Algebraic Approach to Quantum Isomorphisms

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

In very brief, this thesis is a study of quantum isomorphisms. We have started with two pairs of quantum isomorphic graphs and looked for generalizations of those.

We have learned that those two pairs of graphs are related by Godsil-McKay switching and one of the graphs is an orthogonality graph of lines in a root system. These two observations lead to research in two directions.

First, since quantum isomorphisms preserve coherent algebras, we studied a question of when Godsil-McKay switching preserved coherent algebras. In this way, non isomorphic graphs related by Godsil-McKay switching with isomorphic coherent algebras are candidates to being quantum isomorphic and non isomorphic.

Second, while it was known that one of the graphs in a pair was an orthogonality graphs of the lines in a root system E_8, we showed that a graph from another pair is also an orthogonality graph of the lines in a root system F_4. We have studied orthogonality graphs of lines in root systems $B_{2^d}, C_{2^d}, D_{2^d}$ and showed that they have quantum symmetry.

Finally, we have touched upon structures of quantum permutations, relationships between fractional and quantum isomorphisms as well as connection to quantum independence and chromatic numbers.

Acknowledgements

The time has come to add another dimension of meaning to this thesis.

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Dedication

To that eternal wisdom, deep love and higher consciousness,
To life, to world, to people,
To endless days ahead and numerous behind,
To infinite, to finite,
To joy and sorrow, success and better 'morrow,
To every fall, that makes us standing stronger,
To everything, to nothing,
To roses in the evening,
To stars, to night, to sunshine,
To sparkles on the shoreline.
To freshest breeze, to the most pouring rain,
To crunchy snow and time not spent in vain,
To pain.
To meaning, to your choice, to your creation,
To you, reflection of myself, and to imagination.
To everything there is, to everything you find,
To way that leads to light, makes you one of a kind.

But really to my Grandmother Valentina.

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Chapter 1

Introduction

Quantum automorphisms actually preceded quantum isomorphisms, so we start with those. Classical automorphism groups are well-studied. Recently, quantum generalizations have emerged.

In 2003, Bichon [9] introduced a quantum automorphism group of a graph in the language of C^*-algebras. Shortly after, in [4] Banica slightly modified Bichon's definition, and the updated definition has become standard in the area of quantum automorphisms. We won't be working with it, but for reference, we include it here.

We often use X and Y to denote graphs. For us graphs are simple, without loops and multiple edges. $E(X)$ denotes edges of the graph X and $V(X)$ stands for its vertices. We say two vertices $i, j \in V(X)$ are adjacent in X if $\{i, j\} \in E(X)$. The notation that we use is $i \sim_X j$. On the other hand, if $\{i, j\} \notin E(X)$, we denote it by $i \nsim_X j$. From now on, unless otherwise specified, $A(X)$ is the *adjacency matrix* of a graph X. It is defined as a binary matrix with $(A(X))_{ij} = 1$ if $\{i, j\} \in E(X)$ and zero otherwise. If the graph is clear from context, we simply write A. For more reference on terminology see [28].

Unless otherwise specified, n denotes the number of vertices in the graph in consideration and m denotes the number of edges.

Definition 1.0.1 **([4])** *Let $X = (V, E)$ be a finite graph on n vertices $V = \{1, ..., n\}$. The quantum automorphism group $G^+(X)$ is a compact matrix quantum group $(C(G^+(X)), u)$, where $C(G^+(X))$ is the universal C^*-algebra with generators $u_{ij}, 1 \leq i, j \leq n$ and relations*

(a) $u_{ij} = u_{ij}^* = u_{ij}^2, \quad 1 \leq i, j \leq n$

(b) $\sum_{j=1}^{n} u_{ij} = \sum_{j=1}^{n} u_{ji}, \quad 1 \leq i \leq n$

(c) $\sum_j u_{ij} A_{jk} = \sum_j A_{ij} u_{jk}, \quad 1 \leq i, k \leq n.$

Those rare cases when we will appeal to this definition would be times when we reference earlier work. In that context we will also use term *magic unitary* which denotes the $n \times n$ matrix with those generators u_{ij}'s as entries.

For this thesis we use the following finite-dimensional definition. Before the definition, we are going to familiarize the reader with some notation. For a ring R, and a natural number d, we define $M_d(R)$ to be the set of all $d \times d$ matrices with entries from R. And I_d, or I, whenever clear from the context, denotes identity matrix.

Definition 1.0.2 *The quantum automorphism group of a graph X is the set of all matrices $Q \in M_{|V(X)|}(M_d(\mathbb{C})), d \in \mathbb{N}$ satisfying the following conditions:*

- $\sum_i Q_{ij} = I_d = \sum_j Q_{ij}$ *when* $1 \leq i, j \leq |V(X)|$.
- $Q_{ij} = Q_{ij}^* = Q_{ij}^2$.
- $Q(A \otimes I_d) = (A \otimes I_d)Q$.

Matrices satisfying the first two points out of three are called *quantum permutations*. So another way to restate the definition is that the quantum automorphism group is the set of all quantum permutations with finite-dimensional projections as entries that commute with the adjacency matrix of the graph. Note that when we restrict the above definition to $d = 1$, we just get a classical automorphism group of the graph.

We say that a graph has *quantum symmetry* if there is a quantum automorphism of the graph that has non-commutative entries. To the best of my knowledge of existing literature, proving that a graph has quantum symmetry involved either finding a quantum permutation with finite-dimensional projections as entries or determining the quantum automorphism group and showing it is different from classical automorphism group. A lot of study of quantum automorphism groups of graphs was done first by Banica [7] and extensively developed by Schmidt in [62], [62], [61] and [63].

At the same time, quantum automorphisms are intimately tied to quantum isomorphisms. However, since quantum isomorphisms do not form a group, their history is quite different and starts with quantum homomorphisms. In 2013 David Roberson has published his PhD thesis [59] and the results about quantum graph homomorphisms were later published in the paper with Laura Mančinska [43]. Three years later in [2], Mančinska and Roberson together with other authors formulated a definition of quantum isomorphisms in the form we use it in this thesis.

Definition 1.0.3 *A quantum permutation $Q \in M_{|V(X)|}(M_d(\mathbb{C}))$ is a quantum isomorphism between two graphs X and Y on the same number of vertices if*

$$Q(A_X \otimes I_d)Q^* = A_Y \otimes I_d$$

From now on, whenever we mention quantum isomorphism between graphs X and Y, it always assumed that $|V(X)| = |V(Y)|$.

Analogously, if in the definition above $d = 1$, then Q is just a classical isomorphism between the graphs. The question in the area of quantum isomorphisms is to find graphs that are quantum isomorphic and not isomorphic. This has been the initial motivation for all of this thesis.

Originally, at our disposal, we had only graphs from [2]. While there is an infinite family of non-isomorphic quantum isomorphic graphs, we were working with the tangible 24-vertex pair of graphs, and observed that they are related by Godsil-McKay switching (See 2.3).

Theorem 1.0.4 (5.1.1) *The 24-vertex graphs from [2] are related by Godsil-McKay switching.*

Later, when Schmidt's paper [65] with 120-vertex non-isomorphic quantum isomorphic strongly regular graphs came out in September 2023, we verified that those two graphs are also related by Godsil-McKay switching.

Theorem 1.0.5 (5.2.1) *The 120-vertex graphs from [65] are related by Godsil-McKay switching.*

From there on, I have been on a quest to try to generalize these two examples. That inevitably led to identifying more commonalities between these two pairs of graphs.

Godsil-McKay switching graphs are often non isomorphic. At the same time, we have a fact from [2] that quantum isomorphic graphs are cospectral and have isomorphic coherent algebras. Combining all this information and driven by the goal to find more non-isomorphic quantum isomorphic graphs, we asked this question:

Question 1 *When does Godsil-McKay switching preserve coherent algebras?*

In the process of answering this question, we found that conjugation by quantum isomorphism matrix is an isomorphism of coherent algebras.

Orbital algebras are defined in Section 6.5 and coherent algebra of the graph in 6.6).

Theorem 1.0.6 (6.10.2) *Suppose Q is a quantum isomorphism between X and Y. If $\mathcal{O}_X, \mathcal{O}_Y$ denote the orbital algebras of X and Y respectively, then*

$$Q(\mathcal{O}_X \otimes I)Q^* = (\mathcal{O}_Y \otimes I).$$

□

The question about the converse arises.

Question 2 *Suppose, X and Y are graphs with orbital algebras $\mathcal{O}_X$ and $\mathcal{O}_Y$. Suppose, Q is a quantum permutation satisfying*

$$Q(\mathcal{O}_X \otimes I)Q^* = (\mathcal{O}_Y \otimes I).$$

Is Q a quantum isomorphism between X and Y?

Theorem 1.0.7 (6.11.2) *Suppose Y is a graph with adjacency matrix B and coherent algebra $\mathcal{C}(Y)$. Suppose X is a graph with adjacency matrix A and coherent algebra $\mathcal{C}(X)$ that has 01-basis $B_1, ..., B_k$. Suppose X is related to Y by Godsil-McKay switching with partition $(C_1, ..., C_\ell, D)$ and S is like in the definition of Godsil-McKay switching with $SA = BS$. Then if for each $1 \leq i \leq k$ we have that B_i can be partitioned into a Godsil-McKay partition $(C_1, ..., C_\ell, D)$, then conjugation by S is an isomorphism between coherent algebras $\mathcal{C}(X)$ and $\mathcal{C}(Y)$.*

Corollary 1.0.8 (6.11.5) *Suppose X and Y are related by Godsil-McKay switching with the Godsil-McKay switching matrix Q, partition (C, D), and Godsil-McKay switching matrix S. Suppose that conjugation by S is an isomorphism of coherent algebras. Suppose further that at least one of X or Y satisfies all of the following:*

(a) each vertex in D is adjacent to $\frac{|C|}{2}$ vertices for all $1 \leq i \leq k$.

(b) is connected with connected complements.

(c) has a homogeneous coherent algebra.

Then directed graphs corresponding to basis matrices of the coherent algebra satisfying the third condition can be partitioned according to the Godsil-McKay partition.

A second similarity that we identified is that one of the graphs from each pair is an orthogonality graph of lines in a root system. Now one of the Schmidt's graphs was originally presented in his paper [65] as an orthogonality graph of lines in the root system E_8. We have observed that one of the 24-vertex graphs also is isomorphic to an orthogonality graph of the root system F_4. The following question begged to be answered:

Question 3 What can be said about orthogonality graphs of the lines in the other root systems?

As it turned out, this question tied quantum isomorphisms and automorphisms together. While we were looking for quantum isomorphic, non isomorphic graphs, we have found graphs with quantum symmetry instead.

Theorem 1.0.9 (7.7.2,7.6.10,7.7.1) *The orthogonality graphs of lines in the root systems $B_{2^n}, C_{2^n}, D_{2^n}$ have quantum symmetry.*

Another common point between the two pairs of graphs is that one graph in each pair, in particular, the one that is the orthogonality graph of the lines in the appropriate root system, has quantum independence and chromatic numbers that differ from the classical counterparts, while the other graph has equal classical and quantum parameters. Kochen-Specker sets, are known to be intimately connected to the gap in quantum and classical parameters of the graphs [60],[67]. First, we define them. A set $S = S_1 \sqcup ...S_k$ consisting of sets S_i of projections in $M_d(\mathbb{C})$ that sum to I_d is a *projective Kochen-Specker set* if we cannot select k pairwise nonorthogonal projections from it. We explored the connection between graphs being quantum isomorphic and nonisomorphic, at least one of the graphs having different quantum and classical parameters and the entries of the quantum isomorphism forming a projective Kochen-Specker set. Some results on this topic include:

- (4.5.2) A sufficient condition on graphs X, Y and a quantum isomorphism Q between them for $\alpha(X) < \alpha_q(X)$.

- (4.5.4) A sufficient condition on graphs X, Y and a quantum isomorphism Q between them for $\chi(X) > \chi_q(X)$.

Theorem 1.0.10 (4.2.1) *Suppose X and X' are non isomorphic distinct graphs. Suppose Q is a quantum permutation with rank-one or rank-zero entries and is a quantum isomorphism between them. Then entries of Q form a projective Kochen-Specker set.*

Last, but not least, another similarity that we have identified between the 24-vertex and 120-vertex examples is a connection to Cayley graphs. In [2], the authors mention that both 24-vertex graphs are Cayley, and in [67], we find the groups for those Cayley graphs, S_4, and the connection sets, which turn out to be unions of conjugacy classes.

For one of the 120-vertex graphs, I showed that it is Cayley for S_5 and identified a connection set for it, which, however, was not a union of conjugacy classes.

Nonetheless, the following is an interesting question:

Question 4 *When are two normal Cayley graphs cospectral?*

We provide a sufficient condition:

Theorem 1.0.11 (3.4.2) *Let $X = Cay(S_n, M)$ and $Y = Cay(S_n, N)$. Suppose M and N are disjoint and each consist of conjugacy classes of odd permutations. Assume further that $M \cup N$ are the set of all odd permutations in S_n and $|M| = |N|$. Then X and Y are cospectral.*

Chapter 2

Background on quantum isomorphisms

This chapter contains definitions and known properties of quantum permutations, quantum automorphisms and quantum isomorphisms. Throughout the thesis we use trace inner product between matrices $C, D \in M_n(\mathbb{C})$:

$$\operatorname{tr}\langle C, D\rangle = \operatorname{tr} C^* D.$$

2.1 Definition of quantum isomorphisms and quantum automorphism group

In this thesis we usually denote a graph by X with vertices $V = \{1, ..., n\}$ and edges $E \subseteq (V, V)$. The letter A is usually reserved for adjacency matrices. The ij^{th} entry of adjacency matrix $A(X)$ is one if $(i, j) \in E$, and zero otherwise. We work with undirected simple graphs, unless stated otherwise.

Now, quantum permutations are one of the central objects in this thesis. We define them and explore some of their properties.

Definition 2.1.1 A *quantum permutation* matrix $P \in M_n(M_d(\mathbb{C}))$ is an $n \times n$ matrix with the ij-entry a projection P_{ij}, satisfying the following conditions.

$$P_{ij} = P_{ij}^* = P_{ij}^2, \qquad 1 \leq i, j \leq n, \tag{2.1.1}$$

$$\sum_{\ell=1}^{n} P_{i\ell} = I_d = \sum_{\ell=1}^{n} P_{\ell i}, \qquad 1 \leq i \leq n, \tag{2.1.2}$$

Next we provide a common knowledge lemma to derive orthogonality relations on the projections in 2.1.2. More can theory be found in [76].

Lemma 2.1.2 *Suppose* $\{P_l\}_{l\in[n]}$ *are projections in* $M_d(\mathbb{C})$*, such that* $\sum_{l=1}^n P_l = I_d$*, then* $P_l P_k = P_k P_l = 0$*, when* $l \neq k$*.*

Proof.

$$P_k = P_k P_k = P_k \left(\sum_{l=1}^{n} P_l \right) P_k = P_k + \sum_{\substack{l=1,\\ l\neq k}}^{n} P_k P_l P_k,$$

from which it follows that

$$\sum_{\substack{l=1,\\ l\neq k}}^{n} P_k P_l P_k = 0.$$

Take traces of both sides and use cyclic property of trace to get:

$$\sum_{\substack{l=1,\\ l\neq k}}^{n} \mathrm{tr}(P_k P_l P_k) = 0$$

$$\sum_{\substack{l=1,\\ l\neq k}}^{n} \mathrm{tr}(P_l P_k) = 0$$

Now the trace of two positive semidefinite matrices is nonnegative. Thus, it follows that for each $1 \leq l \neq k \leq n$,

$$\mathrm{tr} P_l P_k = 0.$$

□

Above lemma is also true for infinite-dimensional projections. From this it easily follows that quantum permutations are unitary matrices.

Corollary 2.1.3 *If $P \in M_n(M_d(\mathbb{C}))$ is a quantum permutation, then $PP^T = I$.* □

Recall that a classical automorphism of X can be represented as a permutation matrix Q such that $QA = AQ$. The set of permutation matrices that commute with the adjacency matrix is denoted $\mathrm{Aut}(X)$.

Now, we use the following definition of the quantum isomorphism:

Definition 2.1.4 Suppose X and Y are graphs with adjacency matrices A and B respectively. We say X and Y are quantum isomorphic if there is an $n \times n$ quantum permutation matrix $Q \in M_n(M_d(\mathbb{C}))$ such that

$$(A \otimes I_d)Q = Q(B \otimes I_d).$$

In this case, we call Q a *quantum isomorphism* between X and Y. When the graphs X and Y denote the same graph, Q is a *quantum automorphism*.

We will often write $A \otimes I_d$ as just A, when d is clear from the context.

Next we introduce conditions on the entries of the quantum automorphism that are equivalent to the condition in 2.1.4.

Lemma 2.1.5 *(adapted from [23]) Let P be a quantum permutation, and X and Y be graphs with adjacency matrices A and B respectively. Then $P(B \otimes I_d) = (A \otimes I_d)P$ if and only if:*

$$P_{ji}P_{lk} = P_{lk}P_{ji} = 0, \qquad \text{for all } (i,k) \notin E(Y), \quad (j,l) \in E(X), \tag{2.1.3}$$

$$P_{ij}P_{kl} = P_{kl}P_{ij} = 0, \qquad \text{for all } (i,k) \notin E(X), \quad (j,l) \in E(Y), \tag{2.1.4}$$

Proof. For the forward direction, since $P(B \otimes I) = (A \otimes I)P$ and $PP^T = I$, we can use the equality of the ij entries of $P(B \otimes I)P^T$ and A to get the desired relations. We have:

$$\sum_{k,m} P_{i,k} b_{km} P^T_{mj} = a_{ij}.$$

In the left hand side only the terms with $k \sim m$ do not vanish, which leaves us with:

$$\sum_{k \sim m \text{ in } Y} P_{ik} P_{jm} = a_{ij}$$

Now, if $(i, j) \notin E(X)$, clearly $a_{ij} = 0$. Take traces of both sides and use the fact that trace of a product of positive semidefinite matrices is nonnegative to obtain:

$$P_{ik} P_{jm'} = 0,$$

which proves Condition 2.1.3. Condition 2.1.4 are proved similarly.

On the other hand, suppose $P \in M_n(M_d(\mathbb{C}))$ is a quantum permutation satisfying the conditions 2.1.4 and 2.1.3. To show that this implies that $P(B \otimes I) = (A \otimes I)P$, we will look at the ij-entry of the left and right hand sides once again:

$$\begin{aligned}
\sum_l P_{il} b_{lj} &= \sum_l b_{lj} P_{il} \\
&= \sum_l b_{lj} P_{il} \sum_k P_{kj} \\
&= \sum_{l,k} b_{lj} P_{il} P_{kj} \\
&= \sum_{k,l} b_{lj} a_{ik} P_{il} P_{kj},
\end{aligned}$$

where the last equality follows from the fact that if $a_{lj} = 0$, then $P_{il} P_{kj} = 0$ by hypothesis. Similarly:

$$\begin{aligned}
\sum_k a_{ik} P_{kj} &= \sum_l P_{il} \sum_k a_{ik} P_{kj} \\
&= \sum_{l,k} a_{ik} P_{il} P_{kj} \\
&= \sum_{l,k} a_{ik} b_{lj} P_{il} P_{kj}.
\end{aligned}$$

Therefore, the ij-entries of $P(B \otimes I)$ and $(A \otimes I)P$ are the same, and P commutes with $(A \otimes I)$. $\square$

An immediate corollary follows for graph automorphisms.

Corollary 2.1.6 *[23] Let P be a quantum permutation, and X a graph with adjacency matrix A. Then $P(A \otimes I_d) = (A \otimes I_d)P$ if and only if:*

$$P_{ji}P_{lk} = P_{lk}P_{ji} = 0, \qquad (i,k) \notin E(X), (j,l) \in E(X), \tag{2.1.5}$$
$$P_{ij}P_{kl} = P_{kl}P_{ij} = 0, \qquad (i,k) \notin E(X), (j,l) \in E(X) \tag{2.1.6}$$

Note that every classical automorphism is also a quantum one with dimension of the projection $d = 1$.

Definition 2.1.7 Let $X = (V, E)$ be a finite graph on n vertices $V = \{1, ..., n\}$. The *quantum automorphism group of X*, denoted $\mathrm{QAut}(X)$, is the set of quantum permutation matrices $P = (P_{ij})_{ij} \in M_n(M_d(\mathbb{C})), d \in \mathbb{N}$ such that $PA = AP$.

We summarise all the properties of the matrices $P = (P_{ij})_{ij}$ that qualify as quantum automorphisms of a graph X with the adjacency matrix A.

$$P_{ij} = P_{ij}^* = P_{ij}^2, \qquad 1 \leq i, j \leq n, \tag{2.1.7}$$
$$\sum_{j=1}^{n} P_{ij} = I_d = \sum_{i=1}^{n} P_{ij}, \qquad 1 \leq i \leq n, \tag{2.1.8}$$
$$P_{ji}P_{lk} = P_{lk}P_{ji} = 0, \qquad (i,k) \notin E, (j,l) \in E, \tag{2.1.9}$$
$$P_{ij}P_{kl} = P_{kl}P_{ij} = 0, \qquad (i,k) \notin E, (j,l) \in E, \tag{2.1.10}$$

The first two conditions ensure that P is a quantum permutation, while the last two conditions are equivalent to $PA = AP$.

2.2 Quantum symmetries and compositions of quantum automorphisms

If for all quantum permutations Q commuting with the adjacency matrix of a graph X it holds that $P_{ij}P_{kl} = P_{kl}P_{ij} \quad \forall i, j, k, l \in V(X)$, then we say that the graph has *no quantum symmetry* [62]. Otherwise, if there is at least one quantum automorphism of X with non-commuting entries, we say that X has *quantum symmetry*. We'll show in the next two lemmas how to decompose quantum automorphisms with commuting entries into different classical automorphisms.

First, following personal communication with Godsil, we will define an operation $\boxplus$, similar to a general direct sum operation on matrices, that combines two quantum permutations in the following manner. Suppose $P \in M_n(M_d(\mathbb{C}))$ and $M_n(M_t(\mathbb{C}))$ are quantum permutations. Say, $P = (P_{ij})_{ij}$ and $Q = (Q_{ij})_{ij}$. Then $P \boxplus Q \in M_n(M_{d+t}(\mathbb{C}))$ with entries

$$(P \boxplus Q)_{ij} = (P_{ij} \oplus Q_{ij}).$$

Note that $P \boxplus Q \neq Q \boxplus P$ in general. At the same time, $\boxplus$ is well-defined for any pair of quantum permutations.

The lemma below says that if entries in a quantum permutation commute, it is a "direct sum" of classical automorphisms.

Lemma 2.2.1 (Godsil) *Suppose $Q \in M_n(M_d(\mathbb{C}))$ is a quantum automorphism of a graph X on n vertices. If all projections Q_{ij} pairwise commute, then there are classical automorphisms $P_1, ..., P_d$ such that $P = \boxplus_{i=1}^d Q_i$*

Proof. Suppose Q is as in the statement. Since $\{Q_{ij}\}$ is a set of commuting Hermitian matrices, there is an orthonormal basis $\{v_k\}_{k=1}^d$ in which all those projections are diagonal. Let u be the matrix with the $v_1.,,,v_d$ as columns. Let U be a block-diagonal matrix with n matrices u on the diagonal. Then UQU^{-1} is a quantum permutation with each entry being a diagonal matrix, which we call block-diagonalized Q.

Now we will construct the classical quantum automorphisms from this block-diagonalized Q. For convenience we refer to the n rows of projections as block rows, and to the nd rows as single rows. Thus, block-rows of the block-diagonalized Q are now projections that sum to identity. Therefore, each block-row entry has eigenvalues of projections, which are ones and the zeroes on the diagonal. Meanwhile, each single row out of nd rows has only one 1 in it, since projections in the same block-row are orthogonal. We are ready to define $P_1, ..., P_d$. Let $(P_i)_{ab}$ be 1 if there is a one 1 in the i^{th} single row of a projection P_{ab}, and zero otherwise.

Then since $\sum_b Q_{ab} = I$, there will be at most one 1 entry in the a^{th} row of P_i. Similarly, since $\sum_b Q_{ab} = I$, there will be at most one 1 entry in the b^{th} column of Q_i. Finally, the product of diagonal matrices is 0 if and only if they have disjoint support. Therefore, since $Q_{ab}Q_{k\ell} = 0$ if $(a, k) \in E(X)$, but $(b, \ell) \in E(X)$, we have that also $(P_i)_{ab}(P_i)_{k\ell} = 0$ when $(a, k) \in E(X)$, but $(b, \ell) \in E(X)$. Analogously, $(P_i)_{ba}(P_i)_{\ell k} = 0$ when $(a, k) \in E(X)$, but $(b, \ell) \in E(X)$. Hence, $P_1, ..., P_d$ are classical automorphisms of X.

At the same time, from construction, $Q = \boxplus_{i=1}^d P_i$. □

The next two lemmas illustrate why the quantum automorphism group might not be equal to the classical automorphism group for any graph.

Lemma 2.2.2 (Godsil) *Suppose X is an undirected graph, and $P = (P_{ij})_{ij}$ and $Q = (Q_{ij})_{ij}$ are quantum automorphisms of X, then $P \boxplus Q$ is also a quantum automorphism of X.*

Proof. We will check conditions 2.1.7,...,2.1.10 one by one.

- To check that $P \boxplus Q$ has projections as entries, we check that

$$(P \boxplus Q)_{ij} = P_{ij} \oplus Q_{ij} = P_{ij}^* \oplus Q_{ij}^* = (P_{ij} \oplus Q_{ij})^* = (P \boxplus Q)_{ij}^*.$$

  In addition,

$$(P \boxplus Q)_{ij}^2 = P_{ij}^2 \oplus Q_{ij}^2 = P_{ij} \oplus Q_{ij} = (P \boxplus Q)_{ij}.$$

- We are going to check that entries in each row of $P \boxplus Q$ sum to identity. The proof that entries in each column sum to identity is analogous.

$$\sum_{j=1}^{n} (P \boxplus Q)_{ij} = \sum_{j=1}^{n} (P_{ij} \oplus Q_{ij}) = \sum_{j=1}^{n} (P_{ij} \oplus \sum_{j=1}^{n} Q_{ij}) = I \oplus I = I$$

- Suppose $(i,k) \in E$ while $(j,l) \notin E$, then verify

$$(P \boxplus Q)_{ij}(P \boxplus Q)_{kl} = (P_{ij} \oplus Q_{ij})(P_{kl} \oplus Q_{kl}) = P_{ij}P_{kl} \oplus Q_{ij}Q_{kl} = 0.$$

  Similarly,

$$(P \boxplus Q)_{ji}(P \boxplus Q)_{lk} = 0.$$

□

Now define a composition $*$ of quantum permutations as follows. Provided that $P \in M_n(M_d(\mathbb{C}))$ and $Q \in M_n(M_t(\mathbb{C}))$ are quantum permutations, it turns out that $(P * Q)_{ij} \in M_n(M_{dt}(\mathbb{C}))$ with entries

$$(P * Q)_{ij} = \sum_{k} P_{ik} \otimes Q_{kj}$$

is a quantum permutation as well. Note that such composition is well-defined even when $d \neq t$.

Lemma 2.2.3 ([59]) *Provided that $P \in M_n(M_d(\mathbb{C}))$ and $Q \in M_n(M_t(\mathbb{C}))$ are quantum automorphisms of X, $(P * Q)$ is a quantum automorphism too.*

Proof. We will check conditions 2.1.7,...,2.1.10 one by one.

- To check that $P * Q$ has projections as entries, we check that

  $$(P * Q)_{ij} = \sum_k P_{ik} \otimes Q_{kj} = \sum_k P_{ik}^* \otimes Q_{kj}^* = \sum_k (P_{ik} \otimes Q_{kj})^* = (P * Q)_{ij}^*.$$

  In addition,

  $$(P * Q)_{ij}^2 = \sum_{k,\ell} (P_{ik} P_{i\ell} \otimes Q_{kj} Q_{\ell j}) = \sum_k (P_{ik} \otimes Q_{kj}) = (P * Q)_{ij},$$

  where the second equality is a consequence of the fact that in the i^{th} row all projections are orthogonal, and so $P_{ik} P_{i\ell} = 0$ for all $k \neq \ell$.

- We are going to check that entries in each row of $P * Q$ sum to identity. The proof that entries in each column sum to identity is analogous.

  $$\begin{aligned} \sum_{j=1}^n (P * Q)_{ij} &= \sum_{j=1}^n \sum_k P_{ik} \otimes Q_{kj} \\ &= \sum_k \sum_{j=1}^n (P_{ik} \otimes Q_{kj}) \\ &= \sum_k (P_{ik} \otimes I) \\ &= I \otimes I \\ &= I. \end{aligned}$$

  By now we have demonstrated that $P * Q$ is a quantum permutation. It remains to prove it commutes with $A(X)$, which we do by checking that the proper orthogonality relations hold.

- Suppose $(i, k) \in E$ while $(j, \ell) \notin E$, then verify

  $$\begin{aligned} (P * Q)_{ij} (P * Q)_{kl} &= \sum_r (P_{ir} \otimes Q_{rj}) \sum_m (P_{km} \otimes Q_{m\ell}) \\ &= \sum_{r,m} (P_{ir} P_{km} \otimes Q_{rj} Q_{m\ell}) \\ &= 0. \end{aligned}$$

The last equality follows since if $r \sim m$, then $Q_{rj}Q_{m\ell} = 0$, and if $r \nsim m$, then $P_{ir}P_{km} = 0$. Similarly,

$$(P * Q)_{ji}(P * Q)_{lk} = 0.$$

□

2.3 Godsil-McKay switching

In this section we introduce Godsil-McKay switching from [30], a well-known procedure to produce cospectral but often not isomorphic graphs. By J_d we denote the $d \times d$ all-ones square matrix. If its dimension is clear from the context, we omit the subscript.

Theorem 2.3.1 *[Godsil-McKay switching][30] Let X be a graph and let*

$$\{C_1, \ldots, C_k, D\}$$

be a partition of the vertex set $V(X)$ of X. Suppose that for every vertex $d \in D$ and every $i \in \{1, \ldots, k\}$, d has either 0, $\frac{1}{2}|C_i|$ or $|C_i|$ neighbors in C_i. Moreover, suppose that for all $i, j \in \{1, \ldots, k\}$ every vertex $x \in C_i$ has the same number of neighbors in C_j. Make a new graph X' as follows. For each $d \in D$ and $i \in \{1, \ldots, k$ such that x has $\frac{1}{2}|C_i|$ neighbors in C_i delete the corresponding $\frac{1}{2}|C_i|$ edges and join x instead to the $\frac{1}{2}|C_i|$ other vertices in C_i. Then X and X' are cospectral.

Proof. Suppose the adjacency matrices of X and X' are A and B respectively. We are going to find an orthogonal matrix Q such that

$$QAQ^* = B,$$

which will prove that A and B are similar, and thus have the same eigenvalues.

Let Q be block diagonal with square blocks of size $m_1 := |C_1|, ..., m_k := |C_k|, m_{k+1} = |D|$. Let the first k blocks be of the form

$$\frac{2}{m_i} J_{m_i} - I_{m_i} \text{ for } 1 \leq i \leq k,$$

and the last block be

$$\frac{1}{m_{k+1}} I_{m_{k+1}}.$$

Note that we can also view A as a block matrix, such that the blocks on the diagonal are square and are of the same size as diagonal blocks of Q: $m_1, ..., m_k, m_{k+1}$.

Now, the $\{i, j\}$-block of QAQ^* is (with indexing referring to blocks):

$$Q_{ii}A_{ij}Q^*_{jj}$$

Consider cases depending on whether Q_{ii} and Q_{jj} are the identity block of Q or not:

(a) If $1 \leq i, j \leq k$, then letting c denote the column sums in A_{ij} and r denote the row sums of A_{ij} and observing that $cm_j = rm_i$ equals the sum of the entries of A_{ij}.

$$\begin{aligned}
Q_{ii}A_{ij}Q^*_{jj} &= (\frac{2}{m_i}J_{m_i} - I_{m_i})A_{ij}\left(\frac{2}{m_j}J_{m_j} - I_{m_j}\right) \\
&= \frac{4}{m_i m_j}J_{m_i}A_{ij}J_{m_j} - \frac{2}{m_i}J_{m_i}A_{ij} - \frac{2}{m_j}A_{ij}J_{m_j} + A_{ij} \\
&= \frac{4}{m_i m_j}\frac{cm_j + rm_i}{2}J_{m_i m_j} - \frac{2}{m_i}cJ_{m_i m_j} - \frac{2}{m_j}rJ_{m_i m_j} + A_{ij} \\
&= A_{ij}
\end{aligned}$$

(b) If, without loss of generality, $1 \leq i \leq k$ and $j = k + 1$, then

$$\begin{aligned}
Q_{ii}A_{ij}Q^*_{jj} &= \left(\frac{2}{m_i}J_{m_i} - I_{m_i}\right)A_{ij} \\
&= \frac{2}{m_i}J_{m_i}A_{ij} - A_{ij}
\end{aligned}$$

If vertex $d \in D$ is adjacent to $|C_i|/2$ vertices in C_i, then the column corresponding to the d^{th} column of A_{ij}, call it $\mathcal{C}_d$ has $m_i/2$ ones. By $\mathbf{1}$ we denote the all-ones vector. So,

$$\frac{2}{m_i}J_{m_i}\mathcal{C}_d - \mathcal{C}_d = \mathbf{1} - \mathcal{C}_d,$$

which corresponds to switching operation on the d^{th} vertex.

If vertex d of D is adjacent to $|C_i|$ vertices in C_i, then the column corresponding to the d^{th} column of A_{ij}, call it $\mathcal{C}_d$, has m_i ones and $\mathcal{C}_d = \mathbf{1}$. So,

$$\frac{2}{m_i}J_{m_i}\mathcal{C}_d - \mathcal{C}_d = 2 \cdot \mathbf{1} - \mathbf{1} = \mathbf{1} = \mathcal{C}_d,$$

which means that no switching occurred on the d^{th} vertex.

Finally, if vertex $d \in D$ is adjacent to 0 vertices in C_i, then the column corresponding to d column of A_{ij}, call it $\mathcal{C}_d$ is $\mathbf{0}$. So,

$$\frac{2}{m_i} J_{m_i} \mathcal{C}_d - \mathcal{C}_d = \mathbf{0} - \mathbf{0} = \mathbf{0} = \mathcal{C}_d,$$

which also corresponds to no switching on the d^{th} vertex.

The other cases are analogous. □

We call the partition $\{C_1, ..., C_k, D\}$ of the graph X satisfying conditions of Theorem 2.3.1, *Godsil-McKay partition*. We will refer the matrix Q from the proof sometimes by *switching matrix*.

Chapter 3

Known non-isomorphic quantum isomorphic graphs

Since motivation for results in this thesis is to find graphs that are quantum ismorphic and non isomorphic, we start by providing an overview of the known examples of graphs that are quantum isomorphic but not isomorphic. The first pair of quantum isomorphic and nonisomorphic graphs together with an infinite family of such graphs appeared in [2]. In this section we describe a procedure of how to construct the quantum isomorphism matrix between the smallest pair of graphs from this infinite family and provide the matrix. While this procedure is known, we did not see the quantum isomorphism matrix appear explicitely anywhere before.

In September 2023 Schmidt [65] constructed a quantum isomorphism between two 120-vertex strongly regular non isomorphic graphs. In the same month, Chan and Martin [15] and Gromada [32] proved that all Hadamard graphs of the same order are quantum isomorphic.

3.1 Hadamard graphs

A matrix $H \in M_n(\{\pm 1\})$ is called *Hadamard* if $HH^T = nI$. We associate a *Hadamard graph* to a Hadamard matrix H by letting:

- The $4n$ vertices be $\{r_i^+\}_{i\in[n]} \cup \{r_i^-\}_{i\in[n]} \cup \{c_i^+\}_{i\in[n]} \cup \{c_i^-\}_{i\in[n]}$.
- For each i, j such that $H_{ij} = 1$, add two edges $\{r_i^+, c_j^+\}$ and $\{r_i^-, c_j^-\}$.

- For each i, j such that $H_{ij} = -1$, add two edges $\{r_i^+, c_j^-\}$ and $\{r_i^-, c_j^+\}$.

We start with the result of Chan and Martin [15] and Gromada [32], both published in September 2023. The first paper uses association schemes and the second one employs a special form of diagrammatic calculus, but both of them arrive at the conclusion that Hadamard graphs of the same order are quantum isomorphic.

Theorem 3.1.1 ([15],[32]) *All Hadamard graphs of the same order are quantum isomorphic.*

3.2 Mančinska-Roberson and games graphs

In [2] Atserias et al use so-called linear binary constraint systems to construct quantum isomorphic graphs. In this section we explain what linear binary constraint systems are and why they are relevant to quantum isomorphisms. We will give an example of a linear binary constraint system with six constraints and, following [2], we'll construct two non-isomorphic quantum isomorphic graphs on twenty-four vertices from it. From this example, it will be clear how one constructs quantum isomorphic graphs in general from a linear binary constraint system. Using the game strategy associated to this linear binary constraint system, we will construct an explicit quantum isomorphism matrix, which we have not seen anywhere written explicitly before.

To begin, the graphs are arising from a linear binary constraint system.A linear binary constraint system consists of linear constraints $C_1, ..., C_m$ and variables $x_1, ..., x_n$. Each constraint is an equation of the form $\sum_{x_i \in S_\ell} x_i = b_\ell \in \mathbb{F}_2$. One example of a linear binary constraint system, and the one we will use to construct the 24-vertex graphs is

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ x_1 + x_4 + x_7 &= 0 \\ x_4 + x_5 + x_6 &= 0 \\ x_2 + x_5 + x_8 &= 0 \\ x_7 + x_8 + x_9 &= 0 \\ x_3 + x_6 + x_9 &= 1 \end{aligned}$$

Here there are six constraints and nine variables.

Now we describe a game associated to a linear binary constraint system as it is done in [2]. The players are Alice and Bob, who are allowed to agree on a strategy beforehand and are not allowed to communicate during the game. They are trying to convince a referee that the linear binary constraint system is satisfiable. The game is as follows:

- Referee gives a constraint C_ℓ to Alice.
- Referee gives a constraint C_k to Bob.
- Alice responds with a satisfying assignment $f_\ell : S_\ell \mapsto \mathbb{F}_2$ to her constraint.
- Bob responds with a satisfying assignment $f_k : S_k \mapsto \mathbb{F}_2$ to his constraint.
- Alice and Bob have to agree on the variables common to their assignments, in other words, on variables in $S_\ell \cap S_k$.

To win such a game for Alice and Bob means to win it with probability one. The players are allowed to agree on a strategy beforehand, but can't communicate during the game. They just play one round and to win, they must respond correctly with probability one.

It turns out that there is a perfect classical strategy for a linear binary constraint system game if and only if the linear binary constraint system is satisfiable [16]. A strategy for the linear binary constraint system game, where Alice and Bob are allowed to share an entangled state between them and can only communicate before the game starts is called *quantum LBCS strategy*. In the light of this, we say a linear binary constraint system is *quantum satisfiable* if and only if there is a perfect quantum strategy for the linear binary constraint system game.

Interestingly, there are linear binary constraint systems that do not have a classical winning strategy, but have a quantum winning strategy. In particular, the system described above is of this type. We will first show how to construct the 24-vertex graphs from this linear binary constraint system. Afterwards, we will show why this linear binary constraint system is not classically satisfiable, we will deduce a winning quantum strategy for this linear binary constraint system and then use this strategy to come up with an explicit quantum isomorphism matrix between the two constructed graphs.

In [2] Atserias et al use these linear binary constraint systems to construct graphs in the following way. Let $f_\ell : S_\ell \mapsto \mathbb{F}_2$ be a satisfying assignment to the constraint C_ℓ. For example, in the above linear binary constraint system,

$$C_1 = (x_1 + x_2 + x_3 = 0), \quad S_1 = \{x_1, x_2, x_3\}$$

and f_1 could be

$$f_1(x_1) = 1, \quad f_1(x_2) = 1, \quad f_1(x_3) = 0.$$

Clearly, the choice for f_1 is not unique. A function

$$f_1'(x_1) = 1, \quad f_1'(x_2) = 0, \quad f_1'(x_3) = 1$$

would also be a satisfying assignment. In particular, for C_1 there are four possible satisfying assignments that assign value one to even number of variables from $S_1 = \{x_1, x_2, x_3\}$. We call these satisfying assignments $f_1^1.f_1^2, f_1^3, f_1^4$. Similarly, in this example, there are four satisfying assignments to other constraints as well.

