

Measurements of quantum fields with particle detectors

by

José Polo Gómez

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Examining Committee Membership

The following served on the Examining Committee for this thesis. The decision of the Examining Committee is by majority vote.

External Examiner: Kasia Rejzner
 Dept. of Mathematics, University of York

Supervisor: Eduardo Martín-Martínez
 Dept. of Applied Mathematics, University of Waterloo

Internal Members: Graeme Smith
 Dept. of Applied Mathematics, University of Waterloo

 Robert B. Mann (cross-appointed)
 Department of Physics and Astronomy, University of Waterloo

Internal-External Member: Doreen Fraser
 Dept. of Philosophy, University of Waterloo

Author's Declaration

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of Contributions

The content of this thesis is based on the following published works:

- [1] **J. Polo-Gómez**, L. J. Garay, and E. Martín-Martínez, “A detector-based measurement theory for quantum field theory”, [Phys. Rev. D **105**, 065003 \(2022\)](#);
- [2] **J. Polo-Gómez** and E. Martín-Martínez, “Nonperturbative method for particle detectors with continuous interactions”, [Phys. Rev. D **109**, 045014 \(2024\)](#);
- [3] D. Grimmer, I. Melgarejo-Lermas, **J. Polo-Gómez**, and E. Martín-Martínez, “Decoding quantum field theory with machine learning”, [J. High Energy Phys. **2023**, 31 \(2023\)](#);
- [4] H. Maeso-García, **J. Polo-Gómez**, and E. Martín-Martínez, “Entanglement harvesting: State dependence and covariance”, [Phys. Rev. D **106**, 105001 \(2022\)](#);
- [5] H. Maeso-García, **J. Polo-Gómez**, and E. Martín-Martínez, “How measuring a quantum field affects entanglement harvesting”, [Phys. Rev. D **107**, 045011 \(2023\)](#);

The results of [1] appear in Secs. 5.1 and 5.4. I was responsible for the calculations and the first draft of this paper. The final version is a collaborative effort by all authors.

The results of [2] appear in Ch. 6. I was responsible for the calculations and the first draft of this paper. The final version is a collaborative effort by all authors.

The results of [3] appear in Ch. 7. The first draft of this paper was written by D. Grimmer. I was responsible for the final version of the lattice QFT calculations and the final execution of the machine learning algorithms. The final version of the paper is a collaborative effort by all authors.

The results of [4] and [5] appear in Ch. 8. The calculations and the first draft of these papers were a collaborative effort between H. Maeso-García and myself. The final version of the paper is a collaborative effort by all authors.

I am fully responsible for the content and contributions of Chs. 3, 4, and 5, which were originally written for this thesis (except for Sec. 5.4, see above) and constitute the first versions of the following forthcoming works, respectively:

- [6] **J. Polo-Gómez**, “Localized causal channels in real scalar QFT revisited”, (forthcoming);

- [7] **J. Polo-Gómez**, T. R. Perche, and E. Martín-Martínez, “Useful qubits in relativistic quantum information”, (forthcoming);
- [8] **J. Polo-Gómez**, L. J. Garay, and E. Martín-Martínez, “A detector-based measurement theory for QFT in curved spacetimes”, (forthcoming).

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- [9] T. R. Perche, **J. Polo-Gómez**, B. d. S. L. Torres, and E. Martín-Martínez, “Particle detectors from localized quantum field theories”, [Phys. Rev. D **109**, 045013 \(2024\)](#).

Other works published during my PhD but not included in this thesis are:

- [10] B. d. S. L. Torres, K. Wurtz, **J. Polo-Gómez**, and E. Martín-Martínez, “Entanglement structure of quantum fields through local probes”, [J. High Energy Phys. **2023**, 58 \(2023\)](#);
- [11] **J. Polo-Gómez**, “Thermodynamic bound on quantum state discrimination”, [Phys. Rev. E **109**, 014119 \(2024\)](#);
- [12] T. R. Perche, **J. Polo-Gómez**, B. d. S. L. Torres, and E. Martín-Martínez, “Fully relativistic entanglement harvesting”, [Phys. Rev. D **109**, 045018 \(2024\)](#);
- [13] Y. Li, **J. Polo-Gómez**, and E. Martín-Martínez, “A proposal to characterize and quantify superoscillations”, [J. Phys. A Math. Theor. **57**, 135202 \(2024\)](#);
- [14] Á. Álvarez-Domínguez, L. J. Garay, E. Martín-Martínez, and **J. Polo-Gómez**, “No Black Holes from Light”, [Phys. Rev. Lett. **133**, 041401 \(2024\)](#);
- [15] Á. Álvarez-Domínguez, L. J. Garay, E. Martín-Martínez, and **J. Polo-Gómez**, “Reply to Comment on “No Black Holes from Light” [arXiv:2408.06714]”, [arXiv:2408.11097 \[gr-qc\]](#);
- [16] I. Agullo, B. Bonga, E. Martín-Martínez, S. Nadal-Gisbert, T. R. Perche, **J. Polo-Gómez**, P. Ribes-Metidieri, and B. de S. L. Torres, “Multimode nature of spacetime entanglement in QFT”, [Phys. Rev. D **111**, 085013 \(2025\)](#);
- [17] J. Arrechea, C. Barceló, G. García-Moreno, and **J. Polo-Gómez**, “Beyond Buchdahl’s limit: bilayered stars and thin-shell configurations”, [arXiv:2411.14018 \[gr-qc\]](#).

Abstract

This thesis aims to provide a working measurement theory for quantum fields built upon measurements using localized non-relativistic quantum systems, widely known as particle detectors.

In Chs. 3 to 5, we focus on using detector-based schemes to carefully formulate a measurement theory that is compatible with relativity. To do so, we provide a rigorous analysis of the causal behaviour of the induced non-selective channels, as well as an update rule that is consistent with relativistic causality. In the process, we establish a body of results, including a characterization of localized causal channels in real scalar QFT, and the formulation of a formalism that allows a consistent treatment of non-relativistic multipartite systems in relativistic setups.

In Chs. 6 to 8, we focus on verifying that the measurement theory formulated in the first part is a *working* measurement theory, meaning that it can actually be used in practice to measure specific targeted features of the quantum field. To ensure this, we introduce a measurement strategy where particle detectors always undergo the same measurement protocol—regardless of the quantity or feature of interest—and then are subjected to a tomography process. The resulting data is fed into a trained neural network that is capable of inferring the targeted quantity or feature from the detector’s readings. This strategy has the potential to be applicable to measure any quantity of the field that is accessible through local measurements. More tangentially, we also introduce a method beyond perturbation theory that uses trains of delta couplings to efficiently approximate the final state of a detector undergoing a continuous interaction. Additionally, we apply the detector-based measurement theory to analyze the effect of measurements on the protocol of entanglement harvesting.

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Dedication

A todos mis maestros, y en especial a los primeros que tuve: mis padres.

To all my teachers, and especially to the first ones I had: my parents.

Table of Contents

Examining Committee	ii
Author’s Declaration	iii
Statement of Contributions	iv
Abstract	vi
Acknowledgements	vii
Dedication	ix
List of Figures	xv
List of Tables	xxi
Preamble	xxii
Overture	1
1 Introduction	2
2 Background	7
2.1 Quantum theory of measurement	7

2.2	Quantum theory of a free real scalar field	12
2.3	Particle detector models	24
Act 1: Measurements and causality		30
3	Causal maps in quantum field theory	31
3.1	Sorkin’s impossible measurements	31
3.2	Localized causal maps	34
3.3	Examples	46
3.3.1	Unitary channels generated by a smeared field	47
3.3.2	Spectral decomposition of the smeared field	48
3.3.3	Fewster-Verch framework	49
3.4	Non-trace-preserving operations	50
4	Causal update rules in relativistic quantum mechanics	52
4.1	Update rules in relativistic quantum mechanics	52
4.2	Updating a Bell pair	53
4.3	A relativistic formalism for multipartite systems	58
4.3.1	Statistical interpretation	62
4.4	Observer-dependent framework	63
4.5	Covariance and conservation of charges	64
4.6	Summary	65
5	A detector-based measurement theory for QFT	67
5.1	UDW-based measurement schemes	67
5.2	Causal behaviour of non-selective measurements	72
5.3	Update rule	74
5.4	A measurement theory for QFT	77

Intermezzo: Particle detectors beyond perturbation theory 79

6 Non-perturbative method for particle detectors with continuous interactions	80
6.1 Perturbative v. Non-perturbative	80
6.2 Detectors undergoing multiple delta couplings	81
6.2.1 Single detector case	82
6.2.2 Multiple detectors case	83
6.3 Approximation via multiple delta couplings	84
6.3.1 Single detector case	84
6.3.2 Multiple detector case	86
6.4 Convergence and efficiency of the approximation	88
6.5 Examples	95
6.5.1 Single detector example	96
6.5.2 Two detectors example	101
6.6 Summary	106

Act 2: Measuring quantum fields with particle detectors 109

7 Extracting information from quantum fields with machine learning	110
7.1 From probing to measuring	111
7.2 Measurement protocol and data processing	113
7.2.1 Simple fixed measurement procedure	114
7.2.2 Data preprocessing	116
7.2.3 Neural network training	117
7.3 Two methods for simulating data generation	118
7.3.1 Method 1: Lattice approximation	119
7.3.2 Method 2: Non-Gaussian Wigner functions	122
7.4 Examples	124

7.4.1	Remote boundary sensing	124
7.4.2	Thermometry	129
7.4.3	Discrimination between Fock and coherent states	130
7.5	Summary	135
8	Entanglement harvesting and measurements	137
8.1	A witness of the entanglement structure	138
8.2	Entanglement harvesting without measurements	139
8.2.1	Low mixedness states	142
8.3	Entanglement harvesting with measurements	143
8.3.1	Non-orthogonal regime	148
8.3.2	Orthogonal regime	149
8.3.3	Transition regime	150
8.4	Case study: Inertial pointlike detectors	151
8.4.1	Orthogonal regime	153
8.4.2	Transition regime	158
8.5	Summary	161
	Finale	164
9	Summary and future directions	165
	References	168
	Appendices	188
A	The n-dimensional Riemann-Stieltjes integral	189

B	Generation and processing of simulated data	195
B.1	Preprocessing, Neural Network Architecture and Training Details	195
B.2	Total Variation and Hellinger Distances for Binary Boundary Classification	197
B.3	Moments of non-Gaussian states	200
B.3.1	The effect of Gaussian unitaries on non-Gaussian states	200
B.3.2	Two delta pulse interaction	203
B.4	Moments of Fock and Phase-averaged states	205
B.4.1	Second moments	206
B.4.2	Fourth moments	207
B.4.3	Eighth moments	208
C	Real scalar quantum field theory on the lattice	212
D	Setups with multiple detectors	219
D.1	General final state of two initially uncorrelated detectors	219
D.2	State of two detectors after a measurement performed on a third	221
D.3	Negativity	224

List of Figures

3.1	Sorkin's setup: Projective measurements may allow faster-than-light signalling.	32
3.2	Localized projective measurements may still allow faster-than-light signalling.	33
4.1	Spacetime diagram showing the worldlines of A and B, along with the proper time τ_A^* at which the measurement on A is performed. The diagram also indicates the proper times τ_B^- and τ_B^+ , which correspond to the intersections of B's worldline with the past and future light cones of the measurement, respectively. An analogous construction for some $\tau_B^* \in (\tau_B^-, \tau_B^+)$ is also sketched.	55
5.1	(1+1) diagram representing in a slice of spacetime the interaction region D and the hypersurface P when the projective measurement is performed on the detector. In blue, the causal future of the projective measurement; in pale blue, the causal future of the measurement that is not already in the future of the projective measurement, $\mathcal{J}^+(D) \setminus \mathcal{J}^+(P)$	76
6.1	a) Excitation probability of a two-level UDW detector coupled to a quantum scalar field through a train of delta couplings approximating a Heaviside switching, as a function of the number of couplings N . Here we set the relevant parameters γ and s to 1. The constant red dashed line marks the value of the excitation probability for the exact Heaviside switching. b) Logarithmic relative error of the excitation probability for the Heaviside switching approximated by delta couplings, as a function of the number of couplings N . The function represented by the solid cyan line is proportional to $1/N^2$, marking the rate of convergence of the approximation.	99

6.2	a) Excitation probability of a two-level UDW detector coupled to a quantum scalar field through a train of delta couplings approximating a Gaussian switching truncated at $q = 1$ variances, as a function of the number of couplings N . Here we set the relevant parameters γ and s to 1. The constant red dashed line marks the value of the excitation probability for the exact truncated Gaussian switching. b) Logarithmic relative error of the excitation probability for the Gaussian switching approximated by delta couplings, as a function of the number of couplings N . The function represented by the solid cyan line is proportional to $1/N^2$, marking the rate of convergence of the approximation.	100
6.3	a) Excitation probability of a two-level UDW detector coupled to a quantum scalar field through a train of delta couplings approximating a bump switching, as a function of the number of couplings N . Here we set the relevant parameters γ and s to 1. The constant red dashed line marks the value of the excitation probability for the exact bump switching. b) Logarithmic relative error of the excitation probability for the bump switching approximated by delta couplings, as a function of the number of couplings N . The function represented by the solid cyan line is proportional to $1/N^{5.6}$, upper bounding the rate of convergence of the approximation.	102
6.4	a) $\mathcal{L}_{AA} = \mathcal{L}_{BB} \equiv \mathcal{L}_{ii}$ term for a pair of identical two-level UDW detectors coupled to a quantum scalar field in spacelike-separated regions through a train of delta couplings approximating a Heaviside switching, as a function of the number of couplings N . Here we set the relevant parameters to $\gamma = 1$, $d = 1.2$, and $r = 0.1$. The constant red dashed line marks the value of the $\mathcal{L}_{ii}$ term for the exact Heaviside switching. b) Logarithmic relative error of the $\mathcal{L}_{ii}$ term for the Heaviside switching approximated by delta couplings, as a function of the number of couplings N . The function represented by the solid cyan line is proportional to $1/N^{2.8}$, upper bounding the rate of convergence of the approximation.	105

6.5	a) $\mathcal{L}_{AB}$ term for a pair of identical two-level UDW detectors coupled to a quantum scalar field in spacelike-separated regions through a train of delta couplings approximating a Heaviside switching, as a function of the number of couplings N . Here we set the relevant parameters to $\gamma = 1$, $d = 1.2$, and $r = 0.1$. The constant red dashed line marks the value of the $\mathcal{L}_{AB}$ term for the exact Heaviside switching. b) Logarithmic relative error of the $\mathcal{L}_{AB}$ term for the Heaviside switching approximated by delta couplings, as a function of the number of couplings N . The function represented by the solid cyan line is proportional to $1/N^2$, marking the rate of convergence of the approximation.	106
6.6	a), b), and c) Respectively, real part, imaginary part, and absolute value of the $\mathcal{M}$ term for a pair of identical two-level UDW detectors coupled to a quantum scalar field in spacelike-separated regions through a train of delta couplings approximating a Heaviside switching, as a function of the number of couplings N . Here we set the relevant parameters to $\gamma = 1$, $d = 1.2$, and $r = 0.1$. The constant red dashed lines mark the values of the real part, imaginary part, and absolute value of the $\mathcal{M}$ term for the exact Heaviside switching. d) Logarithmic relative error of the $\mathcal{M}$ term for the Heaviside switching approximated by delta couplings, as a function of the number of couplings N . The function represented by the solid cyan line is proportional to $1/N$, marking the rate of convergence of the approximation.	107
7.1	A schematic example of a neural network for processing local probe data to learn about a global feature of a QFT.	118

7.2	We trained a neural network to predict (through classification) the position of the boundary of a quantum field in a cavity from local probe data gathered far from the boundary. The network was asked 1) to detect a signal sent from the boundary (green triangles) and 2) to detect a modification of the field's boundary position (blue circles). The network's accuracy (solid) along with upper and lower bounds on the theoretical optimal accuracy (dashed) are plotted as a function of the duration of the probe's interaction with the field. A point plotted at time t indicates the network's accuracy given measurements taken at $N_{\text{TIMES}} = 10$ measurement times between t and the previous plot point. The network was trained on $N_{\text{train}} = 11250$ examples. Each example summarizes $N_{\text{TOM}} = 10^{22}$ measurements of each of the probe's quadratures ($\hat{q}_{\text{D}}$, $\hat{r}_{\text{D}}$ and $\hat{p}_{\text{D}}$) at each measurement times. The inset shows details of the causal response of the detector to the signal. The vertical red line is at the edge of detector to boundary light-crossing time.	126
7.3	A neural network trained to predict (through regression) the temperature of a quantum field from local probe data. The network was trained on labelled data corresponding to field temperatures from a range $T \pm 10\%$. The fraction of the validation data which the network labelled correctly to within $\pm 1\%$ is plotted as a function of the duration of the probe's interaction with the field. A point plotted at time t indicates the network's accuracy given measurements taken at $N_{\text{TIMES}} = 10$ measurement times between t and the previous plot point. The network was trained on $n_{\text{train}} = 7500$ examples from each range. Each example summarizes $N_{\text{TOM}} = 10^{20}$ measurements of each of the probe's quadratures ($\hat{q}_{\text{D}}$, $\hat{r}_{\text{D}}$ and $\hat{p}_{\text{D}}$) at each measurement time. The vertical red line is the probe's Heisenberg time ω_{D}^{-1}	130
7.4	a) The marginal distributions for the $N = 4$ Fock state (solid) and for the phase averaged coherent state with $ \alpha ^2 = 4$ (dashed). The distributions are not Gaussian and they have the same mean and variance, making them impossible to distinguish with simple statistical analysis of first and second moments. b) The validation accuracy of a neural network trained to distinguish two field states from the measurements of a local detector coupled to the field. In particular the network differentiates vacuum cavity states with the following two modifications: 1) the lowest field mode is in an N -particle Fock state or 2) the lowest field mode is in a coherent state with expectation $\langle \hat{n}_1 \rangle = N$ particles and unknown phase.	132

7.5	a) Validation accuracies attained for different tomographic sizes N_{TOM} , for expected number of particles $N = 1$, and detector frequency $\omega_D = \omega_1$. b) Evolution of network validation accuracies with the number of training iterations, plotted for different tomographic sizes. As before, the expected number of particles is $N = 1$, and the frequency of the detector is that of the first mode of the lattice, $\omega_D = \omega_1$.	133
8.1	Configuration in a slice of spacetime of the interaction regions (in blue) of the three detectors involved in the first example protocol, i.e., with parameters $L_{AC} = L_{BC} = L_{AB}/2 = 2.5T$ and $\Delta_{AB} = 0$.	154
8.2	(Solid lines) Negativity in the orthogonal regime of $\hat{\rho}_{AB}^s$ as a function of the delay Δ_{AC} for different values of the energy gap ΩT , and parameters $L_{AC} = L_{BC} = L_{AB}/2 = 2.5T$ and $\Delta_{AB} = 0$. This plot corresponds to the scheme in Fig. 8.1. (Dashed lines) Negativity of $\hat{\rho}_{AB}$ for the same values of L_{AB} and Δ_{AB} , with ΩT matching the value indicated by the colour of the corresponding solid line.	155
8.3	(Solid lines) Negativity in the orthogonal regime of $\hat{\rho}_{AB}^s$ as a function of the delay Δ_{AC} for different values of the energy gap ΩT , and parameters $L_{AC} = L_{BC} = L_{AB}/2 = 5T$ and $\Delta_{AB} = 0$. This plot corresponds to the scheme in Fig. 8.1. (Dashed lines) Negativity of $\hat{\rho}_{AB}$ for the same values of L_{AB} and Δ_{AB} , with ΩT matching the value indicated by the colour of the corresponding solid line.	157
8.4	Configuration in a slice of spacetime of the interaction regions (in blue) of the three detectors involved in the second example protocol, i.e., with parameters $L_{AB} = L_{AC} = L_{BC} = 0$ and $\Delta_{BA}, \Delta_{AC} > 0$.	158
8.5	(Solid lines) Negativity in the orthogonal regime of $\hat{\rho}_{AB}^s$ as a function of the delay $\Delta_{AC} > 0$ for different values of the energy gap ΩT , and parameters $L_{AC} = L_{BC} = L_{AB} = 0$ and $\Delta_{BA} = 5T$. This plot corresponds to the scheme in Fig. 8.4. (Dashed lines) Negativity of $\hat{\rho}_{AB}$ for the same values of L_{AB} and Δ_{AB} , with ΩT matching the value indicated by the colour of the corresponding solid line.	159
8.6	Configuration in a slice of spacetime of the interaction regions (in blue) of the three detectors involved in the third example protocol, i.e., with parameters $L_{AC} = L_{BC} = L_{AB} = 0$ and $\Delta_{BA} > \Delta_{CA} > 0$.	160

8.7	(Solid lines) Negativity in the transition regime of $\hat{\rho}_{AB}^s$ as a function of Δ_{AC} for different values of ξ , and $\Omega T = 2.5$, $L_{AC} = L_{BC} = L_{AB}/2 = 2.5T$ and $\Delta_{AB} = 0$. This plot corresponds to the scheme in Fig. 8.1. (Dashed line) Negativity of $\hat{\rho}_{AB}$ for the same values of L_{AB} , Δ_{AB} and ΩT	161
8.8	(Solid lines) Negativity in the transition regime of $\hat{\rho}_{AB}^s$ as a function of Δ_{AC} for different values of ξ , and $\Omega T = 2.5$, $L_{AC} = L_{BC} = L_{AB} = 0$ and $\Delta_{BA} = 5T$. This plot corresponds to the scheme in Fig. 8.4. (Dashed line) Negativity of $\hat{\rho}_{AB}$ for the same values of L_{AB} , Δ_{BA} and ΩT	162
8.9	(Solid lines) Negativity in the transition regime of $\hat{\rho}_{AB}^s$ as a function of the delay $0 < \Delta_{CA} < \Delta_{BA}$ for different values of the phase ξ , and parameters $\Omega T = 2.5$, $L_{AC} = L_{BC} = L_{AB} = 0$ and $\Delta_{BA} = 5T$. This plot corresponds to the scheme in Fig. 8.6. (Dashed line) Negativity of $\hat{\rho}_{AB}$ for the same values of L_{AB} , Δ_{BA} and ΩT	162
C.1	In subfigure a) we show the probe's coupling strength to the field modes, $\tilde{F}(k)$, (blue Gaussian) as a function of the mode's wavenumber, k , for $\sigma = 1$. The field's dispersion relation $\hbar\omega_k$ is also plotted (yellow hyperbola) for $m = 1$. Taking a UV cutoff at $K = 16$ (vertical red dashed line) yields a modified dispersion relation, $\hbar\omega'_k$, (green dashed) at high frequencies. In subfigure b) we plot the average relative error in $\hbar\omega_k$ as a function the cutoff K and the field mass m . This error decreases polynomially as $K\sigma$ increases. The error also decreases as the mass of the field increases. The black dashed line is a mass independent upper-bound on this error. In both subfigures we have taken $\hbar = c = 1$	217

List of Tables

7.1	Modifications of $\hat{H}_{\text{last}}$ connecting the last spatial mode to the rest of the lattice. $g = \hbar^2/ma^2$ is the site-to-site coupling strength.	124
B.1	Statistics for the N particle Fock states.	206
B.2	Statistics for the PAC state with $\langle \hat{n} \rangle = N$	206

Preamble

This thesis follows the structure of an opera, with an Overture setting the tone, Acts developing the core themes, an Intermezzo providing a transition between them, and a Finale bringing the work to its conclusion—though, unlike an opera, it contains no arias; it is all recitative.

Within the Overture, Ch. 1 introduces the measurement problem in quantum field theory, and motivates the work carried out in the thesis. Ch. 2 provides the reader with a brief overview of the theoretical background necessary to understand the thesis. I would still invite the reader familiar with these topics to go over them; firstly, because in this chapter we set a good fraction of the notation that is used in the chapters that follow (although it will nevertheless be reminded very often); but secondly because, taking the liberty of manipulating Fermi’s words, one should not underestimate the joy you can derive from reading something you already know.

Act 1 is devoted to reconciling measurements of quantum fields with relativistic causality. Ch. 3 develops a rigorous framework to understand under what conditions a measurement respects relativistic causality or not, and Ch. 4 is devoted to finding an update rule for relativistic setups that respects causality while still accounting for correlations across spacelike-separated regions. The frameworks laid down in these two chapters are put to work together to achieve the goal of formulating a measurement theory for QFT where the probes are particle detectors and that is consistent, at least within a controlled regime, with relativistic causality.

The Intermezzo shifts the focus from the quantum field to the particle detector, reminding us that to be able to use these as measurement probes we still need to be able to make predictions about them, which is not always a trivial endeavour since their interaction theories are generally not exactly solvable.

Act 2 is devoted to demonstrating that beyond its theoretical formulation, the detector-based measurement theory has the potential to become a practical measurement theory

that we can use in actual experiments to extract local information about a quantum field. A strong argument in favour of this is put forward in Ch. 7, where we show that neural networks can assist particle detectors to extract information from the field. Ch. 8 shows that the detector measurement schemes can be used to analyze the impact of measurements in quantum information protocols such as entanglement harvesting.

The Finale brings the thesis to an end, summarizing the main takeaways of each chapter, and pointing to potential next steps to take in each line of research.

Notation and conventions

In everything that follows, spacetime is always described by a pair $(\mathcal{M}, \mathbf{g})$, where $\mathcal{M}$ is a $(1+d)$ -dimensional smooth connected (paracompact and Hausdorff) manifold, and $\mathbf{g}$ is a Lorentzian metric on $\mathcal{M}$. Regarding index notation, we follow the conventions of Section 2.4 in [18]. Hence, $\mathbf{g}$ has components g_{ab} and determinant g . Although both of them depend on the point of the manifold in general, that dependency will be omitted at all times for the sake of conciseness. The signature of the metric is chosen so that $g < 0$, regardless of the number of spatial dimensions d . The spacetime is assumed to be globally hyperbolic and time-oriented, and therefore admits a global time function t whose level hypersurfaces form a foliation $\{\Sigma_t\}$ of $\mathcal{M}$. We denote with $\mathbf{h}$ the metric induced by $\mathbf{g}$ on Σ_t , with components h_{ab} and determinant h . We also define n to be the timelike unit normal to Σ_t that is oriented to the future, i.e., $n^a n_a = -1$, $g_{ab} = -n_a n_b + h_{ab}$, and $n^a \nabla_a t > 0$, where ∇_a is the Levi-Civita connection associated with $\mathbf{g}$. Both n and $\mathbf{h}$ depend on t , but once again we will omit this dependency to avoid overloading the notation.

Arbitrary points of the spacetime manifold are denoted with sans-serif font (i.e., $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$), and this notation is also used interchangeably to denote the point's spacetime coordinates. In particular, $d\mathbf{x}$ represents the corresponding coordinate volume element, and therefore $d\mathbf{x}\sqrt{-g}$ is the *invariant* volume form associated with $\mathbf{g}$. Bold fonts are reserved to denote the spatial coordinates of a point with respect to a coordinate system whose first component is timelike and the rest are spacelike (e.g., $\mathbf{z} = (\tau, \mathbf{z})$, for some time coordinate τ).

Given a subset $\mathcal{O} \subset \mathcal{M}$, a point $\mathbf{x}$ belongs to the causal past [resp. future] of $\mathcal{O}$ if there exists a smooth causal curve (i.e., whose derivative is timelike or null at all points) that is future-directed [resp. past-directed] going from $\mathbf{x}$ to some point in $\mathcal{O}$. The causal past [resp. future] of $\mathcal{O}$ is denoted $\mathcal{J}^-(\mathcal{O})$ [resp. $\mathcal{J}^+(\mathcal{O})$]. The union $\mathcal{J}(\mathcal{O}) := \mathcal{J}^-(\mathcal{O}) \cup \mathcal{J}^+(\mathcal{O})$ is the causal support of $\mathcal{O}$, and the intersection $\mathcal{J}^-(\mathcal{O}) \cap \mathcal{J}^+(\mathcal{O})$ is the causal hull. A set is causally convex if it coincides with its causal hull. There are two notions of causal

complement, $\mathcal{O}^\perp := \mathcal{M} \setminus \mathcal{J}(\mathcal{O})$ and $\mathcal{O}' := \mathcal{M} \setminus \text{cl} [\mathcal{J}(\mathcal{O})]$. We will primarily be concerned with the first one. In particular, a subset of $\mathcal{M}$ is said to be spacelike separated from $\mathcal{O}$ if it is included in $\mathcal{O}^\perp$.

Finally, given a smooth compactly-supported function $f \in \mathcal{C}_0^\infty(\mathcal{M})$, its support is given by

$$\text{supp } f := \text{cl}\{\mathbf{x} \in \mathcal{M} : f(\mathbf{x}) \neq 0\}.$$

In some contexts, the symbols g and h will also denote smooth compactly-supported functions. However, we will always define them explicitly in such cases, so no ambiguity should arise regarding their distinction from the determinants of the metric and the induced metric, respectively.

Overture

Chapter 1

Introduction

The first half of the twentieth century saw the birth of modern physics with the development of quantum mechanics and general relativity. Attempts to reconcile quantum mechanics with special relativity led to the abandonment of the notion of ‘particles’ in favour of the notion of ‘fields.’ This insight—that not only classical fields but also massive physical systems like electrons, which admit well-defined non-relativistic limits where they can be safely treated as particles, must instead be described as fields to ensure a consistent relativistic quantum theory—lies at the core of quantum field theory (QFT), one of the most successful physical theories formulated up to this date, responsible for some of the most precise predictions in the history of physics [19]. Today, much of modern physics is built upon the pillars of QFT and general relativity. Although these two theories have yet to be fully unified, they can still be brought together in certain regimes through a ‘semiclassical approximation’, where gravity remains a classical (often fixed) background on which quantum fields evolve. This approach—known as QFT in curved spacetimes—has led to remarkable predictions, including the renowned Hawking radiation, and has deepened our understanding of extreme phenomena such as black holes and the early universe, though its predictions remain, for now, largely beyond the reach of direct experimental verification.

However, any physical theory—no matter how successful—must be able to describe how we access information about the systems it models; in other words, it must provide a framework to model measurements. An accurate description of measurements is needed to establish the crucial dictionary that translates the theory’s predictions into what an experimentalist observes. Without such a dictionary, the theory cannot be fully tested, which ultimately undermines its scientific character.

In classical physics, the description of measurements is often left implicit, relying on

the (mostly safe) assumption that classical systems can be observed without significant disturbance. However, in quantum mechanics, describing measurement processes has been problematic and a subject of discussion since its very inception (see, e.g., [20]). This can be traced back, for instance, to the way quantum systems are intrinsically disturbed by measurements, as well as to the fundamentally different nature of quantum information, which is the kind of information that quantum systems hold, and classical information, which is the kind of information that, ultimately, experimentalists can interpret and represent.

Despite these difficulties, even if measurements cannot be derived from first principles, the problem can be bypassed operationally by introducing a *measurement theory*: a framework that, rather than using a bottom-up approach to derive a model that describes measurements, simply prescribes an ‘effective’ model. Along with this prescription, a proper measurement theory should 1) model measurements of all physically meaningful (and operationally reasonable) quantities; 2) provide computational rules to predict the statistics of measurement outcomes within the model; and 3) specify the post-measurement description of the system, identifying how a measurement affects its state. This last ingredient, the so-called *update rule*, should not be underestimated: Not all measurements are destructive—a quantum field most certainly does not vanish after we measure it—and therefore a complete measurement model must account for how the system evolves after a measurement is performed. In particular, update rules are essential for experiments that involve sequences of measurements, ensuring that later outcomes remain predictable and compatible with earlier ones.

The basic ingredients of the measurement theory in non-relativistic quantum mechanics are the Born rule and the Lüders rule (also known as the projection postulate). These principles, important enough that they are often included among the axioms of quantum mechanics in undergraduate courses, define the standard model of ideal measurements, which form a restricted subclass of all possible measurements. The modern understanding of the quantum theory of measurement considers a wider class of measurements, including, e.g., weak measurements, which allow partial information to be extracted while minimally disturbing the system.

For decades, the same measurement theory that applied to quantum mechanics was thought to apply to quantum field theory as well. This belief began to unravel when, in 1987, a collective effort culminated in a proof that, under very reasonable conditions, local algebras in QFT are hyperfinite factors of type III_1 [21–23]. One immediate consequence of this result is that finite-rank projectors are not local operators in QFT, and therefore they cannot accurately represent the measurement operations we carry out in practice. The situation deteriorated further in 1993 when Sorkin pointed out that, even if one were to allow finite-rank projective measurements, these would, in general, enable superluminal

communication between spacelike-separated regions [24], a direct violation of relativistic causality, and thus of QFT’s fundamental principles. These two findings made it clear that the measurement theory of QFT needed to be fundamentally reconsidered.

Given the essential role of measurement in physics, one might expect Sorkin’s result to have sent shockwaves through the community, exposing the unsettling fact that the most successful theory of modern physics lacked a consistent measurement model to connect theoretical predictions with observations. Yet, remarkably, the result was largely ignored—to the point that Dowker later had to bring it back to the community’s attention [25]. A likely reason for this oversight is that the result did not have a direct impact on the most experimentally active areas of QFT, namely, high-energy physics, quantum optics, and condensed matter. In high-energy physics, the Standard Model—which condenses our current best understanding of elementary particles and their interactions—is formulated as a relativistic quantum field theory, but most of its successes (including the aforementioned prediction of the gyromagnetic ratio of the free electron) are tied to scattering theory (S-matrix formalism), which describes asymptotic ‘in/out’ states rather than local field measurements, making the role of locality and detection less central. On the other hand, while measurements are crucial in quantum optics and condensed matter physics, these areas were concerned almost exclusively with non-relativistic regimes, where using finite-rank projections to model measurements seemed to work well enough for practical purposes.

Ironically, despite QFT being fundamentally a theory of *fields*, the concept of particle is used ubiquitously in quantum field theoretical contexts. Particle physicists work with ‘particle colliders’, condensed matter physicists study ‘quasiparticles’, and quantum optics routinely manipulates ‘photons’—despite the fact that the electromagnetic field does not even have a well-defined non-relativistic limit, as Lamb emphatically pointed out [26]. The widespread use of Fock states as ‘ n -particle states’ reinforces the intuition that detecting n particles amounts to performing a projective measurement on the corresponding Fock state. In many practical scenarios, this particle-based intuition still works, largely because the experimental applications of QFT are still mostly confined to regimes where this picture, and the intuition it carries along, remain reasonably accurate approximations.

Nevertheless, in the past decade, the interest in the problem of measurements in QFT has revitalized, driven mainly by two different research communities: algebraic quantum field theory (AQFT) and relativistic quantum information (RQI).

In AQFT, quantum field theories are formulated in terms of algebraic relations between local operations. This formalism has proven especially well-suited for defining QFTs in curved spacetimes and has led to remarkable insights into the mathematical structure of quantum fields, including the aforementioned result regarding the universal type of lo-

cal von Neumann algebras. In [27], Fewster and Verch proposed a measurement scheme formulated entirely within AQFT, in which the target quantum field is probed by another quantum field. Building on this work, Fewster, Bostelmann, and Ruep later demonstrated that superluminal signalling ‘à la Sorkin’ does not occur in this measurement scheme [28]. In other words, if one assumes that all physical measurement apparatus must ultimately be described by QFT (even if the precise formulation remains unknown), then any measurement process that enables faster-than-light signalling—such as finite-rank projections—is simply impossible.

RQI, on the other hand, is concerned with how quantum information concepts can be extended to QFT, including curved spacetime scenarios, and conversely, how quantum information techniques can provide new insights into QFT itself. An important research line within RQI involves adapting quantum information protocols to relativistic settings, often exploiting the relativistic nature of QFT to enhance them. Many of these protocols rely fundamentally on measurements, making a well-defined measurement theory for QFT essential within this field. It is in this context that particle detectors—one of the most widely used tools in RQI—have emerged as strong candidates for modelling measurements of quantum fields. Particle detectors are localized non-relativistic quantum systems that couple to quantum fields to gather information about them. Crucially, because they are described with non-relativistic quantum mechanics, they closely resemble the actual measurement apparatus used in experimental settings: indeed, many practical detection systems, from LHC detectors to photodetectors, are most naturally described with standard quantum mechanics.

The main goal of this thesis is to provide a (somewhat incomplete) guide to measuring real scalar quantum fields using particle detectors. Specifically, we present contributions to three different aspects of detector-based measurements in QFT, aiming to lay the groundwork for a complete, systematic measurement framework.

In the first ‘act’, we focus on providing a rigorous formulation of detector-based measurement schemes for QFT. This requires an analysis of their causal behaviour (to ensure they do not enable superluminal signalling à la Sorkin, or at least that any violations occur in a controlled, well-understood way) and a prescription of an update rule (analogous to Lüders’ rule in standard quantum mechanics). To achieve this, in Ch. 3 we conduct a thorough analysis of localized causal channels in real scalar QFT, and in Ch. 4 we develop a ‘relativistic formalism’ that allows us to formulate an update rule for relativistic quantum mechanics that is covariant, causal, accounts for correlations, and ensures the compatibility of sequential measurements. Standing on the results of these two chapters, in Ch. 5 we achieve our goal by building a measurement theory from detector-based measurement schemes.

The ‘intermezzo’, consisting solely of Ch. 6, addresses what we identify as a weakness of particle detector models: their heavy reliance on perturbation theory. Making predictions about QFT using detector-based measurements requires computing the state of the detectors after their interaction with the field. However, these calculations are almost always performed perturbatively, even though detectors in real experiments may not always be weakly coupled to the field, and some effects of interest may only be captured non-perturbatively. Moreover, many of the phenomena detected by particle detectors are already weak (e.g., entanglement harvesting), meaning that perturbative methods are unlikely to reveal stronger effects. To address these issues, in Ch. 6 we present an alternative approximation method that, while still an approximation, does not rely on perturbation theory and, unlike perturbative approaches, has a well-characterized rate of convergence to the exact result. It is in this sense that we describe this approximation as ‘non-perturbative.’

In the second ‘act’, we explore applications of our measurement theory. Ch. 8 examines how measurements affect entanglement harvesting, and Ch. 7 addresses a deeper issue: While it is fundamental to have a well-defined measurement theory, for it to be useful we need to know *how to use it*. This does not reduce to understanding what observables of the field we are measuring when we measure a detector: it requires being able to design a protocol that tells us what to measure and how, depending on what particular observable or feature of a quantum field we may be interested in. In Ch. 7, we propose a novel approach: instead of tailoring measurement schemes to specific observables, we fix a simple, untargeted measurement protocol, where a detector interacts with the field for different time intervals before undergoing tomographic reconstruction to extract all the information it holds. The apparent “blindness” of this protocol is compensated for by a previously trained neural network, which analyzes and interprets the data to produce a definite answer about the feature of interest—without fine-tuning the measurement setup.

The ‘finale’ concludes with a brief summary of the main takeaways of this work, and some thoughts on the next questions to tackle.

Chapter 2

Background

In this chapter, we set up the scene for the following chapters by providing a brief overview of the three main theoretical ingredients that will be (almost) omnipresent in the rest of this thesis: the standard measurement theory of quantum mechanics, the quantum theory of a free real scalar field, and the fundamentals of particle detector models. In doing so, we will also establish some of the notation that will be used in subsequent chapters—although very often it will also be reminded where it is used for the sake of clarity.

2.1 Quantum theory of measurement

In this section, we review how measurements are modelled in standard quantum mechanics, loosely following the concise exposition provided in [29]. We begin by defining essential notions from quantum information, followed by the concepts of measurement, instrument, and measurement scheme. For further details, we refer the reader to [30] and [31], where they will find comprehensive introductions to the theory of quantum information and the quantum theory of measurement, respectively.

Let us consider a (possibly multipartite) quantum system, and let $\mathcal{H}$ be the (complex separable) Hilbert space associated with it. We denote with $\mathcal{L}(\mathcal{H}) \supset \text{Pos}(\mathcal{H}) \supset \text{Proj}(\mathcal{H})$ the space of (linear) operators on $\mathcal{H}$, the cone of positive-semidefinite operators, and the set of (orthogonal) projections, respectively. We also denote with $\mathcal{B}(\mathcal{H}) \supset \mathcal{B}_2(\mathcal{H}) \supset \mathcal{B}_1(\mathcal{H})$ the von Neumann algebra of bounded linear operators, the Hilbert space of Hilbert-Schmidt operators, and the Banach space of trace-class operators—i.e., the operators with finite spectral norm, finite Hilbert-Schmidt norm (a.k.a. 2-norm), and finite trace norm (a.k.a.

1-norm), respectively. The possible states of the quantum system are represented by positive-semidefinite operators with unit trace, which form the convex set of density operators, $\mathcal{D}(\mathcal{H})$. The transformations undergone by the quantum system are described by completely positive (CP) maps $\Phi : \mathcal{B}_1(\mathcal{H}) \rightarrow \mathcal{B}_1(\mathcal{K})$ called *operations*, where $\mathcal{K}$ is a (complex separable) Hilbert space, possibly different from $\mathcal{H}$ in order to account for the possibility of adding or tracing out an ancillary subsystem in the process. Operations do not necessarily preserve the trace (of trace-class operators), but they do not increase it. Operations that are also trace-preserving (CPTP maps) are called *channels*. Given an operation $\Phi : \mathcal{B}_1(\mathcal{H}) \rightarrow \mathcal{B}_1(\mathcal{K})$, it is possible to define its adjoint with respect to the Hilbert-Schmidt inner product, $\Phi^\dagger : \mathcal{B}(\mathcal{K}) \rightarrow \mathcal{B}(\mathcal{H})$, which satisfies that

$$\text{Tr}[\hat{A}\Phi(\hat{B})] = \text{Tr}[\Phi^\dagger(\hat{A})\hat{B}], \quad \forall \hat{A} \in \mathcal{B}(\mathcal{K}), \hat{B} \in \mathcal{B}_1(\mathcal{H}), \quad (2.1)$$

where it was implicitly used that $\mathcal{B}_1(\mathcal{H})$ and $\mathcal{B}_1(\mathcal{K})$ are two-sided $*$ -ideals in $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}(\mathcal{K})$, respectively. It then holds that Φ is an operation if and only if $\Phi^\dagger$ is CP and $\Phi^\dagger(\mathbb{1}_{\mathcal{K}}) \leq \mathbb{1}_{\mathcal{H}}$. It is always possible to represent any operation by a (non-unique) countable set of bounded *Kraus operators* [32] $\{\hat{K}_j\}_{j=1}^\infty \subset \mathcal{B}(\mathcal{H}, \mathcal{K})$ satisfying

$$\Phi(\hat{A}) = \sum_{j=1}^\infty \hat{K}_j \hat{A} \hat{K}_j^\dagger, \quad \forall \hat{A} \in \mathcal{B}_1(\mathcal{H}). \quad (2.2)$$

This holds even when $\mathcal{H}$ and $\mathcal{K}$ are infinite-dimensional, as long as $\Phi^\dagger$ is continuous in the weak operator topology [33, 34]. In the finite-dimensional case, there exist representations with a finite number N of Kraus operators, where, in the minimal case (i.e., when the Kraus operators are linearly independent), N is equal to the rank of the operation's Choi representation. Moreover, for all $\hat{A} \in \mathcal{B}(\mathcal{K})$,

$$\Phi^\dagger(\hat{A}) = \sum_{j=1}^\infty \hat{K}_j^\dagger \hat{A} \hat{K}_j, \quad \text{and} \quad \Phi^\dagger(\mathbb{1}_{\mathcal{K}}) = \sum_{j=1}^\infty \hat{K}_j^\dagger \hat{K}_j \leq \mathbb{1}_{\mathcal{H}}, \quad (2.3)$$

with the latter inequality becoming an equality if and only if Φ is a channel.

Consider now an observable (i.e., a self-adjoint operator) of a quantum system, $\hat{A} \in \mathcal{L}(\mathcal{H})$. By the spectral theorem, there exists a projection-valued measure¹ defined on the Borel

¹A (normalized) *projection-valued measure* $\hat{m}$ is a map defined from the Borel σ -algebra of a set Ω to the projections of a Hilbert space $\mathcal{H}$, such that

1. $\hat{m}(\emptyset) = 0$ and $\hat{m}(\Omega) = \mathbb{1}_{\mathcal{H}}$, and

2. Given a sequence of disjoint Borel sets $\{X_n\}_{n=1}^\infty$, $\hat{m}(\bigcup_{n=1}^\infty X_n) = \sum_{n=1}^\infty \hat{m}(X_n)$.

In particular, given any two Borel sets X and Y , $\hat{m}(X \cap Y) = \hat{m}(X)\hat{m}(Y)$.

σ -algebra of the spectrum of $\hat{A}$, $\hat{m}_A : \text{Bor}(\text{sp}(\hat{A})) \rightarrow \text{Proj}(\mathcal{H})$, such that

$$\hat{A} = \int_{\text{sp}(\hat{A})} x \, d\hat{m}_A(x). \quad (2.4)$$

A *projective* or *ideal measurement* of $\hat{A}$ is a measurement whose set of possible outcomes is a countable set of Borel subsets $\{X_n\}_{n=1}^\infty \subset \text{Bor}(\text{sp}(\hat{A}))$ that are mutually disjoint and cover $\text{sp}(\hat{A})$, i.e., $X_n \cap X_m = \emptyset$ for $n \neq m$, and $\bigcup_{n=1}^\infty X_n = \text{sp}(\hat{A})$. Their probability distribution only depends on the state $\hat{\rho} \in \text{D}(\mathcal{H})$ of the system, and is given by the Born rule:

$$\text{Prob}(X_n) = \text{Tr}[\hat{\rho} \hat{m}_A(X_n)], \quad \forall n. \quad (2.5)$$

After a projective measurement is performed, the state of the system gets *updated* according to Lüders' update rule [35]. Namely, if the measurement is *selective*, i.e., if the outcome $X_n \in \text{Bor}(\text{sp}(\hat{A}))$ has been recorded,² then the state after the measurement is given by

$$\hat{\rho}_{X_n} = \frac{\hat{m}_A(X_n) \hat{\rho} \hat{m}_A(X_n)}{\text{Prob}(X_n)} \in \text{D}(\mathcal{H}), \quad (2.6)$$

where the numerator is the *update operation*, which does not preserve the trace in general. If, alternatively, the measurement outcome has not been recorded, i.e., if the measurement is *non-selective*, then the update does preserve the trace, and the updated state is given by the probabilistic mixture of the possible selective updates,

$$\hat{\rho}_{\text{ns}} = \sum_{n=1}^\infty \hat{m}_A(X_n) \hat{\rho} \hat{m}_A(X_n). \quad (2.7)$$

In practice, most measurements are not projective. More often than not, to measure a system (the *target*), a second system (the *probe*) is put to interact with the first, and it is on this second system that, after the interaction, a projective measurement is performed. It is also common for measurement apparatus to have some degree of uncertainty (especially when the observable has a continuum spectrum), so that different outcomes may be associated with overlapping subsets of the spectrum. To account for both of these circumstances, a *general measurement* is described by a positive-operator-valued measure³ (POVM) defined on the Borel σ -algebra of the set of possible outcomes Ω , $\hat{E} : \text{Bor}(\Omega) \rightarrow \text{Pos}(\mathcal{H})$, so

²This, in particular, implicitly assumes that $\text{Prob}(X_n) > 0$.

³A (normalized) *positive-operator-valued measure* $\hat{E}$ is a map defined from the Borel σ -algebra of a set Ω to the positive-semidefinite operators of a Hilbert space $\mathcal{H}$, such that

1. $\hat{E}(\emptyset) = 0$ and $\hat{E}(\Omega) = \mathbb{1}_{\mathcal{H}}$, and

that the probability distribution of the outcomes is given by

$$\text{Prob}(X) = \text{Tr}[\hat{\rho} \hat{E}(X)], \quad \forall X \in \text{Bor}(\Omega), \quad (2.8)$$

where $\hat{\rho} \in \text{D}(\mathcal{H})$ is the state of the system when the measurement is performed. Among general measurements, those for which Ω is a countable set are called *discrete*, and for them the measurement can be characterized by its *POVM elements*, $\{\hat{E}(\{x\}) \equiv \hat{E}(x) : x \in \Omega\}$. If a discrete measurement satisfies $\hat{E}(x)\hat{E}(y) = \delta_{x,y}\hat{E}(x)$, it is called *sharp*, and it holds that $\hat{E}(x) \in \text{Proj}(\mathcal{H})$. Projective measurements are a particular kind of sharp measurement.

Unlike projective measurements, general measurements do not have a unique associated update rule to determine the change that they induce in the state of the system: this depends on the *instrument* [36]. An instrument is an operation-valued measure I defined on the Borel σ -algebra of the set of possible outcomes Ω of the measurement it implements, such that I_Ω is a channel. For each outcome $X \in \text{Bor}(\Omega)$, I_X represents the operation that takes the initial state of the system, $\hat{\rho} \in \text{D}(\mathcal{H})$, to its (unnormalized) post-measurement state, with

$$\text{Tr}[I_X(\hat{\rho})] = \text{Prob}(X). \quad (2.9)$$

Normalizing the resulting operator, we find that for a selective measurement with recorded outcome $X \in \text{Bor}(\Omega)$, the updated state of the system is given by

$$\hat{\rho}_X = \frac{I_X(\hat{\rho})}{\text{Tr}[I_X(\hat{\rho})]} \in \text{D}(\mathcal{H}), \quad (2.10)$$

and for a non-selective measurement,

$$\hat{\rho}_{\text{ns}} = I_\Omega(\hat{\rho}) \in \text{D}(\mathcal{H}). \quad (2.11)$$

where we used that I_Ω is a channel to guarantee that the result has unit trace.

Given a general measurement represented by a POVM $\hat{E}$, from Eqs. (2.8) and (2.9) we find that an instrument that implements it must satisfy

$$\text{Prob}(X) = \text{Tr}[\hat{\rho} \hat{E}(X)] = \text{Tr}[I_X(\hat{\rho})], \quad \forall X \in \text{Bor}(\Omega), \hat{\rho} \in \text{D}(\mathcal{H}). \quad (2.12)$$

2. given a sequence of disjoint Borel sets $\{X_n\}_{n=1}^\infty$, $\hat{E}(\bigcup_{n=1}^\infty X_n) = \sum_{n=1}^\infty \hat{E}(X_n)$.

In particular, unlike for projection-valued measures, it is not true in general that $\hat{E}(X \cap Y) = \hat{E}(X)\hat{E}(Y)$, but monotonicity still holds: given two Borel sets X and Y such that $X \subset Y$, we have that $\hat{E}(X) \leq \hat{E}(Y)$.

We say that I is an $\hat{E}$ -compatible instrument. A given instrument can only be compatible with one measurement, since Eq. (2.12) implies that $\hat{E}(X) = I_X^\dagger(\mathbb{1}_{\mathcal{H}})$ for all $X \in \text{Bor}(\Omega)$, and hence the instrument fully determines the POVM. The converse, however, does not hold: let $\Phi : \mathcal{B}_1(\mathcal{H}) \rightarrow \mathcal{B}_1(\mathcal{H})$ be an arbitrary channel, and let I be an $\hat{E}$ -compatible instrument. Then, for all $X \in \text{Bor}(\Omega)$,

$$\hat{E}(X) = I_X^\dagger(\mathbb{1}_{\mathcal{H}}) = I_X^\dagger(\Phi^\dagger(\mathbb{1}_{\mathcal{H}})), \quad (2.13)$$

where in the last step we used that, because Φ is a channel, $\Phi^\dagger(\mathbb{1}_{\mathcal{H}}) = \mathbb{1}_{\mathcal{H}}$. We conclude that $\Phi \circ I$ is also an $\hat{E}$ -compatible instrument. Remarkably, the set of all $\hat{E}$ -compatible instruments can be characterized for discrete measurements [37]: let I^L be the Lüders instrument, defined as

$$I_x^\text{L}(\hat{\rho}) = \sqrt{\hat{E}(x)} \hat{\rho} \sqrt{\hat{E}(x)}, \quad \forall x \in \Omega, \quad (2.14)$$

by analogy with the update rule, Eq. (2.6). Then, any other $\hat{E}$ -compatible instrument I can be written as $I_x = \Phi_x \circ I_x^\text{L}$, for some channel $\Phi_x : \mathcal{B}_1(\mathcal{H}) \rightarrow \mathcal{B}_1(\mathcal{H})$, and for each $x \in \Omega$.

One way to specify the instrument for a particular measurement is to model it as the result of the coupling between the target and an ancillary probe that acts as measurement apparatus. Namely, let $\Psi : \mathcal{B}_1(\mathcal{H} \otimes \mathcal{K}) \rightarrow \mathcal{B}_1(\mathcal{H} \otimes \mathcal{K})$ be the channel representing the interaction through which the target and the probe get correlated, where $\mathcal{K}$ is the Hilbert space of the probe. Then, given an initially uncorrelated joint state of the target and the probe, $\hat{\rho} \otimes \hat{\chi} \in \text{D}(\mathcal{H} \otimes \mathcal{K})$, if a measurement $\hat{F}$ is performed on the probe after the interaction, the (unnormalized) state of the target system after an outcome X has been recorded is given by

$$I_X(\hat{\rho}) = \text{Tr}_{\mathcal{K}}\{[\mathbb{1}_{\mathcal{H}} \otimes \hat{F}(X)] \Psi(\hat{\rho} \otimes \hat{\chi})\}, \quad (2.15)$$

where $\text{Tr}_{\mathcal{K}}$ is the partial trace over the probe's sector of the joint Hilbert space. The tuple that specifies the probe, its initial state, the correlating channel, and the measurement on the probe, $\{\mathcal{K}, \hat{\chi}, \Psi, \hat{F}\}$, is called *measurement scheme*. From Eq. (2.15), a measurement scheme fully specifies the instrument I , and therefore the measurement induced on the target, $\hat{E}$. In particular, $I_\Omega(\hat{\rho}) = \text{Tr}_{\mathcal{K}}[\Psi(\hat{\rho} \otimes \hat{\chi})]$, showing that a non-selective measurement on the probe does not affect the partial state of the target, which is fully determined by the interaction. The partial state of the probe after the interaction, on the other hand, is given by $\Theta(\hat{\rho}) = \text{Tr}_{\mathcal{H}}[\Psi(\hat{\rho} \otimes \hat{\chi})]$, and we say that the channel $\Theta : \mathcal{B}_1(\mathcal{H}) \rightarrow \mathcal{B}_1(\mathcal{K})$ is *conjugate* to I_Ω .

Every general measurement admits a measurement scheme. In fact, every $\hat{E}$ -compatible instrument admits not only infinitely many measurement schemes, but infinitely many where the initial state of the probe is pure, the channel Ψ is unitary, and the measurement

$\hat{F}$ is projective [38]. In particular, as can be deduced directly from Naimark's dilation theorem, any general measurement can be seen as a projective measurement on an enlarged Hilbert space.

2.2 Quantum theory of a free real scalar field

In this section, we review the description of real scalar quantum fields in curved spacetimes. We begin with the canonical approach, which employs Fock quantization and depends on the selection of a preferred complex structure on the symplectic manifold of solutions to the equations of motion. This is followed by the algebraic approach, which avoids specifying a Hilbert-space representation of the theory and instead focuses on the structure of the $*$ -algebra of observables.

Although these two approaches are formally distinct, they are ultimately equivalent, and both will be used in subsequent chapters. For further details on the canonical approach, we refer the reader to the books by Fulling [39] and Birrell & Davies [40], as well as Garay's lecture notes [41]. Comprehensive accounts of the algebraic formalism can be found in Haag's book [42] and a recent collaborative work [43], while a pedagogical introduction is provided in Fewster & Rejzner's review [44]. Finally, Wald's book [45] can serve as a bridge between both frameworks.

Let $(\mathcal{M}, \mathbf{g})$ be a $(1+d)$ -dimensional globally hyperbolic connected spacetime equipped with a Lorentzian metric, and let t and $\{\Sigma_t\}_{t \in \mathbb{R}}$ be a global time function and its associated foliation of $\mathcal{M}$ into Cauchy hypersurfaces of constant t . The action of a minimally coupled free real scalar field is given by

$$S = \int_{\mathcal{M}} \mathrm{d}\mathbf{x} \sqrt{-g} \left(-\frac{1}{2} \nabla_a \phi \nabla^a \phi - \frac{m^2}{2} \phi^2 \right), \quad (2.16)$$

from which one can find its equation of motion to be given by the Klein-Gordon equation,

$$(\nabla_a \nabla^a - m^2) \phi \equiv (\square - m^2) \phi = 0, \quad (2.17)$$

and the canonical momentum of ϕ on Σ_t to be

$$\pi(t, \mathbf{x}) = \left. \frac{\delta S}{\delta \partial_t \phi} \right|_{\mathbf{x}=(t, \mathbf{x})} = \sqrt{h} \nabla_{\mathbf{n}} \phi(t, \mathbf{x}) = \sqrt{h} n^a \nabla_a \phi(t, \mathbf{x}), \quad (2.18)$$

where we recall that h is the determinant of the metric $\mathbf{h}$ that $\mathbf{g}$ induces on Σ_t , and n^a is the future-oriented unit normal⁴ to Σ_t . The Poisson brackets are then given by

$$\{\phi(t, \mathbf{x}), \pi(t, \mathbf{x}')\} = \delta(\mathbf{x} - \mathbf{x}'), \quad \{\phi(t, \mathbf{x}), \phi(t, \mathbf{x}')\} = \{\pi(t, \mathbf{x}), \pi(t, \mathbf{x}')\} = 0. \quad (2.19)$$

Because the spacetime $\mathcal{M}$ is globally hyperbolic, and since the Klein-Gordon equation is hyperbolic, the values of ϕ and π along one Σ_t (or any other Cauchy hypersurface) uniquely determine the values of ϕ everywhere. One way to show this is using the equation of motion's Green functions: let E^+ and E^- be the retarded and advanced Green functions, which satisfy

$$(\square_{\mathbf{x}} - m^2)E^\pm(\mathbf{x}, \mathbf{x}') = \frac{\delta(\mathbf{x} - \mathbf{x}')}{\sqrt{-g}}, \quad (2.20)$$

so that $E^\pm(\mathbf{x}, \mathbf{x}') = 0$ whenever $\mathbf{x} \notin \mathcal{J}^\pm(\mathbf{x}')$. We define $E := E^+ - E^-$, which is a solution of the (homogeneous) Klein-Gordon equation, satisfying that $E(\mathbf{x}, \mathbf{x}') = 0$ whenever $\mathbf{x} \notin \mathcal{J}(\mathbf{x}')$, and that given any solution ϕ such that $\phi|_{\Sigma_t}, \nabla_{\mathbf{n}}\phi|_{\Sigma_t} \in \mathcal{C}_0^\infty(\Sigma_t)$,

$$\phi(\mathbf{x}') = \int_{\Sigma_t} d\mathbf{x} \sqrt{h} [E(\mathbf{x}, \mathbf{x}') \nabla_{\mathbf{n}}\phi(\mathbf{x}) - \phi(\mathbf{x}) \nabla_{\mathbf{n}}E(\mathbf{x}, \mathbf{x}')] \quad (2.21)$$

$$= \int_{\Sigma_t} d\mathbf{x} [\pi(\mathbf{x}) E(\mathbf{x}, \mathbf{x}') - \phi(\mathbf{x}) \sqrt{h} \nabla_{\mathbf{n}}E(\mathbf{x}, \mathbf{x}')], \quad (2.22)$$

where we denoted $(t, \mathbf{x}) \equiv \mathbf{x}$. In particular, the ‘equal-time’ Poisson brackets of Eq. (2.19) can be expressed in a covariant manner as

$$\{\phi(\mathbf{x}), \phi(\mathbf{x}')\} = E(\mathbf{x}, \mathbf{x}'). \quad (2.23)$$

The fact that solutions can be put in one-to-one correspondence with the values of the field amplitude and its conjugate momentum along a Cauchy hypersurface Σ_t implies that the space of solutions can be identified with the field's phase space on Σ_t , and in particular it can inherit its symplectic structure. Specifically, given two solutions ϕ_1 and ϕ_2 , we define

$$\Omega(\phi_1, \phi_2) := \int_{\Sigma_t} d\mathbf{x} [\pi_1(t, \mathbf{x}) \phi_2(t, \mathbf{x}) - \pi_2(t, \mathbf{x}) \phi_1(t, \mathbf{x})]. \quad (2.24)$$

To ensure that the symplectic form is well defined and independent of the hypersurface Σ_t where it is computed, we constrain ourselves to the space of solutions whose restrictions to each leaf and their normal derivatives are smooth and compactly supported, which we

⁴This means that $n^a n_a = -1$, $n^a \nabla_a t > 0$, and $g_{ab} = -n_a n_b + h_{ab}$ along Σ_t .

denote $\mathcal{S}$. The correspondence between the solutions and a leaf's phase space implies that, for all $t \in \mathbb{R}$,

$$\{(\phi, \pi) : \phi, \pi \in \mathcal{C}_0^\infty(\Sigma_t)\} \cong \mathcal{S}. \quad (2.25)$$

A *complex structure* J compatible with Ω is a linear map defined on $\mathcal{S}$ such that $J^2 = -\mathbb{1}$, and

$$(\phi_1, \phi_2) := \Omega(J\phi_1, \phi_2) \quad (2.26)$$

defines an inner product.⁵ Because the only eigenvalues of J are $\{+i, -i\}$, its eigenspaces can only be defined in the complexified space of solutions, $\mathcal{S}^\mathbb{C}$. We refer to these eigenspaces as the spaces of solutions of positive and negative frequency, denoted by $\mathcal{S}_\pm^\mathbb{C}$, with associated orthogonal projections $\Pi_\pm = (\mathbb{1} \mp iJ)/2$. On the positive-frequency subspace $\mathcal{S}_+^\mathbb{C}$, we can define the *Klein-Gordon inner product*,

$$(\varphi, \psi)_{\text{KG}} := i \int_{\Sigma_t} d\mathbf{x} \sqrt{h} [\varphi^*(t, \mathbf{x}) \nabla_{\mathbf{n}} \psi(t, \mathbf{x}) - \psi(t, \mathbf{x}) \nabla_{\mathbf{n}} \varphi^*(t, \mathbf{x})]. \quad (2.27)$$

This definition can be used to extend the Klein-Gordon product to all $\mathcal{S}^\mathbb{C}$, although this extension is not positive-definite. In fact, if ψ is a non-zero positive-frequency solution, then its complex conjugate ψ^* is a negative-frequency solution, and $(\psi^*, \psi^*)_{\text{KG}} = -(\psi, \psi)_{\text{KG}} < 0$.

The Klein-Gordon product endows $\mathcal{S}_+^\mathbb{C}$ with a metric structure, and by completing it we obtain a Hilbert space, $\mathcal{H}_+$, which we call the *one-particle Hilbert space*. Let $\{\psi_j(\mathbf{x})\}$ be an orthonormal basis⁶ of $\mathcal{H}_+$, then any (real) solution $\phi \in \mathcal{S}$ can be written as

$$\phi(\mathbf{x}) = (\Pi_+ \phi)(\mathbf{x}) + (\Pi_- \phi)(\mathbf{x}) = (\Pi_+ \phi)(\mathbf{x}) + (\Pi_+ \phi)^*(\mathbf{x}) = \sum_j [a_j \psi_j(\mathbf{x}) + a_j^* \psi_j^*(\mathbf{x})], \quad (2.28)$$

where a_j and a_j^* are the creation and annihilation variables, respectively, which satisfy that

$$a_j = (\psi_j, \Pi_+ \phi)_{\text{KG}} = i \int_{\Sigma_t} d\mathbf{x} [\pi(t, \mathbf{x}) \psi_j^*(t, \mathbf{x}) - \phi(t, \mathbf{x}) \sqrt{h} \nabla_{\mathbf{n}} \psi_j^*], \quad (2.29)$$

and therefore, as functions of ϕ and π , their Poisson brackets are given by

$$\{a_j, a_k\} = \{a_j^*, a_k^*\} = 0, \quad \{a_j, a_k^*\} = -i\delta_{jk}. \quad (2.30)$$

⁵It can be deduced from its definition that J must be a symplectomorphism.

⁶The construction that follows can be straightforwardly extended to generalized orthonormal bases, i.e., bases whose elements are not technically part of the Hilbert space. For instance, if the equation of motion can be written in the form $-\partial_t^2 \phi = D\phi$, for some differential operator D , then the set of eigenfunctions of D can be used to construct a basis of complex positive-frequency solutions.

This is precisely the symplectic structure of the creation and annihilation variables of an infinite set of decoupled harmonic oscillators, which can be irreducibly represented in a Hilbert space via *Fock quantization*: the Hilbert space of the real scalar quantum field, $\mathcal{H}_\phi$, is given by the bosonic Fock space constructed from the one-particle Hilbert space $\mathcal{H}_+$, i.e.,

$$\mathcal{H}_\phi := \mathcal{F}_b(\mathcal{H}_+) = \bigoplus_{n=0}^{\infty} \mathcal{H}_+^{\odot n}, \quad (2.31)$$

where $\odot$ denotes the symmetric tensor product. The $n = 0$ term in the direct sum is isomorphic to $\mathbb{C}$ and corresponds to the linear span of the vacuum state $|0\rangle$. Given a general state

$$|\Psi\rangle = \sum_{n=0}^{\infty} \Psi^{(n)} = \sum_{n=0}^{\infty} \sum_{j_1, \dots, j_n} \Psi_{j_1, \dots, j_n}^{(n)} \psi_{j_1} \odot \dots \odot \psi_{j_n}, \quad (2.32)$$

the creation and annihilation operators, $\hat{a}_j^\dagger$ and $\hat{a}_j$, are defined so that

$$\hat{a}_j^\dagger |\Psi\rangle = \sum_{n=0}^{\infty} \sum_{j_1, \dots, j_n} \sqrt{n+1} \Psi_{j_1, \dots, j_n}^{(n)} \psi_j \odot \psi_{j_1} \odot \dots \odot \psi_{j_n}, \quad (2.33)$$

and

$$\hat{a}_j |\Psi\rangle = \sum_{n=0}^{\infty} \sum_{j_1, \dots, j_n} \sqrt{n+1} \Psi_{(j, j_1, \dots, j_n)}^{(n+1)} \psi_{j_1} \odot \dots \odot \psi_{j_n}. \quad (2.34)$$

In particular, $\hat{a}_j |0\rangle = 0$ for all j , and they satisfy the canonical commutation relations

$$[\hat{a}_j, \hat{a}_k] = [\hat{a}_j^\dagger, \hat{a}_k^\dagger] = 0, \quad [\hat{a}_j, \hat{a}_k^\dagger] = \delta_{jk} \mathbb{1}. \quad (2.35)$$

From Eq. (2.28), the field operator is given in the Heisenberg picture by

$$\hat{\phi}(\mathbf{x}) = \sum_j [\hat{a}_j \psi_j(\mathbf{x}) + \hat{a}_j^\dagger \psi_j^*(\mathbf{x})], \quad (2.36)$$

and, from Eq. (2.23), it satisfies

$$[\hat{\phi}(\mathbf{x}), \hat{\phi}(\mathbf{x}')] = iE(\mathbf{x}, \mathbf{x}') \mathbb{1}. \quad (2.37)$$

However, $\hat{\phi}(\mathbf{x})$ is not a well-defined operator in the usual sense, but rather an *operator-valued distribution*. To obtain a well-defined operator, we must smear $\hat{\phi}(\mathbf{x})$ against a smooth compactly-supported function $f \in \mathcal{C}_0^\infty(\mathcal{M})$, yielding a *smear*ed field operator,

$$\hat{\phi}(f) = \int_{\mathcal{M}} d\mathbf{x} \sqrt{-g} f(\mathbf{x}) \hat{\phi}(\mathbf{x}), \quad (2.38)$$

which is self-adjoint and unbounded. From Eq. (2.37), smeared field operators satisfy

$$[\hat{\phi}(f), \hat{\phi}(g)] = iE(f, g)\mathbb{1} = i \left[\int_{\mathcal{M} \times \mathcal{M}} dx dx' \sqrt{gg'} f(x)g(x')E(x, x') \right] \mathbb{1}, \quad (2.39)$$

for all $f, g \in \mathcal{C}_0^\infty(\mathcal{M})$.

The quantization process described above heavily relies on the choice of one-particle Hilbert space, $\mathcal{H}_+$, and therefore on the choice of complex structure J . Let J' be another complex structure, and let $\mathcal{H}'_+$ be its associated one-particle Hilbert space. Given an orthonormal basis of $\mathcal{H}'_+$, $\{\psi'_k(x)\}$, we can use the orthogonal projections associated with J to decompose each ψ'_k into a sum of positive-frequency and negative-frequency solutions:

$$\psi'_k(x) = (\Pi_+ \psi'_k)(x) + (\Pi_- \psi'_k)(x) = \sum_j [\alpha_{kj} \psi_j(x) + \beta_{kj} \psi_j^*(x)]. \quad (2.40)$$

Along with Eq. (2.28), this implies that the creation and annihilation variables associated with J and J' are related by

$$a_j = \sum_k (\alpha_{kj} a'_k + \beta_{kj}^* a'^*_k). \quad (2.41)$$

The coefficients α_{kj} and β_{kj} and the linear maps they define are called *Bogoliubov coefficients* and *Bogoliubov transformations*, respectively, and they satisfy the following consistency conditions (expressed in matrix notation):

$$\alpha \alpha^\dagger - \beta \beta^\dagger = \mathbb{1}, \quad \alpha^* \beta^\dagger - \beta^* \alpha^\dagger = 0, \quad (2.42)$$

$$\alpha^\dagger \alpha - \beta^\dagger \beta^* = \mathbb{1}, \quad \alpha^\dagger \beta - \beta^\dagger \alpha^* = 0. \quad (2.43)$$

Conversely, any transformation of the form of Eq. (2.40) which satisfies Eqs. (2.42) and (2.43) is associated with a change of complex structure.

Given two Fock quantizations obtained from two different complex structures, J and J' , they will yield the same physical predictions if and only if there exists a unitary $\hat{U} : \mathcal{H}_\phi \rightarrow \mathcal{H}'_\phi$ that takes one field operator to the other, i.e., such that

$$\hat{U} \hat{\phi}(x) \hat{U}^\dagger = \hat{\phi}'(x), \quad (2.44)$$

or, equivalently, by Eq. (2.41),

$$\hat{U} \hat{a}_j \hat{U}^\dagger = \sum_k (\alpha_{kj} \hat{a}'_k + \beta_{kj}^* \hat{a}'^\dagger_k). \quad (2.45)$$

It can be shown that such a unitary exists if and only if:

1. There exist $c_1, c_2 > 0$ such that, for all $\phi \in \mathcal{S}$,

$$c_1(\phi, \phi) \leq (\phi, \phi)' \leq c_2(\phi, \phi), \quad (2.46)$$

where $(\cdot, \cdot)$ and $(\cdot, \cdot)'$ denote the inner products defined by J and J' , respectively.

2. The Bogoliubov transformation that relates J and J' is such that β is a Hilbert-Schmidt operator, i.e., $\text{Tr}(\beta^\dagger \beta) = \sum_{kj} |\beta_{kj}|^2 < \infty$.

These two conditions are always satisfied when the dimension of the one-particle Hilbert space (i.e., the number of degrees of freedom) is finite. However, they fail to be met in general in the context of quantum field theory, where we can indeed find inequivalent quantizations. In some cases, the class of allowed quantizations can be restricted by requiring the vacuum state to respect all the symmetries of the spacetime, as in Minkowski space, where Poincaré symmetry leads to a unique vacuum up to unitary equivalence. However, in general spacetimes, it is not possible to single out a unique preferred representation due to the lack of a canonical choice of complex structure.

The existence of inequivalent quantizations is not problematic *per se*—it simply responds to the fact that, in QFT, the natural choice of representation (if there is one) depends on the observer’s state of motion, leading to phenomena such as the Unruh effect [46], where different observers (at rest vs. accelerated) assign different physical properties to the same state (Minkowski’s unique Poincaré-invariant vacuum). However, it is often desirable, both for theoretical reasons (to ensure general covariance) and practical reasons (to handle QFT in diverse scenarios), to have a formalism that does not rely on the choice of a specific representation. This can be achieved by shifting the focus of the theory to its algebraic relations and the local nature of its observables, which is precisely the philosophy of algebraic quantum field theory.

In the algebraic framework, a quantum field theory on the (globally hyperbolic) spacetime $(\mathcal{M}, \mathbf{g})$ is described by a unital $*$ -algebra $\mathcal{A}$, where each open causally convex subset $\mathcal{O} \subset \mathcal{M}$ is assigned a unital $*$ -subalgebra $\mathcal{A}(\mathcal{O})$ whose unit is shared with that of $\mathcal{A}$, satisfying the following axioms:

1. *Isotony*: if $\mathcal{O}_1 \subset \mathcal{O}_2 \subset \mathcal{M}$, then $\mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)$.
2. *Microcausality*: if $\mathcal{O}_1, \mathcal{O}_2 \subset \mathcal{M}$ are spacelike separated, then

$$[A_1, A_2] = 0, \quad \forall A_1 \in \mathcal{A}(\mathcal{O}_1), A_2 \in \mathcal{A}(\mathcal{O}_2). \quad (2.47)$$

3. *Existence of dynamics* (a.k.a. *time slice axiom*): if $\mathcal{O}_1, \mathcal{O}_2 \subset \mathcal{M}$ are such that $\mathcal{O}_1$ contains a Cauchy hypersurface of $\mathcal{O}_2$, then $\mathcal{A}(\mathcal{O}_2) \subset \mathcal{A}(\mathcal{O}_1)$.

The elements of $\mathcal{A}(\mathcal{O})$ are understood as the operators of the field theory that are localizable in $\mathcal{O}$, and in particular those that satisfy $A = A^*$ are interpreted as the local observables. Note that, due to the dynamics of the field, a single element of $\mathcal{A}$ may be localizable in more than one region.

To formulate the theory in the particular case of the real scalar quantum field, we start by constructing the free complex unital $*$ -algebra $\tilde{\mathcal{A}}$, generated by the unit $\mathbb{1}$ and the set of symbols $\{\phi(f), \phi(f)^* : f \in \mathcal{C}_0^\infty(\mathcal{M})\}$, where $\mathcal{C}_0^\infty(\mathcal{M})$ is the set of real smooth functions of compact support in $\mathcal{M}$. The algebra $\tilde{\mathcal{A}}$ consists of all (finite) complex linear combinations of (finite) products of these generators. Here, ϕ is a symbol, indicating that f serves as an argument of the generator $\phi(f)$, rather than being an element of the vector space $\mathcal{C}_0^\infty(\mathcal{M})$. Once $\tilde{\mathcal{A}}$ is constructed, we impose on it the relations that need to be satisfied by the real scalar quantum field:

1. Reality: $\phi(f)^* = \phi(f)$.
2. Equation of motion: $\phi((\square - m^2)f) = 0$.
3. Canonical commutation relations: $[\phi(f), \phi(g)] = iE(f, g)\mathbb{1}$.

To enforce these relations on the algebra, it suffices to construct the two-sided $*$ -ideal $\mathcal{I}$ generated by them, i.e., that contains all (finite) complex linear combinations of terms of the form

$$A[\phi(f)^* - \phi(f)]B, \quad A\phi((\square - m^2)f)B, \quad \text{and} \quad A[[\phi(f), \phi(g)] - iE(f, g)\mathbb{1}]B, \quad (2.48)$$

for all $A, B \in \tilde{\mathcal{A}}$. Then, we define the algebra of the theory by taking the quotient of the free algebra $\tilde{\mathcal{A}}$ over the ideal of relations $\mathcal{I}$, i.e., $\mathcal{A} := \tilde{\mathcal{A}}/\mathcal{I}$, where the product, the involution, and the unit are preserved by the quotient. For each open causally convex set $\mathcal{O} \subset \mathcal{M}$, the subalgebra $\mathcal{A}(\mathcal{O}) \subset \mathcal{A}$ is generated by the unit of $\mathcal{A}$ and those $\phi(f) \in \mathcal{A}$ such that the support of f is contained in $\mathcal{O}$.

Although not required by the framework's axioms, describing the quantum field theory in terms of a C^* -algebra is often useful.⁷ For the real scalar quantum field, this can always be achieved with the *Weyl algebra*, $\mathcal{W}$. To construct it, we first follow the same procedure outlined above to obtain the unital $*$ -algebra generated by the symbols $\{W(f), W(f)^* : f \in \mathcal{C}_0^\infty(\mathcal{M})\}$ satisfying the following relations:

⁷This is simply because C^* -algebras enjoy more stringent properties than general $*$ -algebras.

1. Reality: $W(f)^* = W(-f)$.
2. Equation of motion: $W((\square - m^2)f) = \mathbb{1}$.
3. Weyl relations: $W(f)W(g) = e^{-\frac{i}{2}E(f,g)} W(f+g)$.

There exists a unique norm on this unital $*$ -algebra whose completion yields a C^* -algebra. This C^* -algebra, unique up to the intrinsic isometric ambiguity of completion, is the Weyl algebra. Although one might be tempted to identify $W(f)$ with $e^{i\phi(f)}$, this identification cannot be done at the level of the algebras $\mathcal{A}$ and $\mathcal{W}$. However, motivated by this intuition, in what follows we will denote $W(f)$ as $e^{i\phi(f)}$, understanding this as a purely notational choice for now.

Given the algebra of the theory $\mathcal{A}$, the state of the field is described by a linear map $\omega : \mathcal{A} \rightarrow \mathbb{C}$ that is normalized and positive, i.e., $\omega(\mathbb{1}) = 1$ and $\omega(A^*A) \geq 0$ for all $A \in \mathcal{A}$. A state is mixed if it can be written as a non-trivial convex combination of two different states, and it is pure if it cannot. Let ω be a state of the field, we define its associated (smeared) n -point functions as

$$w_n(f_1, \dots, f_n) := \omega(\phi(f_1) \dots \phi(f_n)), \quad (2.49)$$

for any $f_1, \dots, f_n \in \mathcal{C}_0^\infty(\mathcal{M})$. If the smeared n -point functions are sufficiently regular with respect to their arguments, they define distributions, and we denote with $w_n(x_1, \dots, x_n)$ their associated generalized functions. A special class of states is that of zero-mean Gaussian states (also known as *quasifree*⁸), for which all odd n -point functions vanish, and even n -point functions are given by

$$w_{2k}(f_1, \dots, f_{2k}) = \sum_{\sigma \in S_{2k}} \prod_{j=1}^k \theta(\sigma(2j) - \sigma(2j-1)) w_2(f_{\sigma(2j-1)}, f_{\sigma(2j)}), \quad (2.50)$$

where S_{2k} is the group of permutations of $\{1, \dots, 2k\}$, and θ represents Heaviside's step function. Thus, in a quasifree state, all n -point functions are completely determined by the two-point function. In particular, by Wick's theorem, the vacuum state arising from a Fock quantization is always quasifree.

The connection between the algebraic and the Hilbert space descriptions of quantum field theory can be established via the *GNS construction* (named after Gelfand, Naimark

⁸This term—which has traditionally been more common in the AQFT literature—was introduced by D. W. Robinson in [47], and later became established through subsequent works (e.g., [48, 49]). The terminology was likely influenced by an earlier result of O. W. Greenberg and A. L. Licht [50], which showed that these states do not lead to scattering.

and Segal): let $\mathcal{A}$ be a unital $*$ -algebra, and let $\omega : \mathcal{A} \rightarrow \mathbb{C}$ be a state. Because ω is positive, it satisfies the Cauchy-Schwarz inequality,

$$|\omega(A^*B)|^2 \leq \omega(A^*A)\omega(B^*B), \quad (2.51)$$

which implies that

$$\mathcal{N}_\omega := \{A \in \mathcal{A} : \omega(A^*A) = 0\} \quad (2.52)$$

is a left $*$ -ideal. Therefore, the quotient $\mathcal{D}_\omega := \mathcal{A}/\mathcal{N}_\omega$ is a well-defined unital $*$ -algebra, where we can define an inner product of equivalence classes by

$$\langle [A] | [B] \rangle := \omega(A^*B). \quad (2.53)$$

The completion of $\mathcal{D}_\omega$ with respect to this inner product is a Hilbert space $\mathcal{H}_\omega$ in which $\mathcal{D}_\omega$ is dense. Hence, for each $A \in \mathcal{A}$, the operator

$$\pi_\omega(A) : \mathcal{D}_\omega \rightarrow \mathcal{D}_\omega, [B] \mapsto [AB] \quad (2.54)$$

may not be bounded as an operator in $\mathcal{H}_\omega$, but its adjoint is well defined. Thus, π_ω assigns to each element of $\mathcal{A}$ a bounded operator $\pi_\omega(A) \in \mathcal{B}(\mathcal{D}_\omega)$ (that may be extended to $\mathcal{H}_\omega$ as a possibly unbounded operator), satisfying

$$\pi_\omega(\mathbb{1}) = \mathbb{1}_{\mathcal{D}_\omega}, \pi(A^*) = \pi_\omega(A)^\dagger \Big|_{\mathcal{D}_\omega}, \pi_\omega(A+B) = \pi_\omega(A) + \pi_\omega(B), \pi_\omega(AB) = \pi_\omega(A)\pi_\omega(B), \quad (2.55)$$

as can be straightforwardly verified from Eqs. (2.53) and (2.54). Eq. (2.55) shows that π_ω is a $*$ -representation of $\mathcal{A}$. Moreover, if we define $|\Omega_\omega\rangle := [\mathbb{1}] \in \mathcal{D}_\omega$, then we have

$$\pi_\omega(\mathcal{A})|\Omega_\omega\rangle = \mathcal{D}_\omega \quad \text{and} \quad \langle \Omega_\omega | \pi_\omega(A) | \Omega_\omega \rangle = \omega(A). \quad (2.56)$$

Since $\mathcal{D}_\omega$ is dense in $\mathcal{H}_\omega$, we say that $|\Omega_\omega\rangle$ is *cyclic* for the representation π_ω , and ω acts as the linear form that computes expectation values for $|\Omega_\omega\rangle$. This completes the GNS construction (establishing the *GNS representation theorem*), which is unique up to isometries: given a unital $*$ -algebra $\mathcal{A}$ and a state ω , there exists a Hilbert space $\mathcal{H}_\omega$, a dense subspace $\mathcal{D}_\omega$, a $*$ -representation $\pi_\omega : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{D}_\omega)$, and a state $|\Omega_\omega\rangle \in \mathcal{H}_\omega$ such that Eq. (2.56) holds.

