Non-Generic Analytic Combinatorics in Several Variables and Lattice Path Asymptotics

by

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Author's Declaration

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of Contributions

This thesis is based on work that I authored or co-authored. Chapter 3 is based on Completing the Asymptotic Classification of Mostly Symmetric Short Step Walks in an Orthant, by Kroitor and Melczer [27]. Chapter 4 is based, in part, on work of Greenwood, Kroitor, Melczer, and Simon extending Asymptotics of Weighted Reflectable Walks in A_2 by Greenwood and Simon [22].

Abstract

Finding and analyzing generating functions is a classic and powerful way of studying combinatorial structures. While generating functions are a priori purely formal objects, there is a rich theory treating them as complex-analytic functions. This is the field of *analytic combinatorics*, which exploits classical results in complex and harmonic analysis to find asymptotics of coefficient sequences of generating functions. More recently, the field of *analytic combinatorics in several variables* has been developed to study multivariate generating functions, which are amenable to multivariate analytic results.

When dealing with multivariate asymptotics we first pick a *direction vector*. To find asymptotics using analytic combinatorics, one typically uses the Cauchy integral formula to write the underlying sequence being studied as a complex integral, then uses residue computations to reduce to a so-called *Fourier-Laplace integral*. When the direction vector chosen is *generic*, the Fourier-Laplace integral being studied is well-behaved and asymptotics can be computed. When the direction vector chosen is *non-generic*, the Fourier-Laplace integral has singular amplitude, and asymptotics are harder to compute. This thesis studies singular Fourier-Laplace integrals and their applications to combinatorics.

Determining asymptotics for the number of lattice walks restricted to certain regions is one of the particular successes of analytic combinatorics in several variables. In particular, Melczer and Mishna [28] determined asymptotics for the number of walks in an orthant whose set of steps is a subset of $\{\pm 1, 0\}^d \setminus \{0\}$ and is symmetric over every axis. Melczer and Wilson [31] later determined asymptotics for lattice path models whose set of steps is a subset of $\{\pm 1, 0\}^d \setminus \{0\}$ and is symmetric over all but one axis, except when the vector sum of all the steps is $\mathbf{0}$. Here we determine asymptotics in the zero sum case using asymptotics of singular Fourier-Laplace integrals. We additionally study asymptotics of lattice path models restricted to Weyl chambers, before giving some generalizations of asymptotics of singular Fourier-Laplace integrals.

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- My family and friends, mathematical and otherwise. Thank you for your support over the last two years.

Dedication

Dedicated to the memory of Caroline Beth Norman.

And I will show you something different from either
Your shadow at morning striding behind you
Or your shadow at evening rising to meet you;
I will show you fear in a handful of dust.

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Chapter 1

Introduction

The term enumerative combinatorics often refers to a toolbox of techniques used to count families of discrete structures by size (where the meaning of 'size' can be chosen). Examples include counting the number of graphs with n vertices, or counting the number of bijections between two sets. This problem can be encoded as finding the value of the nth element of a sequence, and difficulty arises when the sequence has no obvious explicit formula. In these cases an extremely powerful tool is the *generating function*, which is a series that encodes a sequence in its coefficients.

While exact formulae for the elements of sequences of interest are not always possible to find, often it suffices to find and classify the *asymptotic behaviour* of such sequences. If we consider a generating function as a complex function, when possible, the techniques of *analytic combinatorics* allow us to find the asymptotics of its coefficient sequence by applying results from complex analysis. This often relies on knowing the *singular points* of the generating function in question, and applying classical theorems from analysis.

A natural generalization of counting objects based off of one parameter is to keep track of multiple parameters at the same time. An example is counting the number of graphs with n vertices and k edges. Similar to the one-variable case, we can derive from this a *multivariate generating function*: a series in several variables, where the coefficient of each monomial counts the number of structures with parameters equal to the exponents in the monomial. Note that asymptotics for multivariate sequences are not, a priori, defined. Instead, one often picks a so-called *direction vector*, and finds asymptotics of a corresponding univariate subsequence of the multivariate sequence.

Generalizing analytic combinatorics techniques to *analytic combinatorics in several variables* (ACSV) is complex, chiefly because the geometry of the set of singularities of a multivariate complex analytic function is much more complicated. This gives rise to a need to define a (generically finite) set of so-called *critical points* that contribute to the asymptotics of our multivariate sequence in a selected direction. The locations of the critical points depend on the direction that we choose and the local geometry of the singular set. When critical points appear with multiplicity, the chosen direction is called *non-generic*. In this case the analysis is trickier and often ad-hoc, as the asymptotics of relevant multivariate Fourier-Laplace integrals are not always known.

1.1 Organization

This thesis introduces analytic combinatorics in one and several variables in Chapter 2. Using this background, we enumerate a new family of lattice path models in Chapter 3. Chapter 4 surveys ongoing future work in both lattice path enumeration and asymptotics of general Fourier-Laplace integrals in non-generic directions. Chapter 3 is sourced from [27]. Chapter 4 is partially taken from ongoing work of Greenwood, Kroitor, Melczer, and Simon following up on [22].

Chapter 2

Analytic Combinatorics

Enumeration is the mathematics of counting. There are two clarifications that must be made.

a) What are we counting?

b) What 'counts' as counting?

The first question will be answered shortly by defining combinatorial classes and their associated generating functions. The essence of counting for us is identifying the coefficients of a power series that encodes the sequence that we want to identify. We thus move to a more general (and perhaps less combinatorial) framework by instead considering the question of computing the behaviour of coefficients in complex power series.

The second question may seem obvious at first glance: we are concerned with finding the number of mathematical objects satisfying certain constraints. Issues arise however when we try to count things that do not admit nice formulas. Consider the three sequences

- a_n = the number of permutations of $\{1, \ldots, n\}$;
- b_n = the number of integer partitions of n;
- c_n = the number of self-avoiding lattice walks of length n.

The first sequence has the nice closed representation $a_n = n!$. The third sequence has no known closed form representation [38]. The second sequence lies somewhere between: there is no nice closed formula, but we can find a different sequence

$$b'_n = \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}$$

The first section of this chapter is dedicated to a brief overview of the theory of generating functions, along with the requisite complex analysis needed for our later results. The second section gives background on analytic combinatorics in a single variable, specifically residues and saddle-point integrals. The third and final section covers analytic combinatorics in several variables, and gives motivation for the next chapter.

The presentation of this chapter is inspired by the notes [30] and the textbook [29].

2.1 Background Material

In this section we briefly review the theory of generating functions and asymptotic notation. We also cover the complex analysis that will be used later.

2.1.1 Combinatorial Classes and Generating Functions

We would like to formalize the theory of enumeration.

Definition 1 (Combinatorial Class)**.** A **combinatorial class** (C, ω) is a set C equipped with a **weight function** $\omega : C \to \mathbb{N}$ such that $|\omega^{-1}(n)| < \infty$ for all $n \in \mathbb{N}$.

The weight function allows us to ask "how many elements are there of size n" while having a well-defined notion of size (we say that the size of an element is its image under ω) and a finite number of objects of any size.

Example 2. Consider the set of all strings of brackets (and) such that:

- there are the same number of opening brackets and closing brackets in each string;
- in each left-to-right prefix the number of opening brackets is greater than or equal to the number of closing brackets.

If we equip this set with the weight function $\omega(s) = \text{length}(s)/2$ then $\omega^{-1}(n)$ is the set of all such strings of length $2n$. This is a valid combinatorial class, and $|\omega^{-1}(n)| = \frac{1}{n+1}\binom{2n}{n}$ is the nth Catalan number. ◁

We use power series to store information about combinatorial classes.

Definition 3. Let R be an integral domain. We make the set

$$R[[z]] = \left\{ \sum_{n\geq 0} a_n z^n : a_n \in R \right\}$$

the **ring of formal power series** (in the variable z over R) by defining addition term-wise and multiplication by the convolution

$$\left(\sum_{n\geq 0} a_n z^n\right)\left(\sum_{n\geq 0} b_n z^n\right) = \sum_{n\geq 0}\left(\sum_{k=0}^{n} a_k b_{n-k}\right) z^n.$$

A **polynomial** is an element of $R[[z]]$ that has only finitely many non-zero coefficients, and we write the collection of polynomials as $R[z]$. The field of fractions of $R[[z]]$ is

$$R((z)) = \left\{ \sum_{n\geq k} a_n z^n : a_n \in \mathrm{Frac}(R), k \in \mathbb{Z} \right\},$$

which is called the **field of formal Laurent series** (in the variable z over R). Here $\mathrm{Frac}(R)$ is the field of fractions of R.

In this thesis we almost always let $R = \mathbb{C}$.

Example 4. In $R[[z]]$ we have $(1-z)\sum_{n\geq 0} z^n = \sum_{n\geq 0} z^n - \sum_{n\geq 0} z^{n+1} = 1$, so we write

$$\sum_{n\geq 0} z^n = \frac{1}{1-z}.$$

This is the formal analogue of the Taylor series expansion of $\frac{1}{1-z}$ at the origin, which is valid when $|z| < 1$. ◁

The generating function of a combinatorial class is a power series related to the class.

Definition 5. Let (C, ω) be a combinatorial class and $c_n = |\omega^{-1}(n)|$ be the number of elements of weight n. We define the **generating function** of (C, ω) as the formal power series

$$C(z) = \sum_{n\geq 0} c_n z^n \in \mathbb{C}[[z]].$$

The condition on the finitude of the preimage of each natural number ensures that C is a valid formal power series.

Example 6. Let C be the class of finite strings using both 0 and 1. Equip this class with the weight function $\omega(0) = 1$, $\omega(1) = 2$, and the weight of a string is the sum of the weights of its constituent elements. Then, recursively decomposing a string based on its last symbol, it can be shown that

$$C(z) = \frac{1}{1 - z - z^2},$$

which is the generating function for the *Fibonacci numbers*. ◁

Generating functions allow us to implicitly encode sequences using mathematical objects.

Definition 7 (Classes of Functions)**.** Let $A(z) \in \mathbb{C}[[z]]$. We say that $A(z)$ is

- **rational** if it can be expressed as the ratio of two polynomials in $\mathbb{C}[z]$;
- **algebraic** if there exists a polynomial $P(z, \zeta) \in \mathbb{Q}[z, \zeta]$ such that $P(z, A(z)) = 0$;
- **D-finite** if $A(z)$ satisfies a linear differential equation with coefficients in $\mathbb{C}[z]$.

Combinatorial enumeration via generating functions usually boils down to the following process.

1. From a combinatorial problem, characterize the generating function $C(z)$ of a class C using properties of its combinatorial structure.
2. Analyze the generating function $C(z)$ to extract information about the underlying counting sequence c_n.

We focus on the second part of this process: extracting information about a sequence from its generating function. In particular, in this thesis we do not often pay much attention to *where* a generating function comes from (see [19] or [41] for further details), although we do recap the so-called *kernel method* for lattice path enumeration in Chapter 3 below. Analytic combinatorics is usually concerned with finding *asymptotics* of c_n using complex analytic methods.

Each set of functions in Definition 7 contains the last one (for example, every rational generating function is also algebraic). It turns out that (aside from certain pathological situations that don't arise in applications, see for example Skolem's Problem [29, Open Problem 2.1]) algebraic generating functions admit *automatic* (as in, algorithmic and implementable) ways of determining asymptotics, while D-finite generating functions have no such guarantee. The methods for analyzing algebraic functions are not relevant to us and would be a long detour, but the methods for analyzing rational generating functions are useful for us. We briefly describe how to approach these generating functions (in a single variable), before moving on to *analytic combinatorics in several variables*, a framework that makes it possible to study a new class of functions (the class of *rational diagonals*) that lies between algebraic and D-finite functions.

2.1.2 Complex Analysis

Our methods for asymptotics rely on complex analysis, and we now review some basic complex analytic definitions and results. Further details can be found in [8, 9, 39].

Definition 8 (Domain)**.** A **domain** $U \subset \mathbb{C}$ is a bounded connected open subset.

Definition 9 (Analytic Functions, Radius of Convergence)**.** A complex function $A(z)$ is **analytic** on a domain U if for each point $\zeta \in U$ there exists a disk $B(\zeta, \rho) \subset U$ of radius $\rho > 0$ centered at ζ such that the series representation

$$A(z) = \sum_{n \geq 0} a_n^{(\zeta)} (z - \zeta)^n \tag{2.1}$$

converges for all $z \in B(\zeta, \rho)$. That is to say, $A(z)$ is analytic at a point if there exists a convergent power series expansion around that point. We say that the supremum of all ρ such that the power series representation in (2.1) holds is the **radius of convergence** of the power series.

From now on, we default to studying analytic functions at the origin. Note that technically $A(z)$ and the power series expansion of $A(z)$ at a point are not the same thing; all we know is that $A(z)$ and a power series of $A(z)$ agree near some point. This is not too harmful for us, however, and we identify $A(z)$ with its power series representation at the origin (doing this formally requires introducing the theory of germs of analytic functions). We call the series coefficients $a_n^{(0)}$ the **power series coefficients** of $A(z)$.

Integrals of analytic functions on curves satisfy a remarkable property: the value of the integral does not depend on the curve, only on the endpoints of the curve.

Lemma 10. *Let γ, γ' be two curves with the same starting and ending point. Assume that there is a homotopy from γ to γ' staying in a domain U. If $A(z)$ is analytic at all points of U then*

$$\int_\gamma A(z)dz = \int_{\gamma'} A(z)dz.$$

Lemma 10 means that we can move curves of integration around as we wish, as long as $A(z)$ remains analytic.

Lemma 11 (Maximum Modulus Bound)**.** *Let γ be a curve of finite length. Then*

$$\left| \int_\gamma f(z)dz \right| \leq \text{length}(\gamma) \cdot \sup_{z \in \gamma} |f(z)|$$

wherever this integral exists.

Our analysis will be heavily reliant on points where a function is not analytic.

Definition 12 (Singularities)**.** We say a function f has an **isolated singularity** at $\zeta \in \mathbb{C}$ if there is an $R > 0$ such that f is analytic in the punctured disk $B(\zeta, R) \setminus \{\zeta\}$. We further say that ζ is a **removable singularity** if there exists a function g which is analytic on $B(\zeta, R)$ such that $f = g$ on $B(\zeta, R) \setminus \{\zeta\}$.

If a function has a removable singularity then it can be "filled in" by an *analytic continuation.* For this reason, we may assume that we never have removable singularities.

Definition 13 (Poles)**.** An isolated singularity ζ of $A(z)$ is a **pole** if $\lim_{z \to \zeta} |A(z)| = \infty$. An isolated singularity that is neither removable nor a pole is called an **essential singularity**.

Note that not all singularities are isolated (the prototypical counter-example being a branch point such as $\sqrt{z}$ around 0) but we do not need to define more general singularities here. We will be entirely concerned with functions that are analytic near the origin with a finite number of non-zero poles.

Definition 14 (Laurent Series, Residues)**.** Let $A(z)$ have a pole at ζ. Then for some $k \in \mathbb{N}_{>0}$ there is a unique convergent expansion

$$A(z) = \sum_{n \geq -k} a_n (z - \zeta)^n$$

for all z in some neighbourhood of ζ, except at $z = \zeta$, where $a_{-k} \neq 0$. We say that k is the **order of the pole at** ζ. This expansions is called the **Laurent series expansion** of $A(z)$ at ζ, and we call the coefficient a_{-1} the **residue** of $A(z)$ at ζ, which is denoted $\operatorname*{res}_{z=\zeta} \big[A(z)\big]$.

Definition 15 (Meromorphic Functions)**.** A function $A(z)$ on a domain $U \subset \mathbb{C}$ is said to be **meromorphic** on U if there exists a (potentially empty) sequence of points $z_1, z_2, \ldots$ with no limit points in U such that $A(z)$ has poles at each z_j and is analytic at every other point of U.

Meromorphic functions are the main focus of this thesis. Functions with essential singularities are 'too badly behaved', and analyticity is the minimum requirement for interesting results using most classical techniques.

One of the most powerful tools in analytic combinatorics, and indeed all of complex analysis, is the residue theorem.

Theorem 16 (Residue Theorem)**.** *Let γ be a piecewise smooth curve (counter-clockwise oriented) in a domain U containing the origin. Let $A(z)$ be a function that is meromorphic in U and has no singularities on γ. Let Σ be the set of singularities of $A(z)z^{-n-1}$ inside U. Then*

$$\frac{1}{2\pi i}\int_\gamma \frac{A(z)}{z^{n+1}}dz = \sum_{\sigma \in \Sigma} \operatorname*{res}_{z=\sigma}\left[\frac{A(z)}{z^{n+1}}\right].$$

Furthermore, if $A(z)$ is analytic at the origin then

$$a_n = -\sum_{\sigma \in \Sigma \setminus 0} \operatorname*{res}_{z=\sigma}\left[\frac{A(z)}{z^{n+1}}\right] + \frac{1}{2\pi i}\int_\gamma A(z)\frac{dz}{z^{n+1}}.$$

Remark 17. Note that we have freedom when defining the contour of integration γ in Theorem 16. Furthermore, if γ contains no singularities of $A(z)z^{-n-1}$ besides the origin, then

$$a_n = \frac{1}{2\pi i}\int_\gamma A(z)\frac{dz}{z^{n+1}}. \tag{2.2}$$

This will be important to us in the multivariate case.

Definition 18. Let $r > 0$ and $a \in \mathbb{C}$. We define respectively circles and disks as

$$S_r(a) = \left\{\ z \in \mathbb{C} : |z-a| = r\ \right\},$$
$$D_r(a) = \left\{\ z \in \mathbb{C} : |z-a| < r\ \right\}.$$

If $a = 0$ then we simply write S_r and D_r, respectively.

2.2 Analytic Combinatorics in a Single Variable

Consider a generating function $A(z) = \sum_{n\geq 0} a_n z^n \in \mathbb{C}[[z]]$. At the heart of analytic combinatorics is the fact that in many cases we can identify $A(z)$ (a priori a strictly algebraic object) with a convergent power series centered at the origin.[1] Doing so lets us use complex analytic techniques to find coefficients of our power series, and thus elements of our sequence. Note that this is not always possible, for instance the generating function $\sum_{n\geq 0} n!z^n$ of permutations does not converge anywhere except the origin.

When a generating function is rational, it has a finite number of polar singularities. We can generalize our analysis to meromorphic generating functions, which have a finite number of singularities inside any disk in the complex plane. While the theory of analytic combinatorics for algebraic generating functions is rich, we are only interested in the theory for meromorphic generating functions; unless otherwise stated, all generating functions are meromorphic and convergent at the origin.

The two relevant methods for us are applications of the residue theorem and reducing to saddle-point type integrals.

2.2.1 The Residue Method

We first demonstrate the residue method before giving more general details.

Example 19. The Fibonacci numbers f_n have the generating function

$$F(z) = \frac{1}{1 - x - x^2}.$$

Solving the denominator for its roots shows that $F(z)$ has poles at

$$\zeta_1 = \frac{\sqrt{5} - 1}{2}$$
$$\zeta_2 = \frac{-\sqrt{5} - 1}{2},$$

[1]Rigorously this requires the theory of *germs of analytic functions*; there is an injection from a subset of $\mathbb{C}[[z]]$ to germs, giving a local power series that has the same coefficients as our original generating function, but is now a function rather than an algebraic object.

and we note that $|\zeta_1| < |\zeta_2| < 2$. Taking $\gamma = S_2$ in the residue theorem shows that

$$\begin{aligned}
a_n &= -\operatorname*{res}_{z=\zeta_1}\left[\frac{1}{1-z-z^2}\frac{1}{z^{n+1}}\right] - \operatorname*{res}_{z=\zeta_2}\left[\frac{1}{1-z-z^2}\frac{1}{z^{n+1}}\right] + \frac{1}{2\pi i}\int_{|z|=2}\frac{1}{1-z-z^2}\frac{1}{z^{n+1}}dz \\
&= -\operatorname*{res}_{z=\zeta_1}\left[\frac{\frac{1}{(1-z/\zeta_2)z^{n+1}}}{1-z/\zeta_1}\right] - \operatorname*{res}_{z=\zeta_2}\left[\frac{\frac{1}{(1-z/\zeta_1)z^{n+1}}}{1-z/\zeta_2}\right] + \frac{1}{2\pi i}\int_{|z|=2}\frac{1}{1-z-z^2}\frac{1}{z^{n+1}}dz \\
&= \frac{1}{\sqrt{5}}\zeta_1^{-n-1} - \frac{1}{\sqrt{5}}\zeta_2^{-n-1} + \frac{1}{2\pi i}\int_{|z|=2}\frac{1}{1-z-z^2}\frac{1}{z^{n+1}}dz.
\end{aligned}$$

Note that the modulus of $\frac{1}{1-z-z^2}$ is bounded above on S_2 by some constant C_2, so the max modulus bound implies

$$\begin{aligned}
\left|\int_{|z|=2}\frac{1}{1-z-z^2}\frac{1}{z^{n+1}}dz\right| &\le 4\pi\cdot\sup_{|z|=2}\frac{1}{1-z-z^2}\frac{1}{|z|^{n+1}} \\
&\le 4\pi C_2 2^{-n-1} \\
&= O(2^{-n}).
\end{aligned}$$

We thus have that

$$a_n = \frac{1}{\sqrt{5}}\zeta_1^{-n-1} - \frac{1}{\sqrt{5}}\zeta_2^{-n-1} + O(2^{-n}).$$

In fact we can expand the circle S_2 out to some arbitrary S_r while keeping the corresponding constant C_r bounded, and so can replace the $O(2^{-n})$ term with $O(r^{-n})$ for any $r \ge 2$. It follows from the possible asymptotic behaviour of sequences with rational generating functions that we can write

$$a_n = \frac{1}{\sqrt{5}}\zeta_1^{-n-1} - \frac{1}{\sqrt{5}}\zeta_2^{-n-1}.$$

Note that this expression also comes from geometric series and partial fraction decomposition for rational functions, but this residue method applies to all meromorphic functions. Furthermore, since $|\zeta_1| < |\zeta_2|$ we have

$$a_n \sim \frac{1}{\sqrt{5}}\zeta_1^{-n-1}.$$

◁

Example 19 is representative of the general approach using the residue theorem for polar singularities, encapsulated in the following theorem.

