Greedy and Lazy β-expansions for PV and Salem Numbers

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

We call a sequence (a_i) a **β-representation** for x provided $\sum_{i=1}^{\infty} a_i \beta^{-i} = x$. We give the name **expansion** to those representations that are lexicographically largest and smallest. We call them the **greedy** and **lazy** expansions for x and denote them by $d_\beta(x)$ and $\ell_\beta(x)$ respectively. A real number, β, is called a **Parry number** if the greedy β-expansion for 1 is finite or eventually periodic. We also introduce the concept of a **lazy Parry number** which is a real number β such that the lazy β-expansion for 1 is eventually periodic.

Two important sets of algebraic integers in relation to Parry numbers are the PV and Salem numbers. It is known that all PV numbers are Parry numbers. We will show that all PV numbers are also lazy Parry numbers. This thesis focuses on finding Salem numbers that are Parry numbers and describing the greedy expansions for 1 that they induce. Further work develops similar results for lazy Parry numbers and their lazy expansions for 1.

Given a PV number θ with minimal polynomial $M(x)$, we know that for sufficiently large m the polynomial $T_m^{\pm}(x) = M(x)x^m \pm M^*(x)$ has a Salem root, $\alpha_m^{\pm}$. We will give criteria that relates the greedy $\alpha_m^{\pm}$-expansions for 1 to $d_\theta(1)$, the greedy θ-expansion for 1. Following our discussion of lazy expansions for PV numbers we develop similar results for lazy $\alpha_m^{\pm}$-expansions.

We consider Salem numbers of degree 4, the minimal such degree. We will prove these are always lazy Parry numbers and give explicit constructions for their lazy expansions for 1. We compare these expansions to the results of Boyd, that proved that they are also Parry numbers.

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Chapter 1

Introduction

This thesis is primarily concerned with representations of the positive real unit 1 in numeration systems with non integer bases. In this chapter, we begin by introducing the key concepts that will be discussed throughout this thesis. By providing a brief history of numeration systems, we motivate the notion of non-integer bases. In general, we use successive powers of a positive integer to represent numbers. For example $12.789 = 10 + 2 \cdot 10^0 + 7 \cdot 10^{-1} + 8 \cdot 10^{-2} + 9 \cdot 10^{-3}$. The question we aim to answer is what happens when we take this base to be a non integer.

In Section 1.1 we define β-representations, the key concept for our results. Finding that multiple representations can exist for the same real number, we define an ordering on then. With this ordering, we take note of the largest and smallest, giving them distinguished names. In Section 1.2 we will introduce two families of algebraic integers, known as Pisot-Vijayaraghavan (PV) numbers and Salem numbers. We will explore why using these numbers as the non integer base for our numeration systems yields interesting results. Section 1.3 first combines these concepts, showing that the distinguished representations of 1 have certain forms under PV bases. Lastly, we conclude this chapter by giving a brief overview of the structure of the remaining chapters of this thesis.

1.1 β-expansions

As different cultures around the world evolved, so did the numeral systems they used. That is, the way they write numbers. In some cases, we see non positional systems, the most well known in our time being the Roman numerals. In this system, we use different symbols to represent different powers of 10, $I = 10^0, X = 10^1, C = 10^2, M = 10^3$. We still see the use of Roman numerals in the present day, most commonly to represent years (MMXXVI = 2026) or on the faces of clocks.

However, in most cases, we find positional numeration systems. Here, one or more bases are chosen, and each digit represents a different power of the base. Remnants of non decimal bases are still in use today. For example, we divide hours into 60 minutes, which originates with the ancient Babylonians. It is believed that they chose base 60 due to its large number of divisors.

We find that most civilizations developed under decimal numeration systems. Each digit was to represent a power of 10. Among these civilizations, we find some of the most famous in history, the ancient Egyptians, Greeks, Indians, Chinese, and Hebrew. In many of these, the radix, or the number of unique digits, is equal to the base.

To this day, the base 10 numbering system is used to represent real numbers conventionally, that is, each digit in the representation is attributed to a specific power of 10. Other integer bases have been used elsewhere in academic studies, most commonly binary (base 2) and hexadecimal (base 16).

We are concerned with numeration systems where the base is not an integer. That is, each digit represents the copies of a different power for our non-integer base β required to construct a desired real number. We must choose a radix for our situation. In the integer case, we let the alphabet of digits $\mathcal{A}_N = \{0, 1, \ldots, N-1\}$ be the set of non negative integers smaller than the base N. We will see why it makes sense to continue with this construction in the non integer case.

These systems with non-integer bases were first written about by Rényi [20] in 1957. Rényi was considering representations of a real number x by iterations of a positive function $y = f(x)$, calling these f-expansions. He writes

$$x = \varepsilon_0 + f(\varepsilon_1 + f(\varepsilon_2 + f(\varepsilon_3 + \cdots)\ldots).$$

Where the digits $\varepsilon_n = \varepsilon_n(x)$ and the remainders $r_n(x) = f(\varepsilon_{n+1} + f(\varepsilon_{n+2} + f(\varepsilon_{n+3} + \cdots)\ldots)$ satisfy the following recurrence relation.

$$\begin{aligned} \varepsilon_0(x) &= \lfloor x \rfloor, \quad r_0(x) = x - \varepsilon_0(x), \\ \varepsilon_{n+1}(x) &= \lfloor g(r_n(x)) \rfloor, \quad r_{n+1}(x) = g(r_n(x)) - \varepsilon_n(x), \quad n \geq 0. \end{aligned} \tag{1.1.1}$$

Where $g(y) = x$ is the inverse function of $f(x) = y$. He considered the following function where $\beta > 1$ is not an integer,

$$f(x) = \begin{cases} x/\beta, & \text{for } 0 \leq x \leq \beta \\ 1, & \text{for } \beta < x. \end{cases} \tag{1.1.2}$$

Then, using the recursive definitions, we have

$$x = \varepsilon_0 + \frac{\varepsilon_1}{\beta} + \frac{\varepsilon_2}{\beta^2} + \frac{\varepsilon_3}{\beta^3} + \cdots + \frac{\varepsilon_n}{\beta^n} + \cdots$$

where except for the possibility of $n = 0$ we have the digits ε_n taking values from the set $\{0, 1, \ldots, \lfloor\beta\rfloor\}$. In the literature such objects have come to be commonly referred to as β-representations, a term introduced by Parry [19] in 1960. Here we use β as the non-integer base for the representations. Parry expanded on the work of Rényi, proving many results. In particular, relating to what are commonly called eventually periodic representations. Here, we give a formal definition for β-representations.