Atserias et al in [2] construct a graph with vertices being tuples (C_ℓ, f_ℓ^i), such that the first entry ranges over all possible constraints and the second entry ranges over all possible satisfying assignments f_ℓ^i for an indicated constraint. Two vertices (C_i, f_i) and (C_j, f_j) are adjacent if they have variables in common and assignments differ on at least one variable.:

(a) $S_i \cap S_j \neq \emptyset$,

(b) There is $x_k \in S_i \cap S_j$, $f_i(x_k) \neq f_j(x_k)$.

Given a linear binary constraint system we can construct a *homogenized linear binary constraint system* by requiring that all constraints sum to zero, i.e. $b_\ell = 0$ for all ℓ.

Now we will show how to construct non isomorphic graphs on 24-vertices with a quantum isomorphism between them, following [2]. Let $X(F)$ be a graph construced as above for the familiar-to-us linear BCS:

$$\begin{aligned}
x_1 + x_2 + x_3 &= 0 \\
x_1 + x_4 + x_7 &= 0 \\
x_4 + x_5 + x_6 &= 0 \\
x_2 + x_5 + x_8 &= 0 \\
x_7 + x_8 + x_9 &= 0 \\
x_3 + x_6 + x_9 &= 1
\end{aligned}$$

and let $X(F_0)$ be a graph for the homogenized linear BCS.

$$\begin{aligned}
x_1 + x_2 + x_3 &= 0 \\
x_1 + x_4 + x_7 &= 0 \\
x_4 + x_5 + x_6 &= 0 \\
x_2 + x_5 + x_8 &= 0 \\
x_7 + x_8 + x_9 &= 0 \\
x_3 + x_6 + x_9 &= 0
\end{aligned}$$

In [2], the authors show that $X(F)$ and $X(F_0)$ are not isomorphic, but are quantum isomorphic. Here we provide an illustration of these graphs from their paper.

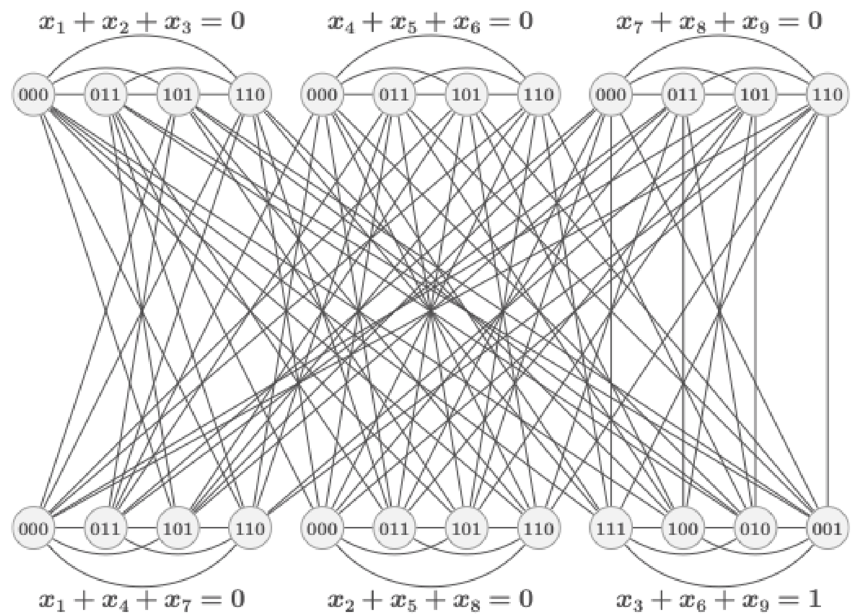

Figure 3.1: $X(F)$, [2]

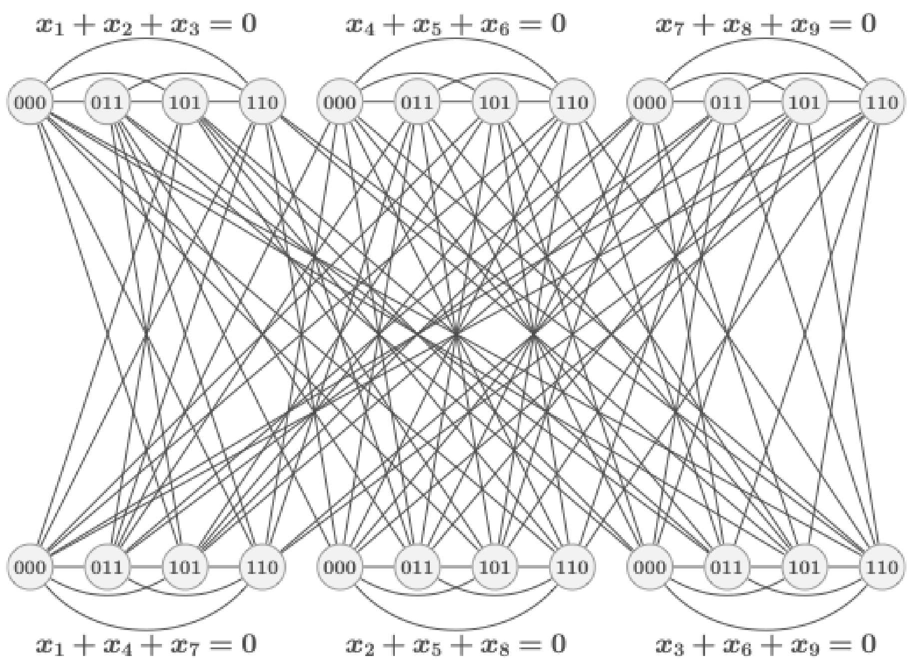

Figure 3.2: $X(F_0)$, [2]

We observe that both of these graphs are related by Godsil-McKay switching in Theorem 5.1.1.

3.2.1 Construction of an explicit isomorphism between 24-vertex graphs

In this section we construct an explicit quantum isomorphism matrix between the graphs $X(F)$ and $X(F_0)$. As the title suggests, we will construct the quantum isomorphism matrix between $X(F)$ and $X(F_0)$. The construction of the quantum permutation relies on the translation into the matrix notation of the proof from [2, Theorem 6.3] that originally used the language of games.

In this section we will use the bra-ket notation. Recall that for a vector $c \in \mathbb{C}^d$

$$|x\rangle = x$$
$$\langle x| = x^T$$

In addition,

$$|0\rangle = [1,0]^T$$
$$|1\rangle = [0,1]^T$$

Sometimes, when we have a tensor product of vectors in the tensor product of Hilbert spaces, $|x\rangle_A \otimes |y\rangle_B \in \mathcal{H}_A \otimes \mathcal{H}_B$, we omit the tensor product sign:

$$|x\rangle_A \otimes |y\rangle_B =: |x\rangle_A |y\rangle_B$$

In games, by assumption Alice is able to observe results of the measurements in $\mathcal{H}_A$.

We will first review the Mermin-Peres magic square game [49], [48]. Here we provide an adapted version of the rules of the Mermin-Peres square game. Alice and Bob can agree on the strategy beforehand, but are not allowed to communicate during the game as in [2]. Below is the schematic representation of the constraints of the previously mentioned linear binary constraint system.

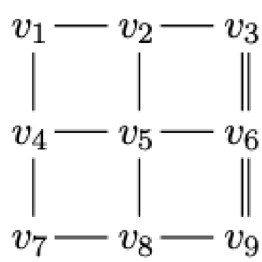

Figure 3.3: [16]. In the picture single lines represent constraints that add up to to 0, and the double-lines the constraint that adds up to 1. In total, there are nine constraints for each row and column. For example, the first row represents constraint $x_1 + x_2 + x_3 = 0$, and the rightmost column represents constraint $x_3 + x_6 + x_9 = 1$.

The rules of the game are:

(a) The referee gives Alice any constraint C_ℓ from the linear binary constraint system on page 19. She responds with a satisfying assignment of 0s and 1s to the variables in the constraint.

(b) The referee gives Bob any constraint C_k from the same system. He responds with a satisfying assignment of 0s and 1s to the variables in the constraint.

In order to win, Alice's and Bob's assignments have to follow the above rules and to agree on the assignment to variables that are common to Alice's and Bob's constraints in each round. It is known that such game can be won with probability one only when Alice and Bob share an entangled quantum state. Nonetheless, they can win using a quantum strategy, when they share an entangled state. Below, we explain how.

Now, Figure 3.4 encodes a quantum winning strategy for Alice an Bob. Below we elaborate on the strategy in detail.

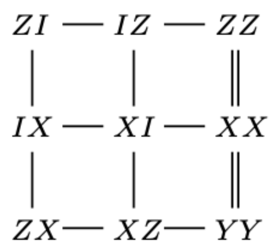

Figure 3.4: [16]. Projections in the winning strategy of Mermin-Peres magic square game.

In the beginning, Alice and Bob share an entangled state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0_A0_B\rangle + |1_A1_B\rangle) \otimes \frac{1}{\sqrt{2}}(|0_A0_B\rangle + |1_A1_B\rangle).$$

When Alice receives one of the six constraints that corresponds to Row 1, Row 2, Row 3, Column 1, Column 2 or Column 3 of the square in Figure 3.3, she measures three two-qubit observables in respectively Row 1, Row 2, Row 3, Column 1, Column 2 or Column 3 of square in Figure 3.4. For example, if Alice receives the constraint $x_1 + x_2 + x_3 = 0$ corresponding to Row 1, she performs measurement of observables

$$\{Z \otimes I_2, I_2 \otimes Z, Z \otimes Z.\}$$

Since the triples of observables in the rows or columns are commuting, she can measure them simultaneously, using their shared eigenprojections as measurement operators. Thus, Alice gets three outcomes as eigenvalues of the three observables for each constraint she was asked. The eigenvalues are 1 or -1. In our example, when Alice receives constraint $x_1+x_2+x_3 = 0$ and measures observables $I \otimes Z$, $Z \otimes I$ and $Z \otimes Z$. She uses measurement operators

$$E_{000} = |00\rangle\langle 00|, E_{101} = |01\rangle\langle 01|, E_{011} = |10\rangle\langle 10|, E_{110} = |11\rangle\langle 11|,$$

which are the shared eigenprojections between $I \otimes Z, Z \otimes I$ and $Z \otimes Z$. Note that indices of each projection are the measurement outcomes of this projection being applied to the operators $I \otimes Z, Z \otimes I$ and $Z \otimes Z$ one by one. Thus, $|10\rangle$ is an eigenvector of $I \otimes Z$ with eigenvalue 1, is an eigenvector of $Z \otimes I$ with eigenvalue -1 and also an eigenvector of $Z \otimes Z$ with eigenvalue -1. Hence, Alice interprets the outcome as an assignment 011 to a constraint $x_1 + x_2 + x_3 = 0$. By $\oplus$ here we indicate entry-wise addition modulo two.

In [2], Atserias et al adapt this game to the quantum isomorphism game between $X(F)$ and $X(F_0)$. Below we show how to win the quantum isomorphism game between the 24-vertex graphs $X(F)$ and $X(F_0)$ using the winning strategy for Mermin-Peres magic square game. Let $\text{hom}(C_i)$ denote the homogenized version of constraint C_i. So for our linear binary constraint system, for all constraints $\text{hom}(C_i) = C_i$, except for the constraint $x_3 + x_6 + x_9 = 1$, for which the homogenized version is $x_3 + x_6 + x_9 = 0$. This procedure generalizes to any other quantum isomorphism game winning strategy for $X(F)$ and $X(F_0)$ constructed in this way from LBCS and a homogenized LBCS.

(a) Suppose the referee gives Alice a vertex $(C_i, f_j^{(i)}) \in X(F)$. She uses the winning Mermin-Peres strategy for the constraint C_i and finds another satisfying assignment of 0s and 1s to the variables in the constraint, say $f_k^{(i)}$. She responds with the vertex $(\text{hom}(C_i), f_j^{(i)} \oplus f_k^{(i)}) \in V(X(F_0))$.

(b) Suppose the referee gives Bob a vertex $(C_k, f_r^{(k)}) \in X(F)$. He applies the winning Mermin-Peres strategy and obtains a satisfying assignment $f_s^{(k)}$ for C_k. He responds with $(\text{hom}(C_k), f_r^{(k)} \oplus f_s^{(k)}) \in V(X(F_0))$.

(c) Suppose the referee gives Alice any vertex $(\text{hom}(C_i), f_j^{(i)}) \in X(F_0)$. Alice uses the winning Mermin-Peres strategy for the constraint C_i and finds a satisfying assignment of 0s and 1s to the variables in the constraint, say $f_k^{(i)}$. She responds with the vertex $((C_i), f_j^{(i)} \oplus f_k^{(i)}) \in V(X(F))$.

(d) Suppose the referee gives Bob a vertex $(\text{hom}(C_k), f_r^{(k)}) \in X(F_0)$. Bob applies the winning Mermin-Peres strategy and obtains a satisfying assignment $f_s^{(k)}$ for C_k. He responds with $(\text{hom}(C_k), f_r^{(k)} \oplus f_s^{(k)}) \in V(X(F_0))$.

To ensure this game is indeed a quantum isomorphism game, one has to check that if Alice and Bob receive adjacent/different non-adjacent/same vertices of one graph, they respond with the adjacent/different non-adjacent/same vertices. The fact that this rule is satisfied is clear from the fact that they are using a winning Mermin-Peres strategy as a subroutine which gives them assignments that agree on common variables. For more details, please check [2, Theorem 6.2], where the authors provide the proof that this game is in fact a quantum isomorphism game.

Now we are ready to translate this quantum isomorphism game strategy into the quantum isomorphism matrix Q. From the fact that the game is a quantum isomorphism game, Bob performs the same measurement as Alice. So, it is sufficient to understand which measurements Alice performs for each vertex $(C_i, f_\ell^{(i)})$ given to her by the referee. For each entry $Q_{(C,f),(C',f')}$, we can think of (C, f) as a vertex of the graph $X(F)$ provided to Alice by the referee, and (C', f') being the vertex of $X(F_0)$, with which Alice responds upon being asked (C, f). From the structure of the quantum isomorphism game, we know that $(C', f') = (\hom(C), f \oplus g)$, where g is the C-satisfying assignment that Alice got after applying her Mermin-Peres game strategy. In particular, g is the outcome of the measurements of the three two-qubit operators that appeared in the row or column representing constraint C. Again, let us consider an example.

To begin, entries of the form $Q_{(C,f),(C',f')}$ where C and C' do not share variables are zero, because Alice responds with a vertex containing the original constraint and the satisfying assignment to this original constraint as per 3.2.1.

Suppose we would like to know which projection should be the entry

$$Q_{(x_3+x_6+x_9=1,001),(x_3+x_6+x_9=0,110)}.$$

We know that since constraint labelling the row (and so the column) of our quantum isomorphism matrix Q has variables from third column of our magic square, we will have one of the common eigenvectors of

$$Z \otimes Z, X \otimes X, Y \otimes Y$$

in the entry. We know that the outcome of that projection should be

$$110 - 001 = 110 \oplus 001 = 111.$$

We use bra-ket notation for compactness. Vectors $|0\rangle$ and $|1\rangle$ indicate standard basis vectors e_0 and e_1 respectively. Vectors $|+\rangle$ and $|-\rangle$ denote vectors $\frac{1}{\sqrt{2}}(e_0 + e_1)$ and $\frac{1}{\sqrt{2}}(e_0 - e_1)$ respectively.

Hence, the projection is the common eigenvector with eigenvalues -1 for each of the three operators which is

$$(|01\rangle - |10\rangle)(\langle 01| - \langle 10|).$$

In this manner we create a block-diagonal 24×24 matrix with 4×4 blocks $Q_1, ..., Q_6$.

Q_1, consisting of the common eigenprojections for $I \otimes Z, Z \otimes I, Z \otimes Z$:

$$\begin{array}{c} \\ {\scriptstyle (x_1+x_2+x_3=0,000)} \\ {\scriptstyle (x_1+x_2+x_3=0,011)} \\ {\scriptstyle (x_1+x_2+x_3=0,101)} \\ {\scriptstyle (x_1+x_2+x_3=0,110)} \end{array} \begin{array}{cccc} {\scriptstyle (x_1+x_2+x_3=0,000)} & {\scriptstyle (x_1+x_2+x_3=0,011)} & {\scriptstyle (x_1+x_2+x_3=0,101)} & {\scriptstyle (x_1+x_2+x_3=0,110)} \end{array}$$

$$\begin{pmatrix} |00\rangle\langle 00| & |10\rangle\langle 10| & |01\rangle\langle 01| & |11\rangle\langle 11| \\ |10\rangle\langle 10| & |00\rangle\langle 00| & |11\rangle\langle 11| & |01\rangle\langle 01| \\ |01\rangle\langle 01| & |11\rangle\langle 11| & |00\rangle\langle 00| & |10\rangle\langle 10| \\ |11\rangle\langle 11| & |01\rangle\langle 01| & |10\rangle\langle 10| & |00\rangle\langle 00| \end{pmatrix}$$

Q_2, consisting of the common eigenprojections for $X \otimes I, I \otimes X, X \otimes X$.

$$\begin{array}{c} \\ {\scriptstyle (x_4+x_5+x_6=0,000)} \\ {\scriptstyle (x_4+x_5+x_6=0,011)} \\ {\scriptstyle (x_4+x_5+x_6=0,101)} \\ {\scriptstyle (x_4+x_5+x_6=0,110)} \end{array} \begin{array}{cccc} {\scriptstyle (x_4+x_5+x_6=0,000)} & {\scriptstyle (x_4+x_5+x_6=0,011)} & {\scriptstyle (x_4+x_5+x_6=0,101)} & {\scriptstyle (x_4+x_5+x_6=0,110)} \end{array}$$

$$\begin{pmatrix} |++\rangle\langle ++| & |+-\rangle\langle +-| & |-+\rangle\langle -+| & |--\rangle\langle --| \\ |+-\rangle\langle +-| & |++\rangle\langle ++| & |--\rangle\langle --| & |-+\rangle\langle -+| \\ |-+\rangle\langle -+| & |--\rangle\langle --| & |++\rangle\langle ++| & |+-\rangle\langle +-| \\ |--\rangle\langle --| & |-+\rangle\langle -+| & |+-\rangle\langle +-| & |++\rangle\langle ++| \end{pmatrix}$$

Q_3, consisting of the common eigenprojections for $X \otimes Z, Z \otimes X, Y \otimes Y$. In the case the rank-one projection is too large, we will just write the corresponding vector:

$$\begin{array}{c} \\ {\scriptstyle (x_7+x_8+x_9=0,000)} \\ {\scriptstyle ((x_7+x_8+x_9=0,011)} \\ {\scriptstyle ((x_7+x_8+x_9=0,101)} \\ {\scriptstyle ((x_7+x_8+x_9=0,110)} \end{array} \begin{array}{cccc} {\scriptstyle (x_7+x_8+x_9=0,000)} & {\scriptstyle (x_7+x_8+x_9=0,011)} & {\scriptstyle (x_7+x_8+x_9=0,101)} & {\scriptstyle (x_7+x_8+x_9=0,110)} \end{array}$$

$$\begin{pmatrix} |0+\rangle + |1-\rangle & |0-\rangle + |1+\rangle & |0+\rangle - |1-\rangle & |0-\rangle - |1+\rangle \\ |0-\rangle + |1+\rangle & |0+\rangle + |1-\rangle & |0-\rangle - |1+\rangle & |0+\rangle - |1-\rangle \\ |0+\rangle - |1-\rangle & |0-\rangle - |1+\rangle & |0+\rangle + |1-\rangle & |0-\rangle + |1+\rangle \\ |0-\rangle - |1+\rangle & |0+\rangle - |1-\rangle & |0-\rangle + |1+\rangle & |0+\rangle + |1-\rangle \end{pmatrix}$$

Q_4, consisting of the common eigenprojections for $I \otimes Z, X \otimes I, X \otimes Z$.

$$\begin{array}{c} \\ {\scriptstyle (x_1+x_4+x_7=0,000)} \\ {\scriptstyle (x_1+x_4+x_7=0,011)} \\ {\scriptstyle (x_1+x_4+x_7=0,101)} \\ {\scriptstyle (x_1+x_4+x_7=0,110)} \end{array} \begin{array}{cccc} {\scriptstyle (x_1+x_4+x_7=0,000)} & {\scriptstyle (x_1+x_4+x_7=0,011)} & {\scriptstyle (x_1+x_4+x_7=0,101)} & {\scriptstyle (x_1+x_4+x_7=0,110)} \end{array}$$

$$\begin{pmatrix} |+0\rangle\langle +0| & |-0\rangle\langle -0| & |+1\rangle\langle +1| & |-1\rangle\langle -1| \\ |-0\rangle\langle -0| & |+0\rangle\langle +0| & |-1\rangle\langle -1| & |+1\rangle\langle +1| \\ |+1\rangle\langle +1| & |-1\rangle\langle -1| & |+0\rangle\langle +0| & |-0\rangle\langle -0| \\ |-1\rangle\langle -1| & |+1\rangle\langle +1| & |0-\rangle\langle 0-| & |+0\rangle\langle +0| \end{pmatrix}$$

Q_5, consisting of the common eigenprojections for $Z \otimes I, I \otimes X, Z \otimes X$.

$$\begin{array}{c|cccc} & {\scriptstyle (x_2+x_5+x_8=0,000)} & {\scriptstyle (x_2+x_5+x_8=0,011)} & {\scriptstyle (x_2+x_5+x_8=0,101)} & {\scriptstyle (x_2+x_5+x_8=0,110)} \\ \hline {\scriptstyle (x_2+x_5+x_8=0,000)} & |0+\rangle\langle 0+| & |0-\rangle\langle 0-| & |1+\rangle\langle 1+| & |1-\rangle\langle 1-| \\ {\scriptstyle (x_2+x_5+x_8=0,011)} & |0-\rangle\langle 0-| & |0+\rangle\langle 0+| & |1-\rangle\langle 1-| & |1+\rangle\langle 1+| \\ {\scriptstyle (x_2+x_5+x_8=0,101)} & |1+\rangle\langle 1+| & |1-\rangle\langle 1-| & |0+\rangle\langle 0+| & |0-\rangle\langle 0-| \\ {\scriptstyle (x_2+x_5+x_8=0,110)} & |1-\rangle\langle 1-| & |1+\rangle\langle 1+| & |0-\rangle\langle 0-| & |0+\rangle\langle 0+| \end{array}$$

Q_6, consisting of the common eigenprojections for $Z \otimes Z, X \otimes X, Y \otimes Y$. In the case the rank-one projection is too large, we will just write the corresponding vector:

$$\begin{array}{c|cccc} & {\scriptstyle (x_3+x_6+x_9=0,000)} & {\scriptstyle (x_3+x_6+x_9=0,011)} & {\scriptstyle (x_3+x_6+x_9=0,101)} & {\scriptstyle (x_3+x_6+x_9,110)} \\ \hline {\scriptstyle (x_3+x_6+x_9=1,111)} & |01\rangle - |10\rangle & |01\rangle + |10\rangle & |00\rangle - |11\rangle & |00\rangle + |11\rangle \\ {\scriptstyle (x_3+x_6+x_9=1,100)} & |01\rangle + |10\rangle & |01\rangle - |10\rangle & |00\rangle + |11\rangle & |00\rangle - |11\rangle \\ {\scriptstyle (x_3+x_6+x_9=1,010)} & |00\rangle - |11\rangle & |00\rangle + |11\rangle & |01\rangle - |10\rangle & |01\rangle + |10\rangle \\ {\scriptstyle (x_3+x_6+x_9=1,001)} & |00\rangle + |11\rangle & |00\rangle - |11\rangle & |01\rangle + |10\rangle & |01\rangle - |10\rangle. \end{array}$$

Each row and column contains four orthogonal rank-one projections, so each row and column sums to identity. Moreover, we need to check that whenever $(C, f) \sim_{(X(F))} (D, g)$ but $(C, f') \nsim_{(X(F))} (D, g')$, then we have

$$Q_{(C,f)(C,f')}Q_{(D,g)(D,g')} = 0$$

and vice versa.

From the adjacency relations, we conclude that constraints C and D share variables, on which f and g disagree and f' and g' agree. Moreover, the projections for the winning strategy for the row/column representing constraint C and row/column representing constraint D in the Mermin-Peres magic square intersect at the common observable, as in Figure 3.4. Thus, the projections in the row (C, f) and in the row (D, g) are eigenvectors for that shared observable.

$$Q_{(C,f)(C,f')} = Q_{(\text{hom}C,f\oplus f')} = v_1 v_1^T, Q_{(D,g)(D,g')} = Q_{(\text{hom}C,g\oplus g')} = v_2 v_2^T,$$

where $v_1 v_1^T$ is a projection with eigenvalues $\{-1^{(f\oplus f')_1}, -1^{(f\oplus f')_2}, -1_3^{(f\oplus f')}\}$ from the measurement that Alice would use as part of her Mermin-Peres winning strategy for constraint C. In other words, v_1 is the eigenvector for the three simultaneously diagonalizable observables in the row/column containing projections for the winning strategy representing constraint

C with eigenvalues for each observable being negative ones to the powers of each of the three entries of $f \oplus f'$ respectively. Similarly, v_2 is the eigenvector for the three simultaneously diagonalizable observables in the row/column representing the winning projective measurement for the homogenized constraint D with three eigenvalues being negative ones to the powers of each three entries of $g \oplus g'$.

However, since f, g disagree on that value of the shared variable and f', g' agree, means that $f \oplus f'$ disagree on the shared variable with $g \oplus g'$. Meaning, v_1 and v_2 are eigenvectors for the shared observable but with different eigenvalues. Hence, v_1 and v_2 are orthogonal. Finally, indeed,

$$Q_{(C,f)(C,f')}Q_{(D,g)(D,g')} = 0.$$

Note that whenever Alice uses a linear binary constraint system game strategy as a subroutine to her strategy for the quantum isomorphism game for $X(F)$ and $X(F_0)$, the quantum isomorphism matrix will be block-diagonal.

As another observation for this particular block-diagonal matrix, we note that each next block is a function of the first block. Each block is a 4x4 matrix of projections in $M_4(\mathbb{C})$. In particular, we can obtain each next block by conjugating the first block in the following way. The matrices below are defined in Section 7.4 and Section 7.5.

(a) The second block as $(H \otimes H)SWAP$ applied to each entry of the top left block.

(b) The third block as $CNOT(H \otimes I)SWAP$ applied to each entry of the top left block.

(c) The fourth block as $(H \otimes I)$ applied to each entry of the top left block.

(d) The fifth block as $(H \otimes I)SWAP$ applied to each entry of the top left block.

(e) The second block as $CNOT(H \otimes H)(X \otimes X)$ applied to each entry of the top left block.

3.3 Strongly regular quantum isomorphic graphs

In this section we explain the construction [65] of two non isomorphic strongly regular graphs G^{E_8} and G^w with parameters $(120, 63, 30, 36)$ that Schmidt used. He relied on this particular construction to find a quantum isomorphism between G^{E_8} and G^w.

At first, we include the definition of G^{E_8}, and then we will construct G^w from G^{E_8}, as is also done in [65]. One construction comes from root systems and is presented in Section 7. The vertices of G^{E_8} are lines in E_8, represented by 120 vectors in $\mathbb{R}^8$ given by

$$e_i \pm e_j, 1 \leq i < j \leq 8$$

and

$$x = (x_1, ..., x_8) \text{ for } x_i \in \{\pm 1\} \text{ and } \Pi_{i=1}^{8} x_i = 1.$$

Two vertices are adjacent in G^{E_8} if the corresponding vectors are orthogonal.

Using built-in functions in Sage, we find that the independence number and clique number are

$$\alpha(G^{E_8}) = \omega(G^{E_8}) = 8.$$

Moreover, Schmidt shows that the vertices of G^{E_8} can be partitioned into 15 cliques of size 8. Vectors of the form $x_S, S \subseteq \{1, ..., 8\}$ stand for $x_S = \sum_{j \in [8] \setminus S} e_j - \sum_{i \in S} e_i$.

$$\begin{aligned}
V_1: &\quad e_1 \pm e_2, e_3 \pm e_4, e_5 \pm e_6, e_7 \pm e_8, \\
V_2: &\quad e_1 \pm e_3, e_2 \pm e_4, e_5 \pm e_7, e_6 \pm e_8, \\
V_3: &\quad e_1 \pm e_4, e_2 \pm e_3, e_5 \pm e_8, e_6 \pm e_7, \\
V_4: &\quad e_1 \pm e_5, e_2 \pm e_6, e_3 \pm e_7, e_4 \pm e_8, \\
V_5: &\quad e_1 \pm e_6, e_2 \pm e_5, e_3 \pm e_8, e_4 \pm e_7, \\
V_6: &\quad e_1 \pm e_7, e_2 \pm e_8, e_3 \pm e_5, e_4 \pm e_6, \\
V_7: &\quad e_1 \pm e_8, e_2 \pm e_7, e_3 \pm e_6, e_4 \pm e_5, \\
V_8: &\quad x_{\{1,2\}}, x_{\{3,4\}}, x_{\{5,6\}}, x_{\{7,8\}}, x_{\{1,4,6,8\}}, x_{\{2,3,6,8\}}, x_{\{2,4,5,8\}}, x_{\{2,4,6,7\}}, \\
V_9: &\quad x_{\{1,3\}}, x_{\{2,4\}}, x_{\{5,7\}}, x_{\{6,8\}}, x_{\{1,4,7,8\}}, x_{\{1,4,5,6\}}, x_{\{1,2,6,7\}}, x_{\{1,2,5,8\}}, \\
V_{10}: &\quad x_{\{1,4\}}, x_{\{2,3\}}, x_{\{5,8\}}, x_{\{6,7\}}, x_{\{1,3,7,8\}}, x_{\{1,3,5,6\}}, x_{\{1,2,5,7\}}, x_{\{1,2,6,8\}}, \\
V_{11}: &\quad x_{\{1,5\}}, x_{\{2,6\}}, x_{\{3,7\}}, x_{\{4,8\}}, x_{\{1,6,7,8\}}, x_{\{2,5,7,8\}}, x_{\{4,5,6,7\}}, x_{\{1,2,4,7\}}, \\
V_{12}: &\quad x_{\{1,6\}}, x_{\{2,5\}}, x_{\{3,8\}}, x_{\{4,7\}}, x_{\{1,5,7,8\}}, x_{\{2,6,7,8\}}, x_{\{3,5,6,7\}}, x_{\{4,5,6,8\}}, \\
V_{13}: &\quad x_{\{1,7\}}, x_{\{2,8\}}, x_{\{3,5\}}, x_{\{4,6\}}, x_{\{1,5,6,8\}}, x_{\{3,6,7,8\}}, x_{\{2,5,6,7\}}, x_{\{4,5,7,8\}}, \\
V_{14}: &\quad x_{\{1,8\}}, x_{\{2,7\}}, x_{\{3,6\}}, x_{\{4,5\}}, x_{\{1,5,6,7\}}, x_{\{4,6,7,8\}}, x_{\{2,5,6,8\}}, x_{\{3,5,7,8\}}, \\
V_{15}: &\quad x_{\emptyset}, x_{\{5,6,7,8\}}, x_{\{3,4,7,8\}}, x_{\{2,4,6,8\}}, x_{\{3,4,5,6\}}, x_{\{2,4,5,7\}}, x_{\{2,3,6,7\}}, x_{\{2,3,5,8\}}.
\end{aligned}$$

Figure 3.5: Partitioning $V(G^{E_8})$ into 15 cliques, [65]

Since we cannot choose fifteen nonorthogonal vectors from these fifteen maximal cliques in $\mathbb{R}^8$, the set of vectors that form $V(G^{E_8})$ is a Kochen-Specker set.

Now, we follow Schmidt and build G^w from G^{E_8}, by $V(G^w) = V(G^{E_8})$. First, $V(G^w) = V(G^{E_8})$. Then we choose arbitrary vectors $w_i \in V_i$ for each $1 \leq i \leq 15$. Now for any $1 \leq i, j \leq 15$, we consider all possible pairs $\{s, t\}$ of vertices with $s \in V_i$ and $t \in V_j$.

(a) If the representatives w_i and w_j are orthogonal, so $\langle w_i, w_j \rangle = 0$, $\{s, t\} \in E(G^w)$ if and only if $\{s, t\} \in E(G^{E_8})$ (if and only if $\langle s, t \rangle = 0$).

(b) If the representatives w_i and w_j are not orthogonal, so $\langle w_i, w_j \rangle \neq 0$, $\{s, t\} \in E(G^w)$ if and only if $\{s, t\} \notin E(G^{E_8})$ (if and only if $\langle s, t \rangle \neq 0$).

Schmidt notes that $\alpha(G^w) = 15$, while $\alpha(G^{E_8}) = 8$, clearly indicating that these two graphs are not isomorphic.

We can choose an arbitrary set of fifteen representatives $w_i, 1 \leq i \leq 15$ because of the following theorem.

Theorem 3.3.1 *[65] Let $w = \{w_1, ..., w_{15}\}$ and $u = \{u_1, ..., u_{15}\}$ be any two sets of representatives from the fifteen orbits. The graphs G^w and G^u are isomorphic for any choice of w and u.* □

In the proof Schmidt finds a Pauli P_i that maps w_i to u_i for each $1 \leq i \leq 15$ and applies it to all vertices in orbit V_i. Turns out, this is the isomorphism between G^w and G^u.

In the same paper [65], Schmidt presents the following result that, given two graphs on the same vertex set that respect the same Godsil-McKay partition determined by the blocks of the quantum ismorphism between them, both switched graphs will also be quantum isomorphic.

Theorem 3.3.2 *Let G_1 and G_2 be quantum isomorphic graphs, where there exists a quantum permutation matrix u with $uA_{G_1} = A_{G_2}u$ of the form*

$$\begin{array}{c} \\ V_1 \\ V_2 \\ \vdots \\ \vdots \\ V_m \end{array} \begin{array}{c} \begin{array}{ccccc} V_1 & V_2 & \dots & \dots & V_m \end{array} \\ \begin{pmatrix} u^{(1)} & 0 & 0 & \dots & 0 \\ 0 & u^{(2)} & 0 & \dots & 0 \\ 0 & 0 & u^{(3)} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & u^{(m)} \end{pmatrix} \end{array} \tag{3.3.1}$$

for some partition $\{V_1, \ldots, V_m\}$ of the vertex sets (we can label both vertex sets by V as quantum isomorphic graphs have the same number of vertices). Let $\{S_1, \ldots, S_{k+1}\}$ be a partition of $[m]$ and define a partition $\pi = \{C_1, \ldots, C_k, D\}$ of the vertex set by setting $C_i := \cup_{s \in S_i} V_j$ and $D := \cup_{s \in S_{k+1}} V_j$. If π is a Godsil-McKay partition for G_1 and G_2, then we can use Godsil-McKay switching and the graphs $G_1^{\pi,D}$ and $G_2^{\pi,D}$ are quantum isomorphic.

3.4 Cayley graphs and cospectrality

One feature of the 24-vertex and 120-vertex graphs is that in the first case both graphs are Cayley for S_4, and in the second case, one is Cayley for S_5. In particular, graphs that have different classical and quantum graph parameters, which coincidentally happen to be orthogonality graphs of the lines in root systems, are Cayley. The fact that the 24-vertex graphs are Cayley has been explored in my Master's thesis [67]. The link to root systems is more closely explored in [**?**]. The connection sets for each 24-vertex graph are made of a union of conjugacy classes. We call a Cayley graph *normal* if its connection set is a union of conjugacy classes. Using Sage, I was able to check that one of the 120-vertex graphs, which is the orthogonality graph of lines in E_8, is Cayley and find an unwieldy connection set for it. For example, one connection set is

[(1,2), (2,4)(3,5), (1,5), (3,4), (1,2)(3,4), (1,5)(3,4), (1,4)(2,5), (1,3), (1,5)(2,3), (2,3), (4,5), (1,2)(4,5), (1,4), (1,3)(2,5), (1,3)(2,4), (2,5), (1,4)(3,5), (3,5), (1,2)(3,5), (2,5)(3,4), (1,5)(2,4), (1,4)(2,3), (1,3)(4,5), (2,4), (2,3)(4,5),

(1,5,2,3,4), (1,2,3,5,4), (1,4,5,3,2), (1,5,3,4,2), (1,3,5,2,4), (1,4,3,2,5), (1,2,4,3,5), (1,2,5,4,3), (1,5,4,2,3), (1,3,2,4,5), (1,4,2,5,3), (1,3,4,5,2), (1,5,3,4), (1,4,3,5), (1,4,5,3), (1,3,5,4), (1,5,4,3), (1,3,4,5), (1,4,2,3), (1,3,2,4), (1,2,4,3), (1,3,4,2), (1,4,3,2), (1,2,3,4), (1,5,2,4), (1,4,2,5), (1,2,5,4), (1,4,5,2), (1,5,4,2), (1,2,4,5),

(1,4,3)(2,5), (1,3,4)(2,5), (1,4,5)(2,3), (1,5,4)(2,3), (1,4,2)(3,5), (1,2,4)(3,5), (1,4)(2,5,3), (1,4)(2,3,5)].

Since the existence of a quantum isomorphism $Q \in M_n(M_d(\mathbb{C}))$ between graphs X and Y implies the adjacency matrices A_X and A_Y are similar, it is a necessary condition for two graphs to be cospectral in order to be quantum isomorphic [2]. Having two quantum isomorphic normal Cayley graphs, we asked a question in general when two normal Cayley graphs are cospectral.

Theorem 3.4.1 *Suppose A, B are sets of equal size. Let X, Y be $|A|/2-$regular graphs with bipartitions A, B that do not have edges in common. In other words, $X \cup Y = K_{|A|,|A|}$. Then X and Y are cospectral.*

Proof. Define the Godsil-McKay partition on X as C being one part of the bipartition and the set on which switching occurs, D, to be the other part of the bipartition. Say, $C := A$ and $D := B$. First, the subgraph induced by C is a coclique, so it is regular. Second, every vertex in B is adjacent to half of the vertices in A by the degree requirement. Perform Godsil-McKay switching. The graph obtained is the bipartite complement of X, which is Y. □

Corollary 3.4.2 *Let X=Cay(S_n, M) and Y=Cay(S_n, N). Suppose M and N are disjoint sets of equal size, each consisting of odd permutations. Moreover, $M \cup N$ is the set of all odd permutations in S_n. Then X and Y are cospectral.*

Proof. We will prove that Y can be obtained from X by Godsil-McKay switching. Partition the vertices of X into two sets: the alternating group A_n of even permutations, and a coset gA_n consisting only of odd permutations. Now we know that multiplying an odd permutation and an odd permutation yields an even permutation, and multiplying an even permutation and an odd one makes an odd permutation. So $\{A_n, gA_n\}$ are the bipartition of X.

Moreover, X is $\frac{1}{2}|A_n|$-regular, since $|M| = \frac{1}{2}|A_n|$ and $|A_n| = |gA_n|$.

Finally, X and Y do not have edges in common. Indeed, it is impossible that

$$pm = pn$$

for two distinct odd cycles $m \in M, n \in N$ and a permutation $p \in S_n$.

Now if we perform switching on X, we obtain a graph Y=Cay(S_n, N). By Theorem 3.4.1, graphs X and Y are cospectral. □

The question arises for which symmetric groups there is a partition into two sets of equal size of conjugacy classes consisting of odd permutations. I have checked all symmetric groups from S_3 to S_{10} and only S_6 and S_3 do not admit such a partition. In addition, it would be interesting to explore which of these graphs are quantum isomorphic. We leave this open question to the reader.

Chapter 4

Quantum graph parameters of quantum isomorphic nonisomorphic graphs

Questions about quantum independence and chromatic numbers have been studied recently in [14], [45], [59]. In [60] and [67] the authors investigated the connection between differences of quantum and classical graph parameters and Kochen-Specker sets. A Kochen-Specker set is a set of k orthogonal bases in $\mathbb{R}^d$ such that one can not select k pairwise nonorthogonal vectors. It is not difficult to see that quantum isomorphism preserves quantum independence and chromatic numbers, but not necessarily their classical counterparts. In this section we, broadly speaking, discover different new ways in which Kochen-Specker sets play a role in quantum isomorphisms.

4.1 Chromatic and quantum independence numbers and relation to Kochen-Specker sets

Before working with them, we define quantum coclique and chromatic numbers. These definitions can be found in any of the above-cited references.

Definition 4.1.1 *[59] For a graph X, quantum coclique number $\alpha_q(X)$ is the maximum integer t for which there exist a $|V(X)| \times t$ block matrix P with each $P^{(u,i)} \in M_d(\mathbb{C})$ being a projection such that*

- $\sum_{v\in V(X)} P^{(v,i)} = I_d$ *for all* $i = 1, ..., t$
- $P^{(u,i)}P^{(v,j)} = 0$ *for any* $u \sim v$ *for all* $i, j = 1, ..., t$
- $P^{(u,i)}P^{(u,j)} = 0$ *for any* $i \neq j$ *for all* $i, j = 1, ..., t$

Note that restricting the definition to 1×1 projections, we obtain a classical coclique. Consider ordering vertices in the α-coclique of a graph X as $1, ..., \alpha$, and for each $u \in V(X)$ making the $P^{(u,i)}$ entry one if u is the i^{th} vertex of the coclique and zero otherwise. Again, P is now just a characteristic matrix of a coclique. The second condition guarantees that classical coclique does not contain adjacent vertices. The first and last are trivially satisfied by each coclique too. It follows that every classical coclique is a quantum coclique and so $\alpha(X) \leq \alpha_q(X)$.

