $T \in \text{CLOS}\,(\mathcal{R}_p)$ if and only if the set*

$$\{\lambda \in \mathbb{C} : \lambda I - T \in \text{FRED}(\mathcal{H}) \text{ and } \text{IND}\,(\lambda I - T) \notin p\mathbb{Z}\}$$

does not separate 0 from ∞.

5.4 Proposition. *Suppose that $T \in \text{CLOS}\,(\mathfrak{C}_{\mathfrak{E}}) \cap \text{FRED}(\mathcal{H})$. Then $T^2 + \frac{1}{4}I \in \text{CLOS}\,(\mathcal{R}_4)$.*

Proof. Let $T \in \text{CLOS}\,(\mathfrak{C}_{\mathfrak{E}}) \cap \text{FRED}(\mathcal{H})$. Since $\text{FRED}(\mathcal{H})$ is open, given $\varepsilon > 0$, there exists $Z \in \mathfrak{C}_{\mathfrak{E}} \cap \text{FRED}(\mathcal{H})$ such that $\max(\|Z - T\|, \|Z^2 - T^2\|) < \varepsilon$. Write $Z = [E, F]$ with $E, F \in \mathfrak{E}$. (Since $[E, F] = [I - F, E]$, as in the proof of Lemma 7.9, we may assume without loss of generality that E has both infinite rank and infinite nullity.) Choose an invertible operator S such that

$$S^{-1}ES = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}.$$

Relative to this decomposition, write

$$S^{-1}FS = \begin{bmatrix} F_1 & X \\ -Y & F_4 \end{bmatrix}.$$

Then

$$W := S^{-1}ZS = \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix} \in \mathfrak{C}_{\mathfrak{E}} \cap \text{FRED}(\mathcal{H}).$$

In particular, X, Y are Fredholm operators. Moreover, since $S^{-1}FS$ is an idempotent, it follows that

$$F_1^2 - XY = F_1.$$

Clearly,

$$XY + \frac{1}{4}I = (F_1 - \frac{1}{2}I)^2 \in \mathcal{R}_2.$$

Note also that

$$W^2 = \begin{bmatrix} XY & 0 \\ 0 & YX \end{bmatrix}.$$

By Lemma 5.2, $\sigma_e(W^2) = \sigma_e(XY) = \sigma_e(YX)$, and

$$\text{IND}(\lambda I - W^2) = 2 \cdot \text{IND}(\lambda I - XY) \text{ for all } \lambda \in \mathbb{C} \setminus \sigma_e(W^2).$$

Since $XY + \frac{1}{4}I \in \mathcal{R}_2 \subset \text{CLOS}\,(\mathcal{R}_2)$, by Theorem 5.3,

$$\{\lambda \in \mathbb{C} : \lambda I - (XY + \frac{1}{4}I) \in \text{FRED}(\mathcal{H}) \text{ and } \text{IND}\,(\lambda I - (XY + \frac{1}{4}I)) \notin 2\mathbb{Z}\}$$

does not separate 0 from ∞.

Thus

$$\{\lambda \in \mathbb{C} : \lambda I - (W^2 + \frac{1}{4}I) \in \text{FRED}(\mathcal{H}) \text{ and } \text{IND}\,(\lambda I - (W^2 + \frac{1}{4}I)) \notin 4\mathbb{Z}\}$$

does not separate 0 from ∞. Applying Theorem 5.3 once again, we find that $W^2 + \frac{1}{4}I \in \text{CLOS}\,(\mathcal{R}_4)$. But $W = S^{-1}ZS$ implies that $W^2 + \frac{1}{4}I = S^{-1}(Z^2 + \frac{1}{4}I)S$, whence $Z^2 + \frac{1}{4}I \in \text{CLOS}\,(\mathcal{R}_4)$, and

$$\| \left(T^2 + \frac{1}{4}I\right) - \left(Z^2 + \frac{1}{4}I\right) \| = \|T^2 - Z^2\| < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, $T^2 + \frac{1}{4}I \in \text{CLOS}(\mathcal{R}_4)$. □

5.5 Corollary. *Let $S \in \mathcal{B}(\mathcal{H})$ denote the unilateral forward shift. If $\mu \in \mathbb{C}$ and $|\mu| > \frac{1}{2}$, then $\mu S \notin \text{CLOS}(\mathfrak{C}_{\mathfrak{E}})$.*

Proof. Since S is unitarily similar to αS, $\alpha \in \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, we may assume $\mu > \frac{1}{2}$. Then

$$\{\lambda \in \mathbb{C} : \lambda - (\mu^2 S^2 + \frac{1}{4}I) \text{ is Fredholm and}$$
$$\text{ind}(\lambda - (\mu^2 S^2 + \frac{1}{4}I)) \notin 4\mathbb{Z}\} = \{z \in \mathbb{C} : |z - \frac{1}{4}| < \mu^2\}$$

does separate 0 from ∞. By Theorem 5.3, $\mu^2 S^2 + \frac{1}{4}I \notin \text{CLOS}\,(\mathcal{R}_4)$. By Proposition 5.4, $\mu S \notin \text{CLOS}\,(\mathfrak{C}_{\mathfrak{E}})$. □

Using the next Lemma, one can exhibit a large class of non-quasitriangular operators in $\mathfrak{C}_{\mathfrak{E}}$.

5.6 Lemma. *Let $\mathcal{M}$ be a closed subspace of $\mathcal{H}$. Let $A \in \mathcal{B}(\mathcal{M}, \mathcal{M}^\perp)$, $B \in \mathcal{B}(\mathcal{M}^\perp, \mathcal{M})$, and suppose that $\operatorname{SPR}(AB) < \frac{1}{4}$, where $\operatorname{SPR}(\cdot)$ denotes the spectral radius function. Then*

$$\begin{bmatrix} 0 & B \\ A & 0 \end{bmatrix}$$

is a commutator of idempotents in $\mathcal{B}(\mathcal{H})$.

Proof. Note that we also have $\operatorname{SPR}(BA) < \frac{1}{4}$. Now $AB + \frac{1}{4}I_{\mathcal{M}^\perp}$ has a square root T which may be expressed as a series:

$$T = \frac{1}{2}I_{\mathcal{M}^\perp} + a_1 AB + a_2(AB)^2 + \cdots .$$

Similarly, $BA + \frac{1}{4}I_{\mathcal{M}}$ has a square root R with the same coefficients:

$$R = \frac{1}{2}I_{\mathcal{M}} + a_1 BA + a_2(BA)^2 + \cdots .$$

It is easy to check that

$$AR = TA, \quad BT = RB.$$

These equations show that the operator

$$E = \begin{bmatrix} \frac{1}{2}I_{\mathcal{M}} + R & -B \\ A & \frac{1}{2}I_{\mathcal{M}^\perp} - T \end{bmatrix}$$

is an idempotent. The proof is completed by taking the commutator of E and

$$F = \begin{bmatrix} I_{\mathcal{M}} & 0 \\ 0 & 0 \end{bmatrix}. \quad \square$$

We are now in a position to produce a converse to Corollary 5.5. We thank Peter Rosenthal for showing us the proof that $S+I$ has a square root, where S is the unilateral forward shift.

5.7 Corollary. *Let $S \in \mathcal{B}(\mathcal{H})$ denote the unilateral forward shift on $\mathcal{H}$ and $\mu \in \mathbb{C}$. Then $|\mu| \le \frac{1}{2}$ implies that $\mu S \in \mathfrak{C}_{\mathfrak{E}}$.*

Proof. Since $\mathfrak{E}$ is invariant under unitary conjugation, so is $\mathfrak{C}_{\mathfrak{E}}$, and since S is unitarily equivalent to αS whenever $\alpha \in \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$, we may assume without loss of generality that $0 \le \mu \le \frac{1}{2}$. Let $\{e_n\}_n$ denote an orthonormal basis for $\mathcal{H}$ with respect to which $Se_n = e_{n+1}$ for all $n \ge 1$. If we write $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, where $\mathcal{H}_1 := \vee\{e_{2n}\}_n$ and $\mathcal{H}_2 := \vee\{e_{2n-1}\}_n$, then we see that S is unitarily equivalent to

$$Y := \begin{bmatrix} 0 & I \\ S & 0 \end{bmatrix}.$$

Thus, it suffices to show $\mu Y \in \mathfrak{C}_{\mathfrak{E}}$, $0 \le \mu \le \frac{1}{2}$.