Theorem 20 (Melczer [29, Prop. 2.3])**.** *Suppose that $A(z)$ is analytic on S_r and analytic in $D_r \setminus \{\zeta_1, \dots, \zeta_m\}$, with each ζ_j being a non-zero pole. Then there exist computable polynomials $P_1(n), \dots, P_m(n)$ such that*

$$a_n = \sum_{k=1}^{m} P_k(n)\zeta_k^{-n} + O(r^{-n}),$$

where the degree of $P_k(n)$ is one less than the order of the pole of $A(z)$ at ζ_k.

Theorem 20 hints at the *Principles of Analytic Combinatorics* [19, p. 227].

1. The locations of the singularities of a generating function determine the exponential growth of its coefficients;

2. The types of singularities of a generating function determine the subexponential behaviour of its coefficients.

2.2.2 The Saddle-Point Method

While the residue theorem often works admirably, many generating functions do not admit the poles necessary for the residue theorem to be useful. Applying the residue theorem to a function $A(z)$ that has no singularities yields

$$a_n = \frac{1}{2\pi i}\int_{|z|=r} A(z)\frac{dz}{z^{n+1}} = O(r^{-n})$$

for any $r > 0$. Note that taking $r \to \infty$ does not imply that $a_n = 0$, as the implied constant in $O(r^{-n})$ may blow up as we take $r \to \infty$. Instead, we must extract the asymptotics of our sequence using a different approach. One method is as follows

1. Pick a well-chosen $r > 0$, that will depend on $A(z)$ and n, where $A(z)z^{-n-1}$ has 'saddle points'.

2. Show that the domain of integration $|z| = r$ may be localized to a finite set of small neighbourhoods while picking up only negligible error.

3. On each neighbourhood, re-parameterize to give an integral where the domain of integration is a neighbourhood of the origin.

4. Apply the theory of *Fourier-Laplace integral asymptotics* to find the asymptotic values of these integrals.

This approach is called the *saddle-point method* for asymptotics.

Example 21. We use the saddle-point method to compute the asymptotics of

$$a_n = \frac{1}{2\pi i} \int_{|z|=r} \frac{1}{(1-z)^{n+1}} \frac{dz}{z^{n+1}}.$$

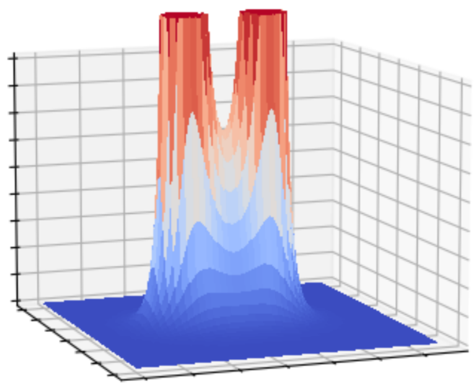


Figure 2.1: *The plot of the modulus of $1/(1-z)^{2+1}z^{2+1}$, with the saddle point at $z = 1/2$ visible between the two poles $z = 0$ and $z = 1$.*

The value of this integral is the same for all $0 < r < 1$, however a good choice is $r = 1/2$ (as we will now see).

Define $\mathcal{N} = \left\{ e^{i\theta}/2 : \theta \in (-\pi/4, \pi/4) \right\}$ and $\mathcal{N}' = S_{1/2} \setminus \mathcal{N}$. For $z \in \mathcal{N}'$

$$\frac{1}{|1-z|} \leq \frac{1}{|1-e^{i\pi/4}/2|} < \frac{3}{2}.$$

The maximum modulus bound thus implies that

$$\left| \frac{1}{2\pi i} \int_{\mathcal{N}'} \frac{1}{(1-z)^{n+1}} \frac{dz}{z^{n+1}} \right| = O\left(3^n\right),$$

and so

$$a_n = \frac{1}{2\pi i} \int_{\mathcal{N}} \frac{1}{(1-z)^{n+1}} \frac{dz}{z^{n+1}} + O(3^n).$$

Define $t_n = \frac{1}{2\pi i}\int_{\mathcal{N}} \frac{1}{(1-z)^{n+1}}\frac{dz}{z^{n+1}}$. Making the change of variables $z = e^{i\theta}/2$ gives

$$\begin{aligned} t_n &\sim \frac{4^n}{2\pi}\int_{-\pi/4}^{\pi/4} \frac{1}{(1-e^{i\theta}/2)} \frac{d\theta}{\left((2-e^{i\theta})e^{i\theta}\right)^n} \\ &= \frac{4^n}{2\pi}\int_{-\pi/4}^{\pi/4} \underbrace{\frac{1}{(1-e^{i\theta}/2)}}_{A(\theta)} \exp\Bigg[-n\cdot \underbrace{\ln\left[(2-e^{i\theta})e^{i\theta}\right]}_{\phi(\theta)}\Bigg]d\theta. \end{aligned}$$

Taylor expanding the *amplitude* A and *phase* ϕ around 0 gives expansions

$$\begin{aligned} A(\theta) &= 2 + 2i\theta - 3\theta^2 + \cdots, \\ \phi(\theta) &= \theta^2 + i\theta^3 + \cdots. \end{aligned}$$

Applying known results on the asymptotics of *Fourier-Laplace integrals* shows that, up to some negligible error, we may replace A and ϕ by the leading terms in their Taylor expansions at 0, so that

$$\begin{aligned} t_n &\sim \frac{4^n}{2\pi}\int_{-\pi/4}^{\pi/4} 2e^{-n\theta^2}d\theta \\ &\sim \frac{4^n}{\pi}\int_{-\infty}^{\infty} e^{-n\theta^2}d\theta \\ &= \frac{4^n}{\sqrt{\pi n}} \end{aligned}$$

and so

$$a_n \sim \frac{4^n}{\sqrt{\pi n.}}$$

$\triangleleft$

Further details on the saddle-point method, including when it applies and how to pick a suitable domain of integration, can be found in [19].

2.3 Analytic Combinatorics in Several Variables

The theory of analytic combinatorics in several variables (ACSV) is a generalization of univariate analytic combinatorics. It is quite natural to consider asymptotics of multivariate sequences (in a manner made precise in Section 2.3.2), and often complicated sequences can be neatly encoded in multivariate sequences[2] (examples of this are seen in Chapters 3 and 4).

Finding asymptotics of a given generating function in n variables usually boils down to

1. finding *critical points*, which roughly generalize the univariate notion of saddle-points,
2. taking residues some $k < n$ times,
3. find asymptotics of a remaining saddle-point integral in $n - k$ variables.

The value k here is calculable and depends on the local geometry of the generating function of interest at its critical points. In chapters 3 and 4 we focus on the case where k is strictly less than the number of smooth sheets meeting at a critical point, which is a non-generic situation. In this case we must put in extra work to find asymptotics of the saddle-point integral that is left after we take residues. We devote the rest of the chapter to the background of analytic combinatorics in several variables.

2.3.1 Multivariate Expansions

Let $z_j \in \mathbb{C}$, $k_j \in \mathbb{R}$, and $d \in \mathbb{N}_{>0}$. For the rest of this thesis we adopt the notation

- $\mathbf{z} := (z_1, \dots, z_d)$;
- $d\mathbf{z} := dz_1 \cdots dz_d$;
- $\mathbf{z}^{\mathbf{k}} = z_1^{k_1} \cdots z_d^{k_d}$.

When appropriate, we also write $\mathbf{z}_{\hat{p}}$ to mean $(z_1, \dots, z_{p-1}, z_{p+1}, \dots, z_d)$.

We can naturally translate from univariate generating functions to multivariate generating functions.[3]

[2] An important fact is that any algebraic generating function can be written as the *main diagonal* of a bivariate rational function [29, Proposition 3.8]. This implies that the class of *rational diagonals* contains the class of algebraic functions.

[3] In a combinatorial setting multivariate series can be used to track different attributes in a combinatorial class. This is not our focus, but an interested reader may consult [30].

Definition 22. Let R be an integral domain. For $d \geq 2$, we inductively define

$$R[[\mathbf{z}]] := R[[z_1, \ldots, z_{d-1}]][[z_d]]$$

as the **ring of formal power series** in the variables $z_1, \ldots, z_d$ over R, where addition is defined term-wise and multiplication is defined by

$$\left(\sum_{\mathbf{i} \in \mathbb{N}^d} a_{\mathbf{i}} \mathbf{z}^{\mathbf{i}} \right) \left(\sum_{\mathbf{j} \in \mathbb{N}^d} b_{\mathbf{j}} \mathbf{z}^{\mathbf{j}} \right) = \sum_{n \in \mathbb{N}^d} \left(\sum_{\mathbf{i}+\mathbf{j}=\mathbf{k}} a_{\mathbf{i}} b_{\mathbf{j}} \right) \mathbf{z}^{\mathbf{k}}.$$

As before, a **polynomial** is an element of $R[[\mathbf{z}]]$ that has only finitely many non-zero coefficients, and we write the collection of polynomials as $R[\mathbf{z}]$. We also inductively define

$$R((\mathbf{z})) := R((z_1, \ldots, z_{d-1}))((z_d))$$

which is called the **field of iterated formal Laurent series**[4] in the variables $z_1, \ldots, z_d$ over R. An element of $R((\mathbf{z}))$ with only finitely many non-zero terms is called a **Laurent polynomial**.

Many definitions in complex analysis in several variables are analogous to their counterparts in univariate complex analysis, though the results are sometimes weaker and the methods are more involved.

Definition 23 (Polydisks and Polytori)**.** Let $\mathbf{r} \in \mathbb{N}^d$ and $\mathbf{z} \in \mathbb{C}^d$. We define respectively the **polydisk centered at a** as the product

$$D_{\mathbf{a}}(\mathbf{r}) = D_{a_1}(r_1) \times \cdots \times D_{a_d}(r_d)$$

and the **polytorus centered at a** as the product

$$S_{\mathbf{a}}(\mathbf{r}) = S_{a_1}(r_1) \times \cdots \times S_{a_d}(r_d)$$

where $D_a(r), S_a(r) \subset \mathbb{C}$ are respectively the open disk and the circle centered at a of radius r. When $\mathbf{a} = \mathbf{0}$ we omit it from our notation and write $D(\mathbf{r})$ and $S(\mathbf{r})$.

Once again we are interested in meromorphic functions.

[4]Note that the order of the variables matters: different orders result in different fields, although the construction of Laurent polynomials does not depend on the ordering used.

Definition 24 (Meromorphicity)**.** We say that a function $f(\mathbf{z})$ is **meromorphic** at a point $\mathbf{a} \in \mathbb{C}^d$ if it can be expressed in a neighbourhood of $\mathbf{a}$ as the ratio $g(\mathbf{z})/h(\mathbf{z})$ of two analytic functions (where h is not identically 0). A function is meromorphic on a domain if it is meromorphic on each point of that domain.

As in univariate analytic combinatorics, asymptotics of sequences depend on the singularities of their associated generating functions. There is a notable jump in difficulty when moving from univariate analytic combinatorics to ACSV, as the singular set is much harder to analyze in several variables.

There is a multivariate version[5] of the Cauchy integral representation (2.2). Through this we are able to recover the coefficients of a meromorphic function via an integral.

Theorem 25. *Suppose that $A(\mathbf{z})$ is analytic on a domain $\Omega \subset \mathbb{C}^d$ containing $\boldsymbol{0}$ and has the expansion*

$$A(\mathbf{z}) = \sum_{\boldsymbol{k} \in \mathbb{N}^d} a_{\boldsymbol{k}} \mathbf{z}^{\boldsymbol{k}}$$

on Ω. Then for any polytorus $S(\boldsymbol{r}) \subset \Omega$ and for any $\boldsymbol{j} \in \mathbb{N}^d$

$$[\mathbf{z}^{\boldsymbol{j}}]A(\mathbf{z}) := a_{\boldsymbol{j}} = \frac{1}{(2\pi i)^d} \int_{S(\boldsymbol{r})} A(\mathbf{z}) \frac{d\mathbf{z}}{\mathbf{z}^{\mathbf{j}+\mathbf{1}}}. \tag{2.3}$$

2.3.2 Coefficient Directions

Given a generating function, we would like to find the asymptotics of a sequence encoded in its coefficients. In a single variable this is straightforward, as there is only one obvious sequence defined by a univariate generating function. In several variables we must be more concrete about what sequences we are interested in.

Definition 26 (Directions)**.** Let $A(\mathbf{z}) = \sum_{\mathbf{k} \in \mathbb{N}^d} a_{\mathbf{k}} \mathbf{z}^{\mathbf{k}}$. We call $\mathbf{r} \in \mathbb{Q}^d_{>0}$ a **direction**, and define the **r-diagonal** of $A(\mathbf{z})$ to be the univariate generating function

$$(\Delta_{\mathbf{r}} A)(t) = \sum_{n \geq 0} a_{n\mathbf{r}} t^n.$$

When $n\mathbf{r} \notin \mathbb{N}^d$ the quantity $a_{n\mathbf{r}}$ is undefined, and all our formulas are interpreted to hold only if $n\mathbf{r} \in \mathbb{N}^d$. In the case that $\mathbf{r} = \mathbf{1}$ we write Δ instead of $\Delta_{\mathbf{1}}$ and call the generating function in this direction the **main diagonal**.

[5]This is not quite an exact generalization, as the domain of integration is a polytorus and not an arbitrary boundary like in one dimension.

We ultimately care about asymptotics of the integral expression

$$[\mathbf{z}^{n\mathbf{r}}]A(\mathbf{z}) = a_{n\mathbf{r}} = \frac{1}{(2\pi i)^d} \int_{S(\epsilon)} A(\mathbf{z}) \frac{d\mathbf{z}}{\mathbf{z}^{n\mathbf{r}+\mathbf{1}}}, \tag{2.4}$$

where $\epsilon = (\epsilon_1, \ldots, \epsilon_d)$ for any $\epsilon_j > 0$ sufficiently small. Recall that in Example 21 our analysis started by finding a saddle point of $A(z)z^{-n-1}$. In the multivariate case, the new notion of 'critical point' depends on the direction chosen, and is more delicate to define. For the rest of this thesis, we assume that

$$F(\mathbf{z}) = \frac{G(\mathbf{z})}{H(\mathbf{z})}$$

where G and H are two coprime polynomials with $H(\mathbf{0}) \neq 0$, and write

$$\mathcal{V} = \left\{ \mathbf{z} \mid H(\mathbf{z}) = 0 \right\}$$

for the **singular variety**[6] of $F(\mathbf{z})$. Note that in one variable $\mathcal{V}$ is a finite set of points, while in several variables it is now a continuous 'surface' of singularities (as long as F is not a polynomial).

Although $\mathcal{V}$ will often not be a manifold, it is possible to (algorithmically) decompose it into a finite set of smooth *strata* that define a so-called *Whitney stratification*. Each stratum can be defined by a finite set of polynomial equalities and non-equalities. We now fix a Whitney stratification for $\mathcal{V}$ (which in our uses below will be trivial to compute).

Definition 27 (Height Functions)**.** Let $\mathbb{C}_\star = \mathbb{C} \setminus \{0\}$. The **height function** corresponding to the direction $\mathbf{r}$ is the function $h_\mathbf{r} : \mathbb{C}_\star^d \to \mathbb{R}$ defined by

$$h_\mathbf{r}(\mathbf{z}) = -\sum_{j=1}^{d} r_j \log |z_j|.$$

Note that

$$-\nabla h_\mathbf{r}(\mathbf{z}) = (r_1/z_1, \ldots, r_d/z_d).$$

Definition 28. A **critical point** of $h_\mathbf{r}$ on a stratum $\mathcal{V}_\alpha$ is a point where the differential of the restricted map $h_\mathbf{r}\big|_{\mathcal{V}_\alpha}$ is 0. More explicitly, if the closure $\overline{\mathcal{V}_\alpha}$ has codimension m and

[6]The results discussed below can be extended to *coprime analytic functions*, once a suitable notion of coprime has been defined.

$\mathbf{z}$ is a point of $\overline{\mathcal{V}_\alpha}$ where $\overline{\mathcal{V}_\alpha}$ is locally defined by the vanishing of the functions $H_1, \ldots, H_m$ then $\mathbf{z}$ is a critical point when the matrix

$$M(\mathbf{z}) = \begin{bmatrix} -\nabla H_1(\mathbf{z}) \\ \vdots \\ -\nabla H_m(\mathbf{z}) \\ -\nabla h_{\mathbf{r}}(\mathbf{z}) \end{bmatrix}$$

is rank deficient. When $m = 1$ and there is only one square-free factor H, this becomes equivalent to finding the points $\mathbf{w}$ that satisfy the **smooth critical point equations**:

$$\begin{gathered} H(\mathbf{w}) = 0, \\ r_j w_1 H_{z_1}(\mathbf{w}) - r_1 w_j H_{z_d}(\mathbf{w}) = 0 \quad (2 \le j \le d), \end{gathered} \tag{2.5}$$

and such a point $\mathbf{w}$ is called a **smooth critical point**. If a point is a critical point of a stratum $\mathcal{V}_\alpha$ we say it is a **critical point** of $\mathcal{V}$ as well.

Definition 29. A critical point $\mathbf{w}$ is **minimal** if there is no other point $\mathbf{x}$ in $\mathcal{V}$ such that $|x_j| < |w_j|$ for all $1 \le j \le d$.

Definition 30 (Generic and Non-Generic Directions)**.** Let $\boldsymbol{\sigma}$ be a critical point of $\mathcal{V}_\alpha$. If $\boldsymbol{\sigma} \in \overline{\mathcal{V}_\alpha} \setminus \mathcal{V}_\alpha$ for any stratum $\mathcal{V}_\alpha$ then we say that $\boldsymbol{\sigma}$ is a **non-generic critical point**. Otherwise we say that $\boldsymbol{\sigma}$ is a **generic critical point**. The direction $\mathbf{r}$ is called a **generic direction** if all of its critical points are generic.

Let $\boldsymbol{\sigma}$ be a critical point on a stratum $\mathcal{V}_\alpha$ whose closure is defined locally by the simultaneous vanishing of the square-free analytic functions $H_1, \ldots, H_m$, such that $\boldsymbol{\sigma}$ is not in any stratum of lower dimension. We let $H_j(\mathbf{0}) \ge 0$. If we set

$$\mathbf{b}_j^{(\boldsymbol{\sigma})} := -\nabla H_j(\boldsymbol{\sigma})$$

then the **positive normal cone** of $\mathcal{V}_\alpha$ at $\boldsymbol{\sigma}$ is

$$N(\boldsymbol{\sigma}) = \left\{ \sum_{j=1}^{m} \beta_j \mathbf{b}_j^{(\boldsymbol{\sigma})} \mid \beta_j > 0 \right\}.$$

We call $\boldsymbol{\sigma}$ a **contributing point** if

$$-\nabla h_{\mathbf{r}}(\boldsymbol{\sigma}) \in N(\boldsymbol{\sigma}).$$

If $\boldsymbol{\sigma}$ is contributing and $-\nabla h_{\mathbf{r}}(\boldsymbol{\sigma})$ is in the *interior* of $N(\boldsymbol{\sigma})$, so that

$$\nabla h_{\mathbf{r}}(\boldsymbol{\sigma}) = \beta_1 \nabla H_1(\boldsymbol{\sigma}) + \ldots + \beta_m \nabla H_m(\boldsymbol{\sigma})$$

where $\beta_j > 0$ for all $1 \le j \le m$, then $\boldsymbol{\sigma}$ is generic. If $\boldsymbol{\sigma}$ is contributing and does not lie in the interior of $N(\boldsymbol{\sigma})$ then we can write (up to a reordering of terms)

$$\nabla h_{\mathbf{r}}(\boldsymbol{\sigma}) = \beta_1 \nabla H_1(\boldsymbol{\sigma}) + \ldots + \beta_{m-k} \nabla H_{m-k}(\boldsymbol{\sigma}) + 0 \cdot \nabla H_{m-k+1}(\boldsymbol{\sigma}) + \ldots + 0 \cdot \nabla H_m(\boldsymbol{\sigma})$$

for $\beta_j > 0$, and $\boldsymbol{\sigma}$ is non-generic, lying on a k-dimensional face of $N(\boldsymbol{\sigma})$.