While the GNS construction provides a representation of the quantum theory of the real scalar field in a Hilbert space, given the large freedom available to define an algebraic state, there is no guarantee that the result will look anything like the construction we performed at the beginning of the section. There is, however, one instance where it does: when the

state ω is quasifree. In that case, the Hilbert space resulting from the GNS construction is always a Fock space, and, if ω is pure, it coincides exactly with the Fock quantization for some choice of complex structure J . Moreover, when ω is quasifree, a representation π_ω of $\mathcal{A}$ induces a representation π_ω^W of the Weyl algebra $\mathcal{W}$, such that

$$\pi_\omega^W(e^{i\phi(f)}) = e^{i\pi_\omega(\phi(f))} \equiv e^{i\hat{\phi}(f)} \quad \text{and} \quad \pi_\omega(\phi(f)) = -i\partial_t \pi_\omega^W(e^{i\phi(tf)})|_{t=0} \equiv \hat{\phi}(f), \quad (2.57)$$

as we had desired when we changed the notation from $W(f)$ to $e^{i\phi(f)}$ to identify the generators of the Weyl algebra. When ω is pure, Eq. (2.57) implies that the Hilbert space $\mathcal{H}_\omega$ carries an irreducible and strongly continuous representation of the Weyl CCR algebra, which in turn means it must be separable [51].

If $\mathcal{A}$ is a C^* -algebra, as would be the case if we decided to describe the scalar field with the Weyl algebra, then the dense subspace $\mathcal{D}_\omega$ is no longer needed in the GNS construction, and we can define π_ω directly into $\mathcal{B}(\mathcal{H}_\omega)$. However, it is also often the case that the C^* -norm is restrictive enough that many relevant observables of the theory will be left out of the representation of the local algebras $\pi_\omega(\mathcal{A}(\mathcal{O}))$, since these are only closed under the norm topology. In order to be able to consider a more general family of relevant operators of the QFT—such as arbitrary bounded smooth functions of the smeared fields—we relax the topology from the one induced by the norm to the weak operator topology in $\mathcal{B}(\mathcal{H}_\omega)$. Hence, the actual set of local operators, $\mathcal{R}(\mathcal{O})$, is identified with the weak completion of $\pi_\omega(\mathcal{A}(\mathcal{O}))$, which by von Neumann's bicommutant theorem coincides with its strong completion and its bicommutant, i.e.,

$$\mathcal{R}(\mathcal{O}) := \pi_\omega(\mathcal{A}(\mathcal{O}))'', \quad (2.58)$$

and in particular $\mathcal{R} := \pi_\omega(\mathcal{A})''$. Here, we denoted the commutant of a subalgebra $\mathcal{B}$ as

$$\mathcal{B}' := \{A \in \mathcal{B}(\mathcal{H}_\omega) : [A, B] = 0, \forall B \in \mathcal{B}\}. \quad (2.59)$$

Consequently, $\mathcal{B}''$ is the commutant of $\mathcal{B}'$, i.e., the bicommutant of $\mathcal{B}$, which coincides with $\mathcal{B}$ if and only if $\mathcal{B}$ is a von Neumann algebra.

The net of local von Neumann algebras $\{\mathcal{R}(\mathcal{O})\}$ satisfies the same axioms as the local subalgebras $\{\mathcal{A}(\mathcal{O})\}$. In particular, by the microcausality axiom, if two open causally convex sets, $\mathcal{O}_1$ and $\mathcal{O}_2$, are spacelike separated, Eq. (2.47) implies

$$\pi_\omega(\mathcal{A}(\mathcal{O}_2)) \subset \pi_\omega(\mathcal{A}(\mathcal{O}_1))' \Rightarrow \pi_\omega(\mathcal{A}(\mathcal{O}_2)) \subset \mathcal{R}(\mathcal{O}_2) \subset \mathcal{R}(\mathcal{O}_1)', \quad (2.60)$$

which demonstrates that the elements of $\mathcal{R}(\mathcal{O})$, as weak limits of observables localized in $\mathcal{O}$, should still be regarded as observables localized in $\mathcal{O}$. In fact, if the representation π_ω

is irreducible, then by Schur's lemma, $\mathcal{R} \cap \mathcal{R}' = \mathbb{C}\mathbf{1}$, i.e., $\mathcal{R}$ is a *factor*. In that case, it is usually assumed⁹ that, furthermore,

$$\mathcal{R}(\mathcal{O}) \cap \mathcal{R}(\mathcal{O})' = \mathbb{C}\mathbf{1}, \quad (2.61)$$

i.e., all local subalgebras are factors. Every factor can be classified into three *types*, broadly characterized by the projections they contain. For an irreducible representation, $\mathcal{R} = \mathcal{B}(\mathcal{H}_\omega)$ is of *type I*, meaning that it contains rank-one projections. Remarkably, however, local subalgebras are generically factors of *type III*,¹⁰ which implies that they do not contain any finite-rank projections (in particular, each projection E is such that the factor contains a unitary between $E\mathcal{H}_\omega$ and $\mathcal{H}_\omega$), and therefore that no local observable has finite-dimensional eigenspaces.

A more restrictive condition than the one set by Eq. (2.60) is the so-called *Haag property*, which states that given any compact connected set $L \subset \mathcal{M}$,

$$\bar{\mathcal{R}}(L) = \mathcal{R}(L^\perp)', \quad (2.62)$$

where

$$\bar{\mathcal{R}}(L) := \bigcap_{L \subset \mathcal{O}} \mathcal{R}(\mathcal{O}), \quad (2.63)$$

the intersection running over all open causally convex sets that contain L . This formulation of the Haag property is a weakened version of *Haag duality* [42], and it is satisfied by Fock representations of the real scalar quantum field in general curved spacetimes (see Appendix C in [27] for the general case¹¹). While Haag duality is known to hold for some special classes of sets and curved spacetimes (see, e.g., [60]), the exact conditions under which it is satisfied are not known [61].

Finally, we analyze the existence of inequivalent quantizations from the perspective of AQFT. In this context, two Hilbert space representations π_1 and π_2 of the same algebra $\mathcal{A}$ are unitarily equivalent if there exists a unitary operator $\hat{U} : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ such that

$$\hat{U}\pi_1(A)\hat{U}^\dagger = \pi_2(A), \quad \forall A \in \mathcal{A}. \quad (2.64)$$

In order to analyze under what conditions such a unitary exists, let us introduce first the notion of *normal state*: given a GNS representation associated with a state ω , not all

⁹See [52] for a compelling handwavy argument.

¹⁰More concretely, they are *hyperfinite* factors of type III₁ [21, 22], which are unique up to isometries [23]. See [53] for a pedagogical review.

¹¹See also [54–58] and [59] for the particular cases of (multi-)diamonds in Minkowski spacetime and for Rindler wedges, respectively.

algebraic states can be written in the form

$$\omega_\rho(A) = \text{Tr}(\hat{\rho} \pi_\omega(A)), \quad \forall A \in \mathcal{A}(\mathcal{M}), \quad (2.65)$$

for some $\hat{\rho} \in \mathcal{D}(\mathcal{H}_\omega)$. In the particular case in which $\mathcal{A}$ is a C^* -algebra, the states that can be associated with a density matrix via Eq. (2.65) are called *normal states* of the representation π_ω . We are now ready to state the following two results [62, 63]: let ω and ω' be states over a C^* -algebra $\mathcal{A}$, and let π_ω and $\pi_{\omega'}$ be their associated representations obtained through the GNS construction. Then,

1. *Characterization of pure states:* ω is a pure state if and only if π_ω is irreducible. Moreover, in that case, every normal state that is pure in the representation (i.e., a state whose density matrix is of the form $\hat{\rho} = |\Psi\rangle\langle\Psi|$, for some $|\Psi\rangle \in \mathcal{H}_\omega$) corresponds to a pure state of the algebra via Eq. (2.65).
2. *Unitary equivalence of representations:* if ω and ω' are pure states, then their representations are unitarily equivalent if and only if their sets of normal states coincide.

It is worth noting that this framework allows for the discussion of the more general problem of *quasiequivalence* of representations. However, we will not pursue this further here. The interested reader is referred to Sec. III.2.2 of [42].

We close this section with a remark: we have pointed out above that not all algebraic states can be associated with a density operator in a given representation. Nevertheless, as an abuse of notation, we will sometimes use $\hat{\rho}_\phi$ in future chapters to represent a generic state of the field, even though in general not all physical states will be included in the set of normal states of the selected representation. It is also worth noting that the partial state is always well-defined, even when the state is not given by a density operator: given two unital $*$ -algebras $\mathcal{A}$ and $\mathcal{B}$, and a state¹² $\omega : \mathcal{A} \otimes \mathcal{B} \rightarrow \mathbb{C}$, it is always possible to define the partial state associated with $\mathcal{A}$ as the linear form that satisfies

$$\omega_{\mathcal{A}}(A) = \omega(A \otimes \mathbf{1}_{\mathcal{B}}), \quad \forall A \in \mathcal{A}, \quad (2.66)$$

and analogously for $\mathcal{B}$. Again, in future chapters, we will often abuse the notation of a partial trace to represent the computation of the partial state.

¹²Here for simplicity we consider the algebraic tensor product, although depending on the context, other notions of tensor product that involve different types of closures of the algebraic tensor product, such as the minimal or the von Neumann tensor products, may be more appropriate [64].

2.3 Particle detector models

In this section, we review key aspects of the theoretical framework of particle detector models that will be relevant in later chapters, largely following the brief introductions provided in [2] and [9]. We begin by presenting the concept of a particle detector in its most general form before focusing on the particular case of Unruh-DeWitt models. We summarize the main results concerning their regularity and their tensions with covariance and causality, and highlight some of their many applications. For further details on how particle detectors can be used to study the properties of quantum fields in flat and curved spacetimes, we refer the reader to the book by Birrell & Davies [40] and Takagi’s comprehensive account of the Unruh effect [65]. Details on the applications of particle detectors to study quantum information in QFT settings can also be found in [66]. Several dissertations from the past decade also offer good introductions to the modern treatment of detector models from slightly different perspectives (see, e.g., [67–70]).

To begin with, consider a system moving in a general $(1+d)$ -dimensional globally hyperbolic spacetime $(\mathcal{M}, \mathbf{g})$, whose centre of mass follows a trajectory $\mathbf{z}(\tau)$ parametrized by its proper time τ . In the system’s proper reference frame, its internal dynamics are described by a finite number of non-relativistic quantum degrees of freedom, with associated Hilbert space $\mathcal{H}_\mathrm{D}$ and free Hamiltonian $\hat{H}_\mathrm{free}$. Let ϕ be a quantum field in $\mathcal{M}$ to which the system is coupled. The interaction is assumed to be localized around the system’s trajectory $\mathbf{z}(\tau)$, in a region small enough to be contained within a neighbourhood where Fermi normal coordinates $(\tau, \mathbf{z})$ are well-defined.¹³ Then, in the interaction picture, the coupling can be described by the Hamiltonian weight^{14,15}

$$\hat{h}_\mathrm{int}(\tau, \mathbf{z}) = \lambda \Lambda(\tau, \mathbf{z}) \hat{\mu}_\alpha(\tau) \hat{O}^\alpha(\tau, \mathbf{z}) + \text{H.c.}, \quad (2.68)$$

¹³In this coordinate system, the time coordinate is the trajectory’s proper time, while the spatial coordinates are such that $s^2(\tau) = \delta_{ab} z^a z^b$ generically corresponds to the square of the (spatial) geodesic distance between $(\tau, \mathbf{z})$ and $(\tau, 0)$. In particular, the trajectory is given by $\mathbf{z}(\tau) = (\tau, 0)$. For details on how to construct Fermi normal coordinates, see [71].

¹⁴The Hamiltonian weight is a scalar, which, when multiplied by the geometric factor $\sqrt{-g}$, yields the Hamiltonian *density*. This can also be unambiguously prescribed by providing the interaction action:

$$S_\mathrm{int} = \int_{\mathcal{M}} \mathrm{d}\mathbf{x} \sqrt{-g} \hat{h}_\mathrm{int}(\mathbf{x}). \quad (2.67)$$

¹⁵Notice that while the Fermi normal coordinates $(\tau, \mathbf{z})$ may only be defined within a neighbourhood of $\mathbf{z}(\tau)$, $\hat{h}_\mathrm{int}(\mathbf{x})$ is well defined everywhere in $\mathcal{M}$, as it can be extended by zero given that the support of the spacetime smearing function lies entirely within the region where the Fermi normal coordinates are defined.

where λ is the coupling strength of the interaction, $\Lambda(\tau, \mathbf{z})$ is the (possibly complex) space-time smearing function that modulates the coupling, and $\hat{\mu}_\alpha(\tau)$ and $\hat{O}^\alpha(\mathbf{z})$ are arbitrary tensor operators on $\mathcal{H}_D$ and $\mathcal{H}_\phi$, respectively, with α denoting a general multi-index, e.g., consisting of several spacetime indices. From Eq. (2.68), we can define the interaction Hamiltonian that governs the joint evolution of the system and the field as

$$\hat{H}_{\text{int}}(\tau) := \int_{\mathbb{R}^d} d\mathbf{z} \sqrt{-g} \hat{h}_{\text{int}}(\tau, \mathbf{z}). \quad (2.69)$$

Eqs. (2.68) and (2.69) represent the most general interaction Hamiltonian that describes the local coupling of a quantum field and a system with a finite number of non-relativistic quantum degrees of freedom. Such a system is referred to as a *particle detector*, and the tuple $(\mathcal{H}_D, \hat{H}_{\text{free}}, \hat{H}_{\text{int}})$ is called a *particle detector model*. It is common to assume that we are within the regime in which the particle detector can be considered a *Fermi-Walker rigid* [72–76]. This means that, in the reference frame defined by the set of Fermi normal coordinates for the detector’s centre of mass, its shape is kept constant, and therefore the spacetime smearing function factorizes as $\Lambda(\tau, \mathbf{z}) = \chi(\tau)F(\mathbf{z})$, where χ and F are the switching and (spatial) smearing functions that modulate the coupling in time and space, respectively. Eq. (2.69) then becomes

$$\hat{H}_{\text{int}}(\tau) := \lambda \chi(\tau) \hat{\mu}_\alpha(\tau) \int_{\mathbb{R}^d} d\mathbf{z} \sqrt{-g} F(\mathbf{z}) \hat{O}^\alpha(\tau, \mathbf{z}) + \text{H.c.} \quad (2.70)$$

Besides simplifying the prescription of the spacetime smearing, Fermi-Walker rigidity is regarded as a physically realistic approximation in many experimental setups [77, 78]. In the special case where ϕ is a real scalar quantum field and

$$\hat{h}_{\text{int}}(\tau, \mathbf{z}) = \lambda \chi(\tau) F(\mathbf{z}) \hat{\mu}(\tau) \hat{\phi}(\tau, \mathbf{z}), \quad (2.71)$$

for some monopole operator $\hat{\mu}$ satisfying $[\hat{\mu}, \hat{H}_{\text{free}}] \neq 0$ (to ensure that the interaction induces energy transitions), we recover the family of *Unruh-DeWitt* (UDW) detector models. These models are named after Unruh, who first introduced them in [46], and DeWitt, who later refined them [79] into their modern form. A popular instance of the UDW model that we will make use of in the following chapters corresponds to the case where the detector is a two-level system, with ground and excited states $|g\rangle$ and $|e\rangle$, free Hamiltonian $\hat{H}_{\text{free}} = \Omega|e\rangle\langle e|$, and monopole moment

$$\hat{\mu}(\tau) = |g\rangle\langle e| e^{-i\Omega\tau} + |e\rangle\langle g| e^{i\Omega\tau}, \quad (2.72)$$

where $\Omega \geq 0$ is the detector’s energy gap. Three particular cases are often of interest:

- *Gapless* detectors, where $\Omega = 0$.
- *Pointlike* detectors, where $F(\mathbf{z}) = \delta(\mathbf{z})$.
- *Delta-coupled* detectors, where $\chi(\tau) = \delta(\tau - \tau_0)$, for some fixed $\tau_0 \in \mathbb{R}$.

Particle detectors are called “detectors” because, through their interaction with the quantum field, they can gather information about it. However, to extract this information, we must compute the detector’s state after the interaction has ended. Given an initial detector-field state $\hat{\rho}_0$, which is usually assumed to be uncorrelated, i.e., of the form¹⁶ $\hat{\rho}_0 = \hat{\rho}_{\text{D},0} \otimes \hat{\rho}_{\phi,0}$, the final joint state is given (in the interaction picture) by

$$\hat{\rho} = \hat{U} \hat{\rho}_0 \hat{U}^\dagger, \quad (2.73)$$

where $\hat{U} \equiv \hat{U}(-\infty, \infty)$ is the time evolution operator associated with the complete interaction process. For the case of a UDW particle detector, $\hat{U}$ can be computed from Eqs. (2.69) and (2.71), yielding

$$\hat{U} = \mathcal{T}_\tau \exp \left(-i \int_{-\infty}^{\infty} d\tau \hat{H}_{\text{int}}(\tau) \right) = \mathcal{T}_\tau \exp \left(-i \int_{\mathcal{M}} d\mathbf{x} \sqrt{-g} \hat{h}_{\text{int}}(\mathbf{x}) \right) \quad (2.74)$$

$$= \mathcal{T}_\tau \exp \left(-i\lambda \int_{-\infty}^{\infty} d\tau \chi(\tau) \hat{\mu}(\tau) \int_{\mathbb{R}^d} d\mathbf{z} \sqrt{-g} F(\mathbf{z}) \hat{\phi}(\tau, \mathbf{z}) \right), \quad (2.75)$$

where $\mathcal{T}_\tau \exp$ denotes the time-ordered exponential with respect to the proper time τ , and the limits of the time integral can be extended to $\pm\infty$ because we are computing the joint state after the coupling has been switched off. Notice that time ordering may only become ambiguous for spacelike-separated points. This is why, as long as

$$[\hat{h}_{\text{int}}(\mathbf{x}), \hat{h}_{\text{int}}(\mathbf{x}')] = 0, \quad \forall \mathbf{x} \in \mathcal{M}, \mathbf{x}' \in \mathcal{M} \setminus \mathcal{J}(\mathbf{x}), \quad (2.76)$$

it is not necessary to specify the time parameter with respect to which the order should be implemented. By analogy with the microcausality axiom (Eq. (2.47)), Eq. (2.76) is known as the *microcausality condition*, and it ensures that the dynamical evolution of the theory respects covariance. The Hamiltonian weight of Eq. (2.71), however, may violate it in general, and thus the time ordering in Eq. (2.75) requires a specification of the time parameter. Concretely, given spacelike-separated points $(\tau, \mathbf{z})$ and $(\tau', \mathbf{z}')$, it is straightforward to compute that [80]

$$[\hat{h}_{\text{int}}(\tau, \mathbf{z}), \hat{h}_{\text{int}}(\tau', \mathbf{z}')] = \lambda^2 \chi(\tau) \chi(\tau') F(\mathbf{z}) F(\mathbf{z}') [\hat{\mu}(\tau), \hat{\mu}(\tau')] \hat{\phi}(\tau, \mathbf{z}) \hat{\phi}(\tau', \mathbf{z}'), \quad (2.77)$$

¹⁶Recall from Sec. 2.2 that, as an abuse of notation, we sometimes represent a general state of the field as a density operator.

where we used that $[\hat{\phi}(\tau, \mathbf{z}), \hat{\phi}(\tau', \mathbf{z}')] = 0$, and from Eq. (2.72),

$$[\hat{\mu}(\tau), \hat{\mu}(\tau')] = 2i \sin \Omega(\tau - \tau') (|e\rangle\langle e| - |g\rangle\langle g|). \quad (2.78)$$

Hence, unless the particle detector is gapless, pointlike, or delta-coupled, there will exist pairs of spacelike-separated points for which the commutator in Eq. (2.77) does not vanish.

In general, it is not possible to compute the evolution operator explicitly from Eq. (2.75). However, assuming that the detector is *weakly coupled* to the field, i.e., if $\lambda \ll 1$, we can express $\hat{U}$ as a perturbative series of terms proportional to increasing powers of λ , keeping only a finite number of them, and neglecting the higher-order ones. Namely, the evolution operator can be expanded in a Dyson series as

$$\hat{U} = \mathbb{1} + \sum_{k=1}^{\infty} \hat{U}^{(k)}, \quad (2.79)$$

where $\hat{U}^{(k)}$ is proportional to λ^k . This induces a similar perturbative expansion for the detector-field final joint state,

$$\hat{\rho} = \hat{\rho}_0 + \sum_{k=1}^{\infty} \hat{\rho}^{(k)}, \quad \text{with} \quad \hat{\rho}^{(k)} = \sum_{r=0}^k \hat{U}^{(r)} \hat{\rho}_0 \hat{U}^{(k-r)\dagger}, \quad (2.80)$$

where we denoted $\hat{U}^{(0)} \equiv \mathbb{1}$. In practice, most relevant phenomena in relativistic quantum information manifest at first or second order in perturbation theory. Up to order λ^2 , the partial state of the detector is given by

$$\hat{\rho}_D = \hat{\rho}_{D,0} + i\lambda \int_{-\infty}^{\infty} d\tau \chi(\tau) W_F(\tau) \hat{A}_1(\tau) + \lambda^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\tau d\tau' \chi(\tau) \chi(\tau') W_F(\tau, \tau') \hat{A}_2(\tau, \tau'), \quad (2.81)$$

where

$$W_F(\tau) = \int_{\mathbb{R}^d} d\mathbf{z} \sqrt{-g} F(\mathbf{z}) w_1((\tau, \mathbf{z})), \quad (2.82)$$

$$W_F(\tau, \tau') = \int_{\mathbb{R}^{2d}} d\mathbf{z} d\mathbf{z}' \sqrt{gg'} F(\mathbf{z}) F(\mathbf{z}') w_2((\tau, \mathbf{z}), (\tau', \mathbf{z}')), \quad (2.83)$$

$$\hat{A}_1(\tau) = [\hat{\rho}_{D,0}, \hat{\mu}(\tau)], \quad (2.84)$$

$$\hat{A}_2(\tau, \tau') = \hat{\mu}(\tau') \hat{\rho}_{D,0} \hat{\mu}(\tau) - \theta(\tau - \tau') \hat{\mu}(\tau) \hat{\mu}(\tau') \hat{\rho}_{D,0} - \theta(\tau' - \tau) \hat{\rho}_{D,0} \hat{\mu}(\tau) \hat{\mu}(\tau'). \quad (2.85)$$

For the integrals in Eqs. (2.82) and (2.83) to converge, the smearing function F must exhibit appropriate infrared (IR) and ultraviolet (UV) behaviours [72–74, 81]. In particular, a pointlike detector will always display divergences. Some quantities, such as the excitation probability of the detector, can be regularized in the pointlike case by considering adiabatic switchings [82]; however, the model in general remains singular. It is in this sense that it can be said that UDW models, and particle detectors in general, require “either a singular or a non-local coupling” [83].

Particle detector models are intrinsically an *effective* description. First of all, a particle detector is a *localized* system, meaning its spatial extension is restricted to a compact set, except possibly for negligible tails. Within the regime of validity of the model, the spatial extension of the detector must be small enough that, for the phenomena under analysis, relativistic effects are negligible for its internal dynamics. This is why, at least in the proper reference frame of the detector, it can be described with a non-relativistic quantum theory of a few degrees of freedom.¹⁷

It is due to this non-relativistic description that the model may conflict with general relativity in certain scenarios. Indeed, all inconsistencies with relativity that may arise in UDW models can be traced back to the fact that spatially smeared detectors couple to multiple spacelike-separated points. Specifically, the degrees of freedom of the detector couple simultaneously to the field at all the events in a hypersurface of constant τ inside the support of the spatial smearing F , and evolve influenced by all of them at once—unless the detector is delta-coupled or gapless. This is the physical intuition behind the violation of the microcausality condition shown in Eq. (2.77).

One consequence of the violation of the microcausality condition is the breakdown of covariance. Specifically, recall that the time evolution operator (see Eq. (2.75)) is prescribed using a time-ordered exponential that privileges the proper time τ . The Hamiltonian weight of Eq. (2.71), though covariant itself, does not yield a covariant time evolution operator in general. The discrepancy between evolution operators for different time parameters is controlled by the detector’s characteristic scales and vanishes as its size shrinks [80]. This means that particle detectors give effectively covariant predictions provided that their proper sizes are sufficiently small.

Another consequence of violating microcausality is that it may compromise causality. Indeed, the simultaneous backreaction of the detector on the field at spacelike-separated events can make the model predict faster-than-light propagation of information. However, these violations only arise in setups involving at least three particle detectors [84], appear

¹⁷For a careful analysis of the conditions and regimes under which these approximations are fulfilled, see [76].

at subleading order in perturbation theory [85], and are also controlled by the scale of the smearing function [86].

In summary, the effective non-relativistic nature of spatially-smeared particle detector models can lead to inconsistencies with general relativity, but the regimes where these inconsistencies appear are controlled by the scale of the spatial smearing of the detector and how it compares with other scales in the problem (e.g., interaction duration, or proper acceleration). Ultimately, these scales set the limits of validity of particle detector models—thus delimiting the range of physical scenarios for which the models produce faithful predictions.

While particle detector models have a limited (and well-understood) regime of validity, they are still very useful tools. They have found many applications in studying properties of quantum fields in flat and curved spacetimes. For instance, they have been used to study the Unruh effect [87–91] and Hawking radiation [46, 92, 93], as well as directly measuring the field’s correlation functions [94, 95]. They have also been used to implement numerous quantum information protocols, such as quantum collect calling [96–98], quantum energy teleportation [99–102], entanglement harvesting [103–115] and quantum communication [116–119]. Moreover, despite their simplicity, particle detector models have been shown to provide reliable approximations for localized quantum systems interacting with quantum fields, such as atoms [78, 120, 121] and superconducting qubits [122–124].

Act 1: Measurements and causality

Chapter 3

Causal maps in quantum field theory

In this chapter, we define and characterize localized causal operations in a real scalar quantum field theory, and show that they precisely correspond to the operations satisfying various equivalent no-signalling conditions proposed in previous works. The simplicity of this characterization allows us to define a faithful quantifier of how much a map enables faster-than-light signalling in a given field state. Finally, we apply our results to revisit relevant examples, providing straightforward proofs of their causal behaviour.

3.1 Sorkin’s impossible measurements

When we are first faced with the task of developing a theory of measurement for quantum fields, nothing *a priori* precludes us from using the standard quantum theory of measurement (cf. Sec. 2.1). Indeed, once a representation of the algebra of the QFT is specified, quantum fields—despite their daunting, uncountably infinite number of degrees of freedom—are described within a Hilbert space, which, moreover, is separable if the representation is “sufficiently regular.”¹ Operations, channels, measurements, updates, instruments, and schemes can be defined without encountering any pathological issues. However, in 1993, Sorkin pointed out that performing a projective measurement on a quantum field, even a non-selective one, can violate causality by allowing faster-than-light signalling [24]. The idea is simple: let three experimenters perform operations in different regions of a flat spacetime, A, B, and C, as depicted in Fig. 3.1. The experimenter in B performs a ‘rank-one projective measurement’, i.e., an ideal measurement of an observable

¹See the assumptions made about the representation at the beginning of Sec. 3.2.

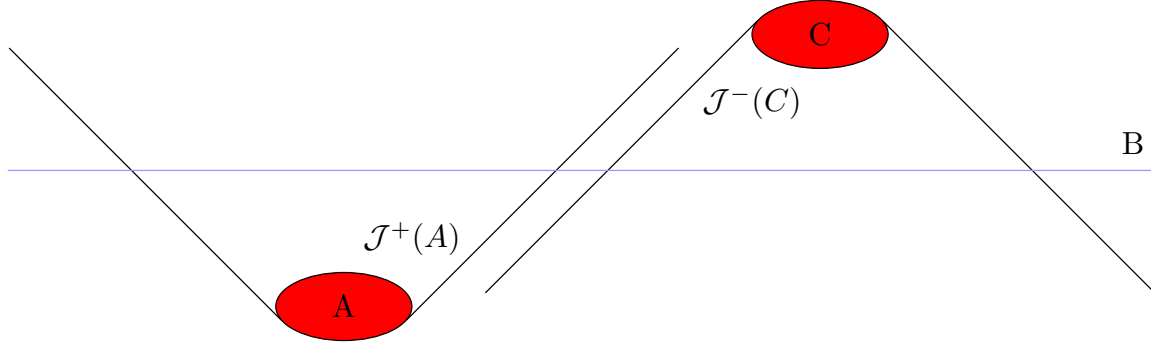


Figure 3.1: Sorkin’s setup: Projective measurements may allow faster-than-light signalling.

with a one-dimensional eigenspace (in Sorkin’s example, they measure whether or not the field is in a particular superposition of the vacuum and a one-particle state). In coordinate time, the measurement occurs after A’s operations and before C’s operations. This ordering of events is consistent as far as all of A is in the past of B, and all of C is in the future of B. It can be shown that with an appropriate choice of parameters for A’s operations and B’s measurement, the outcome of C’s operations depends on A’s operations, even though A and C are spacelike separated and the experimenter at B does not communicate with the rest.

Sorkin’s example highlights a fundamental issue: because relativity—and therefore locality and causality—is at the very core of the framework of quantum field theory, it is not sufficient for operations to be well-defined as maps between operators on a Hilbert space; they must also respect the relativistic structure of the theory. Some operations, though mathematically well-defined, must be discarded as unphysical if they fail to comply with relativistic constraints.

In the process of purging the unphysical operations from our measurement theory, one of the first observations one can make about Sorkin’s example is that the observable measured in B has a one-dimensional eigenspace. This implies that the corresponding projection (and the observable) cannot belong to any local subalgebra of the field (cf. Sec. 2.2). The measurement performed in B is thus *non-localizable* and unfeasible, at least under the assumption that performing an operation along a complete Cauchy hypersurface of spacetime is not realistically possible. The first condition we impose on operations,

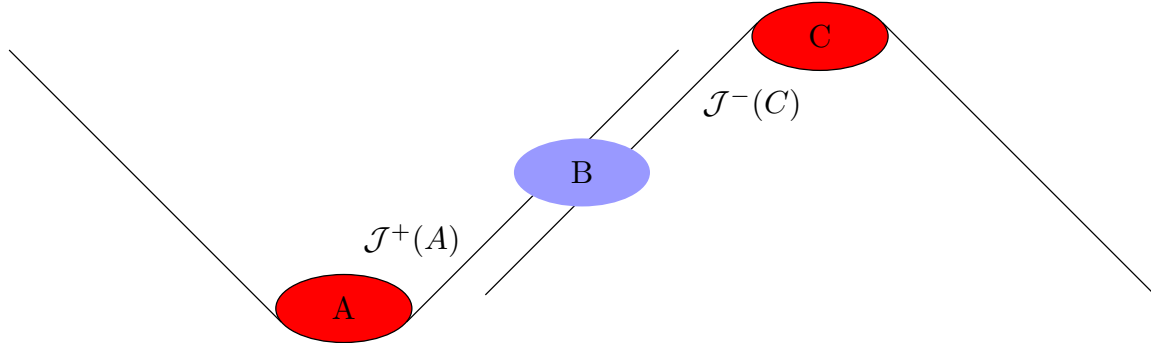


Figure 3.2: Localized projective measurements may still allow faster-than-light signalling.

therefore, is that they must be *localized* in a compact set. In particular, this automatically discards all measurements of observables with finite-dimensional eigenspaces, which, due to the algebraic structure of the theory, are necessarily non-localizable.

However, it would be a mistake to attribute the causality violations in Sorkin’s setup solely to the non-localizability of the measurement. Indeed, one can consider ideal measurements of localized observables, whose spectral resolutions will contain exclusively infinite-rank projectors with a compact support in spacetime [125–127]. For these projectors to be well-defined, their support must be extended (i.e., non-pointlike). As a consequence, one can construct configurations such as the one depicted in Fig. 3.2, for which in general the measurement performed in B still allows faster-than-light signalling [128, 129]. The issue with Sorkin’s measurement was therefore not merely its lack of localizability, but rather its internal dynamics, which allowed information to propagate faster than light—a pathological behaviour that even localizable operations can display.

While we shall keep localizability as a constraint—despite our latter remark, non-localizable operations remain physically unrealizable—it is clear that additional conditions must be imposed on maps for them to be valid operations in QFT. Namely, physical operations must respect the causal structure of the theory.

The task of identifying maps that enable faster-than-light signalling in real scalar quantum field theory has been addressed in previous works [128, 129]. In the following section, we aim to advance this analysis by providing several characterizations and properties of

these maps.

3.2 Localized causal maps

In this section, we refine some of the results in [128, 129], defining the set of localized causal maps, and proving that it coincides with the set of maps that fulfill a series of equivalent no-signalling conditions. We will end the section characterizing the set of localized causal maps in terms of their Kraus operators.

Let $\mathcal{A}$ be the C^* -algebra that describes the theory of the real scalar quantum field in the spacetime $\mathcal{M}$, and let π_ω be the GNS representation induced by a state ω , which we assume satisfies all the properties usually associated with ‘vacuum representations’, namely, that π_ω is irreducible (i.e., ω is pure) and faithful, and that the GNS state $|\Omega_\omega\rangle \in \mathcal{H}_\omega$ is cyclic and separating for all local algebras.² These conditions also imply that the GNS Hilbert space $\mathcal{H}_\omega$ must be separable [131]. As before, we denote the resulting local von Neumann algebras by $\mathcal{R}(\mathcal{O})$, and we assume that the Haag property is satisfied. A general channel acting on the field is then described by a completely positive, trace-preserving $\Phi : \mathcal{B}_1(\mathcal{H}_\omega) \rightarrow \mathcal{B}_1(\mathcal{H}_\omega)$ (cf. Sec. 2.1), pending identification of whether it is a *physical* channel or not. Because quantum fields are more naturally described in the Heisenberg picture, in order to study an operation Φ we will actually analyze its adjoint $\Phi^\dagger : \mathcal{B}(\mathcal{H}_\omega) \rightarrow \mathcal{B}(\mathcal{H}_\omega)$, which we assume to be continuous in the weak operator topology. This sets up the landscape for the following definition:

Definition (Localizable channel). *We will say that a channel Φ is localizable in a compact connected causally convex set $L \subset \mathcal{M}$ if*

$$\Phi^\dagger \Big|_{\mathcal{R}(L^\perp)} = \text{id}. \quad (3.1)$$

In the Schrödinger picture, Eq. (3.1) guarantees that the expectation values of operators whose support is spacelike separated from L do not change after the action of Φ , for any state of the field. The definition above is a slightly modified version of the definition of “local update” given in [129]. Here, we have chosen to use the word ‘localizable’ instead, to avoid confusions with the concept of locality that refers to the nature of interactions in spacetime.

²This is known as the *Reeh-Schlieder property*, which in Minkowski spacetime follows from the Reeh-Schlieder theorem [130]. For pedagogical introductions to its relevance, see [44] and [61].

The typical example of a localizable channel is one whose Kraus operators belong to $\mathcal{R}(\mathcal{O})$, for some $\mathcal{O} \subset L$, in which case, by microcausality, they commute with all the elements of $\mathcal{R}(L^\perp)$, and, because $\Phi^\dagger(\mathbb{1}) = \mathbb{1}$, Eq. (3.1) is satisfied. Under the assumption that π_ω satisfies the Haag property, however, these are essentially the only maps localizable in L :

Proposition 1 (Characterization of localizable channels). *A channel Φ with Kraus operators $\{\hat{K}_j\}$ is localizable in a compact connected causally convex set $L \subset \mathcal{M}$ if and only if $\hat{K}_j \in \bar{\mathcal{R}}(L)$, for all j .*

Proof. The *if* direction of the result is straightforward: let $\hat{A}$ be an operator belonging to the algebra of the causal complement of L , $\hat{A} \in \mathcal{R}(L^\perp)$. If $\hat{K}_j \in \bar{\mathcal{R}}(L) = \mathcal{R}(L^\perp)'$ for all j , then by definition $[\hat{K}_j, \hat{A}] = 0$, and therefore

$$\Phi^\dagger(\hat{A}) = \sum_{j=1}^{\infty} \hat{K}_j^\dagger \hat{A} \hat{K}_j = \sum_{j=1}^{\infty} \hat{K}_j^\dagger \hat{K}_j \hat{A} = \Phi^\dagger(\mathbb{1}) \hat{A} = \mathbb{1} \hat{A} = \hat{A}. \quad (3.2)$$

Conversely, assume that Φ is localizable in L , and let $\hat{A}$ be an operator in $\mathcal{R}(L^\perp)$. Then, by Eq. (3.1),

$$\Phi^\dagger(\hat{A}) = \hat{A}. \quad (3.3)$$

In particular, because the local algebras are closed under involution,

$$\Phi^\dagger(\hat{A}^\dagger \hat{A}) = \hat{A}^\dagger \hat{A} = \Phi^\dagger(\hat{A}^\dagger) \Phi^\dagger(\hat{A}) \quad \text{and} \quad \Phi^\dagger(\hat{A} \hat{A}^\dagger) = \hat{A} \hat{A}^\dagger = \Phi^\dagger(\hat{A}) \Phi^\dagger(\hat{A}^\dagger), \quad (3.4)$$

which means that $\hat{A}$ belongs to the *multiplicative domain* of $\Phi^\dagger$. Now, let $\{|j\rangle\}_{j=1}^\infty$ be an orthonormal basis of an auxiliary Hilbert space $\mathcal{K}$, and define

$$\Psi^\dagger : \mathcal{B}(\mathcal{H}_\omega \otimes \mathcal{K}) \rightarrow \mathcal{B}(\mathcal{H}_\omega), \quad \hat{Y} \mapsto \hat{V}^\dagger \hat{Y} \hat{V}, \quad \text{with} \quad \hat{V} = \sum_{j=1}^{\infty} \hat{K}_j \otimes |j\rangle. \quad (3.5)$$

By definition,

$$\Psi^\dagger(\hat{B} \otimes |i\rangle\langle j|) = \hat{K}_i^\dagger \hat{B} \hat{K}_j, \quad \Psi^\dagger(\hat{B} \otimes \mathbb{1}) = \Phi^\dagger(\hat{B}), \quad \forall \hat{B} \in \mathcal{B}(\mathcal{H}_\omega). \quad (3.6)$$

By Eq. (3.4), $\hat{A} \otimes \mathbb{1}$ belongs to the multiplicative domain of $\Psi^\dagger$, for all $\hat{A} \in \mathcal{R}(L^\perp)$. By Thm. 3.18 in [132], we have that

$$\Psi^\dagger(\hat{K}_i \hat{A} \otimes |i\rangle\langle j|) = \Psi^\dagger(\hat{K}_i \otimes |i\rangle\langle j|) \Psi^\dagger(\hat{A} \otimes \mathbb{1}), \quad (3.7)$$

which, from Eq. (3.6), implies

$$\hat{K}_i^\dagger \hat{K}_i \hat{A} \hat{K}_j = \hat{K}_i^\dagger \hat{K}_i \hat{K}_j \Phi^\dagger(\hat{A}) = \hat{K}_i^\dagger \hat{K}_i \hat{K}_j \hat{A}, \quad (3.8)$$

for all i, j . Summing over i , and since $\sum_i \hat{K}_i^\dagger \hat{K}_i = \mathbb{1}$, we find that, for all j ,

$$\hat{A} \hat{K}_j = \hat{K}_j \hat{A}. \quad (3.9)$$

Therefore, all the Kraus operators commute with all the elements in $\mathcal{R}(L^\perp)$, i.e., they belong to the commutant of $\mathcal{R}(L^\perp)$, which by the Haag property implies that they belong to $\bar{\mathcal{R}}(L)$. $\square$

Notice that, due to the dynamics of the field, a given channel will, in general, be localizable in more than one region. This is simply because multiple compact, connected, causally convex sets may share the same causal complement $L^\perp$. This subtlety motivates the following distinction: we say that a channel is *localized* in $L \subset \mathcal{M}$ if L is the region of spacetime (or, rather, the closure of its causal hull) where the channel is physically implemented. While a *localized* channel is always *localizable* in the same set, the specific region where the operation takes place determines between which regions the map is defined.

Indeed, in a way analogous to how, in non-relativistic quantum mechanics, the adjoint channel establishes a correspondence between operators in the *future* of the time interval where the interaction occurs (in the interaction picture) and operators in the *past* of that interval (in the Heisenberg picture), in quantum field theory the channel maps local operators from sets *not in the past* of the region where it is implemented to local operators from sets *not in the future*. Specifically, given a channel localized in L , we define the ‘in’ and ‘out’ regions [27, 129]:

$$L_{\text{in}} = \mathcal{M} \setminus \mathcal{J}^+(L), \quad L_{\text{out}} = \mathcal{M} \setminus \mathcal{J}^-(L), \quad (3.10)$$

so that the adjoint of the channel is intuitively understood to take operators in $\mathcal{R}(L_{\text{out}})$ to operators in $\mathcal{R}(L_{\text{in}})$.

Both L_{in} and L_{out} are open, causally convex, and globally hyperbolic [133]. Moreover, because L is compact, each of them contains (infinitely many) Cauchy hypersurfaces of $\mathcal{M}$ [134, 135]. In particular, the time slice axiom implies that $\mathcal{R}(L_{\text{in}}) = \mathcal{R}(L_{\text{out}}) = \mathcal{R}$, which may suggest that specifying these domains is redundant. However, while the in and out regions do not affect the formal definition of a channel, they play a crucial role in understanding its causal behaviour.

Now, in order to continue, we must address a subtle yet fundamental question: what makes a channel causal (or not)? To answer this, we can turn our eyes to the concept of

causality in partial differential equations. In that context, the hyperbolicity of the field's dynamical equations implies that the value of a solution at any given point in space and time can be computed exclusively from initial data within its causal past. For the *free* real scalar field, this principle manifests in Eq. (2.21) along with the fact that the commutator $E(\mathbf{x}, \mathbf{x}') = 0$ when $\mathbf{x}$ and $\mathbf{x}'$ are spacelike separated.

In our case, however, the field's evolution is not governed solely by its intrinsic dynamics, but it is also influenced by the action of the channel. Within each 'in' and 'out' region, the field follows its usual dynamical evolution (given by the quantized version of Eq. (2.22)). The role of the channel is therefore to determine how the field in the 'out' region relates to the field in the 'in' region. Specifically, given a Cauchy hypersurface contained in L_{in} , the field operator at a point $\mathbf{x} \in L_{\text{out}}$ (and its higher momenta³) should be expressible purely in terms of the field and momentum operators on the subset of the Cauchy hypersurface that lies within $\mathcal{J}^-(\mathbf{x})$. This can be rigorously formulated as follows:

Definition (Localized causal channel). *We will say that a channel Φ which is localized in a compact connected causally convex set $L \subset \mathcal{M}$ is causal if, for every $g \in \mathcal{C}_0^\infty(\mathcal{M})$ such that $\text{supp } g \subset L_{\text{out}}$, and every $\mathcal{O} \in L_{\text{in}}$ that contains a Cauchy hypersurface of the support of g ,*

$$\Phi^\dagger(e^{i\hat{\phi}(g)}) \in \mathcal{R}(\mathcal{O}). \quad (3.11)$$

The following proposition shows that this notion of causality can be expressed in a number of equivalent ways:

Proposition 2 (Causality conditions). *Let Φ be a channel localized in a compact connected causally convex set $L \subset \mathcal{M}$. The following statements are equivalent:*

- (i) Φ is causal.
- (ii) (Time slice property⁴) *If $\mathcal{O}_1 \subset L_{\text{in}}, \mathcal{O}_2 \subset L_{\text{out}}$ are such that $\mathcal{O}_1$ contains a Cauchy hypersurface of $\mathcal{O}_2$, then $\Phi^\dagger(\mathcal{R}(\mathcal{O}_2)) \subset \mathcal{R}(\mathcal{O}_1)$.*
- (iii) *For all $f, g \in \mathcal{C}_0^\infty(\mathcal{M})$ with $\text{supp } f \subset L_{\text{in}}, \text{supp } g \subset L_{\text{out}}$, and $\text{supp } f \subset (\text{supp } g)^\perp$,*

$$[e^{i\hat{\phi}(f)}, \Phi^\dagger(e^{i\hat{\phi}(g)})] = 0, \quad (3.12)$$

³This condition is necessary in this situation because, unlike dynamical evolution, a general channel need not be unitary, and in particular it is not necessarily multiplicative.

⁴This is the same "past-support non-increasing" (PSNI) property formulated in [129].

or, equivalently,

$$\Upsilon_f^\dagger \circ \Phi^\dagger(e^{i\hat{\phi}(g)}) = \Phi^\dagger(e^{i\hat{\phi}(g)}) = (f \cdot \Phi)^\dagger(e^{i\hat{\phi}(g)}), \quad (3.13)$$

where Υ_f denotes the unitary channel $\Upsilon_f(\hat{A}) := e^{-i\hat{\phi}(f)} \hat{A} e^{i\hat{\phi}(f)}$, and

$$(f \cdot \Phi)(\hat{A}) := \sum_{j=1}^{\infty} [\Upsilon_f(\hat{K}_j)] \hat{A} [\Upsilon_f(\hat{K}_j)]^\dagger \quad (3.14)$$

is the channel that results from applying Υ_f to Φ 's Kraus operators.

Proof. To prove that (i) implies (ii), it suffices to observe that any finite product of exponentials of smeared fields can be expressed in terms of a single exponential possibly multiplied by a complex phase. Explicitly,

$$e^{i\hat{\phi}(g_1+g_2)} = e^{\frac{i}{2}E(g_1,g_2)} e^{i\hat{\phi}(g_1)} e^{i\hat{\phi}(g_2)}, \quad (3.15)$$

$$\begin{aligned} e^{i\hat{\phi}(g_1+g_2+g_3)} &= e^{\frac{i}{2}E(g_1+g_2,g_3)} e^{i\hat{\phi}(g_1+g_2)} e^{i\hat{\phi}(g_3)} \\ &= e^{\frac{i}{2}E(g_1+g_2,g_3)} e^{\frac{i}{2}E(g_1,g_2)} e^{i\hat{\phi}(g_1)} e^{i\hat{\phi}(g_2)} e^{i\hat{\phi}(g_3)}, \end{aligned} \quad (3.16)$$

By induction, this generalizes to arbitrary finite products. Consequently, any linear combination of finite products of exponentials of smeared fields can be rewritten as a linear combination of exponentials of smeared fields alone.

Now, let $\mathcal{O}_1 \subset L_{\text{in}}, \mathcal{O}_2 \subset L_{\text{out}}$ be such that $\mathcal{O}_1$ contains a Cauchy hypersurface of $\mathcal{O}_2$. Then, for every collection $\{g_1, \dots, g_n\} \subset \mathcal{C}_0^\infty(\mathcal{O}_2)$, we have that

$$\Phi^\dagger \left(\sum_{j=1}^n c_j e^{i\hat{\phi}(g_j)} \right) = \sum_{j=1}^n c_j \Phi^\dagger(e^{i\hat{\phi}(g_j)}) \in \mathcal{R}(\mathcal{O}_1), \quad (3.17)$$

where we used the causality of Φ and the fact that $\mathcal{O}_1$ contains a Cauchy hypersurface of $\text{supp } g_j$ for each j . Furthermore, from our initial observation, $\mathcal{R}(\mathcal{O}_2)$ is the weak closure of linear combinations of exponentials of smeared fields. Because $\mathcal{R}(\mathcal{O}_1)$ is closed under weak limits by definition, Eq. (3.17) and the weak continuity of $\Phi^\dagger$ imply

$$\Phi^\dagger(\mathcal{R}(\mathcal{O}_2)) \subset \mathcal{R}(\mathcal{O}_1). \quad (3.18)$$

To prove that (ii) implies (iii), let $f, g \in \mathcal{C}_0^\infty(\mathcal{M})$ such that $\text{supp } f \subset L_{\text{in}}, \text{supp } g \subset L_{\text{out}}$, and their supports are spacelike separated. By Lemma 3 in [28], there exists a Cauchy hypersurface $\Sigma \subset L_{\text{in}} \setminus (\mathcal{J}^-(\text{supp } f) \cup \mathcal{J}^+(\text{supp } g))$. Define

$$\Sigma_g := \Sigma \cap \mathcal{J}^-(\text{supp } g), \quad \Sigma_f := \Sigma \cap \mathcal{J}^+(\text{supp } f). \quad (3.19)$$

Since $\mathcal{M}$ is globally hyperbolic, Σ_g and Σ_f are compact, spacelike separated sets, and there exist open, causally convex sets $\mathcal{O}_g$ and $\mathcal{O}_f$ such that $\Sigma_g \subset \mathcal{O}_g$, $\Sigma_f \subset \mathcal{O}_f$, and $\mathcal{O}_g$ and $\mathcal{O}_f$ are spacelike separated [133]. By the time slice axiom, we have

$$e^{i\hat{\phi}(f)} \in \mathcal{R}(\mathcal{O}_f), \quad (3.20)$$

and because Φ satisfies the ‘time slice property’ (ii),

$$\Phi^\dagger(e^{i\hat{\phi}(g)}) \in \mathcal{R}(\mathcal{O}_g). \quad (3.21)$$

Since $\mathcal{O}_f$ and $\mathcal{O}_g$ are spacelike separated, microcausality implies that

$$[e^{i\hat{\phi}(f)}, \Phi^\dagger(e^{i\hat{\phi}(g)})] = 0, \quad (3.22)$$

as we wanted to show. Notice that expanding the commutator and multiplying on the right by $e^{-i\hat{\phi}(f)}$, we obtain the equivalent expression

$$e^{i\hat{\phi}(f)} \Phi^\dagger(e^{i\hat{\phi}(g)}) e^{-i\hat{\phi}(f)} - \Phi^\dagger(e^{i\hat{\phi}(g)}) = 0, \quad (3.23)$$

which yields Eq. (3.13).

Finally, to show that (iii) implies (i), let $g \in \mathcal{C}_0^\infty(L_{\text{out}})$ and let $\mathcal{O} \subset L_{\text{in}}$ be an open set containing a Cauchy hypersurface of $\text{supp } g$. We assume that $\text{supp } g$ is connected (the case with several connected components generalizes straightforwardly). Define Σ_g as the intersection of this Cauchy hypersurface with $\mathcal{J}^-(\text{supp } g)$, which is compact, connected, and a proper subset of $\mathcal{O}$. Because $\mathcal{M}$ is Hausdorff, paracompact and locally connected, and since $\mathcal{O}$ is causally convex, there exists a compact connected causally convex⁵ set L_g such that $\Sigma_g \subset \text{int } L_g$ and $L_g \subset \mathcal{O}$.

Now, any $f \in \mathcal{C}_0^\infty(L_{\text{in}})$ whose support is spacelike separated from L_g , is also spacelike separated from $\text{supp } g$ (since L_g contains a Cauchy hypersurface of $\text{supp } g$). Then, by condition (iii), we have

$$[e^{i\hat{\phi}(f)}, \Phi^\dagger(e^{i\hat{\phi}(g)})] = 0. \quad (3.24)$$

Applying an argument identical to the one used in proving $[(i) \Rightarrow (ii)]$, we conclude that $\Phi^\dagger(e^{i\hat{\phi}(g)})$ commutes with all the corresponding (finite) linear combinations, and by taking the weak limit we conclude that

$$[\hat{A}, \Phi^\dagger(e^{i\hat{\phi}(g)})] = 0, \quad \forall \hat{A} \in \mathcal{R}(L_{\text{in}} \cap L_g^\perp). \quad (3.25)$$

⁵Namely, because $\mathcal{M}$ is Hausdorff and locally compact, we can guarantee the existence of such a compact set. Because it is locally connected, we can guarantee that Σ_g lies in the compact connected component that contains it. Finally, because $\mathcal{O}$ is causally convex, we can take the causal hull of the resulting set, which will fulfill all the prerequisites.

It remains to show that, in fact, $\mathcal{R}(L_{\text{in}} \cap L_g^\perp) = \mathcal{R}(L_g^\perp)$. Since $L_g^\perp$ is causally convex and open, it is globally hyperbolic. By Proposition 4 in [135], there exists a continuous time function $t : L_g^\perp \rightarrow \mathbb{R}$ that is onto and strictly increasing along causal curves, such that each level set $t^{-1}(a)$ is a Cauchy hypersurface of $L_g^\perp$, for all $a \in \mathbb{R}$.

Using that L is compact and $L_g \subset L_{\text{in}}$, it is easy to see that $t(L \cap L_g^\perp)$ is always bounded from below.⁶ Thus, there exists $b \in \mathbb{R}$ such that $b < t(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{J}^+(L) \cap L_g^\perp$. Consequently, $t^{-1}(b) \subset L_{\text{in}} \cap L_g^\perp$, and by construction it is a Cauchy hypersurface of $L_g^\perp$. By the time slice axiom, we conclude that $\mathcal{R}(L_{\text{in}} \cap L_g^\perp) = \mathcal{R}(L_g^\perp)$, as promised. We can then rewrite Eq. (3.25) as

$$[\hat{A}, \Phi^\dagger(e^{i\hat{\phi}(g)})] = 0, \quad \forall \hat{A} \in \mathcal{R}(L_g^\perp). \quad (3.26)$$

By the Haag property,

$$\Phi^\dagger(e^{i\hat{\phi}(g)}) \in \bar{\mathcal{R}}(L_g) \subset \mathcal{R}(\mathcal{O}), \quad (3.27)$$

which establishes that Φ is causal. $\square$

The causality of a channel is related to, and in fact equivalent to, several no-signalling conditions previously studied in the literature [128, 129]. Some of these equivalences were already proven in [129]; however, here we present a different, straightforward proof:

Proposition 3 (Causality and no-signalling). *Let Φ be a channel localized in a compact connected causally convex set $L \subset \mathcal{M}$. The following statements are equivalent:*

(i) Φ is causal.

(ii) If $\mathcal{O}_1 \subset L_{\text{in}}, \mathcal{O}_2 \subset L_{\text{out}}$ are spacelike separated, then

$$[\hat{B}, \Phi^\dagger(\hat{A})] = 0, \quad \forall \hat{A} \in \mathcal{R}(\mathcal{O}_1), \hat{B} \in \mathcal{R}(\mathcal{O}_2). \quad (3.28)$$

(iii) (Strong no-signalling condition) For any channel Ψ localized in a compact connected causally convex set $S \subset L_{\text{in}}$,

$$\Psi^\dagger \circ \Phi^\dagger(\hat{A}) = \Phi^\dagger(\hat{A}), \quad \forall \hat{A} \in \mathcal{R}(L_{\text{out}} \cap S^\perp). \quad (3.29)$$

⁶If not, then there would exist a sequence $\{\mathbf{x}_n\}_{n=1}^\infty \subset L \cap L_g^\perp$ with $t(\mathbf{x}_n) < -n$ for all $n \in \mathbb{N}$, and a convergent subsequence accumulating at some $\mathbf{x} \in L \cap \partial \mathcal{J}^+(L_g)$. However, choosing a point $\mathbf{y} \in L_g^\perp$ in the past of $\mathbf{x}$, infinitely many elements of the subsequence would be in the future of $\mathbf{y}$. Since $t(\mathbf{y})$ is finite, this contradicts the assumption that $t(\mathbf{x}_n) \rightarrow -\infty$ as $n \rightarrow \infty$.

(iv) (Causal no-signalling condition) *For any causal channel Ψ localized in a compact connected causally convex set $S \subset L_{\text{in}}$,*

$$\Psi^\dagger \circ \Phi^\dagger(\hat{A}) = \Phi^\dagger(\hat{A}), \quad \forall \hat{A} \in \mathcal{R}(L_{\text{out}} \cap S^\perp). \quad (3.30)$$

(v) (Weak no-signalling condition) *For any channel Ψ localized in a compact connected causally convex set $S \subset L_{\text{in}}$ that satisfies the strong no-signalling condition,*

$$\Psi^\dagger \circ \Phi^\dagger(\hat{A}) = \Phi^\dagger(\hat{A}), \quad \forall \hat{A} \in \mathcal{R}(L_{\text{out}} \cap S^\perp). \quad (3.31)$$

Proof. Assume Φ is causal, and let $\mathcal{O}_1 \subset L_{\text{in}}, \mathcal{O}_2 \subset L_{\text{out}}$ be spacelike separated. By causality condition (iii) from Proposition 2,

$$[e^{i\hat{\phi}(f)}, \Phi^\dagger(e^{i\hat{\phi}(g)})] = 0, \quad \forall f \in \mathcal{C}_0^\infty(\mathcal{O}_1), g \in \mathcal{C}_0^\infty(\mathcal{O}_2). \quad (3.32)$$

This relation extends immediately to all (finite) linear combinations of exponentiated smeared fields in both arguments. It suffices then to use again that $\mathcal{R}(\mathcal{O}_1)$ and $\mathcal{R}(\mathcal{O}_2)$ are the weak closures of these expressions, and that $\Phi^\dagger$ is weakly continuous, to conclude that

$$[\hat{B}, \Phi^\dagger(\hat{A})] = 0, \quad \forall \hat{A} \in \mathcal{R}(\mathcal{O}_1), \hat{B} \in \mathcal{R}(\mathcal{O}_2). \quad (3.33)$$

This proves that (i) implies (ii). To show that (ii) implies (iii), let Ψ be any (weakly continuous) channel localized in a compact connected causally convex set $S \subset L_{\text{in}}$. In particular, Ψ is localizable in S , and by the characterization of localizable maps we have that there exists a Kraus representation of Ψ ,

$$\Psi(\hat{B}) = \sum_{j=1}^{\infty} \hat{M}_j \hat{B} \hat{M}_j^\dagger, \quad \forall \hat{B} \in \mathcal{B}_1(\mathcal{H}_\omega), \quad (3.34)$$

such that $\hat{M}_j \in \bar{\mathcal{R}}(S)$ for all j . Now, let $g \in \mathcal{C}_0^\infty(L_{\text{out}} \cap S^\perp)$. Because $\text{supp } g \subset L_{\text{out}} \cap S^\perp$ and $S \subset L_{\text{in}}$ are both compact, there exist $\mathcal{O}_1 \subset L_{\text{in}}, \mathcal{O}_2 \subset L_{\text{out}}$ spacelike separated such that $S \subset \mathcal{O}_1$ and $\text{supp } g \subset \mathcal{O}_2$. By condition (ii), $\Phi^\dagger(e^{i\hat{\phi}(g)})$ commutes with all the elements of $\mathcal{R}(\mathcal{O}_1) \supset \bar{\mathcal{R}}(S)$, and therefore

$$\Psi^\dagger \circ \Phi^\dagger(e^{i\hat{\phi}(g)}) = \sum_{j=1}^{\infty} M_j^\dagger \Phi^\dagger(e^{i\hat{\phi}(g)}) \hat{M}_j = \left(\sum_{j=1}^{\infty} M_j^\dagger \hat{M}_j \right) \Phi^\dagger(e^{i\hat{\phi}(g)}) = \Phi^\dagger(e^{i\hat{\phi}(g)}). \quad (3.35)$$

As before, because this holds for all $g \in \mathcal{C}_0^\infty(L_{\text{out}} \cap S^\perp)$, the linearity and weak continuity of $\Phi^\dagger$ and $\Psi^\dagger$ guarantees that Eq. (3.29) is satisfied.

It is obvious that (iii) implies (iv), which at the same time implies (v). Therefore, it only remains to show that (v) implies (i). We will do this using causality condition (iii) from Proposition 2, as in the beginning of the proof: let $f, g \in \mathcal{C}_0^\infty(\mathcal{M})$ be such that $\text{supp } f \subset L_{\text{in}}$, $\text{supp } g \subset L_{\text{out}}$, and $\text{supp } f \subset (\text{supp } g)^\perp$. We want to show that

$$\Upsilon_f^\dagger \circ \Phi^\dagger(e^{i\hat{\phi}(g)}) = \Phi^\dagger(e^{i\hat{\phi}(g)}). \quad (3.36)$$

This is simply Eq. (3.31) for $\Psi = \Upsilon_f$ and $\hat{A} = e^{i\hat{\phi}(g)}$. Therefore, to conclude it suffices to show that Υ_f satisfies the strong no-signalling condition, which can be done by checking that Υ_f is causal, since we have already proven that (i) implies (iii). Indeed, consider $h \in \mathcal{C}_0^\infty(L_{\text{out}})$, and let $\mathcal{O} \in L_{\text{in}}$ contain a Cauchy hypersurface of the support of h . Then, by the Campbell identity,

$$\Upsilon_f^\dagger(e^{i\hat{\phi}(h)}) = e^{-iE(f,h)} e^{i\hat{\phi}(h)} \in \mathcal{R}(\mathcal{O}), \quad (3.37)$$

where we used the time slice axiom and the fact that $\mathcal{O}$ contains a Cauchy hypersurface of $\text{supp } g$. By definition, Eq. (3.37) means that Υ_f is causal, and therefore it satisfies the strong no-signalling condition, concluding the proof. $\square$

It is worth noting that in [129], updates satisfying the strong and weak no-signalling conditions were dubbed *strongly causal* and *weakly causal*, respectively. Here, we choose to refer to these constraints as ‘no-signalling’ conditions, as they were originally formulated to prevent Φ from enabling faster-than-light signalling in Sorkin-like scenarios.

One of the main takeaways from the results of this section is that a channel is causal if and only if Eq. (3.13) holds, meaning that it does not enable the unitary channel Υ_f to signal faster than light. In previous works, the requirement that a channel must not allow Υ_f to communicate faster than light was used as a necessary condition for causality [129], or as part of a specific setup [24, 136]. Here, we have shown that this requirement is not only necessary but also sufficient: to determine whether a channel permits faster-than-light signalling, it suffices to check whether this happens for the family of unitary channels Υ_f . This observation motivates the following definition:

Definition (Signalling index). *Let Φ be a channel localized in a compact connected causally convex set $L \subset \mathcal{M}$, and let η be an algebraic state. We define the signalling index of Φ for the state η as*

$$v_{\eta,\Phi} := \frac{1}{2} \sup_{f,g} \left\{ \left| \eta \left(\Upsilon_f^\dagger \circ \Phi^\dagger(e^{i\hat{\phi}(g)}) - \Phi^\dagger(e^{i\hat{\phi}(g)}) \right) \right| \right\}, \quad (3.38)$$

where the supremum is taken over all $f \in \mathcal{C}_0^\infty(L_{\text{in}})$, $g \in \mathcal{C}_0^\infty(L_{\text{out}})$ with spacelike-separated supports, and L^2 norms upper-bounded by 1. Similarly, we define the absolute signalling index of Φ as

$$v_\Phi := \frac{1}{2} \sup_{f,g} \left\{ \left\| \Upsilon_f^\dagger \circ \Phi^\dagger(e^{i\hat{\phi}(g)}) - \Phi^\dagger(e^{i\hat{\phi}(g)}) \right\| \right\}, \quad (3.39)$$

where the supremum is taken over the same set of functions as in Eq. (3.38).

As its name suggests, the signalling index quantifies the extent to which a channel allows faster-than-light signalling and, therefore, how far its behaviour deviates from relativistic causality. The index always takes values between 0 and 1, since $\|\Upsilon_f^\dagger\| = \|\Phi^\dagger\| = 1$, with the absolute index upper bounding all the ones relative to a specific state. Furthermore, because it is directly inspired by causality condition (iii) from Proposition 2 (cf. Eq. (3.13)), the signalling index serves as a *faithful witness* of a channel's acausal behaviour: namely, a channel Φ is causal if and only if $v_\Phi = 0$, and, in particular, it allows faster-than-light signalling when the state of the field is η if and only if $v_{\eta,\Phi} > 0$.

We end the section with a result about the structure that the Kraus operators of a localized causal channel can take:

Proposition 4 (Kraus operators of a localized causal channel). *Let Φ be a channel localized in a compact connected causally convex set $L \subset \mathcal{M}$. Then Φ is causal if and only if, for every pair of precompact connected spacelike-separated sets $\mathcal{O}_1 \subset L_{\text{in}}, \mathcal{O}_2 \subset L_{\text{out}}$, there exists a set of Kraus operators of Φ that can be written as*

$$\hat{K}_j = \left(\sum_l \hat{P}_l \hat{Q}_{jl} \right) \hat{U}, \quad (3.40)$$

for some unitary $\hat{U}$, and some sets of operators $\{\hat{P}_l\} \subset \mathcal{R}(\mathcal{O}_1)'$, $\{\hat{Q}_{jl}\} \subset \mathcal{R}(\mathcal{O}_2)'$ that commute with each other and such that

$$\sum_l \hat{P}_l^\dagger \hat{P}_l = \mathbf{1}, \quad \sum_j \hat{Q}_{jl}^\dagger \hat{Q}_{jm} = \delta_{lm} \mathbf{1}, \quad \text{and} \quad \hat{U}^\dagger \left[\sum_l \hat{P}_l^\dagger \mathcal{R}(\mathcal{O}_2) \hat{P}_l \right] \hat{U} \subset \mathcal{R}(\mathcal{O}_1)'. \quad (3.41)$$

Proof. Let $\mathcal{O}_1 \subset L_{\text{in}}, \mathcal{O}_2 \subset L_{\text{out}}$ be precompact connected spacelike-separated sets. Then, there exist open connected causally convex spacelike-separated sets $\tilde{\mathcal{O}}_1, \tilde{\mathcal{O}}_2$ such that

$$\text{cl}(\mathcal{O}_1) \subset \tilde{\mathcal{O}}_1 \subset L_{\text{in}} \quad \text{and} \quad \text{cl}(\mathcal{O}_2) \subset \tilde{\mathcal{O}}_2 \subset L_{\text{out}}. \quad (3.42)$$

Under the assumptions that the representation π_ω has all the properties expected of a vacuum representation of a real scalar quantum field, the inclusions in Eq. (3.42) are *standard split inclusions* [60, 131, 137–139], which means that there exist type I von Neumann factors $\mathfrak{N}_1$ and $\mathfrak{N}_2$ with

$$\mathcal{R}(\mathcal{O}_1) \subset \mathfrak{N}_1 \subset \mathcal{R}(\tilde{\mathcal{O}}_1) \quad \text{and} \quad \mathcal{R}(\mathcal{O}_2) \subset \mathfrak{N}_2 \subset \mathcal{R}(\tilde{\mathcal{O}}_2), \quad (3.43)$$

along with unitaries

$$\hat{U} : \mathcal{H}_\omega \rightarrow \mathcal{H}_\omega \otimes \mathcal{H}_\omega \equiv \mathcal{H}_1 \otimes \mathcal{K}_1, \quad \hat{V} : \mathcal{H}_\omega \rightarrow \mathcal{H}_\omega \otimes \mathcal{H}_\omega \equiv \mathcal{H}_2 \otimes \mathcal{K}_2, \quad (3.44)$$

such that

$$\mathfrak{N}_1 = \hat{U}^\dagger [\mathbb{1}_{\mathcal{H}_1} \otimes \mathcal{B}(\mathcal{K}_1)] \hat{U}, \quad \text{and} \quad \mathfrak{N}_2 = \hat{V}^\dagger [\mathcal{B}(\mathcal{H}_2) \otimes \mathbb{1}_{\mathcal{K}_2}] \hat{V}, \quad (3.45)$$

$$\mathfrak{N}'_1 = \hat{U}^\dagger [\mathcal{B}(\mathcal{H}_1) \otimes \mathbb{1}_{\mathcal{K}_1}] \hat{U}, \quad \text{and} \quad \mathfrak{N}'_2 = \hat{V}^\dagger [\mathbb{1}_{\mathcal{H}_2} \otimes \mathcal{B}(\mathcal{K}_2)] \hat{V}, \quad (3.46)$$

and

$$\hat{U}^\dagger (\hat{B}' \otimes \hat{B}) \hat{U} = \hat{B}' \hat{B}, \quad \hat{V}^\dagger (\hat{A} \otimes \hat{A}') \hat{V} = \hat{A} \hat{A}'. \quad (3.47)$$

Given this extra structure, we can define a channel⁷ $\tilde{\Phi}$ with adjoint

$$\tilde{\Phi}^\dagger : \mathcal{B}(\mathcal{H}_2 \otimes \mathcal{K}_2) \rightarrow \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{K}_1), \quad \hat{Y} \mapsto \hat{U} \Phi^\dagger (\hat{V}^\dagger \hat{Y} \hat{V}) \hat{U}^\dagger. \quad (3.48)$$

Because for every $\hat{A} \in \mathcal{B}(\mathcal{H}_2)$, $\hat{V}^\dagger (\hat{A} \otimes \mathbb{1}_{\mathcal{K}_2}) \hat{V} \in \mathfrak{N}_2 \subset \mathcal{R}(\tilde{\mathcal{O}}_2)$, the causality of Φ implies that

$$\Phi^\dagger (\hat{V}^\dagger (\hat{A} \otimes \mathbb{1}_{\mathcal{K}_2}) \hat{V}) \in \mathcal{R}(\tilde{\mathcal{O}}_1)' \subset \mathfrak{N}'_1, \quad (3.49)$$

where we used the condition (ii) from Proposition 3 and the fact that $\tilde{\mathcal{O}}_1 \subset L_{\text{in}}$ and $\tilde{\mathcal{O}}_2 \subset L_{\text{out}}$ are spacelike separated. From Eq. (3.46), we conclude that

$$\tilde{\Phi}^\dagger (\hat{A} \otimes \mathbb{1}_{\mathcal{K}_2}) = \Psi^\dagger (\hat{A}) \otimes \mathbb{1}_{\mathcal{K}_1} \in \mathcal{B}(\mathcal{H}_1) \otimes \mathbb{1}_{\mathcal{K}_1}, \quad \forall \hat{A} \in \mathcal{B}(\mathcal{H}_2), \quad (3.50)$$

for some map $\Psi^\dagger : \mathcal{B}(\mathcal{H}_2) \rightarrow \mathcal{B}(\mathcal{H}_1)$, which from the equation above can be verified to be completely positive and unital (i.e., Ψ is a channel). This implies that, for all product trace-class operators $\hat{\rho} \otimes \hat{\sigma} \in \mathcal{B}_1(\mathcal{H}_1 \otimes \mathcal{K}_1)$,

$$\text{Tr} \left[\tilde{\Phi}(\hat{\rho} \otimes \hat{\sigma}) (\hat{A} \otimes \mathbb{1}_{\mathcal{K}_2}) \right] = \text{Tr} \left[(\hat{\rho} \otimes \hat{\sigma}) \tilde{\Phi}^\dagger (\hat{A} \otimes \mathbb{1}_{\mathcal{K}_2}) \right] = \text{Tr}_{\mathcal{H}_1} [\hat{\rho} \Psi^\dagger (\hat{A})] \text{Tr}_{\mathcal{K}_1} \hat{\sigma} \quad (3.51)$$

$$= \text{Tr}_{\mathcal{H}_2} [\Psi(\hat{\rho}) \hat{A}] \text{Tr}_{\mathcal{K}_1} \hat{\sigma} = \text{Tr}_{\mathcal{H}_2} \left[(\text{Tr}_{\mathcal{K}_1} \hat{\sigma}) \Psi(\hat{\rho}) \hat{A} \right], \quad (3.52)$$

⁷ $\tilde{\Phi}$ is itself a channel as a result of being the composition of a unitary channel, Φ , and another unitary channel.

for all $\hat{A} \in \mathcal{B}(\mathcal{H}_2)$. On the other hand, by definition of the partial trace,

$$\mathrm{Tr} \left[\tilde{\Phi}(\hat{\rho} \otimes \hat{\sigma})(\hat{A} \otimes \mathbb{1}_{\mathcal{K}_2}) \right] = \mathrm{Tr}_{\mathcal{H}_2} \left[\mathrm{Tr}_{\mathcal{K}_2} \tilde{\Phi}(\hat{\rho} \otimes \hat{\sigma}) \hat{A} \right]. \quad (3.53)$$

Hence, we conclude that

$$\mathrm{Tr}_{\mathcal{K}_2} \tilde{\Phi}(\hat{\rho} \otimes \hat{\sigma}) = (\mathrm{Tr}_{\mathcal{K}_1} \hat{\sigma}) \Psi(\hat{\rho}). \quad (3.54)$$

This implies that $\tilde{\Phi}$'s Choi isomorphism satisfies

$$\mathrm{Tr}_{\mathcal{K}_2} J(\tilde{\Phi}) = J(\Psi) \otimes \mathbb{1}_{\mathcal{K}_1}, \quad (3.55)$$

for the channel Ψ defined in Eq. (3.49). It is not difficult (although a bit lengthy⁸) to show that, if Eq. (3.55) is satisfied, there exists a separable Hilbert space $\mathcal{V}$ and channels $\Phi_0 : \mathcal{B}_1(\mathcal{H}_1) \rightarrow \mathcal{B}_1(\mathcal{H}_2 \otimes \mathcal{V})$ and $\Phi_1 : \mathcal{B}_1(\mathcal{V} \otimes \mathcal{K}_1) \rightarrow \mathcal{B}_1(\mathcal{K}_2)$ such that [30]

$$\tilde{\Phi} = (\mathbb{1} \otimes \Phi_1)(\Phi_0 \otimes \mathbb{1}), \quad (3.56)$$

where, moreover, Φ_0 is an isometric channel. Let $\hat{A} \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2 \otimes \mathcal{V})$ be an isometry that is also a Kraus operator for Φ_0 , and let $\{\hat{M}_j\} \subset \mathcal{B}(\mathcal{V} \otimes \mathcal{K}_1, \mathcal{K}_2)$ be a set of Kraus operators for Φ_1 . Then,

$$\hat{K}_j = (\mathbb{1}_{\mathcal{H}_2} \otimes \hat{M}_j)(\hat{A} \otimes \mathbb{1}_{\mathcal{K}_1}) = \sum_v (\mathbb{1}_{\mathcal{H}_2} \otimes \hat{M}_j |v\rangle)(\langle v| \hat{A} \otimes \mathbb{1}_{\mathcal{K}_1}) \equiv \sum_v \hat{A}_v \otimes \hat{M}_{jv} \quad (3.57)$$

forms a set of Kraus operators for $\tilde{\Phi}$, where $\{\hat{A}_v\} \subset \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $\{\hat{M}_{jv}\} \subset \mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)$ with the property that

$$\sum_v \hat{A}_v^\dagger \hat{A}_v = \mathbb{1}_{\mathcal{H}_1} \quad \text{and} \quad \sum_j \hat{M}_{jv}^\dagger \hat{M}_{jw} = \delta_{vw} \mathbb{1}_{\mathcal{K}_1}. \quad (3.58)$$

From Eq. (3.48), we obtain that

$$\hat{K}_j = \hat{V}^\dagger \sum_v \hat{A}_v \otimes \hat{M}_{jv} \hat{U} \quad (3.59)$$

is a set of Kraus operators for Φ . Now, because $\hat{V}$ is a unitary, we can write

$$\hat{K}_j = \left[\sum_v \hat{V}^\dagger (\hat{A}_v \otimes \hat{M}_{jv}) \hat{V} \right] \hat{V}^\dagger \hat{U}, \quad (3.60)$$

⁸The proof is the straightforward infinite-dimensional generalization of the solution to Exercise 2.5 in [30]. Out of consideration for the teachers who, like myself, have used and intend to continue using this problem in their classes, we leave the proof to the reader—perhaps to the disappointment of the students who will have to tackle it!

where by construction

$$\hat{V}^\dagger(\hat{A}_v \otimes \hat{M}_{jv})\hat{V} = \hat{B}_v \hat{N}_{jv} \quad (3.61)$$

for some $\hat{B}_v \in \mathfrak{N}_2 \subset \mathcal{R}(\mathcal{O}_1)'$ and $\hat{N}_{jv} \in \mathfrak{N}'_2 \subset \mathcal{R}(\mathcal{O}_2)'$. Finally, let $\hat{\mathcal{U}} := \hat{V}^\dagger \hat{U}$. $\hat{\mathcal{U}}$ is a unitary in $\mathcal{H}_\omega$, with the property that $\hat{\mathcal{U}}^\dagger \mathfrak{N}_2 \hat{\mathcal{U}} = \mathfrak{N}'_1$. In particular,

$$\hat{\mathcal{U}}^\dagger \left[\sum_v \hat{B}_v^\dagger \mathcal{R}(\mathcal{O}_2) \hat{B}_v \right] \hat{\mathcal{U}} \subset \hat{\mathcal{U}}^\dagger \mathfrak{N}_2 \hat{\mathcal{U}} = \mathfrak{N}'_1 \subset \mathcal{R}(\mathcal{O}_1)'. \quad (3.62)$$

Since different sets of Kraus operators are related by a linear transformation, this concludes one implication of the proof. The reverse implication is straightforward: assume that the Kraus operators of Φ can be written as

$$\hat{K}_j = \left(\sum_l \hat{P}_l \hat{Q}_{jl} \right) \hat{U}, \quad (3.63)$$

for some unitary $\hat{U}$, and some sets of operators $\{\hat{P}_l\} \subset \mathcal{R}(\mathcal{O}_1)'$, $\{\hat{Q}_{jl}\} \subset \mathcal{R}(\mathcal{O}_2)'$ that commute with each other and such that

$$\sum_l \hat{P}_l^\dagger \hat{P}_l = \mathbb{1}, \quad \sum_j \hat{Q}_{jl}^\dagger \hat{Q}_{jm} = \delta_{lm} \mathbb{1}, \quad \text{and} \quad \hat{U}^\dagger \sum_l \hat{P}_l^\dagger \mathcal{R}(\mathcal{O}_2) \hat{P}_l \hat{U} \subset \mathcal{R}(\mathcal{O}_1)'. \quad (3.64)$$

Then, given $\hat{A} \in \mathcal{R}(\mathcal{O}_2)$, we have that

$$\Phi^\dagger(\hat{A}) = \hat{U}^\dagger \sum_{jlm} \hat{P}_l^\dagger \hat{Q}_{jl}^\dagger \hat{A} \hat{P}_m \hat{Q}_{jm} \hat{U} = \hat{U}^\dagger \sum_{lm} \hat{P}_l^\dagger \hat{A} \hat{P}_m \left(\sum_j \hat{Q}_{jl}^\dagger \hat{Q}_{jm} \right) \hat{U} \quad (3.65)$$

$$= \hat{U}^\dagger \sum_{lm} \hat{P}_l^\dagger \hat{A} \hat{P}_m \delta_{lm} \hat{U} = \hat{U}^\dagger \left(\sum_l \hat{P}_l^\dagger \hat{A} \hat{P}_l \right) \hat{U} \in \mathcal{R}(\mathcal{O}_1)', \quad (3.66)$$

which means that Φ satisfies condition (ii) from Proposition 3, and therefore it is causal. $\square$

3.3 Examples

In this section, we use the results from Sec. 3.2 to briefly analyze some examples of causal maps previously considered in the literature.

3.3.1 Unitary channels generated by a smeared field

Given $h \in \mathcal{C}_0^\infty(\mathcal{M})$, let Φ_ξ be the channel defined by

$$\Phi_\xi^\dagger(\hat{A}) = e^{i\xi(\hat{\phi}(h))} \hat{A} e^{-i\xi(\hat{\phi}(h))}, \quad \forall \hat{A} \in \mathcal{B}(\mathcal{H}_\omega), \quad (3.67)$$

for some bounded analytic function $\xi : \mathbb{R} \rightarrow \mathbb{R}$. This channel is localized in the support of h , or, rather, in its causal hull, which is compact and causally convex.

Let $f \in \mathcal{C}_0^\infty(\mathcal{M})$ be such that $\text{supp } f \cap \mathcal{J}^+(\text{supp } h) = \emptyset$, and $E(f, h) \neq 0$. We will consider the family of functions $\{tf : t \in \mathbb{R}\}$, which shares the same support as f . Then,

$$\Upsilon_{tf}^\dagger(e^{i\xi(\hat{\phi}(h))}) = \exp \left[i\xi \left(e^{it\hat{\phi}(f)} \hat{\phi}(h) e^{-it\hat{\phi}(f)} \right) \right] = \exp \left[i\xi(\hat{\phi}(h) - tE(f, h)) \right]. \quad (3.68)$$

Now, let $g \in \mathcal{C}_0^\infty(\mathcal{M})$ be such that $\text{supp } g \cap \mathcal{J}^-(\text{supp } h) = \emptyset$, is spacelike separated from the support of f , and $E(g, h) \neq 0$. Again, we consider the family $\{sg : s \in \mathbb{R}\}$. Then, the modified channel $(tf.\Phi_\xi)$ (given in Eq. (3.14)) that results from applying Υ_{tf} to Φ_ξ 's Kraus operators (in this case, a single unitary) satisfies

$$(tf.\Phi_\xi)^\dagger(e^{is\hat{\phi}(g)}) = e^{i\xi(\hat{\phi}(h) - tE(f, h))} e^{is\hat{\phi}(g)} e^{-i\xi(\hat{\phi}(h) - tE(f, h))} \quad (3.69)$$

$$= e^{i\xi(\hat{\phi}(h) - tE(f, h))} e^{is\hat{\phi}(g)} e^{-i\xi(\hat{\phi}(h) - tE(f, h))} e^{-is\hat{\phi}(g)} e^{is\hat{\phi}(g)} \quad (3.70)$$

$$= e^{i\xi(\hat{\phi}(h) - tE(f, h)) - i\xi(\hat{\phi}(h) - tE(f, h) - sE(g, h))} e^{is\hat{\phi}(g)}. \quad (3.71)$$

By the causality condition (iii) from Proposition 2 (Eq. (3.13)),

$$(tf.\Phi_\xi)^\dagger(e^{is\hat{\phi}(g)}) = \Phi_\xi^\dagger(e^{is\hat{\phi}(g)}) = e^{i\xi(\hat{\phi}(h)) - i\xi(\hat{\phi}(h) - sE(g, h))} e^{is\hat{\phi}(g)}, \quad (3.72)$$

where the last equality can be obtained right away from Eq. (3.71) by taking $t = 0$. Multiplying by $e^{-is\hat{\phi}(g)}$ from the right, we see that this equality holds if and only if

$$\xi(\hat{\phi}(h) - tE(f, h)) - \xi(\hat{\phi}(h) - tE(f, h) - sE(g, h)) = \xi(\hat{\phi}(h)) - \xi(\hat{\phi}(h) - sE(g, h)) \quad (3.73)$$

holds for all $t, s \in \mathbb{R}$, which, since we picked f and g to satisfy $E(f, h), E(g, h) \neq 0$, implies that $\xi(z) - \xi(z - sE(g, h))$ is a constant function of z , for all $s \in \mathbb{R}$. But that can only be the case if $\xi'(z) = \xi'(z - sE(g, h))$ for all $s \in \mathbb{R}$, i.e., if ξ' is a constant, and therefore ξ must be a linear function.