Definition 1.1.1. *Let $\beta \in (M, M+1)$. Consider the expansion*

$$x = \sum_{i=1}^{\infty} a_i \beta^{-i} \tag{1.1.3}$$

*where $a_i \in \{0, 1, \ldots, M = \lfloor\beta\rfloor\} = \mathcal{A}_\beta$. Then the sequence $(a_i) = a_1 a_2 \ldots$ is a **β-representation** for x. If there exists $k \in \mathbb{N}$ with $a_k \neq 0$ such that $a_i = 0$ for all $i \geq k$ then we say that the representation is **finite** and write it as $a_1 a_2 \ldots a_k$. We call the set $\mathcal{A}_\beta$ the alphabet from which the digits a_i are selected.*

Unlike Rényi we restrict ourselves to the case $a_0 = 0$. This restricts the real numbers x for which we can write a valid β-representation. In practice, we are mainly concerned with finding β-representations for 1 where, in any case $a_0 = 0$. If we wanted to find representations for larger real numbers, we need only to allow $a_0 \neq 0$. Here we introduced the concept of eventually periodic representations originally studied by Parry. If a sequence is not finite, this is the next best structure which allows for algebraic manipulation.

Definition 1.1.2. *If $a_1 a_2 \ldots$ is a β-representation and suppose that there exists $k, \ell \in \mathbb{N}$ such that $a_n = a_{n+\ell}$ for all $n > k$ we say that $a_1 a_2 \ldots$ is **eventually periodic** and we write it as $a_1 \ldots a_k (a_{k+1} \ldots a_{k+\ell})^\omega$. If we take k and ℓ to be minimal, then we call the string $a_1 \ldots a_k$ the preperiodic part of the expansion, and we say that the preperiod has length k. The string $a_{k+1} \ldots a_{k+\ell}$ is called the periodic part of the expansion and has length ℓ. Where the notation w^n indicates the concatenation of the word w with itself n times. w^ω is the infinite concatenation of w with itself.*

In general, we can define β-representations where the base β could be from a large array of mathematical objects. In the literature there exist explorations for real β larger than 1, complex β with modulus greater than 1 and even β that are square matrices. Most often in the literature, we see the restriction to real $\beta \in (1, 2)$ so that the set of viable digits in $\{0, 1\}$.

In this work, we define β-representations so that the base is a positive real non-integer, say $\beta \in (M, M+1)$. We select digits from $\mathcal{A}_\beta = \{0, 1, \ldots, \lfloor\beta\rfloor\}$ so that the radix of our numeral system is $M + 1 = \lfloor\beta\rfloor + 1$. Unlike the case where the base is an integer, it is common to have multiple representations for the same real number. In such a case, we can define an ordering on all such representations.

Definition 1.1.3. *Let* $\mathbf{a} = a_1 a_2 \ldots$ *and* $\mathbf{b} = b_1 b_2 \ldots$ *be two different* β*-representations for a real number* x*. Then we say that* $\mathbf{a}$ *is* ***lexicographically less*** *than* $\mathbf{b}$ *if there exist* $k \in \mathbb{N}$ *such that* $a_k < b_k$ *and* $a_i = b_i$ *for all* $i < k$*. If this is the case, we write* $\mathbf{a} <_{lex} \mathbf{b}$*. When comparing a finite representation, we use the equivalent infinite sequence by appending an infinite number of zeros to the end of the sequence.*

Example 1.1.4. *Consider the representations of 1 under base* $\phi \approx 1.61803$*, the Golden Ratio and root of* $x^2 - x - 1$*. It is known that the finite representations of* 1 *in base* ϕ *can be written as* $1(01)^n 1$ *for each* $n \in \mathbb{Z}_{\geq 0}$*. Infinite expansions* $1(01)^\omega$ *and* 01^ω *are also valid. From the above definition, we have*

$$11 >_{lex} 1011 >_{lex} 1(01)^2 1 >_{lex} \cdots >_{lex} 1(01)^\omega >_{lex} 01^\omega.$$

Notice if we consider preperiodic parts of non minimal length then we can write the same representation in multiple ways. For example, we have the following equality of sequences.

$$1(01)^\omega =_{\text{lex}} 101(01)^\omega =_{\text{lex}} 10101(01)^\omega.$$

One reason why we often consider the case $\beta \in (1, 2)$ is to restrict the allowable set of digits to a binary set. The following distinguished representation becomes easier to compute and work with in this case.

Definition 1.1.5. *If* $a_1 a_2 \ldots$ *is the maximal lexicographic representation for* x *with base* β*, then we say that* $a_1 a_2 \ldots$ *is the* ***greedy expansion*** *for* x *with base* β*, or* ***greedy*** β***-expansion****. This is commonly denoted by* $d_\beta(x)$*.*

We describe the following algorithm, which for a given value x and base β produces the greedy expansion of x in base β. The basic components of this algorithm were first given by Rényi [20] in his original paper on the subject.

Algorithm 1.1.6. *Set* $r_0 := x$*. Then let* $a_n = \lfloor \beta \cdot r_{n-1} \rfloor$ *and* $r_n = \beta \cdot r_{n-1} - a_n$*. Then* $a_1 a_2 \cdots = d_\beta(x)$ *is the greedy* β*-expansion for* x*.*

The result of this algorithm is that at each step a_n is the largest element of $\mathcal{A}_\beta$ such that $\sum_{i=1}^{n} a_i \beta^{-i} < x$. If a_n were taken to be any larger, Equation (1.1.3) which defines the β-representation fails. Therefore, the constructed sequence could not be the greedy β-expansion. In this case, the partial sum exceeds the real number which we are trying to represent. All terms are non negative, so this is not a possible β-representation. Notice that the algorithm does not rely on the fact $\beta \in (1, 2)$, this restriction would ensure that if a given digit a_n cannot be 1 it must be 0.