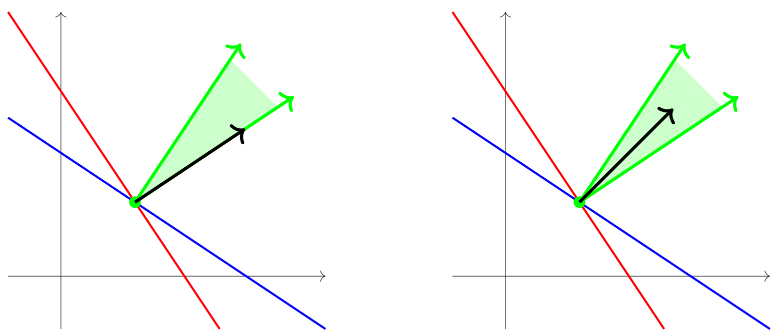


Figure 2.2: *The real part of two strata with a critical point at their intersection, which itself is a stratum, along with the corresponding normal cone. The images correspond to a non-generic direction and a generic one, with $\nabla h_{\mathbf{r}}(\sigma)$ in black.*

Note that the critical point locations, as well as the vector $\nabla h_{\mathbf{r}}$ that must be in the normal cone at each critical point, depend on the direction $\mathbf{r}$.

Intuitively, near non-generic critical points we will not be able to take ‘enough’ residues to compute asymptotics with the standard saddle-point method. Instead, we will get more complicated saddle-point-like integrals that need to be further analyzed.

Finally, we call a critical point $\boldsymbol{\sigma}$ of $F(\mathbf{z})$ **degenerate** if at $\boldsymbol{\sigma}$ the *phase*, a certain function depending on the geometry of the singular set of F, has a singular Hessian matrix (see Melczer [29, Proposition 5.2] for an explicit formula). Otherwise $\boldsymbol{\sigma}$ is called **non-degenerate**. While we will only ever deal with non-degenerate critical points, this is an important definition in analytic combinatorics; most of the results we discuss do not hold unless the relevant critical points are non-degenerate. When critical points are non-generic it becomes more difficult to use the methods of ACSV to find asymptotics.

2.3.3 Asymptotics

We now have the definitions needed to apply techniques from ACSV. We demonstrate the methods on an example.

Example 31. Consider the generating function

$$F(x,y) = \frac{1}{1-x-y} = \sum_{i,j\geq 0} \binom{i+j}{i} x^i y^j$$

in the direction $(1,1)$. The singular variety of F is a manifold defined by the vanishing of the polynomial $H(x,y) = 1-x-y$. Consulting the smooth critical point equations (2.5), we compute that F admits the single smooth critical point $(1/2,1/2)$ (which is minimal and generic). Using (2.3), we can write

$$\binom{2n}{n} = \frac{1}{(2\pi i)^2} \int_{\substack{|x|=1/2 \\ |y|=1/2-\epsilon}} \frac{1}{1-x-y} \frac{dxdy}{x^{n+1}y^{n+1}}.$$

One can show that restricting x near the positive real axis doesn't affect asymptotic behaviour, so

$$\binom{2n}{n} \sim \frac{1}{(2\pi i)^2} \int_{\substack{x\in\mathcal{N} \\ |y|=1/2-\epsilon}} \frac{1}{1-x-y} \frac{dxdy}{x^{n+1}y^{n+1}},$$

where $\mathcal{N} = \left\{ e^{i\theta}/2 : \theta \in (-\pi/4, \pi/4) \right\}$. Similarly, one can show that extending $|y|$ past $\frac{1}{2}$ introduces an exponentially smaller integral, so

$$\begin{aligned}
\binom{2n}{n} &\sim \frac{1}{(2\pi i)^2} \int_{\substack{x\in\mathcal{N} \\ |y|=1/2-\epsilon}} \frac{1}{1-x-y} \frac{dxdy}{x^{n+1}y^{n+1}} - \frac{1}{(2\pi i)^2} \int_{\substack{x\in\mathcal{N} \\ |y|=1/2+\epsilon}} \frac{1}{1-x-y} \frac{dxdy}{x^{n+1}y^{n+1}} \\
&= \frac{1}{2\pi i} \int_{x\in\mathcal{N}} \left(\frac{1}{2\pi i} \int_{|y|=1/2-\epsilon} \frac{1}{1-x-y} \frac{dy}{y^{n+1}} - \frac{1}{2\pi i} \int_{|y|=1/2+\epsilon} \frac{1}{1-x-y} \frac{dy}{y^{n+1}} \right) \frac{dx}{x^{n+1}}.
\end{aligned}$$

Applying the univariate residue theorem to the inner integrand with respect to the variable y gives

$$\begin{aligned}
\binom{2n}{n} &\sim \frac{1}{2\pi i} \int_{x\in\mathcal{N}} \left(\operatorname*{res}_{y=1-x} \left[\frac{1}{1-x-y} \frac{1}{y^{n+1}} \right] \right) \frac{dx}{x^{n+1}} \\
&= \frac{1}{2\pi i} \int_{x\in\mathcal{N}} \frac{1}{(1-x)^{n+1}} \frac{dx}{x^{n+1}}.
\end{aligned}$$

It can be shown that this saddle-point integral is asymptotically equivalent to the one over the domain $|x| = \frac{1}{2}$, already computed in Example 21, implying that

$$\binom{2n}{n} \sim \frac{4^n}{\sqrt{\pi n}}.$$

$\triangleleft$

Chapter 3

Symmetric Lattice Path Models

This chapter is adapted from Kroitor and Melczer [27].

Lattice path enumeration is a classical problem in enumerative combinatorics. Given a (typically finite) set of steps $\mathcal{S} \subset \mathbb{Z}^d$, a *restricting region* $\mathcal{R} \subset \mathbb{Z}^d$, a set of *starting positions* $\mathcal{A} \subset \mathcal{R}$, and a set of *ending positions* $\mathcal{B} \subset \mathcal{R}$, the goal is to enumerate the number of walks taking n steps in $\mathcal{S}$ that start in $\mathcal{A}$, end in $\mathcal{B}$, and always stay in $\mathcal{R}$. Lattice path enumeration has a long history in combinatorics and probability theory (see, for instance, the textbooks [32, 33, 26] and the survey [24]), and finds application to a wide variety of problems in combinatorics and broader fields. There are a vast number of approaches to the enumeration of walks in convex cones: a full accounting would comprise a survey paper, but to illustrate the breadth of work on the subject we note (in addition to the analytic viewpoint here) techniques involving computer algebra [4, 25, 5, 3], differential Galois theory [13, 14], potential theory [11], boundary value problems on Riemann surfaces [36], probabilistic methods [16, 12], and elegant power series manipulations [6, 2]. More recent work has also studied walks in non-convex cones [37, 15, 35, 40, 7].

Much attention in the enumeration literature has focused on walks restricted to an orthant $\mathcal{R} = \mathbb{N}^d$. Using the *kernel method*, described in Section 3.1 below, it is often possible to represent generating functions enumerating walk models as *diagonals* of explicit multivariate series, to which the theory of ACSV can be applied.

3.1 The Kernel Method

For the rest of this chapter we write $\overline{z} := 1/z$ (we will not refer to the complex conjugate). Furthermore, if $A(\mathbf{z}) = \sum_{\mathbf{j}\in\mathbb{Z}^d} a_{\mathbf{j}}\mathbf{z}^{\mathbf{j}}$ is an iterated formal Laurent series then we write

$$[\mathbf{z}^{\geq 0}]A(\mathbf{z}) = \sum_{\mathbf{j}\in\mathbb{N}^d} a_{\mathbf{j}}\mathbf{z}^{\mathbf{j}}.$$

Given a finite set of steps $\mathcal{S} \subset \mathbb{Z}^d$, we would like to enumerate the number of *lattice walks* of length n using the steps in $\mathcal{S}$, starting at the origin, that are restricted to the positive orthant $\mathbb{N}^d$ (at no point is the walk allowed to leave $\mathbb{N}^d$). As an example we will find the generating function for so-called *Dyck prefixes.*

Example 32. A **Dyck prefix** is a walk in $\mathbb{Z}$ that uses the steps $\{-1, +1\}$ which is restricted to $\mathbb{N}$.

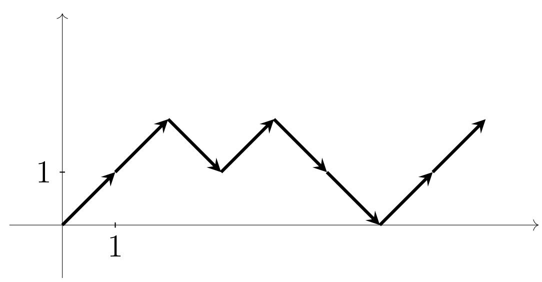


Figure 3.1: *A Dyck prefix. The horizontal axis tracks the number of steps.*

Define

$$F(x,t) = \sum_{n\geq 0}\left(\sum_{i\geq 0} f_{i,n}x^i\right)t^n,$$

where $f_{i,n}$ is the number of Dyck prefixes of length n ending at i. Note that

- $F(0,t)$ is the generating function for Dyck prefixes that end at the origin,
- $F(1,t)$ is the single-variate generating function for Dyck prefixes that does not track ending location.

We can encode the step +1 in the monomial x, and the step -1 in the monomial $\overline{x}$. Then decomposing a walk of length n as a walk of length $n-1$ followed by a single step gives the functional equation

$$F(x,t) = 1 + (x+\overline{x})tF(x,t) - \overline{x}tF(0,t),$$

and rearranging the relation yields

$$(1 - t(x+\overline{x}))F(x,t) = 1 - \overline{x}tF(0,t). \tag{3.1}$$

Solving $1 - t(x+\overline{x}) = 0$ for x gives

$$x = \frac{1 \pm \sqrt{1-4t^2}}{2t},$$

of which only $\frac{1-\sqrt{1-4t^2}}{2t}$ is a power series. Plugging this into (3.1) gives

$$F(0,t) = \frac{1-\sqrt{1-4t^2}}{2t^2},$$

and substituting back into (3.1) finally results in

$$F(x,t) = \frac{1 - 2xt - \sqrt{1-4t^2}}{2t(t+tx^2-x)}.$$

Thus,

$$F(1,t) = \frac{1-2t-\sqrt{1-4t^2}}{2t(2t-1)} = \frac{1}{2t}\left(\sqrt{\frac{1+2t}{1-2t}} - 1\right)$$

is the generating function for Dyck prefixes. From this generating function we can compute

$$[t^n]F(1,t) \sim \frac{\sqrt{2}}{\sqrt{\pi}} \cdot \frac{2^n}{n^{1/2}}$$

using the methods of analytic combinatorics. $\triangleleft$

What if we want to generalize this approach to higher dimensions? Here we use the *kernel method*, introduced separately by Knuth and Malyshev; our modern incarnation of the method, developed by Bousquet-Mélou, is used to solve certain systems of linear functional equations that seem to have 'too many' unknowns.

We demonstrate the kernel method for lattice walk models by using it on a simple example.

Example 33. Consider the lattice walk model using the steps

$$\mathcal{S} = \{(-1,0),(1,0),(0,-1),(0,1)\},$$

starting at the origin, that remain in $\mathbb{N}^2$.

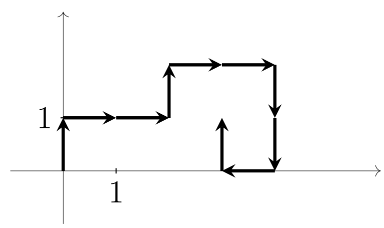


Figure 3.2: *A lattice path using the step set* $\mathcal{S} = \{(\pm1,0),(0,\pm1)\}$ *that remains in* $\mathbb{N}^2$.

We start by encoding the step set $\mathcal{S}$ using the Laurent polynomial

$$S(x,y) = \overline{x} + x + \overline{y} + y,$$

and define a multivariate generating function that keeps track of end points:

$$Q(x,y,t) = \sum_{n\geq0}\left(\sum_{i,j\geq0} q_{i,j,n}x^iy^j\right)t^n,$$

where $q_{i,j,n}$ is the number of walks in the model of length n that end at (i,j). Note that

- $Q(0,y,t)$ is the GF for walks ending on the x-axis, and
- $Q(x,0,t)$ is the GF for walks ending on the y-axis,

so decomposing a walk of length n as a walk of length $n-1$ followed by a single step gives

$$\begin{aligned} Q(x,y,t) &= 1 + S(x,y)tQ(x,y,t) - \overline{y}tQ(x,0,t) - \overline{x}tQ(0,y,t) \\ \implies xy(1-tS(x,y))Q(x,y,t) &= xy - xtQ(x,0,t) - ytQ(0,y,t). \end{aligned}$$

So far we have not done anything clever, and have three "unknowns" $Q(x,y,t)$, $Q(x,0,t)$, and $Q(0,y,t)$ in this equation. We would really like to isolate $Q(x,y,t)$, but we can't. The key of the kernel method comes from realizing the two involutions

$$\begin{aligned} \psi &: (x,y) \mapsto (\overline{x},y), \\ \phi &: (x,y) \mapsto (x,\overline{y}), \end{aligned}$$

keep $S(x,y)$, and thus also $K(x,y) = (1 - tS(x,y))$, fixed. The group generated (under composition) by ψ and ϕ is $\{id, \psi, \phi, \psi \circ \phi\}$, and applying these 4 transformations to the last equation above gives

$$
\begin{aligned}
&(1) \qquad xyK(x,y)Q(x,y,t) = xy - xtQ(x,0,t) - ytQ(0,y,t),\\
&(2) \qquad \overline{x}yK(x,y)Q(\overline{x},y,t) = \overline{x}y - \overline{x}tQ(\overline{x},0,t) - ytQ(0,y,t),\\
&(3) \qquad x\overline{y}K(x,y)Q(x,\overline{y},t) = x\overline{y} - xtQ(x,0,t) - \overline{y}tQ(0,\overline{y},t),\\
&(4) \qquad \overline{xy}K(x,y)Q(\overline{x},\overline{y},t) = \overline{xy} - \overline{x}tQ(\overline{x},0,t) - \overline{y}tQ(0,\overline{y},t).
\end{aligned}
$$

Looking closely we notice that each unknown term on the right hand side appears exactly twice, and taking a signed sum (1) − (2) − (3) + (4) gives

$$xyQ(x,y,t) - \overline{x}yQ(\overline{x},y,t) - x\overline{y}Q(x,\overline{y},t) + \overline{xy}Q(\overline{x},\overline{y},t) = \frac{xy - \overline{x}y - x\overline{y} + \overline{xy}}{K(x,y)}.$$

Each of the terms on the left hand side except $xyQ(x,y,t)$ is a power series where either x or y (or both) have strictly negative powers. Thus

$$
\begin{aligned}
xyQ(x,y,t) &= [x^{\geq 0}y^{\geq 0}]\left(\frac{xy - \overline{x}y - x\overline{y} + \overline{xy}}{K(x,y)}\right)\\
\implies Q(x,y,t) &= [x^{\geq 0}y^{\geq 0}]\left(\frac{(x-\overline{x})(y-\overline{y})}{xy(1-tS(x,y))}\right).
\end{aligned}
$$

Through algebraic manipulations (see Melczer [29, Proposition 3.14] for a proof), this implies the diagonal representation

$$Q(1,1,t) = \Delta\left(\frac{(1+x)(1+y)}{1-txyS(x,y)}\right),$$

and the methods of ACSV give

$$[t^n]Q(1,1,t) \sim \frac{4}{\pi} \cdot \frac{4^n}{n}.$$

◁

The kernel method can be generalized to many large families of models. We now compute asymptotics for one such family that correponds to an ACSV analysis in a non-generic direction.

3.2 Models with Highly and Mostly Symmetric Step Sets

Fix a dimension $d \in \mathbb{N}$ and a step set $\mathcal{S} \subset \{\pm 1, 0\}^d \setminus \{\mathbf{0}\}$. Walk models whose steps have coordinates equal to 0 or ± 1 are known as *short step models*. To rule out redundant cases, we assume that for each $1 \le j \le d$ there is a step in $\mathcal{S}$ moving forward in the jth coordinate, and a step in $\mathcal{S}$ moving backwards in the jth coordinate. We also allow our steps to have positive weights, and define the *weight* of a lattice walk to be the product of the weights of its steps. The *(weighted) characteristic polynomial* of $\mathcal{S}$ is the Laurent polynomial

$$S(\mathbf{z}) = \sum_{\mathbf{i} \in \mathcal{S}} w_{\mathbf{i}} \mathbf{z}^{\mathbf{i}}$$

whose exponents encode the entries in $\mathcal{S}$, where each $w_{\mathbf{i}}$ is a positive real weight. Define the notation

$$\mathbf{z}_{\hat{k}} = (z_1, \ldots, z_{k-1}, z_{k+1}, \ldots, z_d)$$

for any $1 \le k \le d$ and $\widehat{\mathbf{z}} = \mathbf{z}_{\hat{d}} = (z_1, \ldots, z_{d-1})$.

Remark 34. In the *unweighted* case when each weight has the value 1, the value $S(\mathbf{1}) = |\mathcal{S}|$ is the cardinality of the step set and the total weight of walks of length n equals the number of paths of length n. When each weight is a positive integer then we can imagine counting paths where there are (potentially) multiple copies of each step.

We say $\mathcal{S}$ is *symmetric* over a coordinate axis if $\mathcal{S}$ is unchanged by reflection over the axis *and* the weight of any step equals the weight of the step obtained by reflecting over the axis. In this chapter we restrict to the cases where $\mathcal{S}$ is either symmetric over every coordinate axis or all but one axis. We may assume without loss of generality that the axis of non-symmetry (if it exists) is z_d, so that

$$S(\mathbf{z}) = \overline{z}_d A(\widehat{\mathbf{z}}) + Q(\widehat{\mathbf{z}}) + z_d B(\widehat{\mathbf{z}})$$

for Laurent polynomials A, B, and Q that are symmetric in their variables. For all $1 \le k \le d$ let b_k be the total weight

$$b_k = \sum_{\substack{\mathbf{i} \in \mathcal{S} \\ i_k = 1}} w_{\mathbf{i}}$$

of the steps moving forward in the kth coordinate.

Theorem 35 (Highly Symmetric Asymptotics [28, Theorem 3.4]). *Let $\mathcal{S} \subset \{-1, 0, 1\}^d \setminus \{\mathbf{0}\}$ be a set of steps that is symmetric over every axis and moves forwards and backwards in* *each coordinate. Then the total weight s_n of walks of length n taking steps in $\mathcal{S}$, beginning at the origin, and never leaving $\mathbb{N}^d$ satisfies*

$$s_n = S(\mathbf{1})^n \cdot n^{-d/2} \cdot \left[\left(\frac{S(\mathbf{1})}{\pi} \right)^{d/2} \frac{1}{\sqrt{b_1 \cdots b_d}} + O\left(\frac{1}{n}\right) \right].$$

Theorem 35 is obtained by applying the techniques of ACSV to the diagonal expression

$$W(t) = \Delta\left(\frac{G(\mathbf{z},t)}{H(\mathbf{z},t)} \right) = \Delta\left(\frac{(1+z_1)\cdots(1+z_d)}{1 - t(z_1 \cdots z_d)S(\mathbf{z})} \right), \tag{3.2}$$

where $W(t)$ is the generating function enumerating walks in the model defined by $\mathcal{S}$. Expression (3.2) is derived using the kernel method in almost the exact same manner as Example 33. We note that the singular set $\mathcal{V}$ of the rational function G/H consists of the zeroes of the polynomial $H(\mathbf{z},t) = 1 - t(z_1 \cdots z_d)S(\mathbf{z})$. Because this polynomial and its partial derivative with respect to t never simultaneously vanish, $\mathcal{V}$ forms a manifold and only the results of *smooth* ACSV (the simplest case, where we do not need to appeal to a stratification) are needed.

Asymptotics in the mostly symmetric case depends on the **drift** $B(\mathbf{1}) - A(\mathbf{1})$ of a walk with respect to the z_d-axis, which is the weight of the steps in $\mathcal{S}$ with positive z_d coordinate minus the weight of the steps in $\mathcal{S}$ with negative z_d coordinate. In the positive drift case, the number of walks of length n satisfies a formula similar to the highly symmetric case.

Theorem 36 (Positive Drift Asymptotics [31, Theorem 2.1])**.** *Let $\mathcal{S}$ be a step set that is symmetric over all but the dth axis and takes a step forwards and backwards in each coordinate. If $\mathcal{S}$ has positive drift then the total weight s_n of walks of length n taking steps in $\mathcal{S}$, starting at the origin, and never leaving $\mathbb{N}^d$ satisfies*

$$s_n = S(\mathbf{1})^n \cdot n^{-(d-1)/2} \cdot \left[\left(\frac{S(\mathbf{1})}{\pi} \right)^{\frac{d-1}{2}} \frac{B(\mathbf{1}) - A(\mathbf{1})}{B(\mathbf{1})\sqrt{b_1 \cdots b_{d-1}}} + O\left(\frac{1}{n}\right) \right].$$

The asymptotic behaviour of a model with a negative drift step set is slightly messier. Let $\rho = \sqrt{A(\mathbf{1})/B(\mathbf{1})}$, define

$$B_k(\mathbf{z}_{\hat{k}}) = [z_k]S(\mathbf{z}) = [z_k^{-1}]S(\mathbf{z})$$

for each $1 \le k \le d-1$, and let

$$C_\rho = \frac{S(\mathbf{1},\rho)\,\rho}{2\,\pi^{d/2}\,A(\mathbf{1})(1-1/\rho)^2} \cdot \sqrt{\frac{S(\mathbf{1},\rho)^d}{\rho\,B_1(\mathbf{1},\rho)\cdots B_{d-1}(\mathbf{1},\rho)\cdot B(\mathbf{1})}}.$$

Furthermore, define the constant $C_{-\rho}$ by replacing ρ with $-\rho$ in C_ρ (the term under the square-root will always be real and positive when $C_{-\rho}$ is referenced).