We have just shown that Φ_ξ is causal if and only if ξ is linear. This generalizes the analysis of [129], where it was shown that Φ_ξ does not allow signalling when $\xi(z) = z$, but

it does when $\xi(z) = z^2$. As a consistency check, one can verify that the signalling index of Φ_{z^2} is nonzero: Indeed, from Eq. (3.71) one can find that

$$(f \cdot \Phi_{z^2})^\dagger(e^{i\hat{\phi}(g)}) - \Phi_{z^2}^\dagger(e^{i\hat{\phi}(g)}) = C(f, g) e^{iE(g, h)^2} e^{-2iE(g, h)\hat{\phi}(h)} e^{i\hat{\phi}(g)}, \quad (3.74)$$

where $\|f\|_{L^2}, \|g\|_{L^2} \leq 1$, and

$$C(f, g) = e^{2iE(f, h)E(g, h)} - 1. \quad (3.75)$$

Now, let η be any state, then

$$\frac{1}{2} \left| \eta \left(C(f, g) e^{iE(g, h)^2} e^{-2iE(g, h)\hat{\phi}(h)} e^{i\hat{\phi}(g)} \right) \right| = \frac{1}{2} |C(f, g)| \left| \eta \left(e^{-2iE(g, h)\hat{\phi}(h)} e^{i\hat{\phi}(g)} \right) \right|. \quad (3.76)$$

Since neither factor vanishes in general, it follows that $\nu_\eta > 0$. Moreover, *a priori*, without explicit knowledge of E , this quantity can only be upper-bounded by the trivial bound of 1. This demonstrates that, in addition to not being causal, Φ_{z^2} lacks a characteristic scale to control the extent of its acausal behaviour.

3.3.2 Spectral decomposition of the smeared field

In [140], Oeckl uses the Weyl quantization of the exponentiated smeared field to express it as

$$e^{i\hat{\phi}(h)} = \int_{-\infty}^{\infty} dq e^{iq} \hat{B}_\epsilon(q), \quad (3.77)$$

where, for any $\epsilon > 0$,

$$\hat{B}_\epsilon(q) = \frac{1}{\pi} \int_{-\infty}^{\infty} dt e^{-\epsilon^2 t^2 - 2iqt} e^{2it\hat{\phi}(h)} = \frac{1}{\sqrt{\pi}\epsilon} \exp \left[-\frac{1}{\epsilon^2} (\hat{\phi}(h) - q)^2 \right] \quad (3.78)$$

are positive-semidefinite operators on the field's Hilbert space $\mathcal{H}_\omega$. The mapping

$$\text{Bor}(\mathbb{R}) \ni X \mapsto \hat{B}_\epsilon[X] = \int_X dq \hat{B}_\epsilon(q) \in \text{Pos}(\mathcal{H}_\omega) \quad (3.79)$$

defines a POVM, and, as suggested by Eq. (3.77), it approximates the spectral measure of $\hat{\phi}(h)$ in the weak operator topology. Furthermore,

$$A_{\epsilon, X} : \mathcal{B}_1(\mathcal{H}_\omega) \rightarrow \mathcal{B}_1(\mathcal{H}_\omega), \quad \hat{\rho} \mapsto \sqrt{2\pi}\epsilon \int_X dq \hat{B}_{\sqrt{2}\epsilon}(q) \hat{\rho} \hat{B}_{\sqrt{2}\epsilon}(q) \quad (3.80)$$

defines a $\hat{B}_\epsilon$ -compatible instrument. Remarkably, it turns out that the update associated with the non-selective measurement $A_{\epsilon, \mathbb{R}}$ is a causal channel. This is proven in [140] using the properties of $\hat{B}_\epsilon$; here we offer another straightforward proof.

First, observe that each $\hat{B}_\epsilon(q)$, as a function of $\hat{\phi}(h)$, is localized in the support of h . Let $f \in \mathcal{C}_0^\infty(\mathcal{M})$ be such that $\text{supp } f \cap \mathcal{J}^+(\text{supp } h) = \emptyset$. From the definition of $\hat{B}_\epsilon$ (Eq. (3.78)), we have that

$$\Upsilon_f^\dagger(\hat{B}_{\sqrt{2}\epsilon}(q)) = \frac{1}{\sqrt{2\pi}\epsilon} \exp \left[-\frac{1}{2\epsilon^2}(\hat{\phi}(h) - E(f, h) - q)^2 \right] = \hat{B}_{\sqrt{2}\epsilon}(q + E(f, h)). \quad (3.81)$$

Now, let $g \in \mathcal{C}_0^\infty(\mathcal{M})$ be such that $\text{supp } g \cap \mathcal{J}^-(\text{supp } h) = \emptyset$, and it is spacelike separated from the support of f . Then, because Υ_f is a unitary channel, we have that

$$\Upsilon_f^\dagger \circ A_{\epsilon, \mathbb{R}}^\dagger(e^{i\hat{\phi}(g)}) = \sqrt{2\pi}\epsilon \int_{-\infty}^{\infty} dq \Upsilon_f^\dagger(\hat{B}_{\sqrt{2}\epsilon}(q)) e^{i\hat{\phi}(g)} \Upsilon_f^\dagger(\hat{B}_{\sqrt{2}\epsilon}(q)) \quad (3.82)$$

$$= \sqrt{2\pi}\epsilon \int_{-\infty}^{\infty} dq \hat{B}_{\sqrt{2}\epsilon}(q + E(f, h)) e^{i\hat{\phi}(g)} \hat{B}_{\sqrt{2}\epsilon}(q + E(f, h)) \quad (3.83)$$

$$= \sqrt{2\pi}\epsilon \int_{-\infty}^{\infty} dq \hat{B}_{\sqrt{2}\epsilon}(q) e^{i\hat{\phi}(g)} \hat{B}_{\sqrt{2}\epsilon}(q) = A_{\epsilon, \mathbb{R}}^\dagger(e^{i\hat{\phi}(g)}), \quad (3.84)$$

where we used that $\Upsilon_f^\dagger(e^{i\hat{\phi}(g)}) = e^{i\hat{\phi}(g)}$. By the causality condition (iii) from Proposition 2 (Eq. (3.13)), we conclude that $A_{\epsilon, \mathbb{R}}$ is causal.

Notice that the causality of the non-selective measurement crucially relies on having the entire real line as its set of possible outcomes. Any attempt to partition these outcomes into a countable collection of mutually disjoint subsets would result in a non-selective measurement that fails to satisfy Eq. (3.13), thereby violating causality.⁹ Thus, while this shows that a measurement of $\hat{\phi}(h)$ is possible, it cannot be projective—even more so, it cannot be repeatable [31].

3.3.3 Fewster-Verch framework

In [27, 141], Fewster and Verch use algebraic methods to formulate a measurement scheme for quantum fields where the probe is another quantum field. The target and the probe interact in a compact causally convex coupling region L , where they are jointly modelled by a *coupled theory* which is described by the C^* -algebra $\mathcal{C}$. Outside of L , the coupled

⁹This was also illustrated through different methods in [136].

theory should reduce to the *uncoupled theory*, described by $\mathcal{U} \cong \mathcal{A} \otimes \mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are the corresponding C^* -algebras of the target and the probe, respectively. Under this assumption, there must exist two natural $*$ -isomorphisms

$$\tau_{\text{in}} : \mathcal{U}(L_{\text{in}}) \rightarrow \mathcal{C}(L_{\text{in}}) \quad \text{and} \quad \tau_{\text{out}} : \mathcal{U}(L_{\text{out}}) \rightarrow \mathcal{C}(L_{\text{out}}). \quad (3.85)$$

Because L_{in} and L_{out} contain Cauchy hypersurfaces of $\mathcal{M}$, by the time slice axiom we have that each algebra associated with L_{in} and L_{out} simply coincides with the total algebra. Therefore, τ_{in} and τ_{out} are isomorphisms of $\mathcal{U}$ and $\mathcal{C}$. This allows us to define the *scattering (auto)morphism* [27]

$$\Theta = \tau_{\text{in}}^{-1} \circ \tau_{\text{out}} : \mathcal{U} \rightarrow \mathcal{U}, \quad (3.86)$$

By construction, Θ preserves the identity and, because it is a $*$ -homomorphism, it is completely positive [132]. That is, once a representation is chosen, Θ corresponds precisely to the adjoint of the channel that represents the interaction of the target and the probe, taking operators in L_{out} to operators in L_{in} . Part (c) of Proposition 3.1 in [27] then states that if $\mathcal{O}_1 \subset L_{\text{in}}, \mathcal{O}_2 \subset L_{\text{out}}$ are such that $\mathcal{O}_2$ is in the domain of dependence of $\mathcal{O}_1$, i.e., if $\mathcal{O}_1$ contains a Cauchy hypersurface of $\mathcal{O}_2$, then $\Theta(\mathcal{R}(\mathcal{O}_2)) \subset \mathcal{R}(\mathcal{O}_1)$. In other words, Θ satisfies the ‘time slice property’, a.k.a. causality condition (ii) from Proposition 2, that we proved here to be equivalent to the channel being causal. This is precisely the crux of [28], *impossible measurements require impossible apparatus*: Because the coupling of a probe to a target quantum field can only induce causal channels, any Sorkin-like scenario that leads to faster-than-light signalling, any “impossible measurement”, must require an “impossible apparatus”, i.e., one that is, at the very least, not described by QFT.

3.4 Non-trace-preserving operations

So far in this chapter, we have only been concerned with studying the causal behaviour of channels, i.e., completely positive trace-preserving maps. However, general operations are described by completely positive trace non-increasing maps (cf. 2.1). Non-trace-preserving maps arise in quantum information as update operations associated with selective measurements, where the updated trace represents the probability of the outcome, and the state only recovers unit trace once it is normalized by this probability. As such, selective updates are the result of a combination of a physical interaction (the measurement process, represented by the non-selective update channel) and a probabilistic conditioning based on newly acquired knowledge.

At first glance, one might attempt to analyze the causal behaviour of selective measurements in the same way we did for channels in previous sections. However, a key

complication arises: even if the measurement operation (and its associated non-selective update) is localized in a compact region L , the corresponding selective update is almost always *not* localizable [1, 27, 142, 143]. In other words, it generally acts non-trivially on operators that belong to the causal complement of L . This is not surprising, as it can be traced back to the fact that any sufficiently regular state in QFT is generically found to be highly entangled across spacelike-separated regions [144, 145]. The Reeh-Schlieder theorem [130, 146], as well as the ultraviolet divergence of entanglement entropy between adjacent regions [147, 148], are well-known manifestations of this phenomenon [61].

The non-locality of the selective update is thus a consequence of entanglement and does not, by itself, enable superluminal signalling in QFT. Just as in standard quantum mechanics, faster-than-light communication would require the spacelike-separated observer to have access to the measurement outcome used for post-selection. Since this outcome remains inaccessible outside the causal future of the measurement event, selective measurements do not introduce violations of relativistic causality.

Thus, the observed non-locality of the selective measurement stems from the convention of *where* in spacetime we apply the selective update, rather than from any genuine causality violation. To properly analyze the causal behaviour of selective measurements, it is therefore crucial to describe their update rule in a way that accounts for this spurious non-locality. The next chapter is devoted precisely to this issue: prescribing a causal update rule that allows us to rigorously distinguish between non-localities arising from entanglement and those that signal true causality violations.

Chapter 4

Causal update rules in relativistic quantum mechanics

After analyzing how causality and locality constrain the class of trace-preserving operations that we can consider *physical*—including non-selective measurements—this chapter turns to how relativity should also constrain the way we update the quantum state following a selective measurement.

4.1 Update rules in relativistic quantum mechanics

While the challenges of updating the quantum state after a measurement become particularly explicit in quantum field theory [125, 149], many of them persist even in relativistic setups that only involve non-relativistic systems,¹ as long as we take into account their worldlines and the causal structure of the spacetime they live in. Even in this simplified context, there is no clear consensus on how update rules should be implemented in relativistic settings.

In [125], Hellwig and Kraus justify that in relativistic quantum theories, an ideal measurement should alter the state everywhere outside its causal past. This aligns with how we analyzed channels (cf. Ch. 3), which were understood to affect the portion of the spacetime $\mathcal{M}$ that lies outside the causal past of their region of implementation. However, a decade later, Aharonov and Albert found that there are no quantum states that transform

¹We use this term here in the same sense as for particle detectors, i.e., meaning that these systems can be described in their proper reference frames using non-relativistic quantum mechanics

covariantly *and* yield correct predictions (e.g., conserve the total charge) for all frames of reference. In particular, they state that “no relativistically satisfactory version of the collapse postulate can be found” (discarding, in particular, Hellwig & Kraus’s proposal), and conclude that “a description of the physical system in terms of its observables simply cannot consistently be written down” [149].

At first, it may seem that the inconsistencies pointed out by Aharonov and Albert are not really problematic if one adopts a purely epistemic interpretation of the quantum state: as Fewster and Verch put it, there seems to be “no reason to invoke a physical process of state-reduction occurring at points or surfaces in spacetime, rather, the updated state reflects the observer’s filtering of the system by conditioning on measurement outcomes” [27]. “This is an exercise in book-keeping—in fact, a way of arranging the calculation of conditional probabilities—that provides an effective description of a physical process. [...] There is no physical change in the state—indeed, QFT is most naturally described in the Heisenberg picture—but rather a shift in which state is appropriate for making predictions, given the information obtained from a measurement” [142]. However, even under this epistemic interpretation—where the quantum state lacks ontological status, and different update rules merely reflect different conventions with no empirical consequences—relativistic quantum mechanics demands a *covariant* update rule. That is, even if state updates are purely conventional, they must still be formulated in a way that respects the fundamental relativistic principles of the theory. Only once this is done we can go beyond non-selective measurements and begin to analyze the causal behaviour of selective measurements as well.

In the following sections, we aim to provide such a covariant framework. First, we argue that only an update along the future light cone is consistent with the causal structure of relativistic setups. We propose a framework for describing pointlike non-relativistic quantum systems in relativistic contexts that respects this causal structure by construction. Furthermore, we show how this framework can be connected with observer-dependent formulations that can be generalized to systems with spatial extension and quantum field theories. Finally, we discuss the extent to which these modifications address the obstructions raised by Aharonov and Albert in [149].

4.2 Updating a Bell pair

In this section, we use the simple case study of two entangled qubits undergoing timelike trajectories in spacetime to discuss the adequacy of different possible update rules in a relativistic context.

Let the spacetime manifold $\mathcal{M}$ host two qubits, A and B, which we assume to be pointlike for simplicity. They follow timelike trajectories $x_A(\tau_A)$ and $x_B(\tau_B)$, parametrized by their proper times τ_A and τ_B , respectively. We assume A and B to be initially in the entangled state²

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|\uparrow_A\rangle |\uparrow_B\rangle + |\downarrow_A\rangle |\downarrow_B\rangle) \in \mathcal{H}_A \otimes \mathcal{H}_B, \quad (4.1)$$

where $|\uparrow_J\rangle$ and $|\downarrow_J\rangle$ are the eigenstates of the Pauli operator $\hat{\sigma}_z$ for $J = \{A, B\}$.

Now, assume that at $\tau_A = \tau_A^*$, an ideal measurement of $\hat{\sigma}_z$ is performed on A, with outcome $+1$. Because in non-relativistic quantum mechanics there is an absolute time that parametrizes the evolution of all systems at once, the state update follows Lüders' rule: the joint state of A and B after the measurement must be $|\uparrow_A\rangle |\uparrow_B\rangle$, and in particular their partial states are $|\uparrow_A\rangle$ and $|\uparrow_B\rangle$, respectively. However, in a relativistic setup, the meaning of “after the measurement” becomes ambiguous. While we still need to update the joint state following Lüders' rule to guarantee the compatibility between measurements that are sequentially applied on the pair of qubits, we must choose the hypersurface along which this update shall be performed.

A natural analogue to using an absolute time parameter is to update the state along the hypersurface that contains the measurement event $x_A(\tau_A^*)$, relative to a chosen foliation of spacetime. This prescription is clearly non-covariant, since it depends on the chosen foliation. In fact, the only hypersurfaces along which the update can be performed while maintaining covariance are the future and past light cones of the measurement event. These are the only hypersurfaces that can be generically associated with a point based solely on the spacetime structure³ (and vice versa, modulo a conformal factor [18]).

In order to decide which choice is more adequate, let τ_B^- and τ_B^+ be the proper times at which B's worldline intersects the past and future light cones of A's measurement, respectively⁴ (see Fig. 4.1). If one chooses the update to occur along the past light cone, the partial state of B changes from $\hat{\rho}_B = \text{Tr}_A |\Phi^+\rangle\langle\Phi^+| = \mathbb{1}_B/2$ for $\tau_B < \tau_B^-$, to $|\uparrow_B\rangle$ for $\tau_B > \tau_B^-$. If we choose the future light cone instead, the change would happen at $\tau_B = \tau_B^+$.

²If one were uncomfortable with the idea that A and B are *initially* entangled, we could assume instead that they interact at some point along their worldlines, becoming entangled as a result.

³More precisely, for any given point $x \in \mathcal{M}$, the only set of curves passing through x that generate a d -dimensional manifold—without introducing additional structure—are the null geodesics. Depending on whether the tangent vector at x is future-directed or past-directed, these null geodesics generate the future light cone, $\partial\mathcal{J}^+(x)$, or the past light cone, $\partial\mathcal{J}^-(x)$, respectively. Any other hypersurface requires invoking a preferred foliation or additional structure, such as Killing vector fields, whose existence is not generally guaranteed in a globally hyperbolic and simply connected $(\mathcal{M}, g)$.

⁴If B remains outside the future of the measurement indefinitely, the following discussion still applies setting $\tau_B^+ = +\infty$.

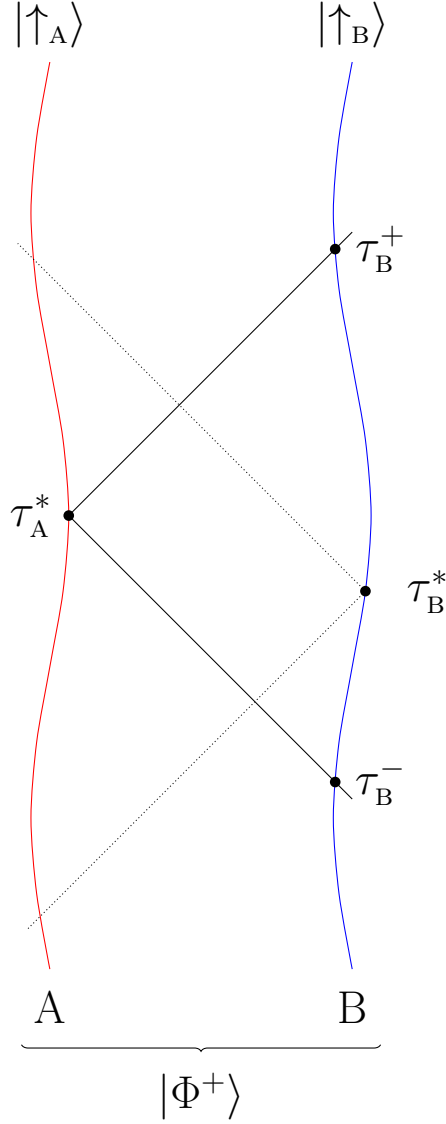


Figure 4.1: Spacetime diagram showing the worldlines of A and B, along with the proper time τ_A^* at which the measurement on A is performed. The diagram also indicates the proper times τ_B^- and τ_B^+ , which correspond to the intersections of B's worldline with the past and future light cones of the measurement, respectively. An analogous construction for some $\tau_B^* \in (\tau_B^-, \tau_B^+)$ is also sketched.

In previous work [143], the following three-step argument was used to justify that updating along the past light cone is the only viable choice:

- (1) Because the initial joint state of A and B is $|\Phi^+\rangle$, once $\hat{\sigma}_z$ is measured for A with outcome +1, a measurement of $\hat{\sigma}_z$ performed on B can only possibly yield +1, even if performed before τ_B^+ .
- (2) Even if B has no access to A's measurement outcome, its partial state should still reflect (1) and yield correct predictions.
- (3) To satisfy (1) and (2), the partial state of B *must be* $|\uparrow_B\rangle$, anytime after τ_B^- . Since the update along the future light cone prescribes that B is in the maximally mixed state for $\tau_B < \tau_B^+$, it yields the wrong statistics, and should be discarded.

However, we find this reasoning to be unsatisfactory, because it can be used to logically justify a retroactive update:

- (4) If the partial state of B is $|\uparrow_B\rangle$ for $\tau_B^* \in (\tau_B^-, \tau_B^+)$, then again, because the initial joint state of A and B is $|\Phi^+\rangle$, a measurement of $\hat{\sigma}_z$ performed on A can only possibly yield +1, even if performed before τ_A^* .
- (5) The partial state of A should reflect (4), and predict outcomes correctly.
- (6) To satisfy (4) and (5), the partial state of A must be $|\uparrow_A\rangle$ *even before the measurement on A is performed*.

This does not lead to a contradiction: once we condition our predictions on obtaining a +1 outcome for $\hat{\sigma}_z$ at τ_A^* , updating A's state to $|\uparrow_A\rangle$ even for $\tau_A < \tau_A^*$ ensures the correct statistics for earlier measurements, provided the measured observables commute with $\hat{\sigma}_{z,A}$, which all measurements on B satisfy.

The argument above suggests that updating the state along the past light cone is equivalent to updating it everywhere—provided one assumes that no incompatible observables were measured in the causal past. While retroactive updates will not lead to inconsistencies under these conditions, they imply accepting that either the measurement result was already predetermined before it happened, or that the whole universe including its past has been post-selected a posteriori to be compatible with it—in which case, the state update loses its meaning as a physical process in the system going from a pre-measurement to a post-measurement state.

Ultimately, the problem lies in conditions (2) and (5), which rely on state updates that are independent of where in spacetime the information is available. In this sense, the ignorance of an observer at B’s site is equated to that of any observer who has incomplete information about the quantum state of a system [143]. This is problematic because, strictly speaking, from B’s perspective, the measurement on A “has not happened” until $\tau_B = \tau_B^+$. Assuming that B’s ignorance is merely epistemic, we were able to deduce that A’s ignorance about their own future measurement outcome must also be epistemic.

An update along the future light cone, in contrast, respects the causal structure of the setup by fiat, and in particular it faithfully represents where the information about the outcome of the measurement is available—and therefore where the update can possibly be performed by an observer. However, it fails to account for the correlations between the two systems: Consider, for instance, a second measurement of $\hat{\sigma}_z$ performed on A at some $\tau_A^{**} > \tau_A^*$, and a measurement of $\hat{\sigma}_z$ performed on B before it enters the future light cone (at some $\tau_B^* \in (\tau_B^-, \tau_B^+)$). Then, an update along the future light cone predicts the following expectation values:

$$\langle \hat{\sigma}_z^{(A)} \rangle = 1, \quad \langle \hat{\sigma}_z^{(B)} \rangle = 0, \quad (4.2)$$

and yet, because A and B are initially correlated, it must hold that

$$\langle \hat{\sigma}_z^{(A)} \otimes \hat{\sigma}_z^{(B)} \rangle = 1. \quad (4.3)$$

Clearly, no joint density operator $\hat{\rho}_{AB} \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$ can simultaneously satisfy Eqs. (4.2) and (4.3).

It seems, therefore, that a representation of multipartite systems in terms of tensor products does not leave room for a covariant update rule that

- (i) is *fully predictive*, i.e., incorporates knowledge of the measurement outcome into the description of the multipartite system, accounts for correlations, and ensures compatibility between sequential measurements applied to both the joint system and its subsystems,

and

- (ii) is *causal*, i.e., it represents the result of a dynamical physical process that occurs in spacetime and is constrained by relativity. This implies that if a measurement is performed locally, then both (a) the propagation of the perturbation it induces on the state and (b) the transmission of the information it gathers must be constrained by the causal structure of spacetime.

To accommodate these two conditions requires going beyond the standard tensor product formalism. The aim of the next section is precisely to provide a minimal extension that allows to formulate a covariant, causally consistent, fully predictive update rule.

4.3 A relativistic formalism for multipartite systems

In section 4.2, we showed that a ‘causal’ update rule must be implemented along the future light cone. Therefore, prescribing an update rule that is also ‘fully predictive’ requires addressing the apparent contradiction posed by Eqs. (4.2) and (4.3). One way to resolve this is by distinguishing between predicting the outcomes of measurements performed *individually* on A or B (i.e., computing expectation values of observables of *only* A or B); and predicting the results of experiments performed *jointly* (i.e., computing expectation values of observables of A *and* B). ‘Individual’ observables can be measured independently on A or B, and their expectations can be computed using local states $\hat{\rho}_A$ and $\hat{\rho}_B$. Meanwhile, joint observables potentially measure correlations between A and B, predicting their outcomes requires knowledge of the composite system (A, B) , and their expectations can only be computed faithfully with a joint state $\hat{\rho}_{AB}$.

Implementing this distinction requires considering that $\hat{A} \in \mathcal{L}(\mathcal{H}_A)$, as an *individual* observable of A, and $\hat{A} \otimes \mathbb{1}_B \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$, as a *joint* observable of A and B, are different operators, even though $\hat{A} \otimes \mathbb{1}_B$ acts trivially on the B sector of the joint system—and similarly for $\hat{B}$ and $\mathbb{1}_A \otimes \hat{B}$. This can be done using an ‘extended’ joint Hilbert space

$$\tilde{\mathcal{H}}_{AB} := \mathcal{H}_A \oplus \mathcal{H}_B \oplus (\mathcal{H}_A \otimes \mathcal{H}_B), \quad (4.4)$$

where each summand acts as a superselection sector, so that the space of ‘physical’ operators is given by

$$\mathcal{L}(\tilde{\mathcal{H}}_{AB})_{\text{phys}} := \mathcal{L}(\mathcal{H}_A) \oplus \mathcal{L}(\mathcal{H}_B) \oplus \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B). \quad (4.5)$$

This construction can be rigorously justified using an operator-algebraic approach, but its structure also admits a straightforward intuitive interpretation: $\mathcal{L}(\mathcal{H}_A)$ and $\mathcal{L}(\mathcal{H}_B)$ are the spaces of individual observables of A and B, respectively, while $\mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$ is the space of joint observables. In particular, $\hat{A}$ and $\hat{A} \otimes \mathbb{1}_B$ act on different sectors of the extended Hilbert space, and similarly for $\hat{B}$ and $\mathbb{1}_A \otimes \hat{B}$. The full state of the system is described by an operator of the form

$$\tilde{\rho}_{AB} := \hat{\rho}_A \oplus \hat{\rho}_B \oplus \hat{\rho}_{AB}, \quad (4.6)$$

where $\hat{\rho}_A \in D(\mathcal{H}_A)$, $\hat{\rho}_B \in D(\mathcal{H}_B)$, and $\hat{\rho}_{AB} \in D(\mathcal{H}_A \otimes \mathcal{H}_B)$ are all density operators. However, in general, $\hat{\rho}_A$ and $\hat{\rho}_B$ will *not* coincide with the corresponding partial traces of $\hat{\rho}_{AB}$.

Hence, given individual observables $\hat{A} \in \mathcal{L}(\mathcal{H}_A)$ and $\hat{B} \in \mathcal{L}(\mathcal{H}_B)$, and a joint observable $\hat{C} \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$, their expectation values can be computed from $\tilde{\rho}$ as

$$\langle \hat{A} \rangle_{\tilde{\rho}} = \text{Tr} [(\hat{\rho}_A \oplus \hat{\rho}_B \oplus \hat{\rho}_{AB}) \hat{A}] = \text{Tr}_A [\hat{\rho}_A \hat{A}], \quad (4.7)$$

$$\langle \hat{B} \rangle_{\tilde{\rho}} = \text{Tr} [(\hat{\rho}_A \oplus \hat{\rho}_B \oplus \hat{\rho}_{AB}) \hat{B}] = \text{Tr}_B [\hat{\rho}_B \hat{B}], \quad (4.8)$$

$$\langle \hat{C} \rangle_{\tilde{\rho}} = \text{Tr} [(\hat{\rho}_A \oplus \hat{\rho}_B \oplus \hat{\rho}_{AB}) \hat{C}] = \text{Tr}_{AB} [\hat{\rho}_{AB} \hat{C}]. \quad (4.9)$$

This ‘(minimal) relativistic’ formalism allows $\tilde{\rho}$ to depend on two time parameters, with the proper times τ_A and τ_B being the natural choices. This removes the need for a preferred foliation to parametrize the system’s time evolution, making it particularly convenient for incorporating measurement updates along light cones, which would always be at odds with a unique description of the quantum state across all hypersurfaces of a spacelike foliation. Thus, in general, the state $\tilde{\rho}(\tau_A, \tau_B)$ can be written in the following form:

$$\tilde{\rho}(\tau_A, \tau_B) = \hat{\rho}_A(\tau_A) \oplus \hat{\rho}_B(\tau_B) \oplus \hat{\rho}_{AB}(\tau_A, \tau_B). \quad (4.10)$$

Notice that the partial states $\hat{\rho}_A$ and $\hat{\rho}_B$ only depend (in the Schrödinger or interaction pictures) on τ_A and τ_B , respectively. This is because they determine expectation values of individual observables, which depend solely on local knowledge of A or B and should therefore rely only on the respective proper time at which the measurement is performed.

To describe $\tilde{\rho}(\tau_A, \tau_B)$, let us define a family of completely positive maps

$$\{\Psi_S : S \subset \mathcal{M} \text{ causally convex}\}, \quad (4.11)$$

where each map $\Psi_S : \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B) \rightarrow \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$ encodes all the transformations undergone by A and B in the (not necessarily open) causally convex set $S \subset M$. This includes the operations they went through, if we are working in the interaction picture, and the unitary evolution, if we are working in the Schrödinger picture. Because we have prescribed that updates should be implemented along the future light cone of the corresponding measurement event—reflecting the fact that the outcome is only available in its causal future—the only transformations that are relevant to determine the partial states $\hat{\rho}_A(\tau_A)$ and $\hat{\rho}_B(\tau_B)$ lie in the causal past of $x_A(\tau_A)$ and $x_B(\tau_B)$, respectively. Specifically, given an initial state of the system

$$\tilde{\rho} = \hat{\rho}_A \oplus \hat{\rho}_B \oplus \hat{\rho}_{AB}, \quad (4.12)$$

where, for simplicity, we assume that $\hat{\rho}_A$ and $\hat{\rho}_B$ are the corresponding partial traces of $\hat{\rho}_{AB}$, we have that

$$\hat{\rho}_A(\tau_A) \propto \text{Tr}_B \Psi_{\mathcal{J}^-(x_A(\tau_A))}(\hat{\rho}_{AB}), \quad (4.13)$$

$$\hat{\rho}_B(\tau_B) \propto \text{Tr}_A \Psi_{\mathcal{J}^-(x_B(\tau_B))}(\hat{\rho}_{AB}), \quad (4.14)$$

where the proportionality constants are determined by the condition that $\hat{\rho}_A(\tau_A)$ and $\hat{\rho}_B(\tau_B)$ have unit trace.

To complete the description of $\tilde{\rho}(\tau_A, \tau_B)$ for arbitrary (τ_A, τ_B) , we need to obtain the density operator used to compute expectations of joint observables, $\hat{\rho}_{AB}(\tau_A, \tau_B)$. Measuring observables of the composite system requires performing measurements on both A and B separately (“non-locally”) and aggregating the results in a spacetime region that must lie in the future of both operations, which we call the *processing region* [1, 150]. Given the processing region $\mathcal{P}$, the corresponding joint state is given by

$$\hat{\rho}_{AB}(\tau_A, \tau_B; \mathcal{P}) \propto \Psi_{\mathcal{J}^-(\mathcal{P})}(\hat{\rho}_{AB}), \quad (4.15)$$

where again the proportionality constant ensures normalization. It remains to specify $\mathcal{P}$ given (τ_A, τ_B) . The only constraint on the processing region, as discussed above, is that it must lie in the future of both $x_A(\tau_A)$ and $x_B(\tau_B)$, leaving still considerable freedom in its choice. This freedom resembles the one we have in standard quantum mechanics: If an experiment is performed at a time t , its results could be analyzed immediately after it took place, or at a later time. If they are analyzed at a later time, the outcomes of measurements that took place after the experiment under study could be taken into account to compute the corresponding predictions. Nevertheless, when it comes to assigning a physical state to the system at time t , we do it only taking into account the transformations of the system up to the point when the experiment is performed. By the same reasoning, the physical joint state assigned to A and B should account only for transformations within *both* of their causal pasts, namely,

$$\hat{\rho}_{AB}(\tau_A, \tau_B) \propto \Psi_{\mathcal{J}^-(x_A(\tau_A)) \cup \mathcal{J}^-(x_B(\tau_B))}(\hat{\rho}_{AB}). \quad (4.16)$$

With this prescription, $\hat{\rho}_{AB}(\tau_A, \tau_B)$ encodes information about the transformations undergone by A and B that *any* processing region will have access to. In fact, for certain spacetimes, it encodes the *maximum* information that *all* processing regions will have access to.⁵

The full prescription of $\tilde{\rho}(\tau_A, \tau_B)$ for any pair of proper times (τ_A, τ_B) allows for the computation of predictions for arbitrary experiments at different points along A and B’s

⁵The ‘maximum information that all processing regions will have access to’ corresponds to the information contained in their *common past*, which in this case is given by

$$\bigcap \{ \mathcal{J}^-(y) : y \in \mathcal{J}^+(x_A(\tau_A)) \cap \mathcal{J}^+(x_B(\tau_B)) \}, \quad (4.17)$$

i.e., the common past of all points in the intersection of the causal futures of $x_A(\tau_A)$ and $x_B(\tau_B)$. While this set will generally be strictly larger than $\mathcal{J}^-(x_A(\tau_A)) \cup \mathcal{J}^-(x_B(\tau_B))$, the two sets will coincide in some special spacetimes, such as the Lorentzian cylinder $\mathbb{R} \times \mathbb{S}^1$.

worldlines. Returning to the Bell pair case analyzed earlier, it is easy to see that this formalism naturally accommodates the predictions associated with an update rule that applies only in the future of selective measurements. In this setup (see Fig. 4.1 for a schematic reminder), we only need to distinguish two maps:

$$\Psi_{\mathcal{S}}(\hat{\rho}) = \begin{cases} \hat{\rho} & \text{if } \mathbf{x}_A(\tau_A^*) \notin \mathcal{S}, \\ |\uparrow_A\rangle\langle\uparrow_A| \hat{\rho} |\uparrow_A\rangle\langle\uparrow_A| & \text{if } \mathbf{x}_A(\tau_A^*) \in \mathcal{S}. \end{cases} \quad (4.18)$$

Hence, from Eqs. (4.13), (4.14), and (4.16), we find that if the joint system is initially in the Bell state $|\Phi^+\rangle$, then

$$\tilde{\rho}(\tau_A^*, \tau_B) = |\uparrow_A\rangle\langle\uparrow_A| \oplus \frac{\mathbb{1}_B}{2} \oplus |\uparrow_A\uparrow_B\rangle\langle\uparrow_A\uparrow_B|, \quad (4.19)$$

for any $\tau_B \in (\tau_B^-, \tau_B^+)$. This state correctly reproduces the expectation values in Eqs. (4.2) and (4.3).

It is worth remarking that, although here we have worked in the Schrödinger and interaction pictures, the Heisenberg picture can be used just as easily by employing the adjoint maps $\Psi_{\mathcal{S}}^\dagger$, as we did in Ch. 3. Finally, also note that it is straightforward to generalize this formalism from two to n pointlike non-relativistic systems following timelike trajectories $\mathbf{x}_i(\tau_i)$, parametrized by their proper times τ_i , for $i \in \{1, \dots, n\}$. In that case, the extended joint Hilbert space is

$$\tilde{H} := \bigoplus_i \mathcal{H}_i \oplus \bigoplus_{i < j} (\mathcal{H}_i \otimes \mathcal{H}_j) \oplus \dots \oplus (\mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_n), \quad (4.20)$$

and the space of physical operators

$$\mathcal{L}(\tilde{H})_{\text{phys}} := \bigoplus_i \mathcal{L}(\mathcal{H}_i) \oplus \bigoplus_{i < j} \mathcal{L}(\mathcal{H}_i \otimes \mathcal{H}_j) \oplus \dots \oplus \mathcal{L}(\mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_n). \quad (4.21)$$

The joint system is described by

$$\tilde{\rho}(\tau_1, \dots, \tau_n) = \bigoplus_i \hat{\rho}_i(\tau_i) \oplus \bigoplus_{i < j} \hat{\rho}_{ij}(\tau_i, \tau_j) \oplus \dots \oplus \hat{\rho}_{1\dots n}(\tau_1, \dots, \tau_n), \quad (4.22)$$

where, for any ordered subset of indices $I \subset \{1, \dots, n\}$,

$$\hat{\rho}_I(\{\tau_i : i \in I\}) \propto \text{Tr}_{I^c} \Psi_{\bigcup_{i \in I} \mathcal{J}^-(\mathbf{x}_i(\tau_i))}(\hat{\rho}_{1\dots n}), \quad (4.23)$$

where $\hat{\rho}_{1\dots n}$ is the initial joint state, and I^c is the complementary set of I in $\{1, \dots, n\}$.

4.3.1 Statistical interpretation

A key virtue of this formalism is that, far from being a convoluted framework designed *ad hoc* to accommodate Eqs. (4.2) and (4.3), it has a clear *statistical* interpretation.

The standard statistical interpretation asserts that the quantum state describes the statistical properties of an ensemble of identically prepared systems [151]. While relinquishing the ambition of an intrinsic, ontological description of the physical state, it has the advantage that—unlike one-shot experiments—it offers a framework where selective updates have a clear operational meaning: that of a filter, where systems that do not yield the prescribed measurement outcome are discarded.

To illustrate this, we revisit the Bell pair scenario: let Alba and Blanca be two experimenters who follow timelike trajectories $\mathbf{x}_A(\tau_A)$ and $\mathbf{x}_B(\tau_B)$ while holding one component of an ensemble of N pairs of qubits initially prepared in the Bell state $|\Phi^+\rangle$. At $\tau_A = \tau_A^*$, Alba performs a measurement of $\hat{\sigma}_z$ on the qubits she is holding, and keeps only the subset of them for which the outcome was $+1$. Thus, the ensemble of qubits she retains is described by the density operator $|\uparrow_A\rangle\langle\uparrow_A|$, precisely matching $\hat{\rho}_A(\tau_A^*)$ in Eq. (4.19). Meanwhile, at any time $\tau_B < \tau_B^+$, and without performing any measurements on the qubits she is holding, Blanca cannot know which pairs were selected after Alba’s measurements, and which ones were discarded—this is, in essence, the reason why regardless of the sort of update rule that we decide to apply, in practice updates can never allow superluminal signalling. Until Blanca receives information from Alba at $\tau_B \geq \tau_B^+$ regarding which qubits to discard and which to keep, she must hold on to all N of them. Consequently, her ensemble is described by $\mathbb{1}_B/2$. This is precisely $\hat{\rho}_B(\tau_B)$ in Eq. (4.19), for any $\tau_B \in (\tau_B^-, \tau_B^+)$. Finally, to measure correlations between their components, Alba and Blanca must consider only the pairs that remain complete (recall that Alba discarded the qubits that returned -1 after her measurement, even though Blanca retained their initially entangled counterparts). At any point where their results can be combined (i.e., in any processing region), the remaining pairs will display the appropriate correlations. Accordingly, they are described by $|\uparrow_A\uparrow_B\rangle\langle\uparrow_A\uparrow_B|$, which coincides with $\hat{\rho}_{AB}(\tau_A^*, \tau_B)$ in Eq. (4.19).

This analysis assigns a clear statistical meaning to each sector of $\tilde{\rho}$ and can be readily generalized to arbitrary initial joint states, general transformations and protocols, and systems involving more parties and experimenters.

4.4 Observer-dependent framework

In this section, we show that the minimally extended formalism introduced in Sec. 4.3 is equivalent to an observer-dependent framework. The latter perspective has the advantage of naturally generalizing to a relativistic formalism for (spatially smeared) particle detectors and quantum field theories.

First, we observe that a direct consequence of Eqs. (4.13), (4.14), and (4.16) is that

$$\hat{\rho}_A(\tau_A) = \text{Tr}_B \hat{\rho}_{AB}(\tau_A, \tau'_B), \quad (4.24)$$

$$\hat{\rho}_B(\tau_B) = \text{Tr}_A \hat{\rho}_{AB}(\tau'_A, \tau_B), \quad (4.25)$$

where τ'_A and τ'_B are the proper times of A and B at which their worldlines intersect the past light cone of $\mathbf{x}_A(\tau_A)$ and $\mathbf{x}_B(\tau_B)$, respectively. This is because, ultimately, the computation of $\hat{\rho}_A(\tau_A)$ is based on the fact that, from system A's perspective, the joint state of A and B at time τ_A is

$$\Psi_{\mathcal{J}^-(\mathbf{x}_A(\tau_A))}(\hat{\rho}_{AB}) \propto \hat{\rho}_{AB}(\tau_A, \tau'_B), \quad (4.26)$$

and it is the partial trace of *this* joint state the one that yields $\hat{\rho}_A(\tau_A)$ (which, in general, will not coincide with the partial trace of $\hat{\rho}_{AB}(\tau_A, \tau_B)$ for a different τ_B , as Eq. (4.19) exemplifies). The same argument applies to the computation of $\hat{\rho}_B(\tau_B)$.

The above observation, along with Eq. (4.15), which prescribes the joint state as observed by a given processing region, points towards a framework that further generalizes the extended formalism taking the state to be an operator-valued function of the spacetime point. Namely, given an initial state $\hat{\rho}_0$, and $\mathbf{x} \in \mathcal{M}$, we define

$$\hat{\rho}(\mathbf{x}) := \frac{\Psi_{\mathcal{J}^-(\mathbf{x})}(\hat{\rho}_0)}{\text{Tr} [\Psi_{\mathcal{J}^-(\mathbf{x})}(\hat{\rho}_0)]}, \quad (4.27)$$

which is nothing but the state of the system as it would be described by an observer at $\mathbf{x}$. This generalization, while unnecessary in the case of pointlike systems (where the extended formalism presented here suffices), is particularly convenient for spatially extended non-relativistic systems and quantum field theories.

Notably, this observer-dependent approach allows us to recover a description of the evolution of the physical system in terms of *one* time parameter, as in non-relativistic quantum mechanics. The fundamental difference lies in that this account of the evolution depends entirely on the observer that recalls it, or, more concretely, on their worldline: given a timelike trajectory $\mathbf{z}(\tau)$ parametrized by its proper time τ , from Eq. (4.27) we have

$$\hat{\rho}_{\mathbf{z}}(\tau) := \hat{\rho}(\mathbf{z}(\tau)) \propto \Psi_{\mathcal{J}^-(\mathbf{z}(\tau))}(\hat{\rho}_0), \quad (4.28)$$

where $\hat{\rho}_0$ is the initial state of the system.⁶ There exist as many recollections $\hat{\rho}_z$ as timelike trajectories in $\mathcal{M}$, and they are guaranteed to be consistent with each other as long as there exists a single operator-valued function $\hat{\rho}(\mathbf{x})$ such that Eq. (4.28) is satisfied for all trajectories.

4.5 Covariance and conservation of charges

In this section, we examine whether the obstructions identified in [149] to defining a well-behaved state in relativistic quantum mechanics are mitigated within the formalism introduced in Sec. 4.3.

First, we note that the operator $\tilde{\rho}$ (Eqs. (4.22) and (4.23)), which represents the state in this formalism, is covariant by construction. This follows from the fact that it is defined solely in terms of the causal structure of spacetime and otherwise transforms as a scalar, i.e., it does not depend on the frame of reference. The same applies to $\hat{\rho}(\mathbf{x})$ (Eq. (4.27)).

Second, let $\{\Sigma_t\}_{t \in \mathbb{R}}$ be a foliation of $\mathcal{M}$, associated with a global time function t , and let us focus on the case of a bipartite system (the generalization to more parts is straightforward). In principle, we could define $\tau_A(t)$ and $\tau_B(t)$ as the proper times at which A's and B's worldlines intersect the leaf Σ_t and then identify the state of the system on Σ_t as $\tilde{\rho}(\tau_A(t), \tau_B(t))$. However, to make predictions in this particular foliation means to compute expectations using all the information available on each leaf *globally*, i.e., using only the joint (“non-local”) sector of the state, $\hat{\rho}_{AB}(\tau_A(t), \tau_B(t))$. Equivalently, the evolution of the quantum state associated with the foliation in the observer-dependent framework is given by

$$\hat{\rho}_\Sigma(t) := \frac{\Psi_{\mathcal{J}^-(\Sigma_t)}(\hat{\rho}_0)}{\text{Tr} [\Psi_{\mathcal{J}^-(\Sigma_t)}(\hat{\rho}_0)]}. \quad (4.30)$$

This guarantees that, in particular, this formalism correctly predicts that globally conserved quantities will always remain unchanged, because any transformation performed on one part of the system will be taken into account to update the state of the full system

⁶For simplicity, we assume that the system has always existed, and that

$$\hat{\rho}_0 = \lim_{\tau \rightarrow -\infty} \hat{\rho}(\tau). \quad (4.29)$$

This assumption can be easily lifted by considering that the system was (jointly) prepared at some point in spacetime, $\mathbf{x}_0$, and therefore its history as recalled from the trajectory $\mathbf{z}(\tau)$ would start at some τ_0 , which corresponds with the intersection of the future light cone of $\mathbf{x}_0$ with the worldline $\mathbf{z}(\tau)$.

across the corresponding leaf of the foliation. The reason this does not contradict the findings of [149] is that, while $\tilde{\rho}$ is covariant, the foliation-dependent state $\hat{\rho}_\Sigma(t)$, as given by Eq. (4.30), depends explicitly on the choice of foliation $\{\Sigma_t\}_{t \in \mathbb{R}}$. This distinction between $\tilde{\rho}$ and $\hat{\rho}_\Sigma(t)$ highlights the key difference between the relativistic framework introduced here and frame-dependent formulations [152, 153], which in general still draw on non-local information even when computing expectation values of local observables.

In the particular case of the conservation of the electric charge discussed in [149], the remark leading to Eq. (4.30) amounts to realizing that the charge density $\hat{q}(\mathbf{x})$ is a *local* operator, while the total charge of the system in a particular leaf, $\hat{Q} = \int_{\Sigma_t} d\mathbf{x} \hat{q}(\mathbf{x})$, is a *non-local* operator.⁷ As a consequence, in the relativistic formalism, the density operators used to compute the expectation values $\langle \hat{q}(\mathbf{x}) \rangle$, and the one used to compute $\langle \hat{Q} \rangle$, are generically different, i.e., in general,

$$\left\langle \int_{\Sigma_t} d\mathbf{x} \hat{q}(\mathbf{x}) \right\rangle_{\tilde{\rho}} \neq \int_{\Sigma_t} d\mathbf{x} \langle \hat{q}(\mathbf{x}) \rangle_{\tilde{\rho}}. \quad (4.31)$$

It is also worth noting that the problem of the conservation of charge can be addressed from an observer-dependent perspective: the set of all possible trajectory-dependent states $\hat{\rho}_z(\tau)$ encompasses all the possible accounts of the evolution of the system. Regardless of what we consider to be the intrinsic description of the state, everything that can possibly be observed is encoded in one of these recollections. In particular, the expectation value of the total charge, $\langle \hat{Q}(\tau) \rangle \equiv \text{Tr}(\hat{\rho}_z(\tau) \hat{Q})$ remains conserved for any worldline $z(\tau)$. Thus, no charge violation is observed, irrespective of the chosen state description.

We conclude that the relativistic formalism provides a representation of the quantum state that is both covariant and fully predictive across all possible foliations of $\mathcal{M}$ (and, in particular, across all reference frames). These two properties, which were found to be incompatible within standard quantum mechanics [149], are thus reconciled within the extended framework presented here.

4.6 Summary

In this chapter, we have addressed the problem of updating multipartite non-relativistic quantum systems in relativistic setups in a way that (1) ensures consistency with obser-

⁷This was explicitly acknowledged by Aharonov and Albert, who clarified in a footnote: “The charge, that is, measured nonlocally.” [149].

vations and subsequent measurements, and (2) respects the causal structure of the setup, and in particular where the information about the outcome is accessible.

We showed that for condition (2) (‘causality’) to hold, the update must be performed along the future light cone of the measurement event. In contrast, global updates along the past light cone—such as those prescribed in [27, 125, 143], among others—are effectively equivalent to retroactive updates. To satisfy condition (1) (full predictivity), we extended the standard non-relativistic formalism to describe the quantum state not by a single density operator, but by a collection of them, each encoding the information available to different subsystems. This modified formalism fits naturally into the statistical interpretation of the quantum state [151], and can be reformulated as an observer-dependent framework. Finally, we showed that the quantum state, as described in the proposed relativistic formalism, is both covariant and respects the conservation of global charges in all frames of reference—something that cannot be achieved in the standard quantum mechanical formalism [149].

Chapter 5

A detector-based measurement theory for QFT

In this chapter, we propose a measurement theory for quantum fields based on measurement schemes where the probe is an Unruh-DeWitt particle detector. First, we introduce the class of UDW-based measurement schemes, analyzing the general perturbative case and the special non-perturbative cases of delta-coupled and gapless detectors. We then examine the causal behaviour of the non-selective measurement channels that these measurement schemes induce, showing that, in the general case, the signalling index has order λ^2 , while the maps induced by gapless and delta-coupled detectors are fully causal. Finally, we complete the measurement theory by prescribing an update rule that is consistent with relativity, ensures the compatibility of sequential measurements, and accounts for correlations across spacelike-separated regions. We conclude by discussing the key features that, in our view, make this framework a proper measurement theory for QFT.

5.1 UDW-based measurement schemes

In Ch. 3, we saw that the standard quantum theory of measurement cannot be straightforwardly generalized to quantum fields. Additional conditions must be imposed on the operations we consider ‘physical’, namely, that realistic maps should be both *localizable* in spacetime and *causal*. As Sorkin put it, “this problem leaves the Hilbert space formulation of quantum field theory with no definite measurement theory” [24].

One way to approach this issue is to determine which instruments are acceptable within

QFT, identifying their associated POVMs and, consequently, the set of (generalized) measurements that we can possibly perform (cf., e.g., the example in Sec. 3.3.2). This is essentially the approach taken in Ch. 3, as well as in other previous works [128, 129, 136, 140].

An alternative avenue consists of specifying the measurement not by directly selecting an instrument but by defining a measurement scheme (cf. Sec. 2.1). That is, if we specify the probe that couples to our target system and the interaction they jointly undergo, this fully determines the instrument and its associated POVM. One instance of this approach is the Fewster-Verch framework [27, 83, 141] (cf. Sec. 3.3.3), where the probe is itself a quantum field, and the interaction channel is given by the scattering morphism. However, because in this framework the probe is another quantum field, we are still left with the problem of how to access the information of that second ancillary field systematically [154]. This is because low-energy measurement apparatus, such as atoms, photodetectors, photomultipliers, and human retinas, are not well described by a free field theory, and the treatment of bound states in QFT is still very much an open problem [155]. While this framework may still be applicable to certain experimental setups where the probe is a quantum field, for now, modelling the measurements on the probe seems to require a case-by-case analysis.

A pragmatic approach is to use particle detectors as probes. Unlike free fields, we have a very good understanding of the measurements we can perform on non-relativistic systems, and how to model them. While particle detectors generally introduce acausal effects due to their effective nature (cf. Sec. 2.3), these deviations from causality are controlled by the detector’s scales [75, 80] and by the perturbative order at which these effects emerge [84–86]. Consequently, under appropriate conditions, detector models can become “effectively causal”. Given that they can also realistically represent the way fields are measured experimentally [78, 120, 121], particle detectors make an especially appealing candidate for modelling measurements in QFT.

In this section, we focus on the particular case in which the particle detector is a two-level Unruh-DeWitt detector, although an identical analysis can be performed for a general Unruh-DeWitt detector, as well as other particle detector models.

Specifically, we consider a spatially smeared two-level Unruh-DeWitt detector, with ground and excited states $|g\rangle$ and $|e\rangle$ separated by an energy gap Ω , coupled to a real scalar quantum field in a $(1+d)$ -dimensional globally hyperbolic connected spacetime with a Lorentzian metric, $(\mathcal{M}, g)$. The coupling between the detector and the field is given in Eq. (2.71), and their joint evolution is described by the unitary in Eq. (2.75), which we

reproduce here for completeness:

$$\hat{U} = \mathcal{T}_\tau \exp \left(-i\lambda \int_{-\infty}^{\infty} d\tau \chi(\tau) \hat{\mu}(\tau) \int_{\mathbb{R}^d} d\mathbf{z} \sqrt{-g} F(\mathbf{z}) \hat{\phi}(\tau, \mathbf{z}) \right), \quad (5.1)$$

where recall from Eq. (2.72) that

$$\hat{\mu}(\tau) = |g\rangle\langle e| e^{-i\Omega\tau} + |e\rangle\langle g| e^{i\Omega\tau}. \quad (5.2)$$

The channel that represents the interaction through which the field and the detector get correlated is then simply given by

$$\Psi : \mathcal{B}_1(\mathcal{H}_D \otimes \mathcal{H}_\phi) \rightarrow \mathcal{B}_1(\mathcal{H}_D \otimes \mathcal{H}_\phi), \quad \hat{\rho} \mapsto \hat{U} \hat{\rho} \hat{U}^\dagger, \quad (5.3)$$

where $\mathcal{H}_D$ and $\mathcal{H}_\phi$ are the Hilbert spaces associated with the detector and the field, respectively. Without loss of generality, we assume that the detector is initially in a pure state $|\psi\rangle \in \mathcal{H}_D$, and that the measurement performed on it is an ideal one, described by the PVM $\hat{F}$, with $\hat{F}(0) = |\varphi_0\rangle\langle\varphi_0|$ and $\hat{F}(1) = |\varphi_1\rangle\langle\varphi_1| = \mathbf{1} - \hat{F}(0)$, where $|\varphi_0\rangle, |\varphi_1\rangle \in \mathcal{H}_D$ are mutually orthogonal. This completes the specification of the measurement scheme, given by the quadruple [67]:

$$\left\{ \mathcal{K} = \mathcal{H}_D \cong \mathbb{C}^2, \hat{\chi} = |\psi\rangle\langle\psi|, \Psi(\hat{\rho}) = \hat{U} \hat{\rho} \hat{U}^\dagger, \hat{F}(s) = |\varphi_s\rangle\langle\varphi_s| (s = 0, 1) \right\}. \quad (5.4)$$

Throughout this chapter, sometimes we will mention the special case where $|\psi\rangle = |g\rangle$, $|\varphi_0\rangle = |g\rangle$ and $|\varphi_1\rangle = |e\rangle$, i.e., the detector starts in its ground state, and the measurement is performed in its energy eigenbasis.

Once the measurement scheme is specified, we can determine the instrument and the POVM it is compatible with:

$$I_{s,\psi}(\hat{\rho}_\phi) = \text{Tr}_D \left\{ [\hat{F}(s) \otimes \mathbf{1}_\phi] \Psi(|\psi\rangle\langle\psi| \otimes \hat{\rho}_\phi) \right\} = \text{Tr}_D \left[(|\varphi_s\rangle\langle\varphi_s| \otimes \mathbf{1}_\phi) \hat{U} (|\psi\rangle\langle\psi| \otimes \hat{\rho}_\phi) \hat{U}^\dagger \right] \quad (5.5)$$

$$= (\langle\varphi_s| \otimes \mathbf{1}_\phi) \hat{U} (|\psi\rangle\langle\psi| \otimes \hat{\rho}_\phi) \hat{U}^\dagger (|\varphi_s\rangle \otimes \mathbf{1}_\phi) = \langle\varphi_s| \hat{U} |\psi\rangle \hat{\rho}_\phi \langle\psi| \hat{U}^\dagger |\varphi_s\rangle, \quad (5.6)$$

where $\hat{K}_{s,\psi} := \langle\varphi_s| \hat{U} |\psi\rangle \in \mathcal{B}(\mathcal{H}_\phi)$ is a Kraus operator for the operation I_s . In particular, the non-selective measurement is given by

$$I_{\text{ns},\psi}(\hat{\rho}_\phi) = \sum_{s=0,1} \hat{K}_{s,\psi} \hat{\rho}_\phi \hat{K}_{s,\psi}^\dagger = \langle\varphi_0| \hat{U} |\psi\rangle \hat{\rho}_\phi \langle\psi| \hat{U}^\dagger |\varphi_0\rangle + \langle\varphi_1| \hat{U} |\psi\rangle \hat{\rho}_\phi \langle\psi| \hat{U}^\dagger |\varphi_1\rangle \quad (5.7)$$

$$= \text{Tr}_D [\Psi(|\psi\rangle\langle\psi| \otimes \hat{\rho}_\phi)]. \quad (5.8)$$

This confirms the result anticipated in Sec. 2.1, where we noted that the non-selective measurement does not affect the partial state of the target, which is fully determined by the interaction. Finally, using Eq. (2.13), we obtain the POVM elements from the instrument,

$$\hat{E}_\psi(s) = I_{s,\psi}^\dagger(\mathbb{1}_\phi) = \hat{K}_{s,\psi}^\dagger \hat{K}_{s,\psi} = \langle \psi | \hat{U}^\dagger | \varphi_s \rangle \langle \varphi_s | \hat{U} | \psi \rangle. \quad (5.9)$$

The pair $\{\hat{E}_\psi(0), \hat{E}_\psi(1)\}$ specifies the observable of the field that is effectively measured when we perform a measurement on the detector.

Assuming that the particle detector and the field are weakly coupled, i.e., $\lambda \ll 1$, the unitary $\hat{U}$ can be expanded in a Dyson series, as in Eq. (2.79), with

$$\hat{U}^{(0)} = \mathbb{1}, \quad (5.10)$$

$$\hat{U}^{(1)} = -i\lambda \int_{-\infty}^{\infty} d\tau \chi(\tau) \hat{\mu}(\tau) \hat{\phi}(\tau, F), \quad (5.11)$$

$$\hat{U}^{(2)} = -\lambda^2 \int_{\mathbb{R}^2} d\tau d\tau' \theta(\tau - \tau') \chi(\tau) \chi(\tau') \hat{\mu}(\tau) \hat{\mu}(\tau') \hat{\phi}(\tau, F) \hat{\phi}(\tau', F), \quad (5.12)$$

where we have denoted

$$\hat{\phi}(\tau, F) \equiv \int_{\mathbb{R}^d} d\mathbf{z} \sqrt{-g} F(\mathbf{z}) \hat{\phi}(\tau, \mathbf{z}). \quad (5.13)$$

From the Dyson series of the evolution operator we can obtain explicit perturbative expansions for the Kraus operators and the POVM elements. In this chapter, we will only be concerned with the general expressions for the Kraus operators up to first order in perturbation theory, but we also compute them up to second order for the special case where $|\psi\rangle = |g\rangle$, $|\varphi_0\rangle = |g\rangle$ and $|\varphi_1\rangle = |e\rangle$:

$$\hat{K}_{s,\psi}^{(0)} = \langle \varphi_s | \psi \rangle \mathbb{1}_\phi \quad (5.14)$$

$$\hat{K}_{s,\psi}^{(1)} = -i\lambda \int_{-\infty}^{\infty} d\tau \chi(\tau) \langle \varphi_s | \hat{\mu}(\tau) | \psi \rangle \hat{\phi}(\tau, F), \quad (5.15)$$

and, in particular,

$$\hat{K}_{s,g}^{(0)} = \delta_{s,g} \mathbb{1}_\phi, \quad (5.16)$$

$$\hat{K}_{s,g}^{(1)} = -i\lambda \delta_{s,e} \int_{-\infty}^{\infty} d\tau \chi(\tau) e^{i\Omega\tau} \hat{\phi}(\tau, F), \quad (5.17)$$

$$\hat{K}_{s,g}^{(2)} = -\lambda^2 \delta_{s,g} \int_{\mathbb{R}^2} d\tau d\tau' \theta(\tau - \tau') \chi(\tau) \chi(\tau') e^{-i\Omega(\tau - \tau')} \hat{\phi}(\tau, F) \hat{\phi}(\tau', F). \quad (5.18)$$

For a detailed calculation of the POVM elements in the general case up to second order in perturbation theory, we refer the reader to Section 2.3 in [67].

Beyond perturbation theory, there are two instances where the evolution operator can be easily computed exactly: delta-coupled and gapless detectors (cf. Sec. 2.3). For the delta-coupled detector, $\chi(\tau) = \eta \delta(\tau)$, and therefore

$$\hat{U} = \exp \left(-i\lambda \eta \hat{\mu} \hat{\phi}(0, F) \right) = e^{-i\lambda \eta \hat{\phi}(0, F)} |+\rangle\langle +| + e^{i\lambda \eta \hat{\phi}(0, F)} |-\rangle\langle -|, \quad (5.19)$$

where $\hat{\mu} = \hat{\mu}(0) = |g\rangle\langle e| + |e\rangle\langle g|$ and $|\pm\rangle = (|g\rangle \pm |e\rangle)/\sqrt{2}$ are its eigenvectors, associated with the eigenvalues ± 1 . Similarly, for a gapless detector, we can apply the Magnus expansion [156] to express the evolution operator as

$$\hat{U} = \exp \left(-i\lambda \hat{\mu} \int_{-\infty}^{\infty} d\tau \chi(\tau) \hat{\phi}(\tau, F) - \frac{i\lambda^2}{2} \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\tau} d\tau' \chi(\tau) \chi(\tau') E(F, F) \mathbb{1} \right). \quad (5.20)$$

Since the second term is proportional to the identity, it only adds a global phase that can be safely neglected. Thus, for both the delta-coupled and gapless detectors, the evolution operator takes the compact form:

$$\hat{U}_{\Lambda} := e^{-i\lambda \hat{\phi}(\Lambda)} |+\rangle\langle +| + e^{i\lambda \hat{\phi}(\Lambda)} |-\rangle\langle -|, \quad (5.21)$$

where $\Lambda(\tau, \mathbf{z}) = \chi(\tau) F(\mathbf{z})$ is the spacetime smearing in each case, which reduces to $\eta \delta(\tau) F(\mathbf{z})$ for the delta coupling. From Eq. (5.21), we can immediately obtain the Kraus operators and POVM elements, which take a particularly simple form when the detector is initially in its ground state, $|\psi\rangle = |g\rangle$. Specifically, for $|\varphi\rangle = \alpha |+\rangle + \beta |-\rangle$, we get

$$\hat{K}_{\varphi, g} = \langle \varphi | \hat{U}_{\Lambda} | g \rangle = \frac{\alpha^*}{\sqrt{2}} e^{-i\lambda \hat{\phi}(\Lambda)} + \frac{\beta^*}{\sqrt{2}} e^{i\lambda \hat{\phi}(\Lambda)}, \quad (5.22)$$

$$\hat{E}(\varphi) = \hat{K}_{\varphi, g}^{\dagger} \hat{K}_{\varphi, g} = \frac{1}{2} + \text{Re}(\alpha^* \beta) \cos 2\lambda \hat{\phi}(\Lambda) + \text{Im}(\alpha^* \beta) \sin 2\lambda \hat{\phi}(\Lambda). \quad (5.23)$$

Even simpler is the case when the measurement is performed in the energy eigenbasis, i.e., $|\varphi_0\rangle = |g\rangle$ and $|\varphi_1\rangle = |e\rangle$:

$$\hat{K}_{g, g} = \langle g | \hat{U}_{\Lambda} | g \rangle = \cos[\lambda \hat{\phi}(\Lambda)], \quad \hat{E}(g) = \cos^2[\lambda \hat{\phi}(\Lambda)], \quad (5.24)$$

$$\hat{K}_{e, g} = \langle e | \hat{U}_{\Lambda} | g \rangle = -i \sin[\lambda \hat{\phi}(\Lambda)], \quad \hat{E}(e) = \sin^2[\lambda \hat{\phi}(\Lambda)]. \quad (5.25)$$

5.2 Causal behaviour of non-selective measurements

In this section, we use the tools from Ch. 3 to analyze the causal behaviour of the non-selective measurement implemented by $I_{\text{ns},\psi}$ (Eq. (5.7)), which is a channel (cf. Sec. 2.1). In particular, Eq. (5.8) shows that the channel induced on the field by the non-selective measurement on the detector coincides with the one implemented by the interaction alone. As a consequence, analyzing the causal behaviour of the non-selective measurement reduces to analyzing the causal behaviour of the interaction in the field sector.

First, notice that the Kraus operators $\hat{K}_{s,\psi}$ all belong to $\mathcal{R}(D)$, where

$$D := \text{supp}\{\chi(\tau) \cdot F(\mathbf{x})\} \quad (5.26)$$

defines the *interaction region*, i.e., the spacetime domain where the particle detector and the quantum field couple. We assume this region to be compact, except perhaps for negligible tails (cf. Sec. 2.3), and connected. Due to the Fermi-Walker rigidity of the detector, this region is centered around its trajectory, and in particular it will not be causally convex in general. Therefore, we take the region of localization of the map to be the causal hull of D , i.e.,

$$L = \mathcal{J}^+(D) \cap \mathcal{J}^-(D). \quad (5.27)$$

The channel that implements the non-selective measurement is therefore localized in L . In particular, e.g., by Proposition 1,

$$I_{\text{ns},\psi}^\dagger \Big|_{\mathcal{R}(L^\perp)} = \text{id}. \quad (5.28)$$

To analyze the causal behaviour of $I_{\text{ns},\psi}$, we examine whether the channel satisfies condition (iii) from Proposition 2. Specifically, let us consider the case where $|\psi\rangle = |g\rangle$, $|\varphi_0\rangle = |g\rangle$ and $|\varphi_1\rangle = |e\rangle$, whose Kraus operators, expanded up to order λ^2 , are given by Eqs. (5.16)–(5.18). Then, for all $\hat{A} \in \mathcal{B}(\mathcal{H}_\phi)$,

$$I_{\text{ns},g}^\dagger(\hat{A}) = \sum_{s=g,e} \left[(\hat{K}_{s,g}^{(0)\dagger} \hat{A} \hat{K}_{s,g}^{(0)}) + (\hat{K}_{s,g}^{(1)\dagger} \hat{A} \hat{K}_{s,g}^{(0)} + \hat{K}_{s,g}^{(0)\dagger} \hat{A} \hat{K}_{s,g}^{(1)}) \right. \quad (5.29)$$

$$\left. + (\hat{K}_{s,g}^{(2)\dagger} \hat{A} \hat{K}_{s,g}^{(0)} + \hat{K}_{s,g}^{(0)\dagger} \hat{A} \hat{K}_{s,g}^{(2)} + \hat{K}_{s,g}^{(1)\dagger} \hat{A} \hat{K}_{s,g}^{(1)}) \right] + \mathcal{O}(\lambda^3)$$

$$= \hat{A} + \lambda^2 \int_{\mathbb{R}^2} d\tau d\tau' \chi(\tau) \chi(\tau') e^{-i\Omega(\tau-\tau')} \left[\hat{\phi}(\tau, F) \hat{A} \hat{\phi}(\tau', F) \right. \quad (5.30)$$

$$\left. - \theta(\tau - \tau') \hat{A} \hat{\phi}(\tau, F) \hat{\phi}(\tau', F) - \theta(\tau' - \tau) \hat{\phi}(\tau, F) \hat{\phi}(\tau', F) \hat{A} \right] + \mathcal{O}(\lambda^3).$$

Now, let $f \in \mathcal{C}_0^\infty(L_{\text{in}})$ and $g \in \mathcal{C}_0^\infty(L_{\text{out}})$ be functions with spacelike-separated supports. Applying the Campbell identity multiple times, we obtain

$$\begin{aligned} \Upsilon_f^\dagger \circ I_{\text{ns},g}^\dagger(e^{i\hat{\phi}(g)}) - I_{\text{ns},g}^\dagger(e^{i\hat{\phi}(g)}) &= e^{i\hat{\phi}(f)} I_{\text{ns},g}^\dagger(e^{i\hat{\phi}(g)}) e^{-i\hat{\phi}(f)} - I_{\text{ns},g}^\dagger(e^{i\hat{\phi}(g)}) \\ &= 2i\lambda^2 \int_{\mathbb{R}^2} d\tau d\tau' \theta(\tau' - \tau) \sin \Omega(\tau' - \tau) \chi(\tau) \chi(\tau') E(f, (\tau, F)) E(g, (\tau', F)) e^{i\hat{\phi}(g)} + \mathcal{O}(\lambda^3). \end{aligned} \quad (5.31)$$

In general, this expression does not vanish if f and g are chosen such that $E(f, \cdot)E(g, \cdot) \neq 0$ over a nonzero-measure subset of D . By Proposition 2, this establishes that I_{ns} is not a causal channel. However, the expression also reveals that the absolute signalling index satisfies $v_{\text{ns},g} \sim \mathcal{O}(\lambda^2)$. This result is not specific to the case $|\psi\rangle = |g\rangle$. Indeed, using Eqs. (5.14) and (5.15), we find that for any $\hat{A} \in \mathcal{B}(\mathcal{H}_\phi)$,

$$I_{\text{ns},\psi}^\dagger(\hat{A}) = \sum_{s=0,1} \left[(\hat{K}_{s,\psi}^{(0)\dagger} \hat{A} \hat{K}_{s,\psi}^{(0)}) + (\hat{K}_{s,\psi}^{(1)\dagger} \hat{A} \hat{K}_{s,\psi}^{(0)} + \hat{K}_{s,\psi}^{(0)\dagger} \hat{A} \hat{K}_{s,\psi}^{(1)}) \right] + \mathcal{O}(\lambda^2) \quad (5.32)$$

$$= \hat{A} + i\lambda \int_{-\infty}^{\infty} d\tau \chi(\tau) \langle \psi | \hat{\mu}(\tau) | \psi \rangle [\hat{\phi}(\tau, F), \hat{A}] + \mathcal{O}(\lambda^2). \quad (5.33)$$

In particular,

$$\Upsilon_f^\dagger \circ I_{\text{ns},\psi}^\dagger(e^{i\hat{\phi}(g)}) = e^{i\hat{\phi}(g)} + i\lambda \int_{-\infty}^{\infty} d\tau \chi(\tau) \langle \psi | \hat{\mu}(\tau) | \psi \rangle [\hat{\phi}(\tau, F) - E(f, (\tau, F))] \mathbf{1}_\phi + \mathcal{O}(\lambda^2). \quad (5.34)$$

Since the extra term is proportional to the identity, it commutes with $e^{i\hat{\phi}(g)}$, leading to

$$\Upsilon_f^\dagger \circ I_{\text{ns},\psi}^\dagger(e^{i\hat{\phi}(g)}) - I_{\text{ns},\psi}^\dagger(e^{i\hat{\phi}(g)}) \sim \mathcal{O}(\lambda^2), \quad (5.35)$$

Thus, we conclude that, in general, the absolute signalling index satisfies $v_{\text{ns},\psi} \sim \mathcal{O}(\lambda^2)$. This is consistent with the conclusions of [85], where it was shown that a triad of particle detectors arranged in a Sorkin-like setup will not witness any faster-than-light signalling below fourth order in perturbation theory. Indeed, since the interaction is already proportional to λ , any effect involving all three detectors would necessarily be at least of order λ^3 . Given that, as we have shown here, faster-than-light signalling only emerges at second order in perturbation theory for the interaction of one detector, this yields the minimum order λ^4 found in [85]. It is worth remarking that this method—using the signalling index to identify the regimes in which a detector model violates causality—can similarly be applied to examine how the causal behaviour depends on additional scales that may be part of a specific setup, such as the size, energy gap, or acceleration of the detector [80].

Finally, we may notice the resemblance between certain terms in Eq. (5.31) and those in Eqs. (2.77) and (2.78). This resemblance is due to the fact that the source of the acausal behaviour of $I_{\text{ns},g}$ stems in the failure of the detector-field interaction Hamiltonian to satisfy the microcausality condition (Eq. (2.76)). However, in Sec. 2.3, we established that gapless and delta-coupled particle detectors do satisfy microcausality, implying that their interactions should respect causality. This claim can now be explicitly confirmed using the expressions from Sec. 5.1 and the results of Ch. 3: from Eq. (5.21),

$$I_{\Lambda,\psi}(\hat{\rho}_\phi) = \text{Tr}_D \left[\hat{U}_\Lambda(|\psi\rangle\langle\psi| \otimes \hat{\rho}_\phi) \hat{U}_\Lambda^\dagger \right] \quad (5.36)$$

$$= \text{Tr}_D \left[\left(e^{-i\lambda\hat{\phi}(\Lambda)}|+\rangle\langle+| + e^{i\lambda\hat{\phi}(\Lambda)}|-\rangle\langle-| \right) |\psi\rangle\langle\psi| \otimes \hat{\rho}_\phi \left(e^{i\lambda\hat{\phi}(\Lambda)}|+\rangle\langle+| + e^{-i\lambda\hat{\phi}(\Lambda)}|-\rangle\langle-| \right) \right] \quad (5.37)$$

$$= |\langle+|\psi\rangle|^2 e^{-i\lambda\hat{\phi}(\Lambda)} \hat{\rho}_\phi e^{i\lambda\hat{\phi}(\Lambda)} + |\langle-|\psi\rangle|^2 e^{i\lambda\hat{\phi}(\Lambda)} \hat{\rho}_\phi e^{-i\lambda\hat{\phi}(\Lambda)}. \quad (5.38)$$

Hence, $I_{\Lambda,\psi}$ is just a mixture of *causal* unitary channels (cf. Sec. 3.3.1), which we would expect to be causal. To verify this, we consider its Kraus operators:

$$\hat{K}_+ := |\langle+|\psi\rangle| e^{-i\lambda\hat{\phi}(\Lambda)} \quad \text{and} \quad \hat{K}_- := |\langle-|\psi\rangle| e^{i\lambda\hat{\phi}(\Lambda)}. \quad (5.39)$$

Using the Campbell identity, we obtain

$$\Upsilon_f^\dagger(\hat{K}_\pm) = |\langle\pm|\psi\rangle| e^{\pm i\lambda E(f,\Lambda)} e^{\mp i\lambda\hat{\phi}(\Lambda)}. \quad (5.40)$$

Since the additional phase introduced by Υ_f for each Kraus operator precisely cancels with that of its adjoint, we conclude that $(f.I_{\Lambda,\psi}) = I_{\Lambda,\psi}$. This confirms that $I_{\Lambda,\psi}$ satisfies causality condition (iii) from Proposition 2 and is therefore causal. Moreover, from Eq. (5.23), we see that gapless detectors, combined with an appropriate modulation of the coupling strength λ , should enable perfectly causal measurements of the expectation value of $\hat{\phi}(\Lambda)$, as well as all its higher moments.

5.3 Update rule

In this section, we build on the framework developed in Ch. 4 to prescribe an update rule for the measurement schemes introduced in Sec. 5.1.

First, let us observe that the minimal relativistic formalism introduced in Sec. 4.3 extends naturally to quantum field theory. In fact, in the case of QFT, it is not even necessary to extend the standard formalism, since the notion of localization of an observable

is already provided by the net of local algebras. Given an observable $\hat{A} \in \mathcal{R}(\mathcal{O})$ for some causally convex open set $\mathcal{O} \subset \mathcal{M}$, its expectation value—as an element of $\mathcal{R}(\mathcal{O})$ —can be computed as

$$\langle \hat{A} \rangle = \frac{\text{Tr} [\Psi_{\mathcal{J}^-(\mathcal{O})}(\hat{\rho}_\phi) \hat{A}]}{\text{Tr} [\Psi_{\mathcal{J}^-(\mathcal{O})}(\hat{\rho}_\phi)]} \equiv \text{Tr} (\hat{\rho}_\phi^\mathcal{O} \hat{A}), \quad (5.41)$$

where $\hat{\rho}_\phi$ is the initial state of the field, and $\Psi_{\mathcal{J}^-(\mathcal{O})}$ encodes all the transformations undergone by the field in the causal past of $\mathcal{O}$, following the same construction as in Sec. 4.3. Here, $\hat{\rho}_\phi^\mathcal{O}$ plays a role entirely analogous to the density operators in Eqs. (4.15) and (4.27), serving as a ‘region-dependent state’.

Now, in order to be able to use this formalism to prescribe a ‘causal’ update rule (in the sense of Ch. 4) for detector-based measurement schemes, we need to identify where in spacetime the selective measurement is localized. Intuitively, one may think that the selective measurement must be localized in the same region as the non-selective measurement, i.e., the causal hull of the interaction region D , which we denoted with L in Sec. 5.2. However, in UDW-based measurement schemes, the measurement on the field is induced by an ideal measurement performed on the detector, and this measurement may happen right after the interaction with the field is switched off, or much later. To distinguish these scenarios, we define $P \subset \mathcal{J}^+(D)$ as the compact set where the measurement outcome becomes available—meaning that the information about the outcome s is only accessible in $\mathcal{J}^+(P)$ (see Fig. 5.1). Under these conditions, while the adjoint of the non-selective channel $I_{\text{ns},\psi}^\dagger$ applies to any operator in L_{out} , the selective operation $I_{s,\psi}^\dagger$ only applies to observables in $\mathcal{J}^+(P)$. In the language of Eq. (5.41), for any $\mathcal{O} \subset \mathcal{M} \setminus D$, the operation $\Psi_{\mathcal{J}^-(\mathcal{O})}$ should include the non-selective measurement if $D \cap \mathcal{J}^-(\mathcal{O}) \neq \emptyset$ (or, equivalently, if $\mathcal{O} \subset L_{\text{out}}$), but it should only include the selective measurement if $P \cap \mathcal{J}^-(\mathcal{O}) \neq \emptyset$.

To illustrate the prescribed update rule, let us compute the n -point distributions that characterize the state of a quantum field after a selective measurement is performed on it using a UDW particle detector, as described in Sec. 5.1. In particular, recall that the coupling of the field and the detector is localized in an interaction region D with causal hull L , followed by an ideal selective measurement of the detector yielding an outcome s . As above, P denotes the region where the outcome is made available. To begin, we examine the simplest case of the one-point distribution, w_1 . Similarly to Eq. (4.18), for each $\mathbf{x} \in \mathcal{M} \setminus D$,

$$\Psi_{\mathcal{J}^-(\mathbf{x})}(\hat{\rho}) = \begin{cases} I_{s,\psi}(\hat{\rho}) = \hat{K}_{s,\psi} \hat{\rho} \hat{K}_{s,\psi}^\dagger & \text{if } \mathbf{x} \in \mathcal{J}^+(P), \\ I_{\text{ns},\psi}(\hat{\rho}) & \text{if } \mathbf{x} \in \mathcal{J}^+(L) \setminus \mathcal{J}^+(P), \\ \hat{\rho} & \text{otherwise.} \end{cases} \quad (5.42)$$

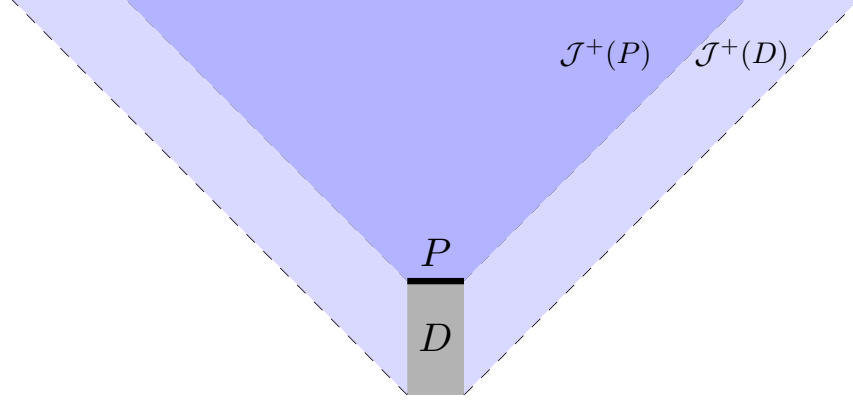


Figure 5.1: (1+1) diagram representing in a slice of spacetime the interaction region D and the hypersurface P when the projective measurement is performed on the detector. In blue, the causal future of the projective measurement; in pale blue, the causal future of the measurement that is not already in the future of the projective measurement, $\mathcal{J}^+(D) \setminus \mathcal{J}^+(P)$.