Now, let us define the quantum generalization of the chromatic number, the quantum chromatic number.

Definition 4.1.2 *[59] For a graph X, the quantum chromatic number $\chi_q(X)$ is the minimum integer s for which there exist a $|V(X)| \times s$ block matrix P such that each $P^{(u,i)} \in M_d(\mathbb{C})$ is a projection, and its entries satisfy the following conditions:*

- $\sum_{i\in 1,...,s} P^{(u,i)} = I_d$ *for all* $u \in V(X)$
- $P^{(u,i)}P^{(v,i)} = 0$ *for all* $u \sim_X v$ *and* $i = 1, ..., s$

To get the intuition for this definition, let the entries of the matrix P be again one-dimensional projections, in other words integers zero or one. Given a classical colouring, let the entry $P^{(u,i)}$ be one if the vertex u is coloured colour i. In this case, P becomes a usual characteristic matrix of a classical colouring. Immediately, it follows that

$$\chi_q(X) \leq \chi(X)$$

for any graph X.

We include the relationship between Kochen-Specker sets and quantum graph parameters to motivate the question about the significance of Kochen-Specker sets in quantum isomorphisms.

First, we define a set $S = S_1 \sqcup ... \sqcup S_k$ consisting of sets S_i of projections in $M_d(\mathbb{C})$ that each sum to I_d to be a *projective Kochen-Specker set* if we cannot select k pairwise nonorthogonal projections from S. It was first defined in Scarpa's thesis [60] as a generalisation of Kochen-Specker sets.

Theorem 4.1.3 *[60] For all graphs G we have that $c = \chi_q(G) < \chi(G)$ if and only if the entries $\{P^{(v,a)}\}_{v \in V(G), a \in [c]}$ of the $|V(G)| \times c$ quantum colouring matrix P form a projective Kochen-Specker set.*

Theorem 4.1.4 *[67] For all graphs G, we have that $k = \alpha_q(G) > \alpha(G)$ if and only if the entries $\{P^{(v,a)}\}_{v \in V(G), a \in [k]}$ of the $|V(G)| \times k$ quantum coclique matrix P form a projective Kochen-Specker set.*

Finally, we include a bound on α_q that we use in the next section to calculate the quantum independence numbers of the quantum isomorphic and non isomorphic graphs. It will be in terms of the Lovász theta function, which we are going to define.

We have that J denotes the all-ones matrix. The Lovász theta function $\vartheta(X)$ of a graph X is the optimum value of the following semidefinite program:

$$\begin{aligned} \vartheta(G) := \max \operatorname{tr}(BJ) & \\ \operatorname{tr} B &= 1 \\ B_{uv} &= 0 \text{ for every edge } uv \in E(G) \\ B &\succeq 0. \end{aligned}$$

Theorem 4.1.5 ([8],[18],[54]) *For any graph G, we have that $\alpha_q(G) \leq \vartheta(G)$.*

4.2 Entries of quantum isomorphisms of non isomorphic graphs and Kochen-Specker sets

We have learned that the difference between quantum and classical independence/chromatic numbers occurs if and only if the entries of the coclique/colouring matrix form a Kochen-Specker set. Could we say something along these lines about entries of matrices that are quantum isomorphisms between non isomorphic graphs? In this section we show an almost if and only if condition.

For a graph X, we define $\operatorname{rel}_X(x, y)$ to stand for relation between vertices x and y in X, which is one of "same", "adjacent" "non-adjacent and distinct".

Theorem 4.2.1 *Let $X \neq X'$ be non isomorphic graphs. Suppose Q is a quantum isomorphism between them with rank-one or rank-zero entries. Then entries of Q form a Kochen-Specker set.*

Proof. We will prove the contrapositive statement. Note that within each row, projections are pairwise orthogonal. Thus, if the set of projections do not form a Kochen-Specker set, then from each row $1 \leq i \leq n$, we are able to select a representative $Q_{ii'}$ such that all n representatives are pairwise nonorthogonal. Clearly, $i' \neq j'$ for all $1 \leq i \neq j \leq n$ since in columns projections are orthogonal. Since Q was a quantum isomorphism, for all $x, y \in V(X), x', y' \in V(X')$, it holds that

$$Q_{xy}Q_{x'y'} = 0 \text{ if } \mathrm{rel}_X(x, y) \neq \mathrm{rel}_{X'}(x', y').$$

Note that if we construct a new matrix of projections Q', the only nonzero entries of which are the selected entries $\{Q_{ii'}\}_{1 \leq i \leq n}$, the above orthogonality relation will still hold, albeit Q' is not a quantum permutation any more. Now we can construct classical isomorphism P where $P_{ii'} = 1$ for all (i, i') indexing these nonzero entries $\{Q_{ii'}\}_{1 \leq i \leq n}$. In case that we can select $\{Q_{ii'}\}_{1 \leq i \leq n}$, such that not all of them are the diagonal entries, P will be an isomorphism between the graphs X, X' since the orthogonality relations hold. □

By definition, Kochen-Specker set is a set of bases. On the other hand, Projective Kochen-Specker set is a set of sets $\{S_k\}_{k=1}^n$ of projections such that within each S_k, projections sum to identity and we cannot select n pairwise nonorthongonal projections (original definition has appeared in [60]). We could have omitted the rank-one or rank-zero requirement in the previous theorem and the same prove would have worked for the following result.

Theorem 4.2.2 *Let $X \neq X'$ be non isomorphic graphs. Suppose Q is a quantum isomorphism between them. Then entries of Q form a projective Kochen-Specker set.*

For the reverse direction, if we have a quantum permutation whose entries form a projective Kochen-Specker set, then we would like to find X and X' (possibly equal) such that Q is a quantum isomorphism between them, and there is no classical isomorphism between them. In general it will not be true, because even the quantum isomorphism matrix that we constructed in Section 3.2.1 is a quantum isomorphism between certain isomorphic graphs, which we verified by running algorithm 6.6.1. So, we need an extra condition.

Theorem 4.2.3 *Let X and X' be graphs. Suppose Q is a quantum isomorphism between X and X', such that entries of Q form a projective Kochen-Specker set. Assume further that for any pair of entries Q_{ik}, Q_{jl} we have*

$$Q_{ik}Q_{jl} = 0 \textit{ if and only if } rel_X(i, j) \neq rel_{X'}(k, l) \qquad (*).$$

Then X and X' are non isomorphic.

Proof. For, suppose there is a classical isomorphism P between X and X', then there is a set $\{P_{ii'}\}$ of mutually nonorthogonal entries. Therefore, for any two nonorthogonal $P_{ii'}$ and $P_{jj'}$, we have:

$$\mathrm{rel}_X(i,j) = \mathrm{rel}_{X'}(i',j').$$

By $(*)$, for any $1 \leq i,j \leq n$, in the quantum isomorphism it holds that $Q_{ii'}Q_{jj'} \neq 0$, contradicting the fact that we can not choose n mutually nonorthogonal projections as entries of Q form a Kochen-Specker set. □

We say a graph is *asymmetric* if it has a trivial automorphism group. Recall that a graph has quantum symmetry if it has a quantum automorphism with non-commutative entries. From [60], for a set S of bases $S \subseteq \mathbb{R}^n$, we call a function $f : S \to \{0,1\}$ a marking function if for each orthonormal basis $b \in S$, we have $\sum_{v \in b} f(v) = 1$.

Theorem 4.2.4 *X is an asymmetric graph with quantum symmetry if and only if the only marking function for the entries of the magic unitary u representing the quantum automorphism group of X is the identity function.*

Proof. Suppose X is an asymmetric graph that has quantum symmetry. Suppose we were able to select n nonorthogonal $u'_{ij}s$, one from each row and column. Then we can create a permutation matrix P with $P_{ij} = 1$ if u_{ij} was selected. P would be a valid classical automorphism.

For the other direction. Suppose entries of u form a Kochen-Specker set. Then entries of every classical automorphism of X obey the same orthogonality relations as those of u. Hence, if there is a non-identity automorphism, there is a non-identity marking function. □

4.3 Quantum isomorphism preserves quantum parameters

Another interesting observation is that quantum isomorphism preserves quantum independence number and quantum chromatic number, but does not necessarily preserve classical versions of these parameters. We will first prove the former and then provide examples of the latter below.

Theorem 4.3.1 *Suppose graphs X and Y have adjacency matrices A and B respectively and $Q \in M_n(M_d(\mathbb{C}))$ is a quantum isomorphism between them such that $Q(A \otimes I_d)Q^* = B \otimes I_d$. Then $\alpha_q(X) = \alpha_q(Y)$.*

Proof. Suppose P is a quantum coclique matrix for X. We will show that $Q * P$ is a quantum coclique matrix for Y. This shows that $\alpha_q(X) \leq \alpha_q(Y)$. Exchanging roles of X and Y and using Q^*, a similar argument would demonstrate that $\alpha_q(X) \geq \alpha_q(Y)$. In the end, we get that $\alpha_q(X) = \alpha_q(Y)$. First,

$$\sum_u (Q * P)_{ui} = \sum_{u,k} Q_{uk} \otimes P_{ki} = \sum_k P_{ki} = I.$$

Second, for $u \sim_Y v$, and $i \neq j$:

$$(Q * P)_{ui}(Q * P)_{vj} = \sum_{k,\ell} Q_{uk}Q_{v\ell} \otimes P_{ki}P_{\ell j} = \sum_{k \sim_X \ell} Q_{uk}Q_{v\ell} \otimes P_{ki}P_{\ell j} = 0,$$

where the last equality follows from P being a quantum coclique matrix for X, and the second to last uses the orthogonality relation on the entries of Q. □

Now we demonstrate an analogous proof for the quantum chromatic numbers.

Theorem 4.3.2 *Suppose graphs X and Y have adjacency matrices A and B and $Q \in M_n(M_d(\mathbb{C}))$ is a quantum isomorphism between them such that $Q(A \otimes I_d)Q^* = B \otimes I_d$. Then $\chi_q(X) = \chi_q(Y)$.*

Proof. Suppose P is a quantum colouring matrix for X. We will show that $Q * P$ is a quantum colouring for Y. This shows that $\chi_q(X) \geq \chi_q(Y)$. A similar proof can be carried our for constructing a quantum colouring of X from a quantum colouring of Y. Therefore, $\chi_q(X) = \chi_q(Y)$. First,

$$\sum_i (Q * P)_{ui} = \sum_{i,k} Q_{uk} \otimes P_{ki} = \sum_k Q_{uk} = I.$$

Second, for $u \sim_X v$,

$$(Q * P)_{ui}(Q * P)_{vj} = \sum_{k,\ell} Q_{uk}Q_{v\ell} \otimes P_{ki}P_{\ell j} = \sum_{k \nsim_Y \ell} Q_{uk}Q_{v\ell} \otimes P_{ki}P_{\ell j} = 0,$$

where the last equality follows from Q being a quantum isomorphism between X and Y, the second to last uses the orthogonality relation on the entries of P. □

For example, $X(F)$ (in the pair of Mančinska-Roberson examples[46] of non isomorphic and quantum isomorphic graphs) has different classical and quantum parameters:

$$\alpha_q(X(F)) = 6 > \alpha(X(F)) = 5.$$

To get the exact quantum independence number of $X(F)$, we verified computationally that $\vartheta(X(F)) = 6$, so $\alpha_q(X(F)) = 6$ by Theorem 4.1.5.

On the other hand, such a separation does not occur in the graph $X(F_0)$ for the homogenized LBCS. It is easy to see that

$$\alpha(X(F_0)) = 6.$$

By Theorem 4.3.1, quantum isomorphc graphs have identical quantum independence numbers, so:

$$\alpha_q(X(F_0)) = \alpha_q(X(F)) = 6.$$

Interestingly, $\vartheta(X(F_0)) = 6$ as well.

We note that the situation when the graph of the homogenized LBCS has the same quantum and classical independence numbers, but the graph of the nonhomogenized LBCS does not, is typical for non-isomorphic quantum isomorphic graphs arising from LBCS games. We will state and prove this as Theorem 4.5.3.

Chris Godsil has suggested to investigate classical and quantum chromatic numbers of these graphs, and there is an interesting observation as well. For the chromatic numbers, using Sage we find that

$$\chi(X(F)) = 5, \chi(X(F_0)) = 4$$

For the second graph, using Hoffman bound [35]

$$4 = 1 + \frac{\lambda_{max}(X(F_0))}{\lambda_{min}(X(F_0))} \leq \chi_q(X(F_0)) \leq \chi(X(F_0)) = 4,$$

implying that

$$\chi_q(X(F_0)) = \chi_q(X(F)) = 4.$$

From Theorem 4.3.2:

$$5 = \chi(X(F)) > \chi_q(X(F)) = \chi_q(X(F_0)) = \chi(X(F_0)) = 4.$$

To summarize,

$$5 = \chi(X(F)) > \chi_q(X(F)) = 4,$$

and

$$6 = \alpha_q(X(F)) > \alpha(X(F)) = 5.$$

4.4 Quantum graph parameters of the 120-vertex quantum isomorphic graphs

Now, let us examine $\alpha, \alpha_q, \chi, \chi_q$ of the two strongly regular graphs G^{E_8} and G^w that Schmidt proved to be quantum isomorphic and non isomorphic [65].

From his paper, we know that

$$\alpha(G^{E_8}) = 8,\ \alpha(G^w) = 15.$$

At the same time, G^{E_8} is an orthogonality graph of 15 bases in $\mathbb{R}^8$. So we may construct block-diagonal quantum coclique matrix of G^{E_8}. The first block will consist of the column of eight projections onto the lines in the first clique and zeroes everywhere else, the second block will contain a column of eight projections onto the lines in the second clique and zeroes everywhere else etc. So,

$$\alpha_q(G^{E_8}) \geq 15 > 8 = \alpha(G^{E_8}).$$

Regarding chromatic numbers, with Hoffman's lower bound for quantum chromatic number [20] and with Sage we have that

$$8 = 1 + \frac{\lambda_{max}(G^w)}{\lambda_{min}(G^w)} \leq \chi_q(G^w) \leq \chi(G^w) = 8.$$

Therefore,

$$\chi_q(G^{E_8}) = \chi_q(G^w) = 8$$

However, from G^{E_8} using the independence number bound,

$$\chi(G^{E_8}) \geq \frac{|V(G^{E_8})|}{\alpha(G^{E_8})} = 15.$$

To summarize,

$$\chi_q(G^{E_8}) = 8 < 15 \leq \chi(G^{E_8})$$

and

$$\alpha_q(G^{E_8}) \geq 15 > 8 = \alpha(G^{E_8}).$$

□

This observation leads to a thought of how to use graphs with $\chi_q < \chi$ or $\alpha_q > \alpha$ to construct quantum isomorphic pairs of graphs. Of course, if in those pairs one of the graphs has equal corresponding quantum and classical parameters while for the other graph these parameters differ, then they will be not isomorphic. We also explore properties of the quantum isomorphism matrix when one of the graphs has different quantum and classical parameters.

4.5 Block-diagonal quantum isomorphism matrices with Kochen-Specker entries and α_q, χ_q

After studying the known examples from the previous sections, questions arise how much of a coincidence is that that two quantum isomorphic, non isomorphic graphs always come in such pairs that one of the graphs has the separation between the classical and quantum parameters and the other does not, whenever the quantum isomorphism between two graphs is block-diagonal. What are the necessary conditions for such separations to occur. In this section we investigate this question in light of block-diagonal quantum isomorphisms and quantum independence and chromatic numbers.

It is important to note that it is not true that when two graphs X and Y are quantum isomorphic and non isomorphic, then, at least for one of the graphs, $\chi(X)$ and $\chi_q(X)$ are different.

Theorem 4.5.1 *Let H_1 and H_2 be two non-isomorphic quantum isomorphic Hadamard graphs. Then $\chi(H_1) = \chi_q(H_1) = \chi_q(H_2) = \chi(H_2) = 2$.*

Proof. Since the graphs H_1, H_2 are Hadamard, they are bipartite, and so

$$\chi(H_1) = \chi(H_2) = 2.$$

We have $1 \leq \chi_q(H_1) \leq \chi(H_1) = 2$. At the same time, $\chi_q(H_1) \neq 1$, since H_1 has at least one edge, and so at least one pair of entries in a column should be orthogonal. Hence,

$$\chi(H_1) = \chi_q(H_1) = \chi_q(H_2) = \chi(H_2).$$

□

From now on, we concentrate on the cases when the quantum isomorphism matrix is block-diagonal.

Theorem 4.5.2 *Suppose X and Y are graphs. Let Q be a block-diagonal quantum isomorphism. Suppose, each diagonal block is $d \times d$ and contains d rank-one $d \times d$ projections as entries. Moreover, suppose the entries of Q form a projective Kochen-Specker set. Then if the independence number of Y is the same as the number of blocks in Q, i.e. $\alpha(Y) = \frac{|V(Y)|}{d}$, then $\alpha(X) < \alpha_q(X)$.*

Proof. First, observe that since blocks are $d \times d$ and contain only d distinct projections, all columns of each block are permutations of the first column of that block.

Next, let $k = \frac{|V(Y)|}{d}$ be the number of blocks and let $\{y_1, ..., y_k\}$ be the coclique in Y. Assume that the columns of Q are labelled by $V(Y)$ and select columns $\{y_1, ..., y_k\}$ of Q labelled by the coclique in Y and denote the selected submatrix by $Q' := Q[y_1, ..., y_k]$. We claim that the entries in Q' form a projective Kochen-Specker set. Now, entries of Q form a Kochen-Specker set, but the entries in each block pairwise commute. So, if it were that entries in Q' did not form a projective Kochen-Specker set, and we were able to select k pairwise nonorthogonal projections, we would be able to select kd pairwise nonorthogonal projections from Q, which is a contradiction. So, the entries in Q' form a projective Kochen-Specker set. At the same time, the entries of Q' satisfy the requirement for being a quantum coclique for X. Combining these two facts, we conclude that $\alpha_q(X) > \alpha(X)$. □

Recall the construction of X and X_0, the non-isomorphic quantum isomorphic graphs constructed from an LBCS and its homogenized version respectively as demonstrated on the example of the 24-vertex graphs in Section 3.2.

Corollary 4.5.3 *Let $\mathcal{F}$ be a linear binary constraint system where each constraint has the same number of variables. Suppose X and X_0 are non-isomorphic quantum isomorphic graphs constructed from an LBCS and its homogenized version respectively. Then $\alpha_q(X) > \alpha(X)$ and $\alpha_q(X_0) = \alpha(X_0)$.*

Proof. Let d denote the number of satisfying assignments to constraints in the LBCS. X_0 is a union of cliques of size d. Consequently, the quantum isomorphism matrix Q between X and X_0 that is constructed from a strategy to an LBCS game, is block-diagonal with $d \times d$ blocks. By Theorem 4.2.1, entries of Q form a projective Kochen-Specker set.

Next, for a constraint C_i let f_i^0 denote a zero assignment. In other words, for each variable $x_k \in S_i$ $f_i^0 : x_k \mapsto 0$. Vertices of the form $\{(C_i, f_i^0)\}$ form a maximal coclique in X_0 of the size equal to the number of blocks of Q (or by [2, Theorem 6.3]), $\alpha_q(X_0) = \alpha(X_0)$ and by Theorem 4.5.2, $\alpha_q(X) > \alpha(X)$. □

Theorem 4.5.4 *Suppose X and Y are non-isomorphic quantum isomorphic graphs with adjacency matrices X and Y. Suppose Q is a block-diagonal quantum isomorphism between them, such that $QBQ^* = A$, and each block is $d \times d$ and contains d distinct projections. Moreover, suppose that the entries of Q form a projective Kochen-Specker set. If Y is a union of d-cliques that can be partitioned into $\frac{|V(Y)|}{d}$ cocliques, then $\chi(X) > \chi_q(X) = \chi_q(Y)$, also $\chi_q(Y) = \chi_q(X) \leq d$, and $\chi(Y) = d \leq \chi(X)$.*

Proof. First, note that since Y is a union of d-cliques that can be partitioned into $\frac{|V(Y)|}{d}$ cocliques, then $\chi(Y) = d$.

In addition, since Y is a union of cliques, and Q is block-diagonal, X is a union of cliques as well. Indeed, for an arbitrary i, the i^{th} diagonal block of A is:

$$(Q(B \otimes I_d)Q^*)_{ii} = \sum_{k,r} Q_{ik}A_{kr}Q_{ir} = Q_{ii}((J-I) \otimes I_d)Q_{ii} = (J-I) \otimes I_d.$$

So, $\chi(X) \geq d$.

Next, we will first form a $|V(X)| \times d$ matrix C by stacking d blocks of Q onto each other. We will show that C is a quantum colouring for X, implying that $\chi_q(X) \leq d$. At the same time, since entries of Q form a projective Kochen-Specker set, entries of the flattened version of Q also form a projective Kochen-Specker set, so $\chi_q(X) < \chi(X)$.

Now, let us verify that C is a quantum coloring of X. Suppose $x \sim_X x'$, and $i \in [d]$ is a column index of C. Note that we can label the vertices of Y that label the columns of Q in such manner that vertices that are equivalent modulo d are in one coclique together. Together these cocliques form a partition of $V(Y)$. We can view column labels of C as remainders from division by d.

Now, given entries C_{xi}, $C_{x'i}$ of C, there are $y, y' \in V(Y)$ in the same coclique of Y, such that

$$C_{xi}C_{x'i} = Q_{xy}Q_{x'y'} = 0.$$

At the same time, for each $x \in V(X)$,

$$\sum_{i \in [d]} C_{xi} = I,$$

so C is indeed a quantum colouring for X. The fact that $\chi_q(X) = \chi_q(Y)$ follows from Theorem 4.3.2. □

Corollary 4.5.5 *Let $\mathcal{F}$ be a linear binary constraint system where each constraint has the same number of variables. Suppose X and X_0 are non-isomorphic quantum isomorphic graphs constructed from an LBCS and its homogenized version respectively. Then $\chi_q(X) > \chi(X)$.*

Proof. The proof is omitted as it is very similar to the one of Corollary 4.5.3. □

4.6 Block-diagonal quantum isomorphism matrices and quantum cocliques

In the previous sections we have seen that quantum isomorphism matrices whose entries from Kochen-Specker sets contain a quantum coclique matrix of one of the graphs as a submatrix, given some conditions. Recall definition of α_q appeared in Definition 4.1.1. This section is dedicated to a reverse question. Given a graph with $\alpha_q > \alpha$, can we use the quantum coclique matrix to construct a quantum isomorphism from the given graph to some other graph.

Between two graphs that are related by Godsil-McKay switching (see Section 2.3), we construct a quantum isomorphism matrix using the quantum coclique matrix of one of the graphs with $\alpha_q > \alpha$, provided a number of conditions hold.

When two graphs $X = (V, E(X))$ and $Y = (V, E(Y))$ are related by Godsil-McKay switching, we say an edge $\{u, v\} \in K_{|V|}$ is *switched* if

$$\{u, v\} \in E(X), \{u, v\} \notin E(Y) \text{ or } \{u, v\} \notin E(X), \{u, v\} \in E(Y).$$

If the first condition holds, we say $\{u, v\}$ is a switched edge in X, while if the second condition does, then $\{u, v\}$ is a switched edge in Y. A switched edge in X is a *switched non-edge* of Y and vice versa. Edges that are present in both X and Y are called *common* edges.

Also, if $W \subseteq V$, we denote by $X[W]$ the subgraph of X induced by W.

Additionally, recall that $\boxtimes$ denotes the strong product of graphs. For graphs X and Y, their strong product $X \boxtimes Y$ denotes a graph with vertices $V(X) \times V(Y)$ and vertices (x, y) and (x', y') are adjacent if:

- $x \sim_X x'$ and $y = y'$ or
- $y \sim_Y y'$ and $x = x'$ or
- $x \sim_X x'$ and $y \sim_Y y'$.

For example, $X \boxtimes K_n$ can be visualized as replacing vertices of K_n with copies of X and placing all possible edges between "adjacent" copies.

Finally, given a quantum coclique matrix P for $\alpha'_q(X) \leq \alpha_q(X)$, we denote by

$$h_z := h_z(P)$$

the integer of our choice such that we can build a quantum block-diagonal coclique P' of X', with blocks of size $h_z \times 1$ for some graph X' such that $\alpha(X) = \alpha(X')$ and $\alpha_q(X') \geq \alpha'_q(X)$. There are at least two possibilities for the choice of X'. If the quantum coclique matrix P is already in block-diagonal form, like for the 24-vertex graph constructed from the non-homogenized LBCS with 6 constraints, then we can let h_z be the height of the blocks (which is 4 in this case) and $X' = X$. Otherwise, we can always choose the strong product

$$X' = X \boxtimes K_{\alpha'_q(X)}.$$

In other words, for this section we define

$$X' = \begin{cases} X \text{ if quantum coclique matrix of X is block diagonal} \\ X \boxtimes K_{\alpha'_q(X)} \text{ otherwise.} \end{cases}$$

When $X = X'$, then the quantum coclique matrix for X' is the same as the one for X and h_z is the height of the blocks in the quantum coclique matrix. Otherwise, when $X \neq X'$, we describe how we construct a quantum coclique for X'.

Let

$$M_{(1,\ldots,|V(X)|)}$$

denote a matrix representing a cyclic permutation $(1, \ldots, |V(X)|) \in S_{|V(X)|}$. Let

$$P_{\text{col}(i)}$$

denote the i^{th} column of P.

In case that $X \neq X'$, we can construct a quantum coclique matrix P' for X' to be a block-diagonal block $\alpha'_q(X) \times \alpha'_q(X)$ matrix, with blocks of size $|V(X)| \times 1$, with the i^{th} diagonal entry being the permutation of the i^{th} column of P:

$$M^{(i-1)} P_{\text{col}(i)}.$$

In this case we set $h_z = V(X)$.

Lastly, whenever we have such a quantum coclique matrix P' for X' in the block-diagonal form, we can always label vertices of X as (v_i, j) whenever v_i is in the block j of P'.

Now, I clarify why in the following lemma there is a parameter α'_q. This theorem is meant to apply in the cases when we have a quantum coclique matrix with α'_q columns that satisfies requirements for being a quantum coclique, but not necessarily the maximal one. We may not know the value of $\alpha(X)$ or have a quantum coclique matrix with $\alpha_q(X)$ columns. This lemma below, however, applies to the case when we have a quantum coclique with $\alpha < \alpha'_q \leq \alpha_q(X)$ columns. An illustration of how this lemma works will be given below with the quantum isomorphism between the 24-vertex graphs $X(F)$ and $X(F_0)$.

Lemma 4.6.1 *Suppose $\alpha_q(X) \geq \alpha'_q > \alpha(X)$, and P is the quantum coclique matrix for $\alpha'_q(X)$. Let $X' = (V, E(X'))$ and h_z be as explained above. Suppose $Y = (V, E(Y))$ satisfies:*

1. *Y is Godsil-McKay-switching-equivalent to X' with the cells in the Godsil-McKay partition $\{C_1, ..., C_k, D\}$ consisting of the sets of vertices with the same second coordinate.*
2. *For any i, either all edges between C_i and D were switched or none.*
3. *From each of the sets $C_1, ..., C_k, D$, there are representatives $c_1, ..., c_k, d$ respectively, such that $c_i \sim_X d$ if and only if the switching occurred between D and C_i.*
4. *$V(Y)$ can be partitioned into α'_q sets $T_1, ...T_{\alpha'_q}$ of size h_z*
5. *There are h_{z-1} automorphisms $R_2, ..., R_{h_z}$ of Y that*
    - (a) *Have disjoint support: for every $1 \leq j, k \leq h_z$ and $(v_i, \ell) \in V(Y)$, we have $R_j((v_i, \ell)) \neq R_k((v_i, \ell))$*
    - (b) *For all a, each R_i maps vertices of X' with the second coordinate a to the vertices with the second coordinate a.*
    - (c) *Any pair maps switched edges in Y to switched edges in Y and switched non-edges of Y to switched non-edges of Y*
    - (d) *Any pair maps common edges $E(X) \cap E(Y)$ to common edges of $E(X) \cap E(Y)$.*
    - (e) *We can relabel $R_2, ..., R_{h_z}$, so that $R_{col(w,a)}(v, a) = a$,*
    - (f) *Actions of R_i are the same on each block.*

Then X' and Y are quantum isomorphic.

Proof. We are going to construct a block-diagonal quantum isomorphism matrix Q from the quantum coclique P for X between X' and Y.

First, note that $\alpha(X) = \alpha(X')$ by construction of X'. The quantum coclique matrix P' of X' has $h_z\alpha'_q(X) = V(X')$ rows and $\alpha'_q(X)$ columns. In other words, $\alpha(X') < \alpha_q(X')$.

As described in the definition of h_z, we know that P' will be block diagonal with $\alpha'_q(X)$ diagonal blocks of size $h_z \times 1$.

Then, the vertices of X' are tuples $\{(v_i, j)\}$ with each v_i being in a tuple with unique j, the index of the block in which v_i is in P'. Even though we can view $V(X')$ and $V(Y)$ as the same set, think of rows of Q as labelled by $V(X') = \{(v_i, j)\}$ and columns labelled by $V(Y) = \{(v_i, j)\}$. Q will have $h_z \times h_z$ blocks $T_1, ..., T_{\alpha'_q(X)}$ on the diagonal, and all other entries will be zero projections.

Finally, we can start constructing the block-diagonal quantum isomorphism Q with all entries being projections of same dimension as entries of P' and with $h_z \times h_z$ diagonal blocks of projections. Let the first column of each block of Q be the $h_z \times 1$ diagonal entry of P_z.

Recall that our Godsil-McKay partition is $\{C_a\}_a \cup \{D\}$. Denote representatives of blocks from point three in the theorem statement as $a := c_a$ for blocks indexed by vertices in C'_as and choose d for the remaining block indexed by D.

Let c be an arbitrarily chosen block index from $\{c_a\}_a \cup \{d\}$. Let w be an arbitrary vertex of Y from the c^{th} block. For a vertex (w, c) of Y, we know w is in the c^{th} block. Let $1 \leq \text{col}(w, c) \leq h_z$ denote in which column of the c^{th} block (w, c) is in. Now, let the i^{th} column of the a^{th} block of Q be formed as follows:

$$Q_{(v,a)(w,c)} = \begin{cases} P'_{(R_{\text{col}(w,a)}(v,a),a)}, & \text{if } a = c \text{ and } w \in i^{th} \text{ column of the block } a \\ 0, & \text{if } a \neq c \end{cases}$$

$$Q_{(w,c)(v,a)} = \begin{cases} P'_{(a,(R_{\text{row}(w,a)}(v,a))}, & \text{if } a = c \text{ and } w \in i^{th} \text{ column of the block } a \\ 0, & \text{if } a \neq c \end{cases}$$

Note that we can have $\text{col}(w, a) = \text{row}(w, a)$ by ordering vertices of X' and Y simultaneously, so

$$R_{\text{col}(w,a)}(v, a) = R_{\text{row}(w,a)}(v, a)$$

Moreover, if necessary, we can relabel $R_2, ..., R_{h_z}$, so that

$$R_{\text{col}(w,a)}(v, a) = a,$$

so blocks of Q are symmetric. We will use either definition, whichever one is more convenient. Remembering that the second coordinate of a vertex indicates in which block of P'

it was, (and so in which block of Q it is), and that we can view graphs Y and X' as being on the same vertex set, verify that

$$\sum_{(w,b)\in V(Y)} Q_{(v,a)(w,b)} = \sum_{(w,b)\in V(X')} Q_{(v,a)(w,b)}$$

Now, by definition of Q, since Q is block-diagonal,

$$\sum_{(w,b)\in V(X')} Q_{(v,a)(w,b)} = \sum_{w\in V(X)} Q_{(v,a)(w,a)} = \sum_{w\in V(Y)} P'_{(R_{\mathrm{col}(w,a)}(v,a),a)} = \sum_{i=2}^{h_z} P'_{(R_i(v,a),a)},$$

where the last equality follows because there are only h_z columns per block. Finally, since no R_i fixes any (v, a) and there are $h_z - 1$ of R_i's,

$$\sum_{i=2}^{h_z} P'_{(R_i(v,a),a)} = \sum_{u\in V(X) \text{ and in block } a} P'_{((u,a),a)} = I$$

Also, suppose for an arbitrary j, b we have that w is in the j^{th}column of the block b, then

$$\begin{aligned}\sum_{(v,a)\in V(X')} Q_{(v,a)(w,b)} &= \sum_{v\in V(X)} Q_{(v,b)(w,b)} \\ &= \sum_{v\in V(X)} P'_{(R_{\mathrm{col}(w,b)}(v,b),b)} \\ &= \sum_{u\in V(X)} P'_{((u,b),b)} \\ &= I\end{aligned}$$

similarly to above.

Now, we have to check that whenever $(v, a) \sim_{X'} (w, b)$ and $(k, i) \nsim_Y (\ell, j)$, then

$$Q_{(v,a),(k,i)} Q_{(w,b),(\ell,j)} = 0.$$

However, since Q is block-diagonal, if $a \neq i$ or $b \neq j$, then at least one of the following entries is zero, so the product is zero:

$$Q_{(v,a),(k,i)} Q_{(w,b),(\ell,j)} = 0.$$

Therefore, we may assume $i = a, j = b$.

Case 1: $(v,a) \sim_Y (w,b)$, $(v,a) \sim_{X'} (w,b)$ and $(k,a) \nsim_Y (\ell,b)$, so

$$Q_{(v,a),(k,a)}Q_{(w,b),(\ell,b)} = P'_{(R_{\mathrm{col}(k,a)}(v,a),a)}P'_{(R_{\mathrm{col}(\ell,b)}(w,b),b)} = 0$$

by conditions 5c,d. They guarantee that pairs of automorphisms map non-switched edges in Y to non-switched edges in Y. Since $\{(v,a),(w,b)\}$ is a common edge in Y,

$$\{R_{\mathrm{col}(k,a)}(v,a), R_{\mathrm{col}(\ell,b)}(w,b)\}$$

remains a common edge in X' and Y.

Case 2: $(v,a) \nsim_Y (w,b)$, $(v,a) \sim_{X'} (w,b)$ and $(k,a) \nsim_Y (\ell,b)$, so $\{(v,a),(w,b)\}$ is a switched edge in X, equivalently, a switched non-edge in Y.

$$Q_{(v,a),(k,a)}Q_{(w,b),(\ell,b)} = P'_{(R_{\mathrm{col}(k,a)}(v,a),a)}P'_{(R_{\mathrm{col}(\ell,b)}(w,b),b)} = 0$$

By condition 5d,

$$R_{\mathrm{col}(k,a)}(v,a) \nsim_Y R_{\mathrm{col}(\ell,b)}(w,b),$$

remains a switched edge in X, so

$$R_{\mathrm{col}(k,a)}(v,a) \sim_X R_{\mathrm{col}(\ell,b)}(w,b).$$

Now, we have to check that whenever $(v,a) \nsim_{X'} (w,b)$ and $(k,a) \sim_Y (\ell,b)$, then

$$Q_{(v,a),(k,a)}Q_{(w,b),(\ell,b)} = 0,$$

Case 1: $(v,a) \nsim_Y (w,b)$, $(v,a) \nsim_{X'} (w,b)$ and $(k,a) \sim_Y (\ell,b)$, then

$$Q_{(v,a),(k,a)}Q_{(w,b),(\ell,b)} = Q_{(k,a),(v,a)}Q_{(\ell,b),(w,b)} = P'_{(R_{\mathrm{col}(v,a)}(k,a),a)}P'_{(R_{\mathrm{col}(w,b)}(\ell,b),b)}.$$

Since the switching happened between all or none edges between all blocks, including blocks a and b, we see that in this case switching did not happen between a and b. So, $(k,a) \sim_X (\ell,b)$.

By condition 5d, $\{(k,a),(\ell,b)\}$ remains a common edge between X and Y, so

$$R_{\mathrm{col}(v,a)}(k,a) \sim_X R_{\mathrm{col}(w,b)}(\ell,b).$$

However, since actions are the same on each of the sets, and $R_i^2 = I$, we may choose labelling of $R_2, ..., R_{h_z-1}$ such that each formed block is symmetric. So,

$$P_{(v,a),(k,a)}P_{(w,b),(\ell,b)} = P_{(R_{\mathrm{col}(k,a)}(v,a),a)}P_{(R_{\mathrm{col}(\ell,b)}(w,b),b)}$$
$$= P_{(R_{\mathrm{row}(\ell,b)}(w,b),b)}P_{(R_{\mathrm{row}(k,a)}(v,a),a)} = 0,$$

since $(R_{\mathrm{row}(\ell,b)}(w,b) \sim_X R_{\mathrm{row}(k,a)}(v,a)$.

Case 2: $(v,a) \sim_Y (w,b)$, $(v,a) \nsim_{X'} (w,b)$ and $(k,a) \sim_Y (\ell,b)$, then we have

$$Q_{(v,a),(k,a)}Q_{(w,b),(\ell,b)} = P'_{c_a,(R_{\mathrm{row}(v,a)}(k,a),a)}P'_{c_b,(R_{b,\mathrm{row}(w,b)}(\ell,b)}.$$

Since either all or none edges were switched between each C_i and D, $(v,a),(k,a)$ are in the same block and $\{(v,a),(w,b)\}$ was switched, $\{(k,a),(\ell,b)\}$ was also switched. So, representatives c_a, c_b are also adjacent n X. Clearly, $c_a \neq c_b$, Using the fact that P' is a quantum coclique, we get that

$$P'_{c_a,(R_{\mathrm{row}(v,a)}(k,a),a)}P'_{c_b,(R_{b,\mathrm{row}(w,b)}(\ell,b)} = 0.$$

□

Next we will give a condition that identifies how to find automorphisms that suit the conditions of the preivious theorem. In the end of the section we will provide illustrations of these theorems to the 24-vertex graphs.

Lemma 4.6.2 *Let X and Y be graphs related by Godsil-McKay switching. R is an automorphism of Y that maps switched edges of Y to switched edges of Y and switched nonedges of Y to switched nonedges of Y if and only if it is a common automorphism between X and Y. Moreover, in that case, R also maps switched edges of X to switched edges of X, and same for nonedges of X.*

Proof. First, suppose R is an automorphism of Y mapping switched edges of Y to switched edges of Y and switched nonedges of Y to switched nonedges of Y.

Since each switched edge of Y is a switched non-edge of X, R maps switched non-edges of X to switched non-edges of X. Similarly, R maps switched edges of X to switched edges of X.

Now, $\{u,v\} \in E(X) \cap E(Y)$ is a common edge between X and Y. We would like to show that R maps $\{u,v\}$ to a common edge between X and Y. Towards contradiction suppose it does not. After all, R is an automorphism of Y, so it can only map $\{u,v\} \in E(Y)$

to $\{Ru, Rv\} \in E(Y)$. So if $\{Ru, Rv\}$ is not a common edge, it is a switched edge in Y. By hypothesis, R maps switched edges to switched edges. Let k be the smallest possible integer such that $\{Ru, Rv\}, \{R^2u, R^2v\}, ..., \{R^ku, R^kv\}$ are distinct switched edges in Y. Thus, $k \geq 1$. Now, for $k+1$, we have either

- $\{R^{k+1}u, R^{k+1}v\} = \{R^iu, R^iv\}, i < k$ and it is an already encountered switched edge or
- $\{R^{k+1}u, R^{k+1}v\}$ is another distinct switched edge.

The second condition can't happen since k is the minimal such that

$$\{Ru, Rv\}, \{R^2u, R^2v\}, ..., \{R^ku, R^kv\}$$

are distinct switched edges in Y. Thus, there is an i such that

$$\{R^{k+1}u, R^{k+1}v\} = \{R^iu, R^iv\}.$$

However, R is an invertible permutation matrix, so

$$u = R^{k+1-i}u \text{ and } v = R^{k+1-i}v.$$

Therefore,

$$\{Ru, Rv\}, \{R^2u, R^2v\}, ..., \{R^{k+1-i}u, R^{k+1-i}v\}, ..., \{R^ku, R^kv\}$$

contains a common edge $\{R^{m+1-i}u, R^{m+1-i}v\}$, a contradiction. We have shown that, indeed, a common edge $\{u, v\} \in E(X) \cap E(Y)$ can be only mapped by R to another common edge.

Similarly, we could show that the common non-edge $\{a, b\} \in E(\bar{X}) \cap E(\bar{Y})$ can be only mapped by R to another common non-edge. In other words, R is an automorphism mapping common edges of X and Y to common edges of X and Y, common non-edges of X and Y to common non-edges of X and Y, switched edges of X to switched edges of X and switched non-edges of X to switched non-edges of X. More importantly for this direction of the proof, R maps edges of X to edges of X and non-edges of X to non-edges of X, so R is an automorphism of X.