Theorem 37 (Negative Drift Asymptotics [31, Theorem 2.3])**.** *Let $\mathcal{S}$ be a negative drift step set that is symmetric over all but the dth axis and takes a step forwards and backwards in each coordinate. If $Q(\widehat{\mathbf{z}}) \neq 0$ (i.e., if there are steps in $\mathcal{S}$ having z_d coordinate 0) then the total weight s_n of walks of length n taking steps in $\mathcal{S}$, starting at the origin, and never leaving $\mathbb{N}^d$ satisfies*

$$s_n = S(\mathbf{1}, \rho)^n \cdot n^{-d/2-1} \cdot C_\rho \left(1 + O\left(\frac{1}{n}\right)\right).$$

If $Q(\widehat{\mathbf{z}}) = 0$ then

$$s_n = n^{-d/2-1} \cdot \left[S(\mathbf{1}, \rho)^n \cdot C_\rho + S(\mathbf{1}, -\rho)^n \cdot C_{-\rho}\right]\left(1 + O\left(\frac{1}{n}\right)\right).$$

Theorems 36 and 37 are derived by applying the techniques of ACSV to the diagonal expression

$$W(t) = \Delta\left(\frac{G(\mathbf{z}, t)}{H(\mathbf{z}, t)}\right), \tag{3.3}$$

where

$$\begin{aligned} G(\mathbf{z}, t) &= (1 + z_1)\cdots(1 + z_{d-1})\left(1 - tz_1\cdots z_d\left(Q(\widehat{\mathbf{z}}) + 2z_d A(\widehat{\mathbf{z}})\right)\right) \\ H(\mathbf{z}, t) &= (1 - z_d)\left(1 - tz_1\cdots z_d\overline{S}(\mathbf{z})\right)\left(1 - tz_1\cdots z_d\left(Q(\widehat{\mathbf{z}}) + z_d A(\widehat{\mathbf{z}})\right)\right) \end{aligned}$$

for $\overline{S}(\mathbf{z}) = S(z_1, \ldots, z_{d-1}, \overline{z}_d)$. In comparison to the diagonal expression (3.2) for highly symmetric models, the denominator in (3.3) has multiple irreducible factors, and the singular set $\mathcal{V}$ of G/H is the union of three smooth surfaces. In the negative drift case the asymptotic behaviour of the diagonal sequence is still determined by the behaviour of G/H near smooth points of $\mathcal{V}$. However in the positive drift case the desired asymptotic behaviour is characterized by the behaviour of G/H near non-smooth points of $\mathcal{V}$. This geometric difference explains why positive and negative drift models have quantitatively different asymptotic behaviour (combinatorially, the difference is explained by the fact that positive drift models don't naturally wander into the restricting boundaries as frequently).

Remark 38. The techniques of ACSV allow the asymptotic expansions in Theorems 35-37 to be computed to any desired order. The expressions for higher-order constants are very unwieldy for general models, but can be computed automatically using a computer algebra system for any explicit model.

3.2.1 Our Results

Theorems 35-37 leave a gap, as they do not cover zero drift walks that are mostly symmetric but not highly symmetric. Our main result in this chapter is to fill this gap.

Theorem 39. *Let $\mathcal{S}$ be a step set that is symmetric over all but the dth axis and takes a step forwards and backwards in each coordinate. If $\mathcal{S}$ has zero drift then the total weight s_n of walks of length n taking steps in $\mathcal{S}$, starting at the origin, and never leaving $\mathbb{N}^d$ satisfies*

$$s_n \sim S(\mathbf{1})^n \cdot n^{-d/2} \cdot \frac{S(\mathbf{1})^{d/2}}{\pi^{d/2}\sqrt{b_1 \cdots b_d}}.$$

The dominant asymptotic behaviour of a zero drift mostly symmetric model matches the dominant term in the asymptotics for a highly symmetric model, however such mostly symmetric models only approximate highly symmetric models 'up to the dominant term' and the subdominant terms do not match. In fact, mostly symmetric models have asymptotic expansions in powers of $n^{-1/2}$ compared to the highly symmetric models whose expansions are series in powers of n^{-1} (see, for instance, Example 54 below). To the best of our knowledge, this is the first family of naturally occurring combinatorial classes analyzed using the methods of ACSV to not have a series in powers of n^{-1}.

Remark 40. In any dimension $d \geq 2$ there exist walk models with short step sets symmetric over all but two axes whose generating functions are non-D-finite, and therefore cannot be written as a rational diagonal (see [31, Theorem 3.8]). Thus, mostly symmetric models are the largest class of short step models defined by axial symmetries that can be handled directly by applying the techniques of ACSV to rational diagonal expressions derived from the kernel method; see Table 3.1 for a (now complete) summary.

Remark 41. Melczer and Wilson [31] incorrectly conjectured that the zero drift mostly symmetric models have asymptotic growth of the form $O(S(\mathbf{1})^n n^{-(d+1)/2})$.

Theorem 39 is proven in Section 3.4. First, in Section 3.3, we reframe the methods of ACSV in this case and describe why computing the behaviour of zero drift mostly symmetric models is harder than the other highly and mostly symmetric cases.

3.3 ACSV and Lattice Path Enumeration

We give examples of the calculation of asymptotics of highly symmetric and mostly symmetric models.

Drift	Exponential Rate	Order	Geometry	Series in	Covered By
zero (highly sym.)	$S(\mathbf{1})$	$n^{-d/2}$	smooth	n^{-1}	Theorem 35
zero (mostly sym.)	$S(\mathbf{1})$	$n^{-d/2}$	nonsmooth	$n^{-1/2}$	Theorem 39
positive	$S(\mathbf{1})$	$n^{-(d-1)/2}$	nonsmooth	n^{-1}	Theorem 36
negative	$< S(\mathbf{1})$	$n^{-1-d/2}$	smooth	n^{-1}	Theorem 37

Table 3.1: *Summary of results for short step models with highly or mostly symmetric step sets.*

3.3.1 Highly Symmetric Models

The simplest case geometrically occurs for highly symmetric models. As mentioned above, there is a diagonal expression

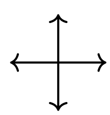
$$W(t) = \Delta\left(\frac{G(\mathbf{z},t)}{H(\mathbf{z},t)}\right) = \Delta\left(\frac{(1+z_1)\cdots(1+z_d)}{1-t(z_1\cdots z_d)S(\mathbf{z})}\right),$$

for the generating function $W(t)$ enumerating walks in the model defined by $\mathcal{S}$, where $S(\mathbf{z}) = \sum_{\mathbf{i}\in\mathcal{S}} w_{\mathbf{i}}\mathbf{z}^{\mathbf{i}}$. Because H and its partial derivative with respect to t never simultaneously vanish, the singular set $\mathcal{V}$ is always a smooth manifold.

Example 42. Consider the (unweighted) highly symmetric model with step set

$$\mathcal{S} = \{(0,\pm1),(\pm1,0)\}$$

defined by the cardinal directions.



Example 33 shows that the generating function of this is the diagonal of

$$F(x,y,t) = \frac{G(x,y,t)}{H(x,y,t)} = \frac{(1+x)(1+y)}{1-txyS(x,y)}$$

where $S(x,y) = x + \overline{x} + y + \overline{y}$. The system of polynomial equations

$$H(x,y,t) = 0 \text{ and } xH_x(x,y,t) = yH_y(x,y,t) = tH_t(x,y,t)$$

defining the critical points has the two solutions $(1,1,1/4)$ and $(-1,-1,-1/4)$, both of which are minimal as if $|x|,|y| < 1$ and $H(x,y,t) = 0$ then $|t| = 1/|xyS(x,y)| > 1/4$.

The Cauchy integral formula implies that the number s_n of walks on $\mathcal{S}$ that start at the origin and stay in $\mathbb{N}^2$ satisfies

$$s_n = \frac{1}{(2\pi i)^2} \int_{\substack{|x|=1\\|y|=1}} \left(\int_{|t|=1/4-\epsilon} \frac{(1+x)(1+y)}{1-txyS(x,y)} \frac{dt}{t^{n+1}} \right) \frac{dxdy}{(xy)^{n+1}}$$

for any $\epsilon > 0$ sufficiently small. Analytic arguments show that we can restrict the circles $|x| = |y| = 1$ to sufficiently small neighbourhoods $\mathcal{N}_\pm$ of $(1,1)$ and $(-1,-1)$, and expand the circle $|t| = 1/4-\epsilon$ to a circle $|t| = 1/4+\epsilon$, while introducing exponentially small errors. Because the numerator of F vanishes at $(-1,-1)$ it turns out that the only point determining dominant asymptotic behaviour of s_n is $(1,1)$, and we can write

$$\begin{aligned} s_n &\sim \frac{-1}{(2\pi i)^2} \int_{\mathcal{N}_+} \left(\int_{|t|=1/4+\epsilon} \frac{(1+x)(1+y)}{1-txyS(x,y)} \frac{dt}{t^{n+1}} - \int_{|t|=1/4-\epsilon} \frac{(1+x)(1+y)}{1-txyS(x,y)} \frac{dt}{t^{n+1}} \right) \frac{dxdy}{(xy)^{n+1}} \\ &= \frac{1}{(2\pi i)^2} \int_{\mathcal{N}_+} \frac{(1+x)(1+y)}{xy} S(x,y)^n \, dxdy, \end{aligned}$$

where the difference of integrals in the first line is computed by taking a residue at the pole $t = \frac{1}{xyS(x,y)}$.

Parametrizing the neighbourhood $\mathcal{N}_+$ of $(1,1)$ by $x = e^{i\theta_1}$ and $y = e^{i\theta_2}$ for θ_1, θ_2 in a neighbourhood $\mathcal{M}$ of the origin, the argument concludes by applying the saddle-point method to approximate

$$\begin{aligned} \frac{1}{(2\pi i)^2} \int_{\mathcal{N}_+} \frac{(1+x)(1+y)}{xy} S(x,y)^n \, dxdy &= \frac{4^n}{(2\pi)^2} \int_{\mathcal{M}} (1+e^{i\theta_1})(1+e^{i\theta_2}) e^{n \log \frac{S(e^{i\theta_1}, e^{i\theta_2})}{S(1,1)}} d\theta_1 d\theta_2 \\ &= \frac{4^n}{(2\pi)^2} \int_{\mathcal{M}} (4 + O(\theta_1+\theta_2)) \cdot \\ &\qquad\qquad e^{-n(\theta_1^2/4+\theta_2^2/4)+O(n(\theta_1+\theta_2)^3)} d\theta_1 d\theta_2 \\ &\sim \frac{4^n}{(2\pi)^2} \int_{\mathbb{R}^2} 4e^{-n(\theta_1^2/4+\theta_2^2/4)} d\theta_1 d\theta_2 \\ &= \frac{4}{\pi} \cdot \frac{4^n}{n}. \end{aligned}$$

◁

Melczer and Mishna [28] combined the techniques of smooth ACSV, which show how to make all steps in the above argument rigorous, with the uniform diagonal expression for highly symmetric models to prove Theorem 35.

3.3.2 Mostly Symmetric Models

Applying the kernel method while exploiting step set symmetry properties show the generating function $W(t)$ for walks in $\mathbb{N}^d$ on a set of mostly symmetric short steps $\mathcal{S}$ has the diagonal representation

$$W(t) = \Delta F(\mathbf{z},t) = \Delta\left(\frac{(1+z_1)\cdots(1+z_{d-1})\left(1-tz_1\cdots z_d\left(Q(\widehat{\mathbf{z}})+2z_dA(\widehat{\mathbf{z}})\right)\right)}{(1-z_d)\left(1-tz_1\cdots z_d\overline{S}(\mathbf{z})\right)\left(1-tz_1\cdots z_d\left(Q(\widehat{\mathbf{z}})+z_dA(\widehat{\mathbf{z}})\right)\right)}\right),$$

and the three factors in the denominator give the singular set $\mathcal{V}$ non-smooth points. The final denominator factor turns out to be irrelevant to the asymptotic analysis, so the points of interest lie on the zero sets $\mathcal{V}_1$ and $\mathcal{V}_2$ of $H_1(\mathbf{z},t) = 1-z_d$ and $H_2(\mathbf{z},t) = 1-tz_1\cdots z_d\overline{S}(\mathbf{z})$. The sets $\mathcal{V}_1$ and $\mathcal{V}_2$ are each manifolds: $\mathcal{V}_1$ does not contain any critical points, but $\mathcal{V}_2$ contains the critical point

$$\boldsymbol{\rho} = \left(\mathbf{1}, \sqrt{\frac{B(\mathbf{1})}{A(\mathbf{1})}}, \frac{\sqrt{A(\mathbf{1})/B(\mathbf{1})}}{2\sqrt{A(\mathbf{1})B(\mathbf{1})}+Q(\mathbf{1})}\right),$$

along with (potentially) other critical points with the same coordinate-wise modulus and non-minimal critical points that do not affect dominant asymptotics. In addition, the intersection $\mathcal{V}_1\cap\mathcal{V}_2$ is a manifold containing the critical point

$$\boldsymbol{\sigma} = \left(\mathbf{1}, 1, S(\mathbf{1})\right)$$

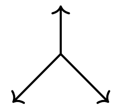
and (once again) potentially other critical points with the same coordinate-wise modulus. The points near which analytic behaviour of F determines dominant asymptotics of s_n depends on the drift of the model.

Negative Drift Models

When $\mathcal{S}$ has negative drift then $A(\mathbf{1}) > B(\mathbf{1})$ and the point $\boldsymbol{\rho}$ is a smooth minimal critical point of $\mathcal{V}$.

Example 43. Consider the (unweighted) mostly symmetric model with step set

$$\mathcal{S} = \{(-1,-1),(1,-1),(0,1)\}$$



whose generating function is the diagonal of

$$F(x,y,t) = \frac{(1+x)(1-2t(x^2y^2+1))}{(1-y)(1-t(x^2y^2+y^2+x))(1-t(x^2y^2+1))}.$$

The critical point $\boldsymbol{\rho} = (1, 1/\sqrt{2}, 1/2)$ is minimal, as are the critical points $(1, -1/\sqrt{2}, 1/2)$ and $(-1, \pm i/\sqrt{2}, -1/2)$, and this time both $\boldsymbol{\rho}$ and $(1, -1/\sqrt{2}, 1/2)$ contribute to dominant asymptotics of s_n. Because H_2 is the only denominator factor vanishing at these points, the smooth analysis used in Example 42 above still applies. Computing a residue and applying the saddle-point method to determine the asymptotic contribution of each of these points ultimately yields

$$s_n \sim \frac{16+12\sqrt{2}}{\pi} \cdot \frac{(2\sqrt{2})^n}{n^2} + \frac{-16+12\sqrt{2}}{\pi} \cdot \frac{(-2\sqrt{2})^n}{n^2} = \begin{cases} \frac{24\sqrt{2}}{\pi} \cdot \frac{(2\sqrt{2})^n}{n^2} & : n \text{ even} \\ \frac{32}{\pi} \cdot \frac{(2\sqrt{2})^n}{n^2} & : n \text{ odd} \end{cases}.$$

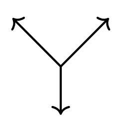
◁

Positive Drift Models

When $\mathcal{S}$ has positive drift then $A(\mathbf{1}) < B(\mathbf{1})$ and the point $\boldsymbol{\rho}$ is no longer minimal, as its dth coordinate is larger than 1. In fact, the only minimal critical point is the non-smooth point $\boldsymbol{\sigma}$ (and potentially other points with the same coordinate-wise modulus that do not affect dominant asymptotics) and a more general argument must be applied.

Example 44. Consider the (unweighted) mostly symmetric model with step set

$$\mathcal{S} = \{(-1,1),(1,1),(0,-1)\}$$



whose generating function is the diagonal of

$$F(x,y,t) = \frac{(1+x)(1-2txy^2)}{(1-y)(1-t(xy^2+x^2+1))(1-txy^2)}.$$

The critical points defined by the vanishing of $H_2(x,y,t) = 1-t(xy^2+x^2+1)$ have coordinate-wise modulus $(1,\sqrt{2},1/4)$ and are not minimal as expected. Because $\boldsymbol{\sigma} = (1,1,1/3)$ is the only minimal critical point, we start by writing

$$s_n = \frac{1}{2\pi i}\int_{|x|=1}\left(\frac{1}{(2\pi i)^2}\int_{T(1-\epsilon_1,1/3-\epsilon_2)}\frac{P(x,y)}{(1-y)(1-t(xy^2+x^2+1))}\frac{dydt}{(xyt)^{n+1}}\right)dx \tag{3.4}$$

where we use the notation $T(\mathbf{w}) = \{\mathbf{z} : |z_j| = |w_j| \text{ for all } j\}$, the constants ϵ_1 and ϵ_2 are sufficiently small, and

$$P(x,y,t) = \frac{(1+x)(1-2txy^2)}{(1-txy^2)}$$

captures the numerator and the denominator factor that does not vanish near $\boldsymbol{\sigma}$. If $\mathcal{N}(\mathbf{w})$ denotes a subset of $T(\mathbf{w})$ where each coordinate is sufficiently close to the positive real axis then an analytic argument shows that we can restrict x to $\mathcal{N}(1)$ and (y,t) to $\mathcal{N}(1-\epsilon_1,1/3-\epsilon_2)$ while introducing only a negligible error.

Our next goal is to compute residues to remove two variables, one for each vanishing denominator factor. To that end, we note that the part of the integrand in (3.4) that depends on n is captured by the function $\phi(x,y,t) = (xyt)^{-1}$. Since

$$\underbrace{(\nabla\phi)(1,1,1/3)}_{(-3,-3,-9)} = \underbrace{(\nabla H_1)(1,1,1/3)}_{(0,-1,0)} + 3\underbrace{(\nabla H_2)(1,1,1/3)}_{(-1,-2/3,-3)}$$

is a linear combination with positive coefficients (thus $\boldsymbol{\sigma}$ is a *generic* critical point), replacing the domain $\mathcal{N}(1-\epsilon_1,1/3-\epsilon_2)$ in (3.4) with the domains $\mathcal{N}(1+\epsilon_1,1/3-\epsilon_2),\mathcal{N}(1-\epsilon_1,1/3+\epsilon_2)$, and $\mathcal{N}(1+\epsilon_1,1/3+\epsilon_2)$ that 'cross' the singular sets $\mathcal{V}_1$ and $\mathcal{V}_2$ results in integrals that are exponentially smaller than the one under consideration. A signed sum of these integrals is computed by taking a 'multivariate residue' at the points where $y = 1$ and $t = 1/(xy^2+x^2+1) = 1/(x^2+x+1)$, ultimately giving that

$$\begin{aligned} s_n &\sim \frac{1}{2\pi i}\int_{\mathcal{N}(1)}\frac{P(x,1)}{x}(x+1+1/x)^n\,dx \\ &= \frac{1}{2\pi i}\int_{\mathcal{N}(1)}\frac{(x^2-x+1)(1+x)}{x(x^2+1)}(x+1+1/x)^n\,dx. \end{aligned}$$

Making the change of variables $x = e^{i\theta}$ for θ in a neighbourhood $\mathcal{M}$ of the origin, then applying the saddle-point method, gives

$$s_n \sim \frac{1}{2\pi}\int_{\mathcal{M}}\frac{(e^{2i\theta}-e^{i\theta}+1)(1+e^{i\theta})}{(e^{2i\theta}+1)}e^{n\log(e^{i\theta}+1+e^{-i\theta})}\,d\theta \sim \frac{1}{2\pi}\int_{\mathbb{R}}e^{n\log(3)-n\theta^2/3}d\theta = \frac{\sqrt{3}}{2\sqrt{\pi}}\cdot\frac{3^n}{\sqrt{n}}.$$

◁

In the mostly symmetric negative drift case the gradient of ϕ at $\boldsymbol{\sigma}$ cannot be written as a positive linear combination of the gradients of H_1 and H_2 at $\boldsymbol{\sigma}$, so the non-smooth minimal critical point $\boldsymbol{\sigma}$ cannot be used to characterize asymptotics. Of course, for negative drift models the smooth critical point $\boldsymbol{\rho}$ is minimal and can be used instead. The fact that smooth minimal critical points arise precisely when non-smooth critical points cannot be used to compute asymptotics holds more generally, and is explained by the theory of ACSV for *multiple points* (see [29, Chapter 9] or [34, Chapter 10]).

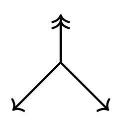
Zero Drift Models

In the zero drift case $A(\mathbf{1}) = B(\mathbf{1})$, so the critical points $\boldsymbol{\rho}$ and $\boldsymbol{\sigma}$ collide. This makes two denominator factors vanish at the minimal critical point determining asymptotics, but unlike the positive drift case we can only take one residue. Computing asymptotics thus requires a more careful analysis, conducted in Section 3.4, to prove Theorem 39.

Example 45. Consider the mostly symmetric zero drift model with step set

$$\mathcal{S} = \{(0,1),(0,1),(-1,-1),(1,-1)\}$$

whose North step appears twice (i.e., has a weight of two).