Thus, for each $\mathbf{x} \in \mathcal{M} \setminus D$, the *updated* 1-point distribution $\tilde{w}_1$ is

$$\tilde{w}_1(\mathbf{x}) = \begin{cases} \text{Tr} [\hat{K}_{s,\psi} \hat{\rho}_\phi \hat{K}_{s,\psi}^\dagger \hat{\phi}(\mathbf{x})] / \text{Tr} [\hat{\rho}_\phi \hat{E}(s)] & \text{if } \mathbf{x} \in \mathcal{J}^+(P), \\ \text{Tr} [I_{\text{ns},\phi}(\hat{\rho}_\phi) \hat{\phi}(\mathbf{x})] & \text{if } \mathbf{x} \in \mathcal{J}^+(L) \setminus \mathcal{J}^+(P), \\ \text{Tr} [\hat{\rho}_\phi \hat{\phi}(\mathbf{x})] = w_1(\mathbf{x}) & \text{otherwise,} \end{cases} \quad (5.43)$$

where $w_1(\mathbf{x})$ denotes the one-point distribution of the initial state $\hat{\rho}_\phi$. The two-point distribution follows analogously: first, we identify the possible maps,

$$\Psi_{\mathcal{J}^-(\mathbf{x}) \cup \mathcal{J}^-(\mathbf{y})}(\hat{\rho}) = \begin{cases} I_{s,\psi}(\hat{\rho}) & \text{if } \mathbf{x} \in \mathcal{J}^+(P) \text{ or } \mathbf{y} \in \mathcal{J}^+(P), \\ I_{\text{ns},\psi}(\hat{\rho}) & \text{if } \mathbf{x}, \mathbf{y} \notin \mathcal{J}^+(P), \text{ and } \mathbf{x} \in \mathcal{J}^+(L) \text{ or } \mathbf{y} \in \mathcal{J}^+(L), \\ \hat{\rho} & \text{otherwise.} \end{cases} \quad (5.44)$$

Therefore, the updated two-point distribution $\tilde{w}_2$ is given by

$$\tilde{w}_2(\mathbf{x}, \mathbf{y}) = \frac{\text{Tr} [\hat{K}_{s,\psi} \hat{\rho}_\phi \hat{K}_{s,\psi}^\dagger \hat{\phi}(\mathbf{x}) \hat{\phi}(\mathbf{y})]}{\text{Tr} [\hat{\rho}_\phi \hat{E}(s)]} \quad (5.45)$$

if $\mathbf{x} \in \mathcal{J}^+(P)$ or $\mathbf{y} \in \mathcal{J}^+(P)$;

$$\tilde{w}_2(\mathbf{x}, \mathbf{y}) = \text{Tr} [I_{\text{ns}, \phi}(\hat{\rho}_\phi) \hat{\phi}(\mathbf{x}) \hat{\phi}(\mathbf{y})] \quad (5.46)$$

if neither $\mathbf{x}$ nor $\mathbf{y}$ belong to $\mathcal{J}^+(P)$, but at least one lies in $\mathcal{J}^+(L)$; and, if none of these circumstances apply, we simply have $\tilde{w}_2(\mathbf{x}, \mathbf{y}) = w_2(\mathbf{x}, \mathbf{y})$, i.e., the two-point distribution of the initial state. This prescription generalizes straightforwardly to the n -point distribution: the updated $\tilde{w}_n$ is given by

$$\tilde{w}_n(\mathbf{x}_1, \dots, \mathbf{x}_n) = \frac{\text{Tr} [\hat{K}_{s, \psi} \hat{\rho}_\phi \hat{K}_{s, \psi}^\dagger \hat{\phi}(\mathbf{x}_1) \cdots \hat{\phi}(\mathbf{x}_n)]}{\text{Tr} [\hat{\rho}_\phi \hat{E}(s)]} \quad (5.47)$$

if at least one argument belongs to $\mathcal{J}^+(P)$; and

$$\tilde{w}_n(\mathbf{x}_1, \dots, \mathbf{x}_n) = \text{Tr} [I_{\text{ns}, \phi}(\hat{\rho}_\phi) \hat{\phi}(\mathbf{x}_1) \cdots \hat{\phi}(\mathbf{x}_n)], \quad (5.48)$$

if none of the arguments belong to $\mathcal{J}^+(P)$, but at least one of them belongs to $\mathcal{J}^+(L)$. In all other cases, we recover the initial distribution, $\tilde{w}_n(\mathbf{x}_1, \dots, \mathbf{x}_n) = w_n(\mathbf{x}_1, \dots, \mathbf{x}_n)$.

5.4 A measurement theory for QFT

In Sec. 5.1, we considered measurement schemes in which particle detectors couple to a quantum field to extract information about its state. After examining their causal behaviour in Sec. 5.2, and providing an update rule in Sec. 5.3, we are now in a position to argue that this framework satisfies all the key characteristics of a proper measurement theory for QFT. Namely:

1. *It is consistent with relativistic QFT.* The measurement process consists of two steps: the interaction between the detector and the field, and the projective measurement on the detector once the interaction has been switched off in order to access the information about the field stored in it. In Sec. 5.2, we saw that the causal behaviour of the non-selective measurement is simply that of the interaction, and that any deviations from causality due to the interaction only arise beyond leading order in perturbation theory. Moreover, we also showed that gapless and delta-coupled particle detectors do not enable faster-than-light signalling at all, and therefore offer a fully causal option for modelling measurements of quantum fields.

2. *It provides an update rule.* In Sec. 5.3, we gave a prescription for the post-measurement field state that respects causality, includes the information obtained from the outcome of the measurement, accounts for correlations, and enforces the compatibility of sequential measurements (i.e., a covariant, causal, fully predictive update rule; cf. Ch. 4).
3. *It produces definite values for the outcome of single-shot measurements.* Since the detectors are measured through standard measurements, the outcome of a measurement is a real number (or an interval) that can be written down in an experimenter's notepad.
4. *It is capable of reproducing experiments.* Indeed, as we mentioned in Sec. 2.3, particle detector models have been shown to capture the essential features of quantum optical phenomena and light-matter interactions [78, 106, 120, 121], as well as the phenomenology of the measurement of other quantum fields, such as neutrinos [157, 158]. Thus, particle detector models provide a direct link to experimentally realistic setups where quantum fields are measured.

By satisfying these four characteristics, we argue that the detector-based framework presented in this chapter constitutes a measurement theory for QFT, at least in practice, and within a well-understood regime of validity.

Intermezzo: Particle detectors beyond perturbation theory

Chapter 6

Non-perturbative method for particle detectors with continuous interactions

In this chapter, we show that detector switching profiles consisting of trains of delta couplings are a useful computational tool to efficiently approximate results involving continuous switching functions, both in setups involving a single detector and multiple ones. The rapid convergence to the continuous results at all orders in perturbation theory for sufficiently regular switchings means that this tool can be used to obtain non-perturbative results for general particle detector phenomena with continuous switching functions.

6.1 Perturbative v. Non-perturbative

In their multiple applications, particle detectors are essentially used either as emitters and receptors of information, or as measurement devices, i.e., local probes that gather information about the field through their interaction (cf. Ch. 5). In both cases, in order to make predictions we need to be able to compute the final state of the detectors after their interaction with the field. However, as it was pointed out in Sec. 2.3, the full theory including the interaction between the detector and the field cannot be exactly solved in general, as is the case for most interacting field theories.

A frequent avenue to circumvent the complications mentioned above is to use perturbation theory, as we did in Ch. 5. In this approach, the final state of the detectors is

expanded in a series of terms proportional to increasingly higher powers of the coupling parameter λ (cf. Eqs. (2.79)–(2.81) in Sec. 2.3). Under the assumption that the detectors are weakly coupled to the field, the higher order contributions can be neglected, leaving a truncated series as the final result. However, despite its popularity, the perturbative approach has its limitations. First, the truncated result is an approximation whose error is not known *a priori*, and can only be trusted to be accurate for “small enough” coupling parameters. Furthermore, there are also many physical situations where one needs to go beyond leading order in perturbation theory, which can be technically challenging (see, e.g., the fourth-order calculations in [105]). Finally, non-perturbative effects in the model cannot be captured by perturbative methods.

These limitations call for the use of non-perturbative techniques. One way to proceed is to work in specific scenarios where the evolution of the detectors and the field can be calculated exactly. One of those scenarios is found in the context of continuous-variable quantum mechanics (see, e.g., [159, 160] for reviews). Specifically, when the quantum field has both infrared and ultraviolet cutoffs, it can be reduced to (or approximated by) a lattice, so that if the detectors and the field are initially in a so-called *Gaussian state* (i.e., the Wigner function describing their joint state is Gaussian), and if their coupling is linear, then the evolution can be solved exactly using Gaussian methods¹ [161, 162]. Another scenario where calculations can be carried out non-perturbatively is when the coupling between the detectors and the field consists of sudden interactions, i.e., one or several delta couplings [116, 163–168]. In this case, as we will review in Sec. 6.2, the evolution can always be written as a sequence of known unitaries, and, moreover, these unitaries can often be exactly evaluated (see, e.g., [99, 169–171]).

In the following sections, we present an efficient non-perturbative method that applies to particle detectors that couple to the quantum field continuously in time. Given the physical relevance of these scenarios, and the fact that so far they have been mostly analyzed perturbatively, these techniques may allow us to explore possible new non-perturbative phenomena in the study of the dynamics induced by time-extended couplings, such as, e.g., in the Unruh effect or entanglement harvesting, among many others.

6.2 Detectors undergoing multiple delta couplings

In this section, we review how to compute the final state of one or multiple particle detectors after their interaction with a quantum field through a sequence of delta couplings.

¹We will see an example of this in Ch. 7.

6.2.1 Single detector case

Recall from Eqs. (2.68)–(2.70) that the coupling of a Fermi-Walker rigid particle detector with a quantum field $\hat{\phi}$ can be described in the interaction picture by the Hamiltonian weight

$$\hat{h}_{\text{int}}(\tau, \mathbf{z}) = \lambda \chi(\tau) [F(\mathbf{z}) \hat{\mu}_\alpha^\dagger(\tau) \hat{O}^\alpha(\tau, \mathbf{z}) + \text{H.c.}]. \quad (6.1)$$

If the detector couples to the field via a train of delta couplings at a sequence of times $\{\tau_j, j = 1, \dots, N\}$, $N \geq 1$, the switching function is given by

$$\chi(\tau) = \sum_{j=1}^N \eta_j \delta(\tau - \tau_j), \quad (6.2)$$

where η_j is the strength with which the detector couples to the field at time τ_j . With the interaction Hamiltonian weight given by Eq. (6.1), from Eq. (2.74) we obtain that the evolution operator in the interaction picture is given by

$$\hat{U} = \mathcal{T}_\tau \exp \left\{ -i\lambda \sum_{j=1}^N \eta_j \left[\hat{\mu}_{\alpha,j}^\dagger \int d\mathbf{z} \sqrt{-g_j} F(\mathbf{z}) \hat{O}_j^\alpha(\mathbf{z}) + \text{H.c.} \right] \right\} \quad (6.3)$$

$$= \mathcal{T}_\tau \prod_{j=1}^N \exp \left\{ -i\lambda \eta_j \left[\hat{\mu}_{\alpha,j}^\dagger \int d\mathbf{z} \sqrt{-g_j} F(\mathbf{z}) \hat{O}_j^\alpha(\mathbf{z}) + \text{H.c.} \right] \right\}, \quad (6.4)$$

where $g_j \equiv \det g_{\mu\nu}(\tau_j, \mathbf{z})$ is the determinant of the spacetime metric at $(\tau_j, \mathbf{z})$, and similarly $\hat{\mu}_{\alpha,j}^\dagger \equiv \hat{\mu}_\alpha^\dagger(\tau_j)$, and $\hat{O}_j^\alpha(\mathbf{z}) \equiv \hat{O}^\alpha(\tau_j, \mathbf{z})$. $\mathcal{T}_\tau \exp$ and $\mathcal{T}_\tau$ respectively denote the time-ordered exponential and the time ordering operation with respect to the detector's proper time τ , which we recall is an acceptable choice of time-ordering parameter as long as we are in the regime of validity of the detector approximation, where the detector model is effectively covariant [80]. In particular, defining

$$\hat{U}_j := \exp \left\{ -i\lambda \eta_j \left[\hat{\mu}_{\alpha,j}^\dagger \int d\mathbf{z} \sqrt{-g_j} F(\mathbf{z}) \hat{O}_j^\alpha(\mathbf{z}) + \text{H.c.} \right] \right\}, \quad (6.5)$$

we can write

$$\hat{U} = \hat{U}_N \cdots \hat{U}_1. \quad (6.6)$$

Notice that the $\hat{U}_j$ operators do not commute with each other, but the action of the time ordering operation in Eq. (6.3) yields Eq. (6.6). After the interaction, the joint state of the detector and the field is

$$\hat{\rho} = \hat{U}_N \cdots \hat{U}_1 \hat{\rho}_0 \hat{U}_1^\dagger \cdots \hat{U}_N^\dagger, \quad (6.7)$$

where $\hat{\rho}_0$ is the initial detector-field state. In particular, the final state of the detector after the interaction results from tracing over the field degrees of freedom, i.e.,

$$\hat{\rho}_D = \text{Tr}_\phi \left(\hat{U}_N \cdots \hat{U}_1 \hat{\rho}_0 \hat{U}_1^\dagger \cdots \hat{U}_N^\dagger \right). \quad (6.8)$$

6.2.2 Multiple detectors case

In complete analogy with the single detector case presented in Subsec. 6.2.1, here we consider D particle detectors in the general globally hyperbolic spacetime $(\mathcal{M}, \mathbf{g})$, whose centres of mass follow trajectories $\mathbf{x}_n(\tau_n)$ parametrized by their proper times τ_n , within the regime in which each one of them can be fairly assumed to be Fermi-Walker rigid.

In order to jointly describe the interaction of all the detectors, we will assume that there exists a set of coordinates $(\tau, \mathbf{z})$ of $\mathcal{M}$ such that, for each detector, there is a neighbourhood of its trajectory $\mathbf{x}_n(\tau_n)$ where the coordinates $(\tau, \mathbf{z})$ are a set of Fermi normal coordinates of the detector's centre of mass. In particular, if these neighbourhoods include the supports of each detector's smearing function, then we can write the interaction Hamiltonian weight as

$$\hat{h}_{\text{int}}(\tau, \mathbf{z}) = \sum_{n=1}^D \lambda_n \chi_n(\tau) \left[F_n(\mathbf{z}) \hat{\mu}_{n,\alpha}^\dagger(\tau) \hat{O}_n^\alpha(\tau, \mathbf{z}) + \text{H.c.} \right], \quad (6.9)$$

where λ_n , χ_n , and F_n are the coupling strength, the switching, and the smearing functions of the n -th detector, while $\hat{\mu}_n^\alpha$ is an arbitrary tensor operator of the n -th detector, and $\hat{O}_n^\alpha$ is the generic tensor field operator it couples to.

In the case of multiple delta couplings, the switching functions can be written as

$$\chi_n(\tau) = \sum_{j=1}^{N_n} \eta_{n,j} \delta(\tau - \tau_{n,j}), \quad (6.10)$$

where N_n is the number of times that the n -th detector couples to the field, and $\eta_{n,j}$ is the strength of the coupling at time $\tau_{n,j}$. With the interaction Hamiltonian weight given by Eq. (6.9), the evolution operator in the interaction picture is given by

$$\hat{U} = \mathcal{T}_\tau \exp \left\{ -i \sum_{n=1}^D \lambda_n \sum_{j=1}^{N_n} \eta_{n,j} \left[\hat{\mu}_{n,j,\alpha}^\dagger \int d\mathbf{z} \sqrt{-g_{n,j}} F(\mathbf{z}) \hat{O}_{n,j}^\alpha(\mathbf{z}) + \text{H.c.} \right] \right\}, \quad (6.11)$$

$$= \mathcal{T}_\tau \prod_{n,j} \exp \left\{ -i \lambda_n \eta_{n,j} \left[\hat{\mu}_{n,j,\alpha}^\dagger \int d\mathbf{z} \sqrt{-g_{n,j}} F(\mathbf{z}) \hat{O}_{n,j}^\alpha(\mathbf{z}) + \text{H.c.} \right] \right\}, \quad (6.12)$$

where in the last equality, n is running from 1 to D , and j runs from 1 to N_n , for each n . Here, in analogy with the notation used for one detector, $g_{n,j} \equiv \det g_{\mu\nu}(\tau_{n,j}, \mathbf{z})$ is the determinant of the metric at $(\tau_{n,j}, \mathbf{z})$, and similarly $\hat{\mu}_{n,j,\alpha}^\dagger \equiv \hat{\mu}_{n,\alpha}^\dagger(\tau_{n,j})$, and $\hat{O}_{n,j}^\alpha(\mathbf{z}) \equiv \hat{O}^\alpha(\tau_{n,j}, \mathbf{z})$. Eq. (6.11) implies that we can write $\hat{U}$ as a product of unitaries in analogy with Eq. (6.6), where each unitary of the sequence is of the form

$$\hat{U}_{n,j} = \exp \left\{ -i\lambda_n \eta_{n,j} \left[\hat{\mu}_{n,j,\alpha}^\dagger \int d\mathbf{z} \sqrt{-g_{n,j}} F(\mathbf{z}) \hat{O}_{n,j}^\alpha(\mathbf{z}) + \text{H.c.} \right] \right\}, \quad (6.13)$$

for some n, j . Notice that this amounts to ordering all the $\tau_{n,j}$ and applying the $\hat{U}_{n,j}$ following the same order. If two times $\tau_{n,j}$ and $\tau_{n',j'}$ coincide, for some $n \neq n'$, then, since we may assume that different detectors do not overlap, the corresponding unitaries $\hat{U}_{n,j}$ and $\hat{U}_{n',j'}$ commute, and therefore it does not matter how we decide to order them. This is because if the supports of detectors n and n' at $\tau_{n,j} = \tau_{n',j'}$ do not overlap, then the field operators in $\hat{U}_{n,j}$ and $\hat{U}_{n',j'}$ have support in spacelike-separated regions, and thus commute. Since the detector operators also commute for being associated with different detectors (and then defined over different Hilbert spaces), the exponents of the exponentials that define $\hat{U}_{n,j}$ and $\hat{U}_{n',j'}$ commute, and hence so do $\hat{U}_{n,j}$ and $\hat{U}_{n',j'}$ themselves. The final state of the field-detectors system is then computed in the same way as in the single detector case, with the detectors' partial state obtained by tracing out the field degrees of freedom.

6.3 Approximation via multiple delta couplings

In this section, we illustrate how the dynamics induced by trains of delta couplings that we reviewed in Sec. 6.2 can be used to approximate the dynamics of continuous couplings. We also state the approximation result that guarantees the convergence and efficiency of the method, leaving the proof for Sec. 6.4.

6.3.1 Single detector case

Let us consider a single detector that couples to the field through a switching function $\xi(\tau)$ that is bounded, and continuous except for maybe a finite number of points. In the following, we call this switching function *regular*. It turns out that this scenario can be efficiently approximated using a train of delta interactions. Specifically, let $[0, T]$ be the interval where $\xi(\tau)$ has its support, where in particular we have $\xi(0) = \xi(T) = 0$. The

idea is to approximate the phenomenology of the regular switching function with that of a sequence of uniformly spaced Dirac delta pulses. Thus, we can define a train of N sudden interactions associated with ξ and a uniform partition $\mathcal{P} = \{t_j = Tj/N, j = 0, \dots, N\}$ of the interval $[0, T]$ as

$$\chi_\xi(\tau; N) = \frac{T}{N} \sum_{j=1}^N \xi\left(\frac{j-1/2}{N}T\right) \delta\left(\tau - \frac{j-1/2}{N}T\right). \quad (6.14)$$

Recalling Eq. (6.2), this amounts to setting

$$\tau_j = \frac{j-1/2}{N}T, \quad (6.15)$$

and

$$\eta_j = \xi(\tau_j) (t_j - t_{j-1}) = \xi(\tau_j) \frac{T}{N}. \quad (6.16)$$

The strength of each sudden interaction is thus determined by both the value of the switching ξ at $\tau = \tau_j$ and the duration T/N of the interval it is effectively “substituting”. Let us denote with $\hat{\rho}_D(N)$ and $\hat{\rho}_D^\xi$ the final states of the detector according to Eq. (6.8) after the train of N sudden interactions $\chi_\xi(\tau; N)$ in Eq. (6.14), and in the original regular scenario, respectively. Assuming that the detector and the field are initially uncorrelated ($\hat{\rho}_0 = \hat{\rho}_{D,0} \otimes \hat{\rho}_{\phi,0}$), and using perturbation theory, as in Eq. (2.80) we can write

$$\hat{\rho}_D(N) = \hat{\rho}_{D,0} + \sum_{k=1}^{\infty} \hat{\rho}_D^{(k)}(N) \quad (6.17)$$

for the train of deltas, and

$$\hat{\rho}_D^\xi = \hat{\rho}_{D,0} + \sum_{k=1}^{\infty} \hat{\rho}_D^{(k)\xi} \quad (6.18)$$

for the regular scenario, where, as before, $\hat{\rho}_D^{(k)}(N)$ and $\hat{\rho}_D^{(k)\xi}$ are proportional to λ^k . Then it can be shown that, for each $k \in \mathbb{Z}^+$, under the assumption that the switching function ξ , the smearing function F , and the initial state of the field $\hat{\rho}_{\phi,0}$ are *sufficiently regular*,²

$$\lim_{N \rightarrow \infty} \hat{\rho}_D^{(k)}(N) = \hat{\rho}_D^{(k)\xi} \quad (6.19)$$

²To expand on what is meant by sufficiently regular, see the proof in Sec. 6.4. In short, for most choices of switching and smearing functions, and for the initial states of the field usually employed in the literature (e.g., Hadamard states, both in flat and curved spacetimes), we expect the regularity condition to be satisfied.

in the weak operator topology. In fact, the convergence is *at least* as fast as $1/N$. Specifically, for any linear functional G ,

$$G[\hat{\rho}_D^{(k)\xi}] = G[\hat{\rho}_D^{(k)}(N)] + \mathcal{O}(1/N). \quad (6.20)$$

Now, when ξ , F , and $\hat{\rho}_{\phi,0}$ are such that the conditions for Eq. (6.20) are fulfilled *for all* $k \in \mathbb{Z}^+$, then the train of deltas approximates the regular scenario *at all orders in perturbation theory*, since

$$G[\hat{\rho}_D^\xi] = G[\hat{\rho}_D(N)] + \mathcal{O}(1/N) \quad (6.21)$$

for any linear functional G . In particular,

$$\lim_{N \rightarrow \infty} \hat{\rho}_D(N) = \hat{\rho}_D^\xi \quad (6.22)$$

in the weak operator topology. Notice that since it is possible to perform non-perturbative calculations with delta couplings (see, e.g., [169–171]), Eq. (6.21) allows to efficiently approximate non-perturbative results, as opposed to the usual perturbative approximations based on truncating the Dyson expansion. While this is still an approximation (that can be made arbitrarily precise by increasing the density of delta couplings), it approximates the full sum of the perturbative expansion, and not only up to a given order. This is why we refer to this technique as a non-perturbative approximation.

6.3.2 Multiple detector case

Let us now consider D particle detectors under the same conditions of Sec. 6.2.2, except for their coupling to the field, which now happens through ‘regular’ switching functions $\xi_n(\tau)$. As in the single detector case, it turns out that this scenario can be efficiently approximated using a train of delta interactions. Specifically, let us denote with $[T_n, T_n + \Delta T_n]$ the interval where $\xi_n(\tau)$ has its support, and in particular $\xi_n(T_n) = \xi_n(T_n + \Delta T_n) = 0$. We can then define the train of N sudden interactions associated with ξ_n and a uniform partition $\mathcal{P}_n = \{t_{n,j} = T_n + j\Delta T_n/N, j = 1, \dots, N\}$ of the interval $[T_n, T_n + \Delta T_n]$ as

$$\chi_{\xi_n}(\tau; N) = \frac{\Delta T_n}{N} \sum_{j=1}^N \xi_n\left(T_n + \frac{j-1/2}{N} \Delta T_n\right) \delta\left(\tau - T_n - \frac{j-1/2}{N} \Delta T_n\right). \quad (6.23)$$

From Eq. (6.10), this means having $N_n = N$ for all n , setting

$$\tau_{n,j} = T_n + \frac{j-1/2}{N} \Delta T_n, \quad (6.24)$$

and

$$\eta_{n,j} = \xi_n(\tau_{n,j})(t_{n,j} - t_{n,j-1}) = \xi_n(\tau_{n,j}) \frac{\Delta T_n}{N}. \quad (6.25)$$

This is in complete analogy with the single detector case: the strength of each sudden interaction is determined by both the value of the switching ξ_n at the corresponding time, and the duration of the interval it stands for.

As in the single detector case, we assume that, initially, the detectors and the field are mutually uncorrelated, and we can use perturbation theory to write

$$\hat{\rho}_D(N) = \sum_{\mathbf{l} \in \mathbb{N}_0^D} \hat{\rho}_D^{(\mathbf{l})}(N) \quad (6.26)$$

for the train of deltas (where $\mathbb{N}_0 = \mathbb{Z}^+ \cup \{0\}$), and

$$\hat{\rho}_D^\xi = \sum_{\mathbf{l} \in \mathbb{N}_0^D} \hat{\rho}_D^{(\mathbf{l})\xi} \quad (6.27)$$

for the regular scenario, where $\hat{\rho}_D^{(\mathbf{l})}(N)$ and $\hat{\rho}_D^{(\mathbf{l})\xi}$ are proportional to $\prod_{n=1}^D \lambda_n^{l_n}$, for each $\mathbf{l} \in \mathbb{N}_0^D$, and the first term for both series is just the initial state. It can be shown that, for each $\mathbf{l} \in \mathbb{N}_0^D$, under the assumption that the switching functions ξ_n , the smearing functions F_n , and the initial state of the field $\hat{\rho}_{\phi,0}$ are *sufficiently regular*,

$$\lim_{N \rightarrow \infty} \hat{\rho}_D^{(\mathbf{l})}(N) = \hat{\rho}_D^{(\mathbf{l})\xi} \quad (6.28)$$

in the weak operator topology, and the convergence is *at least* as fast as $1/N$, i.e., for any linear functional G ,

$$G[\hat{\rho}_D^{(\mathbf{l})\xi}] = G[\hat{\rho}_D^{(\mathbf{l})}(N)] + \mathcal{O}(1/N). \quad (6.29)$$

In particular, when ξ_n , F_n , and $\hat{\rho}_{\phi,0}$ are such that the conditions for Eq. (6.29) are fulfilled *for all* $\mathbf{l} \in \mathbb{N}_0^D$, then

$$G[\hat{\rho}_D^\xi] = G[\hat{\rho}_D(N)] + \mathcal{O}(1/N) \quad (6.30)$$

for any linear functional G . In particular, the dynamics induced by the train of deltas approximates those of the regular scenario *at all orders in perturbation theory*, i.e.,

$$\lim_{N \rightarrow \infty} \hat{\rho}_D(N) = \hat{\rho}_D^\xi \quad (6.31)$$

in the weak operator topology. This means that the same conclusions that we obtained for the single detector scenario—and in particular the possibility of efficiently approximating

non-perturbative results—can be extended to setups involving multiple detectors as well. Notice that the density matrix that we get to approximate describes the joint state of all the detectors, and not just one of them. This means that we can use this technique to extract conclusions regarding not only the state of each individual detector, but also the correlations between them.

6.4 Convergence and efficiency of the approximation

In this section, we prove the approximation results stated in Eqs. (6.19)–(6.22) for a single detector, and in Eqs. (6.29)–(6.31) for multiple detectors. The claim establishes that each term of the Dyson series of the time-evolved density matrix resulting from couplings with trains of sudden interactions converges (as we increase the number of sudden interactions) to the contribution of the same order of the density matrix resulting from couplings with (sufficiently regular) switching functions that the delta interactions aim to approximate, both for setups involving one or more particle detectors.

Let us first consider a one detector scenario with the interaction Hamiltonian weight given by Eq. (6.1), i.e.,

$$\hat{h}_{\text{int}}(\tau, \mathbf{z}) = \lambda \chi(\tau) [F(\mathbf{z}) \hat{\mu}_\alpha^\dagger(\tau) \hat{O}^\alpha(\tau, \mathbf{z}) + \text{H.c.}], \quad (6.32)$$

where the switching function χ can be either the regular switching, ξ , or its approximation with N sudden interactions, $\chi_\xi(N)$, given in Eq. (6.14). For the regular scenario, the joint evolution of the field and the detector is given by the unitary

$$\hat{U}_\xi = \mathcal{T}_\tau \exp \left\{ -i\lambda \int d\tau \xi(\tau) \left[\hat{\mu}_\alpha^\dagger(\tau) \int d\mathbf{z} \sqrt{-g} F(\mathbf{z}) \hat{O}^\alpha(\tau, \mathbf{z}) + \text{H.c.} \right] \right\}, \quad (6.33)$$

with the final state of the detector being

$$\hat{\rho}_\text{D}^\xi = \text{Tr}_\phi \left(\hat{U}_\xi \hat{\rho}_0 \hat{U}_\xi^\dagger \right). \quad (6.34)$$

Each term in the perturbative expansion of Eq. (6.18) is given by (cf. Eq. (2.80))

$$\hat{\rho}_\text{D}^{(k)\xi} = \sum_{r+s=k} \text{Tr}_\phi \left(\hat{U}_\xi^{(r)} \hat{\rho}_0 \hat{U}_\xi^{(s)\dagger} \right). \quad (6.35)$$

Now,

$$\begin{aligned}
& \text{Tr}_\phi \left(\hat{U}_\xi^{(r)} \hat{\rho}_0 \hat{U}_\xi^{(s)\dagger} \right) \\
&= (-i)^{r+s} \lambda^k \int d\tau_1 \dots d\tau_r d\tau'_1 \dots d\tau'_s \prod_{p=1}^{r-1} \theta(\tau_{p+1} - \tau_p) \prod_{q=1}^{s-1} \theta(\tau'_{q+1} - \tau'_q) \prod_{l=1}^r \xi(\tau_l) \prod_{m=1}^s \xi(\tau'_m) \\
&\quad \times \left[W_{\alpha'}^\alpha(\tau'_1, \dots, \tau'_s; \tau_1, \dots, \tau_r; F) \hat{\mu}_{\alpha_r}^\dagger(\tau_r) \dots \hat{\mu}_{\alpha_1}^\dagger(\tau_1) \hat{\rho}_{D,0} \hat{\mu}_{\alpha'_1}(\tau'_1) \dots \hat{\mu}_{\alpha'_s}(\tau'_s) \right. \\
&\quad + W_{\alpha}^{\alpha'}(\tau'_1, \dots, \tau'_s; \tau_1, \dots, \tau_r) \hat{\mu}^{\alpha_r}(\tau_r) \dots \hat{\mu}^{\alpha_1}(\tau_1) \hat{\rho}_{D,0} \hat{\mu}_{\alpha'_1}^\dagger(\tau'_1) \dots \hat{\mu}_{\alpha'_s}^\dagger(\tau'_s) \\
&\quad + W_{\alpha, \alpha'}(\tau'_1, \dots, \tau'_s; \tau_1, \dots, \tau_r) \hat{\mu}^{\alpha_r}(\tau_r) \dots \hat{\mu}^{\alpha_1}(\tau_1) \hat{\rho}_{D,0} \hat{\mu}_{\alpha'_1}(\tau'_1) \dots \hat{\mu}_{\alpha'_s}(\tau'_s) \\
&\quad \left. + W^{\alpha, \alpha'}(\tau'_1, \dots, \tau'_s; \tau_1, \dots, \tau_r) \hat{\mu}_{\alpha_r}^\dagger(\tau_r) \dots \hat{\mu}_{\alpha_1}^\dagger(\tau_1) \hat{\rho}_{D,0} \hat{\mu}_{\alpha'_1}^\dagger(\tau'_1) \dots \hat{\mu}_{\alpha'_s}^\dagger(\tau'_s) \right],
\end{aligned} \tag{6.36}$$

where we denoted

$$W_{\alpha'}^\alpha(t'_1, \dots, t'_s; t_1, \dots, t_r; F) := \text{Tr}_\phi \left[\hat{\rho}_{\phi,0} \hat{O}_{\alpha'_1}^\dagger(t'_1, F) \dots \hat{O}_{\alpha'_s}^\dagger(t'_s, F) \hat{O}^{\alpha_r}(t_r, F) \dots \hat{O}^{\alpha_1}(t_1, F) \right], \tag{6.37}$$

$$W_{\alpha}^{\alpha'}(t'_1, \dots, t'_s; t_1, \dots, t_r; F) := \text{Tr}_\phi \left[\hat{\rho}_{\phi,0} \hat{O}^{\alpha'_1}(t'_1, F) \dots \hat{O}^{\alpha'_s}(t'_s, F) \hat{O}_{\alpha_r}^\dagger(t_r, F) \dots \hat{O}_{\alpha_1}^\dagger(t_1, F) \right], \tag{6.38}$$

$$W_{\alpha, \alpha'}(t'_1, \dots, t'_s; t_1, \dots, t_r; F) := \text{Tr}_\phi \left[\hat{\rho}_{\phi,0} \hat{O}_{\alpha'_1}^\dagger(t'_1, F) \dots \hat{O}_{\alpha'_s}^\dagger(t'_s, F) \hat{O}_{\alpha_r}^\dagger(t_r, F) \dots \hat{O}_{\alpha_1}^\dagger(t_1, F) \right], \tag{6.39}$$

$$W^{\alpha, \alpha'}(t'_1, \dots, t'_s; t_1, \dots, t_r; F) := \text{Tr}_\phi \left[\hat{\rho}_{\phi,0} \hat{O}^{\alpha'_1}(t'_1, F) \dots \hat{O}^{\alpha'_s}(t'_s, F) \hat{O}^{\alpha_r}(t_r, F) \dots \hat{O}^{\alpha_1}(t_1, F) \right], \tag{6.40}$$

and

$$\hat{O}^\alpha(t, F) := \int d\mathbf{z} \sqrt{-g} F(\mathbf{z}) \hat{O}^\alpha(t, \mathbf{z}). \tag{6.41}$$

We can rewrite Eq. (6.36) as

$$\text{Tr}_\phi \left(\hat{U}_\xi^{(r)} \hat{\rho}_0 \hat{U}_\xi^{(s)\dagger} \right) = (-i)^{r+s} \lambda^k \int_{\Delta_r \times \Delta'_s} d\boldsymbol{\tau} d\boldsymbol{\tau}' \Xi(\boldsymbol{\tau}; \boldsymbol{\tau}') \hat{\omega}(\boldsymbol{\tau}; \boldsymbol{\tau}'; F). \tag{6.42}$$

Here,

$$\Delta_r := \{\boldsymbol{\tau} = (\tau_1, \dots, \tau_r) : \tau_1 \in [0, T], \tau_i \in [0, \tau_{i-1}], \forall i = 2, \dots, r\}, \tag{6.43}$$

$$\Delta'_s := \{\boldsymbol{\tau}' = (\tau'_1, \dots, \tau'_s) : \tau'_1 \in [0, T], \tau'_i \in [0, \tau'_{i-1}], \forall i = 2, \dots, s\}, \tag{6.44}$$

are triangular domains; the function Ξ is an abbreviation,

$$\Xi(\boldsymbol{\tau}; \boldsymbol{\tau}') := \prod_{l=1}^r \xi(\tau_l) \prod_{m=1}^s \xi(\tau'_m), \quad (6.45)$$

and $\hat{\omega}$ is the operator corresponding to the expression in square brackets of Eq. (6.36).

On the other hand, if we consider the evolution produced by the coupling through the train of sudden interactions described by $\chi_\xi(N)$, from Eqs. (6.3) and (6.14) we have that the joint field-detector time evolution is given by

$$\begin{aligned} \hat{U}(N) = \mathcal{T}_\tau \exp \left\{ -i\lambda \frac{T}{N} \sum_{j=1}^N \xi \left(\frac{j-1/2}{N} T \right) \right. \\ \left. \times \left[\hat{\mu}_\alpha^\dagger \left(\frac{j-1/2}{N} T \right) \int d\mathbf{z} \sqrt{-g_j} F(\mathbf{z}) \hat{O}^\alpha \left(\frac{j-1/2}{N}, \mathbf{z} \right) + \text{H.c.} \right] \right\}, \end{aligned} \quad (6.46)$$

with the final state of the detector thus being

$$\hat{\rho}_D(N) = \text{Tr}_\phi \left(\hat{U}(N) \hat{\rho}_0 \hat{U}(N)^\dagger \right). \quad (6.47)$$

Each term in the perturbative expansion of Eq. (6.17) is then given by

$$\hat{\rho}_D^{(k)}(N) = \sum_{r+s=k} \text{Tr}_\phi \left(\hat{U}^{(r)}(N) \hat{\rho}_0 \hat{U}^{(s)}(N)^\dagger \right). \quad (6.48)$$

In this case, for $r+s=k$, we have

$$\begin{aligned} & \text{Tr}_\phi \left(\hat{U}^{(r)}(N) \hat{\rho}_0 \hat{U}^{(s)}(N)^\dagger \right) \\ &= (-i)^{r+s} \frac{\lambda^k T^k}{N^k} \sum_{j \in \mathcal{J}_r} \sum_{j' \in \mathcal{J}'_s} \prod_{n=1}^r \xi \left(\frac{j_n-1/2}{N} T \right) \prod_{m=1}^s \xi \left(\frac{j'_m-1/2}{N} T \right) \\ & \times \left[W_{\boldsymbol{\alpha}'}^\alpha \left(\frac{j'_1-1/2}{N} T, \dots, \frac{j'_s-1/2}{N} T; \frac{j_1-1/2}{N} T, \dots, \frac{j_r-1/2}{N} T; F \right) \right. \\ & \quad \times \hat{\mu}_{\alpha_r}^\dagger \left(\frac{j_r-1/2}{N} T \right) \dots \hat{\mu}_{\alpha_1}^\dagger \left(\frac{j_1-1/2}{N} T \right) \hat{\rho}_{D,0} \hat{\mu}_{\alpha'_1} \left(\frac{j'_1-1/2}{N} T \right) \dots \hat{\mu}_{\alpha'_s} \left(\frac{j'_s-1/2}{N} T \right) \\ & \quad + W_{\boldsymbol{\alpha}}^{\alpha'} \left(\frac{j'_1-1/2}{N} T, \dots, \frac{j'_s-1/2}{N} T; \frac{j_1-1/2}{N} T, \dots, \frac{j_r-1/2}{N} T; F \right) \\ & \quad \times \hat{\mu}^{\alpha_r} \left(\frac{j_r-1/2}{N} T \right) \dots \hat{\mu}^{\alpha_1} \left(\frac{j_1-1/2}{N} T \right) \hat{\rho}_{D,0} \hat{\mu}_{\alpha'_1}^\dagger \left(\frac{j'_1-1/2}{N} T \right) \dots \hat{\mu}_{\alpha'_s}^\dagger \left(\frac{j'_s-1/2}{N} T \right) \Big] \end{aligned} \quad (6.49)$$

$$\begin{aligned}
& + W_{\alpha, \alpha'} \left(\frac{j'_1 - 1/2}{N} T, \dots, \frac{j'_s - 1/2}{N} T; \frac{j_1 - 1/2}{N} T, \dots, \frac{j_r - 1/2}{N} T; F \right) \\
& \quad \times \hat{\mu}^{\alpha_r} \left(\frac{j_r - 1/2}{N} T \right) \dots \hat{\mu}^{\alpha_1} \left(\frac{j_1 - 1/2}{N} T \right) \hat{\rho}_{D,0} \hat{\mu}^{\alpha'_1} \left(\frac{j'_1 - 1/2}{N} T \right) \dots \hat{\mu}^{\alpha'_s} \left(\frac{j'_s - 1/2}{N} T \right) \\
& + W^{\alpha, \alpha'} \left(\frac{j'_1 - 1/2}{N} T, \dots, \frac{j'_s - 1/2}{N} T; \frac{j_1 - 1/2}{N} T, \dots, \frac{j_r - 1/2}{N} T; F \right) \\
& \quad \times \hat{\mu}_{\alpha_r}^\dagger \left(\frac{j_r - 1/2}{N} T \right) \dots \hat{\mu}_{\alpha_1}^\dagger \left(\frac{j_1 - 1/2}{N} T \right) \hat{\rho}_{D,0} \hat{\mu}_{\alpha'_1}^\dagger \left(\frac{j'_1 - 1/2}{N} T \right) \dots \hat{\mu}_{\alpha'_s}^\dagger \left(\frac{j'_s - 1/2}{N} T \right) \Big]
\end{aligned}$$

where we have used the same notation as in Eq. (6.36), and

$$\mathcal{J}_r := \{\mathbf{j} = (j_1, \dots, j_r) : j_1 \in \{1, \dots, N\}, j_i \in \{1, \dots, j_{i-1}\}, \forall i = 2, \dots, r\}, \quad (6.50)$$

$$\mathcal{J}'_s := \{\mathbf{j}' = (j'_1, \dots, j'_s) : j'_1 \in \{1, \dots, N\}, j'_i \in \{1, \dots, j'_{i-1}\}, \forall i = 2, \dots, s\}. \quad (6.51)$$

Using the notation employed in Eq. (6.42), we can rewrite Eq. (6.49) as

$$\text{Tr}_\phi \left(\hat{U}^{(r)}(N) \hat{\rho}_0 \hat{U}^{(s)}(N)^\dagger \right) = (-i)^{r+s} \frac{\lambda^k T^k}{N^k} \sum_{(\mathbf{j}, \mathbf{j}') \in \mathcal{J}_r \times \mathcal{J}'_s} \Xi(\boldsymbol{\tau}_{\mathbf{j}}; \boldsymbol{\tau}'_{\mathbf{j}'}) \hat{\omega}(\boldsymbol{\tau}_{\mathbf{j}}; \boldsymbol{\tau}'_{\mathbf{j}'}; F), \quad (6.52)$$

where

$$\boldsymbol{\tau}_{\mathbf{j}} := \left(\frac{j_1 - 1/2}{N} T, \dots, \frac{j_r - 1/2}{N} T \right) \quad \text{and} \quad \boldsymbol{\tau}'_{\mathbf{j}'} := \left(\frac{j'_1 - 1/2}{N} T, \dots, \frac{j'_s - 1/2}{N} T \right). \quad (6.53)$$

To prove the approximation result stated in Sec. 6.3.1, we follow a generalization of the strategy used by Chui in [172, 173]. First, let us define $H_{ab}^{(r,s)} : \mathbb{R}^r \times \mathbb{R}^s \rightarrow \mathbb{C}$ as

$$H_{ab}^{(r,s)}(\boldsymbol{\tau}; \boldsymbol{\tau}') = (-i)^{r+s} \Xi(\boldsymbol{\tau}; \boldsymbol{\tau}') \omega_{ab}(\boldsymbol{\tau}; \boldsymbol{\tau}'; F) \mathbf{I}_{\Delta_r \times \Delta'_s}, \quad (6.54)$$

where the dependence on r and s for Ξ and $\hat{\omega}$ manifests itself in the vectors $\boldsymbol{\tau}$ and $\boldsymbol{\tau}'$ having r and s components, respectively. Here, ω_{ab} is the (a, b) -th matrix component of $\hat{\omega}$ in some basis of the detector Hilbert space $\mathcal{H}_D$, and $\mathbf{I}_X$ is the indicator function for the set X , i.e., a function that equals 1 when the argument belongs to X , and 0 otherwise. With this definition, from Eq. (6.42) we have that

$$\left[\text{Tr}_\phi \left(\hat{U}_\xi^{(r)} \hat{\rho}_0 \hat{U}_\xi^{(s)\dagger} \right) \right]_{ab} = \lambda^k \int_{[0, T]^k} d\mathbf{z} H_{ab}^{(r,s)}(\mathbf{z}), \quad (6.55)$$

where the subindices ab again denote the (a, b) -th matrix element of the trace in the chosen basis of $\mathcal{H}_D$. It is worth remarking that the presence of the indicator function in

the definition of $H_{ab}^{(r,s)}$ allowed us to have here the hypercube $[0, T]^k$ as integration domain, instead of the product of triangular domains of Eq. (6.42). Meanwhile, from Eq. (6.52) we also have that

$$\left[\text{Tr}_\phi \left(\hat{U}^{(r)}(N) \hat{\rho}_0 \hat{U}^{(s)}(N)^\dagger \right) \right]_{ab} = \lambda^k R_N^{(k)} [H_{ab}^{(r,s)}], \quad (6.56)$$

where $R_N^{(k)}$ is the Riemann sum associated with the uniform partition of $[0, T]^k$ into N^k identical cubes with tags placed in their centres, i.e., for an arbitrary $h : \mathbb{R}^k \rightarrow \mathbb{C}$,

$$R_N^{(k)}[h] := \sum_{\mathbf{j}} h\left(\frac{j_1 - 1/2}{N}T, \dots, \frac{j_k - 1/2}{N}T\right) \cdot \frac{T^k}{N^k}, \quad (6.57)$$

with each component of $\mathbf{j}$ running over $\{1, \dots, N\}$. Let us now define $v_N^{(k)} : \mathbb{R}^k \rightarrow \mathbb{C}$ as

$$v_N^{(k)}(\mathbf{z}) = \sum_{\mathbf{j}} T^k \prod_{i=1}^k \mathbf{I}_{\left[\frac{j_i - 1/2}{N}, 1\right]} \left(\frac{z_i}{T} \right) - N^k \prod_{i=1}^k z_i, \quad (6.58)$$

where again each component of $\mathbf{j}$ runs over $\{1, \dots, N\}$. With this definition, we can write

$$R_N^{(k)}[H_{ab}^{(r,s)}] - \int_{[0, T]^k} d\mathbf{z} H_{ab}^{(r,s)}(\mathbf{z}) = \frac{1}{N^k} \int_{[0, T]^k} dv_N^{(k)} H_{ab}^{(r,s)}, \quad (6.59)$$

where the right-hand side should be understood as a k -dimensional Riemann-Stieltjes integral of $H_{ab}^{(r,s)}$ over $[0, T]^k$ with respect to $v_N^{(k)}$, as defined in App. A. Now, $H_{ab}^{(r,s)}$ is a product of functions including Ξ (defined in Eq. (6.45)), which is zero whenever one of its arguments is 0 or T . In particular, $H_{ab}^{(r,s)}$ is zero along the boundary of $[0, T]^k$, so by Lemma 1 in App. A,

$$R_N^{(k)}[H_{ab}^{(r,s)}] - \int_{[0, T]^k} d\mathbf{z} H_{ab}^{(r,s)}(\mathbf{z}) = \frac{(-1)^k}{N^k} \int_{[0, T]^k} dH_{ab}^{(r,s)} v_N^{(k)}. \quad (6.60)$$

Since $v_N^{(k)}$ is bounded in $[0, T]^k$, with bound

$$\|v_N^{(k)}\|_\infty \leq \frac{kT^k}{2} N^{k-1}, \quad (6.61)$$

assuming $H_{ab}^{(r,s)}$ is of bounded variation,³ we can use Lemma 2 in App. A to conclude that

$$\left| R_N^{(k)}[H_{ab}^{(r,s)}] - \int_{[0, T]^k} d\mathbf{z} H_{ab}^{(r,s)}(\mathbf{z}) \right| \leq \frac{k \tilde{C}_{ab}^{(r,s)}}{2N} T^k \equiv \frac{C_{ab}^{(r,s)}}{N}, \quad (6.62)$$

³See Eq. (A.19) in App. A.

where $\tilde{C}_{ab}^{(r,s)}$ is the variation of $H_{ab}^{(r,s)}$ in $[0, T]^k$. If $H_{ab}^{(r,s)}$ is of bounded variation for all pairs (r, s) such that $r + s = k$, from Eqs. (6.35) and (6.48) we get that

$$|[\hat{\rho}_D^{(k)}(N)]_{ab} - [\hat{\rho}_D^{(k)\xi}]_{ab}| \leq \lambda^k \sum_{k=r+s} \frac{C_{ab}^{(r,s)}}{N} = \lambda^k \frac{C_{ab}^{(k)}}{N}, \quad (6.63)$$

for some $C_{ab}^{(k)}$, which proves Eq. (6.20). This also implies

$$\lim_{N \rightarrow \infty} [\hat{\rho}_D^{(k)}(N)]_{ab} = [\hat{\rho}_D^{(k)\xi}]_{ab}, \quad (6.64)$$

which proves Eq. (6.19), as well as the rate of convergence. In particular, if Eq. (6.62) holds at all orders, Eqs. (6.21) and (6.22) are also satisfied.

In order to arrive at the bound given in Eq. (6.62), we needed to assume that $H_{ab}^{(r,s)}$ is of bounded variation as a function of its (time) arguments. Let us now discuss how exotic this condition is.

First, notice that $H_{ab}^{(r,s)}$ depends on Ξ , which is a product of the values of the switching functions at different times, and the matrix elements ω_{ab} , which involve a linear combination of smeared k -point correlation functions of the field operator $\hat{O}^\alpha$ and the corresponding matrix element of an operator that consists of products of $\hat{\mu}$ with $\hat{\rho}_{D,0}$. If the dependence of $\hat{\mu}$ on time in the interaction picture only comes from the free evolution of the detector, as it is often the case, then we can assume that it is a smooth function of time. That way, whether $H_{ab}^{(r,s)}$ is sufficiently regular for Eq. (6.62) to hold or not ends up depending on the switching function (through Ξ) and the state of the field (through the smeared k -point correlation functions for the operator $\hat{O}$). By lemma 3 in App. A, one sufficient (but not necessary) condition for $H_{ab}^{(r,s)}$ to be of bounded variation is that it is of class $\mathcal{C}^k$ in $[0, T]^k$. In particular, this can be straightforwardly extended to the case in which $H_{ab}^{(r,s)}$ is bounded, and $\mathcal{C}^k$ except for maybe a finite number of points in $[0, T]^k$.

Now, the (continuous) switching functions typically employed in particle detector setups (e.g., Heaviside, Gaussian, and Lorentzian couplings) are smooth except for maybe a finite number of points. Therefore, when ξ is one of these common switching functions, it would suffice that the field's k -point correlation functions be of class $\mathcal{C}^k$ in $[0, T]^k$ for $H_{ab}^{(r,s)}$ to be bounded and of class $\mathcal{C}^k$ except for maybe a finite number of points, for all (r, s) such that $r + s = k$.

Let us consider, for instance, the vacuum of a massless real scalar quantum field in $(1 + 3)$ -dimensional Minkowski spacetime. This state is quasifree, so the odd correlation

functions vanish, while the even correlation functions can all be expressed as sums of products of the two-point function, which can be computed from the distribution

$$w_2(\mathbf{x}_1, \mathbf{x}_2) = \int \frac{d\mathbf{k}}{(2\pi)^3 2|\mathbf{k}|} e^{i\mathbf{k} \cdot (\mathbf{x}_1 - \mathbf{x}_2)}. \quad (6.65)$$

If we have a linear coupling with the field amplitude, as in the UDW model, then $\hat{O} = \hat{\phi}$, and the relevant k -point functions can be written in terms of the two-point function

$$W(t; t'; F) = \int d\mathbf{x} d\mathbf{x}' F(\mathbf{x}) F(\mathbf{x}') w_2(\mathbf{x}_1, \mathbf{x}_2) = \int \frac{d\mathbf{k}}{(2\pi)^n 2|\mathbf{k}|} |\tilde{F}(\mathbf{k})|^2 e^{-i|\mathbf{k}|(t-t')}, \quad (6.66)$$

for a given smearing function F . One very popular choice of smearing function in the literature is the Gaussian of variance σ^2 ,

$$G(\mathbf{x}) = \frac{1}{\sqrt{\pi^3} \sigma^3} e^{-\mathbf{x}^2/\sigma^2}. \quad (6.67)$$

With a Gaussian smearing, a straightforward calculation using spherical coordinates yields

$$W(t; t'; G) = \frac{1}{4\pi^2} \int_0^\infty d|\mathbf{k}| |\mathbf{k}| e^{-|\mathbf{k}|^2 \sigma^2/2 - i|\mathbf{k}|(t-t')} = \frac{1 - \sqrt{\frac{\pi}{2\sigma^2}} (t-t') e^{-(t-t')^2/2\sigma^2} [\mathbf{i} + \operatorname{erfi}(\frac{t-t'}{\sqrt{2}\sigma})]}{\sigma^2}, \quad (6.68)$$

which is a smooth function of t and t' . Another common choice is the hard sphere of radius $R > 0$,

$$S(\mathbf{x}) = \theta(R - |\mathbf{x}|). \quad (6.69)$$

Notice that its Fourier transform

$$\tilde{S}(\mathbf{k}) = \frac{4\pi}{k^3} [\sin(kR) - kR \cos(kR)] \quad (6.70)$$

converges to a constant when $k \rightarrow 0$, and behaves like $1/k^2$ as $k \rightarrow \infty$, so that

$$W(t; t'; S) = 4 \int_0^\infty d|\mathbf{k}| \frac{[\sin(|\mathbf{k}|R) - |\mathbf{k}|R \cos(|\mathbf{k}|R)]^2}{|\mathbf{k}|^5} e^{-i|\mathbf{k}|(t-t')} \quad (6.71)$$

is convergent. It can be shown, moreover, that it is of bounded variation, even though in this case

$$\frac{\partial^2}{\partial t \partial t'} W(t; t'; S) = 4 \int_0^\infty d|\mathbf{k}| \frac{[\sin(|\mathbf{k}|R) - |\mathbf{k}|R \cos(|\mathbf{k}|R)]^2}{|\mathbf{k}|^3} e^{-i|\mathbf{k}|(t-t')} \quad (6.72)$$

fails to converge and therefore we cannot use Lemma 3 to reach that conclusion. Thus, we see that for two of the most common smearing functions,⁴ the vacuum of a real scalar quantum field in $(1 + 3)$ -dimensional Minkowski satisfies the condition for Eq. (6.64) to hold, *at all orders* in perturbation theory. Since, for two or more spatial dimensions,⁵ the possible non-smoothness of the field correlation functions can only arise from the ultraviolet behaviour of the field state, we can argue that any Hadamard state (see, e.g., [174–176]), both in flat and curved spacetime (barring pathological geometries), will essentially have the same ultraviolet behaviour as the Minkowski vacuum, and therefore should in principle satisfy the assumptions of the approximation result as well.

Finally, we can turn to the multiple detector scenario. The proof proceeds in just the same way as for the single detector case. For the sake of brevity and simplicity we will not give its details here, but we argue that the procedure is completely analogous. Specifically, the similar structure of Eqs. (6.5) and (6.13) is revealing of how close both scenarios are from the technical point of view. The case of multiple detectors will in general involve products of different switching functions and different detector operators, as well as field correlation functions associated to potentially different field operators, smeared over different spatial profiles. However, one can give homologous definitions of Ξ (cf. Eq. (6.45)) and $\hat{\omega}$ (cf. Eq. (6.36)) that lead to the multiple detectors version of Eq. (6.42). The steps that follow are exactly those followed above for the one detector scenario, adapted to the new integration regions, which for the case of multiple detectors are general rectangles, instead of the hypercubes we dealt with in the case of a single detector.

6.5 Examples

After introducing the approximation method in Sec. 6.3 and proving it in Sec. 6.4, here we will use it in two specific scenarios, involving, respectively, one and two detectors. For each case, we will also analyze the rate of convergence of the approximate density matrices towards the exact ones, illustrating that the approximations are at least as efficient as claimed.

⁴For a more detailed discussion of when and how the smearing function regularizes the two-point distribution, see [73].

⁵Also for one spatial dimension, assuming that the infrared divergences are properly regularized.

6.5.1 Single detector example

In this example, we consider a two-level Unruh-DeWitt particle detector at rest in a $(1+3)$ -dimensional Minkowski spacetime, with ground and excited states $|g\rangle$ and $|e\rangle$, separated by an energy gap Ω , and a Gaussian shape,

$$F(\mathbf{x}) = \frac{1}{\sqrt{\pi^3}\sigma^3} e^{-\mathbf{x}^2/\sigma^2}. \quad (6.73)$$

The detector is coupled to a massless real scalar quantum field via an interaction Hamiltonian

$$\hat{H}_{\text{int}}(t) = \lambda \xi(t) \hat{\mu}(t) \int d\mathbf{x} F(\mathbf{x}) \hat{\phi}(t, \mathbf{x}), \quad (6.74)$$

where λ is the coupling strength as in Eq. (6.1), and $\hat{\mu}$ is the monopole moment of the detector in the interaction picture,

$$\hat{\mu}(t) = |g\rangle\langle e| e^{-i\Omega t} + |e\rangle\langle g| e^{i\Omega t}. \quad (6.75)$$

The field operator $\hat{\phi}$ can be expanded in plane-wave modes as

$$\hat{\phi}(t, \mathbf{x}) = \int \frac{d\mathbf{k}}{\sqrt{(2\pi)^3 2|\mathbf{k}|}} (\hat{a}_{\mathbf{k}} e^{-i(|\mathbf{k}|t - \mathbf{k}\cdot\mathbf{x})} + \text{H.c.}). \quad (6.76)$$

Finally, ξ is a switching function supported in $[0, T]$ that is regular, i.e., bounded and continuous except for maybe a finite number of points, as in Sec. 6.3.

We take the detector and the field to be initially uncorrelated and in their respective ground states,

$$\hat{\rho}_0 = \hat{\rho}_{\text{D},0} \otimes |0\rangle\langle 0|, \quad \hat{\rho}_{\text{D},0} = |g\rangle\langle g|. \quad (6.77)$$

Using perturbation theory, we can write the state of the detector at the end of the interaction process as

$$\hat{\rho}_{\text{D}} = \hat{\rho}_{\text{D},0} + \hat{\rho}_{\text{D}}^{(2)} + \mathcal{O}(\lambda^4), \quad (6.78)$$

where the terms of odd order in the perturbative expansion are zero because the vacuum $|0\rangle$ is a zero-mean Gaussian state, and therefore its odd-point functions are zero. Thus, the final state of the detector is diagonal in the $\{|g\rangle, |e\rangle\}$ basis, and can be completely characterized by the excitation probability,

$$P_e = \langle e | \hat{\rho}_{\text{D}} | e \rangle = \langle e | \hat{\rho}_{\text{D}}^{(2)} | e \rangle + \mathcal{O}(\lambda^4). \quad (6.79)$$

For a massless real scalar quantum field, up to second order in λ (see, e.g., [177]),

$$P_e = \lambda^2 \int \frac{d\mathbf{k}}{(2\pi)^3 2|\mathbf{k}|} |\tilde{\xi}(|\mathbf{k}| + \Omega)|^2 |\tilde{F}(\mathbf{k})|^2, \quad (6.80)$$

where $\tilde{\xi}$ and $\tilde{F}$ are the Fourier transforms⁶ of ξ and F , respectively. For the Gaussian shape specified before, which has spherical symmetry, we get

$$P_e = \frac{\lambda^2}{4\pi^2} \int_0^\infty d|\mathbf{k}| |\mathbf{k}| e^{-|\mathbf{k}|^2 \sigma^2/2} |\tilde{\xi}(|\mathbf{k}| + \Omega)|^2. \quad (6.82)$$

As a particular application of the result stated in Sec. 6.3, Eq. (6.82) can be approximated by the transition probability for a scenario where the detector couples to the field through the train of N sudden interactions given by

$$\chi_\xi(t; N) = \frac{T}{N} \sum_{j=1}^N \xi\left(\frac{j-1/2}{N}T\right) \delta\left(t - \frac{j-1/2}{N}T\right). \quad (6.83)$$

In this case, the excitation probability is given (up to second order in λ) by

$$P_e(N) = \frac{\lambda^2 T^2}{N^2} \sum_{j,j'=1}^N \xi\left(\frac{j-1/2}{N}T\right) \xi\left(\frac{j'-1/2}{N}T\right) \frac{e^{i\Omega T(j-j')/N}}{4\pi^2} \int_0^\infty d|\mathbf{k}| |\mathbf{k}| e^{-|\mathbf{k}|^2 \sigma^2/2} e^{i|\mathbf{k}|T(j-j')/N}. \quad (6.84)$$

In the following we will see, for specific switching functions ξ , how Eq. (6.84) approximates Eq. (6.82) increasingly well as N increases, and does so with the predicted efficiency.

Heaviside switching

The first example that we showcase is a switching function that is constant in its support,

$$\xi(t) = \mathbf{I}_{(0,T)}(t), \quad (6.85)$$

⁶The convention we followed here for the definition of the Fourier transform is that

$$\tilde{G}(\mathbf{v}) := \int d\mathbf{u} G(\mathbf{u}) e^{-i\mathbf{u} \cdot \mathbf{v}}. \quad (6.81)$$

where $I_{(0,T)}$ is the indicator function that equals 1 in the interval $(0, T)$, and zero everywhere else. In this case,

$$\tilde{\xi}(k) = \frac{2e^{-ikT/2}}{k} \sin\left(\frac{kT}{2}\right). \quad (6.86)$$

From Eq. (6.82), we have

$$P_e = \frac{\lambda^2}{\pi^2} \int_0^\infty d|\mathbf{k}| \frac{|\mathbf{k}| e^{-|\mathbf{k}|^2 \sigma^2/2}}{(|\mathbf{k}| + \Omega)^2} \sin^2\left[\frac{(|\mathbf{k}| + \Omega)T}{2}\right] = \frac{\lambda^2}{\pi^2} \int_0^\infty d\kappa \frac{\kappa e^{-\kappa^2 s^2/2}}{(\kappa + \gamma)^2} \sin^2\left(\frac{\kappa + \gamma}{2}\right), \quad (6.87)$$

where the second line explicitly shows that the result only depends on the dimensionless parameters $\gamma = \Omega T$, and $s = \sigma/T$, and we also denoted $\kappa = |\mathbf{k}|T$. Similarly, from Eq. (6.84) we get the excitation probability associated with the train of sudden interactions χ_ξ that approximates the Heaviside switching,

$$P_e(N) = \frac{\lambda^2}{N^2} \sum_{j,j'=1}^N \frac{e^{i\gamma(j-j')/N}}{4\pi^2} \int_0^\infty d\kappa \kappa e^{-\kappa^2 s^2/2} e^{i\kappa(j-j')/N}. \quad (6.88)$$

In Fig. 6.1a we see how $P_e(N)$ as given in Eq. (6.88) converges to the exact value P_e given in Eq. (6.87), for the case $\gamma = s = 1$. Moreover, Fig. 6.1b shows that the rate of convergence of the approximation is as fast as $1/N^2$, which is better than the minimum rate guaranteed by the approximation result of Sec. 6.3.

Truncated Gaussian switching

The second example that we consider is a Gaussian switching where the Gaussian tails are cut beyond q times its variance:

$$\xi(t) = \exp\left[-\left(\frac{t}{qT} - \frac{1}{2q}\right)^2\right] I_{(0,T)}(t). \quad (6.89)$$

Notice that the Gaussian reaches its maximum at $t = T/2$, i.e., the centre of the switching's compact support. Its Fourier transform reads

$$\tilde{\xi}(k) = \sqrt{\pi} q T e^{-ikT/2} e^{-k^2 q^2 T^2/4} \operatorname{Re}\left[\operatorname{erf}\left(\frac{1}{2q} + \frac{ikqT}{2}\right)\right]. \quad (6.90)$$

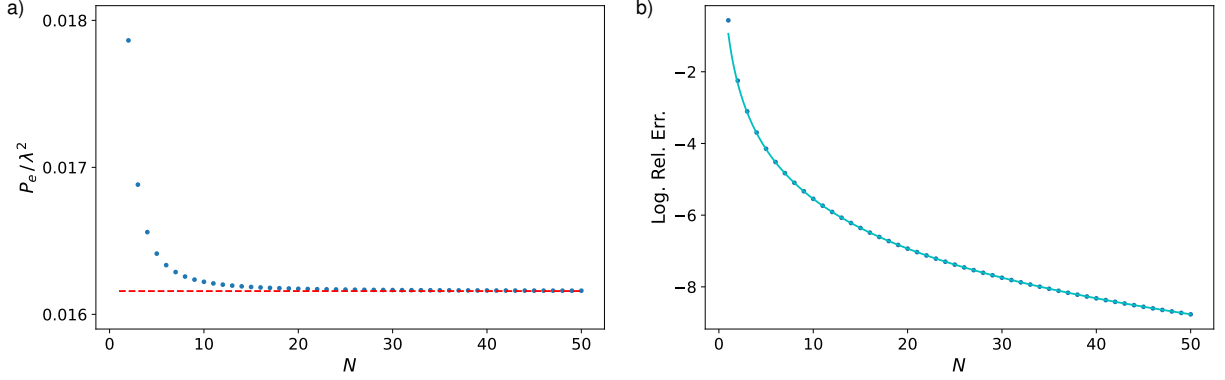


Figure 6.1: a) Excitation probability of a two-level UDW detector coupled to a quantum scalar field through a train of delta couplings approximating a Heaviside switching, as a function of the number of couplings N . Here we set the relevant parameters γ and s to 1. The constant red dashed line marks the value of the excitation probability for the exact Heaviside switching. b) Logarithmic relative error of the excitation probability for the Heaviside switching approximated by delta couplings, as a function of the number of couplings N . The function represented by the solid cyan line is proportional to $1/N^2$, marking the rate of convergence of the approximation.

From Eq. (6.82), we have

$$P_e = \frac{\lambda^2 q^2 T^2}{4\pi} \int_0^\infty d|\mathbf{k}| |\mathbf{k}| e^{-|\mathbf{k}|^2 \sigma^2/2} e^{-(|\mathbf{k}|+\Omega)^2 q^2 T^2/2} \text{Re} \left\{ \text{erf} \left[\frac{1}{2q} + \frac{i(|\mathbf{k}|+\Omega)qT}{2} \right] \right\}^2 \quad (6.91)$$

$$= \frac{\lambda^2 q^2}{4\pi} \int_0^\infty d\kappa \kappa e^{-\kappa^2 s^2/2} e^{-(\kappa+\gamma)^2 q^2/2} \text{Re} \left\{ \text{erf} \left[\frac{1}{2q} + \frac{i(\kappa+\gamma)q}{2} \right] \right\}^2, \quad (6.92)$$

where, as before, we have shown the explicit dependence on the dimensionless parameters $\gamma = \Omega T$, and $s = \sigma/T$, and the integration is performed with respect to the dimensionless variable $\kappa = |\mathbf{k}|T$. Again, from Eq. (6.84) we get the excitation probability associated with the train of delta couplings χ_ξ that yields a transition probability that approximates that of the truncated Gaussian switching,

$$P_e(N) = \frac{\lambda^2}{N^2} \sum_{j,j'=1}^N e^{-(2j-N-1)^2/4q^2 N^2} e^{-(2j'-N-1)^2/4q^2 N^2} \frac{e^{i\gamma(j-j')/N}}{4\pi^2} \int_0^\infty d\kappa \kappa e^{-\kappa^2 s^2/2} e^{i\kappa(j-j')/N}. \quad (6.93)$$

In Fig. 6.2a we see how $P_e(N)$ —as given in Eq. (6.93)—converges to the value of P_e given in Eq. (6.92), for the case $\gamma = s = q = 1$. Moreover, Fig. 6.2b shows that the rate of convergence of the approximation is once more as fast as $1/N^2$, which is again better than the minimum rate guaranteed by the approximation result of Sec. 6.3.

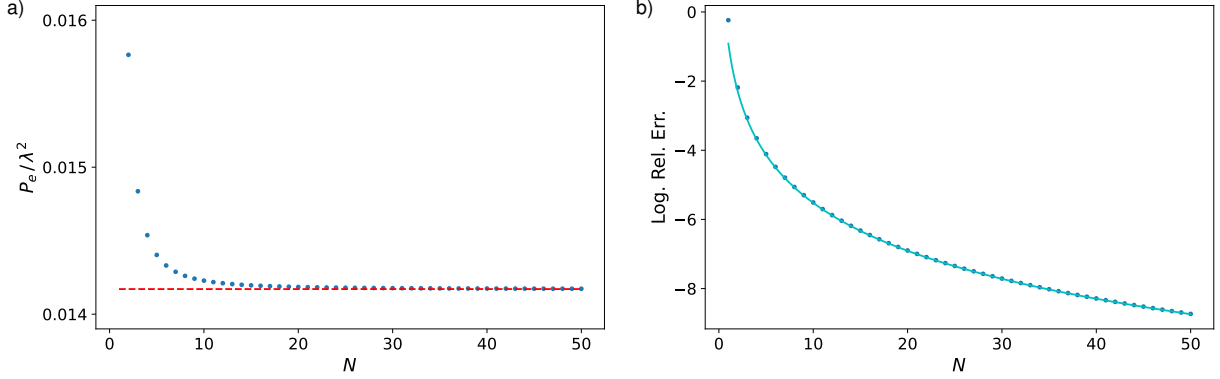


Figure 6.2: a) Excitation probability of a two-level UDW detector coupled to a quantum scalar field through a train of delta couplings approximating a Gaussian switching truncated at $q = 1$ variances, as a function of the number of couplings N . Here we set the relevant parameters γ and s to 1. The constant red dashed line marks the value of the excitation probability for the exact truncated Gaussian switching. b) Logarithmic relative error of the excitation probability for the Gaussian switching approximated by delta couplings, as a function of the number of couplings N . The function represented by the solid cyan line is proportional to $1/N^2$, marking the rate of convergence of the approximation.

Smooth bump switching

As a third example, let us consider the switching to be the C^∞ bump function given by

$$\xi(t) = \begin{cases} \exp \left[-\frac{T^2}{4t(T-t)} \right] & \text{if } 0 < t < T, \\ 0 & \text{otherwise,} \end{cases} \quad (6.94)$$

which is compactly supported in $[0, T]$, infinitely differentiable on $\mathbb{R}$, and reaches its maximum at $t = T/2$. Its Fourier transform does not admit a closed form. However, let us

define

$$\beta(t) = \begin{cases} \exp \left[-\frac{1}{4t(1-t)} \right] & \text{if } 0 < t < 1, \\ 0 & \text{otherwise,} \end{cases} \quad (6.95)$$

which does not depend on T . This function is related to ξ by $\xi(t) = \beta(t/T)$. Thus, their Fourier transforms satisfy

$$\tilde{\xi}(k) = T\tilde{\beta}(kT), \quad (6.96)$$

and hence, from Eq. (6.82), we can write

$$P_e = \frac{\lambda^2}{4\pi^2} \int_0^\infty d\kappa \kappa e^{-\kappa^2 s^2/2} |\tilde{\beta}(\kappa + \gamma)|^2, \quad (6.97)$$

which shows once again the complete dependence of P_e on the dimensionless parameters γ and s . Meanwhile, from Eq. (6.84) we get the excitation probability associated with the train of delta couplings χ_ξ used to approximate that of the bump switching,

$$\begin{aligned} P_e(N) &= \frac{\lambda^2}{N^2} \sum_{j,j'=1}^N \exp \left[-\frac{N^2}{4(j-1/2)(N-j+1/2)} - \frac{N^2}{4(j'-1/2)(N-j'+1/2)} \right] \\ &\quad \times \frac{e^{i\gamma(j-j')/N}}{4\pi^2} \int_0^\infty d\kappa \kappa e^{-\kappa^2 s^2/2} e^{i\kappa(j-j')/N}. \end{aligned} \quad (6.98)$$

In Fig. 6.3a we see how, for the case $\gamma = s = 1$, the approximated values $P_e(N)$ (as given in Eq. (6.98)) converge to the exact value of P_e (given in Eq. (6.97)). Moreover, Fig. 6.3b shows that the error incurred by the approximation is upper bounded by a function that decays faster than $1/N^5$, much faster than the minimum rate guaranteed by the result of Sec. 6.3. In fact, the increase of the rate of convergence of the approximation in this example with respect to the two previous ones is not surprising, since the approximation results in [173] suggest that switching functions of higher differentiability class should admit tighter convergence bounds.

6.5.2 Two detectors example

The setup considered for this example consists of two identical two-level Unruh-DeWitt detectors, A and B, both at rest at positions $\mathbf{x}_A$ and $\mathbf{x}_B$ in a $(1+3)$ -dimensional Minkowski

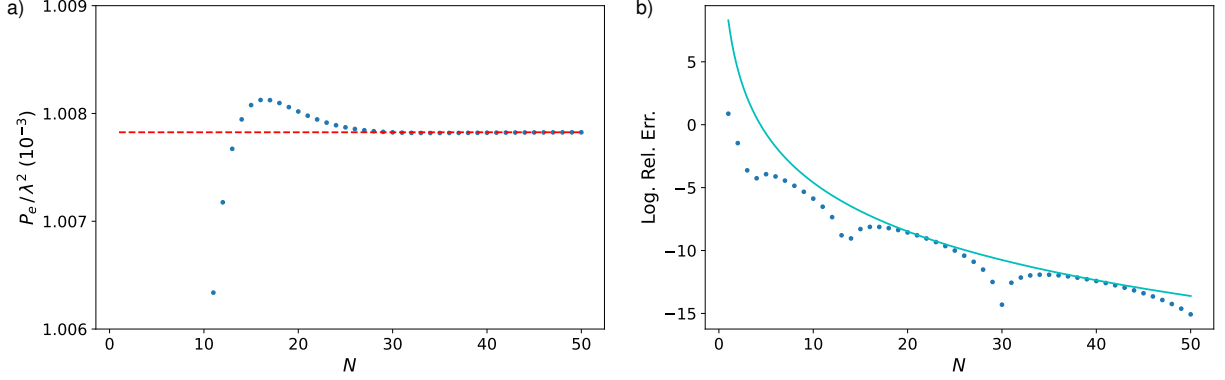


Figure 6.3: a) Excitation probability of a two-level UDW detector coupled to a quantum scalar field through a train of delta couplings approximating a bump switching, as a function of the number of couplings N . Here we set the relevant parameters γ and s to 1. The constant red dashed line marks the value of the excitation probability for the exact bump switching. b) Logarithmic relative error of the excitation probability for the bump switching approximated by delta couplings, as a function of the number of couplings N . The function represented by the solid cyan line is proportional to $1/N^{5.6}$, upper bounding the rate of convergence of the approximation.

spacetime. We consider the ground and excited states of both detectors to be separated by the same energy gap Ω , and their shapes to be (normalized) hard spheres of radius R ,

$$F_i(\mathbf{x}) = \frac{3}{4\pi R^3} \theta(R - |\mathbf{x} - \mathbf{x}_i|), \quad i = A, B. \quad (6.99)$$

As in the single detector example, here we consider a linear coupling between the detectors and a massless scalar quantum field, given in the interaction picture by the interaction Hamiltonian

$$\hat{H}_{\text{int}}(t) = \lambda \xi(t) \left[\hat{\mu}_A(t) \int d\mathbf{x} F_A(\mathbf{x}) \hat{\phi}(t, \mathbf{x}) + \hat{\mu}_B(t) \int d\mathbf{x} F_B(\mathbf{x}) \hat{\phi}(t, \mathbf{x}) \right], \quad (6.100)$$

where λ is the coupling strength as in Eq. (6.1), $\hat{\mu}_i$ is the monopole moment of detector i , which can be written in the interaction picture as

$$\hat{\mu}_i(t) = |g_i\rangle\langle e_i| e^{-i\Omega t} + |e_i\rangle\langle g_i| e^{i\Omega t}, \quad (6.101)$$

the field operator $\hat{\phi}$ can be expanded in plane-wave modes as in Eq. (6.76), and finally ξ is again a switching function supported in $[0, T]$ that is regular (bounded and continuous

except for maybe a finite number of points). Notice that, for simplicity, in this example we have chosen the coupling strengths and the switching functions to be the same for both detectors, although this is of course not necessary in the most general case.

Initially, we consider the detectors and the field to be uncorrelated and in their ground states,

$$\hat{\rho}_0 = |g_A\rangle\langle g_A| \otimes |g_B\rangle\langle g_B| \otimes |0\rangle\langle 0|. \quad (6.102)$$

Perturbation theory allows us to write the joint state of the two detectors after the interaction as

$$\hat{\rho}_{AB} = \hat{\rho}_{AB,0} + \hat{\rho}_{AB}^{(2)} + \mathcal{O}(\lambda^4), \quad (6.103)$$

where $\hat{\rho}_{AB,0} = |g_A\rangle\langle g_A| \otimes |g_B\rangle\langle g_B|$ and, as in the single detector case, the terms of odd order are zero because the odd-point functions of the vacuum state of the field are zero. In the ordered basis $\{|g_A g_B\rangle, |g_A e_B\rangle, |e_A g_B\rangle, |e_A e_B\rangle\}$, the second order contribution can be written in matrix form as (see, e.g., [4, 105])

$$\hat{\rho}_{AB}^{(2)} = \begin{pmatrix} -\mathcal{L}_{AA} - \mathcal{L}_{BB} & 0 & 0 & \mathcal{M}^* \\ 0 & \mathcal{L}_{BB} & \mathcal{L}_{AB}^* & 0 \\ 0 & \mathcal{L}_{AB} & \mathcal{L}_{AA} & 0 \\ \mathcal{M} & 0 & 0 & 0 \end{pmatrix}. \quad (6.104)$$

Now, let us restrict ourselves to the case in which the interaction of the two detectors happens in regions of spacetime that are spacelike separated, i.e., when

$$T < D - 2R, \quad (6.105)$$

where $D = |\mathbf{x}_A - \mathbf{x}_B|$ denotes the spatial distance between the two detectors. Then, we get the following expressions for the terms in Eq. (6.104):

$$\mathcal{L}_{AA} = \mathcal{L}_{BB} = \frac{\lambda^2}{4\pi^2} \int_0^\infty d|\mathbf{k}| |\mathbf{k}| |\tilde{\xi}(|\mathbf{k}| + \Omega)|^2 |\tilde{F}(|\mathbf{k}|)|^2, \quad (6.106)$$

$$\mathcal{L}_{AB} = \frac{\lambda^2}{4\pi^2 D} \int_0^\infty d|\mathbf{k}| \sin(|\mathbf{k}|D) |\tilde{\xi}(|\mathbf{k}| + \Omega)|^2 |\tilde{F}(|\mathbf{k}|)|^2, \quad (6.107)$$

$$\mathcal{M} = -\frac{\lambda^2}{4\pi^2 D} \int_0^\infty d|\mathbf{k}| \sin(|\mathbf{k}|D) \tilde{\xi}(|\mathbf{k}| - \Omega) \tilde{\xi}(|\mathbf{k}| + \Omega)^* |\tilde{F}(|\mathbf{k}|)|^2, \quad (6.108)$$

where $\tilde{\xi}$ and $\tilde{F}$ are the Fourier transforms of the switching function ξ and

$$F(\mathbf{x}) = \frac{3}{4\pi R^3} \theta(R - |\mathbf{x}|), \quad (6.109)$$

respectively. Notice that to arrive at Eqs. (6.106)–(6.108) we used that

$$F(\mathbf{x}) = F_A(\mathbf{x} + \mathbf{x}_A) = F_B(\mathbf{x} + \mathbf{x}_B), \quad (6.110)$$

as well as the spherical symmetry of F , which implies

$$\tilde{F}(\mathbf{k}) = e^{i\mathbf{k} \cdot \mathbf{x}_A} \tilde{F}_A(\mathbf{k}) = e^{i\mathbf{k} \cdot \mathbf{x}_B} \tilde{F}_B(\mathbf{k}) = \tilde{F}(|\mathbf{k}|), \quad (6.111)$$

where

$$\tilde{F}(|\mathbf{k}|) = \frac{3}{(|\mathbf{k}|R)^3} [\sin(|\mathbf{k}|R) - |\mathbf{k}|R \cos(|\mathbf{k}|R)]. \quad (6.112)$$

Finally, we choose the switching function for this example to be of the Heaviside type,

$$\xi(t) = \mathbf{I}_{(0,T)}(t), \quad (6.113)$$

with Fourier transform

$$\tilde{\xi}(k) = \frac{2e^{-ikT/2}}{k} \sin\left(\frac{kT}{2}\right). \quad (6.114)$$

Eqs. (6.106)–(6.108) can then be written in terms of dimensionless parameters as

$$\mathcal{L}_{AA} = \mathcal{L}_{BB} = \frac{\lambda^2}{\pi^2} \int_0^\infty d\kappa \frac{\sin^2[(\kappa + \gamma)/2]}{\kappa} |\tilde{F}(\kappa)|^2, \quad (6.115)$$

$$\mathcal{L}_{AB} = \frac{\lambda^2}{\pi^2 d} \int_0^\infty d\kappa \sin(\kappa d) \frac{\sin^2[(\kappa + \gamma)/2]}{\kappa^2} |\tilde{F}(\kappa)|^2, \quad (6.116)$$

$$\mathcal{M} = -\frac{\lambda^2 e^{i\gamma}}{2\pi^2 d} \int_0^\infty d\kappa \frac{\sin(\kappa d)(\cos \gamma - \cos \kappa)}{\kappa^2 - \gamma^2} |\tilde{F}(\kappa)|^2, \quad (6.117)$$

where $\gamma = \Omega T$, $d = D/T$, $\kappa = |\mathbf{k}|T$, and

$$\tilde{F}(\kappa) = \frac{3}{(\kappa r)^3} [\sin(\kappa r) - \kappa r \cos(\kappa r)], \quad (6.118)$$

with $r = R/T$.

As a particular application of the approximation result of Sec. 6.3, we know that Eqs. (6.115)–(6.117) can be approximated by their counterparts in the scenario where the detectors couple to the field through the train of sudden interactions given by

$$\chi_\xi(t; N) = \frac{T}{N} \sum_{j=1}^N \delta\left(t - \frac{j-1/2}{N}T\right). \quad (6.119)$$

The terms that approximate Eqs. (6.115)–(6.117) can then be written as

$$\mathcal{L}_{\text{AA}}(N) = \mathcal{L}_{\text{BB}}(N) = \frac{\lambda^2}{N^2} \sum_{j,j'=1}^N \frac{1}{4\pi^2} \int_0^\infty d\kappa \kappa |\tilde{F}(\kappa)|^2 \cos \left[\frac{(\kappa + \gamma)(j - j')}{N} \right], \quad (6.120)$$

$$\mathcal{L}_{\text{AB}}(N) = \frac{\lambda^2}{N^2} \sum_{j,j'=1}^N \frac{1}{4\pi^2 d} \int_0^\infty d\kappa \sin(\kappa d) |\tilde{F}(\kappa)|^2 \cos \left[\frac{(\kappa + \gamma)(j - j')}{N} \right], \quad (6.121)$$

$$\mathcal{M}(N) = -\frac{\lambda^2}{N^2} \sum_{j,j'=1}^N \frac{e^{i\gamma(j+j')/T}}{4\pi^2 d} \int_0^\infty d\kappa \sin(\kappa d) |\tilde{F}(\kappa)|^2 \cos \left[\frac{\kappa(j - j')}{N} \right]. \quad (6.122)$$

In Figs. 6.4, 6.5, and 6.6, we see how, for the case $\gamma = 1$, $d = 1.2$, and $r = 0.1$, the approximated values given in Eqs. (6.120)–(6.122) converge to the exact values given in Eqs. (6.115)–(6.117). In particular, Figs. 6.4b, 6.5b, and 6.6d show that the errors committed by the approximations are upper bounded by functions decaying at least as fast as $1/N$, in agreement with the approximation result of Sec. 6.3.