CASE 1. If $0 \le \mu < \frac{1}{2}$, then $\operatorname{SPR}(\mu S \cdot \mu I) < \frac{1}{4}$, and so by Lemma 5.6, $\mu Y \in \mathfrak{C}_{\mathfrak{E}}$.

CASE 2. Suppose that $\mu = \frac{1}{2}$. If we can find an idempotent of the form

$$E = \begin{bmatrix} A & -\frac{1}{2}I \\ \frac{1}{2}S & B \end{bmatrix},$$

then we can complete the proof by exhibiting $\frac{1}{2}Y$ as the commutator of E and

$$F = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}.$$

A necessary condition for E to be an idempotent is that $A^2 - \frac{1}{4}S = A$, or equivalently,

$$(A - \frac{1}{2}I)^2 = \frac{1}{4}(S + I).$$

(It is now hopefully abundantly clear why one might want $S + I$ to have a square root!)

Let H^∞ denote the space of bounded analytic functions on $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. We identify the unilateral forward shift with the Toeplitz operator T_z acting on the Hilbert space $H^2(\mathbb{D})$. Let f denote a holomorphic branch of the square root function defined on $\mathbb{C}$ except at 0 and on the negative real axis. Then the composition $h(z) = f(1 + z)$ is analytic at every point in $\Omega := \mathbb{C} \setminus \{z \in \mathbb{R} : z \le -1\}$.

In particular, $h(z)$ is analytic on the open unit disc, and $h(z)^2 = z + 1$. Since the modulus of $1 + z$ is less than 2 on $\mathbb{D}$, the modulus of $h(z)$ is less than $\sqrt{2}$ on $\mathbb{D}$ and thus $h \in H^\infty$. Hence $S + I = T_{z+1} = T_h^2$, and $T_h S = S T_h$.

Let $A := \frac{1}{2}(I + T_h)$ and $B := \frac{1}{2}(I - T_h)$. A routine computation then shows that the corresponding $E = \frac{1}{2}\begin{bmatrix} (I + T_h) & -I \\ S & (I - T_h) \end{bmatrix}$ is indeed the required idempotent. □

5.8 Remark. We thank the referee for providing a second proof of the fact that $S + I$ admits a square root, namely: the series

$$R := \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} S^n$$

converges absolutely and $R^2 = S + I$.

5.9 Corollary. *Let $S \in \mathcal{B}(\mathcal{H})$ denote the unilateral shift and $\mu \in \mathbb{C}$. Then $\mu S \in \operatorname{CLOS}(\mathfrak{C}_{\mathfrak{E}})$ if and only if $\mu S \in \mathfrak{C}_{\mathfrak{E}}$, and this happens if and only if $|\mu| \le \frac{1}{2}$.*

Of course, since $\mathfrak{C}_{\mathfrak{E}}$ is self-adjoint, we have corresponding results for the unilateral backward shift.

6. $\mathfrak{C}_{\mathfrak{E}}$ is strongly dense in $\mathcal{B}(\mathcal{H})$

6.1. Corollary 5.5 implies (amongst other things) that $\mathfrak{C}_{\mathfrak{E}}$ is not dense in the norm topology of $\mathcal{B}(\mathcal{H})$. In this section, we prove that $\mathfrak{C}_{\mathfrak{E}}$ is, however, dense in the strong operator topology.

6.2 Proposition. *The set* $\text{CLOS}\,(\mathfrak{C}_{\mathfrak{E}})$ *has no interior. Consequently, neither does* $\mathfrak{C}_{\mathfrak{E}}$.

Proof. Let $T \in \text{CLOS}\,(\mathfrak{C}_{\mathfrak{E}})$. By Lemma 4.4, every component of $\sigma(T)$ must intersect $\sigma(-T)$.

Let $\varepsilon > 0$ and choose $0 < \delta < \frac{\varepsilon}{10}$ such that $\|T - X\| < \delta$ implies that $\sigma(X) \subseteq (\sigma(T))_{\varepsilon/4}$; this is possible thanks to the upper semicontinuity of the spectrum. Fix $\alpha \in \sigma(T)$ with $|\alpha| = \text{SPR}(T)$, the spectral radius of T. Then α is an approximate eigenvalue of T, and as such we can find a norm-one vector $x \in \mathcal{H}$ such that $\|(T - \alpha I)x\| < \delta$.

Set $\mathcal{M} := \mathbb{C}x$ and relative to the decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$, write

$$T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}.$$

Note that

$$\| \begin{bmatrix} T_1 - \alpha \\ T_3 \end{bmatrix} \| = \|(T - \alpha I)x\| < \delta.$$

Setting $Y := \begin{bmatrix} \alpha & T_2 \\ 0 & T_4 \end{bmatrix}$, we see that $\|T - Y\| < \delta$ and therefore that

$$\sigma(Y) \subseteq (\sigma(T))_{\varepsilon/4}.$$

Also, since $\mathcal{M}$ is finite-dimensional, it follows from [21, Corollary 8] (alternatively, it can be shown directly) that

$$\{\alpha\} \cup \sigma(T_4) = \sigma(Y) \subseteq (\sigma(T))_{\varepsilon/4}.$$

Let $\beta := \alpha + \dfrac{\varepsilon}{2}\dfrac{\alpha}{|\alpha|}$, so that $\text{DIST}(\beta, \sigma(T)) = \frac{\varepsilon}{2}$. (If $\text{SPR}(T) = 0$, then $\alpha = 0$ and we simply let $\beta = \dfrac{\varepsilon}{2}$.) In particular,

$$|\beta| \geq \text{SPR}(T_4) + \frac{\varepsilon}{4},$$

and so $-\beta \notin \sigma(T_4)$. Set $Z = \begin{bmatrix} \beta & T_2 \\ 0 & T_4 \end{bmatrix}$. Then

$$\|T - Z\| \leq \|T - Y\| + \|Y - Z\| < \delta + \frac{\varepsilon}{2} < \frac{\varepsilon}{10} + \frac{\varepsilon}{2} < \varepsilon.$$

As before, since $\mathcal{M}$ is finite-dimensional, $\sigma(Z) = \{\beta\} \cup \sigma(T_4)$. Let Γ_β denote the connected component of β in $\sigma(Z)$. Since $\{\beta\} = \Gamma_\beta$ does not intersect $\sigma(-Z)$, we conclude from Lemma 4.4 that $Z \notin \text{CLOS}\,(\mathfrak{C}_{\mathfrak{E}})$.

This completes the proof. □

Let us denote by $\mathcal{D}_n$ the set of diagonal matrices in $\mathbb{M}_n(\mathbb{C})$ relative to the standard ONB for $\mathbb{C}^n$. Given a non-empty subset $\Delta \subseteq \mathcal{B}(\mathcal{H})$, we write $\mathcal{S}(\Delta)$ to denote the set

$$\{S^{-1}XS : X \in \Delta, S \in \mathcal{B}(\mathcal{H}) \text{ invertible}\}.$$

The following result is a routine exercise; clearly any matrix $T \in \mathbb{M}_n(\mathbb{C})$ may be upper-triangularised, and by a small perturbation of its diagonal, we may approximate T by a matrix T_0 whose diagonal consists of n distinct entries. But then T_0 is similar to a diagonal matrix.