Parry [19] was the first to further explore the properties of greedy expansions. He presented the following result, which allows us to determine if a sequence is the greedy expansion for 1 under some base β.

Theorem 1.1.7 (Parry, [19]). *Let* $\mathbf{a} = (a_i)_{i\geq 1}$ *be a sequence in* $\{0, 1, \ldots, M\}^{\mathbb{N}}$. *That is,* $\mathbf{a}$ *is a non-empty finite or infinite sequence* $a_1 \ldots a_k$ *or* $a_1 a_2 \ldots$ *with* $a_i \in \{0, 1, \ldots, M\}$. *Then the sequence* $\mathbf{a}$ *is the greedy expansion of* 1 *for some* $\beta > 1$ *if and only if for all* $j \geq 1$

$$\sigma^j(\mathbf{a}) <_{lex} \mathbf{a}.$$

Here $\sigma(a_1 a_2 \ldots) = a_2 a_3 \ldots$.

If $\mathbf{a} = a_1 a_2 \ldots a_k (a_{k+1} \ldots a_{k+\ell})^\omega$, then it suffices to check the above condition only with $1 \leq j \leq k + \ell$. This is because $\sigma^{k+i}(\mathbf{a}) = \sigma^{k+n\ell+i}(\mathbf{a})$ for all $i, n \geq 0$

Example 1.1.8. *The sequence* $\mathbf{a} = 111101(1100)^\omega$ *is the greedy expansion for* 1 *in some base* β. *Consider the shifted sequences* $\sigma^j(\mathbf{a})$.

$$\begin{aligned}
\sigma(\mathbf{a}) = 11101(1100)^\omega &<_{\text{lex}} 1101(1100)^\omega \\
\sigma^2(\mathbf{a}) = 1101(1100)^\omega &<_{\text{lex}} 1101(1100)^\omega \\
\sigma^3(\mathbf{a}) = 101(1100)^\omega &<_{\text{lex}} 1101(1100)^\omega \\
\sigma^4(\mathbf{a}) = 01(1100)^\omega &<_{\text{lex}} 1101(1100)^\omega \\
\sigma^5(\mathbf{a}) = 1(1100)^\omega &<_{\text{lex}} 1101(1100)^\omega \\
\sigma^6(\mathbf{a}) = (1100)^\omega &<_{\text{lex}} 1101(1100)^\omega \\
\sigma^7(\mathbf{a}) = (1001)^\omega &<_{\text{lex}} 1101(1100)^\omega \\
\sigma^8(\mathbf{a}) = (0011)^\omega &<_{\text{lex}} 1101(1100)^\omega \\
\sigma^9(\mathbf{a}) = (0110)^\omega &<_{\text{lex}} 1101(1100)^\omega
\end{aligned}$$

As the sequence of σ *shifts is eventually periodic,*

$$\sigma^{4n+j+6}(\mathbf{a}) = \sigma^{4(n+1)+j+6}(\mathbf{a}). \quad \forall n, j \geq 0.$$

We can see that there are only finitely many unique shifts, and all possible shifts are lexicographically smaller than the expansion.

We will sometimes also consider the lexicographically largest infinite representation, in the case that it is not already infinite. This is called the **quasi-greedy expansion** and we write $d^*_\beta(x)$. For example, the Golden Ratio $\phi \approx 1.61803$ has greedy β-expansion for 1 given by the sequence $d_\phi(1) = 11$. When we apply the greedy algorithm while ensuring that the remainder r_n is never 0, we find that $a_{2k} = 0$ and $a_{2k+1} = 1$ for each $k \in \mathbb{N}$. Following this logic, we find the quasi-greedy expansion $d^*_\phi(1) = 1(01)^\omega$. This concept was first introduced by Daróczy and Kátai [10], who called this type of expansion the quasi-regular expansion. We direct the reader to Chapter 2 of [11] for a more detailed construction and further results on the subject of quasi-greedy expansions.

Definition 1.1.9. *Similarly, if $a_1a_2\ldots$ is the smallest lexicographic representation for x with base β, then we say that $a_1a_2\ldots$ is the* ***lazy expansion*** *for x with base β or* ***lazy β-expansion*** *This is commonly denoted by $\ell_\beta(x)$.*

We have the following similar algorithm to construct lazy expansions.

Algorithm 1.1.10. *Let $\beta \in (M, M+1)$ and set $r_0 := x$. Then let $a_n = \lceil \beta \cdot r_{n-1} - M/(\beta - 1)\rceil \in \mathcal{A}_\beta$ and $r_n = \beta \cdot r_{n-1} - a_n$. Then $a_1a_2\cdots = \ell_\beta(x)$ is the lazy β-expansion for x.*

The result of this algorithm is that at each step a_n is the smallest element of $\mathcal{A}_\beta$ such that $\sum_{i=1}^{n} a_i\beta^{-i} + \sum_{i=n+1}^{\infty}\lfloor\beta\rfloor\beta^{-i} > x$. If a_n were taken any smaller equation (1.1.3) defining the β-representation fails. Therefore, the constructed sequence could not be the lazy β-expansion. In this case, the partial sum fails to reach a value large enough for the remaining digits to meet the real number they are trying to represent. All terms have a maximum value of $\lfloor\beta\rfloor$ so we cannot represent arbitrarily large values.

In what follows, we write greedy β-expansion or lazy β-expansion exclusively to mean the greedy or lazy representations We continue to write β-representation when considering an arbitrary representation that may or may not be one of the greedy expansion or lazy expansion.

We largely restrict our focus to the case $x = 1$. That is, we are interested in finding β-representations for 1. We call a base, β, **univoque** if there exists a unique representation for 1 in the base β. In other words, the greedy and lazy β-expansions for 1 are lexicographically equal, $d_\beta(1) =_{\text{lex}} \ell_\beta(1)$. In his work on the subject Erdős and his collaborators often referred to these two expansions, which can be calculated algorithmically by the common terms used today. Erdős, Joó, and Komornik [12] gave a criterion for when a base in $(1, 2)$ had a unique expansion for 1, in other words it was univoque. Their paper provides some characteristics that all greedy or lazy expansions share. Consider the following result.