For the other direction, suppose R is a common automorphism between X and Y. If $\{u, v\}$, is a switched edge in Y, then $u \nsim_X v$. However, with R being an automorphism of Y, we have $Ru \sim_Y Rv$. Since R is also an automorphism of X we have $R(u) \nsim_X R(v)$, so $\{Ru, Rv\}$ remains a switched edge in Y. The same argument works for non-switched edges of Y and for common edges between X and Y. □

Remark: The above lemma tells us that for Lemma 4.6.1 we should search for R_i's from the set of common automorphisms of X' and Y', since we have a lot of conditions on pairs of automorphisms, and in particular, they have to hold for the pairs of same automorphisms.

Example 4.6.3 The pair of graphs $X(\mathcal{F})$ and $X_0(\mathcal{F})$ from [46] have that $\alpha_q(X_0(\mathcal{F})) > \alpha(X_0(\mathcal{F}))$. Moreover, they satisfy the conditions required for Lemma 4.6.1, and so we can construct the quantum isomorphism matrix from the quantum coclique of the graph with different α and α_q. We will walk through this construction step-by-step below.

First, note that $z = 1$. In Theorem 5.1.1 we proved that these two graphs are related by Godsil-McKay switching. In the Section 3.2.1 we translated proofs from [46] into an explicit quantum isomorphism between this pair of graphs. In [54], Piovesan found an explicit construction of α_q, which is just the 1st, 5th, 8th, 12th, 16th and 20th columns of this explicit quantum isomorphism.

We note that Y can be partitioned into 6 cliques and 4 cocliques, so that each of the cliques contains exactly one vertex from each coclique.

Moreover, there are three automorphisms of Y that map switched edges of Y to switched edges of Y and act transitively on each clique. We can easily uncover these three automorphisms by looking at the first column of each diagonal block of quantum isomorphism matrix and the 2nd, 3d, 4th columnms. For example, looking at the first and second columns of the first block, we notice that $(1, 2)$ entry becomes $(2, 1)$ entry. So we declare vertex $(x_7 + x_8 + x_9 = 0, 000)$ being mapped to $(x_7 + x_8 + x_9 = 0, 110)$. Similarly,

$$(x_7 + x_8 + x_9 = 0, 011) \mapsto (x_7 + x_8 + x_9 = 0, 000),$$
$$(x_7 + x_8 + x_9 = 0, 101) \mapsto (x_7 + x_8 + x_9 = 0, 110),$$
$$(x_7 + x_8 + x_9 = 0, 110) \mapsto (x_7 + x_8 + x_9 = 0, 001).$$

Repeating this procedure for the remaining blocks, we get the first automorphism. Then the second automorphism can be obtained by the similar process by comparing the first and the third column of each block etc. One can check that these are the automorphisms of both $X(\mathcal{F})$ and $X_0(\mathcal{F})$, and that any of their pairs map switched/common edges of Y to switched/common edges of Y and same for non-edges.

An open question is whether there could also be two quantum isomorphic graphs but such that $\alpha_q(X) = \alpha(X)$ and $\alpha_q(Y) = \alpha(Y)$. As an example, consider $X = Y = K_4$. It is known that $\alpha_q(K_4) = \alpha(K_4) = 1$, but it has quantum symmetry. The question becomes if there could be quantum isomorphic, but non isomorphic, graphs such that both graphs have different α and α_q.

Chapter 5

Switching

While studying the known examples of non-isomorphic quantum isomorphic graphs, we have discovered that the 24-vertex graphs from [2] and Schmidt's strongly regular graphs are related by Godsil-McKay switching (see Section 2.3). This observation has lead us to explore when Godsil-McKay switching preserves coherent algebras of graphs as will be explored in the later chapters. In this section we provide the partition

5.1 The 24-vertex graphs are switching related

We observe that both of the graphs from [2] are related by Godsil-McKay switching.

Theorem 5.1.1 *Graph $X(F_0)$ can be obtained from $X(F)$ by Godsil-McKay switching by selecting the following partition of vertices.*

(a) $X_i = ((C_i, f_i^1), ..., (C_i, f_i^4))$ *if* $b_i = 0$.

(b) $Y = ((C_j, f_j^1), ..., (C_j, f_j^4))$ *for the only constraint* C_j *given by*

$$x_3 + x_6 + x_9 = 1$$

Proof. First, we will verify that every vertex $x \in Y$ has either 0, $\frac{1}{2}|X_i|$ or $|X_i|$ neighbors in X_i. As a first case, we will count neighbours of a vertex in Y to a vertex in a subset of vertices $X_i \subseteq V(X(F))$ such that constraint C_i has no variables x_3, x_6 or x_9. In other words, C_i an C_j have no variables in common.

Case 1: $S_i \cap S_j = \emptyset$. Then any vertex $(C_j, f_j) \in Y$ has no neighbours in X_i.

Next, investigate the case where X_i is such that Y and X_i have variables in common. Note that any two constrains have at most one variable in common.

Case 2: $S_i \cap S_j = \{x_k\}$, where $k \in \{3, 6, 9\}$. So the vertices of X_i are $((C_i, f_i^1), ..., (C_i, f_i^4))$. By inspection, half of the assignments from $f_i^1, ..., f_i^4$ will assign value zero to x_k, and the remaining half of the satisfying assignments will asign value one to x_k. Thus, any vertex (C_j, f_j) of Y will be adjacent to half of the vertices of X_i in this case.

Now let us check that every $x \in X_i$ has the same number of the neighbours in X_k for each $1 \leq i, k \leq 5$. For this, fix an X_i and let us start with X_k such that corresponding constraints C_i and C_k have no variables in common.

To verify that the defined partition is indeed a Godsil-McKay partition, we also need to ensure that vertices in X_i have the same number of neighbours in X_j. Consider different cases and suppose first that there is no vertex in common between the associated constraints C_i and C_k.

Case 1: $S_i \cap S_k = \emptyset$. Then any vertex $(C_i, f_i^j) \in X_i, 1 \leq j \leq 4$ has no neighbours in X_k.

Case 2: $S_i \cap S_k = \{x_r\}$. Then, as before, half of the vertices of X_k have assignments that set x_r to zero, and the leftover half assign value one to x_r. So each vertex of X_i is adjacent to half of the vertices of X_k.

Case 3: $X_i = X_k$. The graph induced by X_i is regular.

To conclude the first half of our proof, the conditions of Godsil-McKay partition are satisfied. At this point we are ready to check if by performing the switching itself, we are going to obtain $X(F_0)$.

For this graph, the switching will be the following procedure.

- Choose one of the three possible X_i such that S_i has a variable x_3 or x_6 or x_9.
- For every vertex (C_j, f_j) of Y remove its existent edges to vertices in X_i, and connect it instead to the other vertices of X_i.

We note that given vertices $(C_j, f_j^1), ..., (C_j, f_j^4)$ of Y in $V(X(F))$, we can modify them as follows

$$(C_j, f_j^1) \mapsto (C_j, f_j^1 \oplus (1, 1, 1)).$$

For each k, by negating the evaluation by f_j^k of each variable of C_j we achieve exactly the desired effect of the Godsil-McKay switching. In other words, if (C_j, f_j) and (C_i, f_i) were adjacent before, it was because f_i and f_j agreed on variables in $S_i \cap S_j$. Now, it will be that f_i and $f_j \oplus (1,1,1)$ will, on the constrary, disagree on variables in $S_i \cap S_j$, and so will not be adjacent. Analogously, vertices (C_j, f_j) and (C_i, f_i) that were non-adjacent will become adjacent. And so the procedure of negating the output of f_j for each vertex (C_j, f_j) of Y is equivalent to performing Godsil-McKay switching on the set Y.

Again, $\oplus$ denotes addition modulo two. Denoting by C'_j the homogenized constraint $x_3 + x_6 + x_9 = 0$, we can readily see that $(C'_j, f^1_j \oplus (1,1,1)), ..., (C'_j, f^4_j \oplus (1,1,1))$ are the only vertices of $X(F_0)$ that are not a subset of vertices of $V(X(F))$. In other words, we showed that $X(F)$ and $X(F_0)$ are indeed related by the Godsil-McKay switching. □

5.2 120-vertex strongly regular graphs are switching related

Theorem 5.2.1 *G^{E^8} and G^w are related by Godsil-McKay switching.*

Proof. Refer to Figure 3.5 for the partition $V_1, ..., V_{15}$ of the vertices of G^{E^8}. Our Godsil-McKay partition will consist of fourteen sets of size four and one set of size sixty-four. For $1 \le i \le 7$ define:

$$C_{i+} = \{x \in V_i | x \text{ contains only nonnegative entries 0 and 1}\}$$

$$C_{i-} = \{x \in V_i | x \text{ contains each of the following: } 1, -1 \text{ and } 0\}$$

$$D = V_8 \cup ... \cup V_{15}$$

Since each V_i induces a K_8 in G^{E^8}, it is clear that every $C_{ix}, 1 \le i \le 7, x \in \{+,-\}$ induces a K_4. One can check all vertices in any C_x have each the same number of neighbours in any C_y, namely one of two numbers $\{2,4\}$. Moreover, every vertex in D has either 0, 2 or 4 neighbours in each of the C_x. Recall the construction of G^w from G^{E^8} on page 31. Choose vectors $\{e_1 + e_2, ..., e_1 + e_8, x_{\{1,2\}}, ..., x_{\{1,8\}}, x_\emptyset\}$ from each orbit as representatives to form w without loss of generality. To justify the lack of loss of generality, use Theorem 3.3.1, originally from [65], which says that different choices of w create isomorphic graphs G^w. Now, for any C_{ix} we have that $d \in D, d \in V_k$ is adjacent to half of the vertices in C_{ix} if and only if $\langle w_i, w_k \rangle = 0$. By construction of G^w from G^{E^8}, it means that the switching has occurred between V_k and V_i, which easily generalizes to show that switching G^{E^8} on partition $\{C_{i+}, C_{i+}\}_{i=1}^7 \cup D$ gives G^w. □

Chapter 6

Coherent algebras

In this chapter we provide known classical results connecting properties of automorphism groups of graphs and properties of commutant of an automorphism group of graphs. We generalize these concepts to quantum automorphism groups too.

So far, we have discovered that two examples, the 24-vertex graphs from [2] and 120-vertex strongly regular graphs [65] are related by Godsil-McKay switching. In this section we will see the proof from [2] of the fact that quantum isomorphisms preserve coherent algebras. As a new result, we will show explicit form of this isomorphism of coherent algebras.

Together, these observations would lead us to thinking that potential candidates for quantum isomorphic non-isomorphic graphs could be from the graphs that are related by Godsil-McKay switching and have isomorphic coherent algebras. Thus, to kill two criteria with one answer, we asked when does Godsil-McKay switching preserve coherent algebras. To make it more concrete, we asked when is conjugation by the reflection matrix from Godsil-McKay switching the isomorphism of coherent algebras of graphs that are related by the Godsil-McKay switching. Towards the end of the chapter we will have progress towards these questions.

6.1 Background on coherent configurations and coherent algebras

In this section we explore standard information on coherent configurations and how they relate to coherent algebras. More can be found in, for example, in [27].

A coherent configuration [12] is a finite set X of points and a collection of nonempty binary relations $\mathcal{R} = \{R_i \mid i \in I\}$ on X with the following conditions:

(a) $\mathcal{R}$ is a partition of $X \times X$.

(b) There is a subset H of the index set I such that $\{R_h \mid h \in H\}$ is a partition of the diagonal $\{(x, x) \mid x \in X\}$

(c) For each R_i, its transpose $\{(y, x) \mid (x, y) \in R_i\}$ is also one of the relations in $\mathcal{R}$, say $R_{i'}$.

(d) For $i, j, k \in I$ and $(x, y) \in R_k$, the number of $z \in X$ such that $(x, z) \in R_i$ and $(z, y) \in R_j$ is a constant p^k_{ij} that does not depend on the choice of x, y.

One well-known example is as follows. Consider a distance-regular graph X of diameter d. We can partition $V(X) \times V(X)$ into relations $R_i = \{(x, y) : \text{dist}(x, y) = i\}$. Below we sketch that they form a coherent configuration.

First, if the diameter of the graph is d, any two vertices are at distance $i \in [0, d]$, so $R_0, ..., R_d$ clearly partition $X \times X$. In this case, the whole diagonal $\{(x, x) \mid x \in X\}$ belongs to relation R_0, so $H = \{0\}$.

The distance between x and y is clearly the same as between y and x. Finally, from distance-regularity it follows that the number of vertices at distance i from x and at distance j from y depends only on i, j and the distance between x and y.

We can also think about coherent configurations for a set X in terms of matrices. For this, to each relation R_i, we associate an $|X| \times |X|$ adjacency matrix A_i, with

$$(A_i)_{xy=} \begin{cases} 1 \text{ if } (x, y) \in R_i \\ 0 \text{ otherwise} \end{cases} .$$

Next, we would like to restate the above coherent configuration requirements for relations R_i in terms of these adjacency matrices A_i as well.

In order for the relations to partition the set $X \times X$, we need each pair (x, y) to be in a unique relation. Hence, if we sum over all the matrices A_i, we will get the all ones matrix J :

$$\sum_i A_i = J$$

Second, we know that the relations in H partition the set $\{(x,x)\}_{x\in X}$. So,

$$\sum_{i\in H} A_i = I$$

We require that for any $(x,y) \in R_i$ also $(y,x) \in R_{i'}$, which results in

$$(A_i)^T = A_{i'}$$

To reinterpret the final relation requirement, we look at the xy entry of the product A_iA_j. As expected, it counts the number of elements z that are in relation R_i with x and in relation R_j with y.

For convenience, we restate the observation in concise form.

Proposition 6.1.1 *Suppose $\{A_i\}$ are the adjacency matrices of the relations $\{R_i\}$ arising from a coherent configuration. Then the following hold:*

(a) $\sum_i A_i = J$

(b) $\sum_{i\in H} A_i = I$

(c) $A_i^T = A_{i'}$ *for all* $i \in \mathcal{I}$

(d) $A_iA_j = \sum_k p_{ij}^k A_k$

Coherent configurations are closely related to coherent algebras [51]. Generally, an (associative) *algebra* over a field k is a k-vector space A equipped with an additional binary operation $\times : A \times A \to A$, such that $\times$ is associative, left and right distributive and we have $(xa)(yb) = (xy)ab$ for all $x, y \in k$ and $a, b \in A$. Thus, when talking about matrix algebra over a field k, we mean a vector space of matrices over a field k under addition, equipped with the standard matrix multiplication.

Naturally, a subalgebra of an algebra A is a subset B of A that is closed under addition, multiplication, and scalar products and contains the identity [34].

Definition 6.1.2 A *coherent algebra* W of order n is a subalgebra of $M_n(\mathbb{C})$ such that

(a) $I, J \in W$

(b) $A^* \in W$ for all $A \in W$

(c) For any $A, B \in W$, their Schur product $A \circ B \in W$

(d) For any $A, B \in W$, it holds that $AB \in W$.

We will prove that the span of adjacency matrices of the relations of a coherent configuration is a coherent algebra. For two matrices A and B with the same number of rows and columns $A \circ B$ denotes their entrywise product and is called Schur product.

Lemma 6.1.3 *The adjacency matrices $\{A_i\}$ of the relations R_i of a coherent configuration form a basis of a coherent algebra.*

Proof. Note that the adjacency matrices $\{A_i\}$ are Schur orthogonal and therefore orthogonal:

$$\langle A_i, A_j \rangle = \text{ sum } (A_i \circ A_j^T) = 0.$$

Therefore, these basis matrices form a basis for their linear span of the $\{A_i\}$. Let their linear span be called W. We will prove that W is a coherent algebra. As a linear span, W is a vector space under addition. Now, check that W is closed under multiplication. This follows since $A_i A_j = \sum p_{ij}^k A_k$. Now we will verify that W is coherent. As we concluded earlier, $\sum_{i \in \mathcal{I}} A_i = J$, Moreover, there is a subset $\mathcal{H} \subset \mathcal{I}$, such that $\sum_{i \in \mathcal{H}} A_i = I$. Thus, we are left to prove the Schur-product closure. Take two arbitrary matrices $A, B \in W$. We can write them in terms of the basis elements:

$$A = \sum_k \alpha_k A_k$$
$$B = \sum_j \beta_j A_j$$

Recall that relations $\{R_i\}$ partition the set $X \times X$, so $A_i \circ A_j = 0$ for all $i \neq j$.

$$A \circ B = \sum_{k,j} \alpha_k \beta_j (A_k \circ A_j) = \sum_{k,j} \alpha_k \beta_k A_k \in W.$$

□

Proposition 6.1.4 *Let W be a coherent algebra. Then W has a unique unordered basis consisting of Schur-orthogonal* 01*-matrices.*

Proof. W is an algebra, therefore, it has a basis. Let the basis be $C_1, ..., C_d$. Then for each $C_i, 1 \leq i \leq d$, we do the following.

- Let $\alpha_1, ..., \alpha_k$ be a set (no repetitions) of unique entries of C_i.
- For each $1 \leq j \leq k$, define D_{α_j} to be a matrix that has ones in the same locations as C_i has entries α_j and zeroes everywhere else.

Now we claim that each D_{α_j} is in W. We can define a matrix polynomial where multiplication is Schur multiplication and J is the all-ones matrix to be

$$p_j(C_i) = \circ_{i \neq j}(C_i - \alpha_j J).$$

Note that since $C_i \in W$ and W is closed under Schur multiplication, $p_j(C_i) \in W$ for each i and each j. Notice that up to a scalar,

$$p_j(C_i) = D_{\alpha_j},$$

implying that $D_{\alpha_j} \in W$ for each entry α_j of C_i. Since C_i was arbitrary, we can apply this procedure to all of the basis matrices C_i. Taking union of the resulting 01-matrices, we can select the subset of those that can't be written as the sum of others. We call those minimal idempotents. Clearly, any basis element C_i can be written as a sum of those and they are pairwise orthogonal. Thus, there must be d of them and they also form a basis of W. Since $J \in W$, and minimal idempotents are binary matrices, they should sum to J.

For uniqueness, suppose towards contradiction that there are two Schur-orthogonal bases $\mathcal{A} = \{A_1, ..., A_d\}$ and $\mathcal{B} = \{B_1, ..., B_d\}$. Then we can express elements of one in terms of the elements of another:

$$A_k = \sum_{j=1}^{d} \alpha_j B_j$$

Since $B_j's$ are Schur-orthogonal, and $A_k's$ are 01, the coefficients α_i are either 0 or 1 themselves. We can write

$$A_k = \sum_{j \in \mathcal{I}_k} B_j$$

Then

$$J = \sum_{k=1}^{d} A_k = \sum_{k=1}^{d} \sum_{j \in \mathcal{I}_k} B_j$$

Again, since B_j's are Schur-orthogonal, each of them can occur only once. If it happened that an A_i is a sum of more than one $B_j's$, since $A_k's$ are themselves Schur-orthogonal, by Pigeonhole principle, there will be another A_k, such that all $B_j's$ were already used to express other elements of $\mathcal{A}$. Hence, $\mathcal{B}$ consists of all elements of $\mathcal{A}$. □

Now, we will see that every coherent algebra contains a coherent configuration, and every coherent configuration generates a coherent algebra.

Lemma 6.1.5 *Let W be a coherent algebra. Then the basis of* 01- *matrices $A_1, ..., A_n$ form a coherent configuration.*

Proof. Recall that there is a unique Schur orthogonal 01-basis for the coherent algebra, so we can proceed to work with this basis. Suppose that each A_i has $|X|$ rows and $|X|$ columns for some set X. Since the matrices A_i are binary, they define some relations R_i on $X \times X$ such that

$$R_i(x, y) = \begin{cases} 1 \text{ if } A_{xy} = 1 \\ 0 \text{ otherwise} \end{cases}$$

In other words, the elements x and y of X are in relation R_i, if and only if the corresponding adjacency has entry 1 in the xy-position. We will see that these relations define a coherent configuration. First, observe that since A_i's form a basis of a coherent algebra,

$$0 = \langle A_i, A_j \rangle = \text{sum}(A_i \circ A_j).$$

Therefore, for $i \neq j$, we have that A_i and A_j have nonzero entries in different positions. It follows that $\{R_i\}$ are disjoint relations. Since $J \in W$, the A_i's have to sum to J, ensuring that $\{R_i\}$ partition $X \times X$, as needed. Secondly, since W is a coherent algebra, W contains I. Since $\{A_i\}$ form a basis, there exists some set H such that

$$I = \sum_{h \in H} \alpha_h A_h.$$

However, since the matrices A_h are disjoint, it must be that each coefficient α_h is non-negative. In particular, $\alpha_h = 1$ for all h. Therefore, these sets H are such that $\{R_h\}_{h \in H}$ partition the diagonal $\{(x, x) \,|\, x \in X\}$. Since a coherent algebra is closed under transpositions, we have that if a basis element $A_i \in W$, then $A_i^T \in W$ as well. It remains to show that if A_i is a basis element, then A_i^T is also a basis element. Suppose not, and express A_i^T as linear combination of the basis elements:

$$A_i^T = \sum_j \alpha_j A_j$$

However, then by the nature of Schur-orthogonality of the $A_i's$, we have:

$$A_i^T \circ A_i = \sum_j \alpha_j A_j \circ A_i = \alpha_i A_i \circ A_i = \alpha_i A_i$$

Hence, $A_i^T = A_i$. To sum up, if A_i^T is not in the basis, then $A_i^T = A_i$.

Finally, the existence of constants p_{kj}^i satisfying the equation

$$A_k A_j = \sum_k p_{kj}^i A_i,$$

which is equivalent to Condition (d), follows straight from the fact that $\{A_i\}$ form a basis. We conclude that the unique 01-basis of a coherent algebra defines the coherent configuration. □

6.2 Homogeneity and commutativity of coherent algebras

In this section we will present some general knowledge about certain conditions on homogeneity and commutativity of coherent algebras.

A coherent configuration is called *homogeneous* if one of the relations is

$$\{(x, x) | x \in X\}.$$

Equivalently, a coherent algebra is homogeneous if identity matrix is one of the matrices in the 01-basis. In the coherent algebra arising from the coherent configuration above, we had the relation R_0 describing the vertices at distance 0 from each other. Hence, we see that coherent algebra from the example was homogeneous. A *commutative coherent algebra* is such that any two matrices in it commute.

Proposition 6.2.1 *W is a homogeneous coherent algebra if and only if every matrix in W has a constant diagonal.*

Proof. Suppose W is a homogeneous algebra. Let $\{A_i\}$ be its orthogonal 01-basis, and suppose $A_1 = I$. Since $A_i \circ A_j = 0$ if $i \neq j$, all other basic matrices have zero diagonal. As any matrix in W is a linear combination of these basis matrices, each matrix in W also has constant diagonal.

On the other hand, if in some coherent algebra W, each matrix has constant diagonal, then so do the matrices in the unique orthogonal 01-basis. In particular, since diagonal matrices that sum to identity have to be in every Schur-orthogonal 01-basis of any coherent algebra, it must be that only the identity matrix is in this basis. □

Proposition 6.2.2 *If a coherent algebra is commutative, it is homogeneous.*

Proof. If $\mathcal{A}$ is a commutative coherent algebra, the elements of its basis each commute with J. In other words, they have constant and equal row and column sums. We also know that the matrices in the unique Schur orthogonal binary basis of coherent algebra are adjacency matrices of a coherent configuration. Therefore, there are some elements $\{A_k\}_{k\in\mathcal{K}}$ in the basis that add up to I. If $|\mathcal{K}| > 1$, the set $\{A_k\}_{k\in\mathcal{K}}$ does not contain identity matrix. Matrices A_k are diagonal with 0 and 1 entries. However, that would contradict that they had to have constant and equal row and column sums, as some columns will add up to 1 and others to 0.

Proposition 6.2.3 *Let G be a permutation group acting on the set X. Then X together with the partition R of $X \times X$ into orbits of G (acting on $X \times X$ via $g(x, y) = (gx, gy)$) is a coherent configuration.*

Proof. Clearly, the orbits of G acting on $X\times X$ partition $X\times X$. Additionally, we can easily find a set H such that the relations with index H partition the diagonal $\{(x, x) : x \in X\}$. To find H, we take the union of the disjoint orbits that are obtained starting at different elements of $X \times X$ with the same first and second entries.

Next, suppose a relation R_i is the adjacency matrix of the orbit of (x, y). In other words, R_i has entries 1 in the positions (a, b) if and only if (a, b) are in the orbit of (x, y). Then the orbit of (y, x) will define another relation R_i', which will contain all the transposed elements of R_i. Finally, suppose $(x, y) \in R_k$. We will calculate the number of $z \in X$ such that $(x, z) \in R_i$ and $(z, y) \in R_j$. Since the orbits are disjoint, if $z \neq y$, then we get $p_{ij}^k = 0$. Similarly, if $z \neq x$, we must have that $p_{ij}^k = 0$. Thus, the only possibility when p_{ij}^k is nonzero, is when $z = y$ with $i = k$ and $x = z$ with $k = j$, which forces p_{ij}^k to be 0 for all relations R_k, except for the diagonal relation R_d, in which case p_{ij}^k will be 1 regardless of the choice of the representative $(x, x) \in R_d$. □

6.3 Transitivity of automorphism group and homogeneity of coherent algebras

In this section we will prove that the matrices that commute with all automorphisms of a graph form a coherent algebra, and that this coherent algebra is homogeneous if and only if the action of automorphism group is transitive.

Recall that a *commutant* $\mathcal{W}$ of a set $\mathcal{M} \subseteq M_n(\mathbb{C})$ of matrices:

$$\mathcal{W} := \{N \in M_n(\mathbb{C}) : NM = MN \text{ for all } M \in \mathcal{M}\}$$

We will first prove the result about commutants of a set of permutation matrices. Of course that would be relevant to commutants of automorphism groups of graphs.

Proposition 6.3.1 ([27]) *The commutant $\mathcal{W}$ of a set of permutation matrices $\mathcal{Q}$ is a coherent algebra.*

Proof. Clearly, $I \in \mathcal{Q}$. Additionally, since each row and column of any $Q \in \mathcal{Q}$ contains only one 1,

$$JQ = QJ = J,$$

so $J \in \mathcal{W}$. Now suppose $M \in \mathcal{W}$, and

$$MQ = QM \text{ for all } Q \in \mathcal{Q}.$$

Then, using that $Q^T = Q^{-1}$, we have

$$Q^T M = MQ^T$$

$$(M^T Q)^T = (QM^T)^T$$
$$M^T Q = QM^T,$$

implying that, $M^T \in \mathcal{W}$ as well. Next, we check that for every $M_1, M_2 \in W$, their product belongs to $\mathcal{W}$.

$$(M_2 M_1)Q = (M_2 Q)M_1 = Q(M_2 M_1)$$

Finally, we confirm that for every $M_1, M_2 \in W$, their Schur product is in W. Suppose Q_{kj} is the unique entry in the j^{th} column with a nonzero entry and Q_{il} is the unique nonzero entry in the i^{th} row. Taking a closer look at the ij-entry of the product, we observe:

$$(M_1)_{lj} = (Q_{il}M_1)_{lj} = (QM_1)_{ij} = (M_1Q)_{ij} = (M_1)_{ik}Q_{kj} = (M_1)_{ik}$$

Similarly,

$$(M_2)_{ik} = (M_2Q)_{ij} = (QM_2)_{ij} = (M_2)_{lj}.$$

Now, check:

$$\begin{aligned}(M_2 \circ M_1Q)_{ij} &= (M_2)_{ik}(M_1)_{ik} \\ (QM_2 \circ M_1)_{ij} &= Q_{il}(M_2)_{lj}(M_1)_{lj} = (M_2)_{lj}(M_1)_{lj} = (M_2)_{ik}(M_1)_{ik}.\end{aligned}$$

Hence W is indeed a coherent algebra. □

Since the automorphism group of a graph is a set of permutation matrices, we have the corollary.

Corollary 6.3.2 *Let X be a graph, and let $\mathcal{P}$ be the automorphism group of X. Then the commutant $\mathcal{W}$ of $\mathcal{P}$ forms a coherent algebra.*

We have the connection between transitivity and homogeneity below for automorphism groups of graphs and their commutants.

Corollary 6.3.3 *Let X be a graph, and let $\mathcal{Q}$ be the group of automorphisms of X. Then the commutant $\mathcal{W}$ of $\mathcal{Q}$ is a homogeneous coherent algebra if and only if the automorphism group acts transitively on the graph.*

Proof. The fact that the commutant W of $\mathcal{P}$ forms a coherent algebra was proven above. First we will observe a condition on certain entries of the matrices in the commutant of $\mathcal{P}$ and then proceed to proving homogeneity.

Begin by choosing an arbitrary matrix $M \in W$ and an arbitrary $P \in \mathcal{P}$. Every permutation matrix has a unique one in each row and column. An entry $P_{ij} = 1$ means the automorphism P maps vertex j to vertex i. In other words, in every row $P_{j,P^{-1}(j)}$ entry is nonzero, and in each column, the entry $P_{P(j),j}$ is nonzero. We have

$$MP = PM,$$

and observe that

$$\begin{aligned}(MP)_{ij} &= M_{iP(j)}P_{P(j),j} = M_{i,P(j)} \\ (PM)_{ij} &= P_{i,P^{-1}(i)}M_{P^{-1}(i),j} = M_{P^{-1}(i),j}\end{aligned}$$

So the requirement on M is that for any $P \in \mathcal{P}$, we have

$$M_{i,P(j)} = M_{P^{-1}(i),j} \text{ for all } i, j \in V(X), P \in \mathcal{P}. \tag{6.3.1}$$

Now we proceed to verifying the condition for homogeneity.

For the backward direction, we assume that the automorphism group is transitive, and we would like to show that any $M \in W$ has constant diagonal. Because the group is transitive, for every $i \in V(X)$ and $j \in V$, there is an automorphism P, such that $P(j) = i$. If $P(j) = i$, then

$$M_{ii} = M_{jj} \text{ for all } i, j \in V(X),$$

which implies M has constant diagonal by 6.3.1.

For the forward direction, we know that each M has constant diagonal. Now suppose there is no automorphism P mapping j to i. In such case, the condition

$$M_{ii} = M_{jj}$$

will not be enforced. Therefore, we can find a matrix M where M_{ii} and M_{jj} will be different, but will still commute with all of $\mathcal{P}$, contradicting the assumption. □

6.4 Commutativity of coherent algebras and generous transitivity of automorphism groups

We have one more section on properties of commutants of automorphism groups.

We say an automorphism group of a graph X is *generously transitive* if for any pair of vertices $i, j \in V(X)$, there is an automorphism of X swapping them.

Theorem 6.4.1 (Folklore) *The commutant of a permutation group is a commutative coherent algebra if the group is generously transitive.*

The proof can be found in [27]. We provide a similar proof for the quantum case in Theorem 6.9.1.

Now in the upcoming sections we will show the analogues of the previous two results for quantum automorphism groups. This is a more concrete version of ideas in [40] and [63], due to Chris Godsil.

6.5 Coherent algebras and their relation to quantum automorphism groups

In this section we will now explore relationship between transitivity of the quantum automorphism group of a graph and the commutant coherent algebra of the quantum automorphism group of the graph. This is an enhancement of what has been known in the literature [40], [63].

Before we begin the following lemma, it is important to note that for a matrix $M \in M_n(\mathbb{C})$ to commute with a quantum permutation $Q \in M_n(M_d(\mathbb{C}))$, we mean

$$Q(M \otimes I_d) = (M \otimes I_d)Q.$$

We omit the index d if it is clear from the context.

Lemma 6.5.1 *Let $\mathcal{Q}$ denote all quantum automorphisms of a graph X. The set of matrices $\mathcal{C} \subseteq M_n(\mathbb{C})$ that commute with each matrix in $\mathcal{Q}$ form a coherent algebra.*

Proof. Clearly $I \in \mathcal{C}$. Additionally, $J \in \mathcal{C}$, since each row and column of $P \in \mathcal{Q}$ sums to identity. First observe that if $P \in \mathcal{Q}$ is a quantum permutation, $PP^T = I = P^T P$. We can deduce that $\mathcal{Q}$ is closed under transposes. Indeed, suppose $P \in \mathcal{Q}$, so

$$\begin{aligned} P(A \otimes I) &= (A \otimes I)P \\ (A \otimes I) &= P^T(A \otimes I)P \\ (A \otimes I)P^T &= P^T(A \otimes I), \end{aligned}$$

so $P^T \in \mathcal{Q}$ as well.

We'll show closure of $\mathcal{C}$ under transposes. If we have that $C \in \mathcal{C}$, and P is arbitrary in $\mathcal{Q}$,

$$P(C \otimes I)P^T = C,$$

and

$$P(C \otimes I)^T P^T = (P(C \otimes I)P^T)^T = (C \otimes I)^T.$$

Hence, $\mathcal{C}$ is also closed under taking transposes. Additionally, if $B, C \in \mathcal{C}$,

$$(B \otimes I)(C \otimes I)P = (B \otimes I)(PC) = P((B \otimes I)(C \otimes I)),$$

implying that $BC \in \mathcal{C}$ as well.

We are left to show the closure under Schur multiplication. Suppose $B \otimes I, C \otimes I \in \mathcal{C}$, and $P \in \mathcal{P}$ a quantum automorphism. First note that

$$[(B \otimes I) \circ (C \otimes I)] = [(B \circ C) \otimes I)],$$

which means that we only need to prove that

$$([(B \circ C) \otimes I)]\, P)_{ij} = (P\, [(B \circ C) \otimes I)])_{ij}$$

Or, equivalently,

$$\sum_k b_{ik} c_{ik} P_{kj} = \sum_k b_{kj} c_{kj} P_{ik}$$

In the next step we will demonstrate that

$$([(B \circ C) \otimes I)]\, P)_{ij} = [(B \otimes I)P]_{ij}\, [(C \otimes I)P]_{ij}$$

To start, notice that

$$[(C \otimes I)P]_{ij} = \sum_k c_{ik} P_{kj}$$

$$[(B \otimes I)P]_{ij} = \sum_l b_{il} P_{lj}$$

$$[(B \otimes I)P]_{ij}\, [(C \otimes I)P]_{ij} = \sum_{k,l} b_{il} P_{lj} c_{ik} P_{kj} = \sum_{k,l} b_{il} c_{ik} P_{lj} P_{kj}.$$

Observe that the summands in the last term vanish when $k \neq l$, which brings us to:

$$[(B \otimes I)P]_{ij}\, [(C \otimes I)P]_{ij} = \sum_k b_{ik} c_{ik} P_{kj} = ([(B \circ C) \otimes I]\, P)_{ij}, \qquad (*)$$

as we wanted to show. In the same fashion, we can see that

$$[P(B \otimes I)]_{ij}\, [P(C \otimes I)]_{ij} = (P\, [(B \circ C) \otimes I])_{ij} \qquad (**)$$

The fact that both $B \otimes I$ and $C \otimes I$ are in $\mathcal{C}$ guarantees the commutativity:

$$[(B \otimes I)P]_{ij}\, [(C \otimes I)P]_{ij} = [P(B \otimes I)]_{ij}\, [P(C \otimes I)]_{ij}.$$

Therefore, since the left hand sides are equal in $(*)$ and $(**)$, so are the right hand sides

$$([(B \circ C) \otimes I]\, P)_{ij} = (P\, [(B \circ C) \otimes I)])_{ij},$$

implying that $\mathcal{C}$ is Schur-closed. Since P was arbitrary, and we have demonstrated that $\mathcal{C}$ contains I, J and is closed under products and Schur-products, We are able to deduce that $\mathcal{C}$ is a coherent algebra. □

The coherent algebra defined by the commutant of the magic unitary has been referred to in literature [40] as an *orbital algebra*.

We recall that commutant of a transitively-acting classical automorphism group forms a homogeneous coherent algebra. Now we would like to see if we can characterize when the centralizer of the quantum automorphisms of a graph is homogeneous. To study this question, let us first familiarize ourselves with the notions of quantum orbits, orbitals and the actions of a quantum compact group on a set.

6.6 Background on quantum orbitals, quantum orbital algebra definition

In [47], Laura Mančinska and David Roberson have studied the notion of orbits and orbitals of actions of quantum automorphism groups.

To begin, the action of a classical group G onto a set X induces an equivalence relation on the elements of X. We say $x, x_1 \in X$ are equivalent, or $x \sim_G x_1$ if there is an element $g \in G$ such that $x_1 = gx$. An orbit of an element $x \in G$ is $G_x = \{gx : g \in G\}$. Equivalently, G_x is an equivalence class of x. When extending action of G onto $X \times X$ in a natural manner by $g(x_1, x_2) = (gx_1, gx_2)$, the orbits are called orbitals. We apply Mančinska and Roberson's definition to quantum automorphism group of graphs.

In Lemma 6.5.1, we have seen that the commutant of the set of all of the quantum automorphisms of a graph, also known as *orbital algebra* of a graph is a coherent algebra.

Given that the commutant of a quantum permutation is a coherent algebra, with the help of the Weisfeiler-Leman [38] algorithm we can find the unique orthogonal binary basis for this coherent algebra. Mančinska and Roberson in [40] arrive at the orbitals in another way. They first construct adjacency matrices of the certain equivalence relation, and then show that they form is a coherent configuration. These matrices turn out to be orbitals for the orbital algebra. We will include their explicit formulas for the orbitals below, since it will be useful to us later.

Definition 6.6.1 *[40] Suppose X is a finite set and Q is a quantum automorphism group action on X. Let $U = (u_{xy})_{x,y \in X}$ be a magic unitary defining $C(Q)$. Define the relations $\sim_1, \sim_2$ on X and $X \times X$ as follows:*

- $x \sim_1 y$ *if* $u_{xy} \neq 0$

- $(x, x') \sim_2 (y, y')$ *if* $u_{xy}u_{x'y'} \neq 0$

These relations are generalisations of the classical ones. Whenever we have a permutation group acting on a set, we can represent the action with permutation matrices. So, whenever an xy-entry of a classical permutation is nonzero, we have that it maps standard basis element e_y to standard basis element e_x. We say y is mapped to x. Similarly, as in the example of the graph, if the product of xy and $x'y'$-entries of a classical automorphism are nonzero, then either both $\{x, x'\}$ and $\{y, y'\}$ are edges or both are nonedges. Since y is mapped to x and y' to x', and edges must be mapped to edges, nonedges to nonedges only.

The authors in [40] prove that both of these relations are equivalence relations, which allows them to define quantum orbitals. Given a quantum permutation group $\mathcal{Q}$ acting on a finite set X, the *orbitals* of $\mathcal{Q}$ are the equivalence classes of the relation $\sim_2$ defined earlier.

The next theorem of [40] shows that equivalence classes of $\sim_2$ relation form the basis of the orbital algebra.

Theorem 6.6.2 *[40] Let $\mathcal{Q}$ be the quantum automorphism group of a graph X. Let $U = (u_{xy})_{x,y \in X}$ be the magic unitary with generators of $\mathcal{Q}$, and let $M \in M_{|V(X)|(\mathbb{C})}$. Then $MU = UM$ if and only if M is constant on the orbitals of $\mathcal{Q}$.* □

From Theorem 6.5.1, where we have showed that the commutant of the set of quantum automorphisms is a coherent algebra, we conclude that a matrix $M \in M_n(\mathbb{C})$ commutes with all quantum automorphisms if and only if it is constant on the 01-basis of the commutant of a set of all quantum automorphisms. Theorem 6.6.2 above says that $M \in M_n(\mathbb{C})$ commutes with all quantum automorphisms if and only if it is constant on the adjacency matrices of the $\sim_2$ equivalence relation. Hence, it follows that the 01-basis for the coherent algebra that is the commutant of the set of all quantum automorphisms consists of adjacency matrices of the $\sim_2$ relation.

There is another coherent algebra associated to a graph X, and it is called *coherent algebra of X* and is denoted by $\mathcal{C}(X)$. It is the smallest coherent algebra containing $A(X)$ and J.

As an application, consider the fact that almost all graphs have trivial quantum automorphism group. This proof also relies on the fact that almost all graphs have coherent algebra equal to the full matrix algebra [52]. This fact also gives a different proof of a famous old result that almost all graphs have trivial classical automorphism group, which was originally shown in [21]. Of course, the result for the triviality of the quantum automorphism groups implies its classical counterpart.