6.3 Lemma. *Let $n \in \mathbb{N}$. Then $\mathcal{S}(\mathcal{D}_n)$ is norm-dense in $\mathbb{M}_n(\mathbb{C})$.*

6.4 Proposition. *The set $\mathfrak{C}_{\mathfrak{E}}$ is* SOT*-dense in $\mathcal{B}(\mathcal{H})$, and therefore* WOT*-dense in $\mathcal{B}(\mathcal{H})$.*

Proof. Let $T \in \mathcal{B}(\mathcal{H})$, $\varepsilon > 0$ and $\Omega := \{x_1, x_2, \ldots, x_n\} \subseteq \mathcal{H}$. We shall produce a (finite-rank!) operator $C \in \mathfrak{C}_{\mathfrak{E}}$ such that $\|Tx_k - Cx_k\| < \varepsilon$, $1 \le k \le n$.

Let $\mathcal{M} := \text{span}\,\{x_1, x_2, \ldots, x_n\}$ and let $P_{\mathcal{M}}$ denote the orthogonal projection of $\mathcal{H}$ onto $\mathcal{M}$. Set $F_0 := TP_{\mathcal{M}}$, so that $F_0 \in \mathcal{B}(\mathcal{H})$ is of finite rank. Let $\mathcal{N}$ be a half-space (i.e. a space of infinite dimension and co-dimension) which contains $\mathcal{R} := \text{span}\,\{\mathcal{M}, \text{ran}\, F_0, \text{ran}\, F_0^*\}$, and write the compression F_1 of F_0 to $\mathcal{N}$ as

$$F_1 := \begin{bmatrix} F_{11} & 0 \\ 0 & 0 \end{bmatrix}$$

relative to the decomposition $\mathcal{N} = \mathcal{R} \oplus (\mathcal{N} \ominus \mathcal{R})$. Keep in mind that $\mathcal{R}$ is finite-dimensional. Let $V = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$ relative to the decomposition $\mathcal{H} = \mathcal{N} \oplus \mathcal{N}^\perp$. Relative to the decomposition $\mathcal{H} = \mathcal{R} \oplus (\mathcal{N} \ominus \mathcal{R}) \oplus V\mathcal{R} \oplus V(\mathcal{N} \ominus \mathcal{R})$, consider the operator

$$X = \begin{bmatrix} F_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -F_{11} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let $\eta := \max\{\|x_k\| : 1 \le k \le n\}$. By Lemma 6.3, there exists a normal operator D and an invertible operator S in $\mathcal{B}(\mathcal{R})$ such that $\|S^{-1}DS - F_{11}\| < \frac{\varepsilon}{2\eta+1}$.

Let

$$Y = \begin{bmatrix} S^{-1}DS & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -S^{-1}DS & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then $\|Y - X\| < \varepsilon$ and Y is similar to

$$M := \begin{bmatrix} D & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -D & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Note that M is normal since D is, and in fact, M is finite-rank since D is. By [18, Proposition 3], $M \in \mathfrak{C}_{\mathfrak{E}}$. Since the latter set is clearly invariant under similarity, $Y \in \mathfrak{C}_{\mathfrak{E}}$.

If $1 \leq k \leq n$, then

$$\begin{aligned} \|Tx_k - Yx_k\| &\leq \|Tx_k - F_{11}x_k\| + \|F_{11}x_k - S^{-1}DSx_k\| \\ &\leq 0 + \eta\|F_{11} - S^{-1}DS\| \\ &\leq \frac{\varepsilon}{2} \\ &< \varepsilon. \end{aligned}$$

Thus, every basic SOT-neighbourhood of T contains an element of $\mathfrak{C}_{\mathfrak{E}}$, and as such, the latter is SOT-dense in $\mathcal{B}(\mathcal{H})$. □

7. Factorisation problems

7.1. We now turn our attention to the problem of factoring elements of $\mathcal{B}(\mathcal{H})$ as sums and products of elements of $\mathfrak{C}_{\mathfrak{E}}$ and closely related sets. Given a non-empty subset $\Delta \subseteq \mathcal{B}(\mathcal{H})$ and an integer $n \geq 1$, we define the sets

$$\sum_n \Delta := \{\sum_{k=1}^n T_k : T_k \in \Delta \text{ for all } 1 \leq k \leq n\}$$

and

$$\prod_n \Delta := \{\prod_{k=1}^n T_k : T_k \in \Delta \text{ for all } 1 \leq k \leq n\}.$$

The problem is to find the minimum positive integers m and n (if they exist) such that $\mathcal{B}(\mathcal{H}) = \sum_n \Delta = \prod_m \Delta$. Such decompositions of Hilbert space operators have been studied by several authors for various sets Δ (see, e.g. the papers [37,38] and their references). For example, in the paper [16], it is shown that if $\varnothing \neq \Delta \not\subseteq \mathbb{C}I + \mathcal{K}(\mathcal{H})$ and Δ is invariant under conjugation by invertible operators, then

$$\mathcal{B}(\mathcal{H}) = \sum_8 \Delta.$$

(This applies, in particular, to the case where $\Delta = \mathcal{S}(T)$ is the similarity orbit of a fixed operator $T \notin \mathbb{C}I + \mathcal{K}(\mathcal{H})$.) Of course, the bigger the set Δ is, the smaller we expect the corresponding m and n to be.

7.2 The finite-dimensional case. In the case where $n := \dim \mathcal{H} < \infty$, we note that every element of $\mathfrak{C}_{\mathfrak{E}}$ clearly has trace zero, and as such, for all $m \geq 1$, we have that $\sum_m \mathfrak{C}_{\mathfrak{E}} \subseteq \mathfrak{sl}_n(\mathbb{C}) := \{T \in \mathbb{M}_n(\mathbb{C}) : \mathrm{Tr}(T) = 0\}$. Moreover, if n is odd, then the fact that $T \in \mathfrak{C}_{\mathfrak{E}}$ implies that $T \sim -T$, which in turn implies that $0 \in \sigma(T)$. Thus any product of elements of $\mathfrak{C}_{\mathfrak{E}}$ must be singular. These are the only obstructions to factoring operators as sums and products of elements of $\mathfrak{C}_{\mathfrak{E}}$, as we now prove.

7.3 Proposition. *Let $n := \dim \mathcal{H} < \infty$. Then*

$$\sum_2 \mathfrak{C}_{\mathfrak{E}} = \mathfrak{sl}_n(\mathbb{C}).$$

Proof. Let $T \in \mathfrak{sl}_n(\mathbb{C})$. By the theorem of Albert and Muckenhoupt [1], there exists an ONB $\{e_k\}_{k=1}^n$ relative to which the matrix $[T] = [t_{ij} := \langle Te_j, e_i \rangle]$ for T admits a zero-diagonal; i.e. $t_{kk} = 0$, $1 \leq k \leq n$.

We may then write $T = U + L$, where $U := [u_{ij}]$ and $u_{ij} = \begin{cases} t_{ij} & \text{if } i \leq j \\ 0 & \text{if } i > j \end{cases}$, and $L := T - U$ is lower triangular. Clearly U and L are nilpotent. Since every nilpotent matrix lies in $\mathfrak{C}_{\mathfrak{E}}$ by [18, Proposition 6], we see that $T \in \sum_2 \mathfrak{C}_{\mathfrak{E}}$.

The converse was observed in the paragraph above. □

We also require the following result of Sourour [30].