The generating function for this walk model can be written as the diagonal of

$$F(x,y,t) = \frac{(1+x)(1-2ty^2(x^2+1))}{(1-y)(1-txy(2/y+xy+y/x))(f1-ty^2(x^2+1))},$$

with minimal critical points $\boldsymbol{\rho} = \boldsymbol{\sigma} = (1,1,1/4)$ along with $(1,-1,-1/4)$ and $(-1,\pm i,\mp i)$. Because H_2 is the only denominator factor of H that vanishes at the final three points, the contributions of these points to the asymptotic behaviour of s_n are covered by results on smooth ACSV from Melczer [29, Thm. 5.3]: the fact that the numerator vanishes at each of these points further implies that these contributions are bounded in $O(4^n n^{-2})$, which will turn out not to affect dominant asymptotics of s_n.

Analogously to the positive drift case, we begin by writing

$$s_n = \frac{1}{2\pi i}\int_{T(1)}\left(\frac{1}{(2\pi i)^2}\int_{T(1-\epsilon_1,1/4-\epsilon_2)}\frac{P(x,y)}{(1-y)(1-txy(2/y+xy+y/x))}\frac{dydt}{(xyt)^{n+1}}\right)dx \quad (3.5)$$

where the constants ϵ_1 and ϵ_2 are sufficiently small, and

$$P(x,y,t) = \frac{(1+x)(1-2ty^2(x^2+1))}{1-ty^2(x^2+1)}$$

captures the numerator and the denominator factor that does not vanish near $\boldsymbol{\sigma}$. Again we can replace the domains of integration $T(1)$ and $T(1-\epsilon_1, 1/4-\epsilon_2)$ by subsets $\mathcal{N}(1)$ and $\mathcal{N}(1-\epsilon_1, 1/4-\epsilon_2)$ near the positive real axes while introducing a negligible error. The part of the integrand in (3.5) that depends on n is still captured by the function $\phi(x,y,t) = (xyt)^{-1}$, but now

$$\underbrace{(\nabla\phi)(1,1,1/4)}_{(-4,-4,-16)} = 0\ \underbrace{(\nabla H_1)(1,1,1/4)}_{(0,-1,0)} + 4\ \underbrace{(\nabla H_2)(1,1,1/4)}_{(-1,-1,-4)}$$

is a linear combination with one coefficient equal to zero (and thus $\boldsymbol{\sigma}$ is a *non-generic* critical point lying lying on a codimension 1 face). Because of this, we can replace $\mathcal{N}(1-\epsilon_1, 1/4-\epsilon_2)$ by a domain $\mathcal{N}(1-\epsilon_1, 1/4+\epsilon_2)$ 'crossing' the factor H_2 and get something growing exponentially smaller than s_n, but (unlike the positive drift case) we can no longer introduce the integrals whose domains of integration 'cross' H_1. Performing the residue computation for the pole $t = 1/xy\overline{S}(x,y)$ gives

$$s_n \sim \frac{1}{(2\pi i)^2}\int_{\mathcal{N}(1)}\int_{\mathcal{N}(1-\epsilon_1)} \frac{1+x}{2x^2y}\cdot\frac{2x-y^2(1+x^2)}{1-y}\overline{S}(x,y)^n dxdy,$$

and the $1-y$ factor in the denominator complicates the analysis. We note, however, that the numerator factor $2x-y^2(1+x^2)$ also vanishes at $x=y=1$ and

$$\frac{2x-y^2(1+x^2)}{1-y} = 4 + O(1-x) + O(1-y) + O\left(\frac{(1-x)^2}{1-y}\right).$$

The difficult factor in the denominator can thus be handled by carefully taking $\epsilon_1 \to 0$ at a rate that balances the integral manipulations and approximations above with the need to approximate the integrand by its leading series terms near $(1,1)$, ultimately giving

$$s_n \sim \frac{4}{(2\pi i)^2}\int_{\mathcal{N}(1,1)} \frac{1+x}{2x^2y}\cdot\overline{S}(x,y)^n dxdy.$$

Making the change of variables $x = e^{i\theta_1}$ and $y = e^{i\theta_2}$ and applying the saddle-point method then implies

$$s_n \sim \frac{4}{(2\pi)^2}\int_{\mathbb{R}^2} e^{n\log 4 - n(\theta_1^2/4+\theta_2^2/2)} d\theta_1 d\theta_2 = \frac{4^n}{n}\cdot\frac{2\sqrt{2}}{\pi}.$$

◁

In Section 3.4 we formalize the argument sketched in Example 45 to prove Theorem 39, giving leading asymptotics of s_n. Computing higher order terms in the asymptotic expansion of s_n, and dealing with the general *non-generic* case where less residues can be taken than expected, requires approximating saddle-point-like integrals with amplitudes that are singular at their saddle points. A method handling these integrals in general will be detailed in a forthcoming paper, and we illustrate the approach in Example 54 below to show that

$$s_n = \frac{4^n}{n}\left(\frac{2\sqrt{2}}{\pi} + \frac{1}{\sqrt{\pi}}n^{-1/2} + O(n^{-1})\right)$$

for the lattice path model in Example 45.

3.4 Asymptotics of Zero Drift Mostly Symmetric Models

To prove Theorem 39 we now fix a mostly symmetric zero drift step set $\mathcal{S}$. The diagonal representation (3.3), combined with the Cauchy integral formula, implies that

$$s_n = \frac{1}{(2\pi i)^{d+1}} \int_{T\times C^{in}} \frac{G(\mathbf{z},t)}{H(\mathbf{z},t)} \frac{d\mathbf{z}dt}{(z_1\cdots z_d\, t)^{n+1}}$$

where

$$T = T_\epsilon = \{\mathbf{z}\in\mathbb{C}^d : |z_k| = 1 \text{ for all } 1\le k\le d-1 \text{ and } |z_d| = 1-\epsilon\}$$

$$C^{in} = C^{in}_\gamma = \{t\in\mathbb{C} : |t| = S(\mathbf{1})^{-1}(1-\gamma)\}$$

for any $0<\epsilon,\gamma<1$, the numerator G satisfies

$$G(\mathbf{z},t) = (1+z_1)\cdots(1+z_{d-1})\left(1 - tz_1\cdots z_d\left(Q(\widehat{\mathbf{z}}) + 2z_dA(\widehat{\mathbf{z}})\right)\right),$$

and the denominator H can be written $H = H_1H_2H_3$ for

$$\begin{aligned}
H_1(\mathbf{z},t) &= 1 - z_d\\
H_2(\mathbf{z},t) &= 1 - tz_1\cdots z_d\overline{S}(\mathbf{z})\\
H_3(\mathbf{z},t) &= 1 - tz_1\cdots z_d\left(Q(\widehat{\mathbf{z}}) + z_dA(\widehat{\mathbf{z}})\right).
\end{aligned}$$

We make this choice of T in order to stay in the domain of convergence of the series expansion under consideration, and because – as shown in Melczer and Wilson [31, Proposition

4.2] – the point $(\mathbf{1}, S(\mathbf{1}))$ is the unique minimal zero of $H(\mathbf{z}, t)$ with positive coordinates that minimizes $|z_1 \cdots z_d t|^{-1}$. It will also be of importance that the zeroes of H_2 with the same coordinate wise-modulus as $(\mathbf{1}, S(\mathbf{1}))$ are given by $(\mathbf{w}, t)$ where $t = 1/(w_1 \cdots w_d S(\mathbf{w}, w_d))$ and $\mathbf{w}$ is contained in the set

$$\Gamma = \left\{ (\widehat{\mathbf{w}}, w_d) : \widehat{\mathbf{w}} \in \{\pm 1\}^{d-1},\, w_d \in \{\pm 1, \pm i\}, \quad \left| \frac{1}{w_1 \cdots w_d S(\widehat{\mathbf{w}}, w_d)} \right| = \frac{1}{S(\mathbf{1})} \right\}; \tag{3.6}$$

see [31, Theorem 4.3] (although we note that those authors forgot to allow the possibility that $w_d = \pm i$ in their argument). While this is the set of possible contributing points, we will see that only the real point $(\mathbf{1}, S(\mathbf{1}))$ contributes.

3.4.1 Residue Computations

We use residue computations to simplify the integral under consideration. First, we prove that we can restrict our integral to neighbourhoods of points in Γ and get an exponentially small error. Given $\mathbf{w} \in \Gamma$, we define the product of circle arcs contained in T

$$\mathcal{N}_{\mathbf{w}} = \mathcal{N}_{\mathbf{w}}^{\delta,\epsilon} = \{\mathbf{z} \in \mathbb{C}^d : |z_j| = 1 \text{ for } 1 \le j \le d-1 \text{ with } |z_d| = 1 - \epsilon \text{ and } |\arg(z_j) - \arg(w_j)| < \delta\, \forall j\}$$

which will shrink to $\mathbf{w}$ as $n \to \infty$, and let

$$\mathcal{N} = \bigcup_{\mathbf{w} \in \Gamma} \mathcal{N}_{\mathbf{w}}.$$

Running assumptions on ϵ and δ The asymptotic arguments below yield dominant asymptotics of s_n when $\epsilon = n^{-\alpha}$ and $\delta = n^{-\beta}$ for positive constants α and β such that

$$1/2 < \alpha < 2\beta, \quad \alpha + \beta > 1, \text{ and } 1/3 < \beta < 1/2,$$

and we now assume these inequalities hold. For concreteness, one can take $\epsilon = n^{-7/10}$ and $\delta = n^{-2/5}$.

Lemma 46. *Under our running assumptions on ϵ and δ, if $0 < \gamma < 1$ then*

$$s_n = \frac{1}{(2\pi i)^{d+1}} \int_{\mathcal{N} \times C^{in}} \frac{G(\mathbf{z}, t)}{H(\mathbf{z}, t)} \frac{d\mathbf{z}dt}{(z_1 \cdots z_d\, t)^{n+1}} + O(S(\mathbf{1})^n e_n)$$

where $e_n \to 0$ faster than any fixed power of n.

Proof. By our Cauchy integral representation for s_n, it suffices to bound

$$\frac{1}{(2\pi i)^{d+1}} \int_{(T\setminus\mathcal{N})\times C^{in}} \frac{G(\mathbf{z},t)}{H(\mathbf{z},t)} \frac{d\mathbf{z}dt}{(z_1\cdots z_d\, t)^{n+1}}$$

for suitable values of ϵ, δ, and γ. First, we note that if $\mathbf{z} \in T$ is fixed then C^{in} is contained in the power series domain of convergence of $G(\mathbf{z},t)/H(\mathbf{z},t)$, so

$$\frac{1}{2\pi i} \int_{(T\setminus\mathcal{N})\times C^{in}} \frac{G(\mathbf{z},t)}{H(\mathbf{z},t)} \frac{d\mathbf{z}dt}{(z_1\cdots z_d\, t)^{n+1}} = \int_{(T\setminus\mathcal{N})} \left[\frac{1}{2\pi i} \int_{C^{in}} \frac{G(\mathbf{z},t)}{H(\mathbf{z},t)} \frac{dt}{t^{n+1}}\right] \frac{d\mathbf{z}}{(z_1\cdots z_d)^{n+1}},$$

where

$$\frac{1}{2\pi i} \int_{C^{in}} \frac{G(\mathbf{z},t)}{H(\mathbf{z},t)} \frac{dt}{t^{n+1}} = [t^n] F(\mathbf{z},t)$$

is the coefficient of t^n when $F(\mathbf{z},t) = G(\mathbf{z},t)/H(\mathbf{z},t)$ is expanded as a power series. When each coordinate z_j has unit modulus and $\mathbf{z}$ is bounded away from the elements of Γ then $Q(\widehat{\mathbf{z}}) + z_d A(\widehat{\mathbf{z}})$ and $\overline{S}(\mathbf{z})$ are less than $S(\mathbf{1})$ (see [31, Proposition 4.2]) so that $[t^n]F(\mathbf{z},t) = O(\tau^n)$ for some $0 < \tau < S(\mathbf{1})$. In particular, *if δ was a fixed constant* then

$$\left|\frac{1}{(2\pi i)^{d+1}} \int_{(T\setminus\mathcal{N})\times C^{in}} \frac{G(\mathbf{z},t)}{H(\mathbf{z},t)} \frac{d\mathbf{z}dt}{(z_1\cdots z_d\, t)^{n+1}}\right| \le \frac{1}{(2\pi)^d} \int_{(T\setminus\mathcal{N})} \left|[t^n]F(\mathbf{z},t)\right| \cdot |z_1\cdots z_{d-1} z_d|^{-n-1}\, d\mathbf{z}$$

grows exponentially slower than $S(\mathbf{1})^n$ as $|z_j| = 1$ for all $1 \le j \le d-1$ and $|z_d| = 1 - n^{-\alpha}$ for $\mathbf{z} \in T$.

In actuality $\delta = n^{-\beta} \to 0$ as $n \to \infty$ and we need to show that δ does not go to zero too quickly. By the definition of Γ, the monomials of $\overline{S}(\mathbf{z})$ all have the same argument at $\mathbf{z} = \mathbf{w}$ for $\mathbf{w} \in \Gamma$, so

$$|Q(\widehat{\mathbf{z}}) + z_d A(\widehat{\mathbf{z}})| < |\overline{z}_d B(\widehat{\mathbf{z}}) + Q(\widehat{\mathbf{z}}) + z_d A(\widehat{\mathbf{z}})| = |\overline{S}(\mathbf{z})|$$

for $\mathbf{z}$ sufficiently close to the elements of Γ. Because we can reduce to any arbitrarily small neighbourhoods of the elements of Γ while introducing only an exponentially smaller error, we can thus assume that we are in a small enough neighbourhood such that $\left|[t^n]F(\mathbf{z},t)\right| = O(|\overline{S}(\mathbf{z})|^n)$. We bound $|\overline{S}(\mathbf{z})|^n$ for $\mathbf{z}$ near $\mathbf{1} \in \Gamma$, with the argument for the other elements $\mathbf{w} \in \Gamma$ being analogous. Writing $z_j = e^{i\theta_j}$ for $1 \le j \le d-1$ and $z_d = (1-\epsilon)e^{i\theta}$ we have for θ_j

sufficiently small that

$$
\begin{aligned}
\overline{S}(\mathbf{z})^n &= \exp\Big[n \log \overline{S}\left(e^{i\hat{\boldsymbol{\theta}}}, (1-\epsilon)e^{i\theta_d}\right)\Big] \\
&= S(\mathbf{1})^n \cdot \exp\Big[n \log \overline{S}\left(e^{i\boldsymbol{\theta}}\right) - n \log S(\mathbf{1}) + O(n\epsilon\theta_1) + \cdots + O(n\epsilon\theta_{d-1}) + O(n\epsilon^2)\Big] \\
&= S(\mathbf{1})^n \cdot \exp\big[-n(c_1\theta_1^2 + \cdots + c_d\theta_d^2) \\
&\qquad\qquad + O(n(\theta_1 + \cdots + \theta_d)^3) + O(n\epsilon\theta_1) + \cdots + O(n\epsilon\theta_{d-1}) + O(n\epsilon^2)\big],
\end{aligned}
$$

where the second equality follows from the expansion of $\log \overline{S}(e^{i\hat{\boldsymbol{\theta}}}, (1-\epsilon)e^{i\theta_d})$ derived in the proof of Proposition 50 below, and the c_j are positive constants computed in the proof of Proposition 51 below. When $|\theta_j| \geq \delta$ then, under our assumptions on $\epsilon = n^{-\alpha}$ and $\delta = n^{-\beta}$ and the fact that we can restrict the θ_j to any arbitrarily small neighbourhood of the origin, the big-O terms in the exponential are bounded below the quadratic term so

$$
\left|[t^n]F(\mathbf{z},t)\right| = O(|\overline{S}(\mathbf{z})|^n) = O(S(\mathbf{1})^n e^{-Kn^\rho})
$$

for some $K, \rho > 0$. The claimed result holds as $e^{-Kn^\rho} \to 0$ faster than any fixed power of n. ◻

Having localized near particular singularities of interest, we now introduce an asymptotically negligible integral that allows us to make our desired residue computation. Let

$$
C^{\text{out}} = C_\gamma^{\text{out}} = \{t \in \mathbb{C} : |t| = S(\mathbf{1})^{-1}(1+\gamma)\}.
$$

Lemma 47. *Using the notation and running assumptions above, if $\gamma > 0$ is sufficiently small then*

$$
\frac{1}{(2\pi i)^{d+1}} \int_{\mathcal{N} \times C^{\text{out}}} \frac{G(\mathbf{z},t)}{H(\mathbf{z},t)} \frac{d\mathbf{z}dt}{(z_1 \cdots z_d\, t)^{n+1}} = O(\tau^n)
$$

for some $0 < \tau < S(\mathbf{1})$.

Proof. Bounding the integrand under consideration by the supremum of its modulus, there exists a constant $K_1 > 0$ such that

$$
\left|\int_{\mathcal{N} \times C^{\text{out}}} \frac{G(\mathbf{z},t)}{H(\mathbf{z},t)} \frac{d\mathbf{z}dt}{(z_1 \cdots z_d\, t)^{n+1}}\right| \leq K_1 \sup_{(\mathbf{z},t) \in \mathcal{N} \times C^{\text{out}}} \left|\frac{G(\mathbf{z},t)}{H(\mathbf{z},t)}\right| \left[\frac{S(\mathbf{1})}{(1-\epsilon)(1+\gamma)}\right]^n. \tag{3.7}
$$

When ϵ and δ are sufficiently small then, by the definition of Γ, the only poles of $F(\mathbf{z},t) = G(\mathbf{z},t)/H(\mathbf{z},t)$ with $\mathbf{z} \in \mathcal{N}$ have $|t|$ sufficiently close to $S(\mathbf{1})^{-1}$ or $(Q(\mathbf{1}) + A(\mathbf{1}))^{-1}$. In particular, if γ is fixed sufficiently small and ϵ and δ are sufficiently smaller than γ then there exists $K_2 > 0$ such that

$$\sup_{(\mathbf{z},t)\in\mathcal{N}\times C^{\text{out}}} \left|\frac{G(\mathbf{z},t)}{H(\mathbf{z},t)}\right| \leq K_2 \sup_{|z_d|=1-\epsilon} \left|\frac{1}{1-z_d}\right| = O(n^\alpha),$$

since $\epsilon = n^{-\alpha}$. The right-hand side of (3.7) thus has exponential growth smaller than $S(\mathbf{1})$ whenever $(1-\epsilon)(1+\gamma) > 1$, which occurs for all sufficiently large n under our assumptions. □

Our last result before computing residues shows that the only poles of our integrand that affect our calculations come from the denominator factor H_2 vanishing.

Lemma 48. *The polynomial $H_3(\mathbf{z},t) = 1 - tz_1\cdots z_d\left(Q(\widehat{\mathbf{z}}) + z_dA(\widehat{\mathbf{z}})\right)$ is non-zero when $|z_j| \leq 1$ for all $1 \leq j \leq d$ and $|t| \leq S(\mathbf{1})$. If δ and ϵ are sufficiently small then $B(\widehat{\mathbf{z}}) \neq 0$ for $\widehat{\mathbf{z}} \in \mathcal{N}_\mathbf{w}$.*

Proof. Note that $\overline{S}(\mathbf{z}) = S(z_1,\dots,z_{d-1},\overline{z}_d) = z_dA(\widehat{\mathbf{z}}) + Q(\widehat{\mathbf{z}}) + \overline{z}_dB(\widehat{\mathbf{z}})$ so

$$\left|Q(\widehat{\mathbf{z}}) + z_dA(\widehat{\mathbf{z}})\right| \leq Q(\mathbf{1}) + A(\mathbf{1}) < S(\mathbf{1})$$

when $|z_j| \leq 1$ for all $1 \leq j \leq d$. In particular, if $|z_j| \leq 1$ for all $1 \leq j \leq d$ and $H_3(\mathbf{z},t) = 0$ then

$$|t| = \left|\frac{1}{z_1\cdots z_d\left(Q(\widehat{\mathbf{z}}) + z_dA(\widehat{\mathbf{z}})\right)}\right| > \frac{1}{S(\mathbf{1})}.$$

To prove that $B(\widehat{\mathbf{z}}) \neq 0$ for $\widehat{\mathbf{z}} \in \mathcal{N}_\mathbf{w}$ we note that $B(\widehat{\mathbf{z}})$ does not vanish whenever $\widehat{\mathbf{z}}$ is a maximizer of $|\overline{S}(\widehat{\mathbf{z}})|$ on the unit torus. The points of Γ get arbitrarily close to these maximizers as $\epsilon \to 0$, so by continuity $B(\widehat{\mathbf{z}})$ does not vanish for $\widehat{\mathbf{z}} \in \mathcal{N}_\mathbf{w}$ whenever δ and ϵ are sufficiently small. □

We are now ready to determine a residue integral expression for s_n.

Proposition 49. *With the notations and assumptions above,*

$$s_n = \sum_{\mathbf{w}\in\Gamma} \frac{1}{(2\pi i)^d} \int_{\mathcal{N}_\mathbf{w}} \frac{(1+z_1)\cdots(1+z_{d-1})}{B(\widehat{\mathbf{z}})(z_1\cdots z_d)} \cdot \frac{B(\widehat{\mathbf{z}}) - z_d^2A(\widehat{\mathbf{z}})}{1-z_d} \cdot \overline{S}(\mathbf{z})^n\,d\mathbf{z} + O((S(\mathbf{1})^n e_n) \tag{3.8}$$

where $e_n \to 0$ faster than any fixed power of n.