Theorem 6.6.3 *[47] Let X be a random graph on n vertices. The probability that X has a nontrivial quantum automorphism group goes to 0 as n goes to infinity.*

Proof. The proof relies on the fact that almost all graphs have coherent configuration equal to the full matrix algebra [52]. The basis of the full matrix algebra is formed by the elementary matrices $\{E_{ab}\}_{a,b}$ with the only non-zero entry in the ab-position. Earlier we learned that the basis of the coherent algebra of the graph is also a set of characteristic matrices for the orbitals of the quantum automorphism group. From the above two facts, we conclude that the quantum orbitals of almost all graphs are singletons. This implies that the orbits of quantum automoprhism group are singletons as well. Otherwise, if x and y are different vertices the same orbit, then (x, x) and (y, y) would be in the same orbital, which contradicts the fact that orbitals are singletons. Since orbits are singletons, by definition of the quantum orbit, $u_{xy} \neq 0$ if and only if $x = y$. Hence, the magic unitary u, defining quantum automorphism group of the graph must be a diagonal, so it is identity. □

6.6.1 An algorithm to determine for which graphs a given quantum permutation is a quantum iso-/automorphism

Based on the information we have learned so far, we suggest an easily-implementable algorithm that, given a quantum permutation, will output all pairs of graphs for which this quantum permutation is an isomorphism. Note that the graphs in pairs are not guaranteed to be different. So in the case that a pair contains two same graphs, the input permutation will be a quantum automorphism of the graph. The motivation behind this idea is that instead of studying two graphs and trying to find a quantum isomorphism between them, or to determine after much effort in vain, that there is none, we could look at the problem in reverse. For example, we could start with a quantum permutation that has non-commutative entries and then find graphs for which it will be a quantum automorphism or find pairs of graphs between which this matrix will be a quantum isomorphism. This task is accomplished by the algorithm we present below.

This approach is even more attractive, given that it is quite easy to construct some kinds of quantum permutations. For example, block-diagonal ones are straightforward. All we have to do is to make sure each block is a quantum permutation itself. This can be done by first starting with, say, d rank one projections summing to identity. Then finding an n by n Latin square and associating different projections to different values of the Latin square. So each row and column will have different projections, so the row and column sums will be identity. However, much harder it is to find out for which graphs will such quantum permutation be a quantum isomorphism or automorphism. This is what we would like to explore in this section.

Suppose we are given a quantum permutation Q. We would like to find graphs X and Y between which it will be a quantum isomorphism. Since Q is a square matrix with rows indexed by the vertices of Y and columns indexed by the vertices of X, for convenience we may assume $V(X) = V(Y)$. Now we describe an algorithm to get all possible graphs for which Q is a quantum automorphism and all possible pairs of graphs for which Q is a quantum isomorphism.

I have implemented an algorithm and tested a few of the block-diagonal quantum permutations. We were limited by time and space, so we could not run the algorithm for long enough to covered all the cases. By far, the current implementation of the algorithm has not found a pair where two graphs are non-isomorphic and have these quantum permutations as isomorphisms. Nonetheless, it has output numerous graphs for which these quantum permutations were quantum isomorphisms and identified numerous graphs with quantum symmetries. It would be an interesting direction to try more quantum permutations and eventually use the algorithm to find quantum isomorphic non-isomorphic graphs.

Relations graph algorithm

1. Create an empty bipartite graph H_Q with one bipartition consisting of unordered pairs $V(X) \times V(X)$ and another containing unordered pairs $V(Y) \times V(Y)$. For example, if the first bipartition has a vertex $(u, v)_X$, the second bipartition also has a vertex $(u, v)_Y$.

2. For any two vertices $(u, v)_X$ and $(a, b)_Y$ in different bipartitions, connect $(u, v)_X$ to $(a, b)_Y$ if and only if $Q_{ua}Q_{vb} \neq 0$. This means that $(u, v)_X$ to $(a, b)_Y$ are required to be in the same relation in graph X and graph Y.

3. To get graphs for which Q is a quantum automorphism:

   (a) First connect components that contain vertices with the same labels with an edge. For example, if $(u, v)_X$ is in block B_i and $(u, v)_Y$ in block B_j, then form a new component by adding an edge $(u, v)_X$ and $(u, v)_Y$, say. Suppose now $k \leq p$ distinct complete bipartite blocks remain.

(b) Declare all possible 2^k adjacency/non-adjacency rules for each B_i and get resulting 2^k graphs. For example, for each $(u,v)_X \in B_i$ such that B_i has adjacency rule, let $u \sim_X v$. If B_i had a non-adjacecny rule, let each $(u,v)_X$ mean $u \nsim_X v$.

4. To get pairs of graphs for which Q is a quantum isomorphism (which includes automorphisms):

   (a) Assign one to a block if we decide to declare pairs vertices in the bipartite graph to represent edges, and zero otherwise. In this way, declare all possible 2^p adjacency/non-adjacency rules for each B_i. For example, for each $(u,v)_X \in B_i$ such that B_i has adjacency rule, let $u \sim_X v$. If B_i had a non-adjacency rule, let all each $(u,v)_X, (a,b)_Y \in B_i$ imply $u \nsim_X v$ and $a \nsim_Y b$.

A quick note about the second step. Only the partition into components of the graph H_Q matters, rather than what subgraph each component induces. For if there is a path of length two between vertices $(u,v)_X$ and $(u',v')_X$, i.e $(u,v)_X$ and $(u',v')_X$ have a common neighbour, then all vertices along this path have to be in the same relation. In the connected component of H_Q there is a path between any two vertices, so all vertices in one component will have to be in the same relation.

Theorem 6.6.4 *The relations algorithm produces all possible pairs of graphs for which an input quantum permutation is a quantum isomorphism.*

Proof. Towards a contradiction, suppose there are quantum isomorphic graphs X and Y with Q as their quantum isomorphism, that are not output by this algorithm. Then it must be that there are vertices i,j of X, and a,b of Y, such that $(i,j)_X$ and $(a,b)_Y$ are in different relations in the corresponding graphs, but are in the same component of the Relations algorithm bipartite graph. Then it must hold that

$$Q_{iu}Q_{jv} \neq 0,$$

but this contradicts the fact that Q is the quantum isomorphism between X and Y. Therefore, there always will be an assignment of zeroes and ones in the relations graph that will result into graphs X and Y.

On the other hand, suppose the algorithm finds graphs X and Y to be quantum isomorphic with Q. The only condition graphs X and Y have to satisfy is that whenever entries of $Q_{xy}Q_{x'y'} \neq 0$, it must hold that $\mathrm{rel}_X(x,x') = \mathrm{rel}_Y(y,y')$. This holds by construction of X and Y. □

In the case when there are no two distinct graphs for which Q is an isomorphism and the relations graph algorithm returns only graphs for which Q is a quantum automorphism, we see that the components of the bipartite graph constructed in the step one of the algorithm are orbitals of Q. This is precisely from the fact that the Relations Graph Algorithm produces all possible graphs for which Q is a quantum automorphism. However, we do not have a clear description of what the components of this bipartite graph represent for the graphs for which Q is a quantum isomorphism. This question motivates our next section.

6.7 The coherent algebra $\mathcal{M}_Q$ and its relevance to quantum isomorphisms

This section is motivated by the question, given a quantum permutation Q, what are the components of the relations graph. We study the set of matrices $\mathcal{M}_Q$ such that for every $M \in \mathcal{M}_Q$, there exists an $X \in M_n(\mathcal{C})$ such that

$$Q(M \otimes I_d)Q^* = (X \otimes I_d).$$

Clearly, if A and B are quantum isomorphic with Q, and $Q(A \otimes I_d)Q^* = B \otimes I_d$, then $A \in \mathcal{M}_Q$. It turns out that, $\mathcal{M}_Q$ is a coherent algebra.

Using this generalization, in the next section we will be ready to explore the analogue of the classical result, Lemma 6.3.3, about homogeneity of orbital algebras.

So far, we have learned that $n \times n$ matrices that commute with a quantum permutation $Q \in M_n(M_d(\mathbb{C}))$ form a coherent algebra of Q. Certainly, the adjacency matrices of the graphs for which Q is a quantum automorphism lie in this coherent algebra. Thus, given a quantum permutation, one could hope to find graphs for which Q is a quantum automorphism by searching for binary symmetric matrices with zero diagonal in the commutant of Q. If Q had non-commutative entries, such an approach would find us graphs with quantum symmetry.

Finding quantum isomorphic non-isomorphic graphs is a difficult task. One way to find such graphs would be to start with a quantum permutation that has non-commutative entries and apply the relations algorithm to it in hope to find pairs of graphs for which this quantum permutation is a quantum isomorphism, and later check if these graphs are isomorphic or no.

However, given a quantum permutation Q, if we would like to understand if there are any pairs of graphs at all for which Q is a quantum isomorphism, and not just an automorphism, we won't be able to make use of the commutant coherent algebra, since adjacency matrices of two graphs will be different, and there is no use to consider commutativity. So, we suggest an alternative below.

Theorem 6.7.1 *Let Q be an $n \times n$ quantum permutation and let $\mathcal{M} \subseteq M_n(\mathcal{C})$ denote the set of matrices such that for every $M \in \mathcal{M}$, there exists an $X \in M_n(\mathcal{C})$ such that*

$$Q(M \otimes I_d)Q^* = (X \otimes I_d).$$

Then $\mathcal{M}$ and $Q\mathcal{M}Q^*$ are isomorphic coherent algebras.

Proof. First, we will show that $\mathcal{M}$ is a coherent algebra. Then $Q\mathcal{M}Q^*$ is a coherent algebra as well since it contains matrices such that conjugation by Q results in an $n \times n$ matrix tensored with I_d. From the proof we will see that conjugation by Q respects multiplication and Schur-multiplication identities as well as those two operations, so it automatically becomes the isomorphism between those two coherent algebras.

Clearly, both the multiplicative and Schur-multiplicative identities are in $\mathcal{M}$.

Now we'll demonstrate transpose-closure. Suppose $M \in \mathcal{M}$. So there is an X such that

$$Q(M \otimes I_d)Q^* = (X \otimes I_d).$$

Observe that,

$$Q(M^* \otimes I_d)Q^* = (Q(M \otimes I_d)Q^*)^* = (X \otimes I_d)^* = X^* \otimes I_d,$$

implying that $M^* \in \mathcal{M}_Q$ as well.

Also, if $M, N \in \mathcal{M}$, then there are X, Y such that

$$Q(M \otimes I_d)Q^* = (X \otimes I_d)$$

and

$$Q(N \otimes I_d)Q^* = (Y \otimes I_d).$$

So,

$$Q(MN \otimes I_d)Q^* = Q(M \otimes I_d)Q^*Q(N \otimes I_d)Q = (X \otimes I_d)(Y \otimes I_d) = (XY \otimes I_d).$$

From this it follows that $MN \in \mathcal{M}$.

It remains to show that $M \circ N \in \mathcal{M}$ as well.

Note that

$$\begin{aligned}
[Q((M \circ N) \otimes I_d)]_{ab} &= \sum_k Q_{ak} M_{kb} N_{kb} \\
&= \sum_{k,r} Q_{ak} Q_{ar} M_{kb} N_{rb} \\
&= \sum_k Q_{ak} M_{kb} \sum_r Q_{ar} N_{rb} \\
&= [Q(M \otimes I_d)]_{ab} [Q(N \otimes I_d)]_{ab} \\
&= [(Q(M \otimes I_d)) \circ (Q(N \otimes I_d))]_{ab}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
[((M \circ N) \otimes I_d)Q]_{ab} &= \sum_k M_{ak} N_{ak} Q_{kb} \\
&= \sum_{k,r} M_{ak} N_{ar} Q_{kb} Q_{rb} \\
&= \sum_k M_{ak} Q_{kb} \sum_r N_{ar} Q_{rb} \\
&= [(M \otimes I_d)Q]_{ab} [(N \otimes I_d)Q]_{ab} \\
&= [((M \otimes I_d)Q) \circ ((N \otimes I_d)Q)]_{ab}.
\end{aligned}$$

Therefore, from the first equality,

$$Q((M \circ N) \otimes I_d) = (Q(M \otimes I_d)) \circ (Q(N \otimes I_d)) = ((X \otimes I_d)Q) \circ ((Y \otimes I_d)Q).$$

Now, expanding using the second equality in reverse,

$$((X \otimes I_d)Q) \circ ((Y \otimes I_d)Q) = ((X \circ Y) \otimes I_d)Q.$$

As a result,

$$Q((M \circ N) \otimes I_d)Q^* = (X \circ Y) \otimes I_d,$$

which confirms that $M \circ N \in \mathcal{M}$. □

If $\mathcal{O}_Q$ denotes the orbital algebra of Q and contains all $n \times n$ matrices that commute with Q, we have

$$\mathcal{O}_Q \subseteq \mathcal{M}_Q$$

At the same time,

$$\mathcal{O}_Q \subseteq Q\mathcal{M}_Q Q^*.$$

Certain questions for future research arise:

- When does $\mathcal{O}_Q = \mathcal{M}_Q$, i.e., there are no distinct graphs between which Q is a quantum isomorphism.
- Is it true that $\mathcal{M}_Q \cap Q\mathcal{M}_Q Q^* = \mathcal{O}_Q$?
- Are the bipartitions of the relations graph orbits of $\mathcal{M}_Q$ and $Q\mathcal{M}_Q Q^*$?

Remark 6.7.2 Suppose Q is a quantum permutation and

$$A_1, ..., A_k, A_{k+1}, ..., A_d$$

form a basis for $\mathcal{M}_Q$. Then since conjugation by Q is an isomorphism between coherent algebras $\mathcal{M}_Q$ and $Q\mathcal{M}_Q Q^*$, each QA_iQ^* is a binary matrix. Moreover, each QA_iQ^* contains as many ones as does A_i, which follows from

$$\langle J, QA_iQ^* \rangle = \langle Q^* JQ, A_i \rangle = \langle J, A_i \rangle.$$

Similarly, the traces of QA_iQ^* and A_i the same. □

6.8 Homogeneity of the coherent algebra that is the commutant of a quantum permutation

In this section we expand on this idea of homogeneity of commutants of permutations to commutants of quantum permutations.

Theorem 6.8.1 *If the quantum automorphism group Q acting on vertices of a graph X has only one orbit, then orbital algebra of Q is homogeneous.*

Proof. The orbits of Q partition the diagonal orbital of Q. Since there is only one orbit, there must be an orbital equal to identity. □

Mančinska and Roberson explore transitivity of the quantum automorphism group based on the entries of a magic unitary representing the quantum automorphism group of a graph. They leave the proof to the reader, but it uses the theory of Hopf algebras, and we do not include it here.

Theorem 6.8.2 *[40] Let X be a graph. Let $\mathcal{Q}$ be a quantum automorphism group acting on the vertex set $V(X)$, and let u be the magic unitary representing the quantum automorphism group. The following are equivalent:*

(a) $\mathcal{Q}$ acts transitively on $V(X)$.

(b) $\mathcal{Q}$ has only one orbit on $V(X)$.

(c) there exists $x \in V(X)$ such that $u_{xy} \neq 0$ for all $y \in V(X)$.

(d) $u_{xy} \neq 0$ for all $x, y \in X$.

Below we provide a simplified proof of the fact if a magic unitary has a row of non-zero projections, then its orbital algebra is homogeneous.

Proposition 6.8.3 *Let u be the magic unitary representing the quantum automorphism group of X. Then if there is a quantum automorphism Q with finite dimensional projections as entries such that $Q_{ab} \neq 0$, then $u_{ab} \neq 0$.*

Proof. There is a *-homomorphism from the quantum automorphism group to its finite dimensional representation, mapping u_{ij} to Q_{ij}. Since there are a and b such that $Q_{ab} \neq 0$, then $u_{ab} \neq 0$. □

Proposition 6.8.4 *If $Q \in M_n(M_d(\mathbb{C}))$ contains a row of only rank-one projections, then $\mathcal{M}_Q$ is homogeneous.*

Proof. From Theorem 6.7.1 we learned that $\mathcal{M}_Q$ is a coherent algebra. Let $A_1, ..., A_d$ be a unique orthogonal 01-basis for $\mathcal{M}_Q$. Since $\mathcal{M}_Q$ is a coherent algebra, its unique 01-basis contains matrices summing to identity that each have ones on the diagonal only. Let A_1 represent such diagonal (not necessarily identity) basis element. Then, choose an i with $1 \leq i \leq n$ such that $(A_1)_{ii} = 1$. Finally, $E_{ii} = e_i e_i^T$ and first observe that

$$(Q^*(E_{ii} \otimes I_d)Q)_{ab} = \sum_{k,r} Q_{ka}(E_{ii})_{kr} Q_{rb} = Q_{ia} Q_{ib} = \begin{cases} Q_{ia} \text{ if } a = b \\ 0 \text{ otherwise} \end{cases}$$

Now calculate:

$$\begin{aligned} \langle E_{ii} \otimes I_d, Q(A_1 \otimes I_d)Q^* \rangle &= \langle Q^*(E_{ii} \otimes I_d)Q, A_1 \otimes I_d \rangle \\ &= \langle \text{diag}(\text{row}_i(Q)), A_1 \otimes I_d \rangle. \qquad (*) \end{aligned}$$

The last equality follows from the aforementioned calculation.

Since A_1 is a basis element of coherent algebra $\mathcal{M}_Q$, and conjugation by Q is an isomorphism of coherent algebras, it maps 01-basis to 01-basis, so $Q(A_1 \otimes I_d)Q^*$ is a binary basis matrix of the coherent algebra $Q\mathcal{M}_Q Q^*$. So, either

$$\langle E_{ii} \otimes I_d, Q(A_1 \otimes I_d)Q^* \rangle = 0$$

or

$$\langle E_{ii} \otimes I_d, Q(A_1 \otimes I_d)Q^* \rangle = \mathrm{tr} I_d = d.$$

We will consider those two cases separately, and will show that the first case cannot occur.

In the latter case,

$$d = \mathrm{tr} I_d = \langle \mathrm{diag}(\mathrm{row}_i(Q)), A_1 \otimes I_d \rangle,$$

In the latter case, we want to show that it must be that A_1 is the identity I_n. Since $J \succeq 0$ and for any $0 \preceq D \preceq I$, it holds that

$$D - I \preceq 0,$$

$$-(D - I) \succeq 0,$$

$$0 \leq \langle J, -(D - I) = -\langle J, (D - I) \rangle$$

$$\langle J, D - I \rangle \leq 0,$$

$$\langle J, D \rangle \leq \langle J, I \rangle.$$

Using the fact that sum of the entries in $\mathrm{row}_i(Q)$ is a projection and

$$\sum_j Q_{ij} \preceq I,$$

sum of the entries in $\mathrm{diag}(\mathrm{row}_i(Q))$ is at most

$$\langle J, \sum_j Q_{ij} \rangle \leq \langle J, I_d \rangle = d,$$

so it must be that A_1 selects all the nonzero projections from the i^{th} row of Q for each i. In other words, if $(A_1)_{ii} \neq 0$, then $(A_1)_{jj} \neq 0$, for every j such that $Q_{ij} \neq 0$. In particular, if the i^{th} row of Q consists of rank one projections, then $(A_1)_{jj} \neq 0$ for all $1 \leq j \leq n$, so $\mathcal{M}_Q$ is homogeneous.

In the former case, we have

$$\langle \mathrm{diag}(\mathrm{row}_i(Q)), A_1 \otimes I_d \rangle = 0.$$

Note that the eigenvalues of the block-diagonal matrix $\mathrm{diag}(\mathrm{row}_i(Q))$ are the eigenvalues of the blocks. Since blocks are projections, $\mathrm{diag}(\mathrm{row}_i(Q))$ is a positive-semidefinite matrix. At the same time, $A_1 \otimes I_d$ is positive semidefinite. Therefore, we have

$$\mathrm{diag}(\mathrm{row}_i(Q))(A_1 \otimes I_d) = 0.$$

Since both matrices are block-diagonal, for the product to be zero, A_1 has to "select" only zero projections from $\mathrm{row}_i(Q)$. Formally speaking, if $Q_{ij} \neq 0$, then $A_1 = 0$. However, if the i^{th} row of Q consists only of rank-one (i.e. nonzero) projections, then it would imply that $A_1 = 0$, which is a contradiction. So this case does not occur. □

Corollary 6.8.5 *If $Q \in M_n(M_d(\mathbb{C}))$ contains at least one row of only rank-one projections, then commutant of Q is a homogeneous coherent algebra.*

Proof. Immediately follows from the previous result. The commutant of Q is a coherent subalgebra of $\mathcal{Q}$, therefore, it also must be homogeneous. □

Corollary 6.8.6 *If a graph X has quantum automorphism $Q \in M_n(M_d(\mathbb{C}))$, such Q has a row of nonzero projections, then representation u of its quantum automorphism group has a row of nonzero entries.*

Proof. Follows from Corollary 6.8.6 and Proposition 6.8.3. □

Below are some of my results about vertex transitivity and quantum automorphisms.

Lemma 6.8.7 *Suppose X is a graph such that for fixed i and j, it holds that $Q_{ij} = 0$ for every quantum automorphism Q. Let S be the orbit of j under the classical automorphism group of X, and R the set of pre-images of i:*

$$R = \{r \in \{1..., n\} |\ \textit{there is an automorphism } P \textit{ of } X \textit{ such that } P_{ir} = 1\}.$$

Then for every $r \in R$, the entry $Q_{rj} = 0$ and every $s \in S$, the entry $Q_{is} = 0$ for every automorphism Q.

Proof. Suppose there is a quantum automorphism $Q' \in M_{|V(X)|}(M_d(\mathbb{C}))$ of X such that $Q'_{rj} \neq 0$. Suppose that P is an automorphism with $P_{ir} = 1$. It is easy to see that $(P \otimes I_d)Q'$ is a quantum automorphism of X as well. Thus, by assumption, $(PQ')_{ij} = 0$. However,

$$0 = (PQ')_{ij} = \sum_k P_{ik} Q'_{kj} = P_{ir} Q'_{rj} \neq 0,$$

a contradiction. The second half of the statement follows analogously. □

Corollary 6.8.8 (Follows from [2]) *If X is vertex-transitive, then there is no $1 \leq i, j \leq n$ such that for every quantum automorphism Q, it holds that $Q_{ij} = 0$.*

Proof. Since X is vertex-transitive, there is an automorphism that maps any vertex to i. Hence, if there were i, j such that $Q_{ij} = 0$ for every quantum automorphism X, by previous lemma we would have that the j^{th} column of each quantum automorphism is 0, a contradiction. □

The last corollary implies that in the vertex-transitive graph $u_{xy} \neq 0$ for all $x, y \in V(X)$, i.e. condition (d) holds. Which shows that every vertex-transitive graph is quantum vertex-transitive. It was shown before in [40].

Corollary 6.8.9 *If X is vertex-transitive, X is quantum vertex-transitive.*

6.9 A necessary and sufficient condition for the commutativity of the quantum orbital algebra

Now, we would like to search for some condition analogous to generous transitivity in the classical case (Section 6.4).

To begin, suppose u is a magic unitary representing the quantum automorphism group of a graph X and for all $a, b \in V(X)$, it holds that $u_{ab}u_{ba} \neq 0$. Then (a, b) is in an orbital of commutant of u if and only if (b, a) is the same orbital. And so, just as in the classical case, the orbital algebra will be symmetric and therefore commutative.

Theorem 6.9.1 *Let u be a magic unitary defining the quantum automorphism group of a graph X. Then if $u_{ab}u_{ba} \neq 0$ for all $a, b \in V(X)$, the commutant of u is a commutative coherent algebra.*

Proof. We have observed earlier that the commutant of u is a coherent algebra. Now, from Theorem 6.6, we know that the basis of this algebra is formed by the adjacency matrices of the equivalence relation 6.6.1.

First observe, that if a coherent algebra is symmetric, it is commutative. This follows from the facts that a coherent algebra is closed under multiplication and that the product of two symmetric matrices is symmetric if and only if they commute.

Now, we will show that $u_{ab}u_{ba} \neq 0$ for all $a, b \in V(X)$ if and only if the orbital algebra is symmetric. We see that if for every $a, b \in V(X)$

$$u_{ab}u_{ba} \neq 0.$$

then every orbital has nonzero ab-entry if and only if it has a nonzero ba-entry. Hence, every orbital is symmetric. □

6.10 Quantum isomorphism preserves spectrum and coherent algebras

Since a quantum isomorphism Q is an invertible matrix, and since for two graphs X and Y to be quantum isomorphic, we require that

$$Q(A \otimes I)Q^* = B \otimes I,$$

it is clear that quantum isomorphic graphs are cospectral. First this fact was explored in [2]. This section is, mostly devoted to showing that quantum isomorphisms preserve coherent algebras of quantum isomorphic graphs. Originally, this fact was proven by Atserias et al in the same [2]. However, the explicit isomorphism was not provided. Here we prove that the first natural guess, conjugation by the quantum isomorphism matrix is indeed the isomorphism of the coherent algebras as well.

Let $\mathcal{C}_X, \mathcal{C}_Y$ denote the coherent algebras of X and Y respectively. We will show that if Q is a quantum isomorphism between the graphs X and Y, then

$$Q(\mathcal{C}_X \otimes I)Q^* = (\mathcal{C}_Y \otimes I).$$

That is to say, we slightly abuse notation, and when we say conjugation by Q is an isomorphism of coherent algebras, it is up to tensoring with identity.

We say two coherent algebras $\mathcal{A}, \mathcal{B}$ are *isomorphic* if there exists an algebra isomorphism $h : \mathcal{A} \to \mathcal{B}$, that preserves algebra multiplication, commutes with *, preserves J and Schur product:

(a) $h(AB) = h(A)h(B)$

(b) $h(J) = J$

(c) $h(A^*) = (h(A))^*$

(d) $h(A \circ B) = h(A) \circ h(B)$

We are going to follow the proof of [46] to show that two quantum isomorphic graphs have isomorphic coherent algebras as a corollary of the following result.

Theorem 6.10.1 *[46] Suppose X and Y are quantum isomorphic graphs with adjacency matrices A and B. Then there exists an isomorphism Φ of their quantum orbital algebras such that $\Phi(A) = B$.*

Proof. Remember that the orbitals of a graph H with the magic unitary u representing the quantum automorphism of H are the equivalence classes of the relation on $V(H) \times V(H)$

$$(h, h') \sim_2 (g, g') \text{ if } u_{hg}u_{h'g'} \neq 0.$$

The *quantum orbital algebra* of a graph is a coherent algebra with 01-basis matrices being the characteristic matrices of the above-mentioned equivalence classes. In other words, the 01-basis for the quantum orbital coherent algebra consists of adjacency matrices of the quantum orbitals of the graph. When verifying conditions for the isomorphism of coherent algebras of quantum orbital algebras of X and Y, it suffices to check those conditions just for the basis, the characteristic matrices of the orbitals.

Let $V = (v_{xy})$ be a magic unitary witnessing quantum isomorphism between X and Y. So we know that

- $\sum_{x \in V(X)} v_{xy} = \sum_{y \in V(Y)} v_{xy} = 1$.
- $v_{xy} = v_{xy}^* = v_{xy}^2$.
- $v_{xy}v_{x'y'} = 0$ whenever $x \sim_X x', y \nsim_Y y'$ or $x \nsim_X x', y \sim_Y y'$.

The proof will proceed by first defining a bipartite graph H with bipartition: $V(X) \times V(X)$ and $V(Y) \times V(Y)$. The adjacency relation is given by $(x, x') \sim_H (y, y')$ if $v_{xy}v_{x'y'} \neq 0$. Then we will find a perfect matching in W, which will give a bijection between orbitals of X and the orbitals of Y. What will remain to show is that this bijection is in fact a coherent algebra isomorphism.

First, we will construct a quantum automorphism of X using the quantum isomorphism and the magic unitary defining the quantum automorphism group of Y. Later it will help us to verify Hall's condition for the existence of the perfect matching in the bipartite graph H.

Let $U = (u_{yy'})$ for $y, y' \in V(Y)$ be the magic unitary defining the quantum automorphism group of Y. We will show that the matrix $w = (w_{xx'})$ for $x, x' \in V(X)$ defined by

$$w = v * u * v^*$$

is the magic unitary representing quantum isomorphism of X. The entries of w are

$$w_{xx'} = \sum_{y,y' \ inV(Y)} v_{xy} \otimes u_{yy'} \otimes v_{x'y'}.$$

Since entries of u and v are Hermitian,

$$w_{xx'} = w^*_{xx'}.$$

Now, using the orthogonality of entries of magic unitaries in the same row/column we arrive at:

$$w^2_{xx'} = \sum_{y,y',d,d' \in V(Y)} v_{xy}v_{xd} \otimes u_{yy'}u_{dd'} \otimes v_{x'y'}v_{x'd'} = \sum_{y,y' \in V(Y)} v_{xy} \otimes u_{yy'} \otimes v_{x'y'} = w_{xx'}.$$

Next,

$$\begin{aligned}
\sum_x w_{xx'} &= \sum_{x \in V(X), y,y' \ inV(Y)} v_{xy} \otimes u_{yy'} \otimes v_{x'y'} \\
&= \sum_{y,y' \ inV(Y)} 1 \otimes u_{yy'} \otimes v_{x'y'} \\
&= \sum_{y' \ inV(Y)} 1 \otimes 1 \otimes v_{x'y'} \\
&= 1 \otimes 1 \otimes 1 \\
&= 1.
\end{aligned}$$

A similar argument shows that $\sum_{x'} w_{xx'} = 1$, concluding the proof that w is a magic unitary.

To show that w commutes with A, we will show that

$$w_{ab}w_{a'b'} = 0 \text{ whenever } a \sim_X b, a' \nsim_X b$$

and vice versa. We will just show the former statement, the proof of the latter is analogous. Assume thus, that $a \sim_X a', b \nsim_X b'$.

$$w_{ab}w_{a'b'} = \sum_{y,y',d,d' \in V(Y)} v_{ay}v_{a'd} \otimes u_{yy'}u_{dd'} \otimes v_{by'}v_{b'd'} = 0$$

Since, for a fixed, y, y', d', d, for the summands in the sum to be nonzero we require that

(a) $v_{ay}v_{a'd} \neq 0$, so $y \sim_Y d$, and

(b) $v_{by'}v_{b'd'} \neq 0$, so $y \nsim_Y d'$,

but this inevitably forces $u_{yy'}u_{dd'} = 0$. Therefore, $wA = Aw$, and w is a quantum automorphism of X.

Now we will show that in the bipartite graph H, elements of the same quantum orbital of quantum automorphism group of Y are adjacent to the elements of $V(X) \times V(X)$, which belong to the same quantum orbital of quantum automorphism group of X. For this, let (y_1, y_1') and (y_2, y_2') be in the same quantum orbital of Y. In other words,

$$u_{y_1y_2}u_{y_1'y_2'} \neq 0.$$

Suppose the edges in H include $(y_1, y_1') \sim (x_1, x_1')$ and $(y_2, y_2') \sim (x_2, x_2')$. This means that

$$v_{x_1y_1}v_{x_1'y_1'} \neq 0 \text{ and } v_{x_2y_2}v_{x_2'y_2'} \neq 0.$$

As we are aiming to show that (x_1, x_1') and (x_2, x_2') are in the same orbit of X, equivalently, we have to demonstrate that $w_{x_1x_2}w_{x_1'x_2'} \neq 0$.

$$w_{x_1x_2}w_{x_1'x_2'} = \sum_{y,y',d,d' \in V(Y)} v_{x_1y}v_{x_2d} \otimes u_{yy'}u_{dd'} \otimes v_{x_2y'}v_{x_2d'}.$$

So,

$$\begin{aligned}
&(v_{x_1y_1} \otimes 1 \otimes v_{x_2y_2})(w_{x_1x_2}w_{x_1'x_2'})(v_{x_1'y_1'} \otimes 1 \otimes v'_{x_2'y_2}) \\
&\qquad = v_{x_1y_1}v_{x_1'y_1'} \otimes w_{x_1x_2}w_{x_1'x_2'} \otimes v_{x_2y_2}v_{x_2'y_2'} \\
&\qquad \neq 0
\end{aligned}$$

from the previous conclusions. Thus, we have shown that in W, elements of one quantum orbital of Y are adjacent to the elements of another quantum orbital of Y. Analogously, it can be shown that elements of one quantum orbital of X are adjacent to the elements in one quantum orbital of Y.

Now if we show that every element of the quantum orbital of Y has at least one neighbour, we will have that for any $S \subseteq V(Y) \times V(Y)$, it holds that $|S| \leq |N_G(S)|$. If we then show the Hall's condition for the part of bipartition $V(X) \times V(X)$, we will have a perfect matching in W, that defines a bijection between quantum orbitals of Y and quantum orbitals of X. Hall's condition is immediate, since

$$\sum_{y_1,y_2} v_{xy_1} v_{x'y_2} = \sum_{x_1,x_2} v_{x_1y} v_{x_2y'} = 1.$$

So, if there were a vertex $(y, y') \in V(H)$ not adjacent to any $(x_1, x_2) \in V(H)$, every term in the second sum would have been zero. Conversely, if there were an $(x, x') \in V(H)$ not adjacent to any $(y_1, y_2) \in V(H)$, the first sum would have been zero.

Thus, the graph H has a perfect matching. Vertices in any quantum orbit of Y are matched to some vertices in the same quantum orbit of X. So there is a bijection f between quantum orbits of Y and quantum orbits of X, obtained by perfectly matching representatives of quantum orbits of Y to representatives of quantum orbits of X.

We must verify that this bijection is an isomorphism of coherent algebras. We will show this, relying on the isomorphism of the corresponding coherent configurations that maps intersection numbers in the appropriate way.

The two coherent configurations consist of orbital relations $\{R_i\}$ of X and $\{R'_i\}$ of Y respectively. The first coherent configuration has coherent coefficients $\{p_{ij}^k, i, j, k \in \mathcal{I}\}$ and the second one has $\{q_{ij}^k, i, j, k \in \mathcal{I}'.\}$ We want to show that

$$p_{ij}^k = q_{ij}^{f(k)} \text{ for all } i, j, k \in \mathcal{I}.$$

Suppose (x, x') is in some quantum orbital of X, equivalently, $(x, x') \in R_k$ for some $k \in \mathcal{I}$. Let $i, j \in \mathcal{I}$. Let

$$S := \{r \in V(X) : (x, r) \in R_i, (r, x') \in R_j\}$$

By definition of p_{ij}^k, $|S| = p_{ij}^k$. As established earlier, (x, x') will have at least one neighbour in H, so there is at least one (y, y') such that $v_{xy}v_{x'y'} \neq 0$, and by the choice of f, we have that (y, y') is in the orbit $R'_{f(k)}$.

Now define a set encoding coefficients $q_{ij}^{f(k)}$:

$$S' = \{z \in V(Y) : (y, z) \in R'_{f(i)}, (z, y') \in R'_{f(j)}\},$$

so $|S| = q_{ij}^{f(k)}$. We will use the following calculation to show that $p_{ij}^k = q_{ij}^{f(k)}$.

$$\begin{aligned}
p_{ij}^k v_{xy} v_{x'y'} &= v_{xy} \sum_{r\in S} 1 v_{x'y'} \\
&= v_{xy} \sum_{r\in S, z\in V(Y)} v_{wz} 1 v_{x'y'} \\
&= v_{xy} \sum_{r\in S, z\in S'} v_{wz} v_{x'y'} \text{ since } xy \in R_i \text{ so } yz \text{ must be in } R_{f(i)}, \text{ etc} \\
&= v_{xy} \sum_{r\in V(X), z\in S'} v_{wz} v_{x'y'} \\
&= q_{ij}^{f(k)} v_{xy} v_{x'y'}
\end{aligned}$$

Since $v_{xy}v_{x'y'} \neq 0$ by virtue of being in matched, hence adjacent, quantum orbitals, $p_{ij}^k = q_{ij}^{f(k)}$.

Going back to coherent algebras, let A_i and B_i be characteristic matrices of the relations R_i and R_i' respectively. Let Φ be a map

$$\Phi(A_i) = B_{f(i)},$$

extended linearly. Then Φ is an isomorphism of the quantum orbital coherent algebras. $\Phi(A_X) = \Phi(A_Y)$, since

$$v_{x,x'}v_{yy'} = 0$$

when $\mathrm{rel}(x,x') \neq \mathrm{rel}(y,y')$, so f maps indices of entries of edges of X to the indices of entries of edges of Y and the same for nonedges. □

Corollary 6.10.2 *Suppose Q is a quantum isomorphism between X and Y with adjacency matrices A and B. If $\mathcal{O}_X, \mathcal{O}_Y$ denote orbital algebras of X and Y respectively, then*

$$Q(\mathcal{O}_X \otimes I)Q^* = (\mathcal{O}_Y \otimes I)$$

Proof. Denote by $A_1, ..., A_k$ the 01-basis for $\mathcal{O}(X)$ and by $B_1, ..., B_k'$ the 01-basis for $\mathcal{O}(Y)$. It suffices to show that conjugation by Q maps basis matrix A_i, to some B_j. Because then Q will respect Schur product between basis matrices, and thus respect Schur product between all other matrices in the coherent algebras. The fact that multiplication is preserved, that $J \otimes I$ is mapped to itself and invertibility of Q are obvious. We will see that $j = f(i)$, where f is a bijection from the previous proof.

We know that each A_i is a characteristic matrix of an orbital of $\mathcal{O}(X)$. Indeed, each $A_i \otimes I$ commutes with the magic unitary U_X defining quantum automorphism group of X, and so is constant on the orbitals of U_X. Since A_i is a basis matrix, it cannot be a characteristic matrix of more than one orbit. Similarly, $B_i's$ are characteristic matrices of the orbitals of $\mathcal{O}(Y)$.

Let A_i be a characteristic matrix of the orbital in $\mathcal{O}(X)$. So $(A_i)_{uv} = 1$ if u, v are in the i^{th} orbitals of $\mathcal{O}(X)$.

With f being the bijection defined in the previous proof, we learned from the previous proof that if (a, b) in in the i^{th} orbital of X and $Q_{a'c'}Q_{b'd'} \neq 0$, then (c, d) and (c', d') are in the same orbital of Y, and this orbital is $f(i)^{th}$. Now, suppose (a, b) in in the i^{th} orbital of X.

$$\begin{aligned} I &= \sum_{c} Q_{ac} \\ &= \sum_{c,d} Q_{ac}Q_{bd} \\ &= \sum_{c,d:Q_{ac}Q_{bd}\neq 0} Q_{ac}Q_{bd} \\ &= \sum_{(c,d)\in(f(i))^{th} \text{ orbital of } Y} Q_{ac}Q_{bd} \\ &= \sum_{c,d} Q_{ac}(B_{f(i)})_{c,d}Q^*_{db} \\ &= (Q(B_{f(i)} \otimes I)Q^*)_{ab} \end{aligned}$$

At the same time, from the previous proof we know that if (c, d) and (c', d') are in the j^{th} orbital of Y, and $Q_{a'c}Q_{b'd} \neq 0$, and $Q_{a'c'}Q_{b'd'} \neq 0$, then (a, b) in in the $(f^{-1}(j))^{th}$ orbital of X.

Thus, if $a' = b'$, $f(i) = j, i = f(j)$.

Now suppose (a, b) are not in the i^{th} orbital of X, then

$$\begin{aligned} (Q(B_{f(i)} \otimes I)Q^*)_{ab} &= \sum_{c,d} Q_{ac}(B_{f(i)})_{c,d}Q^*_{db} \\ &= \sum_{(c,d)\in(f(i))^{th} \text{ orbital of } Y} Q_{ac}Q_{bd} \\ &= 0, \end{aligned}$$

since (a, b) in not in the i^{th} orbital of X. $\square$

Lemma 6.10.3 *Suppose, X and Y are graphs with orbital algebras $\mathcal{O}_X$ and $\mathcal{O}_Y$. Suppose, Q is a quantum permutation satisfying*

$$Q(\mathcal{O}_X \otimes I)Q^* = (\mathcal{O}_Y \otimes I).$$

Then Q a quantum isomorphism between X and Y?