7.4 Theorem *(Sourour). Let T be a nonscalar invertible $n \times n$ matrix over a field $\mathbb{F}$ and let α_j and β_j, $1 \leq j \leq n$ be elements of $\mathbb{F}$ such that $\prod_{j=1}^n \alpha_j \cdot \beta_j = \det(T)$. There exist $n \times n$ matrices A and B with eigenvalues $\alpha_1, \alpha_2, \ldots, \alpha_n$ and $\beta_1, \beta_2, \ldots, \beta_n$ respectively such that $T = A \cdot B$.*

7.5 Proposition. *Let $n := \dim \mathcal{H} < \infty$.*

(a) *If n is even, then*

$$\mathbb{M}_n(\mathbb{C}) = \prod_2 \mathfrak{C}_{\mathfrak{E}}.$$

(b) *If n is odd, then*

$$\{T \in \mathbb{M}_n(\mathbb{C}) : 0 \in \sigma(T)\} = \prod_2 \mathfrak{C}_{\mathfrak{E}}.$$

Proof. We reduce the proof to the following two cases:

Case One. $n = 2m$ is even and $T \in \mathbb{M}_n(\mathbb{C})$ is invertible. Suppose first that T is a scalar matrix, say $T = \lambda I_{2m}$. By [18, Proposition 3],

$$T = \begin{bmatrix} \lambda I_m & 0 \\ 0 & -\lambda I_m \end{bmatrix} \cdot \begin{bmatrix} I_m & 0 \\ 0 & -I_m \end{bmatrix} \in \prod_2 \mathfrak{C}_{\mathfrak{E}}.$$

This reduces the problem to the case where T is a nonscalar matrix. It is not hard to see that we may choose $\alpha_1, \cdots, \alpha_m \in \mathbb{C}$ such that all of the numbers $\pm\alpha_1, \pm\alpha_2, \cdots, \pm\alpha_m$ are distinct and

$$(\prod_{k=1}^{m} \alpha_k)^4 = \det T.$$

By Sourour's Theorem 7.4 above, we may factor T as a product of the form $T = A \cdot B$, where

$$\sigma(A) = \sigma(B) = \{\pm\alpha_1, \pm\alpha_2, \cdots, \pm\alpha_m\}.$$

By considering Jordan forms, we see that A and B are then each similar to $D \oplus -D$, where $D = \text{DIAG}(\alpha_1, \alpha_2, \cdots, \alpha_m)$. Then Proposition 3 of [18] asserts that $D \oplus -D \in \mathfrak{C}_{\mathfrak{E}}$, whence $A, B \in \mathfrak{C}_{\mathfrak{E}}$, completing the proof in this case.

Case Two. $n \in \mathbb{N}$ is arbitrary and $T \in \mathbb{M}_n(\mathbb{C})$ is not invertible.

If $n = 2$ and $T = \begin{bmatrix} 0 & t_{12} \\ 0 & 0 \end{bmatrix} \in \mathbb{M}_2(\mathbb{C})$ is nilpotent, then clearly

$$T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 0 & t_{12} \\ 0 & 0 \end{bmatrix} \in \prod_2 \mathfrak{C}_{\mathfrak{E}}.$$

Now suppose that either $n \geq 3$, or $n = 2$ and T is not nilpotent. It follows from [31, Theorem 1] that T may be factored as a product $T = A \cdot B$ where A and B are nilpotent. As already noted above, every nilpotent matrix lies in $\mathfrak{C}_{\mathfrak{E}}$, and so $T \in \prod_2 \mathfrak{C}_{\mathfrak{E}}$.

When n is odd, we have already noted in Section 7.2 that $\prod_k \mathfrak{C}_{\mathfrak{E}} \subseteq \{T \in \mathbb{M}_n(\mathbb{C}) : 0 \in \sigma(T)\}$ for all $k \geq 1$, so this completes the proof. □

7.6 The infinite-dimensional case. We next turn our attention to the case where $\mathcal{H}$ is infinite-dimensional, separable and complex.

7.7 Proposition. *Not every operator in $\mathcal{B}(\mathcal{H})$ is a sum of two commutators of idempotents, but every operator is a sum of at most five of these. In other words,*

$$\sum_2 \mathfrak{C}_{\mathfrak{E}} \subsetneq \mathcal{B}(\mathcal{H}) = \sum_5 \mathfrak{C}_{\mathfrak{E}}.$$

Proof. Suppose that $\mathcal{B}(\mathcal{H}) = \sum_2 \mathfrak{C}_{\mathfrak{E}}$. Then there exist $X, Y \in \mathfrak{C}_{\mathfrak{E}}$ such that

$$I = X + Y,$$

or equivalently,

$$Y = I - X.$$

Since $Y \in \mathfrak{C}_{\mathfrak{E}}$, by Proposition 2.3, $Y \sim -Y$. Then $I - X = Y \sim -Y = X - I$. Applying the spectral mapping theorem,

$$\sigma(X) - 1 = 1 - \sigma(X),$$

and thus $\sigma(X) = 2 - \sigma(X)$.

Choose $\alpha \in \sigma(X)$ such that $\text{Re}\,\alpha = \max\{\text{Re}\,\lambda : \lambda \in \sigma(X)\}$. Then $\alpha \in \sigma(X)$ implies that $-\alpha \in \sigma(X)$ (since $X \sim -X$ because it is a commutator of idempotents) and so $2 - (-\alpha) = 2 + \alpha \in \sigma(X)$. Clearly $\text{Re}(2 + \alpha) > \text{Re}\,\alpha$, a contradiction.

That $\mathcal{B}(\mathcal{H}) = \sum_5 \mathfrak{C}_{\mathfrak{E}}$ follows from the fact that every operator in $\mathcal{B}(\mathcal{H})$ may be written as a sum of at most five nilpotent operators of order 2 [26, Theorem 2], and each nilpotent of order two lies in $\mathfrak{C}_{\mathfrak{E}}$. This holds, since any nilpotent M of order two may be written in the form $M = \begin{bmatrix} 0 & M_2 \\ 0 & 0 \end{bmatrix}$, which is the commutator $[E, F]$ of $E = I \oplus 0$ and $F = \begin{bmatrix} I & M_2 \\ 0 & 0 \end{bmatrix}$. □

7.8 Proposition. *Let $T \in \mathcal{B}(\mathcal{H})$. Then there exist $X, Y \in \text{CLOS}\,(\mathfrak{C}_{\mathfrak{E}})$ such that $T = X + Y$. In other words,*

$$\mathcal{B}(\mathcal{H}) = \sum_2 \text{CLOS}\,(\mathfrak{C}_{\mathfrak{E}}).$$

Proof. Let $T \in \mathcal{B}(\mathcal{H})$ be arbitrary. An immediate corollary of [23, Theorem 5.15] is that there exist operators X, Y, each a limit of nilpotent operators in $\mathcal{B}(\mathcal{H})$, such that $T = X + Y$.

By Theorem 2.13, every nilpotent operator lies in $\text{CLOS}\,(\mathfrak{C}_{\mathfrak{E}})$, from which the result now follows.

As we have seen in Corollary 5.5, if S is the unilateral forward shift, then $S \notin \text{CLOS}\,(\mathfrak{C}_{\mathfrak{E}})$. Thus 2 is the minimum number n for which $\sum_n \text{CLOS}\,(\mathfrak{C}_{\mathfrak{E}}) = \mathcal{B}(\mathcal{H})$. □

We next turn our attention to finding the minimal $m \in \mathbb{N}$ such that $\mathcal{B}(\mathcal{H}) = \prod_m \mathfrak{C}_{\mathfrak{E}}$. We show that $2 \leq m \leq 3$, and identify a large class of operators which lie in $\prod_2 \mathfrak{C}_{\mathfrak{E}}$. Since $I \notin \mathfrak{C}_{\mathfrak{E}}$, the following result is not a complete triviality, though it is not difficult.

7.9 Lemma. *If $X \in \mathfrak{C}_{\mathfrak{E}}$, then $X = Y \cdot Z$, where $Y, Z \in \mathfrak{C}_{\mathfrak{E}}$. In other words,*

$$\mathfrak{C}_{\mathfrak{E}} \subseteq \prod_2 \mathfrak{C}_{\mathfrak{E}}.$$

Proof. Let $X := EF - FE \in \mathfrak{C}_{\mathfrak{E}}$, where $E, F \in \mathfrak{E}$.

Case 1. Consider first the case where E has both infinite rank and infinite nullity. Since both $\mathfrak{C}_{\mathfrak{E}}$ and $\prod_2 \mathfrak{C}_{\mathfrak{E}}$ are invariant under conjugation by invertible operators, it suffices to suppose that $E = P = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$.