Proof. Lemmas 46 and 47 imply that we can write

$$s_n = \frac{-1}{(2\pi i)^d} \int_{\mathcal{N}} \left(\frac{1}{2\pi i} \int_{C^{\text{out}}} \frac{G(\mathbf{z},t)}{H(\mathbf{z},t)} \frac{dt}{(z_1 \cdots z_d\, t)^{n+1}} - \frac{1}{2\pi i} \int_{C^{\text{in}}} \frac{G(\mathbf{z},t)}{H(\mathbf{z},t)} \frac{dt}{(z_1 \cdots z_d\, t)^{n+1}} \right) d\mathbf{z} \\ + O((S(\mathbf{1})^n e_n)$$

under our assumptions. The first statement of Lemma 48 implies that the inner difference of integrals is given by a residue where $t = 1/\left(z_1 \cdots z_d \overline{S}(\mathbf{z})\right)$, and the second statement of Lemma 48 implies that the poles involved are all simple. Computing the residue gives

$$-\frac{(1+z_1)\cdots(1+z_{d-1})\left(1 - \frac{Q(\widehat{\mathbf{z}})+2z_d A(\widehat{\mathbf{z}})}{\overline{z}_d B(\widehat{\mathbf{z}})+Q(\widehat{\mathbf{z}})+z_d A(\widehat{\mathbf{z}})}\right)}{(z_1 \cdots z_d \overline{S}(\mathbf{z}))(1-z_d)\left(1 - \frac{Q(\widehat{\mathbf{z}})+z_d A(\widehat{\mathbf{z}})}{\overline{z}_d B(\widehat{\mathbf{z}})+Q(\widehat{\mathbf{z}})+z_d A(\widehat{\mathbf{z}})}\right)} = -\frac{(1+z_d)\cdots(1+z_{d-1})\left(1 - z_d^2 \frac{A(\widehat{\mathbf{z}})}{B(\widehat{\mathbf{z}})}\right)}{(1-z_d)(z_1 \cdots z_d \overline{S}(\mathbf{z}))}.$$

□

3.4.2 Computing Dominant Asymptotics

To determine asymptotics of s_n it is sufficient to determine asymptotics for the residue integrals in (3.8). If the factor of $1 - z_d$ could be cancelled from the numerator (which occurs in the highly symmetric case, when $A(\widehat{\mathbf{z}}) = B(\widehat{\mathbf{z}})$) then asymptotically evaluating these integrals is a straightforward application of the saddle-point method. The presence of this factor complicates the analysis, which is why the non-highly symmetric zero drift case was left open in [31]. To determine asymptotics, we thus switch to polar coordinates and carefully consider the resulting saddle-point-like integrals.

We start by reducing the contribution of the point $\mathbf{w} = \mathbf{1}$, which turns out to be the only point contributing to dominant asymptotics, to a standard saddle-point integral.

Proposition 50. *Under our running assumptions on ϵ and δ,*

$$\frac{1}{(2\pi i)^d} \int_{\mathcal{N}_1} \frac{(1+z_1)\cdots(1+z_{d-1})}{B\left(\widehat{\mathbf{z}}\right)(z_1 \cdots z_d)} \cdot \frac{B\left(\widehat{\mathbf{z}}\right) - z_d^2 A\left(\widehat{\mathbf{z}}\right)}{1-z_d} \cdot \overline{S}(\mathbf{z})^n \, d\mathbf{z} \sim \frac{1}{\pi^d} \int_{\mathbf{z} \in \mathcal{M}} e^{n \log \overline{S}\left(e^{i\boldsymbol{\theta}}\right)} d\boldsymbol{\theta}$$

where $\mathcal{M} = (-\delta, \delta)^d$ and $e^{i\boldsymbol{\theta}} = \left(e^{i\theta_1}, \ldots, e^{i\theta_d}\right)$.

Proof. Making the change of variables $z_j = e^{i\theta_j}$ for $1 \le j \le d-1$ and $z_d = (1-\epsilon)e^{i\theta_d}$ gives

$$\frac{1}{(2\pi i)^d} \int_{\mathcal{N}_1} \frac{(1+z_1)\cdots(1+z_{d-1})}{B\left(\widehat{\mathbf{z}}\right)(z_1 \cdots z_d)} \cdot \frac{B\left(\widehat{\mathbf{z}}\right) - z_d^2 A\left(\widehat{\mathbf{z}}\right)}{1-z_d} \cdot \overline{S}(\mathbf{z})^n \, d\mathbf{z} = \frac{1}{(2\pi)^d} \int_{\mathbf{z} \in \mathcal{M}} P(\boldsymbol{\theta}) Q(\boldsymbol{\theta}) e^{n\phi(\boldsymbol{\theta})} d\boldsymbol{\theta},$$

where

$$P(\boldsymbol{\theta}) = \frac{(1 + e^{i\theta_1})\cdots(1 + e^{i\theta_{d-1}})}{B\left(e^{i\hat{\boldsymbol{\theta}}}\right)}$$

$$Q(\boldsymbol{\theta}) = \frac{B\left(e^{i\hat{\boldsymbol{\theta}}}\right) - (1-\epsilon)^2 e^{2i\theta_d} A\left(e^{i\hat{\boldsymbol{\theta}}}\right)}{1 - (1-\epsilon)e^{i\theta_d}}$$

$$\phi(\boldsymbol{\theta}) = \log \overline{S}\left(e^{i\hat{\boldsymbol{\theta}}}, (1-\epsilon)e^{i\theta_d}\right)$$

and $e^{i\hat{\boldsymbol{\theta}}} = \left(e^{i\theta_1}, \ldots, e^{i\theta_{d-1}}\right)$. The first factor is easiest to analyze as $\epsilon, \delta \to 0$, with a power series expansion of the numerator and denominator showing that

$$P(\boldsymbol{\theta}) = \frac{2^{d-1}}{B(\mathbf{1})} + O(\epsilon + \theta_1 + \cdots + \theta_{d-1}).$$

Since $A(\widehat{\mathbf{z}})$ and $B(\widehat{\mathbf{z}})$ are invariant under the transformations $z_j = 1/z_j$ for $1 \le j \le d-1$, the functions $A\left(e^{i\hat{\boldsymbol{\theta}}}\right)$ and $B\left(e^{i\hat{\boldsymbol{\theta}}}\right)$ are even. Combined with the fact that $A(\mathbf{1}) = B(\mathbf{1})$, we see that

$$\begin{aligned} Q(\boldsymbol{\theta}) &= \frac{B(\mathbf{1}) - (1-\epsilon)^2 e^{2i\theta_d} A(\mathbf{1}) + O(\theta_1^2 + \cdots + \theta_{d-1}^2)}{1 - (1-\epsilon)e^{i\theta_d}} \\ &= B(\mathbf{1})(1 + (1-\epsilon)e^{i\theta_d}) + O\left(\frac{\theta_1^2 + \cdots + \theta_{d-1}^2}{1 - (1-\epsilon)e^{i\theta_d}}\right) \\ &= 2B(\mathbf{1}) + O(\epsilon + \theta_d + \theta_1^2\epsilon^{-1} + \cdots + \theta_{d-1}^2\epsilon^{-1}). \end{aligned}$$

Finally, we have

$$
\begin{aligned}
\phi(\boldsymbol{\theta}) &= \log\left((1-\epsilon)e^{i\theta_d}A\left(e^{i\hat{\boldsymbol{\theta}}}\right) + Q\left(e^{i\hat{\boldsymbol{\theta}}}\right) + \frac{B\left(e^{i\hat{\boldsymbol{\theta}}}\right)}{(1-\epsilon)e^{i\theta_d}}\right) \\
&= \log\left(e^{i\theta_d}A\left(e^{i\hat{\boldsymbol{\theta}}}\right) + Q\left(e^{i\hat{\boldsymbol{\theta}}}\right) + e^{-i\theta_d}B\left(e^{i\hat{\boldsymbol{\theta}}}\right)\right) \\
&\qquad\qquad + \epsilon\frac{B\left(e^{i\hat{\boldsymbol{\theta}}}\right) - e^{i\theta_d}A\left(e^{i\hat{\boldsymbol{\theta}}}\right)}{e^{i\theta_d}A\left(e^{i\hat{\boldsymbol{\theta}}}\right) + Q\left(e^{i\hat{\boldsymbol{\theta}}}\right) + e^{-i\theta_d}B\left(e^{i\hat{\boldsymbol{\theta}}}\right)} + O(\epsilon^2) \\
&= \log\left(e^{i\theta_d}A\left(e^{i\hat{\boldsymbol{\theta}}}\right) + Q\left(e^{i\hat{\boldsymbol{\theta}}}\right) + e^{-i\theta_d}B\left(e^{i\hat{\boldsymbol{\theta}}}\right)\right) \\
&\qquad\qquad + \epsilon\left(\frac{B\left(e^{i\hat{\boldsymbol{\theta}}}\right) - A\left(e^{i\hat{\boldsymbol{\theta}}}\right)}{A\left(e^{i\hat{\boldsymbol{\theta}}}\right) + Q\left(e^{i\hat{\boldsymbol{\theta}}}\right) + B\left(e^{i\hat{\boldsymbol{\theta}}}\right)} + O(\theta_d)\right) + O(\epsilon^2) \\
&= \log\overline{S}\left(e^{i\boldsymbol{\theta}}\right) + O(\epsilon\theta_1 + \cdots + \epsilon\theta_d + \epsilon^2).
\end{aligned}
$$

Putting everything together gives

$$
\frac{1}{(2\pi)^d}\int_{\mathbf{z}\in\mathcal{M}} P(\boldsymbol{\theta})Q(\boldsymbol{\theta})e^{n\phi(\boldsymbol{\theta})}d\boldsymbol{\theta} \sim \frac{1}{\pi^d}\int_{\mathbf{z}\in\mathcal{M}} e^{n\log\overline{S}\left(e^{i\boldsymbol{\theta}}\right)}d\boldsymbol{\theta},
$$

since each $|\theta_j| < \delta$ on $\mathcal{M}$ and all of $\epsilon, \delta, \delta^2\epsilon^{-1}, n\epsilon\delta, n\epsilon^2 \to 0$ go to zero as $n \to \infty$ under our assumptions on ϵ and δ. □

Having removed the dependence on ϵ for the integral determining dominant asymptotics, we now determine asymptotic behaviour using a saddle-point analysis.

Proposition 51. *Under our running assumptions on ϵ and δ,*

$$
\frac{1}{\pi^d}\int_{\mathbf{z}\in\mathcal{M}} e^{n\log\overline{S}\left(e^{i\boldsymbol{\theta}}\right)}d\boldsymbol{\theta} \sim S(\mathbf{1})^n n^{-d/2}\frac{S(\mathbf{1})^{d/2}}{\pi^{d/2}(b_1\cdots b_d)^{1/2}},
$$

where we recall from above that $b_k = \sum_{\mathbf{i}\in\mathcal{S}, i_k=1} w_{\mathbf{i}}$ is the total weight of the steps in $\mathcal{S}$ moving forward in the kth coordinate.

Proof. We claim that there is an expansion

$$\log \overline{S}\left(e^{i\boldsymbol{\theta}}\right) = \log S(\mathbf{1}) - \sum_{j=1}^{d} \frac{b_j}{S(\mathbf{1})}\theta_j^2 + O((\theta_1 + \cdots + \theta_d)^3). \tag{3.9}$$

Assuming this claim, for n sufficiently large the domain $\mathcal{M}$ is a small enough neighbourhood of the origin that we can write

$$\begin{aligned}
\int_{\mathcal{M}} e^{n \log \overline{S}\left(e^{i\boldsymbol{\theta}}\right)} d\boldsymbol{\theta} &= S(\mathbf{1})^n \int_{\mathbf{z}\in\mathcal{M}} e^{-n\sum_{j=1}^d \frac{b_j}{S(\mathbf{1})}\theta_j^2 + O(n(\theta_1+\cdots+\theta_d)^3)} d\boldsymbol{\theta} \\
&= S(\mathbf{1})^n \left(\int_{\mathbf{z}\in\mathcal{M}} e^{-n\sum_{j=1}^d \frac{b_j}{S(\mathbf{1})}\theta_j^2} d\boldsymbol{\theta}\right)\left(1 + O(n\delta^3)\right) \\
&\sim S(\mathbf{1})^n \int_{\mathbf{z}\in\mathcal{M}} e^{-n\sum_{j=1}^d \frac{b_j}{S(\mathbf{1})}\theta_j^2} d\boldsymbol{\theta} \\
&= S(\mathbf{1})^n \prod_{j=1}^{d} \int_{-\delta}^{\delta} e^{-n\frac{b_j}{S(\mathbf{1})}\theta_j^2} d\theta_j \\
&\sim S(\mathbf{1})^n \prod_{j=1}^{d} \int_{-\infty}^{\infty} e^{-n\frac{b_j}{S(\mathbf{1})}\theta_j^2} d\theta_j \\
&= S(\mathbf{1})^n n^{-d/2} \frac{\pi^{d/2} S(\mathbf{1})^{d/2}}{(b_1\cdots b_d)^{1/2}}.
\end{aligned}$$

To prove the expansion (3.9) we simply need to compute derivatives and evaluate at the origin. First, direct substitution gives $\log \overline{S}(e^{i\mathbf{0}}) = \log \overline{S}(\mathbf{1}) = \log S(\mathbf{1})$. The fact that all first order derivatives of $\log \overline{S}$ vanish is a reflection of the fact that $(\mathbf{1}, 1/S(\mathbf{1}))$ is a critical point. Explicitly, symmetry over the first $(d-1)$-coordinates implies that for each $1 \leq k \leq d-1$ we can write $\overline{S}(\mathbf{z}) = (z_k + \overline{z}_k)B_k(\mathbf{z}_{\hat{k}}) + Q(\mathbf{z}_{\hat{k}})$ so that

$$\overline{S}_{z_k}(\mathbf{z}) = (1 - z_k^{-2})B_k(\mathbf{z}_{\hat{k}})$$

vanishes at $\mathbf{z} = \mathbf{1}$, and

$$\overline{S}_{z_d}(\mathbf{z}) = A(\widehat{\mathbf{z}}) - z_d^{-2}B(\widehat{\mathbf{z}})$$

also vanishes at $\mathbf{z} = \mathbf{1}$ as $A(\mathbf{1}) = B(\mathbf{1})$. The chain rule then implies that all first order partial derivatives of $(\log \overline{S})(e^{i\boldsymbol{\theta}})$ vanish at the origin. These expressions also imply that all mixed second order partial derivatives $\overline{S}_{z_i z_j}(\mathbf{1})$ for $i \neq j$ vanish, so all the mixed second order partial derivatives of $(\log \overline{S})(e^{i\boldsymbol{\theta}})$ vanish at the origin. Finally, we see that

$$\overline{S}_{z_k z_k}(\mathbf{1}) = 2B_k(\mathbf{1}) = 2b_k$$

for each $1 \le k \le d-1$, and

$$\overline{S}_{z_d z_d}(\mathbf{1}) = 2B(\mathbf{1}) = 2b_d,$$

so that the chain rule gives

$$\left(\frac{\partial^2}{\partial\theta_k^2}\log S(e^{i\boldsymbol{\theta}})\right)(\mathbf{0}) = \frac{\overline{S}_{z_k}(\mathbf{1})^2}{S(\mathbf{1})^2} - \frac{\overline{S}_{z_k}(\mathbf{1})}{S(\mathbf{1})} - \frac{\overline{S}_{z_k z_k}(\mathbf{1})}{S(\mathbf{1})} = -2\frac{b_j}{S(\mathbf{1})}$$

for all $1 \le k \le d$. The claimed expansion (3.9) then holds as the quadratic term in the expansion is half the sum of the second order derivatives. □

Remark 52. Although Proposition 51 only concerns the point $\mathbf{1} \in \Gamma$, there is an expansion

$$\log \overline{S}(w_1 e^{i\theta_1}, \dots, w_d e^{i\theta_d}) = \log S(\mathbf{w}) - \sum_{j=1}^{d} \frac{b_j}{S(\mathbf{1})}\theta_j^2 + O((\theta_1 + \cdots + \theta_d)^3)$$

for all $\mathbf{w} \in \Gamma$. Indeed, for $\mathbf{w}$ to lie in Γ all terms $\mathbf{w}^{\mathbf{i}}$ for $\mathbf{i} \in \mathcal{S}$ must have the same argument, which can be used to show the quadratic term is constant for these expansions (this matters for computing higher-order terms in the asymptotic expansion of s_n).

To prove Theorem 39 it is now sufficient to show that the only point of Γ contributing to dominant asymptotics of s_n is $\mathbf{1}$.

Proposition 53. *Under our running assumptions on ϵ and δ, if $\mathbf{w} \in \Gamma$ and $\mathbf{w} \neq \mathbf{1}$ then*

$$\int_{\mathcal{N}_{\mathbf{w}}} \frac{(1+z_1)\cdots(1+z_{d-1})}{B(\widehat{\mathbf{z}})(z_1 \cdots z_d)} \cdot \frac{B(\widehat{\mathbf{z}}) - z_d^2 A(\widehat{\mathbf{z}})}{1 - z_d} \cdot \overline{S}(\mathbf{z})^n \, d\mathbf{z} = o(S(\mathbf{1})^n n^{-d/2}).$$

Proof. For a general point $\mathbf{w} \in \Gamma$ the integral under consideration can be parameterized by $z_j = w_j e^{i\theta_j}$ for $1 \le j \le d-1$ and $z_d = (1-\epsilon) w_d e^{i\theta_d}$ to yield

$$i^d \int_{\mathcal{M}} P(\boldsymbol{\theta}) Q(\boldsymbol{\theta}) e^{n\phi(\boldsymbol{\theta})} d\boldsymbol{\theta},$$

where

$$P(\boldsymbol{\theta}) = \frac{(1 + w_1 e^{i\theta_1}) \cdots (1 + w_{d-1} e^{i\theta_{d-1}})}{B\left(\widehat{\mathbf{w}} e^{i\hat{\boldsymbol{\theta}}}\right)}$$

$$Q(\boldsymbol{\theta}) = \frac{B\left(\widehat{\mathbf{w}} e^{i\hat{\boldsymbol{\theta}}}\right) - (1-\epsilon)^2 w_d^2 e^{2i\theta_d} A\left(\widehat{\mathbf{w}} e^{i\hat{\boldsymbol{\theta}}}\right)}{1 - (1-\epsilon) w_d e^{i\theta_d}}$$

for $\widehat{\mathbf{w}}e^{i\hat{\boldsymbol{\theta}}} = (w_1 e^{i\theta_1}, \dots, w_{d-1}e^{i\theta_{d-1}})$. Roughly speaking, the theory of ACSV (and saddle-point integrals more generally) shows that higher-order vanishing of P and Q corresponds to slower growing asymptotic behaviour.

First, if $w_d = 1$ and $w_j \neq 1$ for some $1 \leq j \leq d-1$ then P vanishes at the origin, and following the argument used to prove Proposition 50 shows that the contribution of $\mathbf{w}$ is

$$\int_{\mathcal{M}} P(\boldsymbol{\theta})Q(\boldsymbol{\theta})e^{n\phi(\boldsymbol{\theta})}d\boldsymbol{\theta} = o\left(\int_{\mathbf{z}\in\mathcal{M}} e^{n\log \overline{S}\left(e^{i\boldsymbol{\theta}}\right)}d\boldsymbol{\theta}\right) = o\big(S(\mathbf{1})^n n^{-d/2}\big).$$

Similarly, if $w_d = -1$ then $Q(\mathbf{w}) = O(\epsilon)$ as the denominator of Q no longer approaches zero as $\epsilon \to 0$, but its numerator still does, and we can again show the contribution of $\mathbf{w}$ lies in $o(S(\mathbf{1})^n n^{-d/2})$. In fact, because only one factor of H vanishes at $\mathbf{w}$ this asymptotic contribution can be computed using the theory of smooth ACSV (in essence each order of vanishing reduces the possible dominating asymptotics by a factor of n^{-1}; see Melczer [29, Thm. 5.3] for details) to see that it lies in $O(S(\mathbf{1})^n n^{-(d+1)/2})$.

Finally, suppose $w_d = \pm i$. The definition of Γ implies that $w_d A(\mathbf{w})$ and $\overline{w}_d B(\mathbf{w})$ have the same argument for any $\mathbf{w} \in \Gamma$, so when $w_d = \pm i$ it must be the case that $A(\mathbf{w}) = -B(\mathbf{w})$. This can only hold if $w_j = -1$ for at least one $1 \leq j \leq d-1$, so again the numerator G and only one factor of H vanishes at $\mathbf{w}$, and the asymptotic contribution of $\mathbf{w}$ is $O(S(\mathbf{1})^n n^{-(d+1)/2}) = o(S(\mathbf{1})^n n^{-d/2})$. □

3.5 Discussion

Theorem 39 completes the determination of dominant asymptotics for mostly symmetric short step models in an orthant. However, unlike the results for highly symmetric and non-zero drift mostly symmetric models, our arguments for the zero drift mostly symmetric case do not allow one to directly compute higher-order terms in an asymptotic expansion for the number of walks. Carefully tracing through the error terms that appear in the analysis shows that we can satisfy our running assumptions on ϵ and δ and take them arbitrarily close to $n^{-1/2}$, resulting in an error term of the form $O(S(\mathbf{1})^n n^{-\gamma})$ for any $\gamma < 1/2$.