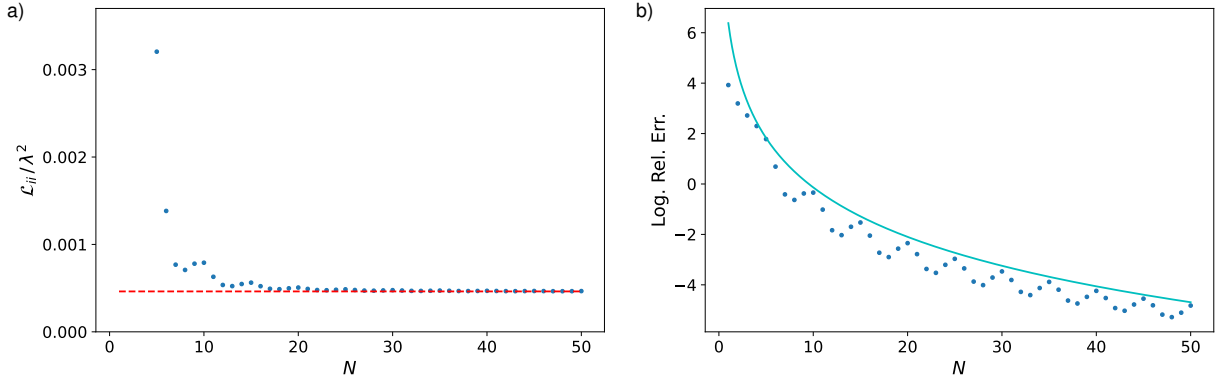


Figure 6.4: a) $\mathcal{L}_{\text{AA}} = \mathcal{L}_{\text{BB}} \equiv \mathcal{L}_{ii}$ term for a pair of identical two-level UDW detectors coupled to a quantum scalar field in spacelike-separated regions through a train of delta couplings approximating a Heaviside switching, as a function of the number of couplings N . Here we set the relevant parameters to $\gamma = 1$, $d = 1.2$, and $r = 0.1$. The constant red dashed line marks the value of the $\mathcal{L}_{ii}$ term for the exact Heaviside switching. b) Logarithmic relative error of the $\mathcal{L}_{ii}$ term for the Heaviside switching approximated by delta couplings, as a function of the number of couplings N . The function represented by the solid cyan line is proportional to $1/N^{2.8}$, upper bounding the rate of convergence of the approximation.

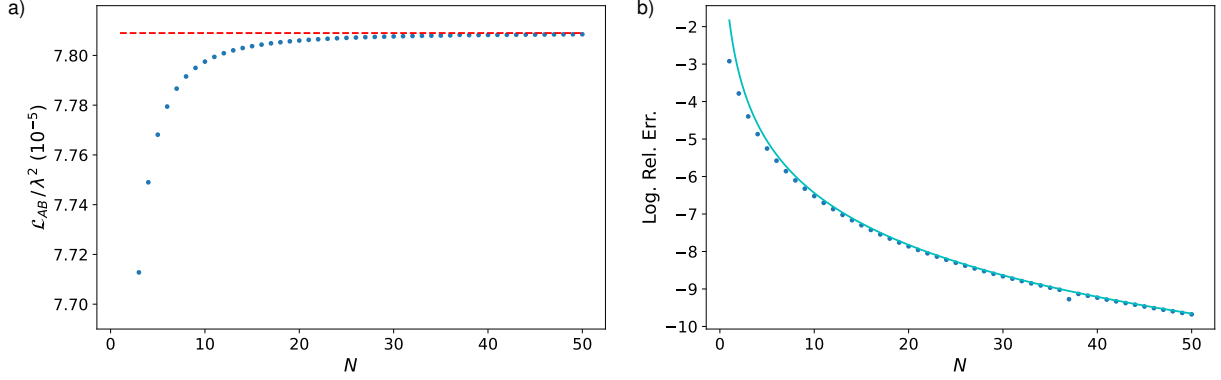


Figure 6.5: a) $\mathcal{L}_{AB}$ term for a pair of identical two-level UDW detectors coupled to a quantum scalar field in spacelike-separated regions through a train of delta couplings approximating a Heaviside switching, as a function of the number of couplings N . Here we set the relevant parameters to $\gamma = 1$, $d = 1.2$, and $r = 0.1$. The constant red dashed line marks the value of the $\mathcal{L}_{AB}$ term for the exact Heaviside switching. b) Logarithmic relative error of the $\mathcal{L}_{AB}$ term for the Heaviside switching approximated by delta couplings, as a function of the number of couplings N . The function represented by the solid cyan line is proportional to $1/N^2$, marking the rate of convergence of the approximation.

6.6 Summary

In this chapter, we have shown how to efficiently approximate non-perturbatively the time evolution of particle detectors interacting with a quantum field for very general compactly-supported switching functions and arbitrary smearing functions.

Specifically, we have described how to approximate the detectors' dynamics when they couple through compactly-supported bounded switching functions, continuous except for maybe a finite number of points, with that of a detector coupled through a sequence of delta couplings, which can be evaluated non-perturbatively. The approximation converges at least as fast as $1/N$ (where N is the number of delta pulses), and often much faster than this for regular enough switchings.

The fast convergence of the approximation is guaranteed at all orders in perturbation theory as long as the detectors' switching and smearing functions and the field's state are 'regular enough'. This, in particular, includes the cases in which the state of the field is Hadamard, and the switching and smearing functions are smooth except for maybe a

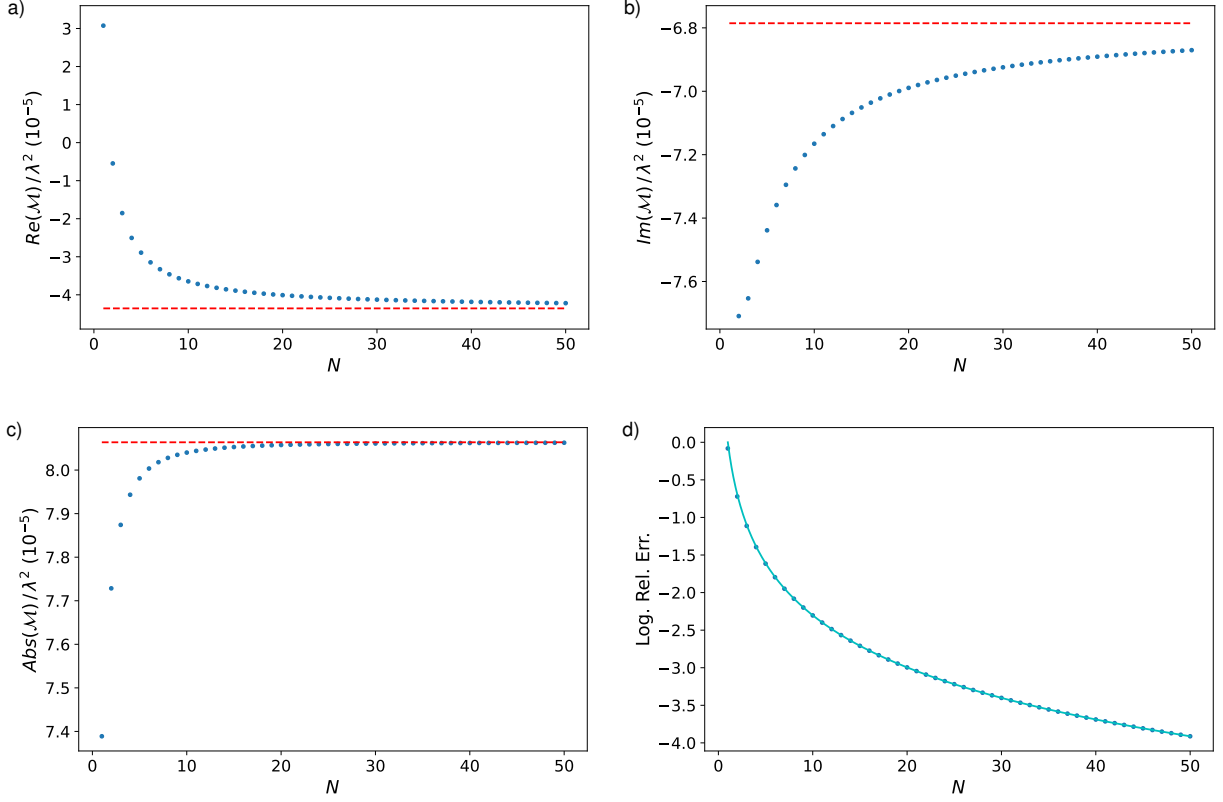


Figure 6.6: a), b), and c) Respectively, real part, imaginary part, and absolute value of the $\mathcal{M}$ term for a pair of identical two-level UDW detectors coupled to a quantum scalar field in spacelike-separated regions through a train of delta couplings approximating a Heaviside switching, as a function of the number of couplings N . Here we set the relevant parameters to $\gamma = 1$, $d = 1.2$, and $r = 0.1$. The constant red dashed lines mark the values of the real part, imaginary part, and absolute value of the $\mathcal{M}$ term for the exact Heaviside switching. d) Logarithmic relative error of the $\mathcal{M}$ term for the Heaviside switching approximated by delta couplings, as a function of the number of couplings N . The function represented by the solid cyan line is proportional to $1/N$, marking the rate of convergence of the approximation.

finite number of points, which covers most of the relevant scenarios in both flat and curved spacetimes.

Since this approximation scheme can be easily (and efficiently) evaluated in current

computers, we expect the method presented here to pave the way for a number of non-perturbative analyses of phenomena involving particle detectors and measurements in quantum field theory.

Act 2: Measuring quantum fields with particle detectors

Chapter 7

Extracting information from quantum fields with machine learning

In this chapter, we demonstrate how one can use machine learning techniques to bypass the technical difficulties of designing an experiment and translating its outcomes into concrete claims about fundamental features of quantum fields. In practice, all measurements of quantum fields are carried out through local probes. Despite measuring only a small portion of the field, such local measurements have the capacity to reveal many of the field's global features, as long as we assume that the field is in equilibrium with its background. We show that neural networks can be trained to unscramble this information from data generated from a very simple one-size-fits-all local measurement protocol. To illustrate this general claim we will consider three non-trivial features of the field as case studies: a) how, as long as the field is in a stationary state, a particle detector can learn about the field's boundary conditions even before signals have time to propagate from the boundary to the detector, b) how detectors can determine the temperature of the quantum field even without thermalizing with it, and c) how detectors can distinguish between Fock states and coherent states even when the first and second moments of all their quadrature operators match. Each of these examples uses the exact same simple fixed local measurement protocol and machine-learning ansatz successfully. This supports the claim that the framework proposed here can be applied to nearly any kind of local measurement on a quantum field to reveal nearly any of the field's properties in a one-size-fits-all manner.

7.1 From probing to measuring

In Ch. 5, we saw that Unruh-DeWitt detector models can serve as fundamental tools in a measurement theory for quantum fields, with the crucial advantage that they can effectively reproduce actual experiments conducted in laboratories and experimental facilities. From high-energy physics experiments at the LHC to the capture of light in the human retina, quantum fields are subject to measurements where data is extracted through interactions with localized probes that can be modelled as particle detectors. While we have seen that measurements performed by particle detectors are not causal in general, their deviations from causality are controlled by the scales of the model and their relation to the experimental setup. Furthermore, in Ch. 6, we introduced a method that allows us to approximate the readings of particle detectors non-perturbatively, opening the possibility of using them beyond the perturbative regime in which they are typically employed.

However, all of this relates to *probing* quantum fields, while we are interested in *measuring* them. The latter requires extracting information about observables of the field, or perhaps properties of the spacetime where it lives. This distinction is more relevant than it might seem at first, as it is not always obvious how to translate the outcomes of measurements performed on local probes into claims about specific features of a quantum field or its background. In Ch. 5, we obtained the POVM of the field induced by the measurement performed on the detector, but it is *a priori* difficult to determine what this POVM represents and what each outcome reveals about the field.

Moreover, we are often interested in global features of the QFT, such as the topology or the boundary conditions of its background. How are we supposed to measure them with particle detectors, which are *localized probes*? Luckily for us, a quantum field *in equilibrium with its environment*¹ stores information about its global structure locally, albeit in a very scrambled way [178–183]. The question is then how sophisticated must our local measurement protocol (and the following data analysis) be in order to determine something of interest about the field from local measurement data. It is thinkable that if one is allowed sufficient measurements on a sufficiently large array of probes, one should be able to resolve any feature of interest of a quantum field, at least in principle. We say ‘in principle’ because—except for very few simple cases—there is usually no direct way of translating the theoretical predictions of particle detectors into specific values for the targeted features or parameters of the field. That makes it even more challenging to translate the readouts of local detectors into concrete claims about the field that they measure. Typically, the best

¹Notice how this is itself a *global* assumption, which crucially may allow *local* information to be extrapolated to infer global features.

one can do when trying to distinguish different field configurations is to look at differences in the detectors' transition probabilities (see, e.g., [65, 184–188]).

Moreover, the probes we use to measure quantum fields are usually simple in nature, and certainly much simpler and with smaller Hilbert spaces than the QFT itself. Additionally, in most cases we will not have access to a large array of probes with which to sample the field, but rather only a handful. Because of this, translating measurement data (e.g., a set of zeros and ones generated by measuring a two-level particle detector) into concrete claims about the field seems, a priori, a very complicated task.

In general, one may not expect a context-free solution to this question. One might expect that for each feature of the QFT we may be interested in, we will need to define 1) a different local measurement protocol and 2) a different data analysis procedure. This assumption, i.e., that in general one needs different experiments and data processing techniques to measure different things, is arguably accepted without question in experimental physics. Our goal here is to show that this is not the case. Concretely, we will show that fixing a simple local measurement protocol and a basic machine-learning data analysis ansatz is enough to unravel a wide variety of non-local features of the QFT with a one-size-fits-all method.

Recently, machine learning has proven effective at processing data from quantum systems (see, e.g., [189–200]) and QFTs in the lattice [201–207]. We will show, beyond lattice theories, how the task of unscrambling non-local information from measurements that are local in space and time is one that machine learning is well suited for.

In the following section, we aim to develop a universal framework to extract information about a QFT through local measurements, which we will later apply to three examples. Our goal is to show that combining machine learning techniques with detector models allows us to avoid the complexity of designing specific local measurement protocols and translating their outcomes into concrete claims about quantum fields. Rather, one can use a simple all-purpose measurement protocol. This offloads all the complexity to the data processing, which is handled via machine learning. The success of this framework in the three substantially different case studies presented here should also be strong evidence of its generality, suggesting that it can be applied regardless of the specific targeted feature of the QFT, hence in a universal, context-free manner.

7.2 Measurement protocol and data processing

In this section, we will introduce the general measurement protocol and data processing framework that constitute the core of the proposal.

Imagine that we are interested in some global property of the field; for instance, we might want to know whether this field lives in a spacetime with an open topology versus a closed one. To determine this, we might carefully design a measurement protocol, $M(\text{TOP})$, where we let a probe (or an array of them) interact with the field in a localized manner, and then perform measurements on the probe (or array of probes) to extract some data $D \in \mathbb{R}^{N_m}$ where N_m is the number of measurements performed on the probe. A very simple example would be if $M(\text{TOP})$ specifies that we are to couple one single probe to the field for 1 second and then measure $\hat{\mu}$, repeating this process N_m times.

In addition to specifying the data collection process, we must also design a data-analysis function, i.e., $f_{\text{TOP}, M(\text{TOP})} : \mathbb{R}^{N_m} \rightarrow \{\text{open}, \text{closed}\}$, which will (hopefully) determine with high accuracy the topology of spacetime from the data. Here, $f_{x, M}$ is a data analysis function which produces claims about X from data generated by the measurement procedure M . Simply put, given a feature of interest X , $M(x)$ is the experimental protocol designed by the experimentalist to inform us about X specifically, and $f_{x, M(x)}$ is the “dictionary” that translates the experimental data collected during the measurement into explicit statements about X , with some degree of accuracy.

The problem now is how we might go about designing a good $M(\text{TOP})$ and $f_{\text{TOP}, M(\text{TOP})}$. In designing $M(\text{TOP})$ one might try to find measurements that are particularly well suited for the identification of the topology. In particular, one might wish to produce data that is, somehow, explicitly revealing of the field’s topology. In other words, if the experimenter does a good job designing $M(\text{TOP})$, then $f_{\text{TOP}, M(\text{TOP})}$ could be relatively trivial, i.e., the data would require little analysis. Unfortunately, this approach is not generally available to us. QFT forces us to use local measurements, and as a consequence some non-trivial data analysis is required to piece these local probe measurements together into a global picture that tells us about topology [179].

However, even if we could find a suitable $M(\text{TOP})$ and $f_{\text{TOP}, M(\text{TOP})}$ to determine the topology of a spacetime through the measurement of a quantum field, their utility would be limited to this particular feature. If we are interested in a different feature, F (the charge of the field, its mass, entanglement structure, the spacetime geometry, etc.), we would likely need to design a very different measurement protocol, $M(F)$ —which is well-suited to F —and a new data analysis function, $f_{F, M(F)}$. While we may not have to design $M(F)$ and $f_{F, M(F)}$ from scratch—many intuitions and previous knowledge may be applicable—this

redesign is likely to be non-trivial. This seemingly uncontroversial statement—that you need different experiments and data processing techniques to measure different things—is one of the central (often unquestioned) tenets of experimental design in physics.

In this light, one may question whether, when we shift our interest to feature F , we can “be lazy” and keep our old measurement protocol, $M(\text{TOP})$. This would of course require a new data-analysis function. If we were able to achieve this, we would be completely transferring the burden from designing a measurement protocol to designing a data-analysis function. In particular, we would now have to construct a function, $f_{F,M(\text{TOP})}$, which extracts information about F from data produced from $M(\text{TOP})$. There are good reasons to think this strategy will not be successful: First, we may have the intuition that since $M(\text{TOP})$ was not designed with feature F in mind, the data collected from $M(\text{TOP})$ might not even have enough information to say anything about F . Moreover, even if $M(\text{TOP})$ somehow does extract a sufficient amount of information to be able to discern a wide variety of features from it, then it seems likely that it will be an excessively complicated measurement procedure, enough to make it impractical.

However, it turns out that both of these concerns can be overcome: in Subsec. 7.2.1, we will present a simple fixed local measurement protocol, M_0 , designed without any specific feature of the QFT in mind. In Subsec. 7.2.2, we will describe how the data produced by M_0 can be compressed to improve the efficiency of its analysis. Finally, in Subsec. 7.2.3 we will outline how the data-analysis functions f_{F,M_0} which produce these conclusions can be easily generated from a basic machine-learning ansatz using standard supervised learning techniques—indeed, extracting non-trivial features from complex data sets is exactly the kind of task that machine learning is well-suited for. Further details of each step of the process can be found in App. B.1.

7.2.1 Simple fixed measurement procedure

In this subsection, we propose a simple measurement protocol (denoted M_0 above) to generate labelled data from a locally coupled probe interacting with a quantum field. This data can then be processed to extract information about various features of QFT. Since we will use it later in explicit examples, we adopt SI units instead of natural units for this subsection and the remainder of the chapter.

To make things concrete, let us consider a real scalar quantum field coupled to an Unruh-DeWitt detector whose proper time coincides with the coordinate time of some coordinate frame $(t, \mathbf{x})$. The coupling can thus be described in the interaction picture by

the Hamiltonian (cf. Eq. (2.71))

$$\hat{H}_{\text{int}}(t) = \lambda \chi(t) \int_{\mathbb{R}^n} d\mathbf{x} \sqrt{-g} F(\mathbf{x}) \hat{\mu}(t) \hat{\phi}(t, \mathbf{x}). \quad (7.1)$$

For illustration purposes, we model the probe as a harmonic oscillator with free Hamiltonian $\hat{H}_{\text{free}} = \hbar\omega_{\text{D}}(\hat{p}_{\text{D}}^2 + \hat{q}_{\text{D}}^2)/2$, where $\hat{q}_{\text{D}}$ and $\hat{p}_{\text{D}}$ are the probe's dimensionless quadrature operators satisfying $[\hat{q}_{\text{D}}, \hat{p}_{\text{D}}] = i\mathbb{1}$. We take the probe to couple to the field via $\hat{\mu} = \hat{q}_{\text{D}}$. Our measurement procedure is then as follows:

1. Initialize the field according to some conditions labelled y . For instance, for a quantum field in an optical cavity, if y labels boundary conditions, this would mean preparing a cavity with those boundary conditions and letting the field equilibrate with the cavity walls. If y labels temperatures, this would mean preparing a cavity at that temperature and letting the field thermalize with the cavity.
2. Initialize the probe to its ground state. Couple the probe locally to the field at time $t = 0$ (i.e., the switching function $\chi(t)$ is zero before $t = 0$).
3. At time $t_{\text{M}} = T_{\text{MIN}} > 0$, perform a projective measurement on the probe's $\hat{q}_{\text{D}}$ quadrature and record the result.
4. Repeat steps 1 to 3 but measuring the probe's $\hat{p}_{\text{D}}$ quadrature. Then repeat steps 1 to 3 but measuring the probe's $\hat{r}_{\text{D}} = (\hat{q}_{\text{D}} + \hat{p}_{\text{D}})/\sqrt{2}$ quadrature.
5. Repeat steps 1 to 4 a total of $N_{\text{TIMES}} - 1$ more times increasing t_{M} by an amount Δt each time.
6. Repeat this whole process N_{TOM} times.

This (simple and untargeted) measurement procedure yields raw data $D_{\text{RAW}} \in \mathbb{R}^{N_m}$, where $N_m = 3 \times N_{\text{TIMES}} \times N_{\text{TOM}}$. The goal of the N_{TOM} repetitions is to increase the precision with which we can calculate the averages $\langle \hat{q}_{\text{D}} \rangle$, $\langle \hat{p}_{\text{D}} \rangle$, and $\langle \hat{r}_{\text{D}} \rangle$. That is, the higher N_{TOM} , the more accurate the state tomography we can perform on the detector. This data comes along with an associated label, y . We collect N_{SAMPLES} of these pairs, (D_{RAW}, y) , where each of these N_{SAMPLES} data points D_{RAW} are associated with generally different labels y .

It is worth remarking that some features of the QFT might be more effectively captured by using more than one detector, so that we can use not only the outcomes of their individual measurements but also the correlations between them. In this sense, note that

the above measurement scheme can be straightforwardly generalized to use *arrays* of local probes.

As a final remark, notice that it is not necessary that the training (and validation) data come from actual experiments as described above. Indeed, it will often be convenient to train the neural network using simulated data from the available theoretical models to prepare the network to identify features in actual experimental data.

7.2.2 Data preprocessing

While we could train our neural network directly on our N_{SAMPLES} labelled data points, (D_{RAW}, y) , in order to improve and speed up the training we may first compress and preprocess the data.

Note that each of the N_{TOM} measurements of $\hat{q}_{\text{D}}(t_{\text{M}})$ (where $t_{\text{M}} = T_{\text{MIN}} + \text{M} \Delta t$, with $\text{M} \in \{0, \dots, N_{\text{TIMES}} - 1\}$) is independent and identically distributed; they each come from identical independent experiments. We can summarize the N_{TOM} measurement outcomes, q_k , using their sample mean, sample variance, and fourth central moment,

$$\bar{q}(t_{\text{M}}) = \frac{1}{N_{\text{TOM}}} \sum_{k=1}^{N_{\text{TOM}}} q_k, \quad (7.2)$$

$$\bar{s}_q(t_{\text{M}}) = \frac{1}{N_{\text{TOM}}} \sum_{k=1}^{N_{\text{TOM}}} (q_k - \bar{q}(t_{\text{M}}))^2, \quad (7.3)$$

$$\bar{s}_{4,q}(t_{\text{M}}) = \frac{1}{N_{\text{TOM}}} \sum_{k=1}^{N_{\text{TOM}}} (q_k - \bar{q}(t_{\text{M}}))^4. \quad (7.4)$$

Our measurements of $\hat{r}_{\text{D}}$ and $\hat{p}_{\text{D}}$ at each time $t_{\text{M}} = T_{\text{MIN}} + \text{M} \Delta t$ can be compressed in exactly the same way. This suffices for the three physical examples addressed at the end of the chapter. In general, depending on our needs, additional moments may be required, though most physically relevant quantities are expected to depend only on finitely many moments. For instance, since quasifree states are Gaussian, experiments involving them may, in principle, retain only the sample means and variances during preprocessing.

Once compressed, the data is described by the time series

$$\left\{ \bar{q}(t_{\text{M}}), \bar{r}(t_{\text{M}}), \bar{p}(t_{\text{M}}), \bar{s}_q(t_{\text{M}}), \bar{s}_r(t_{\text{M}}), \bar{s}_p(t_{\text{M}}), \bar{s}_{4,q}(t_{\text{M}}), \bar{s}_{4,r}(t_{\text{M}}), \bar{s}_{4,p}(t_{\text{M}}) \right\}_{\text{M}=0}^{N_{\text{TIMES}}-1}. \quad (7.5)$$

That is, the compressed data can be represented by a vector $D_c \in \mathbb{R}^d$ where $d = 9 N_{\text{TIMES}}$. Notice that even after compression the data will generally be high-dimensional since N_{TIMES} will be very large.

After compression, we perform standard preprocessing [208]: we center the data, do principal component analysis, and whiten the data. The details of our preprocessing are discussed in App. B.1. Let us call the preprocessed data, D_p . Next we will discuss how our neural network is trained and validated on the N_{SAMPLES} labelled data points (D_p, y) .

7.2.3 Neural network training

In this subsection we lay out how neural networks provide a data-analysis ansatz that we can use to process our data (i.e., what we denoted f_{F, M_0} at the beginning of the section). In particular, we will discuss how we can analyze the data produced by our simple fixed local measurement protocol, M_0 , to arrive at accurate conclusions about a wide variety of (non-local) features of the QFT.²

Neural networks model complicated high-dimensional functions by alternatively applying tunable linear-affine transformations (controlled by weights and biases) and fixed non-linear transformations (i.e., activation functions) to their inputs. Fig. 7.1 illustrates a simple neural network architecture one might use for classifying features of interest (in our example, the topology of the spacetime) based on local probe measurement data on a quantum field. The circles in Fig. 7.1 represent the fixed activation functions and the lines represent the tunable weights and biases. The neural network takes as input the local probe measurement data and outputs a probability assignment for each possible value of the feature of interest (whether the spacetime topology is open or closed).

In order to find the proper settings for the network's weights and biases (in the example in Fig. 7.1, those settings which accurately predict the spacetime topology) we follow a supervised training procedure. Supervised training requires labelled data, i.e., many data points, D_p , each paired with a label, y , indicating the result that our data analysis should produce. In the topology example this is $y \in \{\text{open}, \text{closed}\}$. These labels may also be continuous, e.g., if we were interested in determining the field's mass, m , we would have $y = m \in \mathbb{R}_+$.

Given N_{SAMPLES} labelled data points we divide these into data for training the neural network ($N_{\text{TRAIN}} = 0.75 N_{\text{SAMPLES}}$) and for validating its accuracy ($N_{\text{VALID}} = 0.25 N_{\text{SAMPLES}}$).

²This is under the assumption, of course, that the field is in equilibrium. No faster-than-light signalling is involved in this!

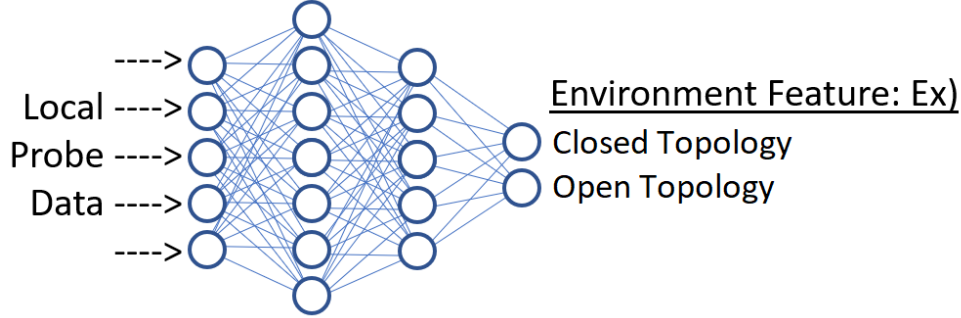


Figure 7.1: A schematic example of a neural network for processing local probe data to learn about a global feature of a QFT.

We then define a cost function to characterize how wrong our network’s predictions are over the training data. The network’s weights and biases are adjusted to minimize this cost function. Once the training is complete, the network’s accuracy is evaluated on the validation data.

The network architecture (fully connected feedforward) and training procedure (stochastic gradient descent) that we use are standard [208]. A summary of our network architecture, data preprocessing, and training process is given in App. B.1.

7.3 Two methods for simulating data generation

In the previous section, we discussed how the probe will interact with the field, how it will be measured, how the resulting data will be processed, how this data will be used to train a neural network, and how the accuracy of this network will be validated. In Sec. 7.4, we will apply this process to three examples: remote boundary sensing, thermometry, and quantum state discrimination, respectively. To be able to do this, first we need to discuss how exactly we produce the labelled data in a general QFT framework. This is what we do in this section.

Since we are in a theoretical setting, rather than having a real-life probe interact with a real-life field to generate our training data, we instead simulate both the probe-field interaction and the probe’s subsequent measurement. Notice however that training and validating with simulated data is not necessary: If one were to work with actual gathered experimental data the exact same analysis would apply. The methods discussed in this

chapter are independent of how the data is generated, be it by experiment or by simulation. Moreover, our methods are independent of exactly how we simulate the data generation process. To demonstrate this we will simulate the probe's response in two different ways: one involving a lattice approximation and one involving Dirac-delta interactions between the probe and the field in a continuous optical cavity. For concreteness (and simplicity), we will consider the case in which the background is a $(1+1)$ -dimensional Minkowski spacetime. The field's free Hamiltonian is then given by

$$\hat{H}_\phi = \frac{1}{2} \int_{\mathbb{R}} dx \left[c^2 \hat{\pi}(t, x)^2 + (\partial_x \hat{\phi}(t, x))^2 + \frac{m^2 c^2}{\hbar^2} \hat{\phi}(t, x)^2 \right], \quad (7.6)$$

where we recall that $[\hat{\phi}(t, x), \hat{\pi}(t, y)] = i\hbar \delta(x - y)\mathbb{1}$.

7.3.1 Method 1: Lattice approximation

For two of the three examples that we will analyze (boundary sensing and thermometry) we simulated the probe's response to the field using a lattice approximation. This approximation is not necessary but it does greatly simplify the analysis. It can be motivated as follows.

From Eq. (7.1), if the region where $F(x)$ is non-negligible has a length scale σ , the probe will not couple to field modes with wavenumber $|k| \gg \sigma^{-1}$ [124]. Taking the probe to have a Gaussian smearing with standard deviation σ , the coupling to the high-frequency field modes is exponentially suppressed. This motivates a UV cutoff [209, 210] at $|k| \leq K := 16/\sigma$ (see App. C for technical details).

Cutting these high-frequency modes out of both the interaction Hamiltonian (Eq. (7.1)) and the field's free Hamiltonian (Eq. (7.6)) strongly simplifies the situation without substantially affecting the probe's response to the field. Typically, UV cutoffs are at odds with locality assumptions, as they break the causality of the theory [84]. However, for our choice of scales, the UV cutoff only introduces a relative error in the probe's response below a fraction of a percent (see again App. C for an explicit estimation) and therefore the field theory remains effectively causal to the degree of accuracy relevant to our setup. Indeed, no acausal behaviour was displayed in our first example (see Sec. 7.4.1).

Taking the UV cutoff can be seen as placing a bandlimit on the field. The Nyquist-Shannon sampling theorem then allows us to exactly reconstruct this bandlimited field from a discrete lattice of sample points. The explicit derivation is included in App. C. This

yields the Hamiltonians

$$\hat{H}_\phi^{\text{UV}} = \sum_n \frac{mc^2}{2} (\hat{p}_n^2 + \hat{q}_n^2) + \frac{\hbar^2}{2ma^2} (\hat{q}_{n+1} - \hat{q}_n)^2, \quad (7.7)$$

$$\hat{H}_{\text{int}}^{\text{UV}} = \lambda_0 \chi(t) \sum_n a F(x_n) \hat{q}_{\text{D}} \otimes \hat{q}_n, \quad (7.8)$$

where $a = \pi/K$ is the lattice spacing, $x_n := na$ and $\lambda_0 = \lambda \hbar / \sqrt{am}$ is strength of the probe-field coupling. We have defined dimensionless field operators $\hat{q}_n := \sqrt{am/\hbar^2} \hat{\phi}(t, x_n)$ and $\hat{p}_n := \sqrt{a/m} \hat{\pi}(t, x_n)$ satisfying the canonical commutation relations $[\hat{q}_i, \hat{p}_j] = i \delta_{ij} \mathbb{1}$. Since the operators $\hat{q}_n$ and $\hat{p}_n$ are associated with the field operators in a localized region, we say that $(\hat{q}_n, \hat{p}_n)$ define *spatial modes*.

It is also possible to restrict the field theory to a finite region of space (think of an optical cavity or a finite-size transmission line). We can incorporate this to our general formalism, which results in an additional IR cutoff after we restrict the field to the region $x \in [0, L]$ where $L = Na$.

Given the above Hamiltonian, we can use Gaussian Quantum Mechanics [159, 211, 212] to efficiently simulate the dynamics. This is because the Hamiltonians controlling the dynamics are all quadratic in the field and probe quadrature operators,

$$\hat{\mathbf{X}} := (\hat{q}_{\text{D}}, \hat{p}_{\text{D}}, \hat{q}_1, \hat{p}_1, \hat{q}_2, \hat{p}_2, \dots)^\top, \quad (7.9)$$

and because the initial states of the probe and field are Gaussian states (i.e., states with Gaussian Wigner functions³). Together, these guarantee that the probe and field states remain in Gaussian states throughout their evolution. As such, their quantum states are fully characterized by the first and second moments of $\hat{\mathbf{X}}$, which we can collect together in a displacement vector

$$\mathbf{X} = \langle \hat{\mathbf{X}} \rangle = (\langle \hat{q}_{\text{D}} \rangle, \langle \hat{p}_{\text{D}} \rangle, \langle \hat{q}_1 \rangle, \langle \hat{p}_1 \rangle, \langle \hat{q}_2 \rangle, \langle \hat{p}_2 \rangle, \dots)^\top, \quad (7.10)$$

and a covariance matrix

$$\sigma_{ij} = \langle \{\hat{X}_i, \hat{X}_j\} \rangle / 2 - \langle \hat{X}_i \rangle \langle \hat{X}_j \rangle. \quad (7.11)$$

Unitary dynamics for the joint density matrix, $\hat{\rho} \rightarrow \hat{U} \hat{\rho} \hat{U}^\dagger$, corresponds to symplectic(-affine) evolution in phase space. In particular, the changes in the displacement vector and

³For an introduction to the phase space quantum mechanics formalism in arbitrary dimensions, see, e.g., Section 4.7 in [213].

the covariance matrix are given by $\mathbf{X} \rightarrow S \mathbf{X} + \mathbf{d}$ and $\sigma \rightarrow S \sigma S^\top$, where S is a symplectic matrix and $\mathbf{d}$ is a vector. Concretely,

$$S(t) = \mathcal{T} \exp \left(\frac{1}{\hbar} \int_0^t dt' \Omega \mathcal{F}(t') \right), \quad (7.12)$$

where $\mathcal{F}(t)$ is the symmetric matrix that satisfies $\hat{H} = \hat{H}_{\text{free}} + \hat{H}_\phi + \hat{H}_{\text{int}}(t) = \frac{1}{2} \hat{\mathbf{X}}^\top \mathcal{F}(t) \hat{\mathbf{X}}$, Ω is the symplectic form, $[\hat{X}_j, \hat{X}_k] = i \Omega_{jk} \mathbf{1}$, and $\mathcal{T} \exp$ is the time-ordered exponential. We note that since there are no terms in $\hat{H}$ that are linear in $\hat{\mathbf{X}}$, we have $\mathbf{d} = 0$. Moreover, S can be calculated non-perturbatively much more efficiently than $\hat{U}$ [159, 212–217].

Once we have computed the evolved joint covariance matrix and displacement vector, we can easily isolate the reduced state of the probe system as $\mathbf{X}_D = (\mu_q, \mu_p)^\top = (\langle \hat{q}_D \rangle, \langle \hat{p}_D \rangle)^\top$ and

$$\sigma_D = \begin{pmatrix} \sigma_{qq} & \sigma_{qp} \\ \sigma_{pq} & \sigma_{pp} \end{pmatrix} = \begin{pmatrix} \langle q_D^2 \rangle & \langle \{\hat{q}_D, \hat{p}_D\} \rangle / 2 \\ \langle \{\hat{q}_D, \hat{p}_D\} \rangle / 2 & \langle p_D^2 \rangle \end{pmatrix} - \mathbf{X}_D \mathbf{X}_D^\top. \quad (7.13)$$

These values determine the distributions that our measurements of $\hat{q}_D$, $\hat{p}_D$ and $\hat{r}_D$ should be drawn from. Namely, the outcomes of the measurements are distributed as

$$q_k \sim \mathcal{N}(\mu_q, \sigma_{qq}), \quad (7.14)$$

$$p_k \sim \mathcal{N}(\mu_p, \sigma_{pp}), \quad (7.15)$$

$$r_k \sim \mathcal{N}\left(\frac{\mu_q + \mu_p}{\sqrt{2}}, \frac{\sigma_{qq} + 2\sigma_{qp} + \sigma_{pp}}{2}\right), \quad (7.16)$$

where $\mathcal{N}(\mu, \sigma)$ is the normal distribution with mean μ and variance σ . This can be straightforwardly justified from the fact that the partial Wigner function of the probe is still Gaussian and that its marginals correspond to the position and momentum distributions.

Moreover, from the evolved probe state and N_{TOM} we can determine the distributions from which the sample mean and sample variance are drawn. For instance,

$$\bar{q} = \frac{1}{N_{\text{TOM}}} \sum_{k=1}^{N_{\text{TOM}}} q_k \sim \mathcal{N}\left(\mu_q, \frac{\sigma_{qq}}{N_{\text{TOM}}}\right), \quad (7.17)$$

$$\bar{s}_q = \frac{1}{N_{\text{TOM}}} \sum_{k=1}^{N_{\text{TOM}}} (q_k - \bar{q})^2 \sim \sigma_{qq} \frac{\chi^2(N_{\text{TOM}} - 1)}{N_{\text{TOM}}}, \quad (7.18)$$

where $\chi^2(k)$ is the chi-squared distribution with k degrees of freedom. Note that while the sample means are distributed normally, the sample variances are not.

7.3.2 Method 2: Non-Gaussian Wigner functions

Although working with the Gaussian formalism when possible is very convenient for calculational purposes, the techniques discussed in the previous subsection can be easily extended to drop the assumption that the field and probe are initially in Gaussian states. We will show this explicitly in our third example scenario, where we will attempt to distinguish between a coherent state (which is Gaussian) and a Fock state (which is not).

Even when the initial state's Wigner function $W(\boldsymbol{\xi})$ is non-Gaussian, we can still understand the dynamics in terms of phase-space evolution [213]. A Gaussian unitary transformation $\hat{U}_G$ (i.e., evolution generated by a quadratic Hamiltonian) still corresponds to a symplectic transformation for the vector of quadrature operators of the form described in Sec. 7.3.1, so that the dynamics induced by our unitary evolution operator, $\hat{\rho} \rightarrow \hat{U}_G(t)\hat{\rho}\hat{U}_G(t)^\dagger$ can be equivalently written in the Heisenberg picture as

$$\hat{\mathbf{X}} \rightarrow \hat{U}_G(t)^\dagger \hat{\mathbf{X}} \hat{U}_G(t) = S(t) \hat{\mathbf{X}} + \mathbf{d}(t)\mathbf{1}, \quad (7.19)$$

for some symplectic matrix $S(t)$ and some vector $\mathbf{d}(t)$. For our particular case, $\mathbf{d}(t) = 0$, since $\hat{H}$ does not have any terms that are linear in $\hat{\mathbf{X}}$.

This transformation entails, in turn, a transformation in the states' Wigner function, independent of the Gaussianity of the states. This transformation can be written as

$$W(\boldsymbol{\xi}) \rightarrow W(S^{-1}(t)\boldsymbol{\xi}) \equiv W(\boldsymbol{\xi}; t). \quad (7.20)$$

We can use this to determine the final reduced probe state from the initial probe-field state by integrating over all of the field variables as

$$W_D(q_D, p_D; t) = \int d\boldsymbol{\zeta} W(q_D, p_D, \boldsymbol{\zeta}; t), \quad (7.21)$$

where $\boldsymbol{\zeta} = (q_1, p_1, q_2, p_2, \dots)$ runs over all the field variables. Note that if the initial probe-field state is non-Gaussian then the final probe state will also be non-Gaussian. That is, we cannot characterize it by its first and second moments alone. In general, this final probe state will have non-trivial higher moments.

However, the higher moments of the final probe distribution can be calculated straightforwardly from the higher moments of the initial probe-field distribution. We can use the central limit theorem to generate the sample means, sample variances and sample fourth central moments of Eqs. (7.2)–(7.4). For instance, from the second, fourth and eighth

central moments of the marginal distribution of the q quadrature of the probe,

$$\sigma_{qq} = \int_{\mathbb{R}^2} dp dq (q - \mu)^2 W_D(q, p), \quad (7.22)$$

$$\gamma_4 = \int_{\mathbb{R}^2} dp dq (q - \mu)^4 W_D(q, p), \quad (7.23)$$

$$\gamma_8 = \int_{\mathbb{R}^2} dp dq (q - \mu)^8 W_D(q, p), \quad (7.24)$$

where

$$\mu = \int_{\mathbb{R}^2} dp dq q W_D(q, p), \quad (7.25)$$

we have, for N_{TOM} sufficiently large,

$$\bar{q} \sim \mu + \sqrt{\frac{\sigma}{N_{\text{TOM}}}} \mathcal{N}(0, 1), \quad (7.26)$$

$$\bar{s}_q \sim \sigma + \sqrt{\frac{\gamma_4 - \sigma^2}{N_{\text{TOM}}}} \mathcal{N}(0, 1), \quad (7.27)$$

$$\bar{s}_{4,q} \sim \gamma_4 + \sqrt{\frac{\gamma_8 - \gamma_4^2}{N_{\text{TOM}}}} \mathcal{N}(0, 1). \quad (7.28)$$

In particular, if we consider states with $\langle \hat{q}_D \rangle = 0$, as we will in the example in Sec. 7.4.3, then

$$\bar{q} \sim \sqrt{\frac{\langle \hat{q}_D^2 \rangle}{N_{\text{TOM}}}} \mathcal{N}(0, 1), \quad (7.29)$$

$$\bar{s}_q \sim \langle \hat{q}_D^2 \rangle + \sqrt{\frac{\langle \hat{q}_D^4 \rangle - \langle \hat{q}_D^2 \rangle^2}{N_{\text{TOM}}}} \mathcal{N}(0, 1), \quad (7.30)$$

$$\bar{s}_{4,q} \sim \langle \hat{q}_D^4 \rangle + \sqrt{\frac{\langle \hat{q}_D^8 \rangle - \langle \hat{q}_D^4 \rangle^2}{N_{\text{TOM}}}} \mathcal{N}(0, 1), \quad (7.31)$$

where

$$\langle \hat{q}_D^n \rangle = \int_{\mathbb{R}^2} dp dq q^n W_D(q, p) \quad (7.32)$$

is the n -th moment of the marginal distribution of the q quadrature of the probe.

7.4 Examples

In this section, we will consider three physical examples to test the degree of success of the measurement protocol and data processing framework that we presented in Sec. 7.2.

7.4.1 Remote boundary sensing

As a first application of our framework, we will use measurements of a local probe near one end of a cavity ($x \approx 0$) to learn about the location of the boundary at the other end of the cavity ($x \approx L$).

In this example we will employ the lattice approximation discussed in the previous section. That is, the field will be approximated as a finite chain of harmonic oscillators which we will call spatial modes. Note from Eq. (7.7) that these spatial modes have a nearest-neighbor coupling. As we will see, the lattice approximation does not significantly impact the relativistic compliance of our setup. Indeed, we will show that, in practice, signals do not propagate superluminally in our lattice.

To simulate different positions of the far boundary, we will modify the coupling in Eq. (7.7) so that we separate the coupling of the two oscillators furthest from the probe from the rest of the Hamiltonian:

$$\hat{H}_\phi^{\text{UV}} = \sum_{n=1}^N \frac{mc^2}{2} (\hat{p}_n^2 + \hat{q}_n^2) + \sum_{n=1}^{N-2} \frac{\hbar^2}{2ma^2} (\hat{q}_{n+1} - \hat{q}_n)^2 + \frac{\hbar^2}{2ma^2} (\hat{q}_N^2 + \hat{q}_{N-1}^2) + \hat{H}_{\text{last}}. \quad (7.33)$$

That is, if there are N harmonic oscillators in the lattice, we will consider modifications to the coupling between oscillator N and oscillator $N - 1$. We summarize the modified couplings under consideration in Table 7.1.

Label	Name	$\hat{H}_{\text{last}}$ for $t < 0$	$\hat{H}_{\text{last}}$ for $t \geq 0$
$y = 1$	Full Bond	$g \quad \hat{q}_{N-1} \otimes \hat{q}_N$	Same as $t < 0$
$y = 2$	Cut Bond	$0 \quad \hat{q}_{N-1} \otimes \hat{q}_N$	Same as $t < 0$
$y = 3$	Signal	$0 \quad \hat{q}_{N-1} \otimes \hat{q}_N$	$g \quad \hat{q}_{N-1} \otimes \hat{q}_N$

Table 7.1: Modifications of $\hat{H}_{\text{last}}$ connecting the last spatial mode to the rest of the lattice. $g = \hbar^2/ma^2$ is the site-to-site coupling strength.

For the first case ($y = 1$), we take $\hat{H}_{\text{last}}$ to be just the same as every other site-to-site coupling, so that the boundary is at the last site of the lattice. In the second case ($y = 2$),

we take $\hat{H}_{\text{last}} = 0$, thus setting the boundary at the second to last site. We added a third case ($y = 3$) to measure the time that it takes for the probe to detect a perturbation coming from the boundary. To do so, we consider a time-dependent coupling between the two last oscillators which turns on at $t = 0$. In each of these three cases we assume that the field has thermalized to the ground state of its $t < 0$ Hamiltonian well before $t = 0$. Recall that $t = 0$ marks the instant when the probe first couples to the field. In this example, we take the switching function $\chi(t)$ to be constant for $t > 0$.

In the first two cases the $t \geq 0$ field Hamiltonian is exactly the same as the $t < 0$ Hamiltonian. All that changes at $t = 0$ in these cases is that the probe couples to the field at one end of the cavity, $x \approx 0$. We thus expect disturbances to begin propagating away from $x = 0$ beginning at $t = 0$. In the third case the field Hamiltonian suddenly changes at $t = 0$. This change is localized around $x \approx L$, and we thus expect disturbances to begin propagating away from both $x = 0$ and $x = L$ beginning at $t = 0$. In this third case, we take the last spatial mode to be in a highly-squeezed state (8 dB [218]) before $t = 0$ so that it produces a notable disturbance when it couples.

By comparing cases 2 and 3 we can explicitly measure the signal-propagation speed on the lattice. In these cases the field is in exactly the same state prior to $t = 0$, but they differ by the local disturbance at $t = 0$ and $x \approx L$ in case 3. For $t > 0$, the disturbance at $x \approx L$ begins to propagate through the lattice towards the probe. We can define the effective signaling time as the time it takes the probe to differentiate between cases 2 and 3. If the probe is able to differentiate cases 1 and 2 in less than this signalling time, then it cannot be due to having received a signal from the boundary. We can compare this signalling time to the light-crossing time of the cavity ($t = L/c$) to see if our probe is receiving any faster-than-light signals due to the approximations employed.

We consider a detector of roughly atomic size, with a Gaussian smearing function of width $\sigma = 53$ pm (the Bohr radius). Taking the UV cutoff $K = 16/\sigma$ gives us a lattice spacing $a = \pi/K = 10.4$ pm. We take the boundary to be at $L = 90\sigma = 457a = 4.7$ nm. We set the detector's excitation energy $\hbar\omega_D = 130$ eV and the field's mass $mc^2 = 1$ eV. The field mass is much smaller than any other energy scale in the problem (i.e., the field is effectively massless). Finally, we investigate the non-perturbative strong-coupling regime where $\lambda_0 = \hbar\omega_D = 130$ eV. Note that these parameters are chosen for illustrative purposes; similar results are obtained for a wide range of values.

In Fig. 7.2 we show the performance of the neural network (solid line). The green triangle lines demonstrate the causal behaviour of the setup: when we send a signal from the far boundary to the detector (coupling a new oscillator at $t = 0$), the neural network accuracy indicates that the probe does not receive the signal before ≈ 15 as. A comparison

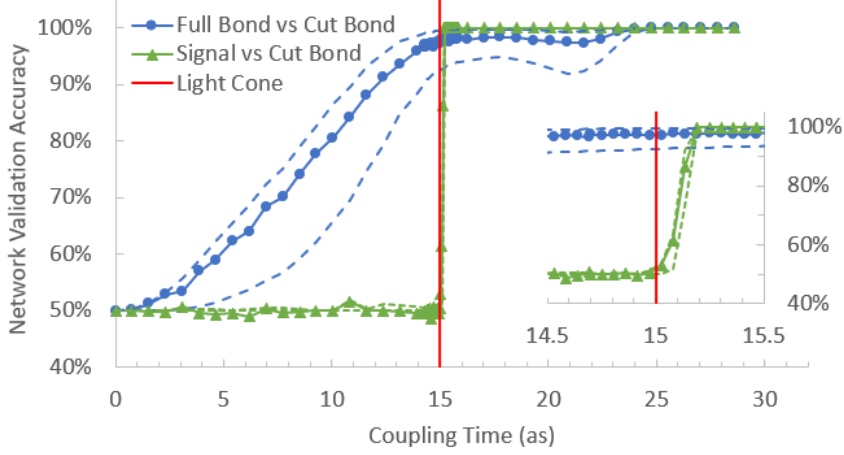


Figure 7.2: We trained a neural network to predict (through classification) the position of the boundary of a quantum field in a cavity from local probe data gathered far from the boundary. The network was asked 1) to detect a signal sent from the boundary (green triangles) and 2) to detect a modification of the field’s boundary position (blue circles). The network’s accuracy (solid) along with upper and lower bounds on the theoretical optimal accuracy (dashed) are plotted as a function of the duration of the probe’s interaction with the field. A point plotted at time t indicates the network’s accuracy given measurements taken at $N_{\text{TIMES}} = 10$ measurement times between t and the previous plot point. The network was trained on $N_{\text{train}} = 11250$ examples. Each example summarizes $N_{\text{TOM}} = 10^{22}$ measurements of each of the probe’s quadratures ($\hat{q}_D$, $\hat{r}_D$ and $\hat{p}_D$) at each measurement times. The inset shows details of the causal response of the detector to the signal. The vertical red line is at the edge of detector to boundary light-crossing time.

between this and a conservative estimate of the signal-to-edge-of-detector light-crossing time, $(L - 5\sigma)/c = 15$ as (vertical red line), shows that our toy model displays a good causal behaviour. This confirms that the lattice approximation did not compromise the relativistic structure that is essential for the model to be a faithful simulation of a quantum field theory in a cavity.

The blue circle line represents the ability of the neural network to sense the boundary at the far end of the cavity. Here, the information about the boundary has had time to spread all over space in the equilibration process before $t = 0$: the ground state knows locally about its boundary conditions. Indeed, the network accuracy shows that the nature of the field boundary can be resolved long before any signal from the boundary propagates

to the detector. This allows the probe to see the boundary “without light”, that is, in the vacuum state of the theory and much before the light-crossing time of the lattice.

Notice that we do observe an increase in the accuracy of the neural network as the time of the measurements is further away from the start of the coupling of the probe. This is not related to signalling from the boundary, as the comparison between cases 2 and 3 shows, but rather to the fact that the more time we let the detector interact with the field, the more knowledge it gathers from the infrared structure of the field state, where the information about the boundary lies.

In itself, this “seeing without light” phenomenon is not a new result. It has been observed and understood in a number of different contexts [185–188]. What is new here is the explicit collection of (simulated) data and the direct translation of these local measurement results into claims about the field’s boundary conditions. Remarkably, the local measurement protocol and data-analysis ansatz was not tailored to the detection of boundaries. In fact (as we will soon see), the exact same local measurement protocol and data-analysis ansatz can be used to make accurate claims about a wide range of other features of the QFT, and not only to “see without light”.

Near Optimality of the Neural Network

Unlike the more complex scenarios in the sections below, this particular example admits a more conventional statistical treatment that will allow us to discuss in some detail how we know that the neural network is behaving near-optimally.

Recall from Eqs. (7.17) and (7.18) that we know the distributions from which the sample first and second moments are drawn from. Thus, ultimately, we know the distributions from which our compressed data D_c is drawn. In particular, we know the distributions $p(D_c|y = 1, \theta)$, $p(D_c|y = 2, \theta)$, and $p(D_c|y = 3, \theta)$, for each of the three cases listed in Table 7.1, where θ contains all the other parameters of the measurement setup (e.g., coupling times). These distributions are simple, yet made out of rather unwieldy combinations of Gaussians and χ^2 distributions.

Suppose that we are presented with some compressed data D_c and asked to guess whether it came from case $y = 1$ or $y = 2$. This is, in fact, the exact question we are repeatedly asking our neural network. For this binary classification task, the optimal solution is known: we ought to guess whichever y makes $p(D_c|y, \theta)$ larger (see App. B.2 for details). The success rate of this strategy is $p_{\text{success}}(\theta) = \frac{1}{2}[1 + \text{TV}_{12}(\theta)]$ where $\text{TV}_{12}(\theta) \in [0, 1]$ is the total variation distance between $p(D_c|y = 1, \theta)$ and $p(D_c|y = 2, \theta)$.

If we could calculate $\text{TV}_{12}(\theta)$, then we would have a tight upper bound on the accuracy achievable by any method aimed to distinguish these two cases. In particular, we would have an upper bound on the validation accuracy achievable by any neural network (with any architecture, training time, training method, etc.). Unfortunately, for the rather cumbersome combination of Gaussians and χ^2 distributions pertaining to this example, TV is not expressible in closed form. However, in the large N_{TOM} regime these distributions each simplify to multi-variate Gaussian distributions. While the total variation distance between multi-variate Gaussian distributions is unknown in general, we can compute the Hellinger distance $H(\theta) \in [0, 1]$ between them. The Hellinger distance bounds the total variation distance above and below as (see App. B.2 for details)

$$H(\theta) \leq \text{TV}(\theta) \leq H(\theta)\sqrt{2 - H(\theta)^2}. \quad (7.34)$$

Thus, from $H(\theta)$ we have upper and lower bounds on the optimal validation accuracy. These bounds are plotted in dashed lines in Fig. 7.2. We have a guarantee that the optimal performance for any network (with any architecture, training time, training method, etc.) lies between the dashed lines. The fact that the network validation accuracy tracks these bounds indicates that the network is near optimal. Moreover, the fact that the upper bound in the signal case (green dashed line) is close to 50% before ≈ 15 as indicates that there is no way to process the data to learn where the boundary is before this time. This shows that the lattice approximations used for this example do not violate relativistic causality; no neural network can extract the signal from the data before relativity says it can, simply because the information is not there yet.

It is worth remarking that the neural network employed here was not designed with this problem in mind, and yet it performs almost optimally. This suggests that the network is good at extracting and processing all the information contained in the data produced by the measurement protocol. Therefore, it is reasonable to expect a near-optimal behaviour in the other examples as well, especially since we have changed neither the network architecture nor the training procedure. Notice in particular that we did not need to feed the neural network any data about the parameters of the experiment ($\sigma, K, a, L, \omega_{\text{D}}$, and m). We did, however, rely on the network being trained on data obtained in the same physical system that it was tested on later. If needed, the network could be trained on data from QFTs with a range of different parameters, so that it also learns to distinguish the dependence on these parameters from the dependence on the feature we are actually interested in.

7.4.2 Thermometry

To showcase the broad applicability of our framework, we consider a very different problem keeping the exact same measurement protocol, the same coupling between probe and field, and the same data-analysis ansatz.

We consider a probe motivated by a superconducting circuit undergoing a long-range interaction with an open transmission line in a thermal state. Such systems do not couple strongly to frequencies above 50 GHz [124, 219, 220]. Assuming a Gaussian profile we can match this behavior by taking $3/\sigma = 50$ GHz/c, i.e., $\sigma = 18$ mm. These numbers are motivated by [124]. Placing our UV cutoff in the field at $K = 16/\sigma = 267$ GHz/c yields a lattice spacing $a = \pi/K = 3.5$ mm. We couple the circuit to the center of a transmission line of length $L = 100a = 19.6\sigma = 353$ mm, with Dirichlet boundary conditions. We take the circuit to have an energy gap of $\omega_{\text{D}} = 10$ GHz, representative of typical values in such systems, and the field to have a mass $mc^2/\hbar = 0.1$ GHz, much smaller than the other energy scales. We again consider strong coupling: $\lambda_0/\hbar = \omega_{\text{D}} = 10$ GHz. Using these parameters we trained the network to estimate the field’s temperature based only on measurements of the local probe. For each base temperature T we generated our labelled data by simulating how the probe would respond to a quantum field of temperature y , selected uniformly from the range $[0.9T, 1.1T]$, i.e., the range $T \pm 10\%$. For each T , we trained a neural network through regression to accurately predict the temperature of the field, y . To validate the accuracy of the network, we determined the fraction of the validation data for which the predicted label fell within $\pm 0.01T$ of the correct value.

In Fig. 7.3 we show the performance of the neural network. By random chance, the network would be expected to guess the correct temperature within this accuracy 10% of the time. This is what one would expect when the coupling time is zero since the probe has not learned anything about the field, which is confirmed by the 10% validation accuracy shown in the plot when the coupling time is zero. As the coupling time increases, the network becomes more accurate. For each base temperature T , the network eventually labels nearly 100% of the validation data correctly (within $\pm 0.01T$). It can do so even before the interaction’s thermalization time, which is lower-bounded by the detector’s Heisenberg time $1/\omega_{\text{D}} = 100$ ps (the smallest timescale that the detector can resolve, marked with a red vertical line in Fig. 7.3). Note that the neural network can determine the temperature very accurately even for very low transmission line temperatures (sub-mK).

It should be noticed that, just as in the previous example of boundary sensing, the possibility of measuring temperature without letting the thermometer thermalize is not a new result in itself. It is known that thermometry in times below the thermalization

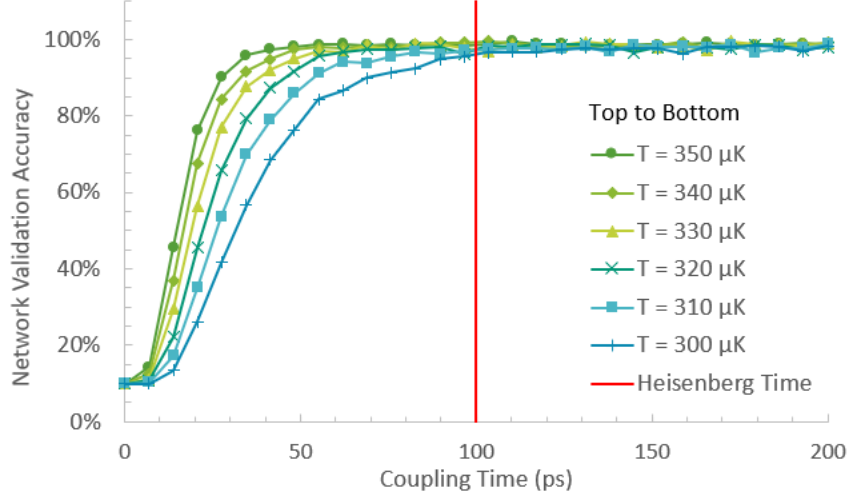


Figure 7.3: A neural network trained to predict (through regression) the temperature of a quantum field from local probe data. The network was trained on labelled data corresponding to field temperatures from a range $T \pm 10\%$. The fraction of the validation data which the network labelled correctly to within $\pm 1\%$ is plotted as a function of the duration of the probe’s interaction with the field. A point plotted at time t indicates the network’s accuracy given measurements taken at $N_{\text{TIMES}} = 10$ measurement times between t and the previous plot point. The network was trained on $n_{\text{train}} = 7500$ examples from each range. Each example summarizes $N_{\text{TOM}} = 10^{20}$ measurements of each of the probe’s quadratures ($\hat{q}_D$, $\hat{r}_D$ and $\hat{p}_D$) at each measurement time. The vertical red line is the probe’s Heisenberg time ω_D^{-1} .

time of the thermometer is possible [221–224]. What this example shows is that the field temperature can indeed be reconstructed for times as short as the Heisenberg time of the thermometer using the exact same local measurement protocol and data-analysis ansatz we used for boundary sensing, thus adding temperature to the potentially long list of features of the QFT that can be reconstructed with our framework using this very innocent choice of data sampling and processing.

7.4.3 Discrimination between Fock and coherent states

The goal of this final example is to showcase the effectiveness of our framework in an example that assumes neither 1) the Gaussianity of the probe/field states, nor 2) a UV

cutoff/bandlimit/lattice discretization. With this in mind, here we apply the proposed measurement framework to the quantum optical problem of distinguishing Fock states (like a single-photon state) from low amplitude coherent states (produced by stimulated emission) when the expectation of the number of photons in the state is the same.

Consider a massless real scalar quantum field in a (1+1)-dimensional cavity of length L with Dirichlet boundary conditions. We can use its mode decomposition to write

$$\hat{\phi}(t, x) = \sqrt{\frac{2\hbar c^2}{L}} \sum_{\ell=1}^{\infty} \frac{\sin(k_{\ell} x)}{\sqrt{\omega_{\ell}}} [\hat{q}_{\ell} \cos(\omega_{\ell} t) + \hat{p}_{\ell} \sin(\omega_{\ell} t)], \quad (7.35)$$

where $c k_{\ell} = \omega_{\ell} = \pi \ell c/L$ and, as before, the dimensionless quadrature operators $\hat{q}_n$ and $\hat{p}_m$ satisfy canonical commutation relations $[\hat{q}_n, \hat{p}_m] = i\delta_{nm}\mathbb{1}$.

We will take the field state to be the vacuum for all modes except for the lowest frequency one (the $\ell = 1$ mode). We will try to determine the initial state of the $\ell = 1$ mode by measuring a probe coupled locally to the field in the center of the cavity. We take the $\ell = 1$ mode to be in either a) a Fock state $|N\rangle$ with N excitations, or b) a phase-averaged coherent state with N excitations on average. That is, a coherent state $|\alpha\rangle$ for some $\alpha \in \mathbb{C}$ with $|\alpha|^2 = \langle \hat{n} \rangle = N$ but with an unknown phase. Namely, we will consider the two following initial probe-field Wigner functions

$$\begin{aligned} W_1 &= W_0(q_D, p_D) W_{\text{Fock}}(q_1, p_1; N) \Pi_{\ell=2}^{\infty} W_0(q_{\ell}, p_{\ell}), \\ W_2 &= W_0(q_D, p_D) W_{\text{PAC}}(q_1, p_1; N) \Pi_{\ell=2}^{\infty} W_0(q_{\ell}, p_{\ell}), \end{aligned} \quad (7.36)$$

where q_D and p_D are the probe variables, and $W_0(q, p) = e^{-q^2 - p^2}/\pi$. Note that neither of these states are Gaussian. The Wigner function of a Fock state is [225]

$$W_{\text{Fock}}(q, p; N) = \frac{(-1)^N}{\pi} L_N(2(q^2 + p^2)) e^{-(q^2 + p^2)}, \quad (7.37)$$

where $L_N(x)$ is the N -th Laguerre polynomial. For the unknown phase coherent state, the fact that we do not know (therefore average over) the phase makes this a non-Gaussian state. The Wigner function of a phase-averaged coherent state (PAC) is [226]

$$W_{\text{PAC}}(q, p; N) = \int_0^{2\pi} \frac{d\theta}{2\pi^2} e^{-(q - \sqrt{2N} \cos \theta)^2 - (p - \sqrt{2N} \sin \theta)^2}. \quad (7.38)$$

Moreover, we note that these two states have exactly the same first and second moments:

$$\begin{aligned} \langle \hat{q} \rangle_{\text{Fock}; N} &= \langle \hat{p} \rangle_{\text{Fock}; N} = 0, & \langle \hat{q} \hat{p} \rangle_{\text{Fock}; N} &= 0, \\ \langle \hat{q}^2 \rangle_{\text{Fock}; N} &= \langle \hat{p}^2 \rangle_{\text{Fock}; N} = \langle \hat{n} + 1/2 \rangle_{\text{Fock}; N} = N + 1/2, \end{aligned} \quad (7.39)$$

and

$$\begin{aligned}\langle \hat{q} \rangle_{\text{PAC};N} &= \langle \hat{p} \rangle_{\text{PAC};N} = 0, & \langle \hat{q} \hat{p} \rangle_{\text{PAC};N} &= 0, \\ \langle \hat{q}^2 \rangle_{\text{PAC};N} &= \langle \hat{p}^2 \rangle_{\text{PAC};N} = \langle \hat{n} + 1/2 \rangle_{\text{PAC};N} = N + 1/2.\end{aligned}\tag{7.40}$$

Hence, no analysis of these two field states in terms of their first and second statistical moments can differentiate them; these field states can only be distinguished by methods which are sensitive to their third and higher order moments.

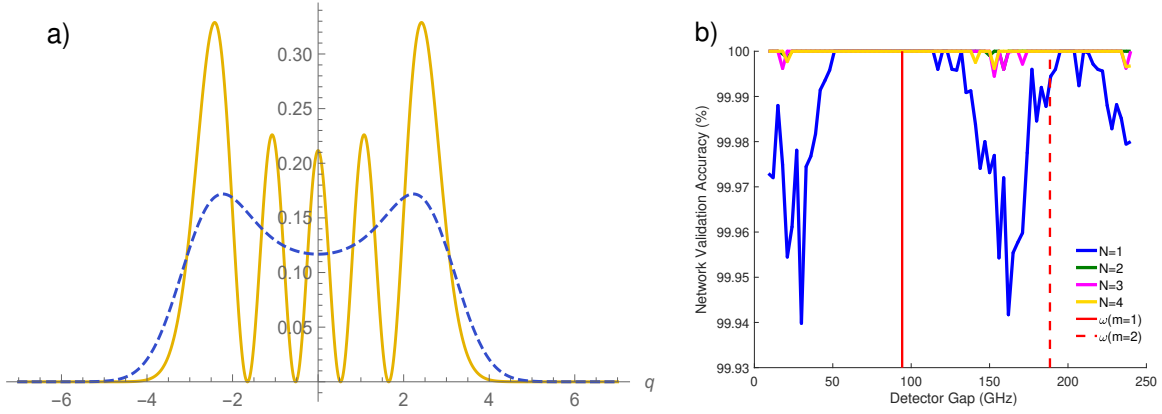


Figure 7.4: a) The marginal distributions for the $N = 4$ Fock state (solid) and for the phase averaged coherent state with $|\alpha|^2 = 4$ (dashed). The distributions are not Gaussian and they have the same mean and variance, making them impossible to distinguish with simple statistical analysis of first and second moments. b) The validation accuracy of a neural network trained to distinguish two field states from the measurements of a local detector coupled to the field. In particular the network differentiates vacuum cavity states with the following two modifications: 1) the lowest field mode is in an N -particle Fock state or 2) the lowest field mode is in a coherent state with expectation $\langle \hat{n}_1 \rangle = N$ particles and unknown phase.

It is a non-trivial task to determine which of these states the field is in. Suppose that (forgoing the localized probe system temporarily) we are somehow able to measure one of the quadrature operators of the lowest mode (e.g., $\hat{q}_1$) directly. The outcome of this measurement would be selected from the marginal distributions of $W_{\text{Fock}}(q, p; N)$ and $W_{\text{PAC}}(q, p; N)$. These are shown in Fig. 7.4a for the case of $N = 4$. The total variation distance between these marginals is $\text{TV} \approx 0.29$, meaning that the best success probability achievable from a single measurement outcome is $(1 + \text{TV})/2 \approx 64.5\%$.

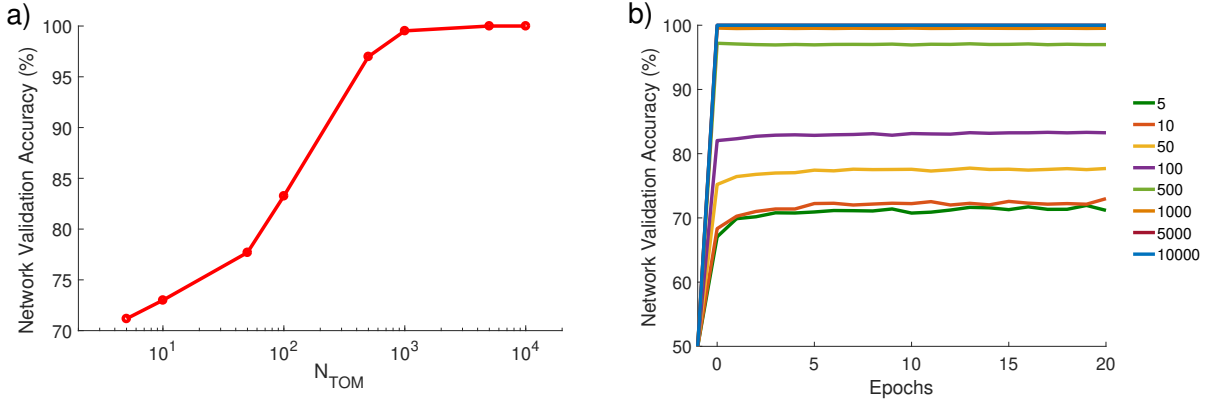


Figure 7.5: a) Validation accuracies attained for different tomographic sizes N_{TOM} , for expected number of particles $N = 1$, and detector frequency $\omega_{\text{D}} = \omega_1$. b) Evolution of network validation accuracies with the number of training iterations, plotted for different tomographic sizes. As before, the expected number of particles is $N = 1$, and the frequency of the detector is that of the first mode of the lattice, $\omega_{\text{D}} = \omega_1$.

In practice, we will attempt to distinguish these field states from a harmonic oscillator probe's response to the field, as explained in Sec. 7.2. This probe will pick up information from all the field modes, much of which is irrelevant to the task at hand. Our machine learning algorithm will need to learn to distinguish the irrelevant noise from the (already weak) signal from the $\ell = 1$ mode.

The only relevant difference with the measurement procedure in the previous examples is the switching function. While the same results would be obtained with the coupling protocol from previous sections, for the sake of an easier analytical treatment, in this case we consider a switching function $\chi(t) = \delta(t) + \delta(t - t_{\text{M}})$, where t_{M} is a time just before we measure the probe. Specifically, the probe undergoes a strong sudden interaction with the field at $t = 0$. Then both the probe and the field evolve freely for a time t_{M} . The probe undergoes another strong sudden interaction with the field at $t = t_{\text{M}}$. Finally, we measure one of the probe operators ($\hat{q}_{\text{D}}$, $\hat{p}_{\text{D}}$ or $\hat{r}_{\text{D}}$). This measurement procedure is repeated N_{TOM} times for each probe operator and at each of the N_{TIMES} different measurement times t_{M} . It is important to note that since the field state is not Gaussian, these probe measurement values will not be distributed normally. The distributions they are drawn from are ultimately derived from the ones in Fig. 7.4a and are much noisier due to vacuum noise from the other field modes that also couple to the probe (we do not carry out any single-mode or rotating-wave approximations).

As in the boundary sensing and thermometry examples, we record the sample means and sample variances of these N_{TOM} measurements in our compressed data. However, as discussed above, we will need more than just first and second moments to handle this problem. Thus, we additionally include the central fourth moments of the distribution of sampled data, as explained in Sec. 7.3.2. In App. B.3 and B.4 we calculate non-perturbatively the second, fourth and eighth moments of the probe's quadrature operators for the two field states that are relevant for this example. Recall that we do not require any more measurements, we just need the network to be able to process higher moments from the same sample of measurements used previously (i.e., less compression). We then train our neural network on many examples of this compressed data until it can accurately classify whether any given data came from an interaction with a Fock state or a coherent state. We use exactly the same neural network architecture, loss function and optimization method as in the two previous examples.

We consider an optical cavity of length $L = 1$ cm and a probe with a Gaussian smearing function of width $\sigma = 0.1$ mm placed at the center of the cavity $x = L/2 = 0.5$ cm. As discussed above, in each run of the experiment the probe strongly interacts with the cavity at two times: first at $t = T_{\text{MIN}} = 0$, then at $t = T_{\text{MIN}} + M \Delta t$ for $M = 1, 2, \dots, N_{\text{TIMES}}$ with $\Delta t = 6.67$ ps. As $N_{\text{TIMES}} = 10$, we have that $T_{\text{MAX}} = 66.7$ ps.

In Fig. 7.4b, we represent the validation accuracy of the neural network given different values for the probe frequency, ω_{D} , and for the same tomography, $N_{\text{TOM}} = 5000$. This shows that the neural network can successfully distinguish between a Fock and a PAC state given the sample fourth moments. The neural network can distinguish the two states with almost 100% accuracy in a wide range of detector gaps. This plot also provides some physical insight: at resonance, i.e., when the probe frequency is the same as the frequency of the mode of interest (in Fig. 7.4b, a solid vertical line corresponds to $\omega_{\text{D}} = \omega_1 = \pi/L = 94.2$ GHz), we obtain an improvement in the accuracy of the neural network. Furthermore, there is a second peak at double the frequency of the first mode (in Fig. 7.4b, the dashed vertical line corresponds to $\omega_{\text{D}} = 2\omega_1 = 188.4$ GHz). This is expected, since when the mode frequency is an integer multiple of the detector energy gap, resonance occurs and the detector becomes more sensitive to field excitations. We also show in Fig. 7.5 how the accuracy a) increases rapidly with the number of measurements, and b) evolves during training. It is worth noticing that as the number of measurements increases, the neural network becomes increasingly fast at reaching a stable validation accuracy. Fig. 7.5a also shows that the validation accuracy almost saturates with a moderate number of measurements ($N_{\text{TOM}} \sim 10^4$), which supports the experimental viability of the proposal. In previous examples, we considered particularly big tomographic sizes ($N_{\text{TOM}} = 10^{22}$ and $N_{\text{TOM}} = 10^{20}$, respectively) to ensure that the validation accuracy that we analyzed was the sole result of the network's

ability to unscramble the information from the local measurements, and that no role was played by the possible inaccuracies of the data it was fed with—even though these potential inaccuracies and the resilience of the protocol against them do play an important role in experiments. Moreover, it should be taken into account that the number of measurements N_{TOM} does not necessarily translate into *sequential* repetitions of the measurement protocol. For instance, collective measurements made on ensembles of particles (e.g., an atomic gas) can be translated into averages of individual observables calculated over the number of particles of the ensemble, which typically is already of the order of the Avogadro number ($N_A \sim 10^{24}$). In this kind of setup, tomographic sizes such as those considered in the two previous examples are clearly within the reach of experiments.

The success of the measurement framework in this last example shows that its applicability is not restricted to simple Gaussian systems. Indeed, we emphasized that here we have recovered a feature of the field without using the Gaussianity of the probe/field states or a UV cutoff/bandlimit/lattice discretization.

7.5 Summary

In this chapter, we have shown that we can use machine learning techniques to unscramble the information about QFTs acquired by localized probes with a one-size-fits-all method, thus eliminating the need to design a specific measurement protocol and data-analysis function for each feature we might be interested in. More concretely, we have demonstrated how to read out non-local features of a QFT from the outcomes of a fixed measurement protocol with local experiments, processed through a neural network with a generic architecture and training procedure.

As particular examples to showcase the power of the proposed machine learning framework we have examined three case studies: i) how a local probe can see a wall far away from it, in the vacuum and without actively sending signals to bounce off it, ii) how a local probe that is not given enough time to thermalize can still accurately determine the temperature of a quantum field, and iii) how detectors can accurately distinguish between Fock states and coherent states even when the first and second statistical moments of their observables match. To do so, in all cases we used the same simple measurement protocol, which was not adapted to the particular toy problems considered in this paper. Yet we were able to distinguish with high accuracy the relevant features of the field we were after in each case. This is evidence of the potential of these methods to accommodate experimental needs. Namely, the use of machine learning techniques in the context of quantum

field theory takes the complexity burden out of the design of experimental protocols and puts it on the data processing, which neural networks can deal with efficiently.

Remarkably, each of the examples we studied involved extracting information about non-local features of the quantum field, echoing the question of how quantum fields in equilibrium encode global information. We were able to track these non-local properties using untargeted local measurements because, once a quantum field reaches equilibrium, it stores global information in a localized (but convoluted) way. While this phenomenon is well-known, our results provided explicit examples of how it can be exploited. Specifically, we demonstrated that under the assumption of equilibrium with the background, simple local measurements can capture global information about a quantum field. While this information does not come out neatly organized, machine learning techniques turn out to be effective in decoding the resulting data, translating them into concrete statements about the field.

Chapter 8

Entanglement harvesting and measurements

In this chapter, we analyze entanglement harvesting for arbitrary initial states of particle detectors and arbitrary quasifree states of the field. Despite the fact that spatially smeared particle detectors are known to break covariance for arbitrary initial states, we show that the entanglement harvested by a pair of detectors from a quasifree state of a scalar field is a covariant quantity up to second order in perturbation theory. Then, we turn to analyze how the protocol of entanglement harvesting is affected by the performance of a measurement on the quantum field. The measurement on the field is modelled as in Ch. 5, with the coupling of a particle detector to the field, followed by a projective measurement performed on the detector. In our analysis, we considered different arrangements for two detectors harvesting entanglement and an ancillary detector used to perform a measurement. We found different regimes for how performing measurements on the field affects the entanglement harvested, depending on the initial and final states of the detector used to measure the field, as well as its coupling strength. We identified the regimes where it is possible to measure the field during the preparation of entanglement harvesting protocols without significantly interfering in the entanglement harvested. We also identify in what regimes the field measurement can degrade or slightly enhance the ability of particle detectors to harvest entanglement.

8.1 A witness of the entanglement structure

Quantum fields are generically entangled across different regions of spacetime [61, 144, 147, 227]. However, quantifying this shared entanglement is a hard problem that remains unsolved for arbitrary pairs of regions [61]. Even for the simpler case of complementary regions, it has only been successfully evaluated in particular cases (e.g., [228–231]).

In this context, particle detector models can provide an efficient means of accessing this entanglement structure. Specifically, a pair of particle detectors can couple to a quantum field extracting a certain amount of entanglement from it [103–105, 232, 233]. When they are spacelike separated, the entanglement that the detectors acquire is coming from pre-existing entanglement in the field between the spacetime regions to which the detectors coupled. This phenomenon is known in relativistic quantum information as *entanglement harvesting*, and it has been profusely studied both in itself (see, among many others, [106–108, 110, 234–244]), and to use it as a resource [245, 246] or as a quasi-local witness of, e.g., spacetime geometry [233] and topology [179].

We have seen in Ch. 5 that some particle detector models may enable a certain degree of faster-than-light signalling, that can manifest in the field at order λ^2 . Moreover, the breakdown of covariance found in [80] for scenarios with multiple detectors does show up in general at second order in perturbation theory, when the initial states of the detectors (as density operators) do not commute with their free Hamiltonians. Since entanglement harvesting is a second order process, it is reasonable to question whether the violations of causality may be tainting the results regarding the entanglement harvested.¹ This question will be addressed in the first part of the chapter, and we will show that this is not the case, and therefore our predictions of the entanglement harvested at leading order can be trusted.

On the other hand, in order to faithfully model realistic entanglement harvesting protocols we need to know the effect of measurements on the harvested correlations, since measurement and state preparation are intrinsic parts of any experimental setup in physics. For instance, the usual setups discussed in the literature often assume the field to be initialized in its vacuum state (see, e.g., [105], among many others) or in coherent states (see, e.g., [237]). Hence, any realistic implementation of these protocols requires a fair understanding of how the very preparation of the setup or even measurements from previous

¹In fact, the breakdown of covariance of particle detector models has been used on various occasions to question whether the phenomenon of entanglement harvesting is a solid prediction (e.g., [150, 247]; see [12, 16, 154] for replies and counterarguments).

experiments affect their results. It is to the analysis of this problem that we dedicate the second part of the chapter.

8.2 Entanglement harvesting without measurements

For the entanglement harvesting protocol, we consider a real scalar field $\hat{\phi}(\mathbf{x})$ in the $(1+d)$ -dimensional globally hyperbolic spacetime $(M, \mathbf{g})$. We consider two two-level UDW particle detectors, which we denote as A and B, undergoing timelike trajectories $\mathbf{z}_A(\tau_A)$ and $\mathbf{z}_B(\tau_B)$ parametrized by their proper times, and with ground states $|g_A\rangle$ and $|g_B\rangle$, excited states $|e_A\rangle$ and $|e_B\rangle$, and proper energy gaps Ω_A and Ω_B , respectively. We consider the UDW detectors to be initially in an arbitrary product state, and uncorrelated with the field:

$$\hat{\rho}_0 = \hat{\rho}_{AB,0} \otimes \hat{\rho}_{\phi,0}, \quad \text{with} \quad \hat{\rho}_{AB,0} = \hat{\rho}_{A,0} \otimes \hat{\rho}_{B,0}, \quad (8.1)$$

where we also assume that the initial field state $\hat{\rho}_{\phi,0}$ is quasifree. In the interaction picture, the particle detectors interact with the field $\hat{\phi}(\mathbf{x})$ via the interaction Hamiltonian weight given in Eq. (2.71), so that the total interaction Hamiltonian weight is given by their sum:

$$\hat{h}_{\text{int}}(\mathbf{x}) = \hat{h}_A(\mathbf{x}) + \hat{h}_B(\mathbf{x}) = \sum_{i \in \{A,B\}} \lambda_i \Lambda_i(\mathbf{x}) \hat{\mu}_i(\tau_i(\mathbf{x})) \hat{\phi}(\mathbf{x}), \quad (8.2)$$

where, as before, λ_i , $\Lambda_i(\mathbf{x})$ and $\hat{\mu}_i$ are the coupling strength, the spacetime smearing, and the monopole moment (cf. Eq. (2.72)) of detector i , respectively. We will assume that the coupling strengths λ_A and λ_B are of the same order of magnitude, and denote their products of order k as $\mathcal{O}(\lambda^k)$. As in Eq. (2.74), the evolution operator is given by

$$\hat{U}_t = \mathcal{T}_t \exp \left(-i \int d\mathbf{x} \sqrt{-g} \hat{h}_{\text{int}}(\mathbf{x}) \right), \quad (8.3)$$

with the key difference that, since we are interested in studying the covariance of entanglement harvesting predictions, we allow t to be a generic time parameter. This enables us to analyze how the results depend on its choice. After the interaction, the final joint density operator of the detectors is obtained tracing out the field degrees of freedom

$$\hat{\rho}_{AB} = \text{Tr}_{\phi}(\hat{U}_t \hat{\rho}^{(0)} \hat{U}_t^\dagger) = \hat{\rho}_{AB}^{(0)} + \hat{\rho}_{AB}^{(2)} + \mathcal{O}(\lambda^4). \quad (8.4)$$

Note that since the odd-point functions of the initial field state vanish, so do the odd terms in the expansion for $\hat{\rho}_{AB}$. Also, recall that $\hat{\rho}_{AB}^{(0)} = \hat{\rho}_{AB,0}$. Then, $\hat{\rho}_{AB}^{(2)}$ is given by

$$\begin{aligned} \hat{\rho}_{AB}^{(2)} = & \sum_{i,j \in \{A,B\}} \lambda_i \lambda_j \int dx dx' \sqrt{gg'} \Lambda_i(\mathbf{x}) \Lambda_j(\mathbf{x}') \left[\hat{\mu}_i(\tau_i) \hat{\rho}_{AB}^{(0)} \hat{\mu}_j(\tau'_j) w_2(\mathbf{x}', \mathbf{x}) \right. \\ & \left. - \hat{\mu}_i(\tau_i) \hat{\mu}_j(\tau'_j) \hat{\rho}_{AB}^{(0)} w_2(\mathbf{x}, \mathbf{x}') \theta(t - t') - \hat{\rho}_{AB}^{(0)} \hat{\mu}_j(\tau'_j) \hat{\mu}_i(\tau_i) w_2(\mathbf{x}', \mathbf{x}) \theta(t - t') \right]. \end{aligned} \quad (8.5)$$

In order to quantify the entanglement in $\hat{\rho}_{AB}$, we use the negativity [248] (which is a faithful entanglement measure for two-qubit systems), defined as

$$\mathcal{N}(\hat{\rho}_{AB}) = \sum_j \frac{|x_j| - x_j}{2}, \quad (8.6)$$

where x_j are the eigenvalues of the partially transposed density matrix.