Write $F = \begin{bmatrix} F_1 & F_2 \\ F_3 & F_4 \end{bmatrix}$, so that $X = \begin{bmatrix} 0 & F_2 \\ -F_3 & 0 \end{bmatrix}$. Now $\mathfrak{C}_{\mathfrak{E}}$ is invariant under unitary conjugation and so

$$Z := \begin{bmatrix} I & 0 \\ 0 & -iI \end{bmatrix} \cdot \begin{bmatrix} 0 & F_2 \\ -F_3 & 0 \end{bmatrix} \cdot \begin{bmatrix} I & 0 \\ 0 & iI \end{bmatrix} = \begin{bmatrix} 0 & iF_2 \\ iF_3 & 0 \end{bmatrix} \in \mathfrak{C}_{\mathfrak{E}}.$$

But $Y := \begin{bmatrix} -iI & 0 \\ 0 & iI \end{bmatrix} \in \mathfrak{C}_{\mathfrak{E}}$ by [18, Proposition 3], and so

$$X = Y \cdot Z \in \prod_2 \mathfrak{C}_{\mathfrak{E}}.$$

Case 2. A similar argument shows that if F has both infinite rank and infinite nullity, then $X \in \prod_2 \mathfrak{C}_{\mathfrak{E}}$. (Here we first reduce to the case where $F = I \oplus 0$.)

Case 3. If neither of E nor F has both infinite rank and infinite nullity, then by replacing E by $(I - E)$ and F by $I - F$ as necessary, we may assume that both E and F have finite rank. Let $\mathcal{M} := \operatorname{span}\{\operatorname{ran} E, \operatorname{ran} F, \operatorname{ran} E^*, \operatorname{ran} F^*\}$. Relative to the decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$, we have that $E = E_0 \oplus 0$ and $F = F_0 \oplus 0$. Since $\mathcal{M}^\perp$ is infinite-dimensional, we can find an orthogonal projection $Q \in \mathcal{B}(\mathcal{M}^\perp)$ such that the rank and nullity of Q are both infinite.

Note that $E_1 := E_0 \oplus Q$ and $F_1 := F_0 \oplus Q$ are idempotents with infinite rank and infinite nullity, and $X = E_1 F_1 - F_1 E_1$. We are therefore in a position to apply Case 1 to complete the proof. □

7.10 Proposition. *Let $A, B \in \mathcal{B}(\mathcal{H})$.*

(a) *If $T \in \mathcal{B}(\mathcal{H})$ is similar to $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$, then $T \in \prod_2 \mathfrak{C}_{\mathfrak{E}}$.*

(b) *If $R \in \mathcal{B}(\mathcal{H})$ is similar to $\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$, then $R \in \prod_2 \mathfrak{C}_{\mathfrak{E}}$.*

Proof. Since $\mathfrak{C}_{\mathfrak{E}}$ is invariant under similarity, it suffices to consider operators T and R that are of the 2×2 operator matrix forms above.

First note that by Proposition 3 of [18],

$$\begin{bmatrix} \alpha I & 0 \\ 0 & -\alpha I \end{bmatrix} \in \mathfrak{C}_{\mathfrak{E}}.$$

(a) Observe that for all $\varepsilon > 0$,

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = \begin{bmatrix} 0 & \varepsilon^{-1} I \\ \varepsilon^{-1} I & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & \varepsilon B \\ \varepsilon A & 0 \end{bmatrix}.$$

Fix $0 < \varepsilon < 1$ such that $\text{SPR}(\varepsilon^2 AB) < \frac{1}{4}$. Then by Lemma 5.6,

$$\begin{bmatrix} 0 & \varepsilon B \\ \varepsilon A & 0 \end{bmatrix} \in \mathfrak{C}_{\mathfrak{E}}.$$

Since $0 < \varepsilon < 1$, $\varepsilon^{-1} > 1$ and so from above,

$$\begin{bmatrix} 0 & \varepsilon^{-1} I \\ \varepsilon^{-1} I & 0 \end{bmatrix} \simeq \begin{bmatrix} \varepsilon^{-1} I & 0 \\ 0 & -\varepsilon^{-1} I \end{bmatrix} \in \mathfrak{C}_{\mathfrak{E}}.$$

From this the result follows.

(b) Similarly,

$$\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} = \begin{bmatrix} \varepsilon^{-1} I & 0 \\ 0 & -\varepsilon^{-1} I \end{bmatrix} \cdot \begin{bmatrix} 0 & \varepsilon A \\ -\varepsilon B & 0 \end{bmatrix}.$$

By Lemma 5.6, the second operator lies in $\mathfrak{C}_{\mathfrak{E}}$ for sufficiently small $0 < \varepsilon < 1$, and as we have just seen, the first operator also lies in $\mathfrak{C}_{\mathfrak{E}}$. This completes the proof. □

7.11 Corollary. *Every element z of the Calkin algebra is of the form $z = x \cdot y$, where x and y are commutators of idempotents in the Calkin algebra.*

Proof. By Voiculescu's non-commutative Weyl-von Neumann Theorem [34] (see also [15, Corollary II.5.5]), given $T \in \mathcal{B}(\mathcal{H})$, there exists $K \in \mathcal{K}(\mathcal{H})$ such that $T - K \simeq \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ for some $A, B \in \mathcal{B}(\mathcal{H})$. By Proposition 7.10, $T - K \in \prod_2 \mathfrak{C}_{\mathfrak{E}}$, say $T = X \cdot Y$ where $X, Y \in \mathfrak{C}_{\mathfrak{E}}$. Passing to the Calkin algebra,

$$\pi(T) = \pi(T - K) = \pi(X) \cdot \pi(Y),$$

and clearly $\pi(X), \pi(Y)$ may each be written as a commutator of two idempotents in the Calkin algebra. □

7.12 Corollary.

(a) *If* $N \in \mathcal{B}(\mathcal{H})$ *is normal, then* $N \in \prod_2 \mathfrak{C}_{\mathfrak{E}}$.
(b) *If* $S \in \mathcal{B}(\mathcal{H})$ *is the unilateral forward shift and* $\kappa \in \mathbb{C}$, *then* $\kappa S \in \prod_2 \mathfrak{C}_{\mathfrak{E}}$.

Proof.
- Let $N \in \mathcal{B}(\mathcal{H})$ be normal. Then N admits a decomposition $N \simeq N_1 \oplus N_2$ where each term acts on an infinite-dimensional subspace of $\mathcal{H}$, and so by Proposition 7.10 (a), $N \in \prod_2 \mathfrak{C}_{\mathfrak{E}}$.
- If $\kappa = 0$, this is trivial. Also, since $\kappa S \simeq |\kappa| S$, it suffices to consider the case where $\kappa > 0$. As we have seen above, $S \simeq \begin{bmatrix} 0 & I \\ S & 0 \end{bmatrix}$, and

$$\kappa S \simeq \begin{bmatrix} 0 & \kappa I \\ \kappa S & 0 \end{bmatrix} = \begin{bmatrix} 4\kappa I & 0 \\ 0 & -4\kappa I \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{4} I \\ -\frac{1}{4} S & 0 \end{bmatrix}.$$

The first normal operator lies in $\mathfrak{C}_{\mathfrak{E}}$ as shown in [18, Proposition 3], and by Lemma 5.6, the second operator lies in $\mathfrak{C}_{\mathfrak{E}}$ as well. □

Our approach to proving that $\mathcal{B}(\mathcal{H}) = \prod_3 \mathfrak{C}_{\mathfrak{E}}$ requires us to show that we can extend the above result for the unilateral shift to arbitrary isometries. The proof that every isometry lies in $\prod_2 \mathfrak{C}_{\mathfrak{E}}$ requires some effort. We begin with a result drawn from the proof of [19, Corollary 4.3].