Interestingly, there are models whose counting sequences have asymptotic series in $n^{-1/2}$ instead of series in n^{-1} as usually occurs from ACSV arguments. To compute higher-order contributions, it is necessary to modify the approach of Proposition 50 to account for higher-order terms involving ϵ. Practically, one shifts the dth coordinate off the real axis and approximates a saddle-point-like integral whose amplitude is singular at the saddle points, which can have an asymptotic series expansion in $n^{-1/2}$. We remark that when the amplitude of a saddle-point integral is non-singular, it still has an asymptotic series expansion in $n^{-1/2}$; the odd-degree terms in the amplitude asymptotically do not contribute, and so half the terms in the series expansion have vanishing coefficients (and thus it appears that it is a series in n^{-1}. An extension of this approach is discussed in Chapter 4; we end our discussion here with an example.

Example 54. As described in Example 45, asymptotic behaviour for the number of walks of length n in the quadrant model defined by the step set $\mathcal{S} = \{(0,1),(0,1),(-1,-1),(1,-1)\}$ (whose North step appears twice) is determined, up to a negligible error, by the integral

$$\frac{1}{(2\pi i)^2}\int_{\mathcal{N}(1,1-\epsilon_1)} \frac{1+x}{2x^2y}\cdot\frac{2x-y^2(1+x^2)}{1-y}\overline{S}(x,y)^n dxdy.$$

Restricting the domain of integration to a neighbourhood of the origin in a precise way allows us to take a series expansion and, up to a negligible error, approximate this integral by

$$\frac{1}{(2\pi)^2}\int_{\mathcal{M}+i(0,\epsilon)} \left(4+\frac{is^2}{t}+\frac{O((s+t)^3)}{t}\right)e^{-n\left(s^2/4+t^2/2+O((s+t)^3)\right)}dsdt,$$

where $\mathcal{M}$ is a neighbourhood of the origin in $\mathbb{R}^2$. The integral of the leading term here is the same as above. However some additional work allows us to compute the second-order term and get

$$\begin{aligned}
&\frac{1}{(2\pi)^2}\int_{\mathcal{M}^2+i(0,\epsilon)} \left(4+\frac{is^2}{t}+\frac{O((s+t)^3)}{t}\right)e^{-n\left(s^2/4+t^2/2+O((s+t)^3)\right)}dsdt \\
&= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2+i(0,\epsilon)} \left(4+\frac{is^2}{t}\right)e^{-n\left(s^2/4+t^2/2\right)}dsdt + O\left(\frac{4^n}{n^2}\right) \\
&= \frac{4^n}{n}\left(\frac{2\sqrt{2}}{\pi}+\frac{1}{\sqrt{\pi}}n^{-1/2}+O(n^{-1})\right).
\end{aligned}$$

Moving to three-dimensions, similar arguments show that the number of orthant walks of length n in the mostly symmetric zero drift model with step set

$$\mathcal{S} = \{(\pm1,0,-1),(0,\pm1,-1),(\pm1,\pm1,1)\}$$

is given (up to negligible error) by

$$\frac{8^n}{(2\pi)^3}\int_{\mathbb{R}^3+i(0,0,\epsilon)}\left(8-\frac{is^2}{u}-\frac{it^2}{u}\right)e^{-n(3/8s^2+3/8t^2+u^2/2)}dsdtdu+O\left(\frac{8^n}{n^{5/2}}\right)$$

$$=\frac{8^n}{n^{3/2}}\left(\frac{8\sqrt{2}}{3\pi^{3/2}}-\frac{8}{9\pi\sqrt{n}}+O(n^{-1})\right).$$

Finally, the number of orthant walks of length n in the mostly symmetric zero drift model with step set

$$\mathcal{S}=\{(\pm1,0,1),(0,\pm1,-1)\}$$

is approximated by

$$\frac{4^n}{(2\pi)^3}\int_{\mathbb{R}^3+i(0,0,\epsilon)}\left(8+\frac{2is^2}{u}-\frac{2it^2}{u}\right)e^{-n(s^2/4+t^2/4+u^2/2)}dsdtdu+O\left(\frac{4^n}{n^{5/2}}\right).$$

In this case, the fact that the terms in $\frac{2is^2}{u}-\frac{2it^2}{u}$ are symmetric and appear with opposite signs means that the second-order term in the asymptotic expansion drops out, giving

$$\frac{4^n}{n^{3/2}}\left(\frac{4\sqrt{2}}{\pi^{3/2}}+O(n^{-1})\right)$$

as the dominant asymptotic behaviour. ◁

Chapter 4

Ongoing Work

In this chapter we present some work in progress that extends the study of non-generic directions.

4.1 Fourier-Laplace Integrals

In Chapters 2 and 3 we saw how computing asymptotics of multivariate sequences can often be reduced to studying multivariate *Fourier-Laplace integrals*

$$I(n) = \int_K A(\mathbf{z}) \exp\left[-n\phi(\mathbf{z})\right] d\mathbf{z}, \tag{4.1}$$

where K is a real neighbourhood of the origin, both the **amplitude** A and the **phase** ϕ are analytic, and ϕ has positive definite Hessian and vanishing gradient at the origin. When the relevant critical point is *not* generic (as in Section 3.4) we are unable to reduce to this case, and instead must find asymptotics of *singular* Fourier-Laplace integrals

$$\int_{K+i\boldsymbol{\epsilon}} \frac{A(\mathbf{z})}{\mathbf{z}^{\mathbf{k}}} \exp\left[-n\phi(\mathbf{z})\right] d\mathbf{z}. \tag{4.2}$$

Note that $\mathbf{z}^{\mathbf{k}}$ does not vanish on $K + i\boldsymbol{\epsilon}$ as the domain is moved off of real space.

We now discuss two approaches to computing asymptotics of Fourier-Laplace integrals, before discussing applications and extensions of the methods to singular integrals of the form (4.2) in Sections 4.2 and 4.3.

Fourier-Laplace Integral Asymptotics via the Morse Lemma

In the classical case when the amplitude is non-singular, there is an expansion

$$I(n) = \int_K A(\mathbf{z}) \exp\left[-n\phi(\mathbf{z})\right] = \sum_{\ell \geq 0} c_\ell n^{(-\ell-d)/2} \tag{4.3}$$

where c_ℓ is computable [1, Theorem 5.2, Lemma 5.16]. One way to prove this expansion relies heavily on the *Morse lemma*.

Lemma 55 (Morse Lemma)**.** *If $\phi(\mathbf{x})$ has vanishing gradient and nonsingular Hessian H at the origin then there is a bi-holomorphic change of variables $\mathbf{x} = \psi(\mathbf{y})$ around $\mathbf{x} = \mathbf{y} = \mathbf{0}$ such that $\phi(\psi(\mathbf{y})) = S(\mathbf{y}) = \sum_{j=1}^d y_j^2$.*

The proof of Lemma 55 is inductive, writing

$$\phi(\mathbf{x}) = \sum_{j,k=1}^{d} x_j x_k \phi_{j,k}(\mathbf{x})$$

for some analytic functions $\phi_{j,k}$, then making the substitution

$$y_d = \sqrt{\phi_{d,d}(\mathbf{x})}\left[x_d + \sum_{k>1} \frac{x_k \phi_{d,k}(\mathbf{x})}{\phi_{d,d}(\mathbf{x})}\right]$$

so that

$$\phi(x_1, \cdots, x_d) = y_d^2 + \tilde{\phi}(x_1, \cdots, x_d)$$

for an analytic function $\tilde{\phi}$ whose series terms of degree 2 do not contain x_d. We call this a **Morse substitution**. Defining ψ as the change of variables coming from defining a similar y_j for each j then gives

$$I(n) = C \int A(\psi^{-1}(\mathbf{y})) \exp\left[-nS(\mathbf{y})\right] d\mathbf{y},$$

where C is an explicit constant. The domain of integration is no longer a real neighbourhood of the origin, but it is possible to show that up to some negligible error we can shift our domain back down to a real neighbourhood of the origin. Once the phase is a sum of squares it is easy to find asymptotics for $I(n)$ using Fubini's theorem to separate the integral into a product of d univariate integrals, whose asymptotic behaviour is known.

Fourier-Laplace Integral Asymptotics via Differentiation

An asymptotic expansion for the singular integral (4.2) has been computed[1] in the $d = 1$ case [1, Lemma 6.4]. Let $\epsilon > 0$ and $k \in \mathbb{N}$. If $A(x) = \sum_{j\geq 0} a_j x^j$ and $\phi(x) = \sum_{j\geq 2} b_j x^j$ satisfy some mild conditions then we have the asymptotic expansion

$$\int_{K+i\epsilon} \frac{A(x)}{x^k} \exp\left[-n\phi(x)\right] dx \sim \int_{K+i\epsilon} \frac{a_0}{y^k} \exp[-nb_2 y^2] dy \sim \frac{a_0(-i)^k (nb_2)^{(k-1)/2}\pi}{\Gamma(\frac{k+1}{2})}. \tag{4.4}$$

The proof of the first equivalence is a classical argument showing the asymptotic contributions from higher-order terms of the amplitude and phase are negligible. The method of evaluating of $I(n) = \int_{K+i\epsilon} \frac{1}{y^k} \exp[-ny^2] dy$ is of interest to us.

Differentiating $I(n)$ with respect to n shows that

$$I'(n) = -\int_{K+i\epsilon} \frac{1}{y^{k-2}} \exp[-ny^2] dy.$$

Each time we differentiate I we reduce the degree of the amplitude by 2, and doing this k times yields an amplitude with a non-negative power. The asymptotics of $I^{(k)}(n)$ can thus be found by appealing to Equation (4.3), and integrating the asymptotic k times gives dominant asymptotics for $I(n)$.

Remark that we have to be careful depending on the parity of k. Since

$$\int_{K+i\boldsymbol{\epsilon}} x \exp\left[-nx^2\right] dx \sim 0n^{-1},$$

we manually find the asymptotics of $\int \frac{1}{x} \exp\left[-nx^2\right] dx$ to find a "base case" when k is odd.

4.2 Walks in Weyl Chambers

This section is taken from work by Greenwood, Kroitor, Melczer, and Simon extending work by Greenwood and Simon [22]. A paper with full details is in preparation.

[1] The presentation of this lemma in [1] has several typos that are corrected here.

4.2.1 Preliminaries

In the last chapter we discussed the enumeration of lattice walks restricted to the positive orthant $\mathbb{N}^d$. There are other natural regions to consider; one such class of regions is the collection of *Weyl chambers*, each of which is the intersection of several half-spaces.[2] There are rich results in the enumeration of lattice walks using *reflectable* step sets in Weyl chambers (step sets that remain the same under reflection in any of the defining half-planes of the Weyl chamber), see [20, 21, 17]. In particular, Feierl [18] found asymptotics for the number of walks restricted to the Weyl chamber of A_n when the step set has *zero-drift* (zero vector sum), and Courtiel, Melczer, Mishna, and Raschel [10] found asymptotics of walks using the *weighted Gouyou-Beauchamps* step set restricted to the Weyl chamber $A_1^2 = \mathbb{R}_+^2$.

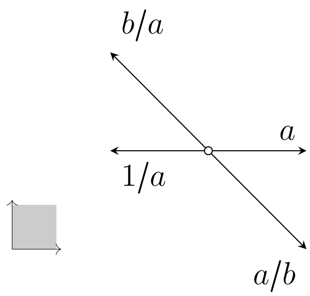


Figure 4.1: *The weighted Gouyou-Beauchamps step set.*

While it is unsurprising that the behaviour of the counting sequences depends on weightings of the steps, it is interesting that the behaviours fall into different *universality classes* where subexponential behaviour is the same as long as the weights stay in a particular region. Notably, the asymptotic behaviour changes on the boundary of two universality classes, where asymptotics transition between two different regimes; models that lie on the boundary are typically called *transitional models*. In Courtiel et al. [10], these universality classes are found for the Gouyou-Beauchamps step set, and the asymptotic behaviour is described for each of them. See Figure 4.2 for a visual representation.

Greenwood and Simon [22] study two similar weighted step sets[3] restricted to the

[2] Each semisimple Lie algebra has an associated *root system* – a collection of real vectors (these associated root systems are used to classify semisimple Lie algebras). There is a classification [23] of irreducible crystallographic root systems into four infinite families A_n, B_n, C_n, D_n, and five sporadic cases E_6, E_7, E_8, F_4, and G_2. Given a root system we are able to define a corresponding *Weyl chamber* as a finite intersection of half-planes corresponding to the roots.

[3] These are the two unique step sets that are invariant under the underlying Weyl group and additionally satisfy an integrality condition (see [22, Def. 2]).

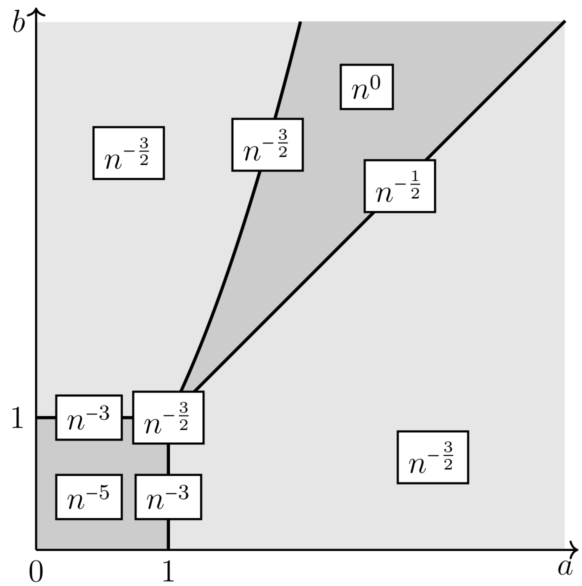


Figure 4.2: *The universality classes for the weighted Gouyou-Beauchamps step-set [10, Fig 2]. The sub-exponential growth for each region is given as a function of a and b.*

Weyl chamber corresponding to A_2. Through a change of variables this is equivalent to finding asymptotics for the number of walks using the (weighted) *Tandem step-set* or the (weighted) *Double Tandem step-set* that are restricted to $\mathbb{N}^2$. They give a similar classification of the asymptotics according to universality class of the weights, but do not find asymptotics in the two transitional cases $a < b = 1$ and $b < a = 1$. In the rest of this section we sketch a method to find asymptotics of these weighted walks, completing the asymptotic classification; see Table 4.1.

Weights	Tandem	Double Tandem
$a < b = 1$	$\frac{27}{8(1-a)^2}\frac{\sqrt{3}}{\sqrt{\pi}}\frac{3^n}{n^{5/2}}$	$\frac{27}{24(1-a)^2}\frac{\sqrt{3}}{\sqrt{\pi}}\frac{6^n}{n^{5/2}}$
$b < a = 1$	$\frac{27}{8(1-b)^2}\frac{\sqrt{3}}{\sqrt{\pi}}\frac{3^n}{n^{5/2}}$	$\frac{27}{24(1-b)^2}\frac{\sqrt{3}}{\sqrt{\pi}}\frac{6^n}{n^{5/2}}$

Table 4.1: *Transitional asymptotics for the Tandem and Double Tandem model.*

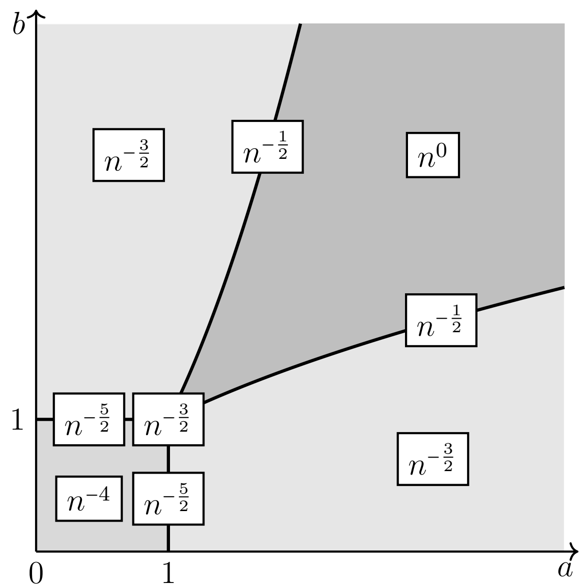


Figure 4.3: *The universality classes for the weighted Tandem and Double Tandem step-sets. The sub-exponential growth is the same for the Tandem and Double Tandem models. The* $n^{-5/2}$ *growths are the two cases* $a < b = 1$ *and* $b < a = 1$ *for which asymptotics are conjectured but not shown by Greenwood and Simon [22].*

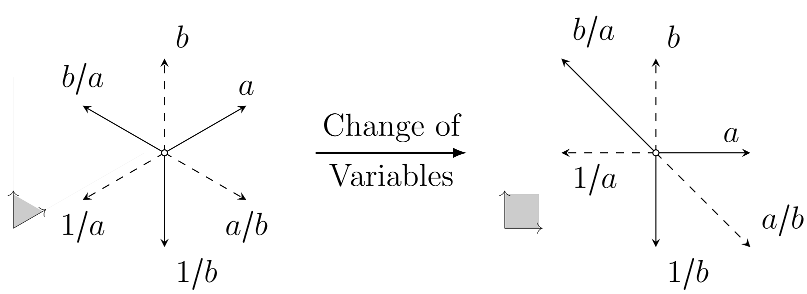


Figure 4.4: *The Tandem step set, along with a shaded copy of* A_2*. Including the dashed arrows gives the Double Tandem step set.*

4.2.2 Comparison to Previous Work

Using the *reflection principle* [20], Greenwood and Simon show that the counting sequence corresponding to restricted walks using the Tandem step-set is the main diagonal of the generating function

$$\frac{(b^2x-ay^2)(bx^2-a^2y)(xy-ab)}{a^2b^2xy(x-1)(y-1)(ab-t(a^2by+b^2x^2+axy^2))}, \tag{4.5}$$

and that the counting sequence corresponding to restricted walks using the Double Tandem step-set is the main diagonal of the generating function

$$\frac{(b^2x-ay^2)(bx^2-a^2y)(xy-ab)}{a^2b^2xy(x-1)(y-1)(ab-t(a^2by+bx^2yb^2x^2+a^2y^2+axy^2+ab^2x))}.$$

In either case, one can compute that there is one minimal critical point that gives dominant asymptotics. In the open cases where $a<b=1$ or $b<a=1$, this critical point is a point where two factors in the denominator vanish. Moreover, this critical point is non-generic (of codimension 1) in both cases, and we are only able to take a single residue, even though the point lies on the vanishing set of two denominator factors.[4] Once we take a residue in t we are left with the task of finding asymptotics of an integral of the form

$$\int_{K+i(\epsilon,0)} \frac{A(x,y)}{x} \exp\left[-n\phi(x,y)\right] dxdy,$$

where A is analytic and K is a neighbourhood of the origin in $\mathbb{R}^2$. This is similar to the situation in the analysis of the generating function for mostly symmetric set steps found in Chapter 3 (Proposition 50), but in that case the leading term of the numerator $A(x,y)$ exactly cancelled with the denominator. This made it possible, after showing that non-leading terms were negligible, to find leading asymptotics through classical results on the asymptotics of Fourier-Laplace integrals. In this new case the leading terms of A do not cancel with the denominator, and we must put in effort to find asymptotics of Fourier-Laplace integrals where the integrand blows up around the critical point being studied.

A further point of difficulty is the fact that the leading series terms of A do not contribute to the dominant asymptotics, due to being odd degree (and thus giving a contribution that has a coefficient of 0). To find asymptotics we must find higher-order asymptotics that come from both the higher-order terms of A and from the higher-order contributions of the leading term of A.

[4]When $a=b=1$ there are now three factors vanishing in the denominator, and a non-generic point of codimension 1 (so we are only able to take 1 residue and are left with two terms in the denominator that vanish) but asymptotics can be found through other means.

4.2.3 Outline of Argument

We present an outline of the methods used to find asymptotics in this particular case. Due to the Tandem model and the Double Tandem model having nearly identical analyses, as well as the two cases $a < b = 1$ and $b < a = 1$ being similar (or identical in the case of the Double Tandem model), we present only a sketch of the proof for the Tandem model when $a < b = 1$.

Consider the generating function

$$F(x,y,t) = \frac{(x - ay^2)(x^2 - a^2y)(xy - a)}{a^2xy(x-1)(y-1)(a - t(a^2y + x^2 + axy^2))},$$

that encodes restricted walks using this Tandem step-set, where $0 < a < 1$. Solving the critical point equations show that this function has three critical points,

$$p_1 = \left(a, 1, \frac{1}{3a}\right),$$
$$p_2 = \left(a\omega^1, \omega^2, \frac{1}{3a}\omega^1\right),$$
$$p_3 = \left(a\omega^2, \omega^1, \frac{1}{3a}\omega^2\right),$$

where $\omega = e^{2\pi i/3}$ is the third root of unity. All of these critical points are minimal and the products of their coordinates have modulus 3.