Corollary 6.10.4 ([40]) *Suppose X and Y are quantum isomorphic graphs with adjacency matrices A and B. Then there exists an isomorphism Φ of their coherent algebras such that $\Phi(A) = B$.*

Proof. From the previous theorem, there exists an isomorphism Φ such that $\Phi(A) = \Phi(B)$ of their orbital algebras. Now, orbital algebra of X, which consists of matrices that commute with the magic unitary representing $\mathrm{Qut}(X)$, naturally contains the adjacency matrix A. Similarly, orbital algebra of Y contains the adjacency matrix B. Φ maps coherent algebra of X to another coherent algebra containing $\Phi(A) = B$. Similarly, Φ^{-1} maps coherent algebra of Y to a coherent algebra containing $\Phi^{-1}(B) = A$. Thus, it must be that Φ maps coherent algebra of A to the coherent algebra of B. □

Corollary 6.10.5 *Suppose Q is a quantum isomorphism between the graphs X and Y. If $\mathcal{C}_X, \mathcal{C}_Y$ denote coherent algebras of X and Y respectively, then*

$$Q(\mathcal{C}_X \otimes I)Q^* = (\mathcal{C}_Y \otimes I)$$

Proof. From the fact that $\mathcal{C}_X \otimes I \subseteq \mathcal{O}_X$ and $\mathcal{C}_Y \otimes I \subseteq \mathcal{O}_Y$, the basis elements of $\mathcal{C}_X \otimes I$ are either elements or unions of elements of basis elements of $\mathcal{O}_X$. Hence, by Corollary 6.10.2 the result follows. □

6.11 When does Godsil-McKay switching preserve coherent algebras?

We have learned that when Q is a quantum isomorphism between two graphs X and Y with coherent algebras $\mathcal{C}_X$ and $\mathcal{C}_Y$, then conjugation by Q maps $\mathcal{C}_X \otimes I$ to $\mathcal{C}_Y \otimes I$. Recall that many of the known examples of quantum isomorphic nonisomorphic graphs are related by Godsil-McKay switching. Therefore, great candidates for quantum isomorphic graphs would be the ones that are related by Godsil-McKay switching and the ones with a priori isomorphic coherent algebras. In the case of relatedness by Godsil-McKay switching we ensure cospectrality of the graphs, and their likelihood of being non-isomorphic. At the same time, if coherent algebras are isomorphic with a matrix S, then $S \otimes I$ will map $\mathcal{C}_X \otimes I$ to $\mathcal{C}_Y \otimes I$.

In this section we explore this question. Specifically, when given two graphs with adjacency matrices A and B, coherent algebras $\mathcal{C}_A$ and $\mathcal{C}_B$ and a switching matrix S such that $SAS = B$, we would like to see when for all M, N in the coherent algebra, say, $\mathcal{C}_A$ the reflection matrix S from Godsil-McKay switching satisfies:

- $S(MN)S = (SMS)(SNS)$
- $SJS = J$
- $S(M^T)S = (SMS)^T$
- $S(M \circ N)S = SMS \circ SNS$.

If those conditions are satisfied, then both $S\mathcal{C}_A S$ and $S\mathcal{C}_B S$ are coherent algebras. We'll show that then $S\mathcal{C}_A S = \mathcal{C}_B$. Indeed, on one hand $S\mathcal{C}_A S$ is a coherent algebra containing B, so

$$\mathcal{C}_B \subseteq S\mathcal{C}_A S.$$

On the other hand, $S\mathcal{C}_B S$ is a coherent algebra containing A, so

$$\mathcal{C}_A \subseteq S\mathcal{C}_B S.$$

Conjugating both sides by S, we get

$$S\mathcal{C}_A S \subseteq \mathcal{C}_B.$$

Hence,

$$S\mathcal{C}_A S = \mathcal{C}_B.$$

Now, the first three properties of conjugation by S are satisfied for all M, N. Also, it is sufficient to prove those four properties hold for the basis of one of, say, $\mathcal{C}_A$, since addition is distributive over the Schur product.

Below we will explore when conjugation by the switching matrix S preserves coherent algebras of the switching-related graphs. The next lemma provides a necessary condition on preservation of Schur product.

Lemma 6.11.1 *Suppose S is the switching matrix for Godsil-McKay partition $(C_1, ..., C_k, D)$. If M, N are 01-matrices such that*

$$S(M \circ N)S = SMS \circ SNS,$$

then the following hold. Below always $j \in D$ and $i, r \leq k$.

(a) *If there is a column j and entries ij, rj with such that*

$$\begin{aligned} M_{ij} = 1, N_{ij} = 1, &\ \textit{but} \\ M_{rj} = 1, N_{rj} = 0, & \end{aligned}$$

then the j^{th} column of M is $\mathbf{1}$.

(b) *If there is a column j and entries ij, rj such that*

$$\begin{aligned} M_{ij} = 0, N_{ij} = 0, &\ \textit{but} \\ M_{rj} = 1, N_{rj} = 0, & \end{aligned}$$

then the whole j^{th} column of N is $\mathbf{0}$.

(c) *If there is a column j and entries ij, rj such that*

$$\begin{aligned} M_{ij} = 1, N_{ij} = 0, &\ \textit{but} \\ M_{rj} = 0, N_{rj} = 1, & \end{aligned}$$

then the j^{th} column of M is a difference of the all ones vector and the j^{th} column of N.

(d) *If there is a column j and entries ij, rj such that*

$$\begin{aligned} M_{ij} = 1, N_{ij} = 1, &\ \textit{and} \\ M_{rj} = 0, N_{rj} = 0, & \end{aligned}$$

then the j^{th} column of M must be equal to the j^{th} column of N and have $m/2$ ones.

The results are symmetric, if we switch M and N.

Proof. Partition M, N according to the block-partition of S. Then we have for the ab^{th} block of the product:

$$(S(M \circ N)S)_{ab} = (SMS \circ SNS)_{ab},$$
$$S_{aa}(M_{ab} \circ N_{ab})S_{bb} = S_{aa}M_{ab}S_{bb} \circ S_{aa}N_{ab}S_{bb}.$$

Consider ab blocks with $a \leq k$ and $b = k+1$. In this case, $S_{bb} = I$ and from the previous equation we obtain:

$$\left(\frac{2}{m}J_m - I_m\right)(M_{ab} \circ N_{ab}) = \left(\frac{2}{m}J_m - I_m\right)M_{ab} \circ \left(\frac{2}{m}J_m - I_m\right)N_{ab}.$$

Now let $R := M_{ab}$ and $T := N_{ab}$ and consider i, j entries of both sides:

$$\frac{2}{m}\sum_k R_{kj}T_{kj} - R_{ij}T_{ij} = \left(\frac{2}{m}\sum_k R_{kj} - R_{ij}\right) \circ \left(\frac{2}{m}\sum_k T_{kj} - T_{ij}\right),$$
$$\frac{2}{m}\sum_k R_{kj}T_{kj} - R_{ij}T_{ij} = \frac{4}{m^2}\sum_k R_{kj} \circ \sum_k T_{kj} + R_{ij} \circ T_{ij} - \frac{2}{m}T_{ij}\sum_k R_{kj} - \frac{2}{m}R_{ij}\sum_k T_{kj},$$
$$\sum_k R_{kj}T_{kj} - \frac{2}{m}\sum_k R_{kj} \circ \sum_k T_{kj} + T_{ij}\sum_k R_{kj} + R_{ij}\sum_k T_{kj} = mR_{kj}T_{kj}.$$

1. In the first case we have if $R_{ij} = 1, T_{ij} = 1$, then the equation becomes

$$\sum_k R_{kj}T_{kj} - \frac{2}{m}\sum_k R_{kj} \circ \sum_k T_{kj} + \sum_k R_{kj} + \sum_k T_{kj} = m.$$

   Also, when $R_{rj} = 1, T_{rj} = 0$, we have

$$\sum_k R_{kj}T_{kj} + \sum_k T_{kj} = \frac{2}{m}\sum_k R_{kj} \circ \sum_k T_{kj},$$

   from which follows that

$$\sum_k R_{kj} = m,$$

   which only can happen when the j^{th} column of R is all one. Since our matrices are binary, $0 \leq \frac{2}{m}\sum_k R_{kj} \leq \frac{2}{m}$

2. In the second case, provided $R_{ij} = 0 = T_{ij}$, but $R_{rj} = 1$ and $T_{rj} = 0$:

$$\sum_k R_{kj}T_{kj} - \frac{2}{m}\sum_k R_{kj} \circ \sum_k T_{kj} = 0,$$

$$\sum_k R_{kj}T_{kj} - \frac{2}{m}\sum_k R_{kj} \circ \sum_k T_{kj} + \sum_k T_{kj} = 0,$$

so $\sum_k T_{kj} = 0$.

3. In the third case, provided $R_{ij} = 1, T_{ij} = 0$, but $R_{rj} = 0$ and $T_{rj} = 1$:

$$\sum_k R_{kj}T_{kj} - \frac{2}{m}\sum_k R_{kj} \circ \sum_k T_{kj} + \sum_k T_{kj} = 0,$$

$$\sum_k R_{kj}T_{kj} - \frac{2}{m}\sum_k R_{kj} \circ \sum_k T_{kj} + \sum_k R_{kj} = 0,$$

we conclude that

$$\sum_k R_{kj} + \sum_k T_{kj} = 0.$$

If it happened that some xj^{th} entries of R and T were the same we would have been in the second or first case. Thus, all entries of R and T must be different, so both have $m/2$ ones in complementary positions.

4. Finally, provided $R_{ij} = 1, T_{ij} = 1$, but $R_{rj} = 0$ and $T_{rj} = 0$:

$$\sum_k R_{kj}T_{kj} - \frac{2}{m}\sum_k R_{kj} \circ \sum_k T_{kj} + \sum_k R_{kj} + \sum_k T_{kj} = m,$$

$$\sum_k R_{kj}T_{kj} - \frac{2}{m}\sum_k R_{kj} \circ \sum_k T_{kj} = 0,$$

yielding

$$\sum_k R_{kj} + \sum_k T_{kj} = m.$$

If R and T have different xj^{th} entries, we would be in the 1st or 2nd case, or in the third case. Therefore, the j^{th} columns of R and T are identical with $m/2$ ones. □

Below we found the conditions to determine whether the switching matrix is an isomorphism of coherent algebras. The conditions are applied to the basis matrices of the coherent algebra. Therefore, we will define Godsil-McKay partition for a binary matrix. It is the same as definition of Godsil-McKay partition of a directed non-simple graph.

We say a matrix A can be partitioned into a Godsil-McKay partition $(C_1, ..., C_k, D)$ if it is a square matrix and after partitioning rows and columns into $(C_1, ..., C_k, D)$, we have that

(a) the block A_{ij} has constant row and column sums for $1 \leq i, j \leq k$,

(b) Each column of $A_{i(k+1)}$ is either all-zero, all-one or have half of the entries zero and half of the entries one for all $i \leq k$.

(c) $A_{(k+1)i}$ has rows that are either all-zero, all-one or have half of the entries zero and half of the entries one for all $i \leq k$.

Theorem 6.11.2 *Suppose Y is a graph with adjacency matrix B and coherent algebra $\mathcal{C}(Y)$. Suppose X is a graph with adjacency matrix A and coherent algebra $\mathcal{C}(X)$ that has* 01*-basis $B_1, ..., B_k$. Suppose X is related to Y by Godsil-McKay switching with partition $(C_1, ..., C_\ell, D)$ and S is in the definition of Godsil-McKay switching with $SA = BS$. Then if for each $1 \leq i \leq k$ we have that B_i can be partitioned into Godsil-McKay partition $(C_1, ..., C_\ell, D)$, then conjugation by S is an isomorphism of coherent algebras $\mathcal{C}(X)$ and $\mathcal{C}(Y)$.*

Proof. Conjugation by S maps J to J, preserves multiplication and transpositions. It remains to show that it preserves Schur multiplication. However, since every matrix in $\mathcal{A}$ can be written in terms of B_i, and conjugation by S is distributive over addition, it only requires us to verify that

$$S(B_i \circ B_j)S = SB_iS \circ SB_jS,$$

equivalently, that

$$0 = SB_iS \circ SB_jS.$$

Note, that now it is sufficient to show that SB_iS is a binary matrix for each $1 \leq i \leq k$. Since then we would have

$$\sum_{i=1}^{k} SB_iS = SJS = J.$$

As binary matrices that sum to J, all SB_iS would have to be pairwise Schur-orthogonal.

Now, let $1 \leq a \leq k$ be arbitrary, and consider SB_aS. We can view both S and B_a as a block matrix according to the vertex partition $(C_1, ..., C_\ell, D)$.

Now, since $(C_1, ..., C_\ell, D)$ is a Godsil-McKay partition for each B_a, each ij-block $(B_a)_{ij}$ with $1 \leq i, j \leq k$ has constant row an column sums. Let the row sums of B_a be r_i and let c_i denote the column sums. Also, because of this partition, each column of each block $(B_a)_{i(k+1)}$ with $1 \leq i \leq k$ is one of three types: a column of zeroes or a column of ones or a column with $\frac{m}{2}$ ones (and $\frac{m}{2}$ zeroes). Therefore, depending on which part of the partition do the rows and columns of a block come from, we may conclude what each block of SB_aS is.

Case 1: $1 \leq i, j \leq k$.

$$(SB_aS)_{ij} = B_a,$$

since B_a has constant row and column sums.

Case 2: $1 \leq i \leq k$.

$$(SB_aS)_{i(k+1)} = S_{ii}B_a,$$

is binary, since S_{ii} preserves $\mathbf{0}, \mathbf{1}$ columns and switches zeros and ones in vectors with $\frac{m}{2}$ ones.

Case 3:

$$(SB_aS)_{(k+1)(k+1)} = B_a.$$

Thus we have demonstrated that for an arbitrary $1 \leq a \leq k$, the matrix SB_aS is binary. And since $\sum_{a=1}^{k} B_a = J$, conjugation by S maps basis of $\mathcal{C}(X)$ to the basis of $S\mathcal{C}S$. At the same time, $S\mathcal{C}(X)S$ contains $\mathcal{C}(Y)$. □

While it is well-known that two strongly regular graphs with the same parameters are cospectral, we provide another proof of the following statement.

Corollary 6.11.3 *If two strongly regular graphs X and Y are related by Godsil-McKay switching with matrix that satisfies $SA_X = A_YS$, then conjugation by S is an isomorphism of their coherent algebras.*

Proof. Since A^2 can be written in terms of A and I, the basis of the coherent algebra of a strongly regular graph is $A, I, J - I - A$. If A satisfies the conditions for Godsil-McKay partition, so does $J - I - A$, and so does I. □

Now we prove an "almost" reverse direction of Theorem 6.11.2 for the case when the Godsil-McKay partition only has two parts.

Theorem 6.11.4 *Suppose graphs X and Y have adjacency matrices A_X and A_Y. Let $\mathcal{C}_X$ be a coherent algebra of X with unique Schur-orthogonal binary basis $A_1, ..., A_s$. Let $\mathcal{C}_Y$ be a coherent algebra of Y with unique ordered Schur-orthogonal binary basis $B_1, ..., B_t$. Suppose X and Y are related by Godsil-McKay switching with partition (C, D). Let S be the Godsil-McKay switching matrix so that*

$$SA_X = A_Y S.$$

Suppose the following conditions hold:

(a) X and Y are connected with connected complements.

(b) The coherent algebra of at least one of X or Y is homogeneous.

Then if conjugation by S is an isomorphism between $\mathcal{C}(A)$ and $\mathcal{C}(B)$, then for all

$$1 \leq a, b \leq k,$$

the ab^{th} blocks of $A_i's$ (and so B_i's too) have the same row and columns sum.

Proof. Suppose $R := A_i$ is a basis element in the decomposition of the adjacency matrix A_X. This means that if $A_{ab} = 0$, then $R_{ab} = 0$ as well. We have for the $\{ab\}^{th}$ block, where $1 \leq a, b \leq k$:

$$[S(A_X \circ R)S]_{ab} = (SAQS)_{ab} \circ (SRS)_{ab}$$
$$S_{aa}(A_X \circ R)_{ab}S_{bb} = S_{aa}(A_X)_{ab}S_{bb} \circ S_{aa}R_{ab}S_{bb}$$
$$S_{aa}(A_X \circ R)_{ab}S_{bb} = (A_X)_{ab} \circ S_{aa}R_{ab}S_{bb},$$

since A_X can be partitioned into a Godsil-McKay partition by hypothesis. Now let $M := (A_X)_{ab}$ and $N := R_{ab}$. Rewriting we get:

$$Q_{aa}(M \circ N)Q_{bb} = M \circ Q_{aa}NQ_{bb}.$$

Since R is in the Schur decomposition of A, $R \circ A = R$, and so $M \circ N = N$ as well:

$$S_{aa}NS_{bb} = M \circ S_{aa}NS_{bb}$$
$$\left(\frac{2}{m}J_m - I_m\right) N \left(\frac{2}{m}J_m - I_m\right) = M \circ \left(\frac{2}{m}J_m - I_m\right) N \left(\frac{2}{m}J_m - I_m\right)$$
$$\frac{4}{m^2}JNJ - \frac{2}{m}JN - \frac{2}{m}NJ + N = M \circ (\frac{4}{m^2}JNJ - \frac{2}{m}JN - \frac{2}{m}NJ + N)$$

Now, compare ij entries of the left and right hand sides. When $M_{ij} = 1$ we have no conditions on M and N since the equation trivially holds. Now, when $M_{ij} = 0$, then we need

$$\frac{4}{m^2}\left(\sum_{q,p} N_{q,p}\right) - \frac{2}{m}\sum_k N_{kj} - \frac{2}{m}\sum_k N_{ik} + N_{ij} = 0.$$

Before we mentioned that since $M_{ij} = 0$, it must be that $N_{ij} = 0$ as well. Thus, for every i, j such that $M_{ij} = 0$, we obtain by rearranging terms and multiplying by $\frac{m}{2}$:

$$\frac{2}{m}\left(\sum_{q,p} N_{q,p}\right) = \sum_k N_{kj} + \sum_k N_{ik}. \qquad (*)$$

Since there is no vertex of degree $n-1$, for every i, there is a column j with $A_{ij} = A_{ji} = 0$, so also

$$\frac{2}{m}\left(\sum_{q,p} N_{q,p}\right) = \sum_k N_{ki} + \sum_k N_{jk}. \qquad (**)$$

Since diagonal of A is 0, the above equation holds for all $1 \leq i \leq n$:

$$\frac{2}{m}\left(\sum_{q,p} N_{q,p}\right) = \sum_k N_{ki} + \sum_k N_{ik}. \qquad (***)$$

Subtracting $(**) - (***)$,

$$\sum_k N_{jk} = \sum_k N_{ik},$$

so sum of entries in the i^{th} and j^{th} rows is equal.

Subtracting $(*) - (***)$,

$$\sum_k N_{kj} = \sum_k N_{ki},$$

so the sum of entries in the i^{th} and j^{th} columns are equal. Since X^c is connected, there is a path between any two vertices v and u in X^c. Say the path is

$$\{i := v_1, j := v_2\}, \{v_2, v_3\}, ..., \{v_{p-1}, v_p =: u.\}$$

Then $A_{ij} = 0, A_{jv_3} = 0, ..., A_{v_{p-1}u} = 0$. So the sum of the entries in the $i^{th}, j^{th,...,u^{th}}$ columns and rows are all equal. Thus, R has constant row and columns sums in blocks under cells $C_1, ..., C_k$.

Now, if $R' := A_{i'}$ is a basis element of $\mathcal{C}(X)$ that is not in the decomposition of A_X, then R' is in the decomposition of $J - A$. However, $\mathcal{C}(X)$ is homogeneous, and we may assume $R' \neq I$, so R' is in the decomposition of $J - I - A = A_{X^c}$.

Now, A^c can also be partition into Godsil-McKay partition (C, D) and X is connected, so all the above arguments apply. So, for $1 \leq a, b \leq k$, each $\{ab\}^{th}$-block of R' has constant row and columns sums too. □

From Theorem 6.11.2 we know that if basis matrices of the coherent algebra of one of the graphs can be partitioned according to a Godsil-McKay partition, then conjugation by the Godsil-McKay switching matrix S will be the isomorphism of coherent algebras of those two graphs related by switching with S as the Godsil-McKay swithcing matrix. Here we produce an almost reverse result.

Corollary 6.11.5 *Suppose X and Y are related by Godsil-McKay switching with the Godsil-McKay partition (C, D) and Godsil-McKay switching matrix S. Suppose that conjugation by S is the isomorphism of coherent algebras. Suppose further that X or Y (or both) satisfies:*

(a) each vertex in D is adjacent to $\frac{|C|}{2}$ vertices.

(b) is connected with connected complements.

(c) has a homogeneous coherent algebra.

Then the basis matrices of the coherent algebra satisfying the third condition can be partitioned according to a Godsil-McKay partition.

Proof. Without loss of generality, out of the two graphs it is X that satisfies these three conditions. Let $\mathcal{C}(X)$ denote the coherent algebra of X and lets its Schur-orthogonal basis matrices be $A_1, ..., A_r$.

The coherent algebra is homogeneous, so it contains I in the basis. First, note that this I basis element can be partitioned into Godsil-McKay partition.

Next, by Theorem 6.11.4, blocks of other basis elements $A_i's$ corresponding to cell C have constant row and column sums.

Now, we will show that for $1 \leq a \leq k$, it is is true that the $(a(k+1))^{st}$ blocks of non-multiplicative-identity basis elements $A_i's$ have half ones and half zeroes in columns. We will appeal to Theorem 6.11.1.

Because of the first condition in the hypothesis, the first two cases of Theorem 6.11.1 can never be satisfied. We can easily ensure ourselves of it. First, suppose A_i is a Schur basis element in the decomposition of A_X. Let a be arbitrary and let $M := (A_i)_{a(k+1)}$ be a block of A_i and $N := (A_X)_{a(k+1)}$. So, each column of N has equal number of zeros and ones. A similar argument applies for any A_j in the decomposition of A_{X^c}. By Theorem 6.11.1 we are in either third or the fourth case of it, guaranteeing that for $1 \leq a \leq k$, it holds that $(a(k+1))^{st}$ blocks of other basis elements $A_i's$ corresponding to the cell D have half ones and half zeroes in columns.

Analogously, for $1 \leq a \leq k$, the $((k+1)a)^{st}$ blocks of other basis elements $A_i's$ corresponding to the cell D have equal number of ones and zeroes in rows.

All of the above shows that basis matrices of coherent algebra of X can be partitioned into Godsil-McKay partition (C, D). □

Chapter 7

Root systems

In this chapter we present our result that orthogonality graphs of root systems B, C, D in a vector space of dimension being a power of two have quantum symmetry. We achieve this by generalizing the construction of quantum isomorphisms between strongly regular graphs from [65]. We also learn how root systems are relevant to the known twenty-four vertex non-isomorphic quantum isomorphic graphs from [2].

7.1 Background on root systems

In this section we provide a brief introduction to root systems. More details can be found in [36]. A subset Φ of the Euclidean space E is called a *root system* in E if the following axioms are satisfied:

(a) Φ is finite, spans E and does not contain 0.

(b) If $\alpha \in \Phi$, the only multiples of α in Φ are $\pm\alpha$.

(c) If $\alpha, \beta \in \Phi$, then $\beta - \frac{2\langle \beta,\alpha \rangle}{\langle \alpha,\alpha \rangle} \in \Phi$.

(d) If $\alpha, \beta \in \Phi$, then $\frac{2\langle \beta,\alpha \rangle}{\langle \alpha,\alpha \rangle} \in \mathbb{Z}$.

Vectors in a root system Φ are called *roots*. Now, let us introduce the known root systems. To specify the dimension of the vector space in which the vectors of the root systems lie, we us the notion of *rank of a root system*.

Suppose Φ is a root system with roots $\{\alpha_i\}_i$. There is a notion of a *dual root system* Φ' to a root system Φ, which contains roots $\{\alpha_i'\}_i$ defined as

$$\alpha_i' = \frac{2}{\langle \alpha_i, \alpha_i \rangle} \alpha_i.$$

We call a root system *irreducible* if Φ cannot be partitioned into two nonzero proper orthogonal subsets. Turns out it is possible to classify all irreducible root systems. Below we list them all. By E we will denote the underlying vector space for each of the root systems.

1. For A_n, the underlying vector space E is the subspace of $\mathbb{R}^{n+1}$ in which coordinates of vectors sum to zero. The roots are the set of integer vectors in E with Euclidean norm $\sqrt{2}$. In other words, these are all possible vectors in $\mathbb{R}^{n+1}$ with two nonzero coordinates from the set $\{1, -1\}$. The total number of roots is $n^2 + n$.
2. For B_n, the underlying vector space $E = \mathbb{R}^n$. The roots are all integer vectors in E of length 1 or $\sqrt{2}$. So $A_{n-1} \subset B_n$. The total number of roots is $2n^2$.
3. For C_n, the underlying vector space is also $E = \mathbb{R}^n$. The roots are all integer vectors in E of length $\sqrt{2}$ together with

$$\{2v : v \in \mathbb{Z}^n, ||v|| = 1\}$$

   So $A_{n-1} \subset C_n$ as well. The total number of roots is $2n^2$.
4. For D_n, the underlying vector space is also $E = \mathbb{R}^n$. The roots are all integer vectors in E of length $\sqrt{2}$. So $D_n \subset C_n$. The total number of roots is $2n(n-1)$.

Below are the isolated root systems that do not form families.

1. For standard basis vectors $\mathbf{e_1}, ..., \mathbf{e_8} \in \mathbb{R}^8$, we have

$$E_8 = D_8 \cup \left\{ \frac{1}{2} \left(\sum_{i=1}^{8} a_i \mathbf{e_i} \right) : a_i \in \{\pm 1\}, \prod_{i=1}^{8} a_i = 1 \right\}$$

2. E_7 can be viewed as a seven-dimensional subspace of $\mathbb{R}^8$ consisting of all permutations of $(1, -1, 0, 0, 0, 0, 0, 0)$ and all permutations of

$$(1/2, 1/2, /1, 1/2, 1/2, -1/2, -1/2, -1/2, -1/2).$$

   Thus there are $8 \times 7 + \binom{8}{4} = 126$ vectors.

3. E_6 can be viewed a subset of E_8, characterized by fixing three coordinates to be equal.

4. For the standard basis vectors $\mathbf{e_1}, ..., \mathbf{e_4} \in \mathbb{R}^4$, we have

$$F_4 = \{\pm\mathbf{e_i}\}_{1\leq i\leq 4} \cup \{\mathbf{e_i} \pm \mathbf{e_j}\}_{1\leq i,j\leq 4} \cup \{\pm\mathbf{e_1} \pm \mathbf{e_2} \pm \mathbf{e_3} \pm \mathbf{e_4}\}.$$

   There are eight vectors in the first subset, twenty four in the second and sixteen in the third.

Given a root system Φ, by a *line* in the root system, we mean a span of a root α. So roots α and $-\alpha$ lie on the same line. We denote the line containing those roots as $\{\alpha, -\alpha\} \in \Phi \times \Phi$. Therefore, the number of lines in the root system is half of the number of all vectors.

7.2 Schmidt's example

In this section we introduce background on how root systems appeared in the study of quantum isomorphisms.

In 2022 in [65], Schmidt proved that the orthogonality graph of the lines in the E_8 root system is quantum isomorphic to a strongly regular graph on 120 vertices, but not isomorphic to it.

Earlier, in 2016, in [2], Atserias et al found a pair of non-isomorphic quantum isomorphic graphs on twenty four vertices. The quantum isomorphism game strategy relied on Mermin-Peres magic square game strategy. Based on this quantum isomorphism game, in the earlier section, we constructed a quantum isomorphism matrix. After studying the Schmidt's root systems example, we decided to check if these twenty-four vertex graphs are isomorphic to an orthogonality graph of lines of some root system with twenty four lines. We discovered that one of the two graphs is an orthogonality graph of lines in the F_4 root system. Moreover, the construction of the quantum isomorphism matrix that Schmidt used in [65] applies also in this case, and results in the same matrix as we have constructed originally. These observations have inspired us to think whether we could consider orthogonality graphs of lines in other root systems. This lead to a generalization of Schmidt's method.

In this section we investigate if other root systems give us any promising information about quantum isomorphisms or automorphisms. We will see that nonorthogonality graphs of D_{2^s}, C_{2^s} and B_{2^s} root systems have quantum automorphisms for any $s \geq 2$. Systems C_{2^s} and B_{2^s} are dual to each other, and vectors in D_{2^s} are a subset of vectors in C_{2^s}. So we will start with C_{2^s}.

For an arbitrary $d \geq 2, d \in \mathbb{N}$, vectors in C_d are given by

$$\{\pm 2e_k\}_{1\leq k\leq d} \cup \{\pm e_i \pm e_j\}_{1\leq i\neq j\leq d}.$$

In this section we restrict attention to d being a power of two and write $d = 2^s$. Let $G^{D_{2^s}}$ denote the non-orthogonality graph of lines in D_{2^s}. For this we identify vectors x and $-x$ into one vertex. For convenience, we fix the labelling of the vertices as $\{e_i \pm e_j\}_{1\leq i\neq j\leq k} \cup \{2e_i\}_{1\leq i\leq d}$. We call this set $\mathcal{V}_d$.

7.3 Pauli group

In this section we include some background about the Pauli group. The Pauli operators are commonly denoted as

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The Pauli group is the group generated by the Pauli matrices. It is easy to see that $X^2 = Y^2 = I$, and so the elements can be listed as

$$\{\pm I, \pm iI, \pm X, \pm iX, \pm Y, \pm iY, \pm Z, \pm iZ\}.$$

Quotioning this group by $\{\pm I, \pm iI\}$, we get a quotient group with representatives

$$\mathcal{P}_2 = \{I, X, Z, XZ\}.$$

This is called a dephased Pauli group. We can also take n-fold tensor products of $\{I, X, Z, XZ\}$ to get tensor products of dephased Pauli groups, which we denote $\mathcal{P}^{\otimes n}$.

$$(\mathcal{P})^{\otimes n} := \underbrace{\mathcal{P} \otimes \mathcal{P} \otimes \ldots \otimes \mathcal{P}}_{n \text{ times}}$$

7.4 A short overview of stabilizers

We define an s-qubit state to be a vector $v \in \mathbb{R}^{2^s}$ with Euclidean norm one $||v|| = 1$. We follow the convention to omit the normalization factor for the ease of perception.

For example, a basis vector $e_0 = [1, 0, ..., 0] \in \mathbb{R}^{2^s}$ is denoted by $|00...0\rangle$ with s zeroes in ket notation and is an s-qubit state. Now, an s-qubit GHZ state is $e_0 + e_d$, where $e_0, e_d \in \mathbb{R}^{2^s}$ are basis vectors. In bra-ket notation, GHZ state is of the form

$$\underbrace{|0...0\rangle}_{s \text{ times}} + \underbrace{|1...1\rangle}_{s \text{ times}}.$$

There is a well-known procedure for constructing GHZ states using repeated applications of the following two gates:

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \text{ and } CNOT = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

CNOT is a two-qubit gate, and one can see that CNOT applies X-gate conditioned on the fact that the first qubit is 1. An illustration of how to construct a CNOT gate is given below. One of two qubits that CNOT acts on, the being-conditioned-on-qubit, is depicted as a black dot.

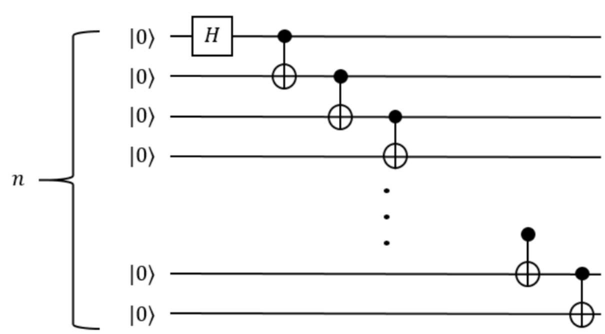

Figure 7.1: Construction of a GHZ state [3]

Gottesman and Knill [31] showed that for certain types of quantum circuits known as stabilizer circuits, efficient simulation on classical computers is possible. Stabilizer circuits are exclusively composed of stabilizer gates, which include Hadamard and CNOT gates, followed by measurements in the computational basis. Such circuits are applied to a computational basis state $|0...0\rangle$ and produce output states called *stabilizer states*. (This paragraph is from [24].)

An equivalent definition is as follows. A *stabilizer state* is a state of an s-qubit system that is a simultaneous eigenvector of a commutative subgroup of $\mathcal{P}^s$. Clearly, GHZ state is an example of a stabilizer state.

Another well-known and important fact about stabilizer states is that the size of the stabilizer of an s-qubit stabilizer state is 2^s. Using this fact we prove the lemma below. But first, let us consider an example.

For illustration, suppose we have the state $|00000\rangle + |11111\rangle$, and we need to construct the state $|00000\rangle + |10000\rangle$ from the GHZ state by applying CNOTs and Hadamards. We identify which qubit is to remain unchanged and CNOT to the remaining ones. In our case, we apply CNOT to the 2nd, 3rd, etc qubits, conditioned on the first qubit (since this system is to remain unchanged). We straightforwardly formalize this procedure in the lemma.

Lemma 7.4.1 *Let $e_1 \pm e_j \in \mathbb{R}^{2^s}$ be two s-qubit states. The size of their stabilizers is 2^s.*

Proof. We are going to demonstrate how to construct s-qubit states $e_i \pm e_j$ from a s-qubit GHZ state. This will prove that the sizes of stabilizers of $e_i \pm e_j$ is also 2^s. To obtain any of $e_1 \pm e_j$, we just choose the qubit which is to remain unchanged in the GHZ state and whose reduced system is $|0\rangle + |1\rangle$. Then we apply CNOT conditioned on this qubit to all the qubits, that are to be changed, as illustrated in the example above. □

7.5 Action of the dephased Pauli group on the vertices of an orthogonality graph

We are going to understand the orbits of lines of a root system under the action of the dephased Pauli group. Some parts of this proof will benefit from the theory of stabilizer states, which we include here.

Lemma 7.5.1 *The group $\mathcal{P}^{\otimes s}$ of s-tensors of Paulis $\{M_1 \otimes \cdots \otimes M_s\}_{M_i \in \{X,Y,Z,I\}}$, acts on the set of vectors in $\mathcal{V}_{2^s}$ and partitions $\mathcal{V}_{2^s}$ into 2^s orbits of size 2^s.*

Proof. As in Schmidt's [65], each generalized Pauli is a signed permutation matrix flipping even number of signs. Applying to vectors of the form $\{e_i\}$, we will get either e_j, or $-e_j$ which still span a line in the root system. Similarly, applying it to $\{\pm e_i \pm e_j\}$ we will get vectors of the form $\pm\{\pm e_a \pm e_b\}$, spanning lines in the root system. So, $\mathcal{P}^{\otimes s}(V_{2^s}) \subseteq V_{2^s}$.

Now we will show that one of the orbits under the action of $\mathcal{P}^{\otimes s}$ is the standard basis orbit consisting of $\{e_1, ..., e_{2^s}\}$. It is very easy to see by writing each e_i in ket-notation. The ket-notation for e_i will contain some zeroes and ones. Therefore, we are able to get any e_i from e_1 by applying the appropriate tensor product of X's and $I's$ to e_1. At the same time, permuting entries of $e_i's$ and changing sign in some of them, never results in a vector of the form $\pm e_i \pm e_j$, so one of the orbits of the $\mathcal{L}$-action is the standard basis orbit.

Now, we will categorize orbits that contain lines represented by vectors of the form $e_i + e_j$ and $e_i - e_j$. First, observe that $e_i + e_j$ and $e_i - e_j$ are in the same orbit. For convenience, rewrite these vectors in bra-ket notation and apply generalized Pauli with only one Z in the location where the binary representation of e_i and e_j do not agree. Such location exists, since $i \neq j$. In this way, we obtain either $e_i - e_j$ or $-e_i + e_j$, both on the same line.

Next, we will show that $e_i + e_j$ and $e_i + e_k$ are in different orbits if $j \neq k$. Suppose not, and there is a $P \in \mathcal{P}^{\otimes s}$, such that

$$Pe_i + Pe_j = P(e_i + e_j) = e_i + e_k.$$

From earlier in the proof, any even-signed permutation maps a standard basis vector to a standard basis vector. Thus, $Pe_i = e_a, Pe_j = e_b$ such that $e_a + e_b = e_i + e_k$. The only mutually exclusive options are

- $Pe_i = e_i$ and $Pe_j = e_k$,
- $Pe_i = e_k$ and $Pe_j = e_i$.

For the first case, to fix e_i, Paulis can only contain Z-matrices acting on the 0's in the binary representation of e_i. Such Paulis, however, are not able to change e_j into e_k, where besides change of sign, change in binary notation is required. The flip in the binary representation can only be achieved with Paulis containing X's. Any such Pauli will not fix e_i.

The second case is not less trivial. Suppose $Pe_i = e_k$. However, if we also require that $Pe_j = e_i$, by self-inverse property of dephased Paulis and multiplying by P on both sides, we see that $e_j = Pe_i$. So, P maps e_i to e_j and e_k, implying that $j = k$.

So, the second case also can't happen. Therefore, we have confirmed that $e_i + e_j$ and $e_i + e_k$ are in different orbits if $j \neq k$..

We observed that if $e_i - e_j$ is in the orbit under the action of $\mathcal{P}^{\otimes s}$, then $e_i + e_j$ is also in the same orbit. Additionally, $e_i - e_j$ is in the orbit together with $e_1 - e_j$, via the application of X gate on the appropriate qubit. Thus, for $j \neq k$, to prove that $e_i + e_j$ and $e_i + e_k$ are in different orbits, it is enough to show that $e_1 \pm e_2, ..., e_1 \pm e_{2^s}$ will all be in $2^s - 1$ different orbits. From above, $e_i + e_j$ and $e_i + e_k$ are in different orbits for $j \neq k$. Thus, $e_1 + e_2, ..., e_1 + 2_{2^s}$ are in $2^s - 1$ different orbits.

Moreover, by Lemma 7.4.1, $e_1 \pm e_2, ..., e_1 \pm e_d$ are stabilizer states and the size of the stabilizer of each of $e_1 \pm e_2$ states is 2^s. Now, the size of orbit of each of those is, by the orbit-stabilizer theorem, the size of the acting group divided by the size of the stabilizer:

$$\frac{4^s}{2^s} = 2^s.$$

Thus, together with the standard basis orbit, we have identified 2^s orbits, each of size 2^s. □

Denote the orbits by $V_1, ..., V_{2^s}$, and let us choose a set $\{w_1, ..., w_{2^s}\}$ of representatives from each orbit. Let V_{2^s} be the orbit containing standard basis vectors. Now we are ready to define a quantum permutation which we will later show to be a quantum symmetry of the orthogonality graphs of line in C_{2^s}.

Definition 7.5.2 *We define the matrix $u_{(w,C^d)}$ to be a $d^2 \times d^2$ matrix of $d \times d$ complex projections. It is block-diagonal with blocks $u^{(1)}, ..., u^{(d)}$, each block having rows and columns indexed by vertices in the orbit V_i. For $a, b \in V_i$, let $M_{a \mapsto b}$ be a Pauli mapping a to b. It exists, since a and b are in one orbit. Then the ab-entry (located in the diagonal block corresponding to the i^{th} orbit) is*

$$u^{(i)}_{ab} = M_{a \mapsto b} w_i w_i^* M^*_{a \mapsto b}.$$

First, let us ensure the matrix above is a quantum permutation, as we shall do in the following lemma.

Lemma 7.5.3 *The matrix $u \in M_{d^2}(M_d(\mathbb{C}))$ defined above is a quantum permutation.*

Proof. We need to show that for each $1 \leq i \leq d$, the row and columns sums are identity:

$$\sum_{b \in V_i} u_{ab}^{(i)} = \sum_{a \in V_i} u_{ab}^{(i)} = I_d.$$

Since each $u_{ab}^{(i)}$ is a rank-one projection onto $M_{a \mapsto b} w_i$, it suffices to show that for each $b \neq b'$ in V_i, each $\langle M_{a \mapsto b} w_i, M_{a \mapsto b'} w_i \rangle = 0$. The same holds for $a, a' \in V_i$. We have proved earlier that each orbit contains d vertices, so d linearly independent projections would form a basis.

$$\begin{aligned} \langle M_{a \mapsto b} w_i, M_{a \mapsto b'} w_i \rangle &= w_i^* M_{a \mapsto b}^* M_{a \mapsto b'} w_i \\ &= w_i^* M_{b \mapsto a} M_{a \mapsto b'} w_i \\ &= w_i^* M_{b \mapsto b'} w_i. \end{aligned}$$

As long as $M_{b \mapsto b'} w_i \neq w_i$, we are done, since we know elements in each orbit are pairwise orthogonal. Suppose towards a contradiction that $M_{b \mapsto b'} w_i = w_i$. Let $M_{w_i \mapsto b'} w_i = b'$, equivalently, $M_{b' \mapsto w_i} b' = w_i$. Then

$$w_i = M_{b \mapsto b'} w_i = M_{b \mapsto b'} M_{b' \mapsto w_i} b' = M_{b \mapsto w_i} b'.$$

We get that

$$M_{b \mapsto w_i} b = M_{b \mapsto w_i} b'.$$

Since Pauli matrices are invertible, it follows that $b = b'$, a contradiction. A similar proof applies to show that columns sum to I_d. Hence, u is a quantum permutation. □

Before we will use this quantum permutation for constructing a quantum automorphism of the orthogonality graphs of lines in the said root system, let us first explore the properties of these graphs.

7.6 Structural properties of the orthogonality graph

From now on, for a vector u, we denote by $\mathrm{Stab}(u)$, the stabilizer of u under the action of $\mathcal{P}^{\otimes s}$.