Let us focus on the case when the detectors are initially in arbitrary pure states,

$$\hat{\rho}_{i,0} = |\psi_i\rangle\langle\psi_i|, \quad (8.7)$$

for $i \in \{A, B\}$. Since the detectors are initially in arbitrary pure states, we can pick states $|\chi_A\rangle$ and $|\chi_B\rangle$ that are orthogonal to $|\psi_A\rangle$ and $|\psi_B\rangle$, respectively. Then, we work in the basis $\{|\psi_A\psi_B\rangle, |\psi_A\chi_B\rangle, |\chi_A\psi_B\rangle, |\chi_A\chi_B\rangle\}$. Without loss of generality, we can assume that the change of basis between $\{|\psi_i\rangle, |\chi_i\rangle\}$ and the energy eigenbasis of the i -th detector takes the form

$$|g_i\rangle = \cos \alpha_i |\psi_i\rangle + e^{i\beta_i} \sin \alpha_i |\chi_i\rangle, \quad (8.8)$$

$$|e_i\rangle = -e^{-i\beta_i} \sin \alpha_i |\psi_i\rangle + \cos \alpha_i |\chi_i\rangle, \quad (8.9)$$

for some $\alpha_i \in [0, \pi)$, $\beta_i \in [0, 2\pi)$, $i \in \{A, B\}$. Using Eq. (8.5), we show in App. D.1 that the matrix form of the time-evolved state of the particle detectors is given by

$$\hat{\rho}_{AB} = \begin{pmatrix} 1 - \mathcal{L}_{AA}^{\text{gen}} - \mathcal{L}_{BB}^{\text{gen}} & \mathcal{X}^* & \mathcal{Y}^* & (\mathcal{M}^{\text{gen}})^* \\ \mathcal{X} & \mathcal{L}_{BB}^{\text{gen}} & (\mathcal{L}_{AB}^{\text{gen}})^* & 0 \\ \mathcal{Y} & \mathcal{L}_{AB}^{\text{gen}} & \mathcal{L}_{AA}^{\text{gen}} & 0 \\ \mathcal{M}^{\text{gen}} & 0 & 0 & 0 \end{pmatrix} + \mathcal{O}(\lambda^4), \quad (8.10)$$

with

$$\begin{aligned} \mathcal{L}_{ij}^{\text{gen}} = & \lambda_i \lambda_j \left[\cos^2 \alpha_i \cos^2 \alpha_j \mathcal{L}_{ij} - \cos^2 \alpha_i \sin^2 \alpha_j e^{-2i\beta_j} \mathcal{P}_{ij} \right. \\ & \left. - \sin^2 \alpha_i \cos^2 \alpha_j e^{2i\beta_i} \mathcal{K}_{ij} + \sin^2 \alpha_i \sin^2 \alpha_j e^{2i(\beta_i - \beta_j)} \mathcal{Q}_{ij} \right], \end{aligned} \quad (8.11)$$

$$\begin{aligned} \mathcal{M}^{\text{gen}} = & \lambda_A \lambda_B \left[\cos^2 \alpha_A \cos^2 \alpha_B \mathcal{M} - \cos^2 \alpha_A \sin^2 \alpha_B e^{2i\beta_B} \mathcal{R} \right. \\ & \left. - \sin^2 \alpha_A \cos^2 \alpha_B e^{2i\beta_A} \mathcal{S} + \sin^2 \alpha_A \sin^2 \alpha_B e^{2i(\beta_A + \beta_B)} \mathcal{V} \right], \end{aligned} \quad (8.12)$$

and

$$\mathcal{L}_{ij} = \int dx dx' \sqrt{gg'} \Lambda_i(x) \Lambda_j(x') w_2(x', x) e^{i(\Omega_i \tau_i - \Omega_j \tau'_j)}, \quad (8.13)$$

$$\mathcal{P}_{ij} = \int dx dx' \sqrt{gg'} \Lambda_i(x) \Lambda_j(x') w_2(x', x) e^{i(\Omega_i \tau_i + \Omega_j \tau'_j)}, \quad (8.14)$$

$$\mathcal{K}_{ij} = \int dx dx' \sqrt{gg'} \Lambda_i(x) \Lambda_j(x') w_2(x', x) e^{-i(\Omega_i \tau_i + \Omega_j \tau'_j)}, \quad (8.15)$$

$$\mathcal{Q}_{ij} = \int dx dx' \sqrt{gg'} \Lambda_i(x) \Lambda_j(x') w_2(x', x) e^{i(-\Omega_i \tau_i + \Omega_j \tau'_j)}, \quad (8.16)$$

$$\mathcal{M} = - \int dx dx' \sqrt{gg'} \Lambda_A(x) \Lambda_B(x') G_F(x, x') e^{i(\Omega_A \tau_A + \Omega_B \tau'_B)}, \quad (8.17)$$

$$\mathcal{R} = - \int dx dx' \sqrt{gg'} \Lambda_A(x) \Lambda_B(x') G_F(x, x') e^{i(\Omega_A \tau_A - \Omega_B \tau'_B)}, \quad (8.18)$$

$$\mathcal{S} = - \int dx dx' \sqrt{gg'} \Lambda_A(x) \Lambda_B(x') G_F(x, x') e^{i(-\Omega_A \tau_A + \Omega_B \tau'_B)}, \quad (8.19)$$

$$\mathcal{V} = - \int dx dx' \sqrt{gg'} \Lambda_A(x) \Lambda_B(x') G_F(x, x') e^{-i(\Omega_A \tau_A + \Omega_B \tau'_B)}. \quad (8.20)$$

The expressions for $\mathcal{X}$ and $\mathcal{Y}$ are more cumbersome and we leave them for App. D.1. In Eqs. (8.17)–(8.20), $G_F(x, x')$ is the Feynman propagator, given by

$$G_F(x, x') = \theta(t - t') w_2(x, x') + \theta(t' - t) w_2(x', x). \quad (8.21)$$

Notice that the Feynman propagator is independent of the time parameter t used for the time-ordering operator $\mathcal{T}_t$. This is because of the following:

1. If x and x' are causally connected, any notion of time-ordering respects their causal order. If, for instance, x' precedes x , then for any time parameter we will have that $t > t'$, and thus $\theta(t - t') = 1$ and $\theta(t' - t) = 0$.
2. If x and x' are spacelike separated, the Wightman function satisfies $W(x, x') = W(x', x)$. This implies

$$G_F(x, x') = w_2(x, x'). \quad (8.22)$$

Therefore, the terms $\mathcal{L}_{ij}^{\text{gen}}$ and $\mathcal{M}^{\text{gen}}$ are covariant, since they do not depend on the time coordinate t . In contrast, as shown in Appendix D.1, the terms $\mathcal{X}$ and $\mathcal{Y}$ are not covariant in general, since they depend on the ordering of t and t' .

Now, by Eq. (8.10), we can conclude that the negativity is given by

$$\mathcal{N}(\hat{\rho}_{AB}) = \max(0, \mathcal{N}^{(2)}) + \mathcal{O}(\lambda^4) , \quad (8.23)$$

with

$$\mathcal{N}^{(2)} = \sqrt{\frac{(\mathcal{L}_{AA}^{\text{gen}} - \mathcal{L}_{BB}^{\text{gen}})^2}{4} + |\mathcal{M}^{\text{gen}}|^2} - \frac{\mathcal{L}_{AA}^{\text{gen}} + \mathcal{L}_{BB}^{\text{gen}}}{2} . \quad (8.24)$$

This proves that the negativity is covariant at leading order, despite the fact that the joint state of the detectors in general displays non-covariant terms at second order in perturbation theory [80], which is the leading order for the entanglement harvested.

It is worth mentioning that Eqs. (8.23) and (8.24), along with Eqs. (8.11)–(8.20), generalize the expression found in [105] for the case in which the detectors are initially in their ground states. Notice that the negativity takes the same functional form in this case, with the replacements $\mathcal{L}_{AA} \mapsto \mathcal{L}_{AA}^{\text{gen}}$, $\mathcal{L}_{BB} \mapsto \mathcal{L}_{BB}^{\text{gen}}$, and $\mathcal{M} \mapsto \mathcal{M}^{\text{gen}}$.

Our results also generalize and reinforce the interpretation of entanglement harvesting as a competition between local noise and non-local correlations, which has been pointed out for detectors initially in their ground states since the first works on entanglement harvesting. Indeed, Eq. (8.24) shows that negativity is a competition between the non-local term $\mathcal{M}^{\text{gen}}$ and the local terms $\mathcal{L}_{AA}^{\text{gen}}$ and $\mathcal{L}_{BB}^{\text{gen}}$: it is straightforward to verify that $\mathcal{N}^{(2)} \leq |\mathcal{M}^{\text{gen}}|$, and therefore $\mathcal{L}_{AA}^{\text{gen}}$ and $\mathcal{L}_{BB}^{\text{gen}}$ act as local noise that can only decrease the entanglement harvested.

8.2.1 Low mixedness states

So far, we have only analyzed the case in which the detectors are initially in pure states. Mixedness present in the initial states of the detectors has been shown to rapidly diminish the ability of the detectors to harvest entanglement to almost zero [154]. Therefore the most important states for which we would like to analyze the covariance of the models of entanglement harvesting are those who only have a small degree of mixedness.

Let us consider that the detectors are initially in mixed states with a small degree of mixedness in comparison with the coupling parameter. Namely, we consider here initial states

$$\hat{\rho}_{A,0} = (1 - p_A) |\psi_A\rangle\langle\psi_A| + p_A |\chi_A\rangle\langle\chi_A| , \quad (8.25)$$

$$\hat{\rho}_{B,0} = (1 - p_B) |\psi_B\rangle\langle\psi_B| + p_B |\chi_B\rangle\langle\chi_B| , \quad (8.26)$$

with $1 \gg p_A, p_B \sim \mathcal{L}_{ij}^{\text{gen}}$. In this case, the joint state of the detectors after the entanglement harvesting protocol has taken place can be readily computed with the same techniques used in App. D.1. We take into account that the terms involving $|\chi_i\rangle$ are weighted by $p_i \sim \mathcal{L}_{ij}^{\text{gen}}$ in the initial state, and therefore are already of second order in perturbation theory. Using the orthogonality $\langle\psi_i|\chi_i\rangle = 0$ we see that all corrections appear on the diagonal terms. The final density matrix in this case is given by

$$\hat{\rho}_{AB}^{\text{gen}} = \begin{pmatrix} 1 - \mathcal{L}_{AA}^{\text{gen}} - \mathcal{L}_{BB}^{\text{gen}} - p_A - p_B & \mathcal{X}^* & \mathcal{Y}^* & (\mathcal{M}^{\text{gen}})^* \\ \mathcal{X} & \mathcal{L}_{BB}^{\text{gen}} + p_B & (\mathcal{L}_{AB}^{\text{gen}})^* & 0 \\ \mathcal{Y} & \mathcal{L}_{AB}^{\text{gen}} & \mathcal{L}_{AA}^{\text{gen}} + p_A & 0 \\ \mathcal{M}^{\text{gen}} & 0 & 0 & 0 \end{pmatrix} + \mathcal{O}(\lambda^4). \quad (8.27)$$

The mixedness in each particle detector acts as a source of additional noise that adds up to the local transition probabilities $\mathcal{L}_{AA}^{\text{gen}}$ and $\mathcal{L}_{BB}^{\text{gen}}$. The negativity in this case is given by

$$\mathcal{N} = \max \left(\sqrt{\frac{(\mathcal{L}_{AA}^{\text{gen}} + p_A - \mathcal{L}_{BB}^{\text{gen}} - p_B)^2}{4} + |\mathcal{M}^{\text{gen}}|^2} - \frac{\mathcal{L}_{AA}^{\text{gen}} + p_A + \mathcal{L}_{BB}^{\text{gen}} + p_B}{2}, 0 \right) + \mathcal{O}(\lambda^4), \quad (8.28)$$

which is simply the expression given in Eqs. (8.23) and (8.24) with the substitutions $\mathcal{L}_{AA}^{\text{gen}} \mapsto \mathcal{L}_{AA}^{\text{gen}} + p_A$, and $\mathcal{L}_{BB}^{\text{gen}} \mapsto \mathcal{L}_{BB}^{\text{gen}} + p_B$. Eq. (8.28) is covariant, since the mixedness only modifies the local noise terms in a way independent of the time-ordering parameter t . We also see that (as discussed in [154]) mixedness competes against entanglement harvesting by increasing the local noise.

8.3 Entanglement harvesting with measurements

To analyze the impact that a measurement performed on a detector can have on the entanglement harvested by two other detectors, we consider three experimenters, Alba, Blanca, and Clara, undergoing timelike trajectories parametrized by their proper times as $\mathbf{z}_A(\tau_A)$, $\mathbf{z}_B(\tau_B)$, and $\mathbf{z}_C(\tau_C)$. Each experimenter is equipped with a probe that we model as a spatially smeared two-level UDW particle detector, weakly coupled to a real scalar quantum field $\hat{\phi}(\mathbf{x})$. Also, let Ω_i be the proper energy gap between the ground state $|g_i\rangle$ and the excited state $|e_i\rangle$, for $i \in \{A, B, C\}$. We assume that the field is initially in a quasifree state, uncorrelated from the particle detectors, which are initialized in their ground states, i.e., the density operator that describes the initial joint state reads

$$\hat{\rho}_0 = \hat{\rho}_{ABC,0} \otimes \hat{\rho}_{\phi,0}, \quad \text{with} \quad \hat{\rho}_{ABC,0} = |g_A\rangle\langle g_A| \otimes |g_B\rangle\langle g_B| \otimes |g_C\rangle\langle g_C|. \quad (8.29)$$

As before,

$$\hat{h}_{\text{int}}(\mathbf{x}) = \hat{h}_A(\mathbf{x}) + \hat{h}_B(\mathbf{x}) + \hat{h}_C(\mathbf{x}) = \sum_{i \in \{A, B, C\}} \lambda_i \Lambda_i(\mathbf{x}) \hat{\mu}_i(\tau_i(\mathbf{x})) \hat{\phi}(\mathbf{x}), \quad (8.30)$$

and the time evolution is given by Eq. (8.3). Following the results of Sec. 8.2, we know that any choice of time parameter can be used to implement the time ordering, provided that we restrict our analysis to second order in perturbation theory. In order to perform the perturbative analysis in a consistent way, we need to specify how the orders of the different parameters are related. Here, we assume that all the coupling strengths are equal, $\lambda_A = \lambda_B = \lambda_C \equiv \lambda$. This assumption allows us to write the perturbative series in terms of a single coupling strength λ , simplifying the analysis significantly.²

To incorporate the effect of measurements in the entanglement harvesting protocol, we use the measurement theory presented in Ch. 5, and assume that once Clara's detector interaction is switched off, she performs a projective measurement on her detector, with associated PVM $\hat{F}(s) = |\varphi_s\rangle\langle\varphi_s|$ ($s = 0, 1$). To understand the effect that the measurement has, we shall compare our results with the entanglement that Alba and Blanca would harvest if Clara were not there to perform her measurement, i.e., when their final joint state is given by

$$\hat{\rho}_{AB} = \text{Tr}_\phi \left(\hat{V} \hat{\rho}_{AB,0} \hat{V}^\dagger \right), \quad (8.31)$$

where $\hat{V}$ is the time-evolution operator associated only to the interactions of detectors A and B, namely

$$\hat{V} = \mathcal{T} \exp \left(-i \int d\mathbf{x} \sqrt{-g} (\hat{h}_A(\mathbf{x}) + \hat{h}_B(\mathbf{x})) \right). \quad (8.32)$$

Up to second order in perturbation theory, the final state is given by [105]

$$\hat{\rho}_{AB} = \hat{\rho}_{AB,0} + \hat{\rho}_{AB}^{(2)} + \mathcal{O}(\lambda^4), \quad (8.33)$$

²In particular, doing this we can jointly analyze the measurement and the harvesting protocol just considering the leading order of the perturbative series. One could indeed analyze the case where λ_A , λ_B , and λ_C are of the same order of magnitude but different from each other using the same procedure. In this case, since the magnitudes of the three strengths are similar, the scale that characterizes the different orders in perturbation theory would be the one that sets the magnitude of the three couplings. However, in this paper we restrict ourselves to identical coupling strengths to avoid the increased calculational overhead that such an analysis would require.

Other cases would in general require considering higher orders in perturbation theory to see the interplay between the measurement and the harvesting protocol. For instance, if $\lambda_A, \lambda_B \sim o(\lambda_C)$, then the effect of the measurement on the field state would be predominant, and we would have to go further than leading order to see how the measurement affects the (now weaker) harvesting protocol. On the contrary, if $\lambda_C^2 \sim o(\lambda_A \lambda_B)$, then the harvesting protocol would dominate, and we would have to consider higher-order terms to analyze the effect of the measurement.

where, with respect to the ordered basis $\{|g_A g_B\rangle, |g_A e_B\rangle, |e_A g_B\rangle, |e_A e_B\rangle\}$, $\hat{\rho}_{AB}^{(2)}$ can be written in matrix form as

$$\hat{\rho}_{AB}^{(2)} = \begin{pmatrix} -\mathcal{L}_{AA} - \mathcal{L}_{BB} & 0 & 0 & \mathcal{M}_{AB}^* \\ 0 & \mathcal{L}_{BB} & \mathcal{L}_{AB}^* & 0 \\ 0 & \mathcal{L}_{AB} & \mathcal{L}_{AA} & 0 \\ \mathcal{M}_{AB} & 0 & 0 & 0 \end{pmatrix}, \quad (8.34)$$

where

$$\mathcal{L}_{IJ} = \lambda^2 \int dx dx' \sqrt{gg'} \Lambda_I(\mathbf{x}) \Lambda_J(\mathbf{x}') e^{i(\Omega_I \tau_I - \Omega_J \tau_J')} w_2(\mathbf{x}', \mathbf{x}), \quad (8.35)$$

and

$$\mathcal{M}_{IJ} = -\lambda^2 \int dx dx' \sqrt{gg'} \Lambda_I(\mathbf{x}) \Lambda_J(\mathbf{x}') e^{i(\Omega_I \tau_I + \Omega_J \tau_J')} [\theta(t - t') w_2(\mathbf{x}, \mathbf{x}') + \theta(t' - t) w_2(\mathbf{x}', \mathbf{x})]. \quad (8.36)$$

In this case, the negativity of $\hat{\rho}_{AB}$, corresponding to the protocol without measurements (wm), reads [105]

$$\mathcal{N}_{\text{wm}} = \max \left(0, \sqrt{|\mathcal{M}_{AB}|^2 + \frac{(\mathcal{L}_{AA} - \mathcal{L}_{BB})^2}{4}} - \frac{\mathcal{L}_{AA} + \mathcal{L}_{BB}}{2} \right) + \mathcal{O}(\lambda^4). \quad (8.37)$$

As mentioned before, we can associate the term $\mathcal{M}_{AB}$ with the quantum correlations acquired by detectors A and B, while $\mathcal{L}_{AA}$ and $\mathcal{L}_{BB}$ are local terms corresponding to the excitation probabilities of each detector, and the expression is clearly bounded by $|\mathcal{M}_{AB}|$, so that the local terms act as noise that can only decrease the entanglement harvested.

When analyzing the effect of the measurement, one of the first things we may notice is that, as in Ch. 5, the partial state resulting from the non-selective measurement reduces to the one resulting from the interaction. Moreover, it is easy to see that this state does not contain any term related to detector C up to second order in λ . Concretely, the state of A and B in a region ‘not to the past of the measurement’ (cf. Chs. 3 to 5) is given by

$$\hat{\rho}_{AB}^{\text{ns}} = \text{Tr}_{C,\phi}(\hat{U} \hat{\rho}_0 \hat{U}^\dagger) = \hat{\rho}_{AB,0} + \hat{\rho}_{AB}^{(2)} + \mathcal{O}(\lambda^4), \quad (8.38)$$

where $\hat{\rho}_{AB}^{(2)}$ is given in Eq. (8.34). This is indeed the same joint state for detectors A and B that we get when there is no detector C involved in the protocol, i.e., $\hat{\rho}_{AB}$ is given by Eq. (8.33), up to second order in perturbation theory. Since we are only interested in

results up to order λ^2 , we can conclude that non-selective measurements do not affect entanglement harvesting at leading order, and focus on the selective measurement case.

Let the measurement performed by Clara be selective, with outcome s . Without loss of generality, we can write

$$|\varphi_s\rangle = \epsilon |g_C\rangle + \sqrt{1 - \epsilon^2} e^{i\xi} |e_C\rangle, \quad (8.39)$$

where ϵ is a non-negative real number, and $\xi \in [0, 2\pi)$. If either A or B are in the future of the measurement event P (i.e., if one of them has access to the outcome of the experiment) their joint state is given by (see App. D.2 for details)

$$\hat{\rho}_{AB}^s \propto \langle \varphi_s | \text{Tr}_\phi (\hat{U} \hat{\rho}_0 \hat{U}^\dagger) | \varphi_s \rangle \quad (8.40)$$

$$= \epsilon^2 (\hat{\rho}_{AB,0} + \hat{\rho}_{AB}^{(2)}) + [\mathcal{L}_{CC}(1 - 2\epsilon^2) + \kappa] \hat{\rho}_{AB}^{(0)} + \hat{\gamma} + \hat{\nu} + \epsilon \mathcal{O}(\lambda^4) + \mathcal{O}(\lambda^6), \quad (8.41)$$

where, again with respect to the ordered basis $\{|g_A g_B\rangle, |g_A e_B\rangle, |e_A g_B\rangle, |e_A e_B\rangle\}$,

$$\hat{\gamma} = \epsilon \sqrt{1 - \epsilon^2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ e^{i\xi} \mathcal{M}_{AC} + e^{-i\xi} \mathcal{L}_{AC} & 0 & 0 & 0 \\ e^{i\xi} \mathcal{M}_{BC} + e^{-i\xi} \mathcal{L}_{BC} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \text{H.c.}, \quad (8.42)$$

$$\hat{\nu} = \mathcal{L}_{CC} \begin{pmatrix} -\tilde{\mathcal{L}}_{AA} - \tilde{\mathcal{L}}_{BB} & 0 & 0 & \tilde{\mathcal{M}}_{AB}^* \\ 0 & \tilde{\mathcal{L}}_{BB} & \tilde{\mathcal{L}}_{AB}^* & 0 \\ 0 & \tilde{\mathcal{L}}_{AB} & \tilde{\mathcal{L}}_{AA} & 0 \\ \tilde{\mathcal{M}}_{AB} & 0 & 0 & 0 \end{pmatrix}. \quad (8.43)$$

Here, for $I, J \in \{A, B, C\}$, the expressions for $\mathcal{L}_{IJ}$ and $\mathcal{M}_{IJ}$ are the same as the ones given in Eqs. (8.35) and (8.36), and for $I, J \in \{A, B\}$,

$$\tilde{\mathcal{L}}_{IJ} = \mathcal{L}_{IJ} + \frac{\mathcal{L}_{IC} \mathcal{L}_{JC}^* + \mathcal{M}_{IC} \mathcal{M}_{JC}^*}{\mathcal{L}_{CC}}, \quad (8.44)$$

$$\tilde{\mathcal{M}}_{AB} = \mathcal{M}_{AB} + \frac{\mathcal{L}_{AC} \mathcal{M}_{BC} + \mathcal{L}_{BC} \mathcal{M}_{AC}}{\mathcal{L}_{CC}}, \quad (8.45)$$

The expression for κ is obtained in App. D.2, and given in Eq. (D.30). The terms $\hat{\rho}_{AB}^{(2)}$ and $\hat{\gamma}$ are of second order in λ , whereas $\hat{\nu}$ and κ are of fourth order in λ . In particular, κ is subleading for all the calculations of interest. Finally, in Eq. (8.40), the density operator

needs to be normalized by its trace, which is also the probability $\text{Prob}(s)$ of measuring the outcome $|\varphi_s\rangle$. Using Eq. (8.41), we can write it as

$$\text{Prob}(s) = \text{Tr}(\langle \varphi_s | \hat{U} \hat{\rho}_0 \hat{U}^\dagger | \varphi_s \rangle) = \epsilon^2 + \mathcal{L}_{\text{cc}}(1 - 2\epsilon^2) + \kappa + \epsilon \mathcal{O}(\lambda^4) + \mathcal{O}(\lambda^6). \quad (8.46)$$

Along with Eq. (8.41), this allows us to find a perturbative expansion for the selectively updated joint density operator $\hat{\rho}_{\text{AB}}^s$. As we will discuss later, it suffices to perform this expansion up to second order in perturbation theory, since this is the leading order for the entanglement harvested. This involves considering a perturbative expansion of Eq. (8.40). Note, however, that the expression for $\hat{\rho}_{\text{AB}}^s$ is not analytic at $(\epsilon, \lambda) = (0, 0)$. This forces us to be specially careful with the expansions when $\epsilon = \langle \varphi_s | g_C \rangle$ approaches zero (i.e., when the outcome state $|\varphi_s\rangle$ is close to the excited state $|e_C\rangle$). One way of making sense of this is to realize that one cannot expect the updated state of the field for $\epsilon = 0$ to converge to the initial state when $\lambda_C \rightarrow 0$, since if the particle detector does not couple to the field at all, then an orthogonal outcome is not possible. However, it is important to analyze the case where $1 \gg \epsilon > 0$, for which we can take a perturbative approach on the parameter ϵ .

In order to gain further intuition, let us analyze the behaviour of $\text{Prob}(s)$, given in Eq. (8.46), at leading order. If $\lambda \ll \epsilon$, i.e., if the inner product $|\langle \varphi_s | g_C \rangle|$ is large compared to the coupling strength,³ $\text{Prob}(s)$ is given at leading order by ϵ^2 . In particular, in Eq. (8.46), the term proportional to ϵ^2 is one order of λ below the rest of the terms as long as $\epsilon = \Theta(\lambda^\delta)$ for $\delta < 1/2$, where we have used the big theta symbol⁴ of the Bachmann-Landau notation [249]. We will refer to the regime $\epsilon = \Theta(\lambda^\delta)$ for $\delta < 1/2$ as the *non-orthogonal regime*. This is because in the perturbative regime $\lambda \ll 1$, $\epsilon = \Theta(\lambda^\delta)$ with $\delta < 1/2$ guarantees that $\epsilon \gg \lambda$ (roughly speaking, the distance to orthogonality is much larger than the displacement that the interaction will cause). If $\epsilon = \Theta(\lambda^\delta)$ with $\delta > 3/2$, the term proportional to ϵ^2 is instead one order in λ above the leading order in Eq. (8.46), which is given by the excitation probability $\mathcal{L}_{\text{cc}} = \Theta(\lambda^2)$. We will call this regime the *orthogonal regime*, since in this case $\epsilon \ll \lambda \ll 1$. Finally, if $\epsilon = \Theta(\lambda^\delta)$ for $1/2 < \delta < 3/2$, the leading order term in λ in Eq. (8.46) is $\epsilon^2 + \mathcal{L}_{\text{cc}}$. We will refer to this regime as the *transition regime*. We will study the entanglement harvested for these three different regimes separately.

³Depending on the number of spatial dimensions d , we may need to define a dimensionless coupling strength. In the particular case when $d = 3$, λ is dimensionless, and therefore its magnitude can be directly compared with the inner product $\epsilon = |\langle \varphi_s | g_C \rangle|$, which is also a dimensionless quantity.

⁴Specifically, we say that $f(x) = \Theta(g(x))$ when $x \rightarrow a$ (for $a \in \bar{\mathbb{R}}$) iff there exist positive constants $0 < c_1 < c_2$ such that

$$c_1 < \liminf_{x \rightarrow a} \frac{f(x)}{g(x)} \leq \limsup_{x \rightarrow a} \frac{f(x)}{g(x)} < c_2.$$

This is in contrast with the big O (or big Omicron) notation, $f(x) = \mathcal{O}(g(x))$, which shall be interpreted as $\limsup_{x \rightarrow a} |f(x)/g(x)| < \infty$.

Once both the numerator and the denominator are taken into account in the perturbative expansion, the updated state $\hat{\rho}_{\text{AB}}^s$ has the following structure in matrix form:

$$\hat{\rho}_{\text{AB}}^s = \begin{pmatrix} 1 - \lambda^2 r_{22} - \lambda^2 r_{33} & \Theta(\lambda^b) & \Theta(\lambda^b) & \lambda^2 r_{41}^* \\ \Theta(\lambda^b) & \lambda^2 r_{22} & \lambda^2 r_{32}^* & 0 \\ \Theta(\lambda^b) & \lambda^2 r_{32} & \lambda^2 r_{33} & 0 \\ \lambda^2 r_{41} & 0 & 0 & 0 \end{pmatrix} + \mathcal{O}(\lambda^3), \quad (8.47)$$

where r_{22} and r_{33} are positive and $b \geq 1$. In the case $b \geq 3/2$, which corresponds to both the orthogonal and the non-orthogonal regime, the negativity is given by

$$\mathcal{N}_s = \lambda^2 \max \left(0, \sqrt{|r_{41}|^2 + \frac{(r_{22} - r_{33})^2}{4}} - \frac{r_{22} + r_{33}}{2} \right) + \mathcal{O}(\lambda^3), \quad (8.48)$$

as seen in App. D.3. The cases with $1 \leq b < 3/2$ correspond to the transition regime. In what follows, we will analyze the non-orthogonal and orthogonal regimes in full generality, and the particular case of the transition regime in which $b = 1$, where the negativity admits a second order power expansion in λ .

8.3.1 Non-orthogonal regime

We analyze the case in which the scalar product between the initial state of detector C, $|g_C\rangle$, and the outcome of the measurement, $|\varphi_s\rangle$, are far from being orthogonal. This translates into the assumption $\epsilon = \langle \varphi_s | g_C \rangle = \Theta(\lambda^\delta)$, with $\delta < 1/2$. Under these conditions, the probability of the outcome $|\varphi_s\rangle$ is at leading order ϵ^2 , and since $\lambda \ll 1$, we have that $\epsilon \gg \lambda$. Note that by setting $\delta = 0$ we recover the case in which ϵ is just some finite constant unrelated to the coupling between the field and detector C. Under these conditions, the state updated after the measurement can be expanded as

$$\begin{aligned} \hat{\rho}_{\text{AB}}^s &= \hat{\rho}_{\text{AB},0} + \frac{1}{\epsilon^2} \hat{\gamma} + \hat{\rho}_{\text{AB}}^{(2)} - \frac{\mathcal{L}_{\text{CC}}}{\epsilon^4} \hat{\gamma} + \mathcal{O}(\lambda^3) \\ &= \begin{pmatrix} 1 - \mathcal{L}_{\text{AA}} - \mathcal{L}_{\text{BB}} & \Theta(\lambda^{2-\delta}) & \Theta(\lambda^{2-\delta}) & \mathcal{M}_{\text{AB}}^* \\ \Theta(\lambda^{2-\delta}) & \mathcal{L}_{\text{BB}} & \mathcal{L}_{\text{AB}}^* & 0 \\ \Theta(\lambda^{2-\delta}) & \mathcal{L}_{\text{AB}} & \mathcal{L}_{\text{AA}} & 0 \\ \mathcal{M}_{\text{AB}} & 0 & 0 & 0 \end{pmatrix} + \mathcal{O}(\lambda^3). \end{aligned} \quad (8.49)$$

All the measurement contributions are proportional to the operator $\hat{\gamma}$, and they are contained in the $\Theta(\lambda^{2-\delta})$ entries of the updated final operator. Using the expression given in Eq. (8.48), we find that the negativity for the updated density operator is given by

$$\mathcal{N}_s = \max \left(0, \sqrt{|\mathcal{M}_{AB}|^2 + \frac{(\mathcal{L}_{AA} - \mathcal{L}_{BB})^2}{4}} - \frac{\mathcal{L}_{AA} + \mathcal{L}_{BB}}{2} \right) + \mathcal{O}(\lambda^3), \quad (8.50)$$

which coincides at leading order in λ with the negativity in the absence of measurements, given in Eq. (8.37). Despite the measurement contributing to the joint state of the particle detectors at leading order, the negativity at leading order is fully determined by the interactions of detectors A and B, and therefore independent of the measurement.

Experimental implementations of entanglement harvesting might imply measuring the quantum field during the preparation of the protocol. These results imply that within the non-orthogonal regime it is possible to perform *weak* measurements on the quantum field without significantly affecting the entanglement harvested in an independent protocol. One easy way to guarantee this is to only measure the particle detector used to probe the field in a basis whose elements are not close to being orthogonal to the initial state of such detector (which in this case is its ground state).

8.3.2 Orthogonal regime

We now focus on the case in which the scalar product is small compared to the coupling parameter, i.e. $\epsilon \ll \lambda \ll 1$. Specifically, we consider $\epsilon = \Theta(\lambda^\delta)$ with $\delta > 3/2$. In this regime, the probe's post-measurement state $|\varphi_s\rangle$ is close enough to the excited state so that the probability of obtaining that outcome, $\text{Prob}(s)$, is dominated by the probability of excitation of detector C, $\mathcal{L}_{CC}$. Under these conditions, the joint state of the detectors can be expressed as

$$\begin{aligned} \hat{\rho}_{AB}^s &= \hat{\rho}_{AB,0} + \frac{\hat{\gamma}}{\mathcal{L}_{CC}} + \frac{\hat{\nu}}{\mathcal{L}_{CC}} + \mathcal{O}(\lambda^3) \\ &= \begin{pmatrix} 1 - \tilde{\mathcal{L}}_{AA} - \tilde{\mathcal{L}}_{BB} & \Theta(\lambda^\delta) & \Theta(\lambda^\delta) & \tilde{\mathcal{M}}_{AB}^* \\ \Theta(\lambda^\delta) & \tilde{\mathcal{L}}_{BB} & \tilde{\mathcal{L}}_{AB}^* & 0 \\ \Theta(\lambda^\delta) & \tilde{\mathcal{L}}_{AB} & \tilde{\mathcal{L}}_{AA} & 0 \\ \tilde{\mathcal{M}}_{AB} & 0 & 0 & 0 \end{pmatrix} + \mathcal{O}(\lambda^3). \end{aligned} \quad (8.51)$$

Since $\delta > 3/2$, we can again use Eq. (8.48) to compute the negativity:

$$\mathcal{N}_s = \max \left(0, \sqrt{|\tilde{\mathcal{M}}_{AB}|^2 + \frac{(\tilde{\mathcal{L}}_{AA} - \tilde{\mathcal{L}}_{BB})^2}{4}} - \frac{\tilde{\mathcal{L}}_{AA} + \tilde{\mathcal{L}}_{BB}}{2} \right) + \mathcal{O}(\lambda^3). \quad (8.52)$$

In this case, the measurement performed on C does modify the leading order of the entanglement harvested by A and B. On the one hand, from Eq. (8.35) we see that the measurement enhances the local noise terms by adding positive contributions of the form $(|\mathcal{L}_{IC}|^2 + |\mathcal{M}_{IC}|^2)/\mathcal{L}_{CC}$, for $I \in \{A, B\}$. On the other hand, from Eq. (8.36) we also see that the measurement modifies the non-local correlation term. Since the entanglement harvested is a competition between local and non-local terms [105], it is not clear from the equation above whether the measurement increases or decreases the entanglement harvested by the detectors. In Sec. 8.4, we will restrict ourselves to a specific case study in order to obtain some explicit results.

8.3.3 Transition regime

We now explore the case where $\epsilon = \Theta(\lambda^\delta)$ with $1/2 < \delta < 3/2$. Specifically, we will analyze the case where ϵ and λ are comparable, i.e., when $\delta = 1$. This is convenient because for the remaining range of values of δ , the negativity does not admit a power series up to second order in λ . In the case $\delta = 1$, the final state of the particle detectors can be expanded as

$$\begin{aligned} \hat{\rho}_{AB}^s &= \hat{\rho}_{AB}^{(0)} + \frac{\hat{\gamma} + \hat{\nu} + \epsilon^2 \hat{\rho}_{AB}^{(2)}}{\epsilon^2 + \mathcal{L}_{CC}} + \mathcal{O}(\lambda^3) \\ &= \begin{pmatrix} 1 - \mathcal{L}'_{AA} - \mathcal{L}'_{BB} & \mathcal{L}'_B & \mathcal{L}'_A & \mathcal{M}'_{AB} \\ \mathcal{L}'_B & \mathcal{L}'_{BB} & \mathcal{L}'_{AB} & 0 \\ \mathcal{L}'_A & \mathcal{L}'_{AB} & \mathcal{L}'_{AA} & 0 \\ \mathcal{M}'_{AB} & 0 & 0 & 0 \end{pmatrix} + \mathcal{O}(\lambda^3), \end{aligned} \quad (8.53)$$

with

$$\mathcal{L}'_1 = \frac{\epsilon}{\epsilon^2 + \mathcal{L}_{CC}} \left(e^{-i\xi} \mathcal{M}_{IC}^* + e^{i\xi} \mathcal{L}_{IC}^* \right), \quad (8.54)$$

$$\mathcal{L}'_{IJ} = \mathcal{L}_{IJ} + \frac{\mathcal{L}_{IC} \mathcal{L}_{JC}^* + \mathcal{M}_{IC} \mathcal{M}_{JC}^*}{\epsilon^2 + \mathcal{L}_{CC}}, \quad (8.55)$$

$$\mathcal{M}'_{AB} = \mathcal{M}_{AB} + \frac{\mathcal{L}_{AC} \mathcal{M}_{BC} + \mathcal{L}_{BC} \mathcal{M}_{AC}}{\epsilon^2 + \mathcal{L}_{CC}}, \quad (8.56)$$

for $I, J \in \{A, B\}$. Note that $\mathcal{L}'_I = \Theta(\lambda)$, $\mathcal{L}'_{IJ} = \Theta(\lambda^2)$, and $\mathcal{M}'_{AB} = \Theta(\lambda^2)$. Thus, in this case the matrix $\hat{\gamma}$, which yields the terms $\mathcal{L}'_I$, contributes to the joint state of the detectors at first order in λ . This is in contrast with both the non-orthogonal and orthogonal regimes, for which the contribution of $\hat{\gamma}$ was always of order $\lambda^{3/2}$ or higher, and we saw that it did not affect the entanglement harvested.

Using the matrix form given in Eq. (8.53), and the methods given in Appendix D.3, we can express the negativity of the post-selected state as

$$\mathcal{N}_s = \max \left(0, \frac{\sqrt{\bar{\alpha}^2 + \beta^2 - 2\bar{\beta}\bar{\alpha} + \zeta}}{2} - \frac{\alpha - \beta}{2} \right) + \mathcal{O}(\lambda^3),$$

where we have denoted

$$\alpha = \mathcal{L}'_{AA} + \mathcal{L}'_{BB}, \quad \bar{\alpha} = \mathcal{L}'_{AA} - \mathcal{L}'_{BB}, \quad (8.57)$$

$$\beta = |\mathcal{L}'_A|^2 + |\mathcal{L}'_B|^2, \quad \bar{\beta} = |\mathcal{L}'_A|^2 - |\mathcal{L}'_B|^2, \quad (8.58)$$

$$\zeta = 8 \operatorname{Re}(\mathcal{L}'_A \mathcal{L}'_B \mathcal{M}'_{AB}^*). \quad (8.59)$$

In the next section, we obtain concrete results by examining specific protocols.

8.4 Case study: Inertial pointlike detectors

We have seen that measurements do not affect the entanglement harvested at leading order in λ in the non-orthogonal regime. On the other hand, for the outcomes s for which $\epsilon = \langle \varphi_s | g_C \rangle$ is sufficiently small, i.e., in the orthogonal and the transition regimes, the measurement has a more disruptive effect on the joint state of detectors A and B. As seen in Eqs. (8.52) and (8.57), in these regimes the measurement affects the entanglement harvested at leading order. In this section, we study an explicit protocol with a massless field in the simplest spacetime background and detector trajectories. Namely, we will consider inertial comoving pointlike detectors in a (1+3)-dimensional Minkowski spacetime with Gaussian switchings.

We consider that the three particle detectors involved in the protocol are at rest in the inertial reference frame $(t, \mathbf{x})$, and therefore their proper times coincide with the coordinate time t . Their trajectories can be parametrized as $\mathbf{z}_I(t) = (t, \mathbf{x}_I)$, where $\mathbf{x}_I$ is constant and $I \in \{A, B, C\}$, and their energy gaps are all the same, $\Omega_A = \Omega_B = \Omega_C \equiv \Omega$. For the pointlike case, the spacetime smearing can be written as

$$\Lambda_I(t, \mathbf{x}) = \chi_I(t) \delta^{(3)}(\mathbf{x} - \mathbf{x}_I), \quad \text{with} \quad \chi_I(t) = \frac{1}{\sqrt{2\pi}} \exp \left(-\frac{(t - t_I)^2}{2T^2} \right), \quad (8.60)$$

where T determines the effective duration and t_i the peaks of the interaction of each detector with the field. With these choices, the interactions of the three detectors are identical in shape, and localized around different points of spacetime $(t_i, \mathbf{x}_i)$.

Although the Gaussian switching function for detector C is non-zero for all $t \in \mathbb{R}$,

$$\int_{t_c+5T}^{+\infty} dt \chi_c(t) < 10^{-6} \int_{-\infty}^{+\infty} dt \chi_c(t) . \quad (8.61)$$

Thus, if we assume that the measurement on detector C is performed at $t = t_0 \geq t_c + 5T$, we can consider that its interaction is effectively decoupled from the field at the time at which the measurement is performed.⁵ This approximation allows us to use all the calculations⁶ given in Section 8.3.

Recall that, for all regimes, the negativity is a function of the terms $\mathcal{L}_{IJ}$ and $\mathcal{M}_{IJ}$ given in Eqs. (8.35) and (8.36). For the specific protocol presented in this Section, the $\mathcal{L}_{IJ}$ terms can be given in closed-form. Due to the translational symmetry of Minkowski spacetime, all the local terms are equal, $\mathcal{L}_{AA} = \mathcal{L}_{BB} = \mathcal{L}_{CC} \equiv \mathcal{L}$. The single momentum integral can be solved explicitly,

$$\mathcal{L} = \frac{\lambda^2 T^2}{2\pi^2} \int_0^\infty \frac{dk}{2} k e^{-(\Omega+k)^2 T^2} = \frac{e^{-\Omega^2 T^2} - \sqrt{\pi} \Omega T \operatorname{erfc}(\Omega T)}{8\pi^2} . \quad (8.63)$$

The non-local terms $\mathcal{L}_{IJ}$, for $I \neq J$, can also be cast as a single momentum integral which

⁵This kind of approximation is common in relativistic quantum information and has been studied rigorously in the context of interaction regions that are effectively spacelike separated, even when the supports of the detectors' switching and smearing functions are not compact (see, e.g., [110, 250]).

⁶In more detail, in this case we should apply the operator $\hat{U}(+\infty, t_0)|\varphi_s\rangle\langle\varphi_s|\hat{U}(t_0, -\infty)$ to obtain the final state, instead of $|\varphi_s\rangle\langle\varphi_s|\hat{U}$, where

$$\hat{U}(t_2, t_1) = \mathcal{T} \exp \left(-i \int dV \hat{h}(\mathbf{x}) \theta(t - t_1) \theta(t_2 - t) \right) . \quad (8.62)$$

However, if Clara's detector is effectively switched off at $t > t_0 > t_c + 45$, we can assume that the time evolution corresponding to detector C contained in $\hat{U}(+\infty, t_0)$ is negligible. Since the time evolution associated with detectors A and B commutes with $|\varphi_s\rangle\langle\varphi_s|$ (they act on different Hilbert spaces), we have $\hat{U}(+\infty, t_0)|\varphi_s\rangle\langle\varphi_s|\hat{U}(t_0, -\infty) \simeq |\varphi_s\rangle\langle\varphi_s|\hat{U}$.

admits a closed form

$$\mathcal{L}_{\text{IJ}} = \frac{\lambda^2 T^2}{2\pi^2} \int_0^\infty \frac{dk}{2} k e^{-(\Omega+k)^2 T^2 + i(\Omega+k)\Delta_{\text{IJ}}} \text{sinc}(kL_{\text{IJ}}) \quad (8.64)$$

$$= \frac{\lambda^2}{16\pi^{3/2}} \frac{T}{L_{\text{IJ}}} e^{-\frac{L_{\text{IJ}}^2}{4T} - \frac{\Delta_{\text{IJ}}^2}{T^2}} \left\{ e^{\frac{L_{\text{IJ}}\Delta_{\text{IJ}}}{2T^2} + i\Omega L_{\text{IJ}}} \left[i + \text{erfi} \left(\frac{L_{\text{IJ}} - \Delta_{\text{IJ}}}{T^2} - i\Omega T \right) \right] \right. \\ \left. + e^{-\frac{L_{\text{IJ}}\Delta_{\text{IJ}}}{2T^2} - i\Omega L_{\text{IJ}}} \left[-i + \text{erfi} \left(\frac{L_{\text{IJ}} + \Delta_{\text{IJ}}}{T^2} + i\Omega T \right) \right] \right\}, \quad (8.65)$$

where $L_{\text{IJ}} = |\mathbf{x}_{\text{I}} - \mathbf{x}_{\text{J}}|$ is the distance between detectors I and J, and $\Delta_{\text{IJ}} = t_{\text{I}} - t_{\text{J}}$ is the delay between the peaks of their switching functions. The $\mathcal{M}_{\text{IJ}}$ terms can also be written as a momentum integral, but they do not admit a closed form:

$$\mathcal{M}_{\text{IJ}} = -\frac{\lambda^2 T^2}{4\pi^2} e^{i\Omega(t_{\text{I}}+t_{\text{J}})} \int_0^{+\infty} \frac{dk}{2} k e^{-(\Omega^2+k^2)T^2} \text{sinc}(kL_{\text{IJ}}) \\ \times \left\{ e^{i\Omega\Delta_{\text{IJ}}} \left[1 - \text{erf} \left(\frac{\Delta_{\text{IJ}}}{2T} + ik \right) \right] + e^{-i\Omega\Delta_{\text{IJ}}} \left[1 - \text{erf} \left(\frac{-\Delta_{\text{IJ}}}{2T} + ik \right) \right] \right\}. \quad (8.66)$$

Nevertheless, these terms can be easily evaluated numerically, which will allow us to analyze the effect of the measurement in the orthogonal and the transition regimes.

8.4.1 Orthogonal regime

In this subsection, we consider several entanglement harvesting scenarios, and we study the effect that the measurement has on the entanglement harvested when we are in the orthogonal regime. Thus, in these scenarios the negativity is given by Eq. (8.52).

First, we consider a protocol in which Alba and Blanca harvest spacelike entanglement from the quantum field. In this setup, their detectors are switched on at the same coordinate time, that is $\Delta_{\text{AB}} = 0$, and we take the proper distance between them to be L_{AB} . We take detector C to be located at the middle point between detectors A and B, setting $L_{\text{AC}} = L_{\text{BC}} = L_{\text{AB}}/2$ (see Fig. 8.1).

If we fix $L_{\text{AB}} = 5T$, the interaction regions of the detectors A and B can be considered to be effectively spacelike separated for entanglement harvesting purposes. Namely, it was shown in [250] that for this specific setup, communication between the detectors has a negligible effect in the entanglement that they harvest (see also [110]). The negativity of $\hat{\rho}_{\text{AB}}^s$ in this configuration is plotted in Fig. 8.2 as a function of the delay $\Delta_{\text{AC}} = \Delta_{\text{BC}}$, for different

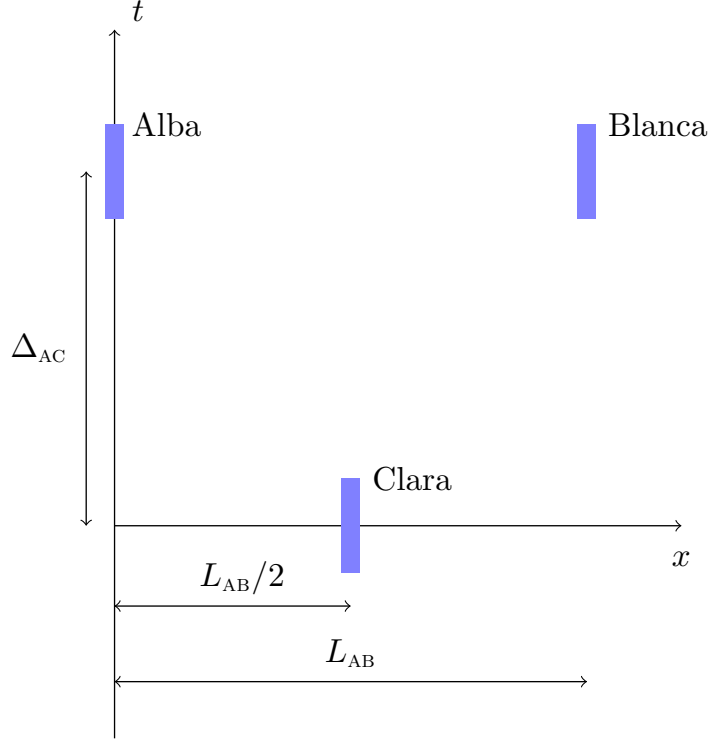


Figure 8.1: Configuration in a slice of spacetime of the interaction regions (in blue) of the three detectors involved in the first example protocol, i.e., with parameters $L_{AC} = L_{BC} = L_{AB}/2 = 2.5T$ and $\Delta_{AB} = 0$.

energy gaps. We consider positive values for the delay, so that this configuration can be understood, e.g., as a simplified model for a spacelike entanglement harvesting protocol in which a measurement is performed during the preparation of the experiment. For comparison, we also plotted with dashed lines the negativity of $\hat{\rho}_{AB}$, which corresponds to the entanglement harvested with the same protocol for Alba and Blanca, but without Clara's intervention, i.e., without measurements. Several values of Ω were chosen to see that the effect of the measurement does not qualitatively change as a function of the energy gap. As a baseline for comparison, we used the value of Ω for which the maximum of the negativity of $\hat{\rho}_{AB}$ is reached, which was found in [250] to be approximately $\Omega_{\max} \approx L_{AB}/2T^2 = 2.5/T$. For each value of ΩT , the negativity of $\hat{\rho}_{AB}^s$ is always below the negativity of $\hat{\rho}_{AB}$. Therefore, the measurement decreases the entanglement harvested. We observe that if the delay between the switching function for C and the switching functions for A and B is small

enough, the measurement destroys the entanglement acquired by A and B up to second order in λ for every examined case. The negativity of $\hat{\rho}_{AB}^s$ increases with Δ_{AC} . For large values of Δ_{AC} , the negativity of $\hat{\rho}_{AB}^s$ tends asymptotically to the negativity of $\hat{\rho}_{AB}$. This is because the field correlations of the interaction region of C with the interaction regions of A and B decay as the delay between them increases. Indeed, as $\Delta_{AC} = \Delta_{BC}$ increases, the non-local terms $\mathcal{L}_{IC}$ and $\mathcal{M}_{IC}$ involving detector C decrease, and we have that $\tilde{\mathcal{L}}_{IJ}$ and $\tilde{\mathcal{M}}_{IJ}$ tend to $\mathcal{L}_{IJ}$ and $\mathcal{M}_{IJ}$, respectively (see Eqs. (8.44) and (8.45)). Therefore, the effect of the measurement on the entanglement harvested becomes negligible for large enough delays.

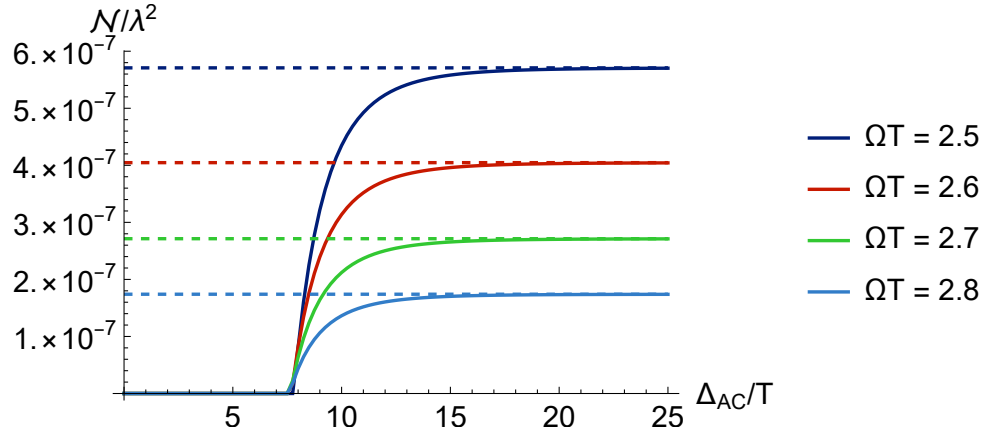


Figure 8.2: (Solid lines) Negativity in the orthogonal regime of $\hat{\rho}_{AB}^s$ as a function of the delay Δ_{AC} for different values of the energy gap ΩT , and parameters $L_{AC} = L_{BC} = L_{AB}/2 = 2.5T$ and $\Delta_{AB} = 0$. This plot corresponds to the scheme in Fig. 8.1. (Dashed lines) Negativity of $\hat{\rho}_{AB}$ for the same values of L_{AB} and Δ_{AB} , with ΩT matching the value indicated by the colour of the corresponding solid line.

The reader might be surprised by the fact that for configurations in which A, B and C are supposed to be spacelike separated, the measurement performed by Clara on detector C can end up not only affecting the entanglement harvested by A and B, but completely destroying it. One can make sense of this taking into account that we are analyzing the state $\hat{\rho}_{AB}^s$, which already assumes a selective update. This, in particular, means that we are analyzing the scenario in which A and B measure their correlations in a ‘processing region’ (cf. Ch. 4; see also [150]) that lies in the future of Clara’s measurement, and therefore has access to its outcome. Since detector C is correlated with the field, which is correlated with detectors A and B, the corresponding update does indeed affect the joint state of A and B, even if both the interaction and the measurement of C are spacelike separated from

the interactions of detectors A and B with the field.

Note that with the choice of parameters analyzed in Fig. 8.2, detectors A and B interact with the field in effectively spacelike separated regions. However, due to its position, C is in effective causal contact with both A and B. Because of this, in this case one might wonder then if the leading order effect of the measurement is due to the interaction of the detector with the field (which then propagates to the effective interaction regions of detectors A and B), rather than to the projective measurement performed on the detector and the field correlations. In order to clarify this, we analyzed the behaviour of the negativity for the same configuration depicted in Fig. 8.1, but with a larger separation between the particle detectors: $L_{AC} = L_{BC} = L_{AB}/2 = 5T$. The results are plotted in Fig. 8.3. For this choice of parameters, we can consider that as $\Delta_{AC} = \Delta_{BC}$ approaches zero, the interaction region of detector C is effectively spacelike separated from those of detectors A and B. We then observe that even when the three interactions are effectively causally disconnected, the measurement does destroy the entanglement harvested. We conclude that the three detectors get correlated due to their interactions with the field, and it is indeed Clara's measurement what sabotages the entanglement between A and B.

Notice that the fact that the selective measurement on C affects A and B does not mean that it can be used to send information from C to A and B. Indeed, the reason the measurement on C affects the joint state of the detectors A and B is that we are post-selecting the state to having obtained a particular outcome of the measurement of C. Thus, just as we discussed in previous chapters, the effect we are witnessing when a spacelike measurement on C affects A and B is not one of faster-than-light signalling, but one of correlations between the state of C (and therefore the outcome of the measurement performed on it) and the joint state of A and B.

We now analyze the effect of the measurement in a timelike entanglement harvesting protocol. Notice that due to the strong Huygens principle, when the detectors are in strictly timelike separation, the entanglement acquired between them is not due to communication but rather is genuine harvesting of the field correlations (see, e.g., [110]). For simplicity, we set $L_{AB} = L_{AC} = L_{BC} = 0$, and we keep the delay between the switching functions of Alba and Blanca constant at $\Delta_{BA} > 0$. For this setup, we vary the delay $\Delta_{AC} > 0$, as depicted in Fig. 8.4.

The negativity of $\hat{\rho}_{AB}^s$ in the orthogonal regime for this configuration is plotted in Fig. 8.5 as a function of the delay $\Delta_{AC} > 0$, for different values of the energy gap. This configuration could model, e.g., a timelike entanglement harvesting protocol for which a measurement has been performed during its preparation. We also plot the negativity of $\hat{\rho}_{AB}$ corresponding to the protocol when Clara does not carry out any measurement. We observe a similar

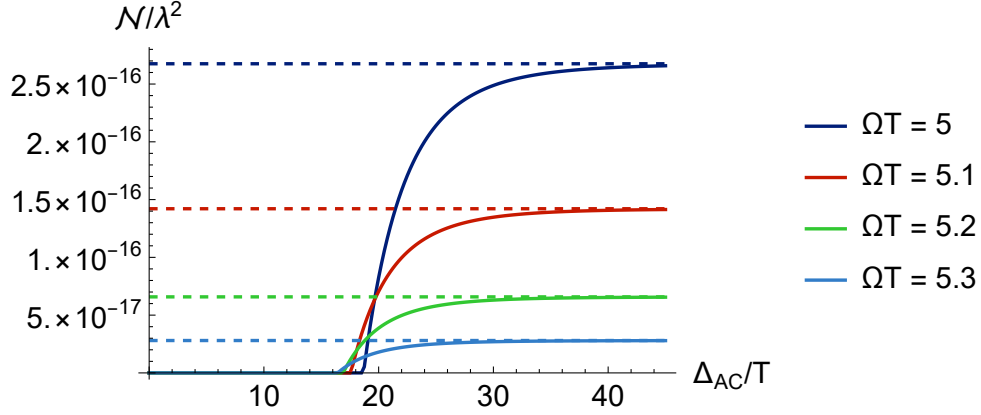


Figure 8.3: (Solid lines) Negativity in the orthogonal regime of $\hat{\rho}_{AB}^s$ as a function of the delay Δ_{AC} for different values of the energy gap ΩT , and parameters $L_{AC} = L_{BC} = L_{AB}/2 = 5T$ and $\Delta_{AB} = 0$. This plot corresponds to the scheme in Fig. 8.1. (Dashed lines) Negativity of $\hat{\rho}_{AB}$ for the same values of L_{AB} and Δ_{AB} , with ΩT matching the value indicated by the colour of the corresponding solid line.

behaviour to the spacelike entanglement harvesting protocol previously analyzed. In particular, the measurement destroys the entanglement harvested at second order by Alba and Blanca for small enough Δ_{AC} . As Δ_{AC} increases, the detectors start harvesting entanglement, and the negativity of $\hat{\rho}_{AB}^s$ tends to that of $\hat{\rho}_{AB}$ for large values of Δ_{AC} , for the same reason as in the case of the spacelike harvesting configuration in Fig. 8.2.

Finally, we also analyze the case in which detector C interacts with the field at a time between the interactions of A and B, as shown in Fig. 8.6. We fixed $\Delta_{BA} = 5T$. For all the values of Δ_{CA} and for all the different energy gaps studied, we find that the measurement cancels the entanglement between detectors A and B up to second order.

Overall, in all the configurations studied we have associated the measurement in the orthogonal regime with a decrease or even a complete cancellation of the entanglement harvested. However, if the delay between the switching functions of detector C and the rest of the detectors is large enough, the effect of the measurement is suppressed. This is because the field correlations between the interaction region of C and the region where harvesting is performed decay with the spacetime separation between them, so that the selective measurement reveals increasingly less information about the field at the harvesting region.

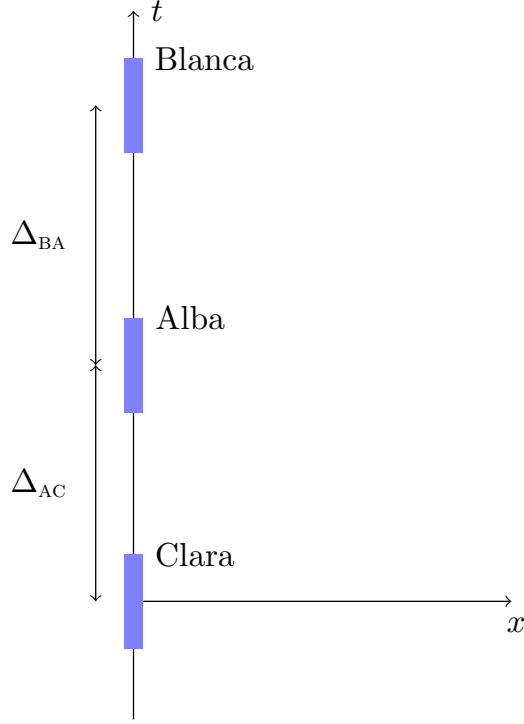


Figure 8.4: Configuration in a slice of spacetime of the interaction regions (in blue) of the three detectors involved in the second example protocol, i.e., with parameters $L_{AB} = L_{AC} = L_{BC} = 0$ and $\Delta_{BA}, \Delta_{AC} > 0$.

8.4.2 Transition regime

In this subsection, we analyze the effect that the measurement has on the entanglement harvested when we are in the transition regime, for the same configurations studied in the previous subsection. Thus, in these scenarios negativity is given by Eq. (8.57).

As discussed in Sec. 8.3, the matrix $\hat{\gamma}$ only yields a leading order contribution to the negativity in the transition regime. We will see that this contribution yields a more complex behaviour than both the non-orthogonal regime, in which the negativity is not affected at leading order, and the orthogonal regime, in which the measurement decreases the negativity. In particular, in this case the entanglement harvested can be increased by a third party performing field measurements.

Recall that in the transition regime we analyzed the case $\epsilon = \Theta(\lambda)$, and note that in

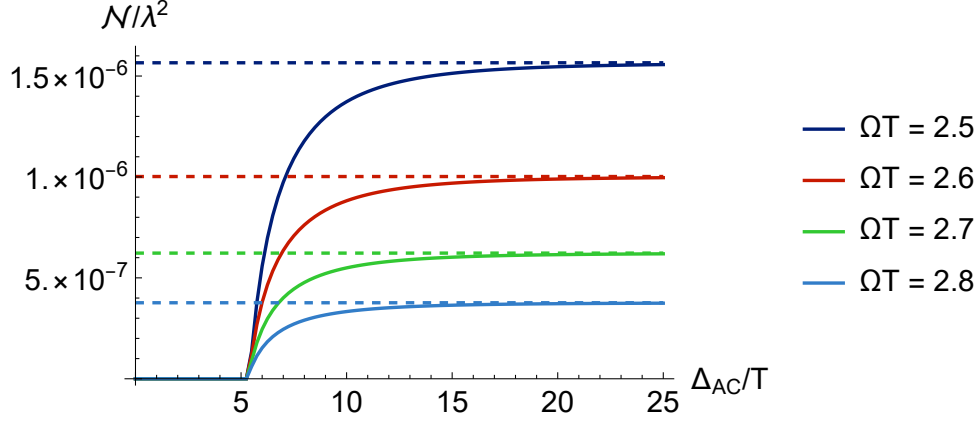


Figure 8.5: (Solid lines) Negativity in the orthogonal regime of $\hat{\rho}_{AB}^s$ as a function of the delay $\Delta_{AC} > 0$ for different values of the energy gap ΩT , and parameters $L_{AC} = L_{BC} = L_{AB} = 0$ and $\Delta_{BA} = 5T$. This plot corresponds to the scheme in Fig. 8.4. (Dashed lines) Negativity of $\hat{\rho}_{AB}$ for the same values of L_{AB} and Δ_{AB} , with ΩT matching the value indicated by the colour of the corresponding solid line.

this case the specific value of ϵ affects $\hat{\rho}_{AB}^s$ at leading order (see Eqs. (8.53)–(8.56)). For simplicity, in what follows we take $\epsilon = \lambda$.

We first analyze the contribution of the measurement in a scenario of spacelike entanglement harvesting (the situation on Fig. 8.1). As before, we set $L_{AC} = L_{BC} = L_{AB}/2 = 2.5T$ and $\Delta_{AB} = 0$. The negativity for this configuration is plotted in Fig. 8.7 as a function of the delay between the switching function of detectors A and C, $\Delta_{AC} > 0$, for different values of the phase ξ (see Eq. (8.39)) and fixed energy gap $\Omega T = 2.5$. We also plot the negativity of $\hat{\rho}_{AB}$ for the same configuration. The measurement introduces small damped oscillations with the parameter Δ_{AC} that converge for large Δ_{AC} to the negativity of $\hat{\rho}_{AB}$, with the phase of the oscillations controlled by ξ . It is worth pointing out that the relative difference between the negativity of $\hat{\rho}_{AB}^s$ and the negativity of $\hat{\rho}_{AB}$ is never larger than a 0.1%. This may however be related to the fact that we are working in a weak-measurement regime, and therefore using a stronger coupling between detector C and the field might yield a higher relative enhancement.

In any case, we see that the effect is small. In fact, we performed the analysis for a different set of parameters, namely, $\Delta_{AB} = 0$ and $L_{AC} = L_{BC} = L_{AB}/2 = 5T$, with a fixed energy gap $\Omega = L_{AB}/2T^2 = 5/T$, observing that in this case it is not possible to distinguish numerically the negativity of $\hat{\rho}_{AB}$ and the negativity of $\hat{\rho}_{AB}^s$. This is related to the fact

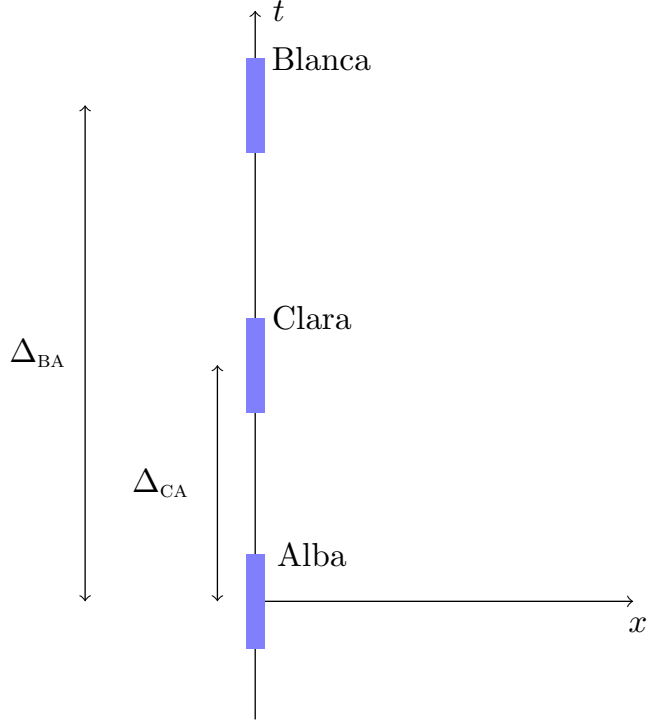


Figure 8.6: Configuration in a slice of spacetime of the interaction regions (in blue) of the three detectors involved in the third example protocol, i.e., with parameters $L_{AC} = L_{BC} = L_{AB} = 0$ and $\Delta_{BA} > \Delta_{CA} > 0$.

that the terms $\mathcal{L}_{IJ}$ and $\mathcal{M}_{IJ}$ decay with the energy gap. As seen in Eqs. (8.54)–(8.57), the measurement contributions to the negativity are given by products of the form $\mathcal{L}_{IC}\mathcal{L}_{JC}^*$, $\mathcal{M}_{IC}\mathcal{M}_{JC}^*$, and $\mathcal{L}_{IC}\mathcal{M}_{JC}$. Meanwhile, the contributions due to the entanglement harvesting protocol performed by A and B, which are present both with and without measurements, are given in terms of $\mathcal{L}_{IJ}$ and $\mathcal{M}_{AB}$. As a result, the measurement contributions decay faster with the energy gap. Since we have considered a value of the gap that is double⁷ the value used for the analysis in Fig. 8.7, the effect of the measurement is suppressed. This is in contrast with the behaviour displayed in the orthogonal regime, when the energy gap does not modify the relative magnitude of the effect of the measurement in the negativity of $\hat{\rho}_{AB}^s$ with respect to $\hat{\rho}_{AB}$.

⁷This is because for larger separation distances one needs to consider larger energy gaps to harvest entanglement [105, 106].

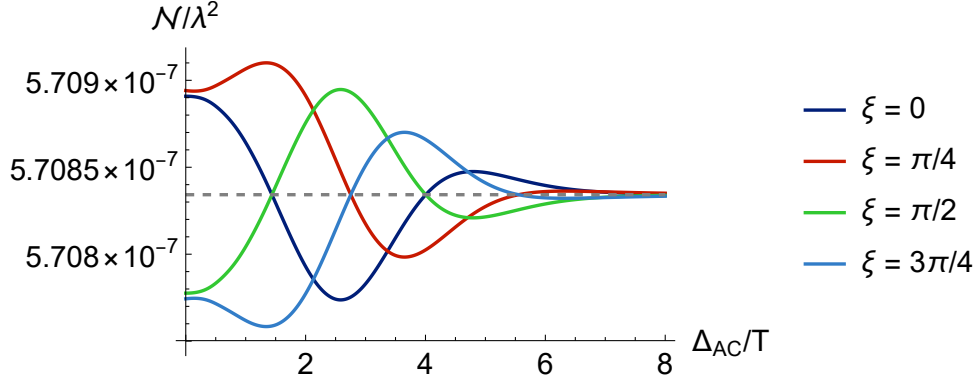


Figure 8.7: (Solid lines) Negativity in the transition regime of $\hat{\rho}_{AB}^s$ as a function of Δ_{AC} for different values of ξ , and $\Omega T = 2.5$, $L_{AC} = L_{BC} = L_{AB}/2 = 2.5T$ and $\Delta_{AB} = 0$. This plot corresponds to the scheme in Fig. 8.1. (Dashed line) Negativity of $\hat{\rho}_{AB}$ for the same values of L_{AB} , Δ_{AB} and ΩT .

We now analyze the effect of the measurement in a timelike harvesting protocol (see Fig. 8.4). As for the orthogonal regime, we set $L_{AB} = L_{AC} = L_{BC} = 0$ and $\Delta_{BA} = 5T$. The negativity of $\hat{\rho}_{AB}$ is plotted in Fig. 8.8 as a function of $\Delta_{AC} > 0$ for different values of ξ and for $\Omega T = 2.5$. As for the spacelike protocol, we observe damped oscillations that converge to the negativity of $\hat{\rho}_{AB}$ for large values of the delay Δ_{AC} . The phase of the oscillations is determined by the parameter ξ .

Finally, Fig. 8.9 shows the analysis for a timelike protocol with $0 < \Delta_{CA} < \Delta_{BA} = 5T$ (see scheme in Fig. 8.6). A similar oscillatory behaviour of the negativity with Δ_{CA} is observed. The amplitude of the oscillations is maximized if the delays Δ_{CA} and Δ_{BC} are equal.

Overall, we have associated the effect of the measurement in the transition regime with small modifications of the negativity at leading order, which are suppressed with the energy gap. The measurement introduces oscillations with the delay between the switching function of detector C and the switching functions of detectors A and B.

8.5 Summary

In this chapter, we have explored how entanglement harvesting is affected by the performance of selective and non-selective measurements of the quantum field—as modelled in

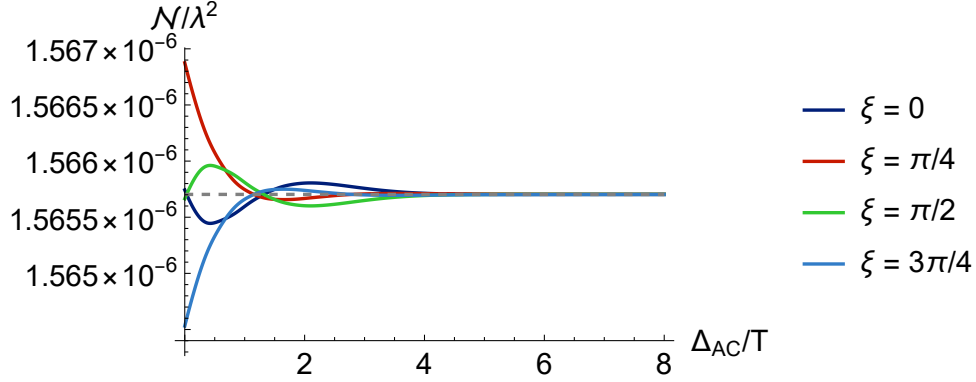


Figure 8.8: (Solid lines) Negativity in the transition regime of $\hat{\rho}_{AB}^s$ as a function of Δ_{AC} for different values of ξ , and $\Omega T = 2.5$, $L_{AC} = L_{BC} = L_{AB} = 0$ and $\Delta_{BA} = 5T$. This plot corresponds to the scheme in Fig. 8.4. (Dashed line) Negativity of $\hat{\rho}_{AB}$ for the same values of L_{AB} , Δ_{BA} and ΩT .

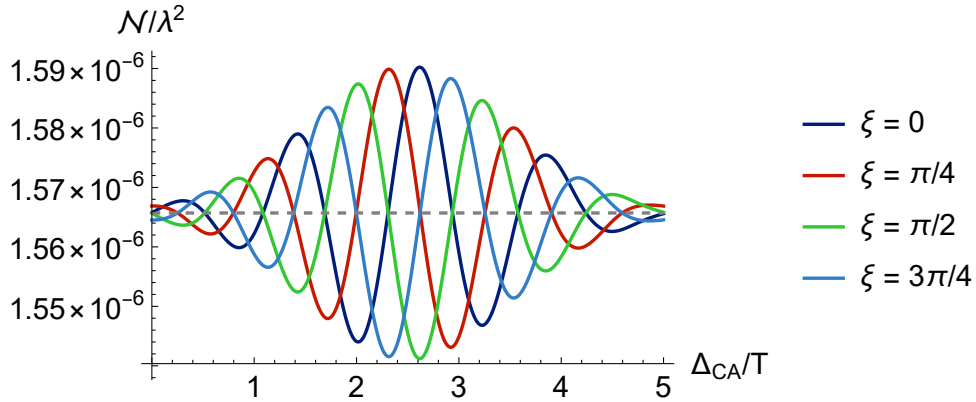


Figure 8.9: (Solid lines) Negativity in the transition regime of $\hat{\rho}_{AB}^s$ as a function of the delay $0 < \Delta_{CA} < \Delta_{BA}$ for different values of the phase ξ , and parameters $\Omega T = 2.5$, $L_{AC} = L_{BC} = L_{AB} = 0$ and $\Delta_{BA} = 5T$. This plot corresponds to the scheme in Fig. 8.6. (Dashed line) Negativity of $\hat{\rho}_{AB}$ for the same values of L_{AB} , Δ_{BA} and ΩT .

Ch. 5, i.e., using particle detectors. For weakly coupled detectors with identical coupling strengths, we found that the effect of the measurement at leading order is strongly dependent on the relation between the coupling parameter of the detector used to measure

the field and the outcome of the measurement. Specifically, we identified three different regimes depending on the relation between the coupling strength λ and the modulus of the scalar product between the post-measurement state and the initial state of the probe used to measure the field, ϵ , which is a measure of how (non-)orthogonal these two states are to each other.

For $\epsilon \gg \lambda$ (*non-orthogonal regime*), we found that the measurement does not affect the entanglement harvested at leading order—even though it does affect the joint state of the particle detectors harvesting entanglement up to second order in λ , which is the leading order of the negativity. For $\epsilon \ll \lambda$ (*orthogonal regime*), we studied numerically specific setups, finding that the measurement acts as a source of noise that sabotages the entanglement harvested. For the same setups, we saw that if $\epsilon \sim \lambda$ (*transition regime*), the measurement has a small effect that can in principle yield small enhancements of the entanglement harvested for an adequate choice of parameters of the protocol. We noted that the effects of the measurement are suppressed as the time interval between the measurement and the entanglement harvesting protocol increases. Finally, we studied the effect of non-selective measurements, to find that they do not affect the entanglement harvested at leading order. This makes it clear that the disruptive effects of the measurement on the entanglement harvested observed in the orthogonal and transition regimes are due to post-selection.

Since our study reveals that the measurement protocol affects the results of entanglement harvesting, it is arguable that the process of state preparation and measurement may be important in an experimental implementation of the protocol. As such, analyses like this may be relevant for future experimental explorations of this relativistic quantum information protocol.

Finale

Chapter 9

Summary and future directions

In this final chapter, we summarize the main findings and takeaways of the thesis (“what we left behind”), and offer some thoughts about follow-up questions to be tackled in the near future (“what lies ahead of us”).

Act 1 of the thesis was concerned with addressing the conflict between relativistic causality and both selective and non-selective measurements in quantum field theory.

- In Ch. 3, we used an algebraic approach to define and characterize *localized causal channels* in real scalar quantum field theory, even providing a characterization of their Kraus operators. The immediate next steps include extending the results to more general QFTs. In the longer term, the ultimate goal is to achieve a complete classification of all possible localized causal maps. To this end, it may be beneficial to refine the characterization of Kraus operators into a more explicit, easily verifiable form.
- In Ch. 4, we argued that, in relativistic setups, updates following measurements should always be applied along the future light cone of the measurement event rather than the past, as has been suggested before. Updating along the past light cone amounts to a retroactive update, which conflicts with the principle that measurement updates should respect the causal structure of the theory—both in terms of the perturbation induced by the measurement and the regions of spacetime where information about the outcome is available. To ensure that the update rule properly accounts for correlations between systems, we propose a minimal extension of the tensor product formalism that explicitly distinguishes between local (or *individual*) observables and non-local (or *joint*) observables. This formalism was shown to have

a clear statistical interpretation and was put into correspondence with an observer-dependent framework that naturally extends to QFT. A key next step is to apply this formalism to analyze the causal behaviour of selective measurements, following the approach taken in Ch. 3 to analyze the causal behaviour of channels.

- In Ch. 5, we introduced a class of measurement schemes for quantum fields where the probe is an Unruh-DeWitt particle detector. We used the results of Ch. 3 to study the causal behaviour of these schemes. In the general case, we analyzed the problem perturbatively and confirmed that the measurement channels induced by particle detectors are not causal in general, although their deviations from causality do not manifest at leading order. We also studied the problem non-perturbatively in the cases of gapless and delta-coupled particle detectors, which we showed are causal and provide a way to measure the statistics of smeared fields. Finally, we prescribed an update rule for this class of schemes, providing the final ingredient needed to establish a complete measurement theory for quantum fields using localized non-relativistic quantum systems as measuring tools. Short-term follow-ups include generalizing the measurement theory to other quantum fields and other particle detector models. A potentially interesting next step is to study the causal behaviour of schemes involving detectors undergoing multiple delta couplings, as well as gapless and delta-coupled detectors when selective measurements are included in addition to non-selective ones.

The intermezzo of the thesis introduced a non-perturbative method to approximate the dynamics of particle detectors, which are generally unsolvable analytically. The core idea was straightforward: to approximate continuous interactions by trains of delta couplings, much like how a Riemann sum approximates a Riemann integral. One of the main advantages it poses is that, unlike perturbative methods, which in general do not guarantee a specific rate of convergence towards the exact result as more terms are included, this approximation method guarantees a weak convergence rate of $1/N$, where N is the number of deltas used in the approximation. Regarding follow-ups, the next immediate step is to apply this approximation method to a relevant subset of quantum information protocols involving particle detectors in RQI, examining whether significant deviations from perturbative predictions arise. Likewise, along the same lines of testing the robustness of the perturbative approach, in an ongoing collaborative project we are analyzing the convergence of the perturbative series for particle detectors. While we have shown that it converges absolutely in some relevant cases, the next steps include generalizing the result and investigating the rate of convergence of the series.

Act 2 of the thesis was concerned with applications of the detector-based measurement theory developed in Ch. 5.

- In Ch. 7, we introduced a measurement strategy that leverages the simplicity of particle detectors to design a straightforward, one-size-fits-all measurement protocol. The detector undergoes a tomography process, and the results are fed to a neural network that has been pre-trained to infer the feature of interest from the detector's readings. This strategy provides a systematic approach to measuring specific features of quantum fields with particle detectors by shifting the complexity of the experimental design from the measurement protocol to the data analysis, which is handled by the neural network. We demonstrated the success of this approach by applying it to three examples involving a field in a cavity in equilibrium with its background. The neural network successfully determined the location of the cavity's boundary, the temperature of the field, and distinguished between two global states of the field. The success of these techniques across diverse examples suggests that this strategy is broadly applicable, paving the way for the use of machine learning techniques in more complex scenarios, such as distinguishing gravitational backgrounds, and detecting entanglement in analogue Hawking radiation. This includes an experimental proposal currently in preparation, aimed at providing direct evidence of the yet untested Unruh effect.
- In Ch. 8, we analyzed the effect of selective measurements on the entanglement harvested by two detectors from the Minkowski vacuum. We found that in many regimes, selective measurements destroy the harvested entanglement, though, somewhat unexpectedly, there appears to be a small window where it is slightly enhanced. Future work includes further exploration of applications of the measurement theory, such as developing a relativistic framework for quantum optics and deriving relativistic corrections to standard quantum optics predictions.