Let w_n be the number of restricted orthant walks of length n using the Tandem step set. It follows that

$$w_n = \frac{1}{(2\pi i)^3} \int_{\substack{|x|=a-\epsilon_x \\ |y|=1-\epsilon_y \\ |t|=(3a)^{-1}-\epsilon_t}} \frac{(x - ay^2)(x^2 - a^2y)(xy - a)}{a^2xy(x-1)(y-1)(a - t(a^2y + x^2 + axy^2))} \frac{dxdydt}{x^{n+1}y^{n+1}t^{n+1}} \tag{4.6}$$

for sufficiently small $\epsilon_x, \epsilon_y, \epsilon_t > 0$. Since the integrand does not blow up when $a - \epsilon_x \le |x| \le a$, we may write

$$w_n = \frac{1}{(2\pi i)^3} \int_{\substack{|x|=a \\ |y|=1-\epsilon_y \\ |t|=(3a)^{-1}-\epsilon_t}} \frac{(x - ay^2)(x^2 - a^2y)(xy - a)}{a^2xy(x-1)(y-1)(a - t(a^2y + x^2 + axy^2))} \frac{dxdydt}{x^{n+1}y^{n+1}t^{n+1}}. \tag{4.7}$$

This integral can be localized to domains of integration near the three critical points. Near the points with non-real coordinates, all three factors in the numerator vanish but only one factor of the denominator vanishes. The theory of smooth ACSV thus implies that the contributions of these points is $O(n^{-3}3^n)$, so

$$w_n = \frac{1}{(2\pi i)^3}\int_{\substack{(x,y)\in\mathcal{M}_\delta \\ |t|=(3a)^{-1}-\epsilon_t}} \frac{(x-ay^2)(x^2-a^2y)(xy-a)}{a^2xy(x-1)(y-1)(a-t(a^2y+x^2+axy^2))}\frac{dxdydt}{x^{n+1}y^{n+1}t^{n+1}} + O\left(n^{-3}3^n\right)$$

where

$$\mathcal{M}_\delta = \left\{\left(ae^{i\theta_1},(1-\epsilon_y)e^{i\theta_2}\right) : \theta_1,\theta_2 \in (-\delta,\delta)\right\}.$$

Taking a residue at

$$t = \frac{a}{a^2y+x^2+axy^2}$$

then shows that

$$\begin{aligned} w_n &= -\frac{1}{(2\pi i)^2}\int_{\mathcal{M}_\delta}\frac{(x-ay^2)(x^2-a^2y)(xy-a)}{a^3x^2y^2(x-1)(y-1)}\left(\frac{(a^2y+x^2+axy^2)}{axy}\right)^n dxdy + O\left(n^{-3}3^n\right) \\ &= -\frac{1}{(2\pi i)^2}\int_{\mathcal{M}_\delta}\frac{(x-ay^2)(x^2-a^2y)(xy-a)}{a^3x^2y^2(x-1)(y-1)}\exp\left[n\ln\left(\frac{(a^2y+x^2+axy^2)}{axy}\right)\right]dxdy + O\left(n^{-3}3^n\right). \end{aligned}$$

Making a change of variables dividing x by a gives

$$w_n = -\frac{1}{(2\pi i)^2}\int_{\mathcal{N}_\delta}\frac{A(x,y)}{xy(y-1)}\exp\left[-n\phi(x,y)\right]dxdy + O\left(n^{-3}3^n\right),$$

where

$$A(x,y) = \frac{(x-y^2)(x^2-y)(xy-1)}{xy(ax-1)},$$

$$\phi(x,y) = -\ln\left(\frac{y+x^2+xy^2}{xy}\right),$$

and

$$\mathcal{N}_\delta = \{\left(e^{i\theta_1},(1-\epsilon_y)e^{i\theta_2}\right) : \theta_1,\theta_2\in(-\delta,\delta)\}.$$

Making another change of variables $x = e^{i\theta_x}$ and $y = (1-\epsilon_y)e^{i\theta_y}$ then yields

$$w_n = -\frac{1}{(2\pi)^2}\int_{x,y\in(-\delta,\delta)}\frac{A\left(e^{i\theta_x},(1-\epsilon_y)e^{i\theta_y}\right)}{(1-\epsilon_y)e^{i\theta_y}-1}\exp\left[-n\phi\left(e^{i\theta_x},(1-\epsilon_y)e^{i\theta_y}\right)\right]d\theta_xd\theta_y + O\left(n^{-3}3^n\right).$$

We can perform a (modified) *Morse substitution* ψ, which in this situation takes the form

$$\begin{aligned}
s &= \frac{\sqrt{3}}{3}\theta_x - \frac{\sqrt{3}}{6}\theta_y - \frac{\sqrt{3}}{12}i\theta_x\theta_y + \frac{\sqrt{3}}{24}i\theta_y^2 + O((\theta_x+\theta_y)^3),\\
t &= -\frac{1}{2}\theta_y - \frac{1}{24}i\theta_y^2 + O((\theta_x+\theta_y)^3).
\end{aligned}$$

Applying this change of variables, it remains to be shown that we are able to shift the domain of integration back onto the real axis in a suitable way. Assuming that this is possible yields

$$w_n = -\frac{3^n}{(2\pi)^2}\frac{2\sqrt{3}}{1}\int_{\substack{s\in(-\delta,\delta)\\ t\in(-\delta,\delta)+i\epsilon}} \frac{\tilde{A}(s,t)}{B(s,t)}\exp\left[-n(s^2+t^2)\right]dsdt + O\left(n^{-3}3^n\right),$$

where

$$\begin{aligned}
\tilde{A}(s,t) &= -\frac{6i\sqrt{3}(s^3-3st^2)}{(a-1)} + \frac{3(-6s^4+\sqrt{3}(9-7a)s^3t+18s^2t^2+\sqrt{3}(13a-19)st^3)}{(a-1)^2} + \cdots\\
B(s,t) &= t\left(-2i - \frac{5}{3}t + \frac{7i}{36}s^2 + \frac{38i}{36}t^2 + \cdots\right),
\end{aligned}$$

so that

$$\frac{1}{B(s,t)} = \frac{\frac{i}{2} - \frac{5}{12}t + \frac{7i}{144}s^2 - \frac{4}{48}t^2 + \cdots}{t},$$

and thus

$$\frac{\tilde{A}(s,t)}{B(s,t)} = \frac{1}{t}\left(\frac{3\sqrt{3}s^3 - 9\sqrt{3}st^2}{a-1} + \frac{9s^4+\sqrt{3}(8a-11)s^3t - 27s^2t^2 - 3\sqrt{3}(4a-7)st^3}{i(a-1)^2} + \cdots\right).$$

The lowest order term in this expansion that does not vanish when integrated is

$$\frac{9}{i(a-1)^2}\cdot\frac{s^4}{t},$$

implying that

$$\begin{aligned}
w_n &= \frac{3^n}{4\pi^2}\frac{2\sqrt{3}}{-i}\frac{9}{(a-1)^2}\int_{\substack{s\in(-\delta,\delta)\\ t\in(-\delta,\delta)+i\epsilon}} \frac{s^4}{t}\exp\left[-n(s^2+t^2)\right]dsdt + O\left(n^{-3}3^n\right)\\
&= \frac{3^n}{4\pi^2}\frac{2\sqrt{3}}{-i}\frac{9}{(a-1)^2}\left(\int_{s\in(-\delta,\delta)} s^4\exp\left[-ns^2\right]ds\right)\left(\int_{t\in(-\delta,\delta)+i\epsilon}\frac{1}{t}\exp\left[-nt^2\right]dt\right) + O\left(n^{-3}3^n\right)\\
&= \frac{27}{8(a-1)^2}\frac{\sqrt{3}}{\sqrt{\pi}}3^n n^{-5/2} + O\left(n^{-3}3^n\right).
\end{aligned}$$

The evaluation of $\int s^4 \exp\left[-ns^2\right] ds$ is known, as it is the fourth moment of the Gaussian distribution, while the evaluation of $\int t^{-1} \exp\left[-nt^2\right] dt$ comes from Equation (4.4) above.

By applying similar arguments we find the results in Table 4.1, which agree with the asymptotics conjectured in Greenwood and Simon [22].

4.3 Higher-Dimensional Singular Fourier-Laplace Integrals

Recall from Section 4.1 the two discussed methods of evaluating Fourier-Laplace integrals, an application of the Morse lemma and differentiating under the integral, and let

$$I(n) = \int_{K+i\epsilon} \frac{A(\mathbf{z})}{\mathbf{z}^{\mathbf{k}}} \exp\left[-n\phi(\mathbf{z})\right] d\mathbf{z}.$$

Applying the Morse lemma to $I(n)$ is no longer feasible, as we are not able to control the amplitude $A(\psi^{-1}(\mathbf{z}))(\psi^{-1}(\mathbf{z}))^{-k}$. We may also not able to differentiate $I(n)$ with respect to n to remove all negative powers. For instance, consider

$$I(n) = \int \frac{1}{x} \exp[-n(x^2 + xy + y^2)] dx dy.$$

Differentiating with respect to n gives

$$I'(n) = -\int \left(x + y + \frac{y^2}{x}\right) \exp[-n(x^2 + xy + y^2)] dx dy$$

and, as we continue to differentiate, we will always have a term of the form y^{2k}/x in the amplitude.

In order to find general asymptotics we combine both ideas, differentiating under the integral until we are able to perform a single substitution step of the Morse lemma. Here we sketch the method in the special case where $\phi(\mathbf{z})$ is a sum of squares. For the remainder of this section we assume that H is a symmetric positive definite real matrix, and define $H(\mathbf{z}) = \mathbf{z}^T H \mathbf{z}$. Furthermore, for any integer vector $\mathbf{p} \in \mathbb{Z}^d$ we let $|\mathbf{p}| = \sum_{j=1}^d p_j$ (note that this can be negative).

Theorem 56. *Let $k_1, \cdots, k_d \in \mathbb{Z}$. Then*

$$I(n) = \int_{K+i\epsilon} z_1^{k_1} \cdots z_d^{k_d} \exp[-nH(\mathbf{z})] d\mathbf{z} = cn^{(-|\mathbf{k}|-d)/2} \log^{\nu} n + o(n^{(-|\mathbf{k}|-d)/2} \log^{\nu} n), \tag{4.8}$$

where $c \in \mathbb{C}$ is computable (possibly zero) and $\nu \le d$ is a computable natural number.

Proof Sketch. We use induction on the number of dimensions d. For $d = 1$ this follows from

$$\int_{K+i\epsilon} z^k \exp[-naz^2]dz \sim cn^{(-k-1)/2} + o(n^{(-k-1)/2})$$

for some computable constant c, which is the content of Equation (4.4).

We now move to the inductive step. If $k_j < 0$ for all $1 \leq j \leq d$ then we differentiate with respect to n. Each time we differentiate the amplitude becomes a sum of monomials, each of degree 2 higher than the original amplitude. Thus, differentiating enough times implies that $k_j \geq 0$ for some j.

We may assume, without loss of generality, that we eventually get $k_1 \geq 0$, and write

$$H(\mathbf{z}) = \sum_{1 \leq a, b \leq d} h_{ab} z_a z_b.$$

Defining the Morse substitution

$$y_1 = \sqrt{h_{11}} \left[z_1 + \frac{h_{12} z_2}{h_{11}} + \cdots + \frac{h_{1d} z_d}{h_{11}} \right],$$

we calculate that

$$H(z_1, \ldots, z_d) = y_1^2 + \tilde{H}(z_2, \ldots, z_d),$$

for some real symmetric positive definite matrix $\tilde{H}$, and

$$I(n) = \frac{1}{\sqrt{h_{11}}} \int R(y_1, z_2, \ldots, z_d) \exp\left[-n(y_1^2 + \tilde{H}(z_2, \cdots, z_d))\right] dy_1 dz_2 \cdots dz_d$$

for some Laurent polynomial R. Notably, we are able to break up the resulting integral to take out all the terms with y_1. This reduces the dimension of the integral we need to consider by 1. By tracking when we take derivatives, we can calculate the final asymptotics – the log factors come from integrating n^{-1}. $\square$

It is not so straightforward to go from purely quadratic phase to a general phase $\phi(\mathbf{z})$. Trying to use the same technique for general phase falls apart when we try to apply a Morse transformation

$$\phi(z_1, \cdots, z_d) \to y_1^2 + \tilde{\phi}(z_1, \cdots, z_d).$$

In the case that ϕ is purely quadratic then $\tilde{\phi}$ is in fact a function of only $z_2, \cdots, z_d$, and so we may split apart the integral to reduce the number of variables. For general ϕ this does not happen.

For our purposes this may seem quite constricting, as the phases of Fourier-Laplace integrals showing up in ACSV are almost never purely quadratic. However the dominant asymptotics of a (non-singular) Fourier-Laplace integral depends only on the terms of lowest order in both the amplitude and the phase. It is reasonable to expect that, with some extra analytic bounding arguments, one can write

$$\begin{aligned} I(n) &= \int_{K+i\boldsymbol{\epsilon}} \mathbf{z}^{\mathbf{k}} \exp\left[-n\phi(\mathbf{z})\right] d\mathbf{z} \\ &\sim \int_{K+i\boldsymbol{\epsilon}} \mathbf{z}^{\mathbf{k}} \exp\left[-nH(\mathbf{z})\right] d\mathbf{z} \\ &= cn^{(-|k|-d)/2} \log^{\nu} n + o(n^{(-|k|-d)/2} \log^{\nu} n). \end{aligned}$$

Once again we emphasize that the coefficient c, while computable, can be 0 (as in Section 4.2). In this case it is necessary to compute higher-order terms to find dominant asymptotics.

Example 57. Consider

$$I(n) = \int_{K+i(\epsilon,\epsilon)} \frac{1}{x^2y^2} \exp\left[-n\left(x^2+xy+y^2\right)\right] dxdy.$$

Since x and y have negative powers, we differentiate twice with respect to n to get

$$I''(n) = \int_{K+i(\epsilon,\epsilon)} \left(\frac{x^2}{y^2} + \frac{y^2}{x^2} + \frac{2x}{y} + \frac{2y}{x} + 3\right) \exp\left[-n\left(x^2+xy+y^2\right)\right] dxdy.$$

Applying the appropriate substitution $s = x + 1/2y$ or $t = y + 1/2x$, depending on the term, we find that

$$I''(n) \sim -\frac{1}{4}\frac{\pi}{n\sqrt{3}} - \frac{1}{4}\frac{\pi}{n\sqrt{3}} - 2\frac{\pi}{n\sqrt{3}} - 2\frac{\pi}{n\sqrt{3}} + 6\frac{\pi}{n\sqrt{3}} = \frac{\sqrt{3}}{2}\frac{\pi}{n}.$$

Integrating implies that

$$I(n) \sim \frac{\sqrt{3}}{2}\pi n \log n,$$

and we note that $(-(2+2)-d)/2 = (4-2)/2 = 1$. $\triangleleft$

4.3.1 Computing ν

It is somewhat dissatisfying to determine the log power ν in Theorem 56 by manually tracking where we pick up log factors. We now describe a streamlined procedure to compute ν when $\mathbf{k} \in \mathbb{N}^d$.

First, we note that making the Morse substitution of $H(\mathbf{z})$ with respect to the variable z_k gives $H(\mathbf{z}) = y_k^2 + \mathbf{z}_{\hat{k}} H_k \mathbf{z}_{\hat{k}}^T$, where H_k is the matrix with entries

$$H_k(i,j) = h_{ij} - \frac{h_{ki}h_{kj}}{h_{kk}}$$

with its kth row and kth column removed, and $\mathbf{z}_{\hat{k}} = \begin{bmatrix} z_1 & \cdots & z_{k-1} & z_{k+1} & \cdots & z_d \end{bmatrix}$.

Definition 58. We say that the matrix H **admits** the step $-e_i - e_j$ if the entry $H(i,j) \neq 0$. Any step that is admitted by H is H**-admissible**.

Definition 59. We say that the matrix H **admits with respect to** k the **Morse step** $+e_k - e_j$ if $H(k,j) \neq 0$. The set of all Morse steps that H admits with respect to k is called the set of H**-admissible Morse steps with respect to** k.

Each admissible step corresponds to a monomial that will be pulled into the amplitude when differentiating, and each admissible Morse step with respect to k corresponds to a term that will appear in the amplitude when making a Morse substitution with respect to z_k.

Now, fix an integer vector $\mathbf{k} \in \mathbb{N}^d$ and a symmetric matrix $H \in M_{d\times d}$. We walk according to the following procedure.

1. Starting at $\mathbf{k}$, take H-admissible steps until some coordinate is non-positive. Denote this path $\mathcal{P}$, and let $\mathbf{u} = \mathbf{k} + \mathcal{P}$ be the ending coordinate of this walk. If, during this walking process, we land on the plane defined by $|\mathbf{z}| = d - 2$ then set $\ell_H^d(\mathcal{P}) = 1$, otherwise we let $\ell_H^d(\mathcal{P}) = 0$.

2. Once there is a coordinate u_j of $\mathbf{u}$ such that $u_j \leq 0$, take $-u_j$ H-admissible Morse steps with respect to j to continue the path $\mathcal{P}$, noting that this ensures that we are now on the plane V consisting of integer points that have j-th coordinate 0. Let $\mathbf{v} = \mathbf{k} + \mathcal{P}$ be the result at the end of these $-u_j$ Morse steps.

3. Continue walking from stage 1 with dimension one lower, our symmetric matrix H now being H_j, and constrained to the plane V. If V is one dimensional, stop the process.

In stage 1, each H-admissible step corresponds to some term in the phase, and taking an H-admissible step corresponds to differentiating and choosing a term pulled out from the phase. Each H-admissible Morse step in stage 2 corresponds to some term that appears when we make a Morse substitution with respect to a variable with positive degree, and taking a step corresponds to picking one such term. We take $-u_j \geq 0$ steps since it is the exponent of z_j, the variable that we are performing the substitution in. Finally, when we arrive on the plane V after performing H-Morse steps, this corresponds to having an integral that can be separated. Pulling it out yields a similar walk in one fewer dimension.

Since we are always able to reduce the dimension of the plane that we are on by 1 with enough steps, this is guaranteed to give a walk $\mathcal{P}$ from $\mathbf{k}$ to the origin. We call such a walk **admissible** with respect to H and let

$$\ell_H(\mathcal{P}) = \ell_H^d(\mathcal{P}) + \ell_H^{d-1}(\mathcal{P}) + \cdots + \ell_H^1(\mathcal{P}) \leq d.$$

Then, in Equation (4.8),

$$\nu = \max_{\mathcal{P}} \ell_H(p)$$

where the maximum is being taken over all walks from $-\mathbf{k}$ to $\mathbf{0}$ that are admissible with respect to H. Each different admissible path corresponds to a term found in the evaluation of $I(n)$, and $\ell_H(\mathcal{P})$ corresponds to the number of times we take a derivative and have a polynomial growth of n^{-1}, giving an extra factor of log when we integrate back up.

We illustrate this with an example.

Example 60. Consider

$$I(n) = \int \frac{1}{x^2 y^2} \exp\left[-n(x^2 + xy + y^2)\right] dxdy.$$

One can calculate that

$$H = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}$$
$$H_1 = H_2 = \begin{bmatrix} 3/4 \end{bmatrix},$$

so the H-admissible steps are $\{(-2, 0), (-1, -1), (0, -2)\}$, because each element of H is non-zero, and both the H_1 and the H_2 admissible steps are $\{(-2)\}$. The only H-admissible Morse step with respect to 1 is $\{(1, -1)\}$ and the only H-admissible Morse step with respect to 2 is $\{(-1,1)\}$. Here $-\mathbf{k} = (2,2)$. The list of all admissible walks is:

$$\begin{aligned}
&\mathcal{P}_1 : (2,2) \to (0,2) \hookrightarrow (2),\\
&\mathcal{P}_2 : (2,2) \to (2,0) \hookrightarrow (2),\\
&\mathcal{P}_3 : (2,2) \to (1,1) \to \underline{(-1,1)} \dashrightarrow (0,0) \hookrightarrow (0),\\
&\mathcal{P}_4 : (2,2) \to (1,1) \to \underline{(1,-1)} \dashrightarrow (0,0) \hookrightarrow (0),\\
&\mathcal{P}_5 : (2,2) \to (1,1) \to \underline{(0,0)} \hookrightarrow (0).
\end{aligned}$$

Here $\to$ indicates a normal step, $\dashrightarrow$ indicates a Morse step, and $\hookrightarrow$ indicates embedding to a plane. The underlined tuples are those that are stepped towards by a normal step (and not a Morse step) that satisfy $|\mathbf{z}| = d-2$. We can see that

$$\nu = \max_{\mathcal{P}} \ell_H(\mathcal{P}) = 1.$$

Indeed, we found in Example 57 that the actual asymptotics are $\frac{\sqrt{3}}{2}\pi n \log n$. $\triangleleft$

4.4 Conclusion

Given a multivariate power series and a direction vector, ACSV gives a powerful framework to find asymptotics of the corresponding sequence by showing that calculating asymptotics is equivalent to finding asymptotics of a Fourier-Laplace type integral. The asymptotics of such Fourier-Laplace integrals are known in many nice cases, but are not known in full generality when the direction vector picked is non-generic. When this is the case the amplitude of the Fourier-Laplace integral is singular, and finding asymptotics is more ad-hoc.

Such non-generic directions arise in the enumeration of restricted lattice walks whose step sets satisfy some symmetry conditions, particularly in 'transitional' cases. In thesis we gave an overview of ACSV before finding asymptotics for the number of walks in an orthant with a so-called 'mostly symmetric zero-drift' step set. A similar problem – finding transitional asymptotics for the number of lattice walks restricted to a Weyl chamber – was discussed. Finally we outlined some generalizations for the asymptotics of singular Fourier-Laplace integrals.

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