Lemma 7.6.1 *If u and v are in the same orbit under $\mathcal{P}^{\otimes s}$, then*

$$Stab(u) = Stab(v).$$

Proof. Same as Schmidt's [65]. Suppose $T \in \text{Stab}(u)$. Since u, v are in the same orbit, there is a Pauli P, such that $Pu = v$. At the same time, because $P^2 = P$, $u = Pv$. Any two Paulis commute or anti-commute, so $Tv = TPTu = \pm Pu = \pm v$. However, v and $-v$ denote the same vertex, as they span the same line. Hence, $T \in \text{Stab}(v)$. Reverse containment is analogous. □

Recall that V_d is the orbit under $P^{\otimes s}$ on the lines in E_8 that contains standard basis vectors in $\mathbb{R}^{2^s}$.

Lemma 7.6.2 *For any $1 \leq i \neq j \leq d-1$ and any two orbits V_i, V_j of $\mathcal{V}_d$ under the action of $\mathcal{P}^{\otimes s}$, we have that each u in the i^{th} orbit is non-orthogonal to four lines in the j^{th} orbit.*

Proof. Let $e_i + e_j$ be an arbitrary vertex with nonnegative entries in the i^{th} orbit. Vertices of the j^{th} orbit are all of the form $e_a \pm e_b$. Note that $e_i + e_j$ is orthogonal to $e_a \pm e_b$ if and only if $i \notin \{a, b\}$ and $j \notin \{a, b\}$. In the previous proof we concluded that there are two vertices $e_i \pm e_x$ containing e_i in each orbit. Similarly, there are only two vertices $e_j \pm e_y$ containing e_j in each orbit. Thus, in total, there are four vertices nonorthogonal to $e_i + e_j$ in the j^{th} orbit. Similar argument applies for $e_i - e_j$. □

Lemma 7.6.3 *For any $1 \leq i \neq j \leq d$, we have that if $e_1 + e_i$ and $e_1 + e_j$ are in different orbits under $\mathcal{P}^{\otimes s}$, they are $+1$-eigenvectors for 2^{s-2} Paulis. In other words,*

$$|\,Stab(e_1 + e_i) \cap Stab(e_1 + e_j)| = 2^{s-2}.$$

Proof. For convenience, write vectors in bra-ket notation with s digits: zeroes and ones. Note that simultaneous stabilizers of $e_1 + e_i$ and $e_1 - e_j$ will be Paulis which are tensor products only of Z's and I's. This is because if a Pauli P contains X gates and fixes any state of the form $e_1 + e_i$, it must switch e_1 and e_i, so $Pe_1 = e_i, Pe_i = e_1$. Clearly, P will not switch e_1 and e_j and thus will not have the latter as a $+1$-eigenvector.

Also observe that such a Pauli P has $e_1 + e_i$ as an eigenvector if and only if $Pe_i = e_i$, since $Pe_1 = e_1$. So, in fact we need to calculate

$$|\,\text{Stab}(e_1 + e_i) \cap \text{Stab}(e_1 + e_j)| = |\,\text{Stab}(e_i) \cap \text{Stab}(e_j)|$$

We can represent each tensor product P_k of Z's and I's with a binary vector p_k of length s, with ones in positions of Z's in P_k. Note that $P_k e_i = e_i$ if and only if $\langle p_k, e_i \rangle = 0(\text{mod } 2)$.

Since e_i, e_j are linearly independent vectors, they span 2^s dimensional subspace of $\mathbb{Z}_2^s$. Therefore, they are orthogonal to the subspace generated by the remaining basis vectors generating $\mathbb{Z}_2^s$. There are $s-2$ such vectors, and so the subspace contains 2^{s-2} vectors orthogonal both to e_i and e_j at the same time. Translating those binary vectors into Z, I-containing Paulis, we obtain the result. □

Corollary 7.6.4 *For any $1 \leq i \neq j \leq d$, we have that if $e_i + e_j$ and $e_a + e_b$ are in different orbits under $\mathcal{P}^{\otimes s}$, they are $+1$-eigenvectors for 2^{s-2} Paulis. In other words,*

$$|\,Stab(e_1 + e_i) \cap Stab(e_1 + e_j)| = 2^{s-2}.$$

Proof. Suppose $e_i + e_j$ is in the obit with $e_1 + e_k$ and $e_a + e_b$ is in the obit with $e_1 + e_c$. Note that by Lemma 7.6.1, $\text{Stab}(e_i + e_j) = \text{Stab}(e_1 + e_k)$ and $\text{Stab}(e_a + e_b) = \text{Stab}(e_1 + e_c)$, from the previous lemma, the result follows. □

Lemma 7.6.5 *For $k, s \in V_i$ and $l, t \in V_j$, if $d(k,l) = d(s,t) = 1$, then there exists $M \in \mathcal{P}^{\otimes s}$ such that $M(k) = s$ and $M(l) = t$.*

Proof. Since k, s are in one orbit, there is P such that $Pv_k = v_s$, and $Pv_l = v_{t'}$. If $t' = t$, we are done. Otherwise, if we find a Pauli P' such that $P's = s$ and $P't' = t$, we would also be done with $M = P'P$.

First, suppose that $1 \leq i, j \leq d-1$. From Corollary 7.6.4, we have that

$$|\,\text{Stab}(s) \cap \text{Stab}(t')| = 2^{s-2}.$$

Let $\text{Stab}(s) \cap \text{Stab}(t')$ be $\{C_1, ..., C_{2^{s-2}}\}$ Now suppose $T_i, T_j \in \text{Stab}(s)$. Clearly, $T_j^{-1}T_i$ is also in $\text{Stab}(s)$. If $T_i t' = T_j t'$, it means that either $T_i = T_j$ or $T_i^{-1}T_j \in \text{Stab}(t') \cap \text{Stab}(s)$, say, $T_j = T_i C_k$.

Moreover, since s and t' are in different orbits,

$$\frac{|\,\text{Stab}(s)|}{|\,\text{Stab}(s) \cap \text{Stab}(t')|} = \frac{2^s}{2^{s-2}} = 4.$$

Therefore, elements of $\text{Stab}(s)$ map t' to four other neighbours of s in the orbit of t'. Since in total s has four neighbours, at least one of the Paulis in the stabilizer of s maps t' to t by pigeonhole principle.

Now, suppose that one of the orbits is V_d with standard basis vectors. Without loss of generality, $i = d$, while $1 \leq j \leq d-1$.

Case 1:

$$k = e_i, \quad l = e_i + e_j,$$
$$s = e_j, \quad t = e_i + e_j.$$

There is a Pauli M such that $Me_i = e_j$. Moreover, multiplying by M on both sides, $e_i = Me_j$, so $M(e_i + e_j) = e_i + e_j$, and we are done.

Case 2:

$$k = e_i, \quad l = e_i + e_j,$$
$$s = e_j, \quad t = e_i - e_j.$$

There is a Pauli M such that $Me_i = e_j$. Multiplying by M on both sides, $e_i = Me_j$, so $M(e_i+e_j) = e_i+e_j$, and we are done. Now, apply Pauli M' with a Z in a coordinate that is one in e_j and zero in e_i (such coordinate exists since $e_i \neq e_j$), to get $M'M(e_i+e_j) = e_i - e_j$ or $M'M(e_i + e_j) = -e_i + e_j$. Moreover, $M'M(e_i) = \pm e_j$ still.

Case 3:

$$k = e_i, \quad l = e_i + e_j,$$
$$s = e_i, \quad t = e_i - e_j.$$

Apply Pauli M' with a Z in a coordinate that is one in e_j and zero in e_i (such coordinate exists since $e_i \neq e_j$), to get $M'M(e_i + e_j) = e_i - e_j$ or $M'M(e_i + e_j) = -e_i + e_j$. Moreover, $M'M(e_i) = \pm e_i$ still.

Case 4:

$$k = e_i, \quad l = e_i \pm e_a,$$
$$s = e_j, \quad t = e_j \pm e_i.$$

This case can't happen, since e_i appears only twice in the orbit V_j as $e_i \pm e_a$ for some a. Therefore, $a = j$, and we are in cases $1, 2$.

Case 5: For $a \neq b$:

$$k = e_i, \quad l = e_i \pm e_a,$$
$$s = e_j, \quad t = e_j \pm e_b.$$

For the same reason as above, $i \neq j$. There exists a Pauli M such that $Me_i = e_j$. There also exists a Pauli N such that $N(e_i \pm e_a) = (\pm e_j \pm e_b)$. As we have learned before, Paulis will map vertices labelled by standard basis vectors to vertices labelled by standard basis vectors. Therefore, there are two possibilities.

- $Ne_i = \pm e_j$ and $Ne_a = \pm e_b$
- $Ne_i = \pm e_b$ and $Ne_a = \pm e_j$.

In the first case, we are done. There is a unique Pauli mapping e_i to e_j, up to stabilizer of e_j, so we can redefine $M := N$.

In the second case, let M' be a Pauli such that $M'e_b = \pm e_a$, so $M'e_a = \pm e_b$ also. M' exists, since as separate vertices, e_b, e_a are in the same orbit.

$$NM'Ne_i = M'e_i = \pm M'Ne_b = \pm NM'e_b = \pm Ne_a = \pm e_j.$$

Additionally,

$$NM'N(e_i \pm e_a) = \pm e_j \pm NM'Ne_a = \pm e_i \pm M'e_a = \pm e_i \pm e_b.$$

Therefore, upon possibly multiplying M' by a Pauli containing I's and Z we get a matrix mapping $\pm e_i$ to $\pm e_j$ and $e_i \pm e_a$ to $e_j \pm e_b$.

We have considered all possible cases, so the result follows. □

Lemma 7.6.6 *For $k, s \in V_i$ and $l, t \in V_j$, such that if $d(k,l) = d(s,t) = 1$, then*

$$u^{(i)}_{ks} u^{(j)}_{lt} = \langle w_i, w_j \rangle.$$

Proof. Note that $M_{k \mapsto l}$ is unique up to a stabilizer of w_i and $M_{s \mapsto t}$ is unique up to a stabilizer of w_j. However, by Lemma 7.6.5, we can choose $M_{k \mapsto l} = M_{s \mapsto t} =: M$ for our calculation.

$$u^{(i)}_{kl} u^{(j)}_{st} = M w_i w_i^* M^* M w_j w_j^* M^* = M w_i w_i^* w_j w_j^* M^* = 0$$

if and only if $\langle w_i w_j \rangle = 0$.

Note, that it is well-defined, for if $d(k,l) = d(s,t') = 1$, then

$$M_{k \mapsto l} = M_{s \mapsto t'} =: MT$$

for some $Tw_i = w_i$, since $M_{k\mapsto l}$ is unique up to a stabilizer of w_i and clearly, $Ms \neq t'$. However, this does not affect the orthogonality relation of the previous case, since $T^*T = I$:

$$u_{kl}^{(i)} u_{st}^{(j)} = MTw_i w_i^* T^* M^* MTw_j w_j^* T^* M^* = MTw_i w_i^* w_j w_j^* T^* M^* = 0.$$

Therefore, orthogonality relations in the quantum permutation do not change depending on which of all possible Paulis mapping s to t, $M_{s\mapsto t}$, and Paulis mapping k to l, $M_{k\mapsto l}$, we choose. □

Fix a vector W of representatives from orbits of action of $\mathcal{P}^{\otimes s}$ on lines in C_{2^s}. Define a graph $G(w, C_d)$, as a nonorthogonality graph of the lines in C_d but with edges between orbits, whose representative $w_i's$ are orthogonal, deleted. We will show these graphs have quantum symmetry.

Theorem 7.6.7 *There is a quantum automorphism of $G(w, C_{2^s})$ with non-commuting entries for every w and $s \in \mathbb{N}$.*

Proof. Throughout the proof, denote the graph $G(w, C_{2^s})$ by G for short.

Let u be the quantum permutation from earlier. By construction, it has non-commutative entries. It is sufficient to show that u is a quantum automorphism for $G(w, C_{2^s})$, which amounts to showing that whenever $k \sim_G l$ and $s \nsim_G v$, and vice versa, it holds that $u_{ks}u_{lv} = 0$. If k, s are in different orbits, $u_{ks} = 0$ and if l, v are in different orbits, $u_{lv} = 0$ from the block-diagonal structure of u. So, suppose $k, s \in V_i$ and $l, t \in V_j$ for some $1 \leq i, j \leq 2^s$.

From Lemma 7.6.6, we exactly get that if $\langle k, l\rangle \neq 0$, equivalently, $k \sim_G l$, and $\langle s, t\rangle \neq 0$, equivalently, $s \sim_G t$, then

$$u_{ks}u_{lt} = \langle w_i, w_j\rangle \neq 0.$$

Indeed, if $\langle w_i, w_j\rangle = 0$, then there are no edges between orbits V_i and V_j by construction of $G(w, C_{2^s})$, and so the case $k \sim_G l$, and $s \sim_G t$ can't happen.

Moreover, from that lemma we know that $u_{ks}u_{lt} \neq 0$ for all the neighbours of s in orbit V_j, i.e. all four $t_1, ..., t_4$ that are nonorthogonal to s. So

$$u_{ks}u_{lt_1} \neq 0, u_{ks}u_{lt_2} \neq 0, u_{ks}u_{lt_3} \neq 0, u_{ks}u_{lt_4} \neq 0.$$

Remember that u_{ks} is a rank-one projection associated to the vector $M_{k\mapsto s}w_i$. And this vector itself is an element of the V_i^{th} orbit. Thus u_{ks} is also nonorthogonal to only four u_{lt_i}, which have been displayed above, and whose column indices are all possible neighbours of s in that orbit. Thus, if there is a $v \in V_j$ such that $\langle s, v \rangle = 0$, so $s \nsim_G v$, then

$$u_{ks}^{(i)} u_{lv}^{(j)} = 0,$$

since $v \notin \{t_1, ..., t_4\}$. So, the first part of the requirement for u to be a quantum isomorphism is satisfied: if the first coordinates are adjacent (nonorthogonal in our case), second coordinates are not adjacent (orthogonal in our case), the product of such entries is 0.

Now suppose we would like to check that also when the first coordinates are not adjacent, and second ones are, then the product of such entries is 0.

For this, let $s' \nsim_G t'$, so $\langle s', t' \rangle = 0$ and $k' \sim_G l'$, so $\langle k', l' \rangle \neq 0$. We have that $u_{ab} = M_{a \mapsto b} w_i w_i^* M_{a \mapsto b}^*$. At the same time, $Ma = b$ if and only if $Mb = a$, since by convenient nature of Paulis, $M^2 = I$. So $M_{a \mapsto b} = M_{b \mapsto a}$. Therefore,

$$u_{ab}^{(i)} = u_{ba}^{(i)}.$$

So,

$$u_{s'k'}^{(i)} u_{v'l'}^{(j)} = u_{k's'}^{(i)} u_{l'v'}^{(j)} = 0,$$

since we are back to the first case with the first coordinates adjacent, and the second ones not.

Finally, observe, that the entries of u are non-commutative. Choose ab arbitrary Pauli $P \in \mathcal{P}^{\otimes s}$.

$$Pw_i w_i^* P^2 w_j w_j^* P = Pw_j w_j^* P^2 w_i w_i^* P$$
$$w_i w_i^* w_j w_j^* = w_j w_j^* w_i w_i^*.$$

Therefore, entries $Pw_i w_i^* P$ and $Pw_j w_j^* P$ commute if an only if $w_i w_i^*$ and $w_j w_j^*$ commute. However, for example, any representative $w_i \in V_d$ from the orbit containing standard basis vectors will not commute with any representative w_j from any other orbit. □

We can go further, and show that purely non-orthogonality graph of the lines in the C_{2^s}-root system has quantum automorphism. The quantum automorphism matrix is going to be the same as before. However, with G^{w,C_d}, we deleted all possible edges between orbits whose representative $w_i's$ were orthogonal.

Note, that orthogonality graph of the lines in the C_{2^s}-root system is a complement of the nonorthogonality graph of the lines in C_{2^s}-root system, so it will also have a quantum automorphism.

What we are going to do next is that we will define two graphs, $X(w, C_d)$ and $U(w, C_d)$ and show that the previously defined matrix u is a quantum isomorphism between them. Then we will prove that $X(w, C_d)$ and $U(w, C_d)$ are also isomorphic. This implies that each of them has quantum symmetry.

Let $X(w, C_{2^s})$ denote the non-orthogonality graph of the lines in a root system C_{2^s}.

Let $u = (u_{ij})_{1 \leq i,j \leq d^2}$ be the block-diagonal quantum permutation described earlier, with d diagonal blocks corresponding to orbits $V_1, ..., V_d$ in that order. Now, let w be the same as in $X(w, C_d)$. From orbits $V_1, ..., V_d$ select d pairwise non-orthogonal vectors. We can do this, since each e_i is present as part of some summand $e_i \pm e_a$ in each orbit. We could just select vectors with e_i component nonzero then. Denote this matrix $a = [a_1, ..., a_{d-1}, a_d]$ with columns being the selected representatives. Let $\mathcal{U}(w, C_d)$ be a non-orthogonality graph of

$$V(\mathcal{U}) := \{\{u_{a_1,x}\}_{x \in V_1} \cup ... \cup \{u_{a_{d-1},x}\}_{x \in V_{d-1}}, \{u_{a_d,d}\}_{d \in V_d}\}.$$

In other words, if $a = [e_1 + e_2, ..., e_1 + e_d, e_1]$ is the matrix whose columns are selected d pairwise non-orthogonal vectors, and each of the columns in the matrix labels the first column of the next block of quantum permutation u, we are just selecting all d^2 entries from the first column of each block.

We will show that there is a quantum isomorphism between $X(w, C_d)$ and $\mathcal{U}(w, C_d)$. But first, we will show that $X(w, C_d)$ is also isomorphic to $\mathcal{U}(w, C_d)$. Therefore, if P is a classical isomorphism, from $X(w, C_d)$ to $\mathcal{U}(w, C_d)$, and $\mathcal{U}$ is a quantum isomorphism between them, then $(P \otimes I_d)\mathcal{U}$ is a quantum automorphism of $X(w, C_d)$.

Lemma 7.6.8 *$X(w, C_d)$ and $\mathcal{U}(w, C_d)$ are isomorphic.*

Proof. Vertices of $\mathcal{U}(w, C_d)$ are just rank one projections associated to vertices of $X(w, C_d)$, just in different order. Hence, those two non-orthogonality graphs are isomorphic.

Note, however, another view of construction of $\mathcal{U}(w, C_d)$ from $X(w, C_d)$.

Suppose $k, s \in V_i$ and $l, t \in V_j$ such that $\langle w_i, w_j \rangle \neq 0, \langle k, l \rangle \neq 0$ and $\langle s, t \rangle \neq 0$. By Lemma 7.6.6, $u_{ks}u_{lt} \neq 0$ for all neighbours of s in V_j, called $\{t_i\}$, that are in V_j and not orthogonal to s. There are four $t_i's$ when $i, j \neq d$, and two in case when i is a standard basis orbit).

Then u_{ks} is also not orthogonal, in other words, adjacent, to $u_{lt_1}, ..., u_{lt_{4(or,2)}}$ in $\mathcal{U}(w, C_d)$.

Now, suppose $k, s \in V_i$ and $l, t \in V_j$ such that $\langle w_i, w_j \rangle = 0, \langle k, l \rangle \neq 0$ and $\langle s, t_i \rangle \neq 0$ for each $1 \leq i \leq 4$. From the same Lemma 7.6.6, it follows that $u_{ks}u_{lt_i} = 0$ for each of four (two) neighbours $t_i \in \{t_1, ..., t_4(or\, 2)\}$ of s in V_j. Moreover, since entries in each block of u are just projections onto vertices in that orbit, so u_{ks} will be orthogonal to this many other distinct projections in the j^{th} block, depending on whether or not $i = d$,

$$|\text{of neighbours in } V_j \text{ of each } s \in V_i| - |\{u_{lt_i} : \langle t_i, s\rangle = 0\}| = (2^s - 4) - 4 \text{ when } i \neq d$$

$$|\text{of neighbours in } V_j \text{ of each } s \in V_i| - |\{u_{lt_i} : \langle t_i, s\rangle = 0\}| = (2^s - 2) - 2 \text{ when } i = d$$

more u_{lv_k}'s. Note that $\langle s, v_k\rangle = 0$ for each k. Finally, there will be four (two) u_{lx_i}'s such that $u_{ks}u_{lx_i} \neq 0$ for each $x_1, ..., x_4(or\, 2)$. They will be the neighbours of u_{ks} in $\mathcal{U}(w, C_d)$.

Observe, that this is well-defined, i.e, not dependent on choice of k, l as long as $k \in V_i$, $l \in V_j$ and $\langle k, l\rangle \neq 0$. Ill-definiteness could occur if there were s, x_i such that $u_{ks}u_{lx_i} \neq 0$. In that case u_{ks} should be adjacent in $\mathcal{U}(w, C_d)$ to u_{lx_i}. And if there were k, l' such that $u_{k's}u_{l'x_i} = 0$, so $u_{k's}$ should not be adjacent in $\mathcal{U}(w, C_d)$ to u_{lx_i}. Note, however, that since both k is nonorthogonal to l and k' is nonorthogonal to l', there is a Pauli M such that

$$Mk = k', Ml = l'.$$

Thus, $u_{ks} = P_{M_{k\mapsto s}w_i} = P_{M_{k\mapsto k'}M_{k'\mapsto s}w_i} = P_{MM_{k'\mapsto s}w_i}$. Similarly, $u_{ltx_i}P_{MM_{l'\mapsto x_i}w_i}$. Hence,

$$\langle u_{ks}, u_{lt}\rangle = \langle MM_{k'\mapsto s}w_i, MM_{l'\mapsto x_i}w_i\rangle = \langle u_{k's}, u_{l't}\rangle.$$

Clearly, the $\mathcal{U}$ constructed is isomorphic to a nonorthogonality graph of the projections in the quantum permutation. □

Later we will show that $X(w, C_d)$ and $\mathcal{U}(w, C_d)$ are also isomorphic. The theorem below, thus, is ket in proving that $X(w, C_d)$ has quantum symmetry.

Theorem 7.6.9 *The quantum permutation u defined above is a quantum isomorphism between $X(w, C_d)$ and $\mathcal{U}(w, C_d)$.*

Proof. From the previous Lemma 7.6.8, there is an isomorphism P between $X(w, C_d)$ and $\mathcal{U}(w, C_d)$ mapping orbits to orbits, such that for each orbit k, where a is the part of the coclique array in $\mathcal{U}$

$$u_{a_k b_i} \mapsto P(b_i)$$

such that

$$\langle u_{a_k b_i} u_{a_l d_r}\rangle = 0 \text{ if and only if } \langle P(b_i)P(d_r)\rangle = 0$$

In other words, the isomorphism P just extracts the column index of each $u_{a_k b_i}$.

Case 1:

Now, suppose with some $a \in V_i$, $b \in V_j$ and $\langle a, b\rangle \neq 0$ we also have $k, u_{ab}^{(i)} \in V_i$ and $l, u_{cd}^{(j)} \in V_j$ such that $\langle w_i, w_j\rangle = 0, \langle k, l\rangle \neq 0$, so $k, l \in E(X(w, C_d))$, but $u_{ab}, u_{cd} \notin E(\mathcal{U}(w, C_d))$. We aim to demonstrate that

$$u_{k,u_{ab}}^{(i)} u_{l,u_{cd}}^{(j)} = 0.$$

Using the fact that as long as the adjacency requirement is satisfied, the choice of representatives k and l from orbits V_i and V_j does not play a role,

$$\begin{aligned}\langle u_{k,u_{ab}}^{(i)} u_{l,u_{cd}}^{(j)}\rangle &= \langle u_{k,P(b)}^{(i)} u_{l,P(d)}^{(j)}\rangle \\ &= \langle u_{a,P(b)}^{(i)} u_{c,P(d)}^{(j)} \\ &= \langle PP(b), PP(d)\rangle \\ &= 0\end{aligned}$$

if and only if $\langle Pb, Pd\rangle = 0$.

At the same time, $u_{ab}, u_{cd} \notin E(\mathcal{U}(w, C_d))$ if and only if $\langle u_{ab}, u_{cd}\rangle = 0$ if and only if $\langle P(b), P(d)\rangle = 0$. Hence,

$$u_{k,u_{ab}}^{(i)} u_{l,u_{cd}}^{(j)} = 0.$$

Case 2:

Now, suppose with some $a \in V_i$, $b \in V_j$ and $\langle a, b\rangle \neq 0$ we also have $k, u_{ab}^{(i)} \in V_i$ and $l, u_{cd}^{(j)} \in V_j$ such that $\langle w_i, w_j\rangle = 0, \langle k, l\rangle = 0$, so $k, l \notin E(X(w, C_d))$, but $u_{ab}, u_{cd} \in E(\mathcal{U}(w, C_d))$. We aim to demonstrate that

$$u_{k,u_{ab}}^{(i)} u_{l,u_{cd}}^{(j)} = 0.$$

We have that u_{ab}, u_{cd} are not orthogonal. Using symmetry and indifference to the choice of nonorthogonal $a \in V_i$ and $c \in V_j$:

$$\begin{aligned}\langle u_{k,u_{ab}}^{(i)} u_{l,u_{cd}}^{(j)}\rangle &= \langle u_{u_{ab},k}^{(i)} u_{u_{cd},l}^{(j)} \\ &= \langle u_{a,k}^{(i)} u_{c,l}^{(j)} \\ &= \langle PP(k), PP(l)\rangle \\ &= 0\end{aligned}$$

If and only if $\langle k, l \rangle = 0$, which it is from the hypothesis.

Case 3: Now, suppose with some $a \in V_i, b \in V_j$ and $\langle a, b \rangle \neq 0$ we also have $k, u_{ab}^{(i)} \in V_i$ and $l, u_{cd}^{(j)} \in V_j$ such that $\langle w_i, w_j \rangle \neq 0, \langle k, l \rangle = 0$, so $\{k, l\} \in E(X(w, C_d))$, but $\{u_{ab}, u_{cd}\} \notin E(\mathcal{U}(w, C_d))$. We aim to demonstrate that

$$u_{k,u_{ab}}^{(i)} u_{l,u_{cd}}^{(j)} = 0.$$

If not, then by construction of $\mathcal{U}$, we would make an edge $\{u_{ab}, u_{cd}\}$, a contradiction.

Case 4:

Now, suppose with some $a \in V_i, b \in V_j$ and $\langle a, b \rangle \neq 0$ we also have $k, u_{ab}^{(i)} \in V_i$ and $l, u_{cd}^{(j)} \in V_j$ such that $\langle w_i, w_j \rangle \neq 0, \langle k, l \rangle = 0$, so $k, l \notin E(X(w, C_d))$, but $u_{ab}, u_{cd} \in E(\mathcal{U}(w, C_d))$. We aim to demonstrate that

$$u_{k,u_{ab}}^{(i)} u_{l,u_{cd}}^{(j)} = 0.$$

Using symmetry and indifference to choice of nonorthogonal vertices $a \in V_i$ and $c \in V_j$:

$$\begin{aligned} \langle u_{k,u_{ab}}^{(i)} u_{l,u_{cd}}^{(j)} \rangle &= \langle u_{u_{ab},k}^{(i)} u_{u_{cd},l}^{(j)} \rangle \\ &= \langle u_{a,k}^{(i)} u_{c,l}^{(j)} \\ &= \langle PP(k), PP(l) \rangle \\ &= 0 \end{aligned}$$

if and only if $\langle k, l \rangle = 0$, which it is. □

Corollary 7.6.10 *The orthogonality graph of C_d has quantum symmetry for $d = 2^s$.*

7.7 Other root systems

The natural question arises if there are other root systems with interesting relations to quantum (iso)automorphisms. Here are some notes about other root systems. The A_d root system contains all integer vectors in $\mathbb{R}^{d+1}$ whose coordinates sum to 0 and whose norm is $\sqrt{2}$. In other words, these are the vectors $\pm e_i \mp e_j$. Note that even if $d + 1$ is a power of two, the Pauli group will not act on such a set of vectors. This is because given $e_i - e_j$, there will be a tensor product of Z's and I's which will map $e_i - e_j \mapsto e_i + e_j$, with the latter vector being not in A_d. If we restrict to only the action of Paulis containing identities and X's, the orbits will not be of length d any more, so the block-diagonal matrix of projections u^{a,A_d} will not be a quantum permutation anymore. Therefore, none of the arguments above apply.

The D_d root system contains all integer vectors in $\mathbb{R}^d$ whose coordinates norm is $\sqrt{2}$. In other words, these are the vectors $\pm e_i \pm e_j$. Note that when d is a power of two, all arguments from lemmas in the previous sections apply for orbits of Paulis acting on C_d, excluding the standard basis vector. Therefore, we will have an analogous result.

Theorem 7.7.1 *The orthogonality graph of D_d has quantum symmetry for $d = 2^s$.*

B_d root system contains all integer vectors in $\mathbb{R}^d$ of norm $\sqrt{2}$ or 1. System B_d is dual to C_d. Which means that each roots of B_d are an image of

$$\alpha \mapsto \frac{2}{\langle \alpha, \alpha \rangle} \alpha$$

applied to each root α of C_d. Clearly, the roots in B_d are different from roots in C_d only up to a factor. Adjacency in the graphs X^{w,B_d} and $X(w, C_d)$ depends only on the orthogonality relations between vectors . So, X^{w,B_d} and $X(w, C_d)$ are isomorphic, and we can arrange vertices of X^{w,B_d} in such order that the adjacency matrices of X^{w,B_d} and $X(w, C_d)$ will be the same.

Theorem 7.7.2 *The graph X^{w,B_d} has quantum symmetry for $d = 2^s$.*

Now, let us discuss root systems which only exist for a fixed dimension. The orthogonality graph of lines in E_8 was covered in [65] and a non-isomorphic quantum isomorphic twin was found. The orthogonality graph of F_4 appeared in [46], even though there it was constructed in a completely different way. Only after [65] was made public, we checked and realised that Schmidt's SRGs are a natural generalisation of the non-isomorphic quantum isomorphic graphs found in [46], since the latter graphs are orthogonality graphs of a root system.

The E_6, E_7 root systems have roots in $\mathbb{R}^6, \mathbb{R}^7$ respectively. Tensor products of Paulis do not act on vectors in such dimension. However, it is possible to view both E_6, E_7 as subsets of E_8. Even then, with help of Sage, we observed that there will be orbits under three-tensor-Paulis of length less than eight. Therefore, if we were to construct u as before, it will not be a quantum permutation.

The orthogonality graph of the lines of G_2 is a bit too simple to be interesting. It is just 3 disjoint edges. Additionally, no tensor of Paulis would acts on vectors in G_2, since the vectors in G_2 are in $\mathbb{R}^3$.

So we have found three infinite families of graphs with quantum symmetry coming from root systems $B_{2^d}, C_{2^d}, D_{2^d}$.

Chapter 8

Fractional isomorphisms and quantum isomorphisms

In this chapter we relate notions of quantum isomorphisms and fractional isomorphisms. In [2] Atserias et al prove that there are graphs that are fractionally isomorphic, but not quantum isomorphic. Chris Godsil observed that with partial tracing each quantum isomorphism gives a fractional isomorphism between two graphs. We restrict our attention to automorphisms and ask if there are graphs that have fractional automorphisms but do not have quantum symmetry. This leads to interesting relations with compact graphs.

A *doubly stochastic* matrix is a square matrix with non-negative entries such that each row and each column sums to one. There is a well established classical notion of fractional isomorphism between two graphs, which can be summarized as follows. We say that the graphs X and Y with adjacency matrices $A(X)$ and $A(Y)$ are *fractionally isomorphic* if there exists a doubly stochastic matrix D such that $DA(X) = A(Y)D$. The notion of quantum isomorphism is similar to fractional isomorphism in a sense that the entries of a quantum permutation that commutes with $A(X)$ and $A(Y)$ are matrices with nonnegative eigenvalues (that are also projections) and sum to identity in each row and column.

Chris Godsil noticed that it is possible to construct a fractional isomorphism from the quantum isomorphism. The next two lemmas illustrate how to do this.

In this section we will use the notion of density matrix. A *density matrix* is a positive-semidefinite matrix of trace one. For example, any rank-one projection is a density matrix.

For any two matrices $Q \in M_n(M_d(\mathbb{C}))$ and $M \in M_d(\mathbb{C})$, we define $F := \langle\langle Q, M \rangle\rangle$ to be the matrix with entries

$$F_{ij} = \langle Q_{ij}, M \rangle.$$

Theorem 8.0.1 (Godsil) *Let P be a quantum isomorphism between the graphs X and Y with adjacency matrices $A(X)$ and $A(Y)$ respectively. Let $M \in \mathbb{C}^{d\times d}$ be a positive semidefinite matrix with $tr\, M = 1$. Then $D = (\langle M, P_{ij}\rangle)_{ij}$ is a fractional isomorphism.*

Proof. Let $D_{ij} = \langle M, P_{ij}\rangle$. Since the inner product of two positive semidefinite matrices is non-negative, $D_{ij} \geq 0$ for all $i, j = 1, ..., n$.

$$\sum_j D_{ij} = \sum_j \langle M, P_{ij}\rangle = \langle M, \sum_j P_{ij}\rangle = \langle M, I\rangle = \mathrm{tr}M.$$

Similarly, $\sum_i D_{ij} = \mathrm{tr}M$. Now, we need to check that $DA(X) = A(Y)D$. We are given that

$$P(A(X) \otimes I_d) = (A(Y) \otimes I_d)P.$$

The above holds if and only if the ij-entries of both sides are equal. For convenience, let the entries of $A(X)$ be x_{ij} and the entries of $A(Y)$ be y_{ij}. In particular,

$$\sum_\ell x_{\ell j} P_{i\ell} = \sum_\ell y_{i\ell} P_{\ell j}.$$

We will now ensure that the ij^{th} entries of $DA(X)$ and $A(Y)D$ are also equal. From above,

$$\begin{aligned}\langle M, \sum_\ell x_{\ell j} P_{i\ell}\rangle &= \langle M, \sum_\ell y_{i\ell} P_{\ell j}\rangle \\ \sum_\ell x_{\ell j}\langle M, P_{i\ell}\rangle &= \sum_\ell y_{i\ell}\langle M, P_{\ell j}\rangle \\ \sum_\ell x_{\ell j} D_{i\ell} &= \sum_\ell y_{i\ell} D_{\ell j} \\ (DA(X))_{ij} &= (A(Y)D)_{ij}.\end{aligned}$$

Therefore, $DA(X) = A(Y)D$ and D is indeed a fractional isomorphism. □

Let us call the above operation *reducing a quantum isomorphism to a fractional isomorphism via measurement*. If we start with a quantum isomorphism P and use inner product with a positive semidefinite matrix M of trace one to construct the fractional isomorphism, we denote the resulting fractional isomorphism by $\langle\langle M, P\rangle\rangle$. Note that his procedure respects two composition operations on the quantum automorphisms, as the following two lemmas illustrate.

Lemma 8.0.2 (Godsil) *Let P, Q be quantum isomorphisms and $M \succeq 0$ with $trM = 1$, then*

$$\langle\langle \frac{M \boxplus M}{2}, P \boxplus Q \rangle\rangle = \frac{1}{2}(\langle\langle M, P \rangle\rangle + \langle\langle M, Q \rangle\rangle).$$

Proof. This follows from the fact that $\mathrm{tr}(M \boxplus M)_{ij}(P \boxplus Q)_{ij} = \mathrm{tr}(M_{ij}P_{ij}) + \mathrm{tr}(M_{ij}Q_{ij})$. □

Note: in the above lemma, since $\langle\langle M, P \rangle\rangle, \langle\langle M, PQ \rangle\rangle$ are fractional isomorphisms, $\langle\langle \frac{M \boxplus M}{2}, P \boxplus Q \rangle\rangle$ is a fractional isomorphism too.

Lemma 8.0.3 (Godsil) *Let P, Q be quantum isomorphisms between graphs X and Y. Suppose M, N are positive semidefinite matrices of trace one such that $\langle M, P \rangle$ and $\langle N, Q \rangle$ make sense. Then $\langle\langle M \otimes N, P * Q \rangle\rangle$ is a fractional isomorphism and*

$$\langle\langle M \otimes N, P * Q \rangle\rangle = \langle\langle M, P \rangle\rangle \langle\langle N, Q \rangle\rangle.$$

Proof. The first part of the statement follows by Lemma 8.0.1 since $M \otimes N$ is a positive semi-definite matrix of trace one and $P * Q$ is a quantum isomorphism.

To prove the equality:

$$\begin{aligned}
(\langle\langle M \otimes N, P * Q \rangle\rangle)_{ij} &= (\langle M \otimes N, (P * Q)_{ij} \rangle)_{ij} \\
&= \langle M \otimes N, \sum_k P_{ik} \otimes Q_{kj} \rangle \\
&= \sum_k \mathrm{tr}(MP_{ik} \otimes NQ_{kj}) \\
&= \sum_k \langle M, P_{ik} \rangle \langle N, Q_{kj} \rangle \\
&= (\langle\langle M, P \rangle\rangle \langle\langle N, Q \rangle\rangle)_{ij} .
\end{aligned}$$

□

8.1 Fractional isomorphisms arising from quantum permutations

Before we were aware of the result in [2] that there are graphs that are not quantum isomorphic and are fractionally isomorphic, we were approaching the same question from another angle. Since we already learned that every quantum isomorphism gives rise to a fractional isomorphism, we were curious whether every fractional isomorphism comes from some quantum isomorphism through the partial tracing procedure. We learned the answer is no, which is an agreement with [2].

Lemma 8.1.1 *Let $F = (a_{ij})_{ij}$ be a fractional isomorphism between graphs X and Y and $Q = (Q_{ij})_{ij}$ be a quantum isomorphism between them. Let there be a subset of indices $I \subseteq \{1, ..., n\} \times \{1, ...n\}$ such that $\sum_{(i,j)\in I} Q_{ij} = I$, but $\sum_{(i,j)\in I} F_{ij} \neq 1$, then there is no density M such that $\langle\langle M, Q\rangle\rangle = F$.*

Proof.

We want an M such that:

(a) $\langle M, Q_{ij}\rangle = a_{ij}$ for all i, j,

(b) $\langle M, I\rangle = 1$,

(c) $M \succeq 0$.

Let $I \subseteq \{1, ..., n\} \times \{1, ...n\}$ be a set of indices such that

$$\sum_{(i,j)\in I} Q_{ij} = I,$$

but

$$\sum_{(i,j)\in I} F_{ij} \neq 1.$$

If there were a density M, such that $\langle\langle M, Q\rangle\rangle = F$, then it wold hold that

$$1 = \langle M, I\rangle = \langle M, \sum_{(i,j)\in I} Q_{ij}\rangle = \sum_{(i,j)\in I} \langle M, Q_{ij}\rangle = \sum_{(i,j)\in I} F_{ij},$$

a contradiction. □

As an example, consider a quantum permutation

$$W = \begin{bmatrix} P & I-P & 0 & 0 & 0 & 0 \\ I-P & P & 0 & 0 & 0 & 0 \\ 0 & 0 & Q & I-Q & 0 & 0 \\ 0 & 0 & I-Q & Q & 0 & 0 \\ 0 & 0 & 0 & 0 & I-Q & Q \\ 0 & 0 & 0 & 0 & Q & I-Q. \end{bmatrix}$$

Just like any $n \times n$ quantum permutation, it commutes with the adjacency matrix of complete graph K_n. Now let F be any 6×6 doubly stochastic matrix with zero diagonal. Just like any doubly stochastic matrix, it will also commute with the adjacency matrix of complete graph K_6. However, since there is a set of indices, in particular $\{(3,3),(5,5)\}$ such that $W_{33} + W_{55} = I$, but $F_{33} + F_{55} = 0$, there is no density matrix M such that $\langle\langle M, W\rangle\rangle = F$.

Note that this lemma does not imply that there is no other quantum automorphism W' of K_6 such that for some density matrix M, it holds that $\langle\langle W', M\rangle\rangle = F$.

8.2 Quantum automorphisms of compact graphs

From Birkhoff's theorem [10] we know that every doubly stochastic matrix can be written as a convex combination of permutation matrices. A graph is *compact* if its every fractional automorphism can be written as a convex combination of its automorphisms. Compact graphs were studied first by Tinhofer [68], from where we know that, for example, trees are compact. Wang and Li studied cubic compact graphs:

Theorem 8.2.1 *[70, Theorem 2.7] Let G be a connected 3-regular graph. If G is compact, then G is one of $Cay(\mathbb{Z}_{2m}, \{1, -1, m\})$, $Cay(\mathbb{Z}_m \times Z_2, \{(1,0),(0,1),(-1,0)\})$ or K_4.*

Automorphism groups of regular compact graphs were shown to be generously transitive by Godsil in [26]. In this section we explore quantum automorphism groups of compact graphs.

The next theorem is similar to above, but we drop the condition of X being compact at the expense of Q being an arbitrary quantum permutation, rather than quantum automorphism of X. The proof is quite similar to the one above.