It is worth remarking that the detector-based approach that is central to this thesis has become one of the two main models of measurement in QFT, as noted in [251, 252], the other one being precisely the Fewster-Verch framework that we have referred to on multiple occasions. It seems clear nevertheless that both models are deeply related, and bridging (or even merging) them seems to be a promising line of research that will hopefully be pursued by both the AQFT and the RQI communities in the following years. A first step was taken in [9], where it was shown that particle detectors can be understood as individual modes of QFTs localized by external potentials. Moreover, trying to extend this analysis to consider potentials that depend on quantum degrees of freedom could even be an entrance gate to address the open problem of describing bounded systems fully within QFT, which would ultimately close the gap between both frameworks once and for all.

References

- [1] J. Polo-Gómez, L. J. Garay, and E. Martín-Martínez, A detector-based measurement theory for quantum field theory, [Phys. Rev. D **105**, 065003 \(2022\)](#).
- [2] J. Polo-Gómez and E. Martín-Martínez, Nonperturbative method for particle detectors with continuous interactions, [Phys. Rev. D **109**, 045014 \(2024\)](#).
- [3] D. Grimmer, I. Melgarejo-Lermas, J. Polo-Gómez, and E. Martín-Martínez, Decoding quantum field theory with machine learning, [J. High Energy Phys. **2023**, 31 \(2023\)](#).
- [4] H. Maeso-García, J. Polo-Gómez, and E. Martín-Martínez, Entanglement harvesting: State dependence and covariance, [Phys. Rev. D **106**, 105001 \(2022\)](#).
- [5] H. Maeso-García, J. Polo-Gómez, and E. Martín-Martínez, How measuring a quantum field affects entanglement harvesting, [Phys. Rev. D **107**, 045011 \(2023\)](#).
- [6] J. Polo-Gómez, Localized causal channels in real scalar QFT revisited, (forthcoming).
- [7] J. Polo-Gómez, T. R. Perche, and E. Martín-Martínez, Useful qubits in relativistic quantum information, (forthcoming).
- [8] J. Polo-Gómez, L. J. Garay, and E. Martín-Martínez, A detector-based measurement theory for QFT in curved spacetimes, (forthcoming).
- [9] T. R. Perche, J. Polo-Gómez, B. d. S. L. Torres, and E. Martín-Martínez, Particle detectors from localized quantum field theories, [Phys. Rev. D **109**, 045013 \(2024\)](#).
- [10] B. de S. L. Torres, K. Wurtz, J. Polo-Gómez, and E. Martín-Martínez, Entanglement structure of quantum fields through local probes, [J. High Energy Phys. **2023**, 58 \(2023\)](#).

- [11] J. Polo-Gómez, Thermodynamic bound on quantum state discrimination, [Phys. Rev. E](#) **109**, 014119 (2024).
- [12] T. R. Perche, J. Polo-Gómez, B. d. S. L. Torres, and E. Martín-Martínez, Fully relativistic entanglement harvesting, [Phys. Rev. D](#) **109**, 045018 (2024).
- [13] Y. Li, J. Polo-Gómez, and E. Martín-Martínez, A proposal to characterize and quantify superoscillations, [J. Phys. A Math. Theor.](#) **57**, 135202 (2024).
- [14] A. Álvarez-Domínguez, L. J. Garay, E. Martín-Martínez, and J. Polo-Gómez, No black holes from light, [Phys. Rev. Lett.](#) **133**, 041401 (2024).
- [15] Álvaro Álvarez Domínguez, L. J. Garay, E. Martín-Martínez, and J. Polo-Gómez, Reply to Comment on "No Black Holes from Light" [arXiv:2408.06714], (2024), [arXiv:2408.11097 \[gr-qc\]](#).
- [16] I. Agullo, B. Bonga, E. Martín-Martínez, S. Nadal-Gisbert, T. R. Perche, J. Polo-Gómez, P. Ribes-Metidieri, and B. d. S. L. Torres, Multimode nature of spacetime entanglement in QFT, [Phys. Rev. D](#) **111**, 085013 (2025).
- [17] J. Arrechea, C. Barceló, G. García-Moreno, and J. Polo-Gómez, Beyond Buchdahl's limit: bilayered stars and thin-shell configurations, (2024), [arXiv:2411.14018 \[gr-qc\]](#).
- [18] R. M. Wald, [General Relativity](#) (Chicago University Press., 1984).
- [19] W. H. Louisell, R. W. Pidd, and H. R. Crane, An experimental measurement of the gyromagnetic ratio of the free electron, [Phys. Rev.](#) **94**, 7 (1954).
- [20] J. A. Wheeler and W. H. Zurek, [Quantum Theory and Measurement](#) (Princeton University Press, 1983).
- [21] K. Fredenhagen, On the modular structure of local algebras of observables, [Commun. Math. Phys.](#) **97**, 79 (1985).
- [22] D. Buchholz, C. D'Antoni, and K. Fredenhagen, The universal structure of local algebras, [Commun. Math. Phys.](#) **111**, 123 (1987).
- [23] U. Haagerup, Conne's bicentralizer problem and uniqueness of the injective factor of type III₁, [Acta Math.](#) **158**, 95 (1987).
- [24] R. D. Sorkin, Impossible Measurements on Quantum Fields, (1993), [arXiv:gr-qc/9302018](#).

- [25] F. Dowker, Useless Qubits in “Relativistic Quantum Information”, (2011), [arXiv:1111.2308 \[quant-ph\]](#).
- [26] W. E. Lamb, Anti-photon, [Appl. Phys. B](#) **60**, 77 (1995).
- [27] C. J. Fewster and R. Verch, Quantum Fields and Local Measurements, [Commun. Math. Phys.](#) **378**, 851 (2020).
- [28] H. Bostelmann, C. J. Fewster, and M. H. Rued, Impossible measurements require impossible apparatus, [Phys. Rev. D](#) **103**, 025017 (2021).
- [29] M. H. Mohammady, T. Miyadera, and L. Loveridge, Measurement disturbance and conservation laws in quantum mechanics, [Quantum](#) **7**, 1033 (2023).
- [30] J. Watrous, *The Theory of Quantum Information* (Cambridge University Press, 2018).
- [31] P. Busch, P. Lahti, J.-P. Pellonpää, and K. Ylinen, *Quantum Measurement* (Springer International Publishing, 2016).
- [32] K. Kraus, General state changes in quantum theory, [Ann. Phys. \(N.Y.\)](#) **64**, 311 (1971).
- [33] A. S. Holevo, Entropy gain and the Choi-Jamiolkowski correspondence for infinite-dimensional quantum evolutions, [Theor. Math. Phys.](#) **166**, 123 (2011).
- [34] S. Friedland, Infinite dimensional generalizations of Choi’s theorem, (2018), [arXiv:1806.06938 \[math.OA\]](#).
- [35] G. Lüders, Concerning the state-change due to the measurement process, [Ann. Phys. \(Berl.\)](#) **15**, 663 (1951).
- [36] E. B. Davies and J. T. Lewis, An operational approach to quantum probability, [Commun. Math. Phys.](#) **17**, 239 (1970).
- [37] J.-P. Pellonpää, Quantum instruments: II. Measurement theory, [J. Phys. A Math. Theor.](#) **46**, 025303 (2012).
- [38] M. Ozawa, [J. Math. Phys.](#) **25**, 79 (1984).
- [39] S. A. Fulling, *Aspects of Quantum Field Theory in Curved Spacetime*, London Mathematical Society Student Texts (Cambridge University Press, 1989).

- [40] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, 1982).
- [41] L. J. Garay, Lecture notes: Quantum fields in curved spacetimes, (2015), available at <https://sites.google.com/site/luisjgaray/lecture-notes>.
- [42] R. Haag, *Local Quantum Physics: Fields, Particles, Algebras* (Springer-Verlag Berlin Heidelberg, 1992).
- [43] R. Brunetti, C. Dappiaggi, K. Fredenhagen, and J. Yngvason, eds., *Advances in Algebraic Quantum Field Theory* (Springer Cham, 2015).
- [44] C. J. Fewster and K. Rejzner, Algebraic Quantum Field Theory, in *Progress and Visions in Quantum Theory in View of Gravity*, edited by F. Finster, D. Giulini, J. Kleiner, and J. Tolksdorf (Birkhäuser, Cham, Germany, 2020).
- [45] R. M. Wald, *Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics* (University of Chicago Press, 1995).
- [46] W. G. Unruh, Notes on black-hole evaporation, *Phys. Rev. D* **14**, 870 (1976).
- [47] D. W. Robinson, The ground state of the Bose gas, *Commun. Math. Phys.* **1**, 159 (1965).
- [48] E. Balslev and A. Verbeure, States on Clifford algebras, *Commun. Math. Phys.* **7**, 55 (1968).
- [49] J. Manuceau and A. Verbeure, Quasi-free states of the C.C.R.—Algebra and Bogoliubov transformations, *Commun. Math. Phys.* **9**, 293 (1968).
- [50] O. W. Greenberg and A. L. Licht, Quantum Field-Theory Model whose Truncated Vacuum Expectation Values Vanished beyond Some Order, *J. Math. Phys.* **4**, 613 (1963).
- [51] J. Earman, *Quantum Physics in Non-Separable Hilbert Spaces*, (2020).
- [52] J. Sorce, Notes on the type classification of von Neumann algebras, *Rev. Math. Phys.* **36**, 2430002 (2024).
- [53] J. Yngvason, The role of type III factors in quantum field theory, *Rep. Math. Phys.* **55**, 135 (2005).

- [54] H. Araki, A Lattice of Von Neumann Algebras Associated with the Quantum Theory of a Free Bose Field, [J. Math. Phys. **4**, 1343 \(1963\)](#).
- [55] H. Araki, Von Neumann Algebras of Local Observables for Free Scalar Field, [J. Math. Phys. **5**, 1–13 \(1964\)](#).
- [56] G. F. Dell’Antonio, Structure of the algebras of some free systems, [Commun. Math. Phys. **9**, 81 \(1968\)](#).
- [57] K. Osterwalder, Duality for free bose fields, [Commun. Math. Phys. **29**, 1 \(1973\)](#).
- [58] J.-P. Eckmann and K. Osterwalder, An application of Tomita’s theory of modular Hilbert algebras: Duality for free Bose fields, [J. Funct. Anal. **13**, 1 \(1973\)](#).
- [59] J. J. Bisognano and E. H. Wichmann, On the duality condition for quantum fields, [J. Math. Phys. **17**, 303 \(1976\)](#).
- [60] R. Verch, Nuclearity, split property, and duality for the Klein-Gordon field in curved spacetime, [Lett. Math. Phys. **29**, 297 \(1993\)](#).
- [61] E. Witten, APS Medal for Exceptional Achievement in Research: Invited article on entanglement properties of quantum field theory, [Rev. Mod. Phys. **90**, 045003 \(2018\)](#).
- [62] G. J. Murphy, [C*-Algebras and Operator Theory](#) (Academic Press, San Diego, 1990).
- [63] R. Honegger and A. Rieckers, [Photons in Fock Space and Beyond: Volume III: Mathematics for Photon Fields](#) (World Scientific, 2014).
- [64] R. Brunetti, K. Fredenhagen, P. Imani, and K. Rejzner, The locality axiom in quantum field theory and tensor products of C*-algebras, [Rev. Math. Phys. **26**, 1450010 \(2014\)](#).
- [65] S. Takagi, Vacuum Noise and Stress Induced by Uniform Acceleration: Hawking-Unruh Effect in Rindler Manifold of Arbitrary Dimension, [Prog. Theor. Phys. Supp. **88**, 1 \(1986\)](#).
- [66] B. L. Hu, S.-Y. Lin, and J. Louko, Relativistic quantum information in detectors–field interactions, [Class. Quantum Gravity **29**, 224005 \(2012\)](#).
- [67] A. R. H. Smith, [Detectors, Reference Frames, and Time](#) (Springer International Publishing, 2019).

- [68] R. Lopp, *Light-Matter Interaction Models in Relativistic Quantum Information*, Ph.D. thesis, University of Waterloo (2021).
- [69] E. Tjoa, *The Roles of Causal Propagator in Relativistic Quantum Information*, Ph.D. thesis, University of Waterloo (2023).
- [70] B. de Souza Leão Torres, *Aspects of Quantum Information in Quantum Field Theory: Particle detector models, entanglement, and complexity*, Ph.D. thesis, University of Waterloo (2024).
- [71] E. Poisson, A. Pound, and I. Vega, The Motion of Point Particles in Curved Space-time, [Living Rev. Relativ.](#) **14** (2011), [10.12942/lrr-2011-7](#).
- [72] S. Schlicht, Considerations on the Unruh effect: causality and regularization, [Class. Quantum Gravity](#) **21**, 4647 (2004).
- [73] J. Louko and A. Satz, How often does the Unruh-DeWitt detector click? Regularization by a spatial profile, [Class. Quantum Gravity](#) **23**, 6321 (2006).
- [74] P. Langlois, Causal particle detectors and topology, [Ann. Phys. \(N.Y.\)](#) **321**, 2027 (2006).
- [75] E. Martín-Martínez, T. R. Perche, and B. de S. L. Torres, General relativistic quantum optics: Finite-size particle detector models in curved spacetimes, [Phys. Rev. D](#) **101**, 045017 (2020).
- [76] T. R. Perche, Localized nonrelativistic quantum systems in curved spacetimes: A general characterization of particle detector models, [Phys. Rev. D](#) **106**, 025018 (2022).
- [77] A. P. Kazantsev, The acceleration of atoms by light, [Zh. Eksp. Teor. Fiz.](#) **66**, 1599 (1974).
- [78] E. Martín-Martínez and P. Rodríguez-Lopez, Relativistic quantum optics: The relativistic invariance of the light-matter interaction models, [Phys. Rev. D](#) **97**, 105026 (2018).
- [79] B. DeWitt, in *General Relativity: an Einstein Centenary Survey*, edited by Hawking, S. and Israel, W. (Cambridge University Press, Cambridge, 1979).
- [80] E. Martín-Martínez, T. R. Perche, and B. de S. L. Torres, Broken covariance of particle detector models in relativistic quantum information, [Phys. Rev. D](#) **103**, 025007 (2021).

- [81] E. Tjoa and F. Gray, The Unruh–DeWitt model and its joint interacting Hilbert space, [J. Phys. A Math. Theor.](#) **57**, 325301 (2024).
- [82] A. Satz, Then again, how often does the Unruh-DeWitt detector click if we switch it carefully? [Class. Quantum Gravity](#) **24**, 1719 (2007).
- [83] C. J. Fewster, I. Jubb, and M. H. Rued, Asymptotic Measurement Schemes for Every Observable of a Quantum Field Theory, [Ann. Henri Poincaré](#) **24**, 1137 (2022).
- [84] E. Martín-Martínez, Causality issues of particle detector models in QFT and quantum optics, [Phys. Rev. D](#) **92**, 104019 (2015).
- [85] J. de Ramón, M. Papageorgiou, and E. Martín-Martínez, Relativistic causality in particle detector models: Faster-than-light signaling and impossible measurements, [Phys. Rev. D](#) **103**, 085002 (2021).
- [86] J. de Ramón, M. Papageorgiou, and E. Martín-Martínez, Causality and signalling in noncompact detector-field interactions, [Phys. Rev. D](#) **108**, 045015 (2023).
- [87] W. G. Unruh and R. M. Wald, What happens when an accelerating observer detects a rindler particle, [Phys. Rev. D](#) **29**, 1047 (1984).
- [88] J. Louko and D. Marolf, Inextendible schwarzschild black hole with a single exterior: How thermal is the hawking radiation? [Phys. Rev. D](#) **58**, 024007 (1998).
- [89] C. J. Fewster, B. A. Juárez-Aubry, and J. Louko, Waiting for Unruh, [Class. Quantum Gravity](#) **33**, 165003 (2016).
- [90] W. G. Brenna, E. G. Brown, R. B. Mann, and E. Martín-Martínez, Universality and thermalization in the unruh effect, [Phys. Rev. D](#) **88**, 064031 (2013).
- [91] R. Carballo-Rubio, L. J. Garay, E. Martín-Martínez, and J. de Ramón, Unruh effect without thermality, [Phys. Rev. Lett.](#) **123**, 041601 (2019).
- [92] L. Hodgkinson, J. Louko, and A. C. Ottewill, Static detectors and circular-geodesic detectors on the Schwarzschild black hole, [Phys. Rev. D](#) **89**, 104002 (2014).
- [93] B. A. Juárez-Aubry and J. Louko, Quantum fields during black hole formation: how good an approximation is the Unruh state? [J. High Energy Phys.](#) **2018**, 140 (2018).
- [94] J. de Ramón, L. J. Garay, and E. Martín-Martínez, Direct measurement of the two-point function in quantum fields, [Phys. Rev. D](#) **98**, 105011 (2018).

- [95] T. R. Perche and E. Martín-Martínez, Geometry of spacetime from quantum measurements, [Phys. Rev. D **105**, 066011 \(2022\)](#).
- [96] R. H. Jonsson, E. Martín-Martínez, and A. Kempf, Information transmission without energy exchange, [Phys. Rev. Lett. **114**, 110505 \(2015\)](#).
- [97] A. Blasco, L. J. Garay, M. Martín-Benito, and E. Martín-Martínez, Violation of the Strong Huygen’s Principle and Timelike Signals from the Early Universe, [Phys. Rev. Lett. **114**, 141103 \(2015\)](#).
- [98] A. Blasco, L. J. Garay, M. Martín-Benito, and E. Martín-Martínez, Timelike information broadcasting in cosmology, [Phys. Rev. D **93**, 024055 \(2016\)](#).
- [99] M. Hotta, Quantum measurement information as a key to energy extraction from local vacuums, [Phys. Rev. D **78**, 045006 \(2008\)](#).
- [100] M. Hotta, Quantum Energy Teleportation: An Introductory Review, (2011), [arXiv:1101.3954 \[quant-ph\]](#).
- [101] M. Hotta, J. Matsumoto, and G. Yusa, Quantum energy teleportation without a limit of distance, [Phys. Rev. A **89**, 012311 \(2014\)](#).
- [102] N. Funai and E. Martín-Martínez, Engineering negative stress-energy densities with quantum energy teleportation, [Phys. Rev. D **96**, 025014 \(2017\)](#).
- [103] A. Valentini, Non-local correlations in quantum electrodynamics, [Phys. Lett. A **153**, 321 \(1991\)](#).
- [104] B. Reznik, A. Retzker, and J. Silman, Violating Bell’s inequalities in vacuum, [Phys. Rev. A **71**, 042104 \(2005\)](#).
- [105] A. Pozas-Kerstjens and E. Martín-Martínez, Harvesting correlations from the quantum vacuum, [Phys. Rev. D **92**, 064042 \(2015\)](#).
- [106] A. Pozas-Kerstjens and E. Martín-Martínez, Entanglement harvesting from the electromagnetic vacuum with hydrogenlike atoms, [Phys. Rev. D **94**, 064074 \(2016\)](#).
- [107] L. J. Henderson, R. A. Hennigar, R. B. Mann, A. R. H. Smith, and J. Zhang, Entangling detectors in anti-de Sitter space, [J. High Energy Phys. **2019**, 178 \(2019\)](#).
- [108] L. J. Henderson and N. C. Menicucci, Bandlimited entanglement harvesting, [Phys. Rev. D **102**, 125026 \(2020\)](#).

- [109] M. P. G. Robbins, L. J. Henderson, and R. B. Mann, Entanglement amplification from rotating black holes, [Class. Quantum Gravity](#) **39**, 02LT01 (2021).
- [110] E. Tjoa and E. Martín-Martínez, When entanglement harvesting is not really harvesting, [Phys. Rev. D](#) **104**, 125005 (2021).
- [111] K. Bueley, L. Huang, K. Gallock-Yoshimura, and R. B. Mann, Harvesting mutual information from BTZ black hole spacetime, [Phys. Rev. D](#) **106**, 025010 (2022).
- [112] J. a. G. A. Caribé, R. H. Jonsson, M. Casals, A. Kempf, and E. Martín-Martínez, Lensing of vacuum entanglement near Schwarzschild black holes, [Phys. Rev. D](#) **108**, 025016 (2023).
- [113] D. Mendez-Avalos, L. J. Henderson, K. Gallock-Yoshimura, and R. B. Mann, Entanglement harvesting of three Unruh-DeWitt detectors, [Gen. Relativ. Gravit.](#) **54**, 87 (2022).
- [114] L. J. Henderson, S. Y. Ding, and R. B. Mann, Entanglement harvesting with a twist, [AVS Quantum Sci.](#) **4**, 014402 (2022).
- [115] C. Gooding, A. Sachs, R. B. Mann, and S. Weinfurtner, Vacuum entanglement probes for ultra-cold atom systems, [New J. Phys.](#) **26**, 105001 (2024).
- [116] R. H. Jonsson, K. Ried, E. Martín-Martínez, and A. Kempf, Transmitting qubits through relativistic fields, [J. Phys. A: Math. Theor.](#) **51**, 485301 (2018).
- [117] R. H. Jonsson, D. Q. Aruquipa, M. Casals, A. Kempf, and E. Martín-Martínez, Communication through quantum fields near a black hole, [Phys. Rev. D](#) **101**, 125005 (2020).
- [118] K. Yamaguchi, A. Ahmadzadegan, P. Simidzija, A. Kempf, and E. Martín-Martínez, Superadditivity of channel capacity through quantum fields, [Phys. Rev. D](#) **101**, 105009 (2020).
- [119] K. Yamaguchi and A. Kempf, Entanglement is better teleported than transmitted, [Phys. Rev. D](#) **108**, 025004 (2023).
- [120] E. Martín-Martínez, M. Montero, and M. del Rey, Wavepacket detection with the Unruh-DeWitt model, [Phys. Rev. D](#) **87**, 064038 (2013).

- [121] R. Lopp and E. Martín-Martínez, Quantum delocalization, gauge, and quantum optics: Light-matter interaction in relativistic quantum information, [Phys. Rev. A](#) **103**, 013703 (2021).
- [122] B. Peropadre, P. Forn-Díaz, E. Solano, and J. J. García-Ripoll, Switchable Ultra-strong Coupling in Circuit QED, [Phys. Rev. Lett.](#) **105**, 023601 (2010).
- [123] P. Forn-Díaz, J. J. García-Ripoll, B. Peropadre, J.-L. Orgiazzi, M. A. Yurtalan, R. Belyansky, C. M. Wilson, and A. Lupascu, Ultrastrong coupling of a single artificial atom to an electromagnetic continuum in the nonperturbative regime, [Nat. Phys.](#) **13**, 39 (2016).
- [124] E. McKay, A. Lupascu, and E. Martín-Martínez, Finite sizes and smooth cutoffs in superconducting circuits, [Phys. Rev. A](#) **96**, 052325 (2017).
- [125] K. E. Hellwig and K. Kraus, Formal Description of Measurements in Local Quantum Field Theory, [Phys. Rev. D](#) **1**, 566 (1970).
- [126] K. E. Hellwig and K. Kraus, Operations and measurements. II, [Commun. Math. Phys.](#) **16**, 142 (1970).
- [127] K. E. Hellwig and K. Kraus, Pure operations and measurements, [Commun. Math. Phys.](#) **11**, 214 (1969).
- [128] L. Borsten, I. Jubb, and G. Kells, Impossible measurements revisited, [Phys. Rev. D](#) **104**, 025012 (2021).
- [129] I. Jubb, Causal state updates in real scalar quantum field theory, [Phys. Rev. D](#) **105**, 025003 (2022).
- [130] H. Reeh and S. Schlieder, Bemerkungen zur Unitäräquivalenz von Lorentzinvarianten Feldern, [Nuovo Cimento](#) **22**, 1051 (1961).
- [131] S. Doplicher and R. Longo, Standard and split inclusions of von Neumann algebras, [Invent. Math.](#) **75**, 493 (1984).
- [132] V. Paulsen, [Completely Bounded Maps and Operator Algebras](#) (Cambridge University Press, 2003).
- [133] C. J. Fewster and R. Verch, Dynamical Locality and Covariance: What Makes a Physical Theory the Same in all Spacetimes? [Ann. Henri Poincaré](#) **13**, 1613 (2012).

- [134] R. Geroch, Domain of Dependence, *J. Math. Phys.* **11**, 437 (1970).
- [135] A. N. Bernal and M. Sánchez, On Smooth Cauchy Hypersurfaces and Geroch’s Splitting Theorem, *Commun. Math. Phys.* **243**, 461 (2003).
- [136] E. Albertini and I. Jubb, Are Ideal Measurements of Real Scalar Fields Causal? (2023), [arXiv:2306.12980 \[quant-ph\]](#).
- [137] D. Buchholz, Product states for local algebras, *Commun. Math. Phys.* **36**, 287 (1974).
- [138] C. J. Fewster, The Split Property for Locally Covariant Quantum Field Theories in Curved Spacetime, *Lett. Math. Phys.* **105**, 1633 (2015).
- [139] C. J. Fewster, The split property for quantum field theories in flat and curved spacetimes, *Abh. Math. Semin. Univ. Hambg.* **86**, 153 (2016).
- [140] R. Oeckl, Spectral decomposition of field operators and causal measurement in quantum field theory, (2024), [arXiv:2409.08748 \[hep-th\]](#).
- [141] C. J. Fewster, A Generally Covariant Measurement Scheme for Quantum Field Theory in Curved Spacetimes, in *Progress and Visions in Quantum Theory in View of Gravity* (Springer International Publishing, 2020) pp. 253–268.
- [142] C. J. Fewster and R. Verch, Measurement in Quantum Field Theory, in *Encyclopedia of Mathematical Physics* (Elsevier, 2025) pp. 335–345.
- [143] N. Pranzini and E. Keski-Vakkuri, Detector-based measurements for QFT: two issues and an AQFT proposal, (2023), [arXiv:2310.06596 \[quant-ph\]](#).
- [144] S. J. Summers and R. Werner, The vacuum violates Bell’s inequalities, *Phys. Lett. A* **110**, 257 (1985).
- [145] S. J. Summers and R. Werner, Bell’s inequalities and quantum field theory. I. General setting, *J. Math. Phys.* **28**, 2440 (1987).
- [146] S. Schlieder, Einige bemerkungen zur zustandsänderung von relativistischen quantenmechanischen systemen durch messungen und zur lokalitätsforderung, *Commun. Math. Phys.* **7**, 305 (1968).
- [147] L. Bombelli, R. K. Koul, J. Lee, and R. D. Sorkin, Quantum source of entropy for black holes, *Phys. Rev. D* **34**, 373 (1986).

- [148] M. Srednicki, Entropy and area, [Phys. Rev. Lett. **71**, 666 \(1993\)](#).
- [149] Y. Aharonov and D. Z. Albert, States and observables in relativistic quantum field theories, [Phys. Rev. D **21**, 3316 \(1980\)](#).
- [150] M. H. Riep, Weakly coupled local particle detectors cannot harvest entanglement, [Class. Quantum Gravity **38**, 195029 \(2021\)](#).
- [151] L. E. Ballentine, The statistical interpretation of quantum mechanics, [Rev. Mod. Phys. **42**, 358 \(1970\)](#).
- [152] S. Malin, Observer dependence of quantum states in relativistic quantum field theories, [Phys. Rev. D **26**, 1330 \(1982\)](#).
- [153] J. Głowacki, Towards Relational Quantum Field Theory, (2024), [arXiv:2405.15455 \[quant-ph\]](#).
- [154] D. Grimmer, B. d. S. L. Torres, and E. Martín-Martínez, Measurements in QFT: Weakly coupled local particle detectors and entanglement harvesting, [Phys. Rev. D **104**, 085014 \(2021\)](#).
- [155] S. Weinberg, [The Quantum Theory of Fields](#), Vol. 1 (Cambridge University Press, 1995).
- [156] S. Blanes, F. Casas, J. A. Oteo, and J. Ros, The Magnus expansion and some of its applications, [Phys. Rep. **470**, 151 \(2009\)](#).
- [157] B. d. S. L. Torres, T. R. Perche, A. G. S. Landulfo, and G. E. A. Matsas, Neutrino flavor oscillations without flavor states, [Phys. Rev. D **102**, 093003 \(2020\)](#).
- [158] T. R. Perche and E. Martín-Martínez, Antiparticle detector models in QFT, [Phys. Rev. D **104**, 105021 \(2021\)](#).
- [159] C. Weedbrook, S. Pirandola, R. García-Patrón, N. J. Cerf, T. C. Ralph, J. H. Shapiro, and S. Lloyd, Gaussian quantum information, [Rev. Mod. Phys. **84**, 621 \(2012\)](#).
- [160] A. Serafini, [Quantum Continuous Variables](#) (CRC Press, 2017).
- [161] E. G. Brown, E. Martín-Martínez, N. C. Menicucci, and R. B. Mann, Detectors for probing relativistic quantum physics beyond perturbation theory, [Phys. Rev. D **87**, 084062 \(2013\)](#).

- [162] D. E. Bruschi, A. R. Lee, and I. Fuentes, Time evolution techniques for detectors in relativistic quantum information, [J. Phys. A Math. Theor.](#) **46**, 165303 (2013).
- [163] A. Pozas-Kerstjens, J. Louko, and E. Martín-Martínez, Degenerate detectors are unable to harvest spacelike entanglement, [Phys. Rev. D](#) **95**, 105009 (2017).
- [164] T. R. Perche and A. Shalabi, Spacetime curvature from ultrarapid measurements of quantum fields, [Phys. Rev. D](#) **105**, 125011 (2022).
- [165] E. Tjoa and K. Gallock-Yoshimura, Channel capacity of relativistic quantum communication with rapid interaction, [Phys. Rev. D](#) **105**, 085011 (2022).
- [166] E. Tjoa, Quantum teleportation with relativistic communication from first principles, [Phys. Rev. A](#) **106**, 032432 (2022).
- [167] E. Tjoa, Fermi two-atom problem: Nonperturbative approach via relativistic quantum information and algebraic quantum field theory, [Phys. Rev. D](#) **106**, 045012 (2022).
- [168] N. K. Kollas, D. Moustos, and M. R. Muñoz, Cohering and decohering power of massive scalar fields under instantaneous interactions, [Phys. Rev. A](#) **107**, 022420 (2023).
- [169] P. Simidzija and E. Martín-Martínez, Nonperturbative analysis of entanglement harvesting from coherent field states, [Phys. Rev. D](#) **96**, 065008 (2017).
- [170] A. Sahu, I. Melgarejo-Lermas, and E. Martín-Martínez, Sabotaging the harvesting of correlations from quantum fields, [Phys. Rev. D](#) **105**, 065011 (2022).
- [171] E. Tjoa, Nonperturbative simple-generated interactions with a quantum field for arbitrary Gaussian states, [Phys. Rev. D](#) **108**, 045003 (2023).
- [172] C. K. Chui, A Convergence Theorem for Certain Riemann Sums, [Can. Math. Bull.](#) **12**, 523 (1969).
- [173] C. K. Chui, Concerning rates of convergence of Riemann sums, [J. Approx. Theory](#) **4**, 279 (1971).
- [174] M. J. Radzikowski, Micro-local approach to the Hadamard condition in quantum field theory on curved space-time, [Commun. Math. Phys.](#) **179**, 529 (1996).

- [175] A. Strohmaier, Microlocal Analysis, in *Quantum Field Theory on Curved Spacetimes* (Springer Berlin Heidelberg, 2009) pp. 85–127.
- [176] C. J. Fewster and R. Verch, The necessity of the Hadamard condition, *Class. Quantum Gravity* **30**, 235027 (2013).
- [177] E. Tjoa, I. López-Gutiérrez, A. Sachs, and E. Martín-Martínez, What makes a particle detector click, *Phys. Rev. D* **103**, 125021 (2021).
- [178] D. N. Page, Information in black hole radiation, *Phys. Rev. Lett.* **71**, 3743 (1993).
- [179] E. Martín-Martínez, A. R. H. Smith, and D. R. Terno, Spacetime structure and vacuum entanglement, *Phys. Rev. D* **93**, 044001 (2016).
- [180] J. Maldacena, S. H. Shenker, and D. Stanford, A bound on chaos, *J. High Energy Phys.* **2016**, 106 (2016).
- [181] B. Yoshida and A. Kitaev, Efficient decoding for the Hayden-Preskill protocol, (2017), [arXiv:1710.03363 \[hep-th\]](#).
- [182] K. A. Landsman, C. Figgatt, T. Schuster, N. M. Linke, B. Yoshida, N. Y. Yao, and C. Monroe, Verified quantum information scrambling, *Nature* **567**, 61 (2019).
- [183] K. Yamaguchi, N. Watamura, and M. Hotta, Quantum information capsule and information delocalization by entanglement in multiple-qubit systems, *Phys. Lett. A* **383**, 1255 (2019).
- [184] L. Hodgkinson and J. Louko, Static, stationary, and inertial Unruh-DeWitt detectors on the BTZ black hole, *Phys. Rev. D* **86**, 064031 (2012).
- [185] K. K. Ng, R. B. Mann, and E. Martín-Martínez, Over the horizon: Distinguishing the Schwarzschild spacetime and the $\mathbb{RP}^3$ spacetime using an Unruh-DeWitt detector, *Phys. Rev. D* **96**, 085004 (2017).
- [186] A. Ahmadzadegan, E. Martín-Martínez, and R. B. Mann, Cavities in curved spacetimes: The response of particle detectors, *Phys. Rev. D* **89**, 024013 (2014).
- [187] A. R. H. Smith and R. B. Mann, Looking inside a black hole, *Class. Quantum Gravity* **31**, 082001 (2014).
- [188] A. Ahmadzadegan, F. Lalegani, A. Kempf, and R. B. Mann, Probing geometric information using the Unruh effect in the vacuum, *Phys. Rev. D* **100**, 085013 (2019).

- [189] A. Hentschel and B. C. Sanders, Efficient Algorithm for Optimizing Adaptive Quantum Metrology Processes, [Phys. Rev. Lett. **107**, 233601 \(2011\)](#).
- [190] C. E. Granade, C. Ferrie, N. Wiebe, and D. G. Cory, Robust online Hamiltonian learning, [New J. Phys **14**, 103013 \(2012\)](#).
- [191] J. Carrasquilla and R. G. Melko, Machine learning phases of matter, [Nat. Phys. **13**, 431 \(2017\)](#).
- [192] Y. Quek, S. Fort, and H. K. Ng, Adaptive quantum state tomography with neural networks, [npj Quantum Inf. **7** \(2021\)](#), [10.1038/s41534-021-00436-9](#).
- [193] G. Torlai, G. Mazzola, J. Carrasquilla, M. Troyer, R. Melko, and G. Carleo, Neural-network quantum state tomography, [Nat. Phys. **14**, 447 \(2018\)](#).
- [194] E. S. Tiunov, V. V. T. (Vyborova), A. E. Ulanov, A. I. Lvovsky, and A. K. Fedorov, Experimental quantum homodyne tomography via machine learning, [Optica **7**, 448 \(2020\)](#).
- [195] Y. Mori, K. Kashiwa, and A. Ohnishi, Application of a neural network to the sign problem via the path optimization method, [Prog. Theor. Exp. Phys. **2018** \(2018\)](#).
- [196] V. Cimini, I. Gianani, N. Spagnolo, F. Leccese, F. Sciarrino, and M. Barbieri, Calibration of Quantum Sensors by Neural Networks, [Phys. Rev. Lett. **123**, 230502 \(2019\)](#).
- [197] A. Lidiak and Z. Gong, Unsupervised Machine Learning of Quantum Phase Transitions Using Diffusion Maps, [Phys. Rev. Lett. **125**, 225701 \(2020\)](#).
- [198] J. Carrasquilla and G. Torlai, How To Use Neural Networks To Investigate Quantum Many-Body Physics, [PRX Quantum **2**, 040201 \(2021\)](#).
- [199] E. Genois, J. A. Gross, A. Di Paolo, N. J. Stevenson, G. Koolstra, A. Hashim, I. Siddiqi, and A. Blais, Quantum-Tailored Machine-Learning Characterization of a Superconducting Qubit, [PRX Quantum **2**, 040355 \(2021\)](#).
- [200] F. S. Luiz, A. d. O. Junior, F. F. Fanchini, and G. T. Landi, Machine classification for probe-based quantum thermometry, [Phys. Rev. A **105**, 022413 \(2022\)](#).
- [201] K. Zhou, G. Endrődi, L.-G. Pang, and H. Stöcker, Regressive and generative neural networks for scalar field theory, [Phys. Rev. D **100**, 011501 \(2019\)](#).

- [202] P. E. Shanahan, D. Trewartha, and W. Detmold, Machine learning action parameters in lattice quantum chromodynamics, [Phys. Rev. D **97**, 094506 \(2018\)](#).
- [203] A. Decelle, V. Martin-Mayor, and B. Seoane, Learning a local symmetry with neural networks, [Phys. Rev. E **100**, 050102 \(2019\)](#).
- [204] M. N. Chernodub, H. Erbin, I. V. Grishmanovskii, V. A. Goy, and A. V. Molochkov, Casimir effect with machine learning, [Phys. Rev. Research **2**, 033375 \(2020\)](#).
- [205] D. L. Boyda, M. N. Chernodub, N. V. Gerasimeniuk, V. A. Goy, S. D. Liubimov, and A. V. Molochkov, Finding the deconfinement temperature in lattice Yang-Mills theories from outside the scaling window with machine learning, [Phys. Rev. D **103**, 014509 \(2021\)](#).
- [206] L. Jiang, L. Wang, and K. Zhou, Deep learning stochastic processes with QCD phase transition, [Phys. Rev. D **103**, 116023 \(2021\)](#).
- [207] S. Shi, K. Zhou, J. Zhao, S. Mukherjee, and P. Zhuang, Heavy quark potential in the quark-gluon plasma: Deep neural network meets lattice quantum chromodynamics, [Phys. Rev. D **105**, 014017 \(2022\)](#).
- [208] I. Goodfellow, Y. Bengio, and A. Courville, *Deep Learning* (MIT Press, Cambridge, US, 2016) <http://www.deeplearningbook.org>.
- [209] J. B. Kogut, An introduction to lattice gauge theory and spin systems, [Rev. Mod. Phys. **51**, 659 \(1979\)](#).
- [210] C. Gattringer and C. B. Lang, Quantum chromodynamics on the lattice, [Lect. Notes Phys. **788**, 1 \(2010\)](#).
- [211] R. Simon, E. C. G. Sudarshan, and N. Mukunda, Gaussian pure states in quantum mechanics and the symplectic group, [Phys. Rev. A **37**, 3028 \(1988\)](#).
- [212] A. Ferraro, S. Olivares, and M. G. A. P. Paris, *Gaussian States in Quantum Information*, Napoli Series on Physics and Astrophysics (Bibliopolis, 2005).
- [213] E. Martín-Martínez, Quantum Mechanics in Phase Space: An introduction, (2022), [arXiv:2208.08682](#).
- [214] G. Adesso, Entanglement of Gaussian states, (2007), [arXiv:quant-ph/0702069 \[quant-ph\]](#).

- [215] G. Adesso, S. Ragy, and A. R. Lee, Continuous Variable Quantum Information: Gaussian States and Beyond, [Open Syst. Inf. Dyn.](#) **21**, 1440001 (2014).
- [216] L. Lami, B. Regula, X. Wang, R. Nichols, A. Winter, and G. Adesso, Gaussian quantum resource theories, [Phys. Rev. A](#) **98**, 022335 (2018).
- [217] D. Grimmer, E. Brown, A. Kempf, R. B. Mann, and E. Martín-Martínez, A classification of open Gaussian dynamics, [J. Phys. A](#) **51**, 245301 (2018).
- [218] H. Vahlbruch, M. Mehmet, K. Danzmann, and R. Schnabel, Detection of 15 dB Squeezed States of Light and their Application for the Absolute Calibration of Photoelectric Quantum Efficiency, [Phys. Rev. Lett.](#) **117**, 110801 (2016).
- [219] D. H. Douglass and R. Meservey, Energy Gap Measurements by Tunneling Between Superconducting Films. I. Temperature Dependence, [Phys. Rev.](#) **135**, A19 (1964).
- [220] P. Forn-Díaz, J. J. García-Ripoll, B. Peropadre, J.-L. Orgiazzi, M. A. Yurtalan, R. Belyansky, C. M. Wilson, and A. Lupascu, Ultrastrong coupling of a single artificial atom to an electromagnetic continuum in the nonperturbative regime, [Nat. Phys.](#) **13**, 39 (2016).
- [221] L. A. Correa, M. Mehboudi, G. Adesso, and A. Sanpera, Individual Quantum Probes for Optimal Thermometry, [Phys. Rev. Lett.](#) **114**, 220405 (2015).
- [222] V. Cavina, L. Mancino, A. De Pasquale, I. Gianani, M. Sbroscia, R. I. Booth, E. Rocca, R. Raimondi, V. Giovannetti, and M. Barbieri, Bridging thermodynamics and metrology in nonequilibrium quantum thermometry, [Phys. Rev. A](#) **98**, 050101 (2018).
- [223] J. Boeyens, S. Seah, and S. Nimmrichter, Uninformed Bayesian quantum thermometry, [Phys. Rev. A](#) **104**, 052214 (2021).
- [224] P. Sekatski and M. Perarnau-Llobet, Optimal nonequilibrium thermometry in Markovian environments, [Quantum](#) **6**, 869 (2022).
- [225] K. E. Cahill and R. J. Glauber, Density operators and quasiprobability distributions, [Phys. Rev.](#) **177**, 1882 (1969).
- [226] A. Allevi, M. Bondani, P. Marian, T. A. Marian, and S. Olivares, Characterization of phase-averaged coherent states, [J. Opt. Soc. Am. B](#) **30**, 2621 (2013).

- [227] A. Higuchi, S. Iso, K. Ueda, and K. Yamamoto, Entanglement of the vacuum between left, right, future, and past: The origin of entanglement-induced quantum radiation, [Phys. Rev. D](#) **96**, 083531 (2017).
- [228] P. Calabrese and J. Cardy, Entanglement entropy and quantum field theory, [J. Stat. Mech. Theory Exp.](#) **2004**, P06002 (2004).
- [229] S. Ryu and T. Takayanagi, Holographic Derivation of Entanglement Entropy from the anti-de Sitter Space/Conformal Field Theory Correspondence, [Phys. Rev. Lett.](#) **96**, 181602 (2006).
- [230] P. Calabrese and J. Cardy, Entanglement entropy and conformal field theory, [J. Phys. A Math. Theor.](#) **42**, 504005 (2009).
- [231] M. Saravani, R. D. Sorkin, and Y. K. Yazdi, Spacetime entanglement entropy in 1 + 1 dimensions, [Class. Quantum Gravity](#) **31**, 214006 (2014).
- [232] B. Reznik, Entanglement from the Vacuum, [Found. Phys.](#) **33**, 167 (2003).
- [233] G. V. Steeg and N. C. Menicucci, Entangling power of an expanding universe, [Phys. Rev. D](#) **79**, 044027 (2009).
- [234] J. Silman and B. Reznik, Long-range entanglement in the Dirac vacuum, [Phys. Rev. A](#) **75**, 052307 (2007).
- [235] E. Martín-Martínez and N. C. Menicucci, Cosmological quantum entanglement, [Class. Quantum Gravity](#) **29**, 224003 (2012).
- [236] G. Salton, R. B. Mann, and N. C. Menicucci, Acceleration-assisted entanglement harvesting and rangefinding, [New J. Phys.](#) **17**, 035001 (2015).
- [237] P. Simidzija and E. Martín-Martínez, All coherent field states entangle equally, [Phys. Rev. D](#) **96**, 025020 (2017).
- [238] P. Simidzija and E. Martín-Martínez, Harvesting correlations from thermal and squeezed coherent states, [Phys. Rev. D](#) **98**, 085007 (2018).
- [239] L. J. Henderson, R. A. Hennigar, R. B. Mann, A. R. H. Smith, and J. Zhang, Harvesting entanglement from the black hole vacuum, [Class. Quantum Gravity](#) **35**, 21LT02 (2018).

- [240] K. K. Ng, R. B. Mann, and E. Martín-Martínez, New techniques for entanglement harvesting in flat and curved spacetimes, [Phys. Rev. D **97**, 125011 \(2018\)](#).
- [241] K. K. Ng, R. B. Mann, and E. Martín-Martínez, Unruh-DeWitt detectors and entanglement: The anti-de Sitter space, [Phys. Rev. D **98**, 125005 \(2018\)](#).
- [242] W. Cong, E. Tjoa, and R. B. Mann, Entanglement harvesting with moving mirrors, [J. High Energy Phys. **2019**, 21 \(2019\)](#).
- [243] W. Cong, C. Qian, M. R. Good, and R. B. Mann, Effects of horizons on entanglement harvesting, [J. High Energy Phys. **2020**, 67 \(2020\)](#).
- [244] Z. Liu, J. Zhang, and H. Yu, Entanglement harvesting in the presence of a reflecting boundary, [J. High Energy Phys. **2021**, 20 \(2021\)](#).
- [245] E. Martín-Martínez, E. G. Brown, W. Donnelly, and A. Kempf, Sustainable entanglement production from a quantum field, [Phys. Rev. A **88**, 052310 \(2013\)](#).
- [246] E. G. Brown, W. Donnelly, A. Kempf, R. B. Mann, E. Martín-Martínez, and N. C. Menicucci, Quantum seismology, [New J. Phys. **16**, 105020 \(2014\)](#).
- [247] I. Agullo, B. Bonga, P. Ribes-Metidieri, D. Kranas, and S. Nadal-Gisbert, How ubiquitous is entanglement in quantum field theory? [Phys. Rev. D **108**, 085005 \(2023\)](#).
- [248] G. Vidal and R. F. Werner, Computable measure of entanglement, [Phys. Rev. A **65**, 032314 \(2002\)](#).
- [249] D. E. Knuth, Big omicron and big omega and big theta, [ACM SIGACT News **8**, 18–24 \(1976\)](#).
- [250] H. Maeso-García, T. R. Perche, and E. Martín-Martínez, Entanglement harvesting: Detector gap and field mass optimization, [Phys. Rev. D **106**, 045014 \(2022\)](#).
- [251] D. Fraser and M. Papageorgiou, Note on episodes in the history of modeling measurements in local spacetime regions using QFT, [Eur. Phys. J. H **48** \(2023\), 10.1140/epjh/s13129-023-00064-1](#).
- [252] M. Papageorgiou and D. Fraser, Eliminating the ‘Impossible’: Recent Progress on Local Measurement Theory for Quantum Field Theory, [Found. Phys. **54**, 26 \(2024\)](#).
- [253] T. H. Hildebrandt, in [Introduction to the Theory of Integration](#) (Academic Press, New York, 1963).

- [254] A. Bhattacharyya, On a measure of divergence between two multinomial populations, [Sankhya](#) **7**, 401 (1946).
- [255] W. G. Cochran, The distribution of quadratic forms in a normal system, with applications to the analysis of covariance, [Math. Proc. Camb. Philos. Soc.](#) **30**, 178 (1934).
- [256] B. L. Schumaker, Quantum mechanical pure states with gaussian wave functions, [Phys. Rep.](#) **135**, 317 (1986).
- [257] R. A. Fisher, On the Mathematical Foundations of Theoretical Statistics, [Philos. Trans. R. Soc. A](#) **222**, 309 (1922).
- [258] H. Weyl, Quantenmechanik und Gruppentheorie, [Z. Phys.](#) **46**, 1 (1927).
- [259] E. Wigner, On the Quantum Correction For Thermodynamic Equilibrium, [Phys. Rev.](#) **40**, 749 (1932).
- [260] M. Hillery, R. O’Connell, M. Scully, and E. Wigner, Distribution functions in physics: Fundamentals, [Phys. Rep.](#) **106**, 121 (1984).
- [261] D. Grimmer, A Discrete Analog of General Covariance: Despite what you’ve heard, a perfectly Lorentzian lattice theory, (2022), [arXiv:2205.07701](#).
- [262] D. Grimmer, A Discrete Analog of General Covariance: Could the world be fundamentally set on a lattice? (2022), [arXiv:2204.02276](#).

APPENDICES

Appendix A

The n -dimensional Riemann-Stieltjes integral

Even though the Riemann-Stieltjes integral is a basic tool in real analysis, its generalization to the n -dimensional case is not as popular in classical texts. Because of that, in this appendix we present a very brief introduction to the n -dimensional Riemann-Stieltjes integral—or rather, its generalization to functions of complex argument—and we prove the three lemmas we referred to in Sec. 6.4.

Let $f, g : \mathbb{R}^n \rightarrow \mathbb{C}$, and let $R = [x_1, y_1] \times \dots \times [x_n, y_n] \subset \mathbb{R}^n$ be a closed rectangle in $\mathbb{R}^n$. Following [253], we define

$$\Delta_R^{(n)} g = \sum_{\mathbf{z}} (-1)^{\#x(\mathbf{z})} g(\mathbf{z}), \quad (\text{A.1})$$

where the sum runs over all $\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{R}^n$ such that $z_i = x_i$ or y_i , for each $i = 1, \dots, n$, and $\#x(\mathbf{z})$ is the number of components of $\mathbf{z}$ that are equal to x_i and not y_i . This definition is motivated by the fact that for the function $p(\mathbf{z}) = \prod_{i=1}^n z_i$, we have

$$\Delta_R^{(n)} p = \prod_{i=1}^n (y_i - x_i) = \text{Vol}(R). \quad (\text{A.2})$$

Now, for each $i = 1, \dots, n$, consider a partition of the interval $[x_i, y_i]$,

$$P_i = \{t_0^{(i)} = x_i, \dots, t_{M_i}^{(i)} = y_i\}, \quad (\text{A.3})$$

with a set of tags

$$T_i = \{u_1, \dots, u_{M_i} : u_j^{(i)} \in [t_{j-1}^{(i)}, t_j^{(i)}], \forall j = 1, \dots, M_i\}. \quad (\text{A.4})$$

These partitions together induce an n -dimensional rectangular partition $P \equiv P_1 \times \dots \times P_n$ of the rectangle R , formed by subrectangles

$$R_j = [t_{j_1-1}^{(1)}, t_{j_1}^{(1)}] \times \dots \times [t_{j_n-1}^{(n)}, t_{j_n}^{(n)}] \subset R \quad (\text{A.5})$$

that can only overlap on their edges. Each subrectangle also has an associated tag $\mathbf{u}_j = (u_{j_1}, \dots, u_{j_n})$, and we can define the norm of P as the maximum among the norms of each P_i , i.e.,

$$\|P\| := \max_{i=1, \dots, n} \|P_i\| = \max\{|t_j^{(i)} - t_{j-1}^{(i)}| : j = 1, \dots, M_i, i = 1, \dots, n\}. \quad (\text{A.6})$$

We define the Riemann-Stieltjes sum associated with f , g , and the n -dimensional tagged partition P of the rectangle R as

$$S_R(f, g; P) = \sum_j f(\mathbf{u}_j) \Delta_{R_j}^{(n)} g. \quad (\text{A.7})$$

Hence, we say that f is Riemann-Stieltjes integrable with respect to g in R when there exists $I \in \mathbb{C}$ such that, for every $\epsilon > 0$, there is some $\delta > 0$ for which

$$|S_R(f, g; P) - I| < \epsilon \quad (\text{A.8})$$

holds whenever $\|P\| < \delta$. In that case, we call I the Riemann-Stieltjes integral of f with respect to g over R ,

$$I \equiv \int_R dg f. \quad (\text{A.9})$$

Lemma 1. *Let $f, g : \mathbb{R}^n \rightarrow \mathbb{C}$, and let $R = [x_1, y_1] \times \dots \times [x_n, y_n]$ be a rectangle in $\mathbb{R}^n$, such that f is integrable over R with respect to g , and either $f = 0$ or $g = 0$ on the boundary ∂R . Then, g is also integrable over R with respect to f , and*

$$\int_R df g = (-1)^n \int_R dg f. \quad (\text{A.10})$$

Proof. Given $\epsilon > 0$, since f is integrable over R with respect to g , there exists $\delta > 0$ such that if $P = P_1 \times \dots \times P_n$ is an n -dimensional tagged partition of R with $\|P\| < \delta$, then

$$\left| S_R(f, g; P) - \int_R dg f \right| < \epsilon. \quad (\text{A.11})$$

Consider now a rectangular partition of R , $P = P_1 \times \dots \times P_n$, such that $\|P\| < \delta/2$. For each one-dimensional partition $P_i = \{t_0^{(i)} = x_i, \dots, t_{M_i}^{(i)} = y_i\}$ of the interval $[x_i, y_i]$, with tags $\{u_1, \dots, u_{M_i}\}$, consider another partition

$$\bar{P}_i := \{t_0^{(i)} = x_i, u_1, \dots, u_{M_i}, t_{M_i} = y_i\}, \quad (\text{A.12})$$

with set of tags $\bar{T}_i = P_i$, i.e.,

$$\bar{T}_i = \{v_1, \dots, v_{M_i}, v_{M_i+1} : v_1 = t_0, v_j = t_{j-1}, \forall j > 1\}. \quad (\text{A.13})$$

Notice in particular that the set of tags of $\bar{P}_i$ includes the limits of the interval, x_i and y_i . We can then define a new partition $\bar{P} = \bar{P}_1 \times \dots \times \bar{P}_n$, with set of tags $\bar{T} = \bar{T}_1 \times \dots \times \bar{T}_n$, which by construction satisfies $\|\bar{P}\| < \delta$. Moreover, a simple calculation shows that

$$S(f, g; \bar{P}) = (-1)^n S(g, f; P) + \text{'Boundary terms'}, \quad (\text{A.14})$$

where the boundary terms only include sums of products of f and g at points of the boundary ∂R . Since either $f = 0$ or $g = 0$ on the boundary, the associated terms cancel out. Eqs. (A.11) and (A.14) thus allow us to conclude that

$$\left| (-1)^n S_R(g, f; P) - \int_R dg f \right| < \epsilon, \quad (\text{A.15})$$

and therefore that g is integrable over R with respect to f , with

$$\int_R df g = (-1)^n \int_R dg f, \quad (\text{A.16})$$

as claimed. $\square$

Lemma 2. *Let $f, g : \mathbb{R}^n \rightarrow \mathbb{C}$, and let $R = [x_1, y_1] \times \dots \times [x_n, y_n]$ be a rectangle in $\mathbb{R}^n$. If f is bounded in R and g is of bounded variation in R , then we have that*

$$\left| \int_R dg f \right| \leq \|f\|_\infty \text{Var}_R(g), \quad (\text{A.17})$$

where

$$\|f\|_\infty = \sup_R |f| \quad (\text{A.18})$$

is the optimal bound for f in R , and

$$\text{Var}_R(g) := \sup_P \sum_j |\Delta_{R_j}^{(n)} g| \quad (\text{A.19})$$

is the variation of g in R , with the supremum taken over all rectangular partitions P of the rectangle R , and the notation used as in Eq. (A.7).

Proof. Since f is integrable over R with respect to g , for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\left| S_R(f, g; P) - \int_R dg f \right| < \epsilon \quad (\text{A.20})$$

whenever $\|P\| < \delta$. Now,

$$|S_R(f, g; P)| = \left| \sum_j f(\mathbf{u}_j) \Delta_{R_j}^{(n)} g \right| \leq \sum_j |f(\mathbf{u}_j)| |\Delta_{R_j}^{(n)} g| \leq \|f\|_\infty \sum_j |\Delta_{R_j}^{(n)} g| \leq \|f\|_\infty \text{Var}_R(g), \quad (\text{A.21})$$

and

$$\left| S_R(f, g; P) - \int_R dg f \right| \geq \left| \int_R dg f \right| - |S_R(f, g; P)|, \quad (\text{A.22})$$

so we conclude from Eq. (A.20) that

$$\left| \int_R dg f \right| \leq \|f\|_\infty \text{Var}_R(g) + \epsilon. \quad (\text{A.23})$$

Since Eq. (A.23) holds for all $\epsilon > 0$, this proves the claim. $\square$

Lemma 3. *Let $g : \mathbb{R}^n \rightarrow \mathbb{C}$, and let $R = [x_1, y_1] \times \dots \times [x_n, y_n]$ be a rectangle in $\mathbb{R}^n$. If g is of class $\mathcal{C}^n(R)$, then g is of bounded variation in R , and*

$$\text{Var}_R(g) = \int_R d\mathbf{z} \left| \frac{\partial^n g}{\partial z_1 \dots \partial z_n} \right|. \quad (\text{A.24})$$

Proof. Consider a rectangle $K = [v_1, w_1] \times \dots \times [v_n, w_n] \subset R$. Singling out the first coordinate, we can write

$$\Delta_K^{(n)}(g) = \sum_{\mathbf{z}} (-1)^{\#v(\mathbf{z})} [g(w_1, \mathbf{z}) - g(v_1, \mathbf{z})] = \sum_{\mathbf{z}} (-1)^{\#v(\mathbf{z})} g(w_1, \mathbf{z}) - \sum_{\mathbf{z}} (-1)^{\#v(\mathbf{z})} g(v_1, \mathbf{z}), \quad (\text{A.25})$$

where $\mathbf{z} = (z_2, \dots, z_n) \in \mathbb{R}^{n-1}$, and $z_i = v_i$ or w_i for each $i = 2, \dots, n$, with $\#v(\mathbf{z})$ being as before the number of components z_i of $\mathbf{z}$ that are equal to v_i . If we define

$$g_1(w) := \sum_{\mathbf{z}} (-1)^{\#v(\mathbf{z})} g(w, \mathbf{z}), \quad (\text{A.26})$$

then g_1 is continuously differentiable, and by the mean value theorem there exists $c_1 \in (v_1, w_1)$ such that

$$\Delta_K^{(n)}(g) = g'_1(c_1)(w_1 - v_1) = (w_1 - v_1) \sum_{\mathbf{z}} (-1)^{\#v(\mathbf{z})} \frac{\partial g}{\partial z_1}(c_1, \mathbf{z}). \quad (\text{A.27})$$

If $n = 1$, we stop here. Otherwise, we can define $K_1 := [v_2, w_2] \times \dots \times [v_n, w_n]$, and realize from Eq. (A.27) that

$$\Delta_K^{(n)}(g) = (w_1 - v_1) \Delta_{K_1}^{(n-1)}\left(\frac{\partial g}{\partial z_1}(c_1, \cdot)\right). \quad (\text{A.28})$$

Since g is of class $\mathcal{C}^n(R)$, the partial derivative is of class $\mathcal{C}^{n-1}(R)$, and in particular so is its restriction to K_1 . We can then repeat the previous argument with the partial derivative instead of g , concluding after this second step that there exists $c_2 \in (v_2, w_2)$ such that

$$\Delta_K^{(n)}(g) = (w_1 - v_1)(w_2 - v_2) \sum_{\mathbf{z}} (-1)^{\#v(\mathbf{z})} \frac{\partial^2 g}{\partial z_1 \partial z_2}(c_1, c_2, \mathbf{z}), \quad (\text{A.29})$$

where now $\mathbf{z} \in \mathbb{R}^{n-2}$ runs only over the last $n - 2$ components (if any). In general, after n iterations we find that there exists $\mathbf{c}$ in the interior of K such that

$$\Delta_K^{(n)}(g) = \left[\prod_{i=1}^n (w_i - v_i) \right] \frac{\partial^n g}{\partial z_1 \dots \partial z_n}(\mathbf{c}) = \text{Vol}(K) \frac{\partial^n g}{\partial z_1 \dots \partial z_n}(\mathbf{c}). \quad (\text{A.30})$$

Now, the variation of g in R is given by

$$\text{Var}_R(g) = \sup_P \sum_j |\Delta_{R_j}^{(n)} g|, \quad (\text{A.31})$$

where the supremum is taken over all possible rectangular partitions of R . By Eq. (A.30), given an arbitrary partition P of R ,

$$\sum_j |\Delta_{R_j}^{(n)} g| = \sum_j \left| \frac{\partial^n g}{\partial z_1 \dots \partial z_n}(\mathbf{c}_j) \right| \text{Vol}(R_j), \quad (\text{A.32})$$

where each $\mathbf{c}_j$ belongs to the interior of R_j . This corresponds to the Riemann sum of $\partial^n g / \partial z_1 \dots \partial z_n$ associated with the partition P with tags $\mathbf{c}_j$. Since $\partial^n g / \partial z_1 \dots \partial z_n$ is continuous, in particular it is Riemann integrable, and therefore given any $\epsilon > 0$, there exists $\delta > 0$ such that if $\|P\| < \delta$, then

$$\left| \int_R d\mathbf{z} \left| \frac{\partial^n g}{\partial z_1 \dots \partial z_n}(\mathbf{z}) \right| - \sum_j \left| \frac{\partial^n g}{\partial z_1 \dots \partial z_n}(\mathbf{c}_j) \right| \text{Vol}(R_j) \right| < \epsilon. \quad (\text{A.33})$$

In particular,

$$\int_R d\mathbf{z} \left| \frac{\partial^n g}{\partial z_1 \dots \partial z_n}(\mathbf{z}) \right| - \epsilon < \sum_j \left| \frac{\partial^n g}{\partial z_1 \dots \partial z_n}(\mathbf{c}_j) \right| \text{Vol}(R_j) \leq \text{Var}_R(g). \quad (\text{A.34})$$

This is true for all $\epsilon > 0$, and thus

$$\int_R d\mathbf{z} \left| \frac{\partial^n g}{\partial z_1 \dots \partial z_n}(\mathbf{z}) \right| \leq \text{Var}_R(g). \quad (\text{A.35})$$

Conversely, for any partition P ,

$$\sum_j |\Delta_{R_j}^{(n)} g| = \sum_j \left| \int_{R_j} d\mathbf{z} \frac{\partial^n g}{\partial z_1 \dots \partial z_n}(\mathbf{z}) \right| \quad (\text{A.36})$$

$$\leq \sum_j \int_{R_j} d\mathbf{z} \left| \frac{\partial^n g}{\partial z_1 \dots \partial z_n}(\mathbf{z}) \right| = \int_R d\mathbf{z} \left| \frac{\partial^n g}{\partial z_1 \dots \partial z_n}(\mathbf{z}) \right|. \quad (\text{A.37})$$

The supremum taken over all possible partitions still verifies the inequality, i.e.,

$$\text{Var}_R(g) \leq \int_R d\mathbf{z} \left| \frac{\partial^n g}{\partial z_1 \dots \partial z_n}(\mathbf{z}) \right|. \quad (\text{A.38})$$

Combining Eqs. (A.35) and (A.38) gives us the desired result. $\square$

Appendix B

Generation and processing of simulated data

B.1 Preprocessing, Neural Network Architecture and Training Details

As we described in Ch. 7, our measurement procedure and compression produces labeled data consisting of data $D_c \in \mathbb{R}^d$ where $d = 9 N_{\text{TIMES}}$ and an associated label y . To begin training we collect $n = N_{\text{SAMPLES}}$ instances of this labeled data into a $d \times N_{\text{SAMPLES}}$ design matrix $\mathbf{X} = (D_1, \dots, D_n)^\top$ and a vector of labels $\mathbf{y} = (y_1, \dots, y_n)^\top$. We then portion off 75% of this data ($N_{\text{TRAIN}} = 0.75 N_{\text{SAMPLES}}$) to be used for training the neural network, $\mathbf{X}_{\text{TRAIN}}$ and $\mathbf{y}_{\text{TRAIN}}$, leaving the other 25% ($N_{\text{VALID}} = 0.25 N_{\text{SAMPLES}}$) as validation data, $\mathbf{X}_{\text{VALID}}$ and $\mathbf{y}_{\text{VALID}}$, which we will ultimately use to test the accuracy of the trained network. Note that the network will not be exposed to any of the validation data during training.

We begin processing our data by subtracting off the mean of the training data, $\mathbf{X} \rightarrow \mathbf{X} - \mathbf{X}_{\text{TRAIN}}^{\text{avg}}$, where $\mathbf{X}_{\text{TRAIN}}^{\text{avg}} = \sum_{k=1}^{N_{\text{TRAIN}}} D_k / N_{\text{TRAIN}}$. Next we do principle component analysis (PCA), which finds a representation of our data without linear correlations. To do this we compute the covariance matrix of our training data and perform a singular value decomposition on it,

$$\frac{1}{N_{\text{TRAIN}} - 1} \mathbf{X}_{\text{TRAIN}}^\top \mathbf{X}_{\text{TRAIN}} = V^\top \Lambda V = \sum_{j=1}^d \lambda_j \mathbf{v}_j \mathbf{v}_j^\top, \quad (\text{B.1})$$

where $V = (\mathbf{v}_1, \dots, \mathbf{v}_d)^\top$ is the matrix of singular vectors, $\mathbf{v}_j$, and $\Lambda = \text{diag}(\lambda_0, \dots, \lambda_d)$ is the matrix of singular values, $\lambda_j \in \mathbb{R}_+$. The singular vectors are the directions in which our data varies independently, and the singular values indicate “how much” variance is in each direction. Using this decomposition we can rewrite our data in this singular basis by taking $\mathbf{X} \rightarrow V\mathbf{X}$. After this transformation, the training data has a diagonal covariance matrix, namely Λ . Finally we can whiten the data by taking $\mathbf{X} \rightarrow \Lambda^{-1/2}\mathbf{X}$. The covariance matrix of the training data is now the identity matrix. We do this in order to force the neural network to take into account all components of the data, since they come from sources that are not supposed to be directly comparable in magnitude. Note that we have not used PCA to compress our data; that is, we have not cut any small singular values out of Λ as is commonly done.

The data is now ready to begin training the neural network. As discussed in the main text, neural networks work by alternatingly applying tuneable linear-affine transformations (controlled by weights and biases) and fixed non-linear transformations (the activation function) to their inputs. See Fig. 7.1 for a schematic of a neural network that can be used to classify the topology of a QFT based on local probe measurement data.

We will now use the architecture in Fig. 7.1 as a basic illustrative example. In this example the network accepts a 5-dimensional input, $\mathbf{x}^{(0)}$, into the left-most layer of the network (note that in the examples discussed in the main text the input dimension is much larger). In passing this data to the next layer of the network, a linear-affine transformation is applied to $\mathbf{x}^{(0)}$, as $\mathbf{x}^{(1)} = A^{(1)}\mathbf{x}^{(0)} + \mathbf{b}^{(1)}$. The weight matrix $A^{(1)}$ here has dimensions 7×5 and the bias vector has a dimension of 7 such that $\mathbf{x}^{(1)}$ is 7-dimensional. The $7 \times 5 + 7 = 42$ values which determine this linear-affine transformation are left as free parameters to be optimized during training. Next, a fixed non-linear function, $\mathcal{G}^{(1)}$ is applied element-wise to each entry of $\mathbf{x}^{(1)}$ yielding $\mathbf{z}^{(1)} = \mathcal{G}^{(1)}(\mathbf{x}^{(1)})$. For instance $\mathcal{G}^{(1)}$ may be the hyperbolic tangent function or a rectified linear unit.

This process is then repeated at each layer. First a linear affine transformation is applied to $\mathbf{z}^{(1)}$ as $\mathbf{x}^{(2)} = A^{(2)}\mathbf{z}^{(1)} + \mathbf{b}^{(2)}$ where $A^{(2)}$ has dimensions 5×7 and $\mathbf{b}^{(2)}$ has dimension 5. Then a fixed non-linear function, $\mathcal{G}^{(2)}$ is applied element-wise to $\mathbf{x}^{(2)}$ yielding $\mathbf{z}^{(2)} = \mathcal{G}^{(2)}(\mathbf{x}^{(2)})$. In the final layer we have $\mathbf{x}^{(3)} = A^{(3)}\mathbf{z}^{(2)} + \mathbf{b}^{(3)}$ where $A^{(3)}$ has dimensions 2×5 and $\mathbf{b}^{(3)}$ has dimension 2 and $\mathbf{z}^{(3)} = \mathcal{G}^{(3)}(\mathbf{x}^{(3)})$ for some non-linear function, $\mathcal{G}^{(3)}$. In total this network computes the function $f(\mathbf{x}^{(1)}; A, b) = \mathbf{z}^{(3)}$ where A and b refer to this network’s 94 free parameters collectively.

There are two different problem types we need to design a network for, classification and regression. In classification, our network is tasked with deciding which of several classes (given by a discrete label y) our data belongs to. In this scenario we take the number of

neurons in the final layer to be equal to the number of classes, and the final activation function to be a softmax. This ensures that the network’s final output is a probability distribution that can be interpreted as the probability that the initial data belongs to each class. In regression, our network is tasked with assigning the data a continuous label y . In this case we take the final layer to have a single neuron.

In the examples discussed in the paper we considered a network consisting of 90 neurons on the input layer, 30 in the intermediate (hidden) layer and either two or one neurons in the final layer depending on which example we are doing. In the boundary sensing and state discrimination examples we have two neurons in the final layer. In the thermometry cases we have only one neuron in the final layer. All of the non-linear activation functions were taken to be leaky rectified linear units [208].

The network’s weights and biases are tuned to minimize error of the network’s predictions over the training set. To quantify this error we define the following cost functions,

$$\text{Classification:} \quad C(A, b) = \frac{-1}{N_{\text{TRAIN}}} \sum_{k=1}^{N_{\text{TRAIN}}} \tilde{y}_k \cdot \log(f(\mathbf{x}_k; A, b)) \quad (\text{B.2})$$

$$\text{Regression:} \quad C(A, b) = \frac{1}{N_{\text{TRAIN}}} \sum_{k=1}^{N_{\text{TRAIN}}} (f(\mathbf{x}_k; A, b) - y_k)^2 \quad (\text{B.3})$$

where $\tilde{y}_k$ is the one-hot encoding of the k^{th} data point’s label. For the classification scenario, our cost function is the relative entropy between the network’s probability assignment and the expected result. For the regression case, the cost function is the mean square error. To help reduce overfitting (that is, an excessively close alignment of the network’s model to the training data that might end up worsening its performance with real—or validation—data) we add an L_2 regularizer to this cost function, $\lambda_2 \|A\|_2^2$, for some chosen λ_2 . This reduces the complexity of the model by penalizing the network for using large weights. Additionally, when training the network, we randomly “drop” some fraction of the neurons. This forces the network to be more robust. The sum of the cost function and the regularizer are then minimized by stochastic gradient descent [208].

B.2 Total Variation and Hellinger Distances for Binary Boundary Classification

In this appendix we add details on the analysis of the near-optimal performance of the neural network in our boundary sensing example (Sec. 7.4.1). Notice that in this case study

the probe remains Gaussian throughout its evolution under the Hamiltonians specified in the main text, as they are quadratic [211, 212, 215]. Therefore, although it is not a necessary feature for our methods to work, in our first physical example the data is distributed normally and therefore the compression step is lossless.

Consider the binary classification problem where we are asked to pick a label $y = 0$ or $y = 1$ for data $\mathbf{x}_0$ drawn from either $r_\theta(\mathbf{x}) = p(\mathbf{x}|y = 0, \theta)$ or from $q_\theta(\mathbf{x}) = p(\mathbf{x}|y = 1, \theta)$ with equal odds, where θ is some free parameter of the problem. In terms of the classification scenario considered in the main text, $\mathbf{x}$ is the local probe data, y labels the field's boundary condition and θ describes the other details of the scenario, for instance T_{MIN} . The distributions $r_\theta(\mathbf{x})$ and $q_\theta(\mathbf{x})$ are then the odds that some particular data was produced given the state of the field and measurement procedure.

The optimal strategy (i.e., the one which maximizes your success probability) for this binary classification problem is to guess $y = 1$ if $q_\theta(\mathbf{x}) > r_\theta(\mathbf{x})$ and $y = 0$ if $r_\theta(\mathbf{x}) > q_\theta(\mathbf{x})$, breaking ties randomly. This strategy succeeds with probability of $p_{\text{success}} = \frac{1}{2}(1 + \text{TV}(r_\theta, q_\theta))$ where

$$\text{TV}(r_\theta, q_\theta) = \frac{1}{2} \int d\mathbf{x} |r_\theta(\mathbf{x}) - q_\theta(\mathbf{x})| \quad (\text{B.4})$$

is the total variation distance between $r_\theta(\mathbf{x})$ and $q_\theta(\mathbf{x})$. If we can compute this distance, we can determine for which values of θ (e.g., for which coupling times) the distributions $r_\theta(\mathbf{x})$ and $q_\theta(\mathbf{x})$ are distinguishable.

The total variation distance is only useful for binary classification problems (such as our remote boundary sensing example in the main text). It cannot be used in this way when there are more than two classes or for regression problems, such as our thermometry example. In the remote boundary sensing scenario, calculating the total variation distance directly is infeasible. An alternate approach is to compute upper and lower bounds on TV using the Hellinger distance, $H(r_\theta, q_\theta)$ [254], as

$$H(r_\theta, q_\theta)^2 \leq \text{TV}(r_\theta, q_\theta) \leq H(r_\theta, q_\theta) \sqrt{2 - H(r_\theta, q_\theta)^2}, \quad (\text{B.5})$$

where

$$H(r, q) = \sqrt{\frac{1}{2} \int d\mathbf{x} [\sqrt{r(\mathbf{x})} - \sqrt{q(\mathbf{x})}]^2}. \quad (\text{B.6})$$

Unfortunately, for the distribution of the compressed data, the Hellinger distance is not easier to compute. However, in the high tomography regime ($N_{\text{TOM}} \gg 1$) we can apply the central limit theorem to approximate r_θ and q_θ by multivariate normal distributions, $r_\theta(\mathbf{x}) = \mathcal{N}(\mathbf{x}; \mu_r, \Sigma_r)$ and $q_\theta(\mathbf{x}) = \mathcal{N}(\mathbf{x}; \mu_q, \Sigma_q)$ for some means, μ_r and μ_q , and some

covariances, Σ_q and Σ_r . The Hellinger distance between two such multivariate normal distributions is given by [254]

$$H(r_\theta, q_\theta)^2 = 1 - \left[\frac{\det(\Sigma_r \Sigma_q)}{\det(\bar{\Sigma}^2)} \right]^{\frac{1}{4}} \exp \left(- \frac{\Delta\mu^\top \bar{\Sigma}^{-1} \Delta\mu}{8} \right), \quad (\text{B.7})$$

where $\Delta\mu = \mu_r - \mu_q$ and $\bar{\Sigma} = (\Sigma_r + \Sigma_q)/2$. Thus if we can compute the means and covariances of our data in the central limit, we can find bounds for the neural network's optimal performance.

Due to the Gaussian nature of our setup all of our measurement results were drawn from normal distributions. Moreover, in the main text, we discussed how our data can be compressed by just considering the sample means and sample variances of each quadrature at each time point. For clarity we will restrict the following discussion to the results, q_k , of our N_{TOM} measurements of $\hat{q}$ at some $t = T_{\text{MIN}} + M\Delta t$, where $M \in \{0, \dots, N_{\text{TIMES}} - 1\}$. The sample mean and variance of these measurement outcomes are distributed as

$$\bar{q} = \frac{1}{N_{\text{TOM}}} \sum_{k=1}^{N_{\text{TOM}}} q_k \sim \mathcal{N} \left(\mu_q, \frac{\sigma_{qq}}{N_{\text{TOM}}} \right), \quad (\text{B.8})$$

and

$$\bar{s}_q = \frac{1}{N_{\text{TOM}}} \sum_{k=1}^{N_{\text{TOM}}} (q_k - \bar{q})^2 \sim \sigma_{qq} \frac{\chi^2(N_{\text{TOM}} - 1)}{N_{\text{TOM}}}, \quad (\text{B.9})$$

where $\chi^2(k)$ is the chi-squared distribution with k degrees of freedom [255] and $\mu_q = \langle \hat{q} \rangle$ and $\sigma_{qq} = \langle \hat{q}^2 \rangle - \langle \hat{q} \rangle^2$ are the probe's first moment and variance in $\hat{q}$ at time t . Due to the Gaussian nature of our setup, these moments can be efficiently computed [214, 256]. Moreover, for independent identically distributed normal data, q_k , the sample mean and variance are sufficient statistics [257] such that this compression is lossless.

Note that the compressed data is not normally distributed. However, for large N_{TOM} , we can apply the central limit theorem yielding

$$\bar{q} \sim \mathcal{N} \left(\mu_q, \frac{\sigma_{qq}}{N_{\text{TOM}}} \right), \quad \text{and} \quad \bar{s}_q \sim \mathcal{N} \left(\sigma_{qq}, \frac{2\sigma_{qq}^2}{N_{\text{TOM}}} \right). \quad (\text{B.10})$$

The same discussion applies equally well for our measurements of $\hat{p}$ and $\hat{r}$ at each time. Thus in the high tomography regime, our compressed data,

$$\mathbf{x} = (\bar{q}(t), \bar{r}(t), \bar{p}(t), \bar{s}_q(t), \bar{s}_r(t), \bar{s}_p(t)) \quad \text{for } t = T_{\text{MIN}}, T_{\text{MIN}} + \Delta t, \dots, T_{\text{MAX}}, \quad (\text{B.11})$$

is distributed as $\mathbf{x} \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$ where

$$\boldsymbol{\mu} = (\langle \hat{q}(t) \rangle, \langle \hat{r}(t) \rangle, \langle \hat{p}(t) \rangle, \sigma_{qq}(t), \sigma_{rr}(t), \sigma_{pp}(t))_{t=T_{\min}, T_{\min}+\Delta t, \dots, T_{\max}}, \quad (\text{B.12})$$

$$\Sigma = \text{diag} \left(\frac{\sigma_{qq}(t)}{N_{\text{TOM}}}, \frac{\sigma_{rr}(t)}{N_{\text{TOM}}}, \frac{\sigma_{pp}(t)}{N_{\text{TOM}}}, \frac{2\sigma_{qq}^2(t)}{N_{\text{TOM}}}, \frac{2\sigma_{rr}^2(t)}{N_{\text{TOM}}}, \frac{2\sigma_{pp}^2(t)}{N_{\text{TOM}}} \right)_{t=T_{\min}, T_{\min}+\Delta t, \dots, T_{\max}}, \quad (\text{B.13})$$

where $T_{\max} = T_{\min} + (N_{\text{TIMES}} - 1) \Delta t$. Knowing this distribution, we can compute the Hellinger distance (Eq. (B.5)) and place bounds on the optimal classification rate.

B.3 Moments of non-Gaussian states

In this appendix we show how to calculate the moments of the non-Gaussian state which results from applying a Gaussian unitary to a non-Gaussian state with known moments.

In particular, recall from Secs. 7.3 and 7.4.3 that we consider a Gaussian unitary interaction between a harmonic oscillator probe system coupled to a quantum field in a (1+1)-dimensional cavity. The probe is characterized by the dimensionless quadrature operators $\hat{q}_{\text{D}}$ and $\hat{p}_{\text{D}}$, and the field operator can be decomposed into modes as in Eq. (7.35),

$$\hat{\phi}(t, \mathbf{x}) = \sqrt{\frac{2\hbar c^2}{L}} \sum_{\ell=1}^{\infty} \frac{\sin(k_{\ell} x)}{\sqrt{\omega_{\ell}}} [\hat{q}_{\ell} \cos(\omega_{\ell} t) + \hat{p}_{\ell} \sin(\omega_{\ell} t)], \quad (\text{B.14})$$

where the dimensionless quadrature operators $\hat{q}_n$ and $\hat{p}_m$ satisfy canonical commutation relations $[\hat{q}_n, \hat{p}_m] = i\delta_{nm}\mathbb{1}$ and where $c k_{\ell} = \omega_{\ell} = \pi \ell c/L$. As in Eq. (7.9), we collect all of these dimensionless quadrature operators together into a operator-valued phase space vector,

$$\hat{\mathbf{X}} := (\hat{q}_{\text{D}}, \hat{p}_{\text{D}}, \hat{q}_1, \hat{p}_1, \hat{q}_2, \hat{p}_2, \dots)^{\text{T}}. \quad (\text{B.15})$$

B.3.1 The effect of Gaussian unitaries on non-Gaussian states

Consider a scenario in which the initial state of the probe-field system is non-Gaussian with well known moments, e.g., the probe in its (Gaussian) ground state and the field in a Fock state. Suppose that the joint system undergoes a generic Gaussian unitary interaction, $\hat{U}$. Such a transformation maps Gaussian states to Gaussian states. In this case there is a well

known relationship between the first and second moments of the initial and final states. Conversely, when applied to a non-Gaussian state, the result of such a transformation would be another non-Gaussian state. Note that in this case, since the field is in a non-Gaussian state, the reduced state of the probe may end up being non-Gaussian even if its initial state was Gaussian.

However, despite this complication, just as in the Gaussian case, there is still a relatively simple systematic relationship between all of the higher-moments before and after this transformation. That is, given the moments of the initial non-Gaussian state, we can efficiently determine all of the moments of the final non-Gaussian state.

To show this, we first note that Gaussian unitary transformations correspond to symplectic transformations in phase space. That is, if we act on each component of the vector of operators, $\hat{\mathbf{X}}$, via some Gaussian unitary, $\hat{U}$, the result is a symplectic(-affine) transformation of the phase space vector $\hat{\mathbf{X}}$ itself. Namely,

$$\hat{U}^\dagger \hat{\mathbf{X}} \hat{U} := \begin{pmatrix} \hat{U}^\dagger \hat{X}_1 \hat{U} \\ \hat{U}^\dagger \hat{X}_2 \hat{U} \\ \hat{U}^\dagger \hat{X}_3 \hat{U} \\ \hat{U}^\dagger \hat{X}_4 \hat{U} \\ \dots \end{pmatrix} = S \begin{pmatrix} \hat{X}_1 \\ \hat{X}_2 \\ \hat{X}_3 \\ \hat{X}_4 \\ \dots \end{pmatrix} + \begin{pmatrix} d_1 \hat{\mathbb{1}} \\ d_2 \hat{\mathbb{1}} \\ d_3 \hat{\mathbb{1}} \\ d_4 \hat{\mathbb{1}} \\ \dots \end{pmatrix} =: S \hat{\mathbf{X}} + \mathbf{d} \hat{\mathbb{1}} \quad (\text{B.16})$$

for some symplectic transformation S with $S\Omega S^\top = \Omega$ (where Ω is the symplectic form) and some displacement vector $\mathbf{d}$. Note that in the above equation $\hat{U}$ acts as a linear map on the system's Hilbert space and acts on $\hat{\mathbf{X}}$ component-wise. On the other hand, S is a linear map on the system's phase space and acts on $\hat{\mathbf{X}}$ as a phase space vector, yielding linear combinations of its (operator-valued) components.