7.13 Theorem *(Guinand-Marcoux). Let* $2 \le n \in \mathbb{N}$ *and* $0 < r_1 < r_2 < \cdots < r_n < 1$. *Let* $S \in \mathcal{B}(\mathcal{H})$ *denote the unilateral shift. Then there exist* $B \in \mathcal{B}(\mathbb{C}^n)$ *with* $\sigma(B) = \{r_1, r_2, \ldots, r_n\}$ *and* $A \in \mathcal{B}(\mathbb{C}^n, \mathcal{H})$ *such that*

$$S \simeq \begin{bmatrix} S & A \\ 0 & B \end{bmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathbb{C}^n).$$

7.14 Corollary. *Let* $T \in \mathcal{B}(\mathbb{C}^{2m}) \simeq \mathbb{M}_{2m}(\mathbb{C})$ *be a nonscalar invertible matrix. Given* $0 < \varepsilon < 1$, *there exist* $A, B \in \mathfrak{C}_{\mathfrak{E}}$ *such that* $\text{SPR}(A) < \varepsilon$, $\text{SPR}(B^{-1}) < \varepsilon$, *and*

$$T = A \cdot B.$$

Proof. By Corollary 9 of [18], if $X \in \mathbb{M}_{2m}(\mathbb{C})$ and $\frac{1}{2} i \notin \sigma(X)$, then $X \in \mathfrak{C}_{\mathfrak{E}}$ if and only if X is similar to $-X$. Let $\delta := \det(T)$, and write $\delta = e^{i\theta}|\delta|$ for an appropriate $\theta \in [0, 2\pi)$. Choose $0 < \alpha_m < \alpha_{m-1} < \cdots < \alpha_1 < \frac{\varepsilon}{3} \min(1, |\delta|^{1/2m})$, and set $\alpha_{m+k} := -\alpha_k$, $1 \le k \le m$. Clearly $|\alpha_k| < \frac{\varepsilon}{3} < \varepsilon$ for all $1 \le k \le 2m$.

Let $\beta_k := e^{i\theta/2m}|\delta|^{1/2m}\alpha_k^{-1}$, $1 \le k \le 2m$. For $1 \le k \le 2m$,

$$|\beta_k^{-1}| = |\alpha_k| \cdot |\delta|^{-1/2m} < \frac{\varepsilon}{3}.$$

Observe furthermore that $\alpha_k, 1 \leq k \leq 2m$ are distinct, as are $\beta_k, 1 \leq k \leq 2m$, and that

$$\prod_{k=1}^{2m} \alpha_k \cdot \beta_k = \prod_{k=1}^{m} (e^{i\theta/2m} |\delta|^{1/2m})^2 = e^{i\theta}|\delta| = \delta = \det(T).$$

Using Theorem 7.4, we can choose matrices A and B with $\sigma(A) = \{\alpha_k\}_{k=1}^{2m}$ and $\sigma(B) = \{\beta_k\}_{k=1}^{2m}$ such that

$$T = A \cdot B.$$

Since all eigenvalues of A are distinct and $\alpha_{m+k} = -\alpha_k$, $1 \leq k \leq m$, A is similar to $-A$. Since $|\alpha_k| < \frac{\varepsilon}{3} < \frac{1}{3}$ for all $1 \leq k \leq 2m$, we see that $\frac{1}{2}i \notin \sigma(A)$ and so as noted in the first paragraph of the proof, $A \in \mathfrak{C}_{\mathfrak{E}}$.

Similarly, since all β_k's are distinct, $1 \leq k \leq 2m$ and $\beta_{m+k} = -\beta_k$, $1 \leq k \leq m$, B is similar to $-B$. Moreover, $|\beta_k^{-1}| < \frac{\varepsilon}{3}$, implying that $|\beta_k| \geq \frac{3}{\varepsilon} > 3$, $1 \leq k \leq 2m$, whence $\frac{1}{2}i \notin \sigma(B)$. Thus $B \in \mathfrak{C}_{\mathfrak{E}}$, and so $T \in \prod_2 \mathfrak{C}_{\mathfrak{E}}$.

Finally, the fact that $\max_{1 \leq k \leq 2m}(|\alpha_k|, |\beta_k|^{-1}) < \frac{\varepsilon}{3}$ implies that

$$\max(\operatorname{SPR}(A), \operatorname{SPR}(B^{-1})) < \frac{\varepsilon}{3} < \varepsilon. \quad \square$$

7.15 Proposition. *Let $\mathcal{H}$ be an infinite-dimensional Hilbert space and $W \in \mathcal{B}(\mathcal{H})$ be an isometry. Let $\kappa > 0$ be a constant. Then $\kappa W \in \prod_2 \mathfrak{C}_{\mathfrak{E}}$.*

Proof. Let $0 < \kappa$ be fixed. By the Wold Decomposition (see, e.g. [15, Theorem V.2.1]), there exist a unitary operator U and a cardinal number $0 \leq \alpha \leq \dim \mathcal{H}$ such that

$$W \simeq U \oplus S^{(\alpha)},$$

where S is the unilateral forward shift. The space upon which U acts may be zero, finite-dimensional, or infinite-dimensional. Also, by Corollary 7.12 (b), we have that $\kappa S \in \prod_2 \mathfrak{C}_{\mathfrak{E}}$, say $\kappa S = Y_2 \cdot Z_2$ with $Y_2, Z_2 \in \mathfrak{C}_{\mathfrak{E}}$.

Case 1. Suppose that U acts on an infinite-dimensional space, or a finite-dimensional space of *even* dimension. By Corollary 7.12(a) or Proposition 7.5(a) (depending upon the dimension of the space), $\kappa U \in \prod_2 \mathfrak{C}_{\mathfrak{E}}$, say $\kappa U = Y_1 \cdot Z_1$ where $Y_1, Z_1 \in \mathfrak{C}_{\mathfrak{E}}$. Thus, for any $0 \leq \alpha$, we have

$$\kappa W \simeq \kappa U \oplus \kappa S^{(\alpha)} = (Y_1 \oplus Y_2^{(\alpha)}) \cdot (Z_1 \oplus Z_2^{(\alpha)}) \in \prod_2 \mathfrak{C}_{\mathfrak{E}}.$$

Case 2. Suppose that U acts on a finite-dimensional space of *odd* dimension n. In this case, $\alpha \geq 1$. Hence we may write $W \simeq (S \oplus U) \oplus S^{(\alpha-1)}$, where $\alpha - 1 := \alpha$ if α is an infinite cardinal. Since $\kappa S \in \prod_2 \mathfrak{C}_{\mathfrak{E}}$, it clearly suffices to prove that $\kappa(S \oplus U) \in \prod_2 \mathfrak{C}_{\mathfrak{E}}$.

Choose $\frac{3}{4} < r_1 < r_2 < \cdots r_n < 1$. By Theorem 7.13, we can find operators $A \in \mathcal{B}(\mathbb{C}^n, \mathcal{H})$, $B \in \mathcal{B}(\mathbb{C}^n)$ with $\sigma(B) = \{r_1, r_2, \ldots, r_n\}$ such that

$$S \simeq \begin{bmatrix} S & A \\ 0 & B \end{bmatrix}.$$

Observe that $\kappa(B \oplus U) \in \mathcal{B}(\mathbb{C}^{2n})$ is nonscalar and invertible, and so by Corollary 7.14, we can find $Y_0 = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}$ and $Z_0 = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \in \mathfrak{C}_{\mathfrak{E}}$ such that $\textsc{spr}(Y_0) < \frac{1}{10}$, $\textsc{spr}(Z_0^{-1}) < \frac{1}{10}$, and $\kappa(B \oplus U) = Y_0 \cdot Z_0$.

It then follows that

$$\kappa(S \oplus U) \simeq \kappa \begin{bmatrix} 0 & S & A_1 & 0 \\ I & 0 & A_2 & 0 \\ 0 & 0 & B & 0 \\ 0 & 0 & 0 & U \end{bmatrix}$$

$$= \begin{bmatrix} 3\kappa I & 0 & 0 & 0 \\ 0 & -3\kappa I & 0 & 0 \\ 0 & 0 & Y_{11} & Y_{12} \\ 0 & 0 & Y_{21} & Y_{22} \end{bmatrix} \cdot \begin{bmatrix} 0 & \frac{1}{3}S & \frac{1}{3}A_1 & 0 \\ -\frac{1}{3}I & 0 & -\frac{1}{3}A_2 & 0 \\ 0 & 0 & Z_{11} & Z_{12} \\ 0 & 0 & Z_{21} & Z_{22} \end{bmatrix}.$$

Clearly the first operator matrix in the product lies in $\mathfrak{C}_{\mathfrak{E}}$, since both $\begin{bmatrix} 3\kappa I & 0 \\ 0 & -3\kappa I \end{bmatrix}$ and Y_0 do.