Lemma 1.1.11 (Erdős, Joó, and Komornik, [12])**.** *Fix $1 < \beta < 2$ and $0 \le x \le 1/(\beta - 1)$ such that*

$$x = \sum_{i=1}^{\infty} a_i\beta^{-1}, \quad a_i \in \{0, 1\} \tag{1.1.4}$$

So that (a_i) a β-representation for x.

1. *Then (a_i) is the greedy β-expansion for x if and only if,*

$$\sum_{i=1}^{\infty} a_{k+i}\beta^{-i} < 1, \quad \text{whenever } a_k = 0.$$

2. *And* (a_i) *is the lazy* β*-expansion for* x *if and only if,*

$$\sum_{i=1}^{\infty}(1-a_{k+i})\beta^{-i}<1, \quad \text{whenever } a_k=1.$$

This pair of results has been used to prove further characterizations of greedy and lazy expansions as well as univoque numbers. Notice how these results in particular require the restriction $1<\beta<2$ as otherwise $x=1$ is not in the allowed interval and we cannot guarantee $a_i \in \{0,1\}$.

We will primarily be interested in β-expansions for 1 that are finite or eventually periodic.

Definition 1.1.12. *A real number* β *is called a* ***Parry number*** *if the greedy* β*-expansion for* 1 *is finite or eventually periodic.*

The primary results of this thesis are concerned with proving whether certain algebraic integers are Parry numbers. To avoid algebraic manipulation with infinite sequences, we introduce the following object. If any representation of 1 is finite or eventually periodic, we can define a polynomial whose coefficients are related to that representation.

Definition 1.1.13. *Consider an arbitrary* β*-representation for 1,* $a_1a_2\dots a_k$ *if the sequence is finite, and* $a_1a_2\dots a_k(a_{k+1}\dots a_{k+\ell})^\omega$ *if the sequence is eventually periodic. Define* $P_j(x)=x^j-a_1x^{j-1}-\cdots-a_j$*. Then the* ***companion polynomial*** *of the representation is defined as*

$$R(x)=\begin{cases}P_k(x) & \text{finite case}\\ P_{k+\ell}(x)-P_k(x) & \text{periodic case.}\end{cases} \tag{1.1.5}$$

The companion polynomial has historically only been associated with the greedy expansion, we find it useful when working with all types of representations. In particular, the companion polynomial for the lazy expansion is well defined. The companion polynomial was first introduced by Parry [19], where he called it the **characteristic equation**. Later in his work on the subject, Boyd [8], [9] referred to this object as the **characteristic polynomial** and later the companion polynomial, which is widely seen to this date in the literature.

Remark 1.1.14. *In both cases, we have* $R(\beta)=0$*. Together with the fact that* $R(x)$ *is a monic integer polynomial, it follows that if a* β*-representation for 1 is finite or eventually periodic, then* β *is an algebraic integer. Furthermore, if* $M(x)$ *is the minimal polynomial for* β *then* $M(x)|R(x)$*.*

We label the remaining factor of the companion polynomial. This object will be used to prove that a given sequence is a β-representation for 1.

Definition 1.1.15. *Suppose β has a minimal polynomial $M(x)$ and $R(x)$ is the companion polynomial for some β-representation for 1. Then we have $M(x)|R(x)$. We write $R(x) = M(x)Q(x)$ and we call $Q(x)$ the* ***co-factor*** *of this β-representation.*

Example 1.1.16. *In base $\Phi \approx 1.61803$, with minimal polynomial $M(x) = x^2 - x - 1$, the greedy Φ-expansion for 1 is $d_\Phi(1) = 11$. In the greedy case,*

$$R(x) = P_2(x) = x^2 - x - 1, \quad Q(x) = 1.$$

Another valid Φ-representation for 1 is the sequence $(a_i) = 101011$. In this case,

$$R(x) = P_6(x) = x^6 - x^5 - x^3 - x - 1, \quad Q(x) = x^4 + x^2 + 1.$$

The lazy Φ-expansion for 1 is $\ell_\Phi(1) = 01^\omega$. In the lazy case,

$$R(x) = P_2(x) - P_1(x) = (x^2 - 1) - (x), \quad Q(x) = 1.$$

We see that not all representations have the same co-factor. However, a finite representation may share co-factor with an eventually periodic representation.

1.2 PV and Salem Numbers

This section introduces the main sets of real numbers which will act as the bases for the representations we consider.

Definition 1.2.1. *A* ***Pisot-Vijayaraghavan number****, hereby referred to as a* ***PV number****, θ is a real algebraic integer $\theta > 1$ such that all conjugates of θ are strictly less than 1 in modulus.*

Example 1.2.2. *The following algebraic integers are PV numbers, we provide their minimal polynomials;*

$$\frac{1+\sqrt{5}}{2} \approx 1.618033\ldots, \quad M(x) = x^2 - x - 1,$$

$$1 + \sqrt{2} \approx 2.414213\ldots, \quad M(x) = x^2 - 2x - 1,$$

$$\approx 1.534157\ldots, \quad M(x) = x^5 - x^3 - x^2 - x - 1.$$

We find that PV numbers can have minimal polynomials of arbitrarily large degrees and others that are arbitrarily large in value.

Often we see examples in the literature that these numbers are labeled simply as Pisot numbers, and we use the term PV number to better recognize the work of both mathematicians who originally worked with them. There is a great deal of study on PV numbers and their structure. For example, we know that the set of PV numbers is closed in $\mathbb{R}$, contains all its limit points, and is countable. The proof of closedness is due to the work of Salem [21]. The real root $\theta_0 \approx 1.324717...$ of $x^3 - x - 1$ is proven to be the smallest PV number [23], which must exist since the set is closed. The book compiled by Bertin and company [4], provides a survey on PV numbers and related Salem numbers, which we will soon discuss.

We further discuss the structure of the set of PV numbers for later use in this thesis. In particular, what is known about the limit points of the set of PV numbers within the interval $(1, 2)$. The following result enumerates these limit points.