We note that this Gaussian unitary (or the equivalent symplectic) relationship is a property of the dynamics alone, independent of the system state. Thus, we can use this characterization of Gaussian unitaries to understand their effect on non-Gaussian states. The effect that a Gaussian unitary has on a state's Wigner function (even a non-Gaussian one) is,

$$U : W_{\hat{\rho}}(\boldsymbol{\xi}) \mapsto W_{\hat{U}\hat{\rho}\hat{U}^\dagger}(\boldsymbol{\xi}) = W_{\hat{\rho}}(S^{-1}\boldsymbol{\xi} - \mathbf{d}). \quad (\text{B.17})$$

That is, the effect of a Gaussian unitary is just to transform the original Wigner function by applying a linear-affine transformation to the joint phase space variables. In the above equation, $\boldsymbol{\xi}$ is the real-valued vector of phase space variables, $\boldsymbol{\xi} := (q_D, p_D, q_1, p_1, q_2, p_2, \dots)^\top$. For simplicity we will now restrict our attention to situations with zero displacement, $\mathbf{d} = 0$, since this is the case relevant for Sec. 7.4.3.

Next, we can use Eq. (B.17) to determine the moments of the final probe distribution from the initial probe field moments. For instance, suppose that we are interested in the fourth moment of the probe's quadrature $\hat{q}_D$ after the interaction with the field. We can calculate this as follows. Let $\mathbf{q}_D = (1, 0, 0, \dots)^\top$ be a phase space vector such that $\mathbf{q}_D^\top \boldsymbol{\xi} = q_D$. That is, $\mathbf{q}_D$ isolates q_D from the vector of phase space variables $\boldsymbol{\xi}$. This allows us to rewrite the desired fourth moment as

$$\langle \hat{q}_D^4 \rangle_{\hat{U} \hat{\rho} \hat{U}^\dagger} = \int d\xi q_D^4 W_{\hat{U} \hat{\rho} \hat{U}^\dagger}(\boldsymbol{\xi}) = \int d\xi (\mathbf{q}_D^\top \boldsymbol{\xi})^4 W_{\hat{\rho}}(S^{-1} \boldsymbol{\xi}). \quad (\text{B.18})$$

Note we have assumed $\mathbf{d} = \mathbf{0}$ for simplicity. Making a canonical change of variables to $\boldsymbol{\xi}' = S^{-1} \boldsymbol{\xi}$ we have

$$\langle \hat{q}_D^4 \rangle_{\hat{U} \hat{\rho} \hat{U}^\dagger} = \int d\xi' (\mathbf{q}_D^\top S \boldsymbol{\xi}')^4 W_{\hat{\rho}}(\boldsymbol{\xi}') = \int d\xi' (\mathbf{Q}_D^\top \boldsymbol{\xi}')^4 W_{\hat{\rho}}(\boldsymbol{\xi}'), \quad (\text{B.19})$$

where we have defined $\mathbf{Q}_D := S^\top \mathbf{q}_D \Rightarrow \mathbf{q}_D = S^{-\top} \mathbf{Q}_D$. Note that for all symplectic transformations, $\det(S) = 1$, such that no Jacobian factor arises in the above change of variables. The fourth moment of q_D in the final state is equal to the fourth moment of $\mathbf{Q}_D$ in the initial state. Since we have assumed that we know all of the moments of the initial non-Gaussian state, we can, at least in principle, calculate $\langle \hat{q}_D^4 \rangle_{\hat{U} \hat{\rho} \hat{U}^\dagger}$. In general, $\mathbf{Q}_D$ will have support over the probe portion of the phase spaces as well as over a great deal of the field phase space. As such, calculating $\langle \hat{q}_D^4 \rangle_{\hat{U} \hat{\rho} \hat{U}^\dagger}$ requires us to know the correlations between all of the field modes. In particular we have

$$\langle \hat{q}_D^4 \rangle_{\hat{U} \hat{\rho} \hat{U}^\dagger} = \int d\xi (\mathbf{Q}_D^\top \boldsymbol{\xi})^4 W_{\hat{\rho}}(\boldsymbol{\xi}) = \int d\xi \left(\sum_{\ell=1}^{\infty} Q_{D,\ell} \xi_\ell \right)^4 W_{\hat{\rho}}(\boldsymbol{\xi}) \quad (\text{B.20})$$

$$= \int d\xi \sum_{i,j,k,\ell=1}^{\infty} Q_{D,i} Q_{D,j} Q_{D,k} Q_{D,\ell} \xi_i \xi_j \xi_k \xi_\ell W_{\hat{\rho}}(\boldsymbol{\xi}) \quad (\text{B.21})$$

$$= \sum_{i,j,k,\ell=1}^{\infty} Q_{D,i} Q_{D,j} Q_{D,k} Q_{D,\ell} \int d\xi \xi_i \xi_j \xi_k \xi_\ell W_{\hat{\rho}}(\boldsymbol{\xi}) \quad (\text{B.22})$$

$$= \sum_{i,j,k,\ell=1}^{\infty} Q_{D,i} Q_{D,j} Q_{D,k} Q_{D,\ell} T_{4,i,j,k,\ell} = T_4(\mathbf{Q}_D, \mathbf{Q}_D, \mathbf{Q}_D, \mathbf{Q}_D) \quad (\text{B.23})$$

where $T_{4,i,j,k,\ell} = \int d\xi \xi_i \xi_j \xi_k \xi_\ell W_{\hat{\rho}}(\boldsymbol{\xi})$ are the fourth moments of the initial Wigner function, and T_4 collects these. The desired probe moment, $\langle \hat{q}_D^4 \rangle_{\hat{U} \hat{\rho} \hat{U}^\dagger}$, is this tensor evaluated on four

copies of the phase space vector $\mathbf{Q}_D$. Similarly, for the fourth moments of $\hat{p}_D$ and $\hat{r}_D$,

$$\langle \hat{p}_D^4 \rangle_{\hat{U}\hat{\rho}\hat{U}^\dagger} = T_4(\mathbf{P}_D, \mathbf{P}_D, \mathbf{P}_D, \mathbf{P}_D), \quad (\text{B.24})$$

$$\langle \hat{r}_D^4 \rangle_{\hat{U}\hat{\rho}\hat{U}^\dagger} = T_4(\mathbf{R}_D, \mathbf{R}_D, \mathbf{R}_D, \mathbf{R}_D), \quad (\text{B.25})$$

where $\mathbf{p}_D := (0, 1, 0, \dots)^\top$ and $\mathbf{P}_D := S^\top \mathbf{p}_D$, $\mathbf{R}_D = (\mathbf{Q}_D + \mathbf{P}_D)/\sqrt{2}$. Any other final probe moments (e.g. $\langle \hat{q}_D^2 \rangle_{\hat{U}\hat{\rho}\hat{U}^\dagger}$ or $\langle \hat{p}_D^8 \rangle_{\hat{U}\hat{\rho}\hat{U}^\dagger}$) can be calculated in analogous ways. All that one needs is an understanding of the moments of the initial non-Gaussian state (i.e., the tensors T_2, T_4, T_8 , etc.) as well as the vectors $\mathbf{Q}_D$ and $\mathbf{P}_D$ which the Gaussian unitary interaction maps into the probe's phase space.

B.3.2 Two delta pulse interaction

As we showed in the previous subsection, in order to construct the desired probe moments from the moments of the initial non-Gaussian state, we need to know the vectors $\mathbf{Q}_D := S^\top \mathbf{q}_D$ and $\mathbf{P}_D := S^\top \mathbf{p}_D$. These are the vectors which the interaction maps to the probe observables $\hat{q}_D$ and $\hat{p}_D$. To identify these vectors we first need to compute the symplectic transformation S which is associated with our Gaussian unitary evolution.

Let us consider an interaction Hamiltonian given by Eq. (7.1) with a switching function which is the sum of two delta functions, specifically, $\chi(t) = \delta(t) + \delta(t - t_M)$. In this scenario the probe undergoes a very strong, very brief interaction with the field once at $t = 0$ and then once again at $t = t_M > 0$. Between these two times, the probe evolves freely. As such, the operator with which it couples to the field, $\hat{\mu} = \hat{q}_D$, evolves in the interaction picture as

$$\hat{\mu}(t) = \hat{q}_D(t) = \hat{q}_D \cos(\omega_D t) + \hat{p}_D \sin(\omega_D t), \quad (\text{B.26})$$

where ω_D is the probe's natural frequency. The full unitary map resulting from both of these sudden interactions is given by

$$\hat{U} = \hat{U}_{\text{int}}(t_M) \hat{U}_{\text{int}}(0), \quad (\text{B.27})$$

where

$$\hat{U}_{\text{int}}(t) = \exp \left(-i\lambda \hat{\mu}(t) \int_{-\infty}^{\infty} dx F(x) \hat{\phi}(t, x) \right). \quad (\text{B.28})$$

Rewriting this in terms of the operator valued phase space vector, $\hat{\mathbf{X}}$, we have

$$\hat{U}_{\text{int}}(t) = \exp \left(-i\lambda \frac{1}{2} \hat{\mathbf{X}}^\top \mathcal{H}(t) \hat{\mathbf{X}} \right), \quad (\text{B.29})$$

with

$$\mathcal{H}(t) = \mathbf{u}(t)\mathbf{v}(t)^\top + \mathbf{v}(t)\mathbf{u}(t)^\top, \quad (\text{B.30})$$

$$\mathbf{u}(t) = (\cos(\omega_{\text{D}}t), \sin(\omega_{\text{D}}t), 0, 0, \dots)^\top, \quad (\text{B.31})$$

$$\mathbf{v}(t) = \left(0, 0, \frac{F_1 \cos(\omega_1 t)}{\sqrt{\omega_1}}, \frac{F_1 \sin(\omega_1 t)}{\sqrt{\omega_1}}, \frac{F_2 \cos(\omega_2 t)}{\sqrt{\omega_2}}, \frac{F_2 \sin(\omega_2 t)}{\sqrt{\omega_2}}, \dots \right)^\top, \quad (\text{B.32})$$

where $F_n = \sqrt{2\hbar c^2/L} \int_{-\infty}^{\infty} dx F(x) \sin(k_n x)$. Note that $\mathbf{u}(t)$ has support only on the probe sector of the phase space. It tracks the evolution of the probe observable, $\hat{\mu} = \hat{q}_{\text{D}}$, through time. Similarly, $\mathbf{v}(t)$ has support only on the field sector of the phase space, and it tracks the evolution of the field observable, $\int_{-\infty}^{\infty} dx F(x) \hat{\phi}(t, x)$.

The symplectic transformation S corresponding to U is given by

$$S = S_{\text{int}}(t_{\text{M}}) S_{\text{int}}(0), \quad (\text{B.33})$$

where

$$S_{\text{int}}(t) = \exp(\lambda \Omega \mathcal{H}(t)). \quad (\text{B.34})$$

In this case, the above symplectic transformations are easy to compute. This is due to the fact that the matrix $\Omega \mathcal{H}(t)$ is nilpotent, and in particular, $[\Omega \mathcal{H}(t)]^2 = 0$. This follows from the four orthogonality relations

$$\mathbf{v}(t)^\top \Omega \mathbf{u}(t) = 0, \quad \mathbf{u}(t)^\top \Omega \mathbf{u}(t) = 0, \quad \mathbf{v}(t)^\top \Omega \mathbf{v}(t) = 0, \quad \mathbf{u}(t)^\top \Omega \mathbf{v}(t) = 0. \quad (\text{B.35})$$

The four equalities hold because Ω is anti-symmetric and because Ω does not mix the probe and field portions of the phase space (therefore, $\Omega \mathbf{u}(t)$ and $\mathbf{v}(t)$ have no common support). The nilpotence of $\Omega \mathcal{H}(t)$ makes the matrix exponential trivial to compute. Indeed since $\Omega \mathcal{H}(t)$ squares to the zero operator we have

$$S_{\text{int}}(t) = \mathbb{1} + \lambda \Omega \mathcal{H}(t) = \mathbb{1} + \lambda \Omega (\mathbf{u}(t)\mathbf{v}(t)^\top + \mathbf{v}(t)\mathbf{u}(t)^\top), \quad (\text{B.36})$$

$$S_{\text{int}}^{-1}(t) = \mathbb{1} - \lambda \Omega \mathcal{H}(t) = \mathbb{1} - \lambda \Omega (\mathbf{u}(t)\mathbf{v}(t)^\top + \mathbf{v}(t)\mathbf{u}(t)^\top). \quad (\text{B.37})$$

Note that these expressions are exact, not perturbative. Applying these transformations to $\mathbf{q}_{\text{D}} := (1, 0, 0, \dots)^\top$ and $\mathbf{p}_{\text{D}} := (0, 1, 0, \dots)^\top$ gives $\mathbf{Q}_{\text{D}}$ and $\mathbf{P}_{\text{D}}$ as desired.

B.4 Moments of Fock and Phase-averaged states

In this appendix we will calculate the second, fourth and eighth moments for Fock and Phase-averaged states. The initial probe-field Wigner function is given by Eq. (7.36) in Sec. 7.4.3, namely,

$$W(q_D, p_D, q_1, p_1, q_2, p_2, \dots; N) = W_{\text{Vac}}(q_D, p_D) W_{\text{Fock}}(q_1, p_1; N) \prod_{n=2}^{\infty} W_{\text{Vac}}(q_n, p_n), \quad (\text{B.38})$$

or

$$W(q_D, p_D, q_1, p_1, q_2, p_2, \dots; N) = W_{\text{Vac}}(q_D, p_D) W_{\text{PAC}}(q_1, p_1; N) \prod_{n=2}^{\infty} W_{\text{Vac}}(q_n, p_n), \quad (\text{B.39})$$

where q_D and p_D are the probe variables, and $W_{\text{Vac}}(q, p) = e^{-q^2 - p^2}/\pi$. Note that the probe and all the modes are uncorrelated from each other. Thus, all mixed moments factorize (e.g., $\langle q_4 p_4^2 q_6^3 p_6^5 \rangle = \langle q_4 p_4^2 \rangle \langle q_6^3 p_6^5 \rangle$). Note also that these averages are taken with respect to the Wigner function, so that when we calculate the average of a certain dynamical function $f(\xi)$, which we will denote $\langle f(\xi) \rangle$, this corresponds to the expectation value of the Weyl quantized operator $\langle \hat{f}(\xi) \rangle$, which is the one that we obtain by using symmetric ordering in the quantization scheme [213, 258–260].

For the vacuum Wigner function,

$$\langle q^2 \rangle_{\text{vac}} = 1/2, \quad \langle q^4 \rangle_{\text{vac}} = 3/4, \quad \langle q^6 \rangle_{\text{vac}} = 15/8, \quad \langle q^8 \rangle_{\text{vac}} = 105/16, \quad (\text{B.40})$$

with $\langle p^n \rangle = \langle q^n \rangle$, $\langle p^n q^m \rangle = \langle p^n \rangle \langle q^m \rangle$, and all odd moments vanishing. We only write explicitly up to the eighth moments that we need for our purposes, but of course for the modes in the vacuum all their odd moments vanish and their even moments are trivial functions of their second moments.

The statistics ($\langle q^2 \rangle_N$, $\langle q^4 \rangle_N$, $\langle q^6 \rangle_N$, $\langle q^8 \rangle_N$) of the Fock and PAC states Wigner functions are given in Tables B.1 and B.2, respectively, with $\langle p^n \rangle = \langle q^n \rangle$, $\langle p^n q^m \rangle = \langle p^n \rangle \langle q^m \rangle$, and the odd moments vanishing. Note that second moments match (first column). Also, the first row is the same for both tables since for $N = 0$ both the Fock state and the PAC reduce to the vacuum.

In App. B.3, we obtained a general formula to calculate the moments of the probe's observables for a probe delta-coupled to a general field at times $t = 0$ and $t = t_M$. We considered a harmonic oscillator probe, $\hat{\mu}_D = \hat{q}_D$, with frequency ω_D . In the next subsection, we will apply those results to the particular cases of Tables B.1 and B.2.

	$\langle q^2 \rangle_N$	$\langle q^4 \rangle_N$	$\langle q^6 \rangle_N$	$\langle q^8 \rangle_N$
N=0	1/2	3/4	15/8	105/16
N=1	3/2	15/4	105/8	945/16
N=2	5/2	39/4	375/8	4305/16
N=3	7/2	75/4	945/8	13545/16
N=4	9/2	123/4	1935/8	33705/16

Table B.1: Statistics for the N particle Fock states.

	$\langle q^2 \rangle_N$	$\langle q^4 \rangle_N$	$\langle q^6 \rangle_N$	$\langle q^8 \rangle_N$
N=0	1/2	3/4	15/8	105/16
N=1	3/2	21/4	215/8	2835/16
N=2	5/2	51/4	715/8	12425/16
N=3	7/2	93/4	1635/8	34755/16
N=4	9/2	147/4	3095/8	77385/16

Table B.2: Statistics for the PAC state with $\langle \hat{n} \rangle = N$.

B.4.1 Second moments

Using the techniques of App. B.3, the general formula for $\langle \hat{q}_D(t)^2 \rangle$, $\langle \hat{p}_D(t)^2 \rangle$ and $\langle \hat{r}_D(t)^2 \rangle$ for $t > t_M$ can be obtained as

$$\langle \hat{q}_D(t)^2 \rangle = \mathbf{Q}_D^\top \langle \hat{\mathbf{X}} \hat{\mathbf{X}}^\top \rangle \mathbf{Q}_D = \mathbf{Q}_D^\top \sigma_0 \mathbf{Q}_D, \quad (\text{B.41})$$

$$\langle \hat{p}_D(t)^2 \rangle = \mathbf{P}_D^\top \langle \hat{\mathbf{X}} \hat{\mathbf{X}}^\top \rangle \mathbf{P}_D = \mathbf{P}_D^\top \sigma_0 \mathbf{P}_D, \quad (\text{B.42})$$

$$\langle \hat{r}_D(t)^2 \rangle = \frac{1}{2}(\mathbf{Q}_D + \mathbf{P}_D)^\top \langle \hat{\mathbf{X}} \hat{\mathbf{X}}^\top \rangle (\mathbf{Q}_D + \mathbf{P}_D) = \frac{1}{2}(\mathbf{Q}_D + \mathbf{P}_D)^\top \sigma_0 (\mathbf{Q}_D + \mathbf{P}_D), \quad (\text{B.43})$$

where $\mathbf{Q}_D = S^\top \mathbf{q}_D$ and $\mathbf{P}_D = S^\top \mathbf{p}_D$ can be calculated as

$$\mathbf{Q}_D = S^\top \mathbf{q}_D = S_{\text{int}}^\top(0) S_{\text{int}}^\top(t_M) \mathbf{q}_D \quad (\text{B.44})$$

$$= (\mathbb{1} - \lambda(\mathbf{u}(0)\mathbf{v}^\top(0) + \mathbf{v}(0)\mathbf{u}^\top(0))\Omega)(\mathbb{1} - \lambda(\mathbf{u}(t_M)\mathbf{v}^\top(t_M) + \mathbf{v}(t_M)\mathbf{u}^\top(t_M))\Omega)\mathbf{q}_D \quad (\text{B.45})$$

$$= (1 - \lambda^2 \beta \sin(\omega_D t_M))\mathbf{q}_D + \lambda \sin(\omega_D t_M)\mathbf{v}(t_M), \quad (\text{B.46})$$

$$\mathbf{P}_D = S^\top \mathbf{p}_D = S_{\text{int}}^\top(0) S_{\text{int}}^\top(t_M) \mathbf{p}_D \quad (\text{B.47})$$

$$= (\mathbb{1} - \lambda(\mathbf{u}(0)\mathbf{v}^\top(0) + \mathbf{v}(0)\mathbf{u}^\top(0))\Omega)(\mathbb{1} - \lambda(\mathbf{u}(t_M)\mathbf{v}^\top(t_M) + \mathbf{v}(t_M)\mathbf{u}^\top(t_M))\Omega)\mathbf{p}_D \quad (\text{B.48})$$

$$= \mathbf{p}_D + \lambda^2 \beta \cos(\omega_D t_M)\mathbf{q}_D - \lambda(\mathbf{v}(0) + \cos(\omega_D t_M)\mathbf{v}(t_M)), \quad (\text{B.49})$$

where

$$\mathbf{u}(t) = (\cos(\omega_D t), \sin(\omega_D t), 0, 0, \dots)^\top, \quad (\text{B.50})$$

$$\mathbf{v}(t) = \left(0, 0, \frac{F_1 \cos(\omega_1 t)}{\sqrt{\omega_1}}, \frac{F_1 \sin(\omega_1 t)}{\sqrt{\omega_1}}, \frac{F_2 \cos(\omega_2 t)}{\sqrt{\omega_2}}, \frac{F_2 \sin(\omega_2 t)}{\sqrt{\omega_2}}, \dots \right)^\top, \quad (\text{B.51})$$

$$\beta = \mathbf{v}(0)^\top \Omega \mathbf{v}(t_M) = \sum_{n=1}^{\infty} \frac{F_n^2}{\omega_n} \sin(\omega_n t_M), \quad (\text{B.52})$$

$$F_n = \sqrt{2\hbar c^2/L} \int_{-\infty}^{\infty} dx F(x) \sin(k_n x). \quad (\text{B.53})$$

In Eqs. (B.41)–(B.52) we observe that the only dependence on the initial state of the field is encoded in σ_0 , the covariance matrix of the initial states. Therefore, the only further particularization that we need to perform is to substitute the covariance matrix of the initial field state for each particular case, which can be computed using the values in Tables B.1 and B.2.

B.4.2 Fourth moments

From App. B.3, we know that

$$\langle \hat{q}_D(t)^4 \rangle = \sum_{i,j,k,\ell=1}^{\infty} Q_i(t) Q_j(t) Q_k(t) Q_\ell(t) T_{4,i,j,k,\ell}, \quad (\text{B.54})$$

$$\langle \hat{p}_D(t)^4 \rangle = \sum_{i,j,k,\ell=1}^{\infty} P_i(t) P_j(t) P_k(t) P_\ell(t) T_{4,i,j,k,\ell}, \quad (\text{B.55})$$

$$\langle \hat{r}_D(t)^4 \rangle = \sum_{i,j,k,\ell=1}^{\infty} \frac{1}{4} [Q_i(t) + P_i(t)] [Q_j(t) + P_j(t)] [Q_k(t) + P_k(t)] [Q_\ell(t) + P_\ell(t)] T_{4,i,j,k,\ell}, \quad (\text{B.56})$$

where $T_{4,i,j,k,\ell} = \int_{-\infty}^{\infty} d\boldsymbol{\xi} \xi_i \xi_j \xi_k \xi_\ell W(\boldsymbol{\xi}) = \langle \xi_i \xi_j \xi_k \xi_\ell \rangle$. We observe that $T_{4,i,j,k,\ell}$ is fully symmetric under index permutation, i.e., $T_{4,i,j,k,\ell} = T_{4,\tau(i),\tau(j),\tau(k),\tau(\ell)}$ for any 4-permutation τ . We also observe that only the terms of the form $T_{4,i,i,j,j} = T_{4,i,j,j,i} = T_{4,i,j,i,j} = T_{2,i,i} T_{2,j,j}$ or $T_{4,i,i,i,i}$ for $i \neq j$ are different from zero. Taking into account the different permutations,

we obtain:

$$\langle \hat{q}_D(t)^4 \rangle = \sum_{i,j,k,\ell=1}^{\infty} Q_i(t) Q_j(t) Q_k(t) Q_\ell(t) T_{4,i,j,k,\ell} \quad (\text{B.57})$$

$$\begin{aligned} &= \sum_{i=1}^{\infty} Q_i^4(t) T_{4,i,i,i,i} + 3 \sum_{i=1, j=1, j \neq i}^{\infty} Q_i(t)^2 Q_j(t)^2 T_{2,i,i} T_{2,j,j} \\ &\quad + 3 \sum_{i=1, j=i}^{\infty} Q_i^4(t) T_{2,i,i} T_{2,i,i} - 3 \sum_{i=1, j=i}^{\infty} Q_i^4(t) T_{2,i,i}^2 \end{aligned} \quad (\text{B.58})$$

$$= \sum_{i=1}^{\infty} Q_i^4(t) (T_{4,i,i,i,i} - 3T_{2,i,i}^2) + 3 \left(\sum_{i=1}^{\infty} Q_i(t)^2 T_{2,i,i} \right)^2. \quad (\text{B.59})$$

We additionally have that, if $i \neq 3, 4$, then $T_{4,i,i,i,i} - 3T_{2,i,i}^2 = \frac{3}{4} - 3\left(\frac{1}{2}\right)^2 = 0$. Therefore,

$$T_{4,i,i,i,i} - 3T_{2,i,i}^2 = \begin{cases} 0 & i \neq 3, 4 \\ \langle q^4 \rangle_N - 3\langle q^2 \rangle_N^2 & i = 3, 4 \end{cases} \quad (\text{B.60})$$

and

$$T_{2,i,i} = \begin{cases} 1/2 & i \neq 3, 4 \\ \langle q^2 \rangle_N & i = 3, 4 \end{cases}. \quad (\text{B.61})$$

Proceeding analogously, and with $R = (Q + P)/\sqrt{2}$, we get

$$\langle \hat{p}_D(t)^4 \rangle = \sum_{i=1}^{\infty} P_i^4(t) (T_{4,i,i,i,i} - 3T_{2,i,i}^2) + 3 \left(\sum_{i=1}^{\infty} P_i(t)^2 T_{2,i,i} \right)^2, \quad (\text{B.62})$$

$$\langle \hat{r}_D(t)^4 \rangle = \sum_{i=1}^{\infty} R_i^4(t) (T_{4,i,i,i,i} - 3T_{2,i,i}^2) + 3 \left(\sum_{i=1}^{\infty} R_i(t)^2 T_{2,i,i} \right)^2. \quad (\text{B.63})$$

B.4.3 Eighth moments

In order to calculate the eight moments, as we did with the fourth moments, we can take advantage of the symmetries of the tensor $T_{8,i,j,k,l,m,n,o,p}$. As before,

$$T_{8,i,j,k,l,m,n,o,p} = T_{8,\tau(i),\tau(j),\tau(k),\tau(l),\tau(m),\tau(n),\tau(o),\tau(p)} \quad (\text{B.64})$$

for any 8-permutation τ . As a consequence, several terms cancel out, yielding:

$$\langle \hat{q}_b(t)^8 \rangle = \sum_{i,j,k,\ell,m,n,o,p=1}^{\infty} Q_i(t) Q_j(t) Q_k(t) Q_\ell(t) Q_m(t) Q_n(t) Q_o(t) Q_p(t) T_{8,i,j,k,\ell,m,n,o,p} \quad (\text{B.65})$$

$$\begin{aligned} &= \sum_{i=1}^{\infty} T_{8,i,i,i,i,i,i,i} Q_i^8(t) + 28 \sum_{i,j=1,j \neq i}^{\infty} T_{8,i,i,i,i,i,j,j} Q_i^6(t) Q_j^2(t) \\ &\quad + 35 \sum_{i,j=1,j \neq i}^{\infty} T_{8,i,i,i,i,j,j,j,j} Q_i^4(t) Q_j^4(t) + 210 \sum_{i,j,k=1,j \neq i,k \neq i,k \neq j}^{\infty} T_{8,i,i,i,i,j,j,k,k} Q_i^4(t) Q_j^2(t) Q_k^2(t) \\ &\quad + 105 \sum_{i,j,k,l=1,j \neq i,k \neq i,j,l \neq i,j,k}^{\infty} T_{8,i,i,j,j,j,k,k,l,l} Q_i^2(t) Q_j^2(t) Q_k^2(t) Q_l^2(t). \end{aligned} \quad (\text{B.66})$$

Now let us look at the terms one by one. The second term's coefficient comes from the possible combinations with six equal indices and two equal indices (these two different from the previous six). Therefore we have to choose six indices from the eight available to be equal to i and then two from the remaining two to set to j : $\binom{8}{6} \binom{2}{2} = 28$. Then, we have to sum over all the pairs (and the order of the pairs matters). Now, since $T_{8,i,i,i,i,i,j,j} = T_{6,i,i,i,i,i,i} T_{2,j,j}$, we get

$$\begin{aligned} 28 \sum_{i,j=1,j \neq i}^{\infty} T_{6,i,i,i,i,i,i} T_{2,j,j} Q_i^6(t) Q_j^2(t) &= 28 \sum_{i,j=1}^{\infty} T_{6,i,i,i,i,i,i} T_{2,j,j} Q_i^6(t) Q_j^2(t) - 28 \sum_{i=1}^{\infty} T_{6,i,i,i,i,i,i} T_{2,i,i} Q_i^8(t) \\ &= 28 \left(\sum_{i=1}^{\infty} T_{6,i,i,i,i,i,i} Q_i^6(t) \right) \left(\sum_{j=1}^{\infty} T_{2,j,j} Q_j^2(t) \right) - 28 \sum_{i=1}^{\infty} T_{6,i,i,i,i,i,i} T_{2,i,i} Q_i^8(t). \end{aligned} \quad (\text{B.67})$$

The third term's coefficient comes from the possible combinations of two quadruples of equal indices different between each other. Therefore, we have to choose four indices from the eight available to be equal to i and then four from the remaining four to set to j : $\binom{8}{4} \binom{4}{4} = 70$. Then, we have to sum over all the ordered pairs, but since in this case order does not matter, we have to divide by two. Now, since $T_{8,i,i,i,i,j,j,j,j} = T_{4,i,i,i,i} T_{4,j,j,j,j}$, we get

$$35 \sum_{i,j=1,j \neq i}^{\infty} T_{4,i,i,i,i} T_{4,j,j,j,j} Q_i^4(t) Q_j^4(t) = 35 \sum_{i,j=1}^{\infty} T_{4,i,i,i,i} T_{4,j,j,j,j} Q_i^4(t) Q_j^4(t) - 35 \sum_{i=1}^{\infty} T_{4,i,i,i,i}^2 Q_i^8(t) \quad (\text{B.68})$$

$$= 35 \left(\sum_{i=1}^{\infty} T_{4,i,i,i,i} Q_i^4(t) \right)^2 - 35 \sum_{i=1}^{\infty} T_{4,i,i,i,i}^2 Q_i^8(t). \quad (\text{B.69})$$

The fourth term's factor comes from the possible combinations of a quadruple and two pairs of equal indices, different between each other. Therefore, we have to choose four indices from the eight available to be equal to i , then two from the remaining four to set to j and then the last two to set to k : $\binom{8}{4}\binom{4}{2}\binom{2}{2} = 420$. Then, we have to sum over all triples, taking into account that we have to divide by two because the order of the pairs does not matter. Carefully handling the combinatorics yields

$$210 \sum_{i,j,k=1, j \neq i, k \neq i, k \neq j}^{\infty} T_{4,i,i,i,i} T_{2,j,j} T_{2,k,k} Q_i^4(t) Q_j^2(t) Q_k^2(t) \quad (\text{B.70})$$

$$= 210 \left[\left(\sum_{i=1}^{\infty} T_{4,i,i,i,i} Q_i^4(t) \right) \left(\sum_{j=1}^{\infty} T_{2,j,j} Q_j^2(t) \right)^2 - 2 \left(\sum_{i=1}^{\infty} T_{4,i,i,i,i} T_{2,i,i} Q_i^6(t) \right) \left(\sum_{j=1}^{\infty} T_{2,j,j} Q_j^2(t) \right) \right. \\ \left. - \left(\sum_{i=1}^{\infty} T_{4,i,i,i,i} Q_i^4(t) \right) \left(\sum_{j=1}^{\infty} T_{2,j,j}^2 Q_j^4(t) \right) + 2 \left(\sum_{i=1}^{\infty} T_{4,i,i,i,i} T_{2,i,i}^2 Q_i^8(t) \right) \right]. \quad (\text{B.71})$$

The fifth term's coefficient comes from the possible combinations with 4 pairs of indices with i, j, k, l : $\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2} = 2520$. Taking into account the symmetries—i.e. the order of the pairs does not matter—we have to divide by the total number of 4-permutations, which is 24. Again, after some careful treatment of the combinatorics, we find

$$105 \sum_{i,j,k,l=1, j \neq i, k \neq i, l \neq i, j, k}^{\infty} T_{8,i,i,j,j,k,k,l,l} Q_i^2(t) Q_j^2(t) Q_k^2(t) Q_l^2(t) \quad (\text{B.72})$$

$$= 105 \left[\left(\sum_{i=1}^{\infty} T_{2,i,i} Q_i^2(t) \right)^4 - 6 \left(\sum_{i=1}^{\infty} T_{2,i,i}^2 Q_i^4(t) \right) \left(\sum_{j=1}^{\infty} T_{2,j,j} Q_j^2(t) \right)^2 + 3 \left(\sum_{i=1}^{\infty} T_{2,i,i}^2 Q_i^4(t) \right)^2 \right. \\ \left. + 8 \left(\sum_{i=1}^{\infty} T_{2,i,i}^3 Q_i^6(t) \right) \left(\sum_{j=1}^{\infty} T_{2,j,j} Q_j^2(t) \right) - 6 \left(\sum_{i=1}^{\infty} T_{2,i,i}^4 Q_i^8(t) \right) \right]. \quad (\text{B.73})$$

Joining everything,

$$\langle \hat{q}_D(t)^8 \rangle = \sum_{i=1}^{\infty} (T_{8,i,i,i,i,i,i,i,i} - 28T_{6,i,i,i,i,i,i} T_{2,i,i} - 35T_{4,i,i,i,i}^2 + 420T_{4,i,i,i,i} T_{2,i,i}^2 - 630T_{2,i,i}^4) Q_i^8(t) \\ + \left(\sum_{i=1}^{\infty} (28T_{6,i,i,i,i,i,i} - 420T_{4,i,i,i,i} T_{2,i,i} + 840T_{2,i,i}^3) Q_i^6(t) \right) \left(\sum_{j=1}^{\infty} T_{2,j,j} Q_j^2(t) \right)$$

$$\begin{aligned}
& + \left(\sum_{i=1}^{\infty} T_{4,i,i,i} Q_i^4(t) \right) \left(\sum_{i=1}^{\infty} (35T_{4,i,i,i} - 210T_{2,i,i}^2) Q_i^4(t) \right) \\
& + \left(\sum_{i=1}^{\infty} (210T_{4,i,i,i} - 630T_{2,i,i}^2) Q_i^4(t) \right) \left(\sum_{j=1}^{\infty} T_{2,j,j} Q_j^2(t) \right)^2 \\
& + 105 \left[\left(\sum_{i=1}^{\infty} T_{2,i,i} Q_i^2(t) \right)^4 + 3 \left(\sum_{i=1}^{\infty} T_{2,i,i}^2 Q_i^4(t) \right)^2 \right]. \tag{B.74}
\end{aligned}$$

Finally, we note that for $i \neq 3, 4$ we have

$$T_{8,i,i,i,i,i,i,i} - 28T_{6,i,i,i,i,i,i} T_{2,i,i} - 35T_{4,i,i,i,i}^2 + 420T_{4,i,i,i,i} T_{2,i,i}^2 - 630T_{2,i,i}^4 = 0, \tag{B.75}$$

$$28T_{6,i,i,i,i,i,i} - 420T_{4,i,i,i,i} T_{2,i,i} + 840T_{2,i,i}^3 = 0, \tag{B.76}$$

$$210T_{4,i,i,i,i} - 630T_{2,i,i}^2 = 0. \tag{B.77}$$

Appendix C

Real scalar quantum field theory on the lattice

As we discussed in the main text, we can motivate a UV cutoff for the field-probe system through the length scale of the probe's smearing function. To see this, let us expand the field-probe interaction Hamiltonian

$$\hat{H}_{\text{int}} = \lambda \chi(t) \int_{-\infty}^{\infty} dx F(x) \hat{q}_{\text{D}}(t) \hat{\phi}(t, x) \quad (\text{C.1})$$

in terms of plane-wave modes as

$$\hat{H}_{\text{int}} = \lambda \chi(t) \hat{q}_{\text{D}}(t) \int_{\mathbb{R}^2} \frac{dx dk}{2\sqrt{\pi\omega_k}} F(x) \left(e^{-i\omega_k t} e^{ikx} \hat{a}_k + \text{H.c.} \right) \quad (\text{C.2})$$

$$= \lambda \chi(t) \hat{q}_{\text{D}}(t) \int_{-\infty}^{\infty} \frac{dk}{2\sqrt{\pi\omega_k}} \left(\tilde{F}(k) e^{-i\omega_k t} \hat{a}_k + \text{H.c.} \right), \quad (\text{C.3})$$

where $\tilde{F}(k) = \mathcal{F}_k[F(x)]$ is the Fourier transform of $F(x)$. Note that $\tilde{F}(k)$ determines how strongly the probe couples to each of the field modes. If the smearing function decays fast enough outside of a region of size $\sim \sigma$ (e.g., $F(x)$ is a Gaussian with standard deviation σ) then $\tilde{F}(k)$ would have an approximate width $\sim 1/\sigma$. That is, the probe would not couple strongly to modes with wavenumber $|k| \gg \sigma^{-1}$. Thus by considering a probe with an effectively finite spatial extent we are automatically considering a soft UV cutoff in the interaction of the field and the probe.

If $\tilde{F}(k)$ decays sufficiently fast, we can neglect the coupling to the modes above some large UV threshold, say $|k| > K$ (e.g. for a Gaussian profile we can take $K = 16/\sigma$). This

yields an effective coupling of $\tilde{F}^{\text{UV}}(k) := \Pi_K(k) \tilde{F}(k)$ where $\Pi_K(k)$ is the rectangle function over $k \in [-K, K]$. By the Nyquist-Shannon sampling theorem we can then reconstruct our UV cutoff smearing function, $F^{\text{UV}}(x) := \mathcal{F}_x^{-1}[\tilde{F}^{\text{UV}}(k)]$, as $F^{\text{UV}}(x) = \sum_n F^{\text{UV}}(x_n) S_n(x/a)$, where $a = \pi/K$ is the spacing of the discrete positions, $x_n = n a$, and we have denoted $S_n(r) \equiv \sin(\pi(r - n))/\pi(r - n)$, which is a displaced normalized sinc function.

Note that in general $F^{\text{UV}}(x_n) \neq F(x_n)$. This means that in order to recover the UV-cutoff smearing function we cannot sample the original smearing function, but its bandlimited version instead. However, precisely because we are assuming that $\tilde{F}(k)$ is effectively bandlimited, we can approximate $F^{\text{UV}}(x_n) \simeq F(x_n)$. Indeed, for the particular case of a Gaussian with standard deviation σ and $K = 16/\sigma$, it can be straightforwardly shown that $|F(x) - F^{\text{UV}}(x)| \lesssim 10^{-59}$ for every real x . Now, $S_n(r)$ decays only polynomially for large r . Thus in general our UV-cutoff smearing function will have polynomial tails, as all bandlimited functions do. This might seem to be in contradiction with our previous approximation, since $F^{\text{UV}}(x_n)$ will decay polynomially, while $F(x_n)$ does so exponentially. However, the bound above shows precisely that these differences in the rhythm of decay are only relatively significant in the regions in which the order of magnitude of both $F(x)$ and $F^{\text{UV}}(x)$ is already negligible. We can therefore faithfully approximate the UV-cutoff smearing function by sampling $F(x)$ instead of $F^{\text{UV}}(x)$, i.e., we redefine $F^{\text{UV}}(x) = \sum_n F(x_n) S_n(x/a)$. This function is still bandlimited and so still has polynomial tails, however the coefficients $F(x_n)$ —which, as we will soon see, tell us how the probe couples to the lattice sites—are sampled directly from the original smearing function.

Since F^{UV} as defined above is bandlimited, we can define the UV-cutoff interaction Hamiltonian as

$$\hat{H}_{\text{int}}^{\text{UV}} := \lambda \chi(t) \int_{-\infty}^{\infty} dx F^{\text{UV}}(x) \hat{q}_{\text{D}}(t) \hat{\phi}(t, x) = \lambda \chi(t) \int_{-\infty}^{\infty} dx F^{\text{UV}}(x) \hat{q}_{\text{D}}(t) \hat{\phi}^{\text{UV}}(t, x), \quad (\text{C.4})$$

where we note that the UV-cutoff smearing function effectively induces a UV cutoff of the field operator,

$$\hat{\phi}^{\text{UV}}(t, x) := \mathcal{F}_x^{-1} [\Pi_K(k) \mathcal{F}_k[\phi(t, x)]] = \int_{-K}^K \frac{dk}{2\sqrt{\pi\omega_k}} \left(\hat{a}_k e^{-i\omega_k t} e^{ikx} + \text{H.c.} \right). \quad (\text{C.5})$$

Next, we observe that, since $\hat{\phi}^{\text{UV}}(t, x)$ is bandlimited, we can express it as a sum of sinc functions as, $\hat{\phi}^{\text{UV}}(t, x) = \sum_n \hat{\phi}^{\text{UV}}(t, x_n) S_n(x/a)$. Recomputing the UV-cutoff interaction Hamiltonian using these sinc representations we find

$$\hat{H}_{\text{int}}^{\text{UV}} = \lambda \chi(t) \sum_n a F(x_n) \hat{q}_{\text{D}}(t) \hat{\phi}^{\text{UV}}(t, x_n) \quad (\text{C.6})$$

where we have used the orthonormality of the collection $\{S_m(r)\}$ in the L^2 norm. Thus, by taking a hard UV cutoff on the probe's smearing function we automatically find that the probe effectively only couples to the field at the discrete positions $x_n = na$.

Notice that so far we are not implying that the field itself has a UV cutoff or that the space it lives on is discretized. We have only discussed an approximation of the probe coupling. We could study the field theory as is without an explicit UV cutoff, but for our purposes it is convenient to consider that the field is also bandlimited. We apply this UV cutoff to the field by removing the field modes with $k > |K|$, yielding

$$\hat{\phi}^{\text{uv}}(t, x) := \mathcal{F}_x^{-1}[\Pi_K(k)\mathcal{F}_k[\hat{\phi}(t, x)]], \quad (\text{C.7})$$

$$\hat{\pi}^{\text{uv}}(t, x) := \mathcal{F}_x^{-1}[\Pi_K(k)\mathcal{F}_k[\hat{\pi}(t, x)]], \quad (\text{C.8})$$

$$\partial_x \hat{\phi}^{\text{uv}}(t, x) := \mathcal{F}_x^{-1}[\Pi_K(k)\mathcal{F}_k[\partial_x \hat{\phi}(t, x)]]. \quad (\text{C.9})$$

Since these operators are now bandlimited, we can express them as

$$\hat{\phi}^{\text{uv}}(t, x) = \sum_n \hat{\phi}^{\text{uv}}(t, x_n) S_n(x/a), \quad (\text{C.10})$$

$$\hat{\pi}^{\text{uv}}(t, x) = \sum_n \hat{\pi}^{\text{uv}}(t, x_n) S_n(x/a), \quad (\text{C.11})$$

$$\partial_x \hat{\phi}^{\text{uv}}(t, x) = \sum_n \partial_x \hat{\phi}^{\text{uv}}(t, x_n) S_n(x/a). \quad (\text{C.12})$$

The UV-cutoff field Hamiltonian is then the free field Hamiltonian for this bandlimited QFT, namely

$$\hat{H}_\phi^{\text{uv}} := \frac{1}{2} \int_{-\infty}^{\infty} dx \left[c^2 \hat{\pi}^{\text{uv}}(t, x)^2 + (\partial_x \hat{\phi}^{\text{uv}}(t, x))^2 + \frac{m^2 c^2}{\hbar^2} \hat{\phi}^{\text{uv}}(t, x)^2 \right] \quad (\text{C.13})$$

$$= \frac{a}{2} \sum_n \left[c^2 \hat{\pi}^{\text{uv}}(t, x_n)^2 + (\partial_x \hat{\phi}^{\text{uv}}(t, x_n))^2 + \frac{m^2 c^2}{\hbar^2} \hat{\phi}^{\text{uv}}(t, x_n)^2 \right], \quad (\text{C.14})$$

where we have again used the operator's sinc representations and the L^2 orthonormality of $\{S_m(r)\}$ to express the integral as a sum. This way, we have *completely* reduced the dynamics to the field amplitudes and momenta at points (t, x_n) .

One may think that the fact that the Hamiltonian $\hat{H}_\phi^{\text{uv}}$ has terms with the field derivatives $\partial_x \hat{\phi}^{\text{uv}}(t, x_n)$ shows that we are still dealing with a continuum space, not a lattice. Surprisingly, this is not the case, as these continuous derivative terms are understandable in terms of the discrete lattice with no approximation. To see this it suffices to recall that

the derivative of a bandlimited function is itself a bandlimited function, and therefore it can be fully described on a lattice. This is facilitated by the following remarkable derivative approximation,

$$\partial_x f(x) = 2 \sum_{m=1}^{\infty} (-1)^{m+1} \frac{f(x + m a) - f(x - m a)}{2 m a}, \quad (\text{C.15})$$

which is exact when f is bandlimited with bandwidth $[-K, K]$ and $a \leq \pi/K$. Moreover, if the Fourier transform of f is mostly supported in $[-K, K]$ with thin tails (e.g, Gaussian tails) outside this region, then this is a very good derivative approximation.

We can apply this logic to the field operator as well. By Eq. (C.10) we have,

$$\partial_x \hat{\phi}(t, x) = \sum_n \partial_x \hat{\phi}(t, x_n) S_n(x/a) = \sum_n \hat{\phi}(t, x_n) \partial_x S_n(x/a). \quad (\text{C.16})$$

The derivative of the sinc profile $\partial_x S_n(x/a)$ is bandlimited and so can be written as a sum of sinc profiles. Carrying this out we find for $k \in \mathbb{Z}$, we get

$$\partial_x \hat{\phi}(t, x_k) = \frac{1}{a} \sum_{n \neq k} \frac{(-1)^{k-n}}{k-n} \hat{\phi}(t, x_n), \quad (\text{C.17})$$

where we have used the L^2 orthonormality of $\{S_m(r)\}$ again. Thus, we see that the derivative $\partial_x \hat{\phi}(t, x_k)$ has an expression in terms of the field operators at (t, x_n) .

If we use Eq. (C.17) in Eq. (C.14), we get an expression for the UV-cutoff Hamiltonian that is fully defined within the 1D lattice $\{x_n\}$. Our bandlimited QFT is thus perfectly representable on a lattice. Indeed, as discussed in [261], there are perfectly Lorentzian lattice theories. However, the derivative understood in terms of the lattice sites has one drawback: it is, in a sense, extremely non-local, involving *all* $n \neq k$. How is it that a perfectly local operation in the continuum (i.e., differentiation) is here being exactly represented by a non-local operation on the lattice sites? This issue is discussed at length in [262], but the ultimate resolution is as follows: the lattice site themselves ought to be thought of as non-local objects, each associated with overlapping sinc-profiles. Thus, our perfectly local differentiation in the continuum is carried out in terms of the lattice via a non-local combination of non-local objects.

If, however, we want to think of the lattice sites themselves as being local objects undergoing nearest-neighbour interactions, then the dynamics must be modified further.

To achieve this, instead of the exact formula Eq. (C.17), we instead take the approximation $\partial_x \hat{\phi}(t, x_n) \approx [\hat{\phi}(t, x_{n+1}) - \hat{\phi}(t, x_n)]/a$, yielding

$$\hat{H}_\phi^{\text{UV}} \approx \frac{a}{2} \sum_n \left[c^2 \hat{\pi}^{\text{UV}}(t, x_n)^2 + \left(\frac{\hat{\phi}^{\text{UV}}(t, x_{n+1}) - \hat{\phi}^{\text{UV}}(t, x_n)}{a} \right)^2 + \frac{m^2 c^2}{\hbar^2} \hat{\phi}^{\text{UV}}(t, x_n)^2 \right], \quad (\text{C.18})$$

where we remark that the bandlimited field still satisfies canonical commutation relations, $[\hat{\phi}^{\text{UV}}(t, x_n), \hat{\pi}^{\text{UV}}(t, x_m)] = i\hbar(\delta_{nm}/a)\mathbb{1}$. Finally, rewriting this Hamiltonian in terms of the dimensionless operators, $\hat{q}_n = \sqrt{am/\hbar^2} \hat{\phi}^{\text{UV}}(t, x_n)$ and $\hat{p}_n = \sqrt{a/m} \hat{\pi}^{\text{UV}}(t, x_n)$, which satisfy $[\hat{q}_i, \hat{p}_j] = i\delta_{ij}\mathbb{1}$, yields the bandlimited field Hamiltonian of Eqs. (7.7) and (7.8).

The implementation of this bandlimitation has introduced essentially three changes to the scenario being considered: 1) a UV cutoff in the interaction Hamiltonian, that is, the probe no longer couples to high frequency modes, 2) a UV cutoff in the field Hamiltonian, that is, the field no longer contains high frequency modes, and 3) a discrete approximation for the derivative. The effects of the first two changes on the probe's response to the field are exponentially suppressed with increasing K and quickly become irrelevant for our calculations. The effect of the third change is more subtle, since the approximation that lies behind it is more drastic in nature, as becomes apparent by comparing the expression for the discretized derivative and the exact expression in Eq. (C.17). The discrete approximation for the derivative changes the dynamics of the field; namely, it modifies its dispersion relation from

$$\hbar\omega_k = \sqrt{(m c^2)^2 + (\hbar c k)^2} \quad \text{to} \quad \hbar\omega'_k = \sqrt{(m c^2)^2 + \left(\frac{2 \hbar c K}{\pi} \right)^2 \sin^2 \left(\frac{\pi k}{2 K} \right)}. \quad (\text{C.19})$$

These two dispersion relations are represented in Fig. C.1a, where we can confirm that $\hbar\omega_k \geq \hbar\omega'_k$. We also observe that the dispersion relation is mostly modified at high frequencies, that is, at frequencies to which the probe does not couple strongly. In practice the probe does not couple to the field modes that behave pathologically.

To quantify how much the dispersion relation has changed *in the modes that couple to the probe* we define the ‘average relative error’ in $\hbar\omega_k$. This error is the average relative difference between the modified and unmodified dispersion relations at each frequency weighted by the strength of the probe's coupling to that frequency. Namely,

$$\text{Avg. Rel. Error} := \int_{-\infty}^{\infty} dk \frac{\hbar\omega_k - \hbar\omega'_k}{\hbar\omega_k} \frac{\tilde{F}(k)}{\|\tilde{F}\|_1}. \quad (\text{C.20})$$

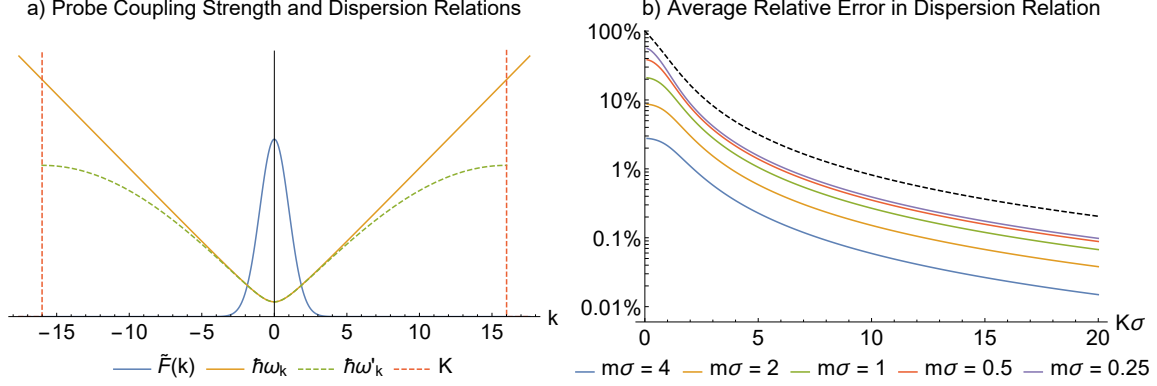


Figure C.1: In subfigure a) we show the probe's coupling strength to the field modes, $\tilde{F}(k)$, (blue Gaussian) as a function of the mode's wavenumber, k , for $\sigma = 1$. The field's dispersion relation $\hbar\omega_k$ is also plotted (yellow hyperbola) for $m = 1$. Taking a UV cutoff at $K = 16$ (vertical red dashed line) yields a modified dispersion relation, $\hbar\omega'_k$, (green dashed) at high frequencies. In subfigure b) we plot the average relative error in $\hbar\omega_k$ as a function of the cutoff K and the field mass m . This error decreases polynomially as $K\sigma$ increases. The error also decreases as the mass of the field increases. The black dashed line is a mass independent upper-bound on this error. In both subfigures we have taken $\hbar = c = 1$.

We have computed the average relative error for various cutoffs, field masses, and probe sizes in Fig. C.1b. To investigate how the average relative error decreases as we increase K we use the following chain of inequalities:

$$\text{Avg. Rel. Error} = \int_{-\infty}^{\infty} dk \frac{\hbar\omega_k - \hbar\omega'_k}{\hbar\omega_k} \frac{\tilde{F}(k)}{\|\tilde{F}\|_1} \leq \int_{-\infty}^{\infty} dk \frac{(\hbar\omega_k)^2 - (\hbar\omega'_k)^2}{(\hbar\omega_k)^2} \frac{\tilde{F}(k)}{\|\tilde{F}\|_1} \quad (\text{C.21})$$

$$\leq \int_{-\infty}^{\infty} dk \frac{(\hbar\omega_k)^2 - (\hbar\omega'_k)^2}{(\hbar c k)^2} \frac{\tilde{F}(k)}{\|\tilde{F}\|_1}, \quad (\text{C.22})$$

where the first inequality follows from $\hbar\omega_k \geq \hbar\omega'_k \geq 0$ and the second from $\hbar\omega_k \geq \hbar c k$. For the case in which the smearing function is a Gaussian of variance σ^2 , the final expression is mass-independent and can be computed in closed form, yielding

$$\text{Avg. Rel. Error} \leq 1 - 2 \frac{K\sigma}{\sqrt{2\pi}} \text{Erf} \left(\frac{\pi}{K\sigma\sqrt{2}} \right) + \frac{2K^2\sigma^2}{\pi^2} \left[1 - \exp \left(\frac{-\pi^2}{2K^2\sigma^2} \right) \right] \quad (\text{C.23})$$

$$\sim O \left(\frac{1}{(K\sigma)^2} \right). \quad (\text{C.24})$$

That is, we expect the error made by the discrete approximation to be polynomially suppressed as we increase $K\sigma$. In Sec. 7.4, we have chosen $K\sigma = 16$ in both the remote boundary sensing and thermometry examples. This places the upper-bound on the average relative error at 0.32%. If we also take into account that (taking $\hbar = c = 1$), $m\sigma = 0.006$ and $m\sigma = 0.00027$, respectively, the average relative error can be computed numerically, yielding 0.16% in both cases.

Appendix D

Setups with multiple detectors

D.1 General final state of two initially uncorrelated detectors

In this appendix, we show that in the basis $\{|\psi_A\psi_B\rangle, |\psi_A\chi_B\rangle, |\chi_A\psi_B\rangle, |\chi_A\chi_B\rangle\}$, the matrix form of the time-evolved state $\hat{\rho}_{AB}$ is given by Eq. (8.10). The matrix form of the initial state of the detectors in this basis is simply

$$\hat{\rho}_{AB,0} = |\psi_A\rangle\langle\psi_A| \otimes |\psi_B\rangle\langle\psi_B| = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (\text{D.1})$$

The second order terms in the coupling parameter are given by Eq. (8.5). In this equation, we can separate the local and the non local terms as follows:

$$\hat{\rho}_{AB}^{(2)} = \hat{r}_A^{(2)} \otimes \hat{\rho}_{B,0} + \hat{\rho}_{A,0} \otimes \hat{r}_A^{(2)} + \hat{r}_{AB}^{(2)}. \quad (\text{D.2})$$

The local terms are given by

$$\begin{aligned} \hat{r}_i^{(2)} = \lambda_i^2 \int d\mathbf{x}d\mathbf{x}' \sqrt{gg'} \Lambda_i(\mathbf{x}) \Lambda_j(\mathbf{x}') & \left[\hat{\mu}_i(\tau_i) \hat{\rho}_{i,0} \hat{\mu}_i(\tau'_i) w_2(\mathbf{x}', \mathbf{x}) \right. \\ & \left. - \hat{\mu}_i(\tau_i) \hat{\mu}_i(\tau'_i) \hat{\rho}_{i,0} w_2(\mathbf{x}, \mathbf{x}') \theta(t - t') - \hat{\rho}_{i,0} \hat{\mu}_i(\tau'_i) \hat{\mu}_i(\tau_i) w_2(\mathbf{x}', \mathbf{x}) \theta(t - t') \right], \end{aligned} \quad (\text{D.3})$$

for $i \in \{A, B\}$. In the basis $\{|\psi_i\rangle, |\chi_i\rangle\}$, the local terms take the form

$$\hat{r}_i^{(2)} = \begin{pmatrix} -\mathcal{L}_{\text{II}}^{\text{gen}} & \mathcal{I}_{\text{II}}^* \\ \mathcal{I}_{\text{II}} & \mathcal{L}_{\text{II}}^{\text{gen}} \end{pmatrix}, \quad (\text{D.4})$$

with $\mathcal{L}_{\text{II}}^{\text{gen}}$ given by Eq. (8.11), and

$$\begin{aligned} \mathcal{I}_{\text{II}} = & \cos^3 \alpha_i \sin \alpha_i \left(-e^{-i\beta_i} \mathcal{S}_{\text{II}} + e^{i\beta_i} (-\mathcal{L}_{\text{II}} + \eta_i - \gamma_i) \right) \\ & + \cos \alpha_i \sin^3 \alpha_i \left(e^{3i\beta_i} \mathcal{K}_{\text{II}} + e^{i\beta_i} (\mathcal{R}_{\text{II}} + \eta_i - \gamma_i) \right), \end{aligned} \quad (\text{D.5})$$

with

$$\gamma_i = \lambda_i^2 \int dx dx' \sqrt{gg'} \Lambda_i(x) \Lambda_i(x') w_2(x', x) e^{i\Omega_i(\tau_i - \tau'_i)} \theta(t - t'), \quad (\text{D.6})$$

$$\eta_i = \lambda_i^2 \int dx dx' \sqrt{gg'} \Lambda_i(x) \Lambda_i(x') w_2(x', x) e^{-i\Omega_i(\tau_i - \tau'_i)} \theta(t - t'). \quad (\text{D.7})$$

Notice that γ_i and η_i depend on the time-ordering parameter t , and therefore they are not covariant in general.

The non-local terms $\hat{r}_{\text{AB}}$ are given by the same expression as in Eq. (8.5), but considering only the terms with $i \neq j$. In the basis $\{|\psi_A \psi_B\rangle, |\psi_A \chi_B\rangle, |\chi_A \psi_B\rangle, |\chi_A \chi_B\rangle\}$, they yield

$$\hat{r}_{\text{AB}}^{(2)} = \begin{pmatrix} 0 & \mathcal{J}_1^* & \mathcal{J}_2^* & (\mathcal{M}^{\text{gen}})^* \\ \mathcal{J}_1 & 0 & (\mathcal{L}_{\text{AB}}^{\text{gen}})^* & 0 \\ \mathcal{J}_2 & \mathcal{L}_{\text{AB}}^{\text{gen}} & 0 & 0 \\ \mathcal{M}^{\text{gen}} & 0 & 0 & 0 \end{pmatrix}, \quad (\text{D.8})$$

where $\mathcal{M}^{\text{gen}}$ and $\mathcal{L}_{\text{AB}}^{\text{gen}}$ are given in Eqs. (8.11) and (8.12), and $\mathcal{J}_1$ and $\mathcal{J}_2$ read

$$\begin{aligned} \mathcal{J}_1 = & \frac{1}{2} e^{-i\beta_A} \sin 2\alpha_A \left[(-\mathcal{M} + e^{2i\beta_A} (-\mathcal{S} - \mathcal{L}_{\text{AB}}^*) - \mathcal{K}_{\text{AB}}^*) \cos^2 \alpha_B \right. \\ & \left. + e^{2i\beta_B} (\mathcal{R} + e^{2i\beta_A} (\mathcal{V} + \mathcal{P}_{\text{AB}}^*) + \mathcal{Q}_{\text{AB}}^*) \sin^2 \alpha_B \right], \end{aligned} \quad (\text{D.9})$$

$$\begin{aligned} \mathcal{J}_2 = & \frac{1}{2} e^{-i\beta_B} \sin 2\alpha_B \left[(-\mathcal{M} + e^{2i\beta_B} (-\mathcal{R} - \mathcal{L}_{\text{AB}}) - \mathcal{P}_{\text{AB}}^*) \cos^2 \alpha_A \right. \\ & \left. + e^{2i\beta_A} (\mathcal{S} + e^{2i\beta_B} (\mathcal{V} + \mathcal{K}_{\text{AB}}) + \mathcal{Q}_{\text{AB}}) \sin^2 \alpha_A \right]. \end{aligned} \quad (\text{D.10})$$

Adding the contributions of the local and the non-local terms in Eq. (D.2) and using $\hat{\rho}_{AB} = \hat{\rho}_{AB,0} + \hat{\rho}_{AB}^{(2)} + \mathcal{O}(\lambda^4)$, we recover the matrix form given in Eq. (8.10), with the definitions

$$\mathcal{X} = \mathcal{I}_{BB} + \mathcal{J}_1, \quad (\text{D.11})$$

$$\mathcal{Y} = \mathcal{I}_{AA} + \mathcal{J}_2. \quad (\text{D.12})$$

Observe that the non-covariance of the terms $\mathcal{X}$ and $\mathcal{Y}$ is due to the local terms $\mathcal{I}_{AA}$ and $\mathcal{I}_{BB}$, since $\mathcal{J}_1$ and $\mathcal{J}_2$ are independent of the time parameter t .

D.2 State of two detectors after a measurement performed on a third

In this appendix, we prove that $\langle s | \text{Tr}_\phi(\hat{U} \hat{\rho}_0 \hat{U}^\dagger) | s \rangle$ admits the expansion used in Eq. (8.41). First, we note that

$$\langle s | \text{Tr}_\phi[\hat{U} \hat{\rho}_0 \hat{U}^\dagger] | s \rangle = \langle s | \hat{\rho}_{ABC} | s \rangle, \quad (\text{D.13})$$

where $\hat{\rho}_{ABC} = \text{Tr}_\phi[\hat{U} \hat{\rho}_0 \hat{U}^\dagger]$. Using the Dyson expansion of the time-evolution operator, $\hat{\rho}_{ABC}$ can be written as

$$\hat{\rho}_{ABC} = \hat{\rho}_{ABC,0} + \hat{\rho}_{ABC}^{(2)} + \hat{\rho}_{ABC}^{(4)} + \mathcal{O}(\lambda^6), \quad (\text{D.14})$$

where $\hat{\rho}_{ABC}^{(k)}$ is proportional to λ^k :

$$\hat{\rho}_{ABC}^{(2)} = \text{Tr}_\phi[\hat{U}^{(2)} \hat{\rho}_0] + \text{Tr}_\phi[\hat{U}^{(1)} \hat{\rho}_0 \hat{U}^{(1)\dagger}] + \text{Tr}_\phi[\hat{\rho}_0 \hat{U}^{(2)\dagger}], \quad (\text{D.15})$$

$$\hat{\rho}_{ABC}^{(4)} = \sum_{i=0}^4 \text{Tr}_\phi[\hat{U}^{(i)} \hat{\rho}_0 \hat{U}^{(4-i)\dagger}]. \quad (\text{D.16})$$

Notice that the odd-order corrections cancel because all the odd-point functions of the vacuum are zero. Using that detector C is initially in its ground state, and Eq. (8.39) for the outcome state $|s\rangle$, we have that

$$\langle s | \hat{\rho}_{ABC,0} | s \rangle = \epsilon^2 \hat{\rho}_{AB,0}. \quad (\text{D.17})$$

Regarding the second order terms, we have that

$$\begin{aligned}
\langle s | \hat{\rho}_{ABC}^{(2)} | s \rangle = & \sum_{I,J \in \{A,B,C\}} \lambda^2 \left[\int dx dx' \sqrt{gg'} \Lambda_I(x) \Lambda_J(x') \langle s | \hat{\mu}_J(\tau'_J) \hat{\rho}_{ABC,0} \hat{\mu}_I(\tau_I) | s \rangle w_2(x, x') \right. \\
& - \int dx dx' \sqrt{gg'} \Lambda_I(x) \Lambda_J(x') \langle s | \hat{\mu}_I(\tau_I) \hat{\mu}_J(\tau'_J) \hat{\rho}_{ABC,0} | s \rangle w_2(x, x') \theta(t - t') \\
& \left. - \int dx dx' \sqrt{gg'} \Lambda_I(x) \Lambda_J(x') \langle s | \hat{\rho}_{ABC,0} \hat{\mu}_J(\tau'_J) \hat{\mu}_I(\tau_I) | s \rangle w_2(x', x) \theta(t - t') \right]. \quad (D.18)
\end{aligned}$$

In the expression above, we analyze separately the terms which only involve the detectors harvesting entanglement, A and B, the local terms associated to the detector used to perform the measurement, C, and the cross-terms involving the pairs (A, C) and (B, C). Namely, we use

$$\sum_{I,J \in \{A,B,C\}} = \sum_{I,J \in \{A,B\}} + \sum_{I,J \in \{C\}} + \sum_{\substack{I,J \in \{A,C\} \\ I \neq J}} + \sum_{\substack{I,J \in \{B,C\} \\ I \neq J}}. \quad (D.19)$$

The terms involving detectors A and B yield $\epsilon^2 \hat{\rho}_{AB}^{(2)}$, with $\hat{\rho}_{AB}^{(2)}$ given by Eq. (8.34). The local terms in C yield $\mathcal{L}_{CC}(1 - 2\epsilon^2) \hat{\rho}_{AB,0}$. The last two summands in the expression above yield the correlation matrix $\hat{\gamma}$ given by Eq. (8.42). $\langle s | \hat{\rho}_{ABC}^{(2)} | s \rangle$ is thus obtained adding all the contributions:

$$\langle s | \hat{\rho}_{ABC}^{(2)} | s \rangle = \epsilon^2 \hat{\rho}_{AB}^{(2)} + \mathcal{L}_{CC}(1 - 2\epsilon^2) \hat{\rho}_{AB,0} + \hat{\gamma}. \quad (D.20)$$

It is left to calculate

$$\langle s | \hat{\rho}_{ABC}^{(4)} | s \rangle = \sum_{i=0}^4 \langle s | \text{Tr}_\phi [\hat{U}^{(i)} \hat{\rho}_0 \hat{U}^{(4-i)\dagger}] | s \rangle. \quad (D.21)$$

In order to compute the negativity at leading order in perturbation theory for the cases in which ϵ is sufficiently small (i.e., in the orthogonal and transition regimes), we only need to calculate up to fourth order terms in λ . In particular, in these regimes the only fourth order terms that influence the negativity are those which are not proportional to any power of ϵ . Since $|\langle s | g_C \rangle| = \epsilon$, we have that $|\langle s | e_C \rangle| = \sqrt{1 - \epsilon^2} = 1 + \mathcal{O}(\epsilon^2)$, and thus we only need to compute the terms for which the time-evolution yields $|e_C\rangle\langle e_C|$. These terms are characterized by having the Hamiltonian weight $\hat{h}_C$ acting an odd number of times on each side of the initial state $\hat{\rho}_{ABC,0}$. Thus, we need not to calculate $\langle s | \text{Tr}_\phi [\hat{U}^{(0)} \hat{\rho}_0 \hat{U}^{(4)\dagger}] | s \rangle$ and $\langle s | \text{Tr}_\phi [\hat{U}^{(4)} \hat{\rho}_0 \hat{U}^{(0)\dagger}] | s \rangle$ since they only yield terms proportional to ϵ . The only terms

that yield contributions of order zero in ϵ are those with a single Hamiltonian weight $\hat{h}_c(\mathbf{x})$ acting on each side of $\hat{\rho}_{ABC}$, which can be written as

$$\begin{aligned} \langle s | \hat{U}^{(2)} \hat{\rho}_{ABC,0} \hat{U}^{(2)\dagger} | s \rangle &= \lambda^4 \sum_{I,J \in \{A,B\}} \int d\mathbf{x} d\mathbf{x}' d\mathbf{x}'' d\mathbf{x}''' \sqrt{g g' g'' g'''} \Lambda_I(\mathbf{x}) \Lambda_C(\mathbf{x}') \Lambda_J(\mathbf{x}'') \Lambda_C(\mathbf{x}''') \quad (\text{D.22}) \\ &\times e^{i\Omega_C(\tau'_C - \tau''_C)} e^{i(\Omega_I \tau_I - \Omega_J \tau_J'')} \\ &\times \left[w_4(\mathbf{x}''', \mathbf{x}'', \mathbf{x}, \mathbf{x}') \theta(t - t') \theta(t'' - t''') + w_4(\mathbf{x}''', \mathbf{x}'', \mathbf{x}', \mathbf{x}) \theta(t' - t) \theta(t'' - t''') \right. \\ &\quad \left. + w_4(\mathbf{x}'', \mathbf{x}''', \mathbf{x}, \mathbf{x}') \theta(t - t') \theta(t''' - t'') + w_4(\mathbf{x}'', \mathbf{x}'', \mathbf{x}, \mathbf{x}') \theta(t' - t) \theta(t''' - t'') \right] \\ &\times (|e_A\rangle\langle g_A|)^{\delta_{AI}} (|e_B\rangle\langle g_B|)^{\delta_{BI}} |g_A\rangle\langle g_A| \otimes |g_B\rangle\langle g_B| (|g_A\rangle\langle e_A|)^{\delta_{AJ}} (|g_B\rangle\langle e_B|)^{\delta_{BJ}} + \epsilon \mathcal{O}(\lambda^4), \end{aligned}$$

where δ_{KL} is the Kronecker delta. Since the field is initially in a quasifree state, the 4-point distribution can be expressed in terms of the two-point distribution as

$$w_4(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) = w_2(\mathbf{x}_1, \mathbf{x}_2) w_2(\mathbf{x}_3, \mathbf{x}_4) + w_2(\mathbf{x}_1, \mathbf{x}_3) w_2(\mathbf{x}_2, \mathbf{x}_4) + w_2(\mathbf{x}_1, \mathbf{x}_4) w_2(\mathbf{x}_2, \mathbf{x}_3). \quad (\text{D.23})$$

Combining this with the relation $\theta(t - t') + \theta(t' - t) = 1$, the expression above can be rearranged as

$$\begin{aligned} \langle s | \hat{U}^{(2)} \hat{\rho}_{ABC,0} \hat{U}^{(2)\dagger} | s \rangle &= \mathcal{L}_{CC} \tilde{\mathcal{L}}_{AA} |e_A\rangle\langle e_A| \otimes |g_B\rangle\langle g_B| + \mathcal{L}_{CC} \tilde{\mathcal{L}}_{BB} |g_A\rangle\langle g_A| \otimes |e_B\rangle\langle e_B| \quad (\text{D.24}) \\ &+ \mathcal{L}_{CC} \tilde{\mathcal{L}}_{AB} |e_A\rangle\langle g_A| \otimes |g_B\rangle\langle e_B| + \mathcal{L}_{CC} \tilde{\mathcal{L}}_{AB}^* |g_A\rangle\langle e_A| \otimes |e_B\rangle\langle g_B| + \epsilon \mathcal{O}(\lambda^4), \end{aligned}$$

where $\mathcal{L}_{CC}$ is the excitation probability of detector C, and $\tilde{\mathcal{L}}_{IJ}$ is given by Eq. (8.44). We now compute the fourth order terms left, namely $\langle s | \hat{U}^{(3)} \hat{\rho}_{ABC,0} \hat{U}^{(1)\dagger} | s \rangle$ and $\langle s | \hat{U}^{(1)} \hat{\rho}_{ABC,0} \hat{U}^{(3)\dagger} | s \rangle$:

$$\begin{aligned} \langle s | \hat{U}^{(3)} \hat{\rho}_{ABC,0} \hat{U}^{(1)\dagger} | s \rangle &= \langle s | \text{Tr}_\phi \sum_{I,J \in \{A,B\}} \int d\mathbf{x} d\mathbf{x}' d\mathbf{x}'' d\mathbf{x}''' \sqrt{g g' g'' g'''} \left[\hat{h}_C(\mathbf{x}) \hat{h}_I(\mathbf{x}') \hat{h}_J(\mathbf{x}'') \right. \\ &\quad \left. + \hat{h}_I(\mathbf{x}) \hat{h}_C(\mathbf{x}') \hat{h}_J(\mathbf{x}'') + \hat{h}_I(\mathbf{x}) \hat{h}_J(\mathbf{x}') \hat{h}_C(\mathbf{x}'') \right] \hat{\rho}_{ABC,0} \hat{h}_C(\mathbf{x}''') \theta(t - t') \theta(t' - t'') | s \rangle \\ &+ \kappa_1 |g_A\rangle\langle g_A| \otimes |e_B\rangle\langle e_B| + \epsilon \mathcal{O}(\lambda^4), \quad (\text{D.25}) \end{aligned}$$

where κ_1 is a local term of C given by

$$\kappa_1 = \langle s | \text{Tr}_\phi \sum_{I,J \in \{A,B\}} \int d\mathbf{x} d\mathbf{x}' d\mathbf{x}'' d\mathbf{x}''' \sqrt{g g' g'' g'''} \hat{h}_C(\mathbf{x}) \hat{h}_I(\mathbf{x}') \hat{h}_J(\mathbf{x}'') \hat{\rho}_{ABC,0} \hat{h}_C(\mathbf{x}''') \theta(t - t') \theta(t' - t'') | s \rangle. \quad (\text{D.26})$$

Finally,

$$\begin{aligned}
& \langle s | \hat{U}^{(3)} \hat{\rho}_{\text{ABC},0} \hat{U}^{(1)\dagger} | s \rangle + \langle s | \hat{U}^{(1)} \hat{\rho}_{\text{ABC},0} \hat{U}^{(3)\dagger} | s \rangle \\
&= \mathcal{L}_{\text{CC}} \tilde{\mathcal{M}}_{\text{AB}} |e_{\text{A}}\rangle\langle g_{\text{A}}| \otimes |e_{\text{B}}\rangle\langle g_{\text{B}}| + \mathcal{L}_{\text{CC}} \tilde{\mathcal{M}}_{\text{AB}}^* |g_{\text{A}}\rangle\langle e_{\text{A}}| \otimes |g_{\text{B}}\rangle\langle e_{\text{B}}| - \mathcal{L}_{\text{CC}} \tilde{\mathcal{L}}_{\text{AA}} |g_{\text{A}}\rangle\langle g_{\text{A}}| \otimes |g_{\text{B}}\rangle\langle g_{\text{B}}| \\
&\quad - \mathcal{L}_{\text{CC}} \tilde{\mathcal{L}}_{\text{BB}} |g_{\text{A}}\rangle\langle g_{\text{A}}| \otimes |g_{\text{B}}\rangle\langle g_{\text{B}}| + \kappa_2 |g_{\text{B}}\rangle\langle g_{\text{B}}| \otimes |g_{\text{A}}\rangle\langle g_{\text{A}}| + \epsilon \mathcal{O}(\lambda^4),
\end{aligned} \tag{D.27}$$

with κ_2 given by

$$\kappa_2 = \text{Tr} \left[\langle s | \hat{U}^{(3)} \hat{\rho}_{\text{ABC},0} \hat{U}^{(1)\dagger} | s \rangle + \langle s | \hat{U}^{(1)} \hat{\rho}_{\text{ABC},0} \hat{U}^{(3)\dagger} | s \rangle \right]. \tag{D.28}$$

The fourth order terms are obtained adding the contributions of Eqs. (D.21), (D.24), and (D.27), yielding

$$\langle s | \hat{\rho}_{\text{ABC}}^{(4)} | s \rangle = \kappa \hat{\rho}_{\text{AB},0} + \hat{\nu} + \epsilon \mathcal{O}(\lambda^4), \tag{D.29}$$

with $\hat{\nu}$ given by Eq. (8.43) and

$$\kappa = \kappa_1 + \kappa_2. \tag{D.30}$$

Finally, we add the terms proportional to λ^0 , λ^2 and λ^4 , which are given by Eqs. (D.17), (D.20), and (D.29), respectively. This completes the proof for the expansion stated in Eq. (8.41).

D.3 Negativity

In this appendix, we show that the negativity of a density operator of the form of Eq. (8.47) is given by Eq. (8.48). The negativity of a two-qubit system is a function of the negative eigenvalues of the partially transposed density operator (partially transposing with respect to either of the subsystems A or B, which yields the same eigenvalues). Concretely, we can write

$$\mathcal{N}(\hat{\rho}_{\text{AB}}) = \sum_j \frac{|x_j| - x_j}{2}, \tag{D.31}$$

where x_j are the eigenvalues of the partially transposed density matrix. Note that the expression above reduces to the absolute value of the sum of negative eigenvalues of the partially transposed density matrix. Given the density matrix in Eq. (8.47), its partial transpose with respect to subsystem B is given by

$$\begin{pmatrix}
1 - \lambda^2 r_{22} - \lambda^2 r_{33} & \Theta(\lambda^b) & \Theta(\lambda^b) & \lambda^2 r_{32}^* \\
\Theta(\lambda^b) & \lambda^2 r_{22} & \lambda^2 r_{41}^* & 0 \\
\Theta(\lambda^b) & \lambda^2 r_{41} & \lambda^2 r_{33} & 0 \\
\lambda^2 r_{32} & 0 & 0 & 0
\end{pmatrix} + \mathcal{O}(\lambda^3). \tag{D.32}$$

The characteristic polynomial of the matrix above reads

$$P(x) = x^4 - x^3 + (r_{22} + r_{33} + \mathcal{O}(\lambda^3)) x^2 + (|r_{41}|^2 - r_{22}r_{33} + \mathcal{O}(\lambda^5)) + \mathcal{O}(\lambda^8). \quad (\text{D.33})$$

In order to find the roots of the above expression, we need to follow a method that is consistent with the perturbative expansion, as in, e.g., [237]. Assume that x is a root of Eq. (D.33). We expand it in λ as

$$x = x^{(0)} + x^{(1)} + x^{(2)} + \mathcal{O}(\lambda^3), \quad (\text{D.34})$$

where $x^{(i)}$ is proportional to λ^i . We impose $P(x) = 0$ and set the terms of order i in λ equal to zero, starting by $i = 0$ and increasing the order until we determine $x^{(0)}$, $x^{(1)}$ and $x^{(2)}$ up to the order we need. In the first step, for $i = 0$, we obtain

$$[x^{(0)}]^3 [x^{(0)} - 1] = 0. \quad (\text{D.35})$$

Hence, either $x^{(0)} = 0$ or $x^{(0)} = 1$. For $i = 1$, we get

$$4[x^{(0)}]^3 x^{(1)} - 3[x^{(0)}]^2 x^{(1)} = 0. \quad (\text{D.36})$$

If $x^{(0)} = 0$, the equation above vanishes. If $x^{(0)} = 1$, the equation allows to find $x^{(1)} = 0$. For $i = 2$, we obtain the equation

$$4[x^{(0)}]^3 x^{(2)} + 6[x^{(0)}x^{(1)}]^2 - 3[x^{(0)}]^2 x^{(2)} - 3[x^{(1)}]^2 x^{(0)} + x^{(0)}(r_{22} + r_{33}) = 0. \quad (\text{D.37})$$

If the zeroth order term is zero, i.e., $x^{(0)} = 0$, all the terms of the equation above also vanish. If $x^{(0)} = 1$ and $x^{(1)} = 0$, the equation above simplifies to $x^{(2)} + (r_{22} + r_{33}) = 0$, which allows us to find the second order term of the eigenvalue. Therefore, we have completely characterized one of the eigenvalues up to second order in perturbation theory:

$$x_1 = 1 - r_{22} - r_{33} + \mathcal{O}(\lambda^3). \quad (\text{D.38})$$

Henceforth, we can restrict to the case in which the zeroth order term vanishes, $x^{(0)} = 0$. For $i = 3$, we obtain $[x^{(1)}]^3 = 0$, which implies $x^{(1)} = 0$. Consequently, the remaining eigenvalues are quadratic in the coupling strength, $x = x^{(2)} + \mathcal{O}(\lambda^3)$. There are no terms in the equation $P(x) = 0$ proportional to λ^4 or λ^5 . For sixth order, we get

$$-[x^{(2)}]^3 + (r_{22} + r_{33})[x^{(2)}]^2 + (|r_{41}|^2 - r_{22}r_{33})x^{(2)} = 0. \quad (\text{D.39})$$

Hence, $x^{(2)}$ is a root of the previous expression. The solutions are

$$x_2^{(2)} = 0, \quad (\text{D.40})$$

$$x_3^{(2)} = \frac{(r_{22} + r_{33}) + \sqrt{(r_{22} + r_{33})^2 + 4(|r_{41}|^2 - r_{22}r_{33})}}{2}, \quad (\text{D.41})$$

$$x_4^{(2)} = \frac{(r_{22} + r_{33}) - \sqrt{(r_{22} + r_{33})^2 + 4(|r_{41}|^2 - r_{22}r_{33})}}{2}. \quad (\text{D.42})$$

Therefore, the eigenvalues left read

$$x_2 = 0 + \mathcal{O}(\lambda^3), \quad (\text{D.43})$$

$$x_3 = \frac{(r_{22} + r_{33}) + \sqrt{(r_{22} + r_{33})^2 + 4(|r_{41}|^2 - r_{22}r_{33})}}{2} + \mathcal{O}(\lambda^3), \quad (\text{D.44})$$

$$x_4 = \frac{(r_{22} + r_{33}) - \sqrt{(r_{22} + r_{33})^2 + 4(|r_{41}|^2 - r_{22}r_{33})}}{2} + \mathcal{O}(\lambda^3). \quad (\text{D.45})$$

The negativity is derived from the negative eigenvalues of the partially transposed density operator. The eigenvalue x_1 is always positive, since we are assuming that λ is a small parameter. The eigenvalue x_2 is zero (and therefore non-negative) to second order in perturbation theory. The eigenvalue x_3 is positive since it is a sum of positive terms. Therefore, the only eigenvalue that can be negative (to second order) is x_4 . Thus the negativity is

$$\begin{aligned} \mathcal{N}_s &= \max \left(0, \sqrt{\frac{(r_{22} + r_{33})^2}{4} + |r_{41}|^2 - r_{22}r_{33}} - \frac{r_{22} + r_{33}}{2} + \mathcal{O}(\lambda^3) \right) \\ &= \max \left(0, \sqrt{\frac{(r_{22} - r_{33})^2}{4} + |r_{41}|^2 - \frac{r_{22} + r_{33}}{2}} + \mathcal{O}(\lambda^3) \right). \end{aligned} \quad (\text{D.46})$$