As for the second operator matrix, observe that $\sigma(\begin{bmatrix} 0 & \frac{1}{3}S \\ -\frac{1}{3}I & 0 \end{bmatrix}) = \{z \in \mathbb{C} : |z| \leq \frac{1}{3}\}$, while $\sigma(Z_0) \subseteq \{z \in \mathbb{C} : |z| \geq 10\}$. It follows from Section 2.10 that

$$\begin{bmatrix} 0 & \frac{1}{3}S & \frac{1}{3}A_1 & 0 \\ -\frac{1}{3}I & 0 & -\frac{1}{3}A_2 & 0 \\ 0 & 0 & Z_{11} & Z_{12} \\ 0 & 0 & Z_{21} & Z_{22} \end{bmatrix} \sim \begin{bmatrix} 0 & \frac{1}{3}S & 0 & 0 \\ -\frac{1}{3}I & 0 & 0 & 0 \\ 0 & 0 & Z_{11} & Z_{12} \\ 0 & 0 & Z_{21} & Z_{22} \end{bmatrix} \simeq \begin{bmatrix} \frac{1}{3}S & 0 \\ 0 & Z_0 \end{bmatrix}.$$

But $\frac{1}{3}S \in \mathfrak{C}_{\mathfrak{E}}$ by Corollary 5.7, and $Z_0 \in \mathfrak{C}_{\mathfrak{E}}$ from above, so the second operator matrix also lies in $\mathfrak{C}_{\mathfrak{E}}$, implying that $\kappa(S \oplus U) \in \prod_2 \mathfrak{C}_{\mathfrak{E}}$. $\square$

7.16 Theorem. *Every operator in $\mathcal{B}(\mathcal{H})$ is a product of at most three elements of $\mathfrak{C}_{\mathfrak{E}}$; that is,*

$$\mathcal{B}(\mathcal{H}) = \prod_3 \mathfrak{C}_{\mathfrak{E}}.$$

Proof. Let $T \in \mathcal{B}(\mathcal{H})$, and consider the polar decomposition $T = V|T|$ of T. Then V is a partial isometry. Let $\mathcal{R} = (\mathrm{ran} T)^{\perp}$ and $\mathcal{K} := \ker T$. Note that $\mathfrak{C}_{\mathfrak{E}}$ is selfadjoint, and thus so is $\prod_3 \mathfrak{C}_{\mathfrak{E}}$.

By replacing T by T^* if necessary, we may therefore assume without loss of generality that $\dim \mathcal{K} \leq \dim \mathcal{R}$.

In this case, let $P_{\mathcal{K}}$ denote the orthogonal projection of $\mathcal{H}$ onto $\mathcal{K}$ and let $V_0 : \mathcal{K} \to \mathcal{R}$ be any isometry. We then define $W := V + V_0 P_{\mathcal{K}}$, so that W is an isometry and $T = W|T|$.

Let $Q \in \mathcal{B}(\mathcal{H})$ be any spectral projection for $|T|$ such that Q has infinite-dimensional range and infinite-dimensional kernel. Relative to $\mathcal{H} = \operatorname{ran} Q \oplus \ker Q$, we may write $|T| = \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix}$. Next, let $U := \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$.

Choose $\varepsilon > 0$ such that

$$\varepsilon U \cdot |T| = \begin{bmatrix} 0 & \varepsilon H_2 \\ \varepsilon H_1 & 0 \end{bmatrix}$$

satisfies $\textsc{spr}(\varepsilon^2 H_1 H_2) < \frac{1}{4}$. By Lemma 5.6, $\varepsilon U \cdot |T| \in \mathfrak{C}_{\mathfrak{E}}$.

Meanwhile, $\varepsilon^{-1} W U^*$ is a multiple of an isometry, and thus $\varepsilon^{-1} W U^* \in \prod_2 \mathfrak{C}_{\mathfrak{E}}$ by the above Proposition. Hence

$$T = (\varepsilon^{-1} W U^*)(\varepsilon U |T|) \in \prod_3 \mathfrak{C}_{\mathfrak{E}}. \quad \square$$

7.17 Remark. Although we have so far been unable to determine whether or not $\mathcal{B}(\mathcal{H}) = \prod_2 \mathfrak{C}_{\mathfrak{E}}$, it is not hard to show that $\prod_2 \mathfrak{C}_{\mathfrak{E}}$ is dense in $\mathcal{B}(\mathcal{H})$.

Indeed, let $T \in \mathcal{B}(\mathcal{H})$ and $\varepsilon > 0$. Let $\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ be the canonical quotient map. By Voiculescu's Theorem, there exists a compact operator $K \in \mathcal{B}(\mathcal{H})$ such that $\|K\| < \varepsilon$ and $T - K \simeq T \oplus \varrho^{(\infty)}(\pi(T))$, where ϱ is a unital, faithful *-representation of $C^*(\pi(T))$ into $\mathcal{B}(\mathcal{H}_\varrho)$, and where $\mathcal{H}_\varrho$ is a separable Hilbert space. By Proposition 7.10 (a), $T \oplus \varrho^{(\infty)}(\pi(T)) \in \prod_2 \mathfrak{C}_{\mathfrak{E}}$. Thus $T \in \textsc{clos}\,(\prod_2 \mathfrak{C}_{\mathfrak{E}})$.

We have seen that $\mathcal{B}(\mathcal{H}) = \sum_5 \mathfrak{C}_{\mathfrak{E}} = \prod_3 \mathfrak{C}_{\mathfrak{E}}$, and in [25, Proposition 2.3] it was shown that every element of $\mathfrak{C}_{\mathfrak{E}}$ is a difference of idempotents, i.e. that $\mathfrak{C}_{\mathfrak{E}} \subseteq \mathfrak{D}_{\mathfrak{E}}$. Moreover, this inclusion is strict (e.g. $I \in \mathfrak{D}_{\mathfrak{E}} \setminus \mathfrak{C}_{\mathfrak{E}}$). It is therefore of interest to see if we can find better estimates for the minimum values of m and $n \in \mathbb{N}$ respectively such that $\mathcal{B}(\mathcal{H}) = \sum_n \mathfrak{D}_{\mathfrak{E}} = \prod_m \mathfrak{D}_{\mathfrak{E}}$. In the case of sums, we can definitely do this.

7.18 Proposition. *Every operator $T \in \mathcal{B}(\mathcal{H})$ may be written as a sum of four or fewer elements of $\mathfrak{D}_{\mathfrak{E}}$. In other words,*

$$\mathcal{B}(\mathcal{H}) = \sum_4 \mathfrak{D}_{\mathfrak{E}}.$$

Proof. First we observe that if $X \in \mathcal{B}(\mathcal{H})$, then

$$\begin{bmatrix} X & 0 \\ 0 & -X \end{bmatrix} = \frac{1}{2} \begin{bmatrix} I + X & -I + X \\ -I - X & I - X \end{bmatrix} - \frac{1}{2} \begin{bmatrix} I - X & -I + X \\ -I - X & I + X \end{bmatrix}.$$

Hence, $X \oplus -X \in \mathfrak{D}_{\mathfrak{E}}$.