Theorem 1.2.3 (Amara, [2])**.** *The limit points of the PV numbers in* $(1, 2)$ *are the following:*

$$\varphi_1 = \psi_1 < \varphi_2 < \psi_2 < \varphi_3 < \chi < \psi_3 < \varphi_4 < \cdots < \psi_r < \varphi_{r+1} < \cdots < 2. \tag{1.2.1}$$

The minimal polynomial of φ_r *is* $\Phi_r(x) = x^{r+1} - 2x^r + x - 1$*, the minimal polynomial of* ψ_r *is* $\Psi_r(x) = x^{r+1} - x^r - \cdots - x - 1$*, and the minimal polynomial of* χ *is* $\chi(x) = x^4 - x^3 - 2x^2 + 1$*.*

We call a PV number **regular** if it is part of a family which approaches a limit point of the PV numbers. The regular PV numbers have a particularly nice structure in the interval $(1, 2)$. The families of regular PV numbers that approach the limit points from Theorem 1.2.3 are given in Table 1.2.1. These polynomials are not generally irreducible; they do however admit a single root greater than 1, which is the regular PV number. The remaining factors of these polynomials are always cyclotomic. As each polynomial in Table 1.2.1 is of the form $M_q(x) = A(x)x^q \pm C(x)$, we can verify that in each case $|A(x)| \geq |C(x)|$ on the unit circle $|x| = 1$. Therefore, by Rouché's Theorem, $A(x)$ and $M_q(x)$ have the same number of roots outside the disk $|x| \leq 1 + \varepsilon$. So $M_q(x)$ has a unique root outside the unit circle and must be the product of a PV polynomial and a cyclotomic polynomial.

PV numbers that are not regular are called **irregular**. It is known that for each PV number in Equation 1.2.1 there exists $\varepsilon > 0$ such that all PV numbers within ε of this limit point are regular PV numbers. Furthermore, it is known that only finitely many irregular PV numbers exist in the interval $[1, 2 - \epsilon]$ for any $\epsilon > 0$, [25].

An extensive study has been conducted on greedy expansions for 1 under PV bases, especially as it refers to regular PV numbers. In fact, we have a complete description of the greedy expansion of 1 in any regular PV base, which has been established by Panju [18]. We complete a similar study for lazy expansions of 1 under regular PV bases later in this thesis.

Limit Points	Defining Polynomials
φ_r	$\Phi^{A\pm}_{(r,q)}(x) = \Phi_r(x)x^q \pm (x^r - x^{r-1} + 1)$
	$\Phi^{B\pm}_{(r,q)}(x) = \Phi_r(x)x^q \pm (x^r - x + 1)$
	$\Phi^{C\pm}_{(r,q)}(x) = \Phi_r(x)x^q \pm (x^r + 1)(x - 1)$
ψ_r	$\Psi^{A\pm}_{(r,q)}(x) = \Psi_r(x)x^q \pm (x^{r+1} - 1)$
	$\Psi^{B\pm}_{(r,q)}(x) = \Psi_r(x)x^q \pm (x^r - 1)/(x - 1)$
χ	$\chi^{\pm}_A(x) = \chi(x)x^q \pm (x^3 + x^2 - x - 1))$
	$\chi^{\pm}_B(x) = \chi(x)x^q \pm (x^4 - x^2 + 1))$

Table 1.2.1: Regular PV polynomials associated from Theorem 1.2.3

The next family of algebraic integers is the main topic of Chapters 2 and 4. We will see that they are closely related to PV numbers.

Definition 1.2.4. *A **Salem number** α is a real algebraic integer $\alpha > 1$ such that all conjugates of α are less than or equal to 1 in modulus and at least one conjugate is equal to 1 in modulus.*

Since there is at least one root with absolute value 1, if $T(x)$ is the minimal polynomial of a Salem number, it must be reciprocal. Hence, if α is the Salem root of $T(x)$, $1/\alpha$ is also a root and all other conjugates have an absolute value of 1. Since $T(x)$ is reciprocal and irreducible, all roots come in pairs which are not self inverse. Therefore, the degree of $T(x)$ must be even.

Example 1.2.5. *The following algebraic integers are Salem numbers; we provide their minimal polynomials;*

$$\approx 1.72208380\ldots, \quad T(x) = x^4 - x^3 - x^2 - x + 1,$$

$$\approx 1.17628083\ldots, \quad M(x) = x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1.$$

Notice that the set of coefficients in each case form a self inverse sequence, this is what it means for a polynomial to be reciprocal.

Comparatively, much less is known about the Salem numbers in relation to the PV numbers. For example, the smallest known Salem number is the real root of $x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$, yet no proof that it is the smallest in the set has been peer reviewed or widely accepted. We will see that Salem numbers can be related to PV numbers. Boyd [6], examined the relation between the size of a given PV number and the size of Salem numbers that relates to it. Using his results, Boyd showed that the smallest Salem number related to θ_0 is the Salem number mentioned above. Boyd provides a list of 39 'small'

Salem numbers, which are classified as Salem numbers less than 1.3. He later improved upon this list adding four more small Salem numbers. The final additions to the list were by Mossinghoff [17], who added four additional small Salem numbers. This list of 47 numbers remains the list of all known small Salem numbers to date.

In terms of β-expansions, we have some known results. As a result of their structure, Salem numbers appear only in even degrees. Then the smallest possible degree for a Salem polynomial is degree four. Boyd [8] proved that the greedy α-expansion for 1 where α is a degree four Salem number is eventually periodic. That is, all degree four Salem numbers are Parry numbers. He does this by enumerating all possible cases and writing down the expansions for each of them. We will provide full details of this result in Section 2.4. The following is one of the cases.

Proposition 1.2.6 (Boyd, [8]). *Let α be a Salem number of degree 4 with minimal polynomial*

$$T(x) = x^4 - ax^3 + bx^2 - ax + 1.$$

Then the greedy α-expansion for 1, $d_\alpha(1) = (c_i)$ is eventually periodic. If $c_1 = a$, so that $-a \leq b \leq 0$, $b \neq -a + 1$, then

$$(c_i) = a, [(-b), (a-1), (a-1)]^\omega$$

The proof of this theorem provides the basic framework for proving that a proposed sequence is the greedy expansion for 1 in a predetermined base. The following four steps were used to prove this result.

1. Boyd first proves that $T(x)$, the minimal polynomial of the Salem base β, divides $R(x)$, the companion polynomial of the proposed periodic sequence. This shows that (c_i) is a valid representation for 1 in the base β. It remains to show that it is the greedy β-expansion.

2. Next, he proves that the period given is of minimal length. This step is not strictly required to prove that the sequence is the greedy expansion. However, it has been a question of study to consider the minimal value of $k + \ell$, the combined length of the preperiod and periodic parts of the representation.

3. Next, he shows that $c_n \geq 0$ for all $n \geq 1$, otherwise the proposed sequence would have digits outside the allowed alphabet. We should also verify $c_n \leq M$, where $\beta \in (M, M + 1)$.

4. Finally, Boyd uses Theorem 1.1.7, the result of Parry, to prove that the sequence is the greedy expansion for 1 under some base.

Combining these results proves that the sequence is the greedy β-expansion for 1. For most of our results, we need only to prove steps 1 and 4. This is because the sequence in question will already assume that it has digits taken from the required range. We do not concern ourselves with the period length since this is not needed to prove that a given real number is a Parry number.

In what follows, we will use the same basic steps to prove that a sequence is the greedy (or lazy) expansion we are trying to find. We will require a lazy analog to Theorem 1.1.7, which will be developed in Chapter 3.

If $M(x)$ is the minimal polynomial of a PV number, then for sufficiently large m, $T_m^{\pm}(x) = M(x)x^m \pm M^*(x)$ admits a Salem number as a root. Here, $M^*(x) = x^{\deg(M)}M(1/x)$ is the reciprocal polynomial of $M(x)$. Moreover, as $m \to \infty$, these Salem numbers approach the PV number as a two-sided limit. Note that $T_m^{\pm}(x)$ may not be the minimal polynomial for the Salem number, as it may have some cyclotomic factors. We will refer to the case with Salem polynomial $T_m^{+}(x)$ as the *positive case* and the case where the Salem polynomial takes the form $T_m^{-}(x)$ as the *negative case*. In the negative case, $T_m^{-}(x)$ always has at least $(x-1)$ as a factor. We will determine the greedy and lazy β-expansions for Salem numbers arising in this way that satisfy certain properties.

It was Salem who first found this way to construct infinite sequences of Salem numbers. Boyd [6] later proved that all Salem numbers arise in this way. Notice that this construction implies the existence of infinitely many Salem numbers smaller than the smallest PV number $\theta_0 = 1.3247\ldots$. We also take the time to note that Salem numbers can appear as the root of multiple polynomials constructed in this way. That is, a Salem number does not need to have such a relationship with a unique PV number.

Example 1.2.7. *Consider the Salem root of the polynomial* $T(x) = x^4 - x^3 - 2x^2 - x + 1$. *The Table 1.2.2 illustrates multiple examples of minimal PV polynomial* $M(x)$ *such that* $S(x)$ *is a factor if* $T_m^{\pm}(x)$ *for some* m.

$M(x)$	m	$\pm$	$T_m^{\pm}(x)$	Extra Factor
$x^2 - x - 1$	2	$+$	$x^4 - x^3 - 2x^2 - x + 1$	1
$x^3 - x - 1$	1	$+$	$x^4 - x^3 - 2x^2 - x + 1$	1
$x^6 - 2x^5 - 1$	5	$+$	$x^{11} - 2x^{10} - x^6 - x^5 - 2x + 1$	$x^7 - x^6 + x^5 + x^2 - x + 1$
$x^3 - 2x^2 - x + 1$	1	$+$	$x^4 - x^3 - 2x^2 - x + 1$	1
$x^3 - x^2 - 2x - 1$	5	$-$	$x^8 - x^7 - 2x^6 - x^5 + x^3 + 2x^2 + x - 1$	$x^4 - 1$
$x^3 - x^2 - x - 1$	3	$+$	$x^6 - x^5 - x^4 - 2x^3 - x^2 - x + 1$	$x^2 + 1$

Table 1.2.2: Different constructions of the Salem polynomial in Example 1.2.7.

The results of Chapters 2 and 4 rely on the underlying PV number, θ, to construct greedy and lazy expansions under the Salem base. The fact that a given Salem number, α, can be

constructed from multiple PV numbers does not affect these results. The greedy and lazy expansions that appear from our theorems first select the underlying PV number and use it as a base to create a family of Salem numbers which have similar expansions for 1.

1.3 Greedy PV Expansions

This section reviews previously known results regarding greedy β-expansions where the base is a PV number. In particular, previously known results focus on the interval $(1, 2)$. We provide generalizations where needed for any non integer $\beta > 1$.

An important result on PV numbers states that if θ is a PV number, then the greedy expansion of 1 in the base θ is finite or eventually periodic [5]. That is, θ is a Parry number. This is not true for a general algebraic integer; for example, $\beta = \sqrt{2}$ does not admit a finite or periodic expansion. To see this, note that if this were the case, then we have $(x^2 - 2)$ divides $R(x)$, the companion polynomial of the expansion. However, $R(x)$ has a constant term of ± 1, as seen by Definition 1.1.13.

It is not a sufficient condition to have a periodic β-expansion for 1 to guarantee that the base is a PV or Salem number. For example $\beta \approx 1.7403$, the root of $x^8 - x^7 - x^6 - x^3 - x - 1$ has $d_\beta(1) = 11001011$ but is not PV or Salem. The following related result is known to be true.

Theorem 1.3.1 (Schmidt, [22])**.** *If $d_\beta(x)$ is periodic or finite for all $x \in \mathbb{Q} \cap [0, 1)$, then β is a PV or Salem number.*

Schmidt also conjectured that the converse of this theorem holds. It is this result, primarily, that leads towards greedy expansions for Salem numbers. One wonders whether all Salem numbers are Parry numbers. To that end, the following condition will be useful when we explore the related Salem numbers as roots of the polynomial $T_m^{\pm}(x)$.

Condition 1.3.2. *We say an algebraic integer θ with $d_\theta(1) = a_1 \dots a_k(a_{k+1} \dots a_{k+\ell})^\omega$ has a **reversibly greedy** θ-expansion if it is greedy and;*

$$a_1 \dots a_{k+\ell} >_{lex} a_{k+\ell-i} a_{k+\ell-i-1} \dots a_2 \quad \textit{for all } i, \ \textit{with } 0 \leq i \leq k + \ell - 2.$$

Here we replace finite sequences by the sequence of appropriate length by appending a number of 0's to the end as needed. In general, we will want to consider the cases where the given algebraic integer is a PV number. A version of this condition was first used by Hare and Tweedle [15] in their paper on greedy expansions of Salem numbers. They had only been working with finite expansions. We have extended the condition to be applied for use in cases where the greedy expansion is not finite.

Example 1.3.3. *Consider the expansion* $\overline{a} = 111101(1100)^\omega$ *that was shown to be greedy in a previous example. We claim that this is a reversibly greedy sequence.*

$$\begin{aligned} a_9 \dots a_2 = 01110111 &<_{\text{lex}} 1111011100 \\ a_8 \dots a_2 = 1110111 &<_{\text{lex}} 1111011100 \\ a_7 \dots a_2 = 110111 &<_{\text{lex}} 1111011100 \\ a_6 \dots a_2 = 10111 &<_{\text{lex}} 1111011100 \\ a_5 \dots a_2 = 0111 &<_{\text{lex}} 1111011100 \\ a_4 \dots a_2 = 111 &<_{\text{lex}} 1111011100 \\ a_3 a_2 = 11 &<_{\text{lex}} 1111011100 \\ a_2 = 1 &<_{\text{lex}} 1111011100 \end{aligned}$$

We see that all required inverted sequence shifts are smaller than $\overline{a}$ *as required, then the sequence is reversibly greedy.*

The following result shows that if a finite greedy expansion for 1 satisfies Condition 1.3.2, then the quasi-greedy expansion for 1 under the same base also satisfies the lexicographic order in Condition 1.3.2.

Proposition 1.3.4. *Suppose* $a_1 \dots a_k$ *is a finite reversibly greedy* β*-expansion. Then* $(b_1 \dots b_k)^\omega$ *satisfies Condition 1.3.2, where* $b_i = a_i$ *for* $i = 1, \dots, k-1$ *and* $b_k = a_k - 1$.

Proof. Let $\mathbf{c_i} = b_{k-i} \dots b_2$ and $\mathbf{b} = b_1 \dots b_k$. We need to show $\mathbf{b} >_{\text{lex}} \mathbf{c_i}$ for each $i \leq k-2$. The length of the string $\mathbf{b_i}$ is less than k, so we can replace $\mathbf{b}$ by the string $b_1 \dots b_{k-1}$. We can do the same for the analogous strings $\mathbf{a} = a_1 \dots a_k$ and $\mathbf{d_i} = a_{k-i} \dots a_2$.

We have $a_1 \dots a_{k-1} = b_1 \dots b_{k-1}$ and $\mathbf{a_i} >_{\text{lex}} \mathbf{b_i}$, the result follows. □

Table A.0.1 contains the greedy β-expansions of the limit points and most regular PV numbers. It also indicates whether they satisfy the reversibly greedy condition and gives the **pseudo co-factor** associated with the given expansion.

Recall that the co-factor of a representation is the polynomial $Q(x)$ such that $Q(x)M(x) = R(x)$, where $M(x)$ is the minimal polynomial for the base and $R(x)$ is the companion polynomial. The pseudo co-factor is the polynomial $\overline{Q}(x)$ such that the companion polynomial $R(x)$ can be written as $R(x) = \overline{M}(x)\overline{Q}(x)$, where $\overline{M}(x)$ is the defining polynomials given from Table 1.2.1 of the previous theorem, which may not be minimal.

1.4 Thesis Overview

In this section, we give an overview of the remaining chapters contained in this thesis.

Chapter 2 will first introduce a result of Hare and Tweedle [15], which relates greedy expansions under Salem bases to the greedy expansion under the related PV base, θ. We then generalize and prove their result for $\theta > 2$. This result covers the case where the greedy θ-expansion is finite. We then expand this result to use PV numbers with eventually periodic greedy expansions. These first sections will be restricted to the case where the Salem number is the root of $T_m^+(x) = M(x)x^m + M^*(x)$. We also consider the expansions for Salem numbers that arise as roots of $T_m^-(x) = M(x)x^m - M^*(x)$ later in the chapter. We will consider how often our results can be applied, as they do not give a complete categorization for greedy expansions of 1 under Salem bases. We finish this chapter by providing a result of Boyd, all degree four Salem numbers are Parry numbers. This result will be used in Chapter 4 to compare and contrast with the lazy expansions.

Chapter 3 begins our work with lazy expansions. We begin with the goal of finding a lazy analog of Theorem 1.1.7, which will allow us to determine whether a given sequence is a lazy expansion for 1. With this in mind, we recall some useful properties of greedy expansions and Parry numbers. We will show that lazy analogs, or general versions, of these results hold true. The most notable of these being that all PV numbers have an eventually periodic lazy expansion for 1. We conclude this chapter by attempting to list and prove what the lazy expansions for 1 are for every regular PV number. As in the greedy case, we find general forms for each family with a few exceptions.

Chapter 4 looks at lazy expansions for 1 under Salem bases. We will show results similar to those found in Chapter 2. We use the newly found lazy PV expansions from Chapter 3 as a basis for these results. With these more general results in mind, we shift our attention to Salem numbers of degree four. We prove, as in the greedy case, that every one of them induces a lazy expansion for 1 which is eventually periodic. Comparing this result to that of Boyd we find that many degree four Salem numbers are univoque.

Many results in Chapters 2 and 4 do not inherently rely on looking at Salem numbers. Rather, we use the construction of Salem, the polynomial $T_m^{\pm}(x) = x^m M(x) \pm M^*(x)$, to prove our results. In Chapter 5 we consider the case where $M(x)$ is not the minimal polynomial of a PV number, so that $T_m^{\pm}(x)$ need not be Salem. Here we also take the opportunity to visit some unexplored problems and questions related to the work in the thesis. Including lazy expansions for 1 under Salem bases related to PV numbers with a lazy expansion for 1 with a period of length 1.

Chapter 2

Greedy Salem Expansions

In this chapter, we describe greedy β-expansions for 1 where β is a Salem number that satisfies certain conditions. We show that there exist relationships between the expansions of a given family of Salem numbers and the greedy θ-expansion for 1, where θ is an underlying PV number which they approach. These results will hold, provided the θ-expansion for 1 satisfies a relatively common condition. We will provide numerical evidence to show that this condition is satisfied for a large proportion of PV numbers. In the final section, we also provide a number of infinite families that always satisfy the condition.

Recall that if $M(x)$ is a minimal PV polynomial, then $T_M^{\pm}(x) = M(x)x^m \pm M^*(x)$ admits a Salem root for sufficiently large m and these roots approach the PV root of $M(x)$ as m increases. We consider different situations; in particular, we consider the difference when $M(x)$ admits a PV number, θ, with finite or eventually periodic greedy expansion for 1. Recall that the greedy θ-expansion for 1 when θ is a PV number is always finite or eventually periodic. We also consider the difference when looking at the polynomials $T_m^+(x)$ and $T_m^-(x)$, which normally yield different Salem numbers.

Our results are first divided into the positive case and negative case, depending on the polynomial $T_m^{\pm}(x)$. We divide each of these sections into subsections based on the nature of the greedy θ-expansion for 1. We first handle the case where this greedy expansion is finite, and then when it is eventually periodic.

Section 2.1 covers the positive case, and Section 2.2 covers the negative case. In each of these sections, we consider the differences when the PV-number has either a finite or eventually periodic greedy θ-expansion. In each of these are some illustrative examples for our new results. The content of Sections 2.1 and 2.2 is covered in the paper by Hare and Orovec, [14]. Section 2.3 of this chapter investigates how often these results are applicable. We look at how frequently the theorems can be applied for PV numbers of degree at most 20.

2.1 The Positive case: $T_m^+(x)$

2.1.1 The Finite Case

Hare and Tweedle [15] considered the case where for a PV number $\theta \in (1,2)$ with minimal polynomial $M(x)$ where the θ-expansion for 1 is finite. They were able to determine a relationship between the greedy θ-expansion for 1 and the greedy expansions for 1 in the Salem bases defined by $T_m^+(x) = M(x)x^m + M^*(x)$, for large enough m.

Theorem 2.1.1 (Hare and Tweedle, [15])**.** *Let $\theta \in (1,2)$ be a PV-number with minimal polynomial $M(x)$. Further, let 1 have a finite reversibly greedy θ-expansion, say $a_1 a_2 \dots a_k$. In addition, assume that the companion polynomial of θ has co-factor polynomial $Q(x)$, which is reciprocal. Let α_m^+ be the Salem number that satisfies the polynomial $M(x)x^m + M^*(x)$. Then for $m > 2k$ we have the greedy α_m^+-expansion of 1 is*

$$\mathbf{a_m^+} = a_1(a_2 \dots a_k 0^{m-k-1} a_k \dots a_2 00)^\omega.$$

Their proof is in two parts. First, they show that $T_m^+(x)$ divides the companion polynomial of the hypothesized expansion, $a_1(a_2 \dots a_k 0^{m-k-1} a_k \dots a_2 00)^\omega$. This proves that it is, in fact, a valid expansion in the required base, and then they show that it is a greedy expansion for 1 in some base β_m^+. Combining these results, they conclude that β_m^+ is the required Salem number α_m^+ and that the hypothesized expansion is the greedy β-expansion.

To aid in our future cases, we can rewrite this case using the following notation. Let θ be the PV-number in question. We let $\gamma = a_1 a_2 \dots a_k$ be the reversibly greedy θ-expansion for 1 and let $\kappa = a_2 \dots a_k = \sigma(\gamma)$ be the shifted sequence beginning with the second entry. In the periodic case, we define $\gamma = a_{k+1} \dots a_{k+\ell}$, so that γ is the periodic part.

Then the theorem states that the greedy α_m^+-expansion for 1 is given by

$$\mathbf{a_m^+} = 1(\kappa 0^{m-k-1} \kappa^* 00)^\omega.$$

Here, $\kappa^* = a_k \dots a_2$ is the inverted string. We were able to generalize this result for all $\theta > 1$ which are non-integer PV numbers. Here we provide the proof, which will illustrate the basic method discussed in Section 1.2 to prove that a given sequence is the greedy expansion for 1 in a desired base.

Theorem 2.1.2. *Let $\theta > 1$ be a non-integer PV-number with minimal polynomial $M(x)$. Further, let 1 have a finite reversibly greedy θ-expansion, say $a_1 a_2 \dots a_k$. In addition,*

assume that the companion polynomial of θ has co-factor polynomial $Q(x)$, which is reciprocal. Let α_m^+ be the Salem number that satisfies the polynomial $M(x)x^m + M^*(x)$. Let $\kappa = a_2 \dots a_k$ and $\delta = a_1 - 1$. Then for $m > 2k$ we have the greedy α_m^+-expansion of 1 is

$$\mathbf{a_m^+} = a_1(a_2 \dots a_k 0^{m-k-1} a_k \dots a_2 \delta\delta)^\omega = a_1(\kappa 0^{m-k-1} \kappa^* \delta\delta)^\omega.$$

Remark 2.1.3. *We can see this as a direct generalization of Theorem 2.1.1. If (a_i) is the greedy θ-expansion for 1 we know $a_1 = \lfloor \theta \rfloor$ by applying the Algorithm 1.1.6. Then in the restricted case, when $\theta \in (1, 2)$ we have $a_1 = 1$ and $\delta = a_1 - 1 = 0$ as in the original theorem.*

Proof of Theorem 2.1.2. We first show that this sequence is in fact an α_m^+-representation for 1. Then we will use Theorem 1.1.7 together with the fact θ is finite reversibly greedy to show that the representation is the greedy expansion. To prove the first part we show that $x^m M(x) + M^*(x) | R_m^+(x)$, where $R_m(x)$ is the companion polynomial induced by the sequence $\mathbf{a}_m^+$. Let $R(x)$ be the companion polynomial of the greedy expansion $d_\theta(1) = a_1 a_2 \dots a_k$ so that $M(x)Q(x) = R(x)$.

The sequence in question has a period of length $N = m + k - 1$. Computing $P_{N+1}(x)$ and $P_1(x)$ from Definition 1.1.13 we see:

$$\begin{aligned} P_{N+1}(x) =& x^{N+1} - a_1 x^N - a_2 x^{N-1} - \dots - a_k x^{N-k+1} \\ & - a_