Theorem 8.2.2 *Let F be a fractional automorphism of a graph X such that F is a convex combination of at most n permutation matrices. Then there is a quantum permutation Q and a density matrix M such that $\langle\langle M, Q\rangle\rangle = F$.*

Proof. For the rest of this proof let F denote fractional automorphism, reserve the letter P for projections and let R be used for permutation matrices. We can decompose a fractional automorphism $F \in M_{n\times n}(\mathbb{R})$ of X into permutations $R_i \in M_{n\times n}(\mathbb{R})$ by Birkhoff decomposition, since F is a doubly-stochastic matrix:

$$F = \sum_{i=1}^{m} \lambda_i R_i, \tag{8.2.1}$$

where $m \leq n$. Some of those permutations may be also automorphisms. Denote them by $\{R_i\}_{i\in\mathcal{I}}$. Let the remaining permutations that are not automorphisms have indices $j \in \mathcal{J}$. So $|\mathcal{I}| + |\mathcal{J}| = m$ and we can rewrite above as

$$F = \sum_{i\in\mathcal{I}} \lambda_i R_i + \sum_{j\in\mathcal{J}} \lambda_j R_j.$$

Provided that F is a convex combination of at most n permutations, i.e. $|I| + |\mathcal{J}| \leq n$, we can find a matrix M and construct a quantum permutation Q such that $\langle\langle M, Q\rangle\rangle = F$ as follows.

Since F is a convex combination of at most n permutations, there are at most n non-zero coefficients λ_i. Always we are able to choose $m \leq n$ non-zero orthogonal symmetric projections $\{P_i\}_{i\in I}$ that sum to identity. For example, we can take a basis $\{x_i\}_{i\in[n]}$ and partition $\{1, ..., n\}$ into m subsets $\mathcal{L}_i$ such that all $\mathcal{L}_i$ are pairwise disjoint, $\cup_{i=1}^{m} \mathcal{L}_i = [n]$. For $i \in \mathcal{L}_i$ be defined as

$$P_i = \sum_i x_i x_i^*,$$

so P_i's are nonzero whose ranks sum to n. This means that $\sum_{k=1}^{m} P_k = I$ and P_k's are all diagonal in the chosen basis. Let λ_i's become the non-zero eigenvalues of M up to a factor with corresponding spectral idempotents P_k :

$$M = \sum_{i=1}^{m} \frac{\lambda_k}{\mathrm{rk}(P_k)} P_k.$$

Note that M is positive-semidefinite since all λ_k's are non-negative. Moreover, M is invertible, since by choice all $\lambda_i > 0$, so it is positive definite. Now define

$$Q = \sum_i (R_i \otimes P_i) = \sum_{i\in\mathcal{I}} (R_i \otimes P_i) + \sum_{j\in\mathcal{J}} (R_j \otimes P_j).$$

The remaining part of the proof is dedicated to showing that Q is indeed the desired quantum automorphism. For this purpose, it suffices to verify these points:

(a) $\langle\langle M, Q\rangle\rangle = F$,

(b) $\sum_\beta Q_{\alpha\beta} = I = \sum_\alpha Q_{\alpha\beta}$,

(c) $Q^2_{\alpha\beta} = Q_{\alpha\beta}$ for all $1 \leq \alpha, \beta \leq n$,

(d) $Q^*_{\alpha\beta} = Q_{\alpha\beta}$ for all $1 \leq \alpha, \beta \leq n$.

For the point (i), we are going to rely on the definition of Q to verify that $\langle\langle M, Q\rangle\rangle = F$:

$$\begin{aligned}
\langle\langle M, Q\rangle\rangle &= \langle\langle M, \sum_{i=1}^{m}(R_i \otimes P_i)\rangle\rangle \\
&= \sum_{i=1}^{m} R_i \otimes \langle M, P_i\rangle \\
&= \sum_{i=1}^{m} R_i \langle M, P_i\rangle \\
&= \sum_{i=1}^{m} R_i \frac{\lambda_i}{\mathrm{rk} P_i} \mathrm{rk} P_i \\
&= F.
\end{aligned}$$

The second to last equation follows from the fact that the P_i are orthogonal and are spectral idempotents of M.

To prove the point (ii), we rely on the fact that every α^{th} row of permutations R_i has only one non-zero entry equal to one and by the choice of our projections we get:

$$\sum_\beta Q_{\alpha\beta} = \sum_{i=1}^{m} \sum_\beta ((R_i)_{\alpha\beta} \otimes P_i) = \sum_i P_i = I.$$

The same reasoning applies if we are summing over α instead, since every β^{th} column of R_i also has only one non-zero entry entry one. Moving onto the point (iii):

$$Q^2_{\alpha\beta} = \sum_{i=1}^{m} \sum_k ((R_i)_{\alpha\beta}(R_k)_{\alpha\beta} \otimes P_i P_k) = \sum_{i=1}^{m} (R_i)_{\alpha\beta} \otimes P_i = Q_{\alpha\beta}.$$

The second to last equation follows from the fact that $P_i P_k = 0$ whenever $i \neq k$. Finally, we are going to show (iv) by making use of the choice of our projections to be symmetric

$P_i = P_i^*$:

$$Q^*_{\alpha\beta} = \sum_{i=1}^{m}((R_i)_{\alpha\beta} \otimes P_i^*) = \sum_{i,k}((R_i)_{\alpha\beta} \otimes P_i) = Q_{\alpha\beta}.$$

Therefore, indeed Q is a quantum permutation that is also a quantum automorphism of X such that $\langle\langle Q, M\rangle\rangle = F$, as desired. □

Corollary 8.2.3 *Let F be a fractional automorphism of a compact graph X such that F is a convex combination of at most n automorphisms of X. Then there is a quantum automorphism Q of X and a density matrix M such that $\langle\langle M, Q\rangle\rangle = F$.*

Proof. Very similar proof applies. Clearly, the matrix Q constructed in the proof is quantum permutation regardless of decomposition of F.

The difference from the previous lemma is that in the Equation 8.2.1 we can let R_i be automorphisms of X. As a result the matrix Q constructed in the proof, will also commute with A_X.

We proceed remembering that R_i is an automorphism of X, so $R_iA = AR_i$ for all $i \in \mathcal{I}$.

$$\begin{aligned} Q(A \otimes I) &= \sum_i (R_i \otimes P_i)(A \otimes I) \\ &= \sum_i (R_iA \otimes P_i) \\ &= \sum_i (AR_i \otimes P_i) \\ &= (A \otimes I)Q. \end{aligned}$$

□

Note that it is not necessarily the case for a compact graph that a fractional automorphism is a combination of at most n automorphisms. In case a compact graph has more than n automorphisms, convex hull of those will contain a fractional automorphism that can't be written as a convex combination of at most n fractional automorphisms.

Note also that the constructed quantum permutation in the proofs of the two previous results has commuting entries.

In the above two lemmas we showed that any fractional automorphism $F = \sum \lambda_i R_i$ which can be written as a convex combination of at most n permutations arises from a quantum permutation of the form $Q = \sum R_i \otimes P_i$ and a density matrix $M = \sum \lambda_i P_i$, via the $\langle\langle M, Q\rangle\rangle$ operation; where P_i are linearly independent commuting projections. We could try to relax the requirement of projections and orthogonality with a hope to generalize above construction. We could try to suppose that every fractional automorphism $F = \sum \lambda_i R_i$ arises from a quantum permutation of the form $Q = \sum R_i \otimes C_i$ and a density matrix $M = \sum x_i^{(F)} C_i$ with some $x_i^{(F)}$ that would depend on the fractional automorphism F. However, the next result shows that if we were to use Q and M of such form, the linear independence requirement is necessary to be able to obtain all fractional automorphisms.

Therefore, the next lemma brings us to the question whether every graph has a quantum automorphism written as $Q = \sum_{i\in\mathcal{I}} R_i \otimes C_i$, where all C_i are linearly independent and R_i are affinely independent for every $1 \leq |\mathcal{I}| \leq n^2 - n + 2$. If the answer were yes, then we would have proved that every fractional automorphism is obtainable from a quantum automorphism and some density matrix, since every fractional automorphism is a convex combination of at most $n^2 - n + 2$ permutation matrices. Since now from [2], the asnwer is no.

Lemma 8.2.4 *Let Q be a quantum automorphism of a graph X written as*

$$Q = \sum_{i\in\mathcal{I}} (R_i \otimes C_i),$$

where R_i are affinely independent permutation matrices and C_i are linearly dependent matrices that sum to identity. If we only consider density matrices of the form

$$M = \sum_{i\in\mathcal{I}} x_i C_i,$$

we can find a fractional automorphism $F = \sum_{j\in\mathcal{I}} \lambda_i R_j$ such that there will be no density matrix M of the above form with the property such that $\langle\langle M, Q\rangle\rangle = F$.

Proof. Suppose it holds that $\langle\langle M, Q\rangle\rangle = F$, then we need

$$\sum_{i\in\mathcal{I}} \lambda_i R_i = F = \langle\langle M, Q\rangle\rangle = \sum_{i\in\mathcal{I}} (R_i \otimes \langle M, C_i\rangle) = \sum_{i\in\mathcal{I}} R_i \sum_{j\in\mathcal{I}} (x_j \langle C_j, C_i\rangle).$$

Since R_i are affinely independent permutations, and F is a doubly stochastic matrix in the convex hull of permutations, there is a unique way to write F as a convex combination of affinely independent permutations. Therefore, $\langle\langle M, Q\rangle\rangle = F$ if and only if the following system has a solution :

$$\sum_{j\in\mathcal{I}}(x_j\langle C_j, C_i\rangle) = \lambda_i \text{ for all } i.$$

This can be compactly rewritten as

$$x^T \begin{bmatrix} \langle C_1, C_1\rangle & \langle C_1, C_2\rangle & \dots & \langle C_1, C_{|\mathcal{I}|}\rangle \\ \langle C_2, C_1\rangle & \langle C_2, C_2\rangle & \dots & \langle C_2, C_{|\mathcal{I}|}\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle C_{|\mathcal{I}|}, C_1\rangle & \langle C_{|\mathcal{I}|}, C_2\rangle & \dots & \langle C_{|\mathcal{I}|}, C_{|\mathcal{I}|}\rangle \end{bmatrix} = \lambda.$$

Denote by C the Gram matrix on the left. It is positive semi-definite and invertible if and only if the matrices $C_1, ..., C_{|\mathcal{I}|}$ are linearly independent. If they are not, then there will be vectors λ not in the range of C^T, and so not all the F will be from such M and Q. □

Theorem 8.2.5 *Let X be any graph. For every fractional automorphism F of X there is a quantum permutation Q and a density matrix M such that $\langle\langle M, Q\rangle\rangle = F$.*

Proof. As a doubly stochastic matrix, F is a convex combination of permutation matrices R_i:

$$F = \sum_{i\in\mathcal{I}}\lambda_i R_i.$$

If $|\mathcal{I}| \le n$, then we are done by Theorem 8.2.2. Therefore, assume $|\mathcal{I}| > n$, and partition the index set into index sets of size at most n:

$$\mathcal{I} = \mathcal{I}_1 \sqcup \mathcal{I}_2 \sqcup ... \sqcup \mathcal{I}_k$$

such that $|\mathcal{I}_1| = ... = |\mathcal{I}_{k-1}| = n$ and $|\mathcal{I}_k| \le n$. In other words,

$$F = \sum_{i\in\mathcal{I}_1}\lambda_i R_i + \sum_{i\in\mathcal{I}_2}\lambda_i R_i + ... + \sum_{i\in\mathcal{I}_k}\lambda_i R_i.$$

Define the summands to be

$$F_j := \sum_{i\in\mathcal{I}_j}\lambda_i R_i \text{ for all } j = 1, ..., k,$$

so that

$$F = F_1 + ... + F_k. \tag{8.2.2}$$

Now, using the construction from Theorem 8.2.2, for $i = 1, ..., k$ we can define quantum permutations Q_i along with the corresponding matrices M_i such that

$$\langle\langle Q_i, M_i \rangle\rangle = F_i.$$

Note that F_i are not fractional automorphisms, since $\sum_{i \in \mathcal{I}_j} \lambda_i \neq 1$. However, Theorem 8.2.2 still applies as the proof was independent of the fact that fractional automorphism was a convex combination of permutations: only the fact that it was a non-negative combination of permutation matrices was used, so we can utilise the same construction for all M_i and Q_i. For the same reason, the M_i are positive semidefinite but not of trace one. However, M is still a density matrix as its eigenvalues are the eigenvalues of each of the blocks and $\sum_{i \in \mathcal{I}} \lambda_i = 1$. Now, let

$$Q = \boxplus_{i=1}^k Q_i = \begin{bmatrix} \bigoplus_{i=1}^k (Q_i)_{11} & \bigoplus_{i=1}^k (Q_i)_{12} & \cdots & \bigoplus_{i=1}^k (Q_i)_{1n} \\ \bigoplus_{i=1}^k (Q_i)_{21} & \bigoplus_{i=1}^k (Q_i)_{22} & \cdots & \bigoplus_{i=1}^k (Q_i)_{2n} \\ \vdots & \vdots & \ddots & \cdots \\ \bigoplus_{i=1}^k (Q_i)_{n1} & \bigoplus_{i=1}^k (Q_i)_{n2} & \cdots & \bigoplus_{i=1}^k (Q_i)_{nn} \end{bmatrix}.$$

By Lemma 2.2.2, Q is a quantum automorphism, since each Q_i is. Additionally, define

$$M = \bigoplus_{i=1}^k M_i = \begin{bmatrix} M_1 & 0 & 0 & \cdots & 0 \\ 0 & M_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots & 0 \\ 0 & 0 & 0 & \cdots & M_k \end{bmatrix}.$$

Below we ensure that Q and M indeed give us the fractional automorphism F:

$$
\begin{aligned}
\langle\langle M, Q\rangle\rangle_{\alpha\beta} &= \langle M, Q_{\alpha\beta}\rangle && (8.2.3)\\
&= \langle M, \bigoplus_{i=1}^{k} (Q_i)_{\alpha\beta}\rangle && (8.2.4)\\
&= \operatorname{tr}\left(M \bigoplus_{i=1}^{k} (Q_i)_{\alpha\beta}\right) && (8.2.5)\\
&= \sum_{i=1}^{k} \operatorname{tr}(M_i (Q_i)_{\alpha\beta}) && (8.2.6)\\
&= \sum_{i=1}^{k} (F_i)_{\alpha\beta} && (8.2.7)\\
&= F_{\alpha\beta}. && (8.2.8)
\end{aligned}
$$

Above, equation 8.2.6 follows from the fact that both M and $Q_{\alpha\beta}$ are block diagonal, and the equality 8.2.7 is by choice of Q_i and M_i such that $\langle\langle M_i, Q_i\rangle\rangle = F_i$. Finally, 8.2.8 is the consequence of the equation 8.2.2. Hence,

$$\langle\langle Q, M\rangle\rangle = F.$$

□

Theorem 8.2.6 (Godsil) *If Q is a quantum automorphism of X with commuting entries and M is a density matrix, then there exist $\lambda_i \geq 0$ and automorphisms R_i such that $\langle\langle Q, M\rangle\rangle = \sum_i \lambda_i R_i$ with $\sum_i \lambda_i = 1$.*

Proof. Recall that since Q has commuting entries, by Lemma 2.2.1, Q can be written as a direct sum of classical automorphisms $Q = \boxplus_{i=1}^{n} R_i$. Recall that for any matrix B, $\operatorname{sum}(B) = \sum_{\alpha,\beta} B_{\alpha\beta}$. Let us define the entry-wise sum for the matrix of matrices:

$$
\begin{aligned}
s : M_n(\mathbb{C}^{d\times d}) &\to M_n(\mathbb{C})\\
(M_{ij})_{ij} &\mapsto (\operatorname{sum} M_{ij})_{ij}.
\end{aligned}
$$

As usual, $\circ$ denotes entry-wise multiplication, so we can write:

$$\begin{aligned}\langle\langle Q, M\rangle\rangle &= s\begin{bmatrix} Q_{11}\circ M & Q_{12}\circ M & \cdots & Q_{1n}\circ M \\ \vdots & \vdots & \dots & \vdots \\ Q_{n1}\circ M & Q_{n2}\circ M & \cdots & Q_{nn}\circ M \end{bmatrix} \\ &= s(M_{11}R_1 \boxplus M_{22}R_2 \boxplus \ldots \boxplus M_{nn}R_n) \\ &= \sum_{i=1}^{n} M_{ii}R_i\end{aligned}$$

As $\text{tr}M = 1$, and diagonal entries of positive-semidefinite matrix are non-negative, letting $\lambda_i = M_{ii}$ achieves the result. □

Corollary 8.2.7 (Godsil) *If for every fractional automorphism of X there exists a quantum automorphism Q and a density matrix M such that $\langle\langle Q, M\rangle\rangle = F$, then at least one of the following is true:*

(a) X is compact,

(b) X has quantum symmetry .

Proof. We will show that if X satisfies the hypothesis and does not have quantum symmetry, then it is compact.

Since X does not have quantum symmetry, all of its quantum automorphisms have commuting entries. By hypothesis, every fractional automorphism F equals $\langle\langle Q, M\rangle\rangle$ for some quantum automorphism Q of X (with commuting entries) and some density M. Thus, by Theorem 8.2.6, F is a convex combination of automorphisms of X, and so X is compact. □

As an example, Chris Godsil suggested Petersen graph. By [61], Petersen graph has no quantum symmetry and by [72], it is not compact. Therefore, we may conclude that some of its fractional automorphisms does not arise from a quantum automorphism.

8.3 Applications

In this section we prove that paths do not have quantum symmetry, as an application of compactness and fractional automorphism results.

Lemma 8.3.1 *[63, Theorem B] Let u be the magic unitary containing generators of the quantum automorphism group of X. For any $i, k, j, l \in V(X)$, if we have $d(i,k) \neq d(j,l)$, then $u_{ij}u_{kl} = 0$.*

Theorem 8.3.2 *Paths do not have quantum symmetry.*

Proof. Let vertices of P be labelled by $\mathbb{Z}_n$ in order. It is a well-known fact that paths have only one nontrivial automorphism, which reverses the order of the vertices. Paths are trees, therefore, they are compact [68]. So, every fractional automorphism is the following convex combination:

$$\lambda \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \dots & \ddots & \vdots \\ 0 & \dots & 0 & 1 \end{bmatrix} + (1-\lambda) \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & & & \vdots \\ 1 & 0 & \cdots & \end{bmatrix}.$$

Therefore, when n is even, the path P_n has an odd number of vertices, and every fractional automorphism will be of the form

$$\begin{bmatrix} \lambda & 0 & \dots & 0 & 0 & 0 & (1-\lambda) \\ 0 & \lambda & \dots & 0 & 0 & (1-\lambda) & 0 \\ \vdots & \cdots & \ddots & \vdots & \vdots & & \\ & \dots & 0 & 1 & 0 & 0 & \\ & \cdots 0 & (1-\lambda) & 0 & \lambda & 0 & \\ & \cdots (1-\lambda) & 0 & 0 & 0 & \ddots & \vdots \\ (1-\lambda) & \cdots 0 & 0 & 0 & 0 & 0 & \lambda \end{bmatrix}$$

with $(\frac{n+1}{2})^{th}$ diagonal entry 1, λ's on the remaining diagonal entries and $(1-\lambda)$ on the remaining off-diagonal entries. Thus, any quantum automorphism Q of an odd-length path should have $Q_{\frac{n-1}{2},\frac{n-1}{2}} = I$, and $Q_{ij} = 0$ if it is not a diagonal or an off-diagonal entry of Q. Therefore, the quantum automorphism of Q for odd Q is of the form

$$\begin{bmatrix} Q_{11} & 0 & 0 & \dots & 0 & 0 & I-Q_{11} \\ 0 & Q_{22} & 0 & \dots & 0 & I-Q_{22} & 0 \\ \vdots & & & & & & \\ 0 & 0 & \dots & I & \dots & 0 & 0 \\ \vdots & & & & & & \\ 0 & I-Q_{22} & 0 & \dots & 0 & Q_{22} & 0 \\ I-Q_{11} & 0 & 0 & \dots & 0 & 0 & Q_{11}. \end{bmatrix}$$

We will show that all nonzero Q_{ij} pairwise commute, which means they are simultaneously diagonalizable. Let $i, j \neq \frac{n+1}{2}$ be arbitrary. Then

$$Q_{ii} Q_{j,-j} = 0$$

by Lemma 8.3.1, since $i \neq \frac{n+1}{2}$, and so i and j are at different distance than i and $-j$.

Thus, all nonzero entries of Q pairwise commute, which is what we aimed to show.

Now, if entries of Q are not 0 or I, we would be able to decompose Q and one of the classical automorphisms will have ones in the locations other than diagonal or off-diagonal, a contradiction, Therefore, it must be that either $Q_{ii} = 0$ for all i or $Q_{i,-i} = 0$ for all i.

When n is even, the argument is analogous, except that then now we do not have a "middle" vertex. □

Chapter 9

Quantum automorphism group of a line graph

In [50] Schmidt and Nechita provide an algorithm predicts whether a graph has quantum symmetry or no by constructing matrices with entries that are "almost" projections and that "almost" commute with the adjacency matrix. In the paper authors include a list comparing the predictions of the algorithm about existence of quantum symmetries in certain graphs and the existent proofs in the literature. For two of the presented graphs the algorithm predicts absence of quantum symmetry, while the proof of that fact is unknown. By a *line graph* of a graph X we mean a graph $L(X)$ with $V(L(X)) = E(X)$ and two vertices are adjacent if corresponding edges are incident to the same vertex. One of the graphs is the line graph of a 6-cycle with chords between any pair of vertices at distance two. That raised the question of whether knowing the quantum automorphism group of a graph can lead to any information about the quantum automorphism group of its line graph, and this is what this chapter is about.

9.1 G^*, a different notion of quantum automorphism group

In the beginning of the quantum automorphisms research, Bichon [9] has defined another variant of quantum automorphism group of a graph X, called $G^*(X)$. While originally defined in the language of C^* algebras, since we use only with finite-dimensional quantum permutations in this thesis, we restrict this definition to finite-dimensional quantum permutations as well.

For a graph X, we define $G^*(X)$ to be a set of all quantum permutations Q satisfying the following conditions:

(a) $QA_X = A_X Q$

(b) $Q_{ik}Q_{jl} = Q_{jl}Q_{ik}$ for all $i \sim_X j$ and $k \sim_X l$.

This is the finite-dimensional version of the definition from [9], but this is not the standard definition of quantum automorphism group. Schmidt in [61] showed that G^* of Petersen graph is the same as its quantum automorphism group and then showed that Petersen graph has no quantum symmetry. In [62], Schmidt also showed that $G^*(X)$ is the same as quantum automorphism group of any graph of girth at least five. Examples of more relations between the two groups can be found in [63, Section 2.5] and [63, page 24] .

When we say quantum permutation is a quantum automorphism of a graph X, we mean a quantum permutation that is not required to satisfy Condition 9.2.2, but satisfies all the other conditions on the list. If we require Condition 9.2.2 (below) to hold for a quantum permutation Q, we say $Q \in G^*(X)$.

9.2 Quantum automorphism of a graph and its line graph

In this section we explore the relationship between the quantum automorphism groups of the line graph $L(X)$ and the original graph X.

Theorem 9.2.1 *Suppose X is a graph. Every quantum automorphism of X in $G^*(X)$ gives a quantum automorphism of $L(X)$.*

Proof. We work with directed graphs. Any undirected graph X can be easily transformed into a directed one by replacing every edge $\{i, j\}$ of X with two arcs (i, j) and (j, i). Therefore, we will use both pieces of notation: $\{i, j\}$ denotes the edge between i and j, and $\{(i, j), (j, i)\}$ denotes the two arcs replacing it. As mentioned in the introduction, our undirected graphs have opposite-facing arcs per each edge.

Suppose u is a quantum automorphism of X and thus satisfies:

$$u_{ij} = u_{ij}^* = u_{ij}^2 \tag{9.2.1}$$
$$u_{ik}u_{jl} = u_{jl}u_{ik} \text{ for all } (i,j) \in E(X), (k,l) \in E(X) \tag{9.2.2}$$
$$\sum_i u_{ij} = \sum_j u_{ij} = I \tag{9.2.3}$$
$$u_{ik}u_{jl} = u_{jl}u_{ik} = 0 \text{ for all } (i,j) \in E(X), (k,l) \notin E(X) \tag{9.2.4}$$
$$u_{ki}u_{lj} = u_{lj}u_{ki} = 0 \text{ for all } (i,j) \in E(X), (k,l) \notin E(X), \tag{9.2.5}$$

where the last two conditions are equivalent to commutativity with adjacency.

We are going to construct a quantum automorphism w that satisfies all of these conditions but Condition 9.2.2 for A_L, thus making w a quantum automorphism of X in our original sense. Since the vertices of $L(X)$ are the edges of X, the rows and columns of A_L are indexed by the edges of X. We will first define w based on the information we have from u and then prove that we obtain a quantum automorphism of $L(X)$ indeed.

Let

$$w_{\{i,j\},\{k,l\}} = u_{ik}u_{jl} + u_{il}u_{jk}.$$

Note that since $u \in G^*_{aut}(X)$, the above expression is well defined. To explain the analogy with the classical case, consider the case when u is the classical automorphism. We would like $w_{\{i,j\},\{k,l\}}$ to be one if the edge $\{k,l\}$ is mapped onto the edge $\{i,j\}$ by u. This happens if one of the following holds: either $u(k) = i$ and $u(\ell) = j$ or $u(k) = i$ and $u(\ell) = i$. The first case happens if and only if $u_{ik}u_{j\ell} = 1$. The second one happens if and only if $u_{il}u_{jk} = 1$.

The rest of the proof is dedicated to showing that w satisfies the conditions of quantum automorphism of $L(X)$.

<u>Claim:</u> $w_{\{i,j\},\{k,l\}} = w^*_{\{i,j\},\{k,l\}} = w^2_{\{i,j\},\{k,l\}}$.

<u>Proof:</u> We have

$$\begin{aligned} w^*_{\{i,j\},\{k,l\}} = (u_{ik}u_{jl} + u_{il}u_{jk})^* = u^*_{jl}u^*_{ik} + u^*_{jk}u^*_{il} &= u_{jl}u_{ik} + u_{jk}u_{il} \\ &= u_{ik}u_{jl} + u_{il}u_{jk} \end{aligned}$$

The last inequality follows from the fact that $\{j,i\}, \{l,k\}$ are the edges of the underlying undirected graph X. Therefore, $(i,j), (l,k) \in E(X)$, making u_{jl} and u_{ik} commute by Condition9.2.2. The same argument applies to the second summand. Additionally,

$$w^2_{\{i,j\},\{k,l\}} = w_{\{i,j\},\{k,l\}}w_{\{i,j\},\{k,l\}} = (u_{ik}u_{jl})^2 + u_{ik}u_{jl}u_{il}u_{jk} + u_{il}u_{jk}u_{ik} + (u_{il}u_{jk})^2$$

By the fact that the u_{ik}, u_{jl} and u_{il}, u_{jk} are projections and by the commutativity relation from Condition 9.2.2, the first and last term are idempotents. By the commutativity of u_{ik} and u_{jl} with the fact that $(i,i) \notin E(X)$, while $(k,l) \in E(X)$, the two middle terms are 0. Therefore,

$$w^2_{\{i,j\},\{k,l\}} = w_{\{i,j\},\{k,l\}}.$$

<u>Claim:</u> $\sum_{\{k,l\} \in E(X)} w_{\{i,j\}\{k,l\}} = I$

<u>Proof:</u>

$$\begin{aligned}
\sum_{\{k,l\} \in E(X)} w_{\{i,j\}\{k,l\}} &= \sum_{\{k,l\} \in E(X)} (u_{ik}u_{jl} + u_{il}u_{jk}) \\
&= \sum_{\{k,l\} \in E(X)} u_{ik}u_{jl} + \sum_{\{k,l\} \in E(X)} u_{il}u_{jk} \\
&= \frac{1}{2} \sum_{k,l \in V(X)} u_{ik}u_{jl} + \frac{1}{2} \sum_{k,l \in V(X)} u_{il}u_{jk} \\
&= \frac{1}{2} \left(\sum_{k \in V(X)} u_{ik} \sum_{l \in V(X)} u_{jl} + \sum_{l \in V(X)} u_{il} \sum_{k \in V(X)} u_{jk} \right) \\
&= \frac{1}{2}(I \cdot I + I \cdot I) \\
&= I
\end{aligned}$$

Note that the third equality follows from two facts. While we will explain only how the first term after the third equality sign was derived, analogous reasoning helps obtain the second term. First, since $\{i,j\} \in E(X)$, whenever $\{k,l\} \notin E(X)$, $u_{ik}u_{jl} = 0$, so automatically only the terms $u_{ik}u_{jl}$ will be non-zero with $\{k,l\} \in E(X)$, as expected. Secondly, when the summation is over all $k, l \in V$ we count each edge twice. In other words, for a fixed k and l, both $u_{ik}u_{jl}$ and $u_{jl}u_{ik}$ will appear in $\sum_{k,l \in V(X)} u_{ik}u_{jl}$. However, from commutativity relations on the entries on u, we know that $u_{ik}u_{jl} = u_{jl}u_{ik}$. Hence, the factor of $\frac{1}{2}$ appeared.

<u>Claim:</u> $wA_L = A_L w$.

<u>Proof:</u> We will be proving commutativity with the adjacency matrix by showing that the relations

- $w_{e_1,e_2} w_{e_3,e_4} = 0$ when $\{e_1, e_3\} \in E(L(X))$ but $\{e_2, e_4\} \notin E(L(X))$
- $w_{e_1,e_2} w_{e_3,e_4} = 0$ when $\{e_1, e_3\} \notin E(L(X))$ but $\{e_2, e_4\} \in E(L(X))$

hold. Again, it suffices to demonstrate the first relation, the second one follows from the same logic. It is more convenient to assume without loss of generality that $e_1 = \{i, j\}, e_2 = \{\ell, k\}, e_3 = \{i, r\}, e_4 = \{s, t\}$, where $i, j, k, \ell, r, s, t \in V(X)$, and $\{s, t\} \cap \{\ell, k\} = \emptyset$. First, let us prove the assertion that when $r = j$, then

$$w_{\{i,j\},\{k,\ell\}} w_{\{i,r\},\{s,t\}} = 0.$$

This is a simple consequence of the fact that the sum $\sum_{\{k,\ell\}} w_{\{i,j\}\{k,\ell\}} = I$ is a projection and each $w_{\{i,j\}\{k,\ell\}}$ is a projection which can happen if and only if $w_{\{i,j\}\{k,\ell\}}$ are pairwise orthogonal for different edges $\{k, \ell\}$. Therefore, we may assume that $r \neq j$. Now,

$$\begin{aligned} w_{\{i,j\},\{k,\ell\}} w_{\{i,r\},\{s,t\}} &= (u_{ik}u_{j\ell} + u_{i\ell}u_{jk})(u_{is}u_{rt} + u_{it}u_{rs}) \\ &= u_{ik}u_{j\ell}u_{is}u_{rt} + u_{ik}u_{j\ell}u_{it}u_{rs} + u_{i\ell}u_{jk}u_{is}u_{rt} + u_{i\ell}u_{jk}u_{it}u_{rs} \end{aligned}$$

Let us analyse each term individually. By virtue of the fact that both (i, j) and (k, ℓ) are in $E(X)$, we have that $u_{ik}u_{j\ell} = u_{j\ell}u_{ik}$, and so

$$u_{ik}u_{j\ell}u_{is}u_{rt} = u_{j\ell}u_{ik}u_{is}u_{rt} = 0,$$

since $k \neq s$ implies that $u_{ik}u_{is} = 0$. In a similar way commutativity is used in showing that the remaining terms are 0:

$$u_{ik}u_{j\ell}u_{it}u_{rs} = u_{j\ell}u_{ik}u_{it}u_{rs} = 0$$

$$u_{i\ell}u_{jk}u_{is}u_{rt} = u_{jk}u_{i\ell}u_{is}u_{rt} = 0$$

$$u_{i\ell}u_{jk}u_{it}u_{rs} = u_{jk}u_{i\ell}u_{it}u_{rs} = 0$$

Hence, w has all the desirable features of being a quantum automorphism of $L(X)$. □

Chapter 10

Quantum asymmetric graphs

In this section we introduce some results proving certain graphs do not have quantum symmetries.

Proposition 10.0.1 *[63, Proposition 2.4.14] Let u be the magic unitary with entries the generators of the quantum automorphism of a graph X with adjacency matrix A. For a pair $\{a, b\} \in V(X)$, if there is $k \in \mathbb{Z}$ such that $(A^k)_{aa} \neq (A^k)_{bb}$, then $u_{ab} = 0$.*

Proof. A^k is in the coherent algebra of X. The coherent algebra of X is the subalgebra of the coherent algebra of X. So, if $(A^k)_{aa} \neq (A^k)_{bb}$, we have that (a, a) and (b, b) are in different quantum orbitals of G. Therefore, $u_{ab} = u_{ab}u_{ab} = 0$. □

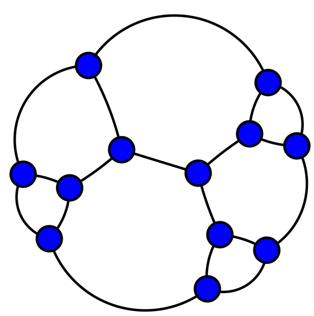

Figure 10.1: Frucht's graph, [1]

We illustrate how this proposition can be applied on the example of proving that Frucht's graph has no quantum symmetry. Similar technique we used to show the next result, that none of the 18 minimal asymmetric graphs have quantum symmetry.

Theorem 10.0.2 *Frucht's graph has no quantum symmetry.*

Proof. Frucht's graph has 12 vertices. Suppose u is the magic unitary representing its quantum automorphism group. Note that if A is the adjacency matrix of Frucht's graph, A^{11} has all pairwise different diagonal entries, except for two entries, say $A^{11}_{ii} = A^{11}_{jj}$. So, by Proposition 10.0.1, $P_{aa} = I$ for $a \notin \{i, j\}$. Thus only nonidentity, nonzero entries can be $P_{ij}, P_{jj}, P_{ii}, P_{ji}$. However, then it must be that there is a projection R such that $P_{ij} = R = P_{ji}, P_{ii} = I - R = P_{jj}$. If neither R nor $I - R$ is zero, P has commutative entries, so there is a classical automorphism of Frucht's graph. Since this is not the case, it must be that $R = 0$. So Frucht's graph has no nontrivial quantum automorphisms. □

In [66] Schweitzers prove that there are 18 minimal by inclusion asymmetric graphs. In other words, every asymmetric graph contains at least one of these as an induced subgraph. We prove that none of those have quantum symmetry.

Theorem 10.0.3 *None of the 18 minimal asymmetric graphs have quantum symmetry.*

Proof. Using Sage we found that diagonal entries of the tenth power of each of the 18 adjacency matrices have distinct diagonal entries. By Proposition 10.0.1, these graphs have no nontrivial quantum symmetries.

10.1 Structure of the graph and the properties of quantum automorphism matrix

In this section we relate the two notions.

In particular, we prove a necessary condition on the graph X to make sure $G^*_{aut}(X)$ contains a quantum automorphism with noncommutative entries.

Lemma 10.1.1 *Let X be a graph. If $G^*_{aut}(X)$ contains a quantum automorphism P such that $P_{ij}P_{k\ell} \neq P_{k\ell}P_{ij}$, then there are $\alpha, \beta, \gamma, \delta \in \{1, ..., n\}$ such that $\{\alpha, \beta, \gamma, \delta\} \neq \{i, j, k, \ell\}$ and $P_{\alpha\beta}P_{\gamma\delta} \neq P_{\gamma\delta}P_{\alpha\beta}$,*

Proof.

$$(\sum_{j'} P_{ij'})(\sum_{\ell'} P_{k\ell'}) = I \cdot I = I.$$

However, if $P_{ij}P_{k\ell} \neq P_{k\ell}P_{ij}$, as in the hypothesis, then the following two expressions are not equal.

$$(\sum_{j'} P_{ij'} - P_{ij})(\sum_{\ell'} P_{k\ell'} - P_{k\ell}) = I - P_{k\ell} - P_{ij} + P_{ij}P_{k\ell} \tag{10.1.1}$$

$$(\sum_{\ell'} P_{k\ell'} - P_{k\ell})(\sum_{j'} P_{ij'} - P_{ij}) = I - P_{k\ell} - P_{ij} + P_{k\ell}P_{ij} \tag{10.1.2}$$

We conclude that $\sum_{j'} P_{ij'} - P_{ij}$ and $\sum_{\ell'} P_{k\ell'} - P_{k\ell}$ do not commute. Therefore, there is a j' and ℓ' in $V(X)$ such that

$$P_{ij'}P_{k\ell'} \neq P_{k\ell'}P_{ij'}$$

Note that in 10.1.1 and 10.1.2 we were summing over the i^{th} and k^{th} rows of P, which allowed us to find noncommutative projections in those two rows. If we were summing over the i^{th} row and ℓ^{th} column instead and would be able to find another pair of non commuting projections.

$$P_{ij''}P_{k'\ell} \neq P_{ij''}P_{k'\ell}.$$

Similarly, there exist two more pairs of non-commuting projections

$$P_{i'j}P_{k''\ell} \neq P_{k''\ell}P_{i'j}$$

and

$$P_{i''j}P_{k\ell''} \neq P_{k\ell''}P_{i''j}.$$

Note that we do not claim that $i \neq i' \neq i''$ or $j \neq j' \neq j''$ or $k \neq k' \neq k''$ or $\ell \neq \ell' \neq \ell''$. □

Corollary 10.1.2 *If X does not have two triples of vertices $\{i, k, k'\}$ and $\{\ell, j, j''\}$ such that*

(a) $\{i, k\} \notin E(X)$ and $\{i, k'\} \notin E(X)$ and

(b) $\{\ell, j\} \notin E(X)$ and $\{\ell, j''\} \notin E(X)$,

then every quantum automorphism in $G^*_{aut}(X)$ has commuting entries.

Proof. If $P \in G^*_{aut}$, then the only non-commuting entries are of the form

$$P_{ij}P_{k\ell} \neq P_{k\ell}P_{ij}$$

where $\{i, k\} \notin E(X)$ and $\{j, \ell\} \notin E(X)$. From the previous lemma, we know that there also exists a pair

$$P_{ij''}P_{k'\ell} \neq P_{ij''}P_{k'\ell}.$$

Therefore, it must be that $\{i, k'\} \notin E(X)$ and $\{j''\ell\} \notin E(X)$. □

Proposition 10.1.3 *The cubic Halin graphs on at least seven vertices are not compact.*

Proof. First, Halin graphs are planar, so they are not of the form $\mathrm{Cay}(\mathbb{Z}_{2m}, \{1, -1, m\})$. Second, graphs $\mathrm{Cay}(\mathbb{Z}_m \times Z_2, \{(1,0), (0,1), (-1,0)\})$ look like two cycles connected by a matching. It is a Halin graph for $m = 3$. It is easy to see that out of all possible spanning trees, none can be encircled by a cycle to create $\mathrm{Cay}(\mathbb{Z}_m \times \mathbb{Z}_2, \{(1,0), (0,1), (-1,0)\})$. □

Here is another proof of the fact that if trees have have two cherries, they have quantum symmetry. The original proof was in [37], our proof gives the form of the general structure of quantum automorphism.

Proposition 10.1.4 *If a tree X has two cherries, it has quantum symmetry.*

Proof. Let the adjacency matrix of X be A. Let quantum automorphism be Q. By Proposition 10.0.1, $Q_{ij} = 0$ if i is a leaf and j is not. Thus, we can order the vertices of X such that first the leaves label rows of A and then non-leaves. Then Q is block-diagonal with the rows of the top block labelled by leaves.

Now we let the bottom block be identity. By *cherry* in a tree we mean two leaves adjacent to a vertex of degree at least two. Suppose leaves $\{a, b\}$ form a cherry and $\{i, j\}$ form a cherry. Then let $Q_{aa} = P, Q_{bb} = P$ and $Q_{ii} = P' = Q_{jj}$. Let $Q_{ab} = I - P = Q_{ba}$ and $Q_{ij} = I - P' = Q_{ji}$. We do not need to check orthogonality within the first block, since no leaves are adjacent to each other. We only need to check that

$$Q_{ab}Q_{xx} = 0 \text{ and } Q_{ij}Q_{yy} = 0$$

for appropriate nonleaves x, y .

But since a and b are leaves sharing the root, there is no x for which we require $Q_{ab}Q_{xx} = 0$. Similarly, no y such that we need to ensure $Q_{ij}Q_{yy} = 0$. Hence, Q is quantum symmetry of X.

In [37] Junk et al show that almost all trees have quantum symmetry because almost all trees have two cherries and in [69] the quantum automorphism groups of trees are constructed.

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