Next, note that $X \oplus 0 \in \sum_2 \mathfrak{D}_{\mathfrak{E}}$ (where the 0-term acts on an infinite-dimensional space). To see this, note that if $Y = X \oplus -X \oplus X \oplus -X \oplus \cdots$ (so that $Y \simeq X^{(\infty)} \oplus -X^{(\infty)}$), then $Y \in \mathfrak{D}_{\mathfrak{E}}$ from above. Clearly $Z = 0 \oplus Y = 0 \oplus X \oplus -X \oplus X \oplus -X \oplus \cdots \in \mathfrak{D}_{\mathfrak{E}}$ as well. But then

$$X \oplus 0 \simeq X \oplus 0 \oplus 0 \oplus 0 \oplus \cdots = Y + Z \in \sum_2 \mathfrak{D}_{\mathfrak{E}}.$$

Finally, it was shown in the proof of [16, Lemma 2.6] that every operator $T \in \mathcal{B}(\mathcal{H})$ admits a tridiagonal operator form

$$T = \begin{bmatrix} T_1 & T_2 & 0 \\ T_4 & T_5 & T_6 \\ 0 & T_8 & T_9 \end{bmatrix}$$

relative to some decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$, where $\dim \mathcal{H}_k = \dim \mathcal{H}$, $k = 1, 2, 3$. Writing

$$T = \begin{bmatrix} T_1 & T_2 & 0 \\ T_4 & T_5 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & T_6 \\ 0 & T_8 & T_9 \end{bmatrix},$$

and applying the argument of the previous paragraph, we conclude that $T \in \sum_4 \mathfrak{D}_{\mathfrak{E}}$. □

In keeping with the terminology from [17], we refer to the class of operators of the form $\lambda I + K$ where λ is a complex number and $K \in \mathcal{K}(\mathcal{H})$ is compact as **thin operators**, and this set will be denoted by $\mathfrak{T}$. It is a classical result of Brown and Pearcy [9] that there exist $A, B \in \mathcal{B}(\mathcal{H})$ such that $T = [A, B]$ if and only if $T \in (\mathcal{B}(\mathcal{H}) \setminus \mathfrak{T}) \cup \mathcal{K}(\mathcal{H})$.

7.19 Lemma. *Let $T \in \mathcal{B}(\mathcal{H})$. If $T \notin \mathfrak{T}$, then there exists a closed subspace $\mathcal{M} \subseteq \mathcal{H}$ of infinite dimension and codimension, such that relative to the decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$,*

$$T = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where B is left-invertible.

Proof. Since T is not a thin operator, by [2] (or [16, Theorem 2.4]), there exists a decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$, where each $\mathcal{H}_k$ is isomorphic to $\mathcal{H}$, and relative to which we may represent T in the form

$$\begin{bmatrix} C_1 & 0 & * \\ 0 & C_2 & * \\ * & * & * \end{bmatrix},$$

where – relative to orthonormal bases $\{e_n\}_n$ and $\{f_n\}_n$ respectively – $C_1 = \text{DIAG}\{\alpha_n\}_{n=1}^\infty$ and $C_2 = \text{DIAG}\{\beta_n\}_{n=1}^\infty$ are diagonal operators with $\lim_{n\to\infty} \alpha_n = \alpha$, $\lim_{n\to\infty} \beta_n = \beta$, and $\alpha \neq \beta$.

Next, fix $N \geq 1$ such that $n \geq N$ implies that

$$\max(|\alpha_n - \alpha|, |\beta_n - \beta|) < \delta := \frac{|\alpha - \beta|}{3}.$$

Setting $\mathcal{L}_1 := \vee\{e_n\}_{n=1}^N$ and $\mathcal{L}_2 := \vee\{f_n\}_{n=1}^N$, we see that relative to the decomposition

$$\mathcal{H} = (\mathcal{H}_1 \ominus \mathcal{L}_1) \oplus (\mathcal{H}_2 \ominus \mathcal{L}_2) \oplus (\mathcal{H}_3 \oplus \mathcal{L}_1 \oplus \mathcal{L}_2),$$

T not only admits a 3×3 decomposition of the form

$$\begin{bmatrix} D_1 & 0 & * \\ 0 & D_2 & * \\ * & * & * \end{bmatrix},$$

but now $D_1 - D_2$ is diagonal and bounded below (by δ), and thus $D_1 - D_2$ is invertible.

Set $g_n = \frac{\sqrt{2}}{2}(e_n + f_n)$, $h_n = \frac{\sqrt{2}}{2}(e_n - f_n)$, $n > N$. Then $\{g_n\}_{n>N}$ is an orthonormal set, as is $\{h_n\}_{n>N}$. Define $\mathcal{K}_1 = \vee\{g_n\}_{n>N}$, $\mathcal{K}_2 = \vee\{h_n\}_{n>N}$ and $\mathcal{K}_3 = (\mathcal{K}_1 \oplus \mathcal{K}_2)^\perp$. Relative to $\mathcal{H} = \mathcal{K}_1 \oplus \mathcal{K}_3 \oplus \mathcal{K}_2$, we find that

$$T = \begin{bmatrix} * & * & \frac{1}{2}(D_1 - D_2) \\ * & * & X \\ * & * & * \end{bmatrix}$$

for an appropriate operator X. Since $D_1 - D_2$ is invertible, it follows that

$$\begin{bmatrix} \frac{1}{2}(D_1 - D_2) \\ X \end{bmatrix}$$

is left-invertible. The proof is completed by setting $\mathcal{M} = \mathcal{K}_1 \oplus \mathcal{K}_3$. □

7.20 Lemma. *Let* $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ *be an operator in* $\mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ *with* B *left-invertible. Then there exists an invertible operator* $R \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ *such that*

$$RTR^{-1} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix},$$

where B_1 *and* C_1 *are both left-invertible.*

Proof. Set $R := \begin{bmatrix} I & 0 \\ tI & I \end{bmatrix}$, where $t \in \mathbb{R} \setminus \{0\}$ is a parameter which will be fixed later. Let $Z \in \mathcal{B}(\mathcal{H})$ be a left inverse of B. Then

$$RTR^{-1} = \begin{bmatrix} A - tB & B \\ tA + C - t(tB + D) & tB + D \end{bmatrix} := \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}.$$

Hence, $B_1 = B$ and

$$C_1 = -(I + \frac{1}{t}DZ - \frac{1}{t}AZ - \frac{1}{t^2}CZ)Bt^2.$$

Since the set of left-invertible operators of $\mathcal{B}(\mathcal{H})$ is open, C_1 is left-invertible for a sufficiently large choice of t. □

7.21 Proposition. *Let $T \in \mathcal{B}(\mathcal{H}) \setminus \mathfrak{T}$, then $T \in \prod_2 \mathfrak{D}_{\mathfrak{E}}$.*

Proof. By Lemma 7.19 and Lemma 7.20, we may assume that

$$T = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

and B, C are left-invertible. Pick Y, Z such that $YB = ZC = I$. Then

$$\begin{aligned} T = \begin{bmatrix} A & B \\ C & D \end{bmatrix} &= \begin{bmatrix} I & -AZ \\ DY & -I \end{bmatrix} \begin{bmatrix} 0 & B \\ -C & 0 \end{bmatrix} \\ &= (\begin{bmatrix} I & 0 \\ DY & 0 \end{bmatrix} - \begin{bmatrix} 0 & AZ \\ 0 & I \end{bmatrix})(\begin{bmatrix} I & B \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} I & 0 \\ C & 0 \end{bmatrix}) \in \prod_2 \mathfrak{D}_{\mathfrak{E}}. \quad \square \end{aligned}$$

7.22. There are certainly many thin operators which lie in $\prod_2 \mathfrak{D}_{\mathfrak{E}}$, but we have been unable to determine whether or not *every* thin operator (and thus every operator) lies in $\prod_2 \mathfrak{D}_{\mathfrak{E}}$. Of course, by Theorem 7.16, $\mathcal{B}(\mathcal{H}) = \prod_3 \mathfrak{D}_{\mathfrak{E}}$. We finish by enumerating a short list of questions which remain unresolved.

(a) Find the minimum integer $n \geq 2$ such that $\mathcal{B}(\mathcal{H}) = \sum_n \mathfrak{D}_{\mathfrak{E}}$. By Proposition 7.18, this minimum integer is at most 4.
(b) Determine whether or not the set $\mathfrak{T}$ of thin operators is contained in $\prod_2 \mathfrak{D}_{\mathfrak{E}}$.
(c) Find the minimum integers $m, n \geq 2$ such that $\mathcal{B}(\mathcal{H}) = \sum_n \mathrm{NEG}_{\mathrm{S}}(\mathcal{H}) = \prod_m \mathrm{NEG}_{\mathrm{S}}(\mathcal{H})$.

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No data was used for the research described in the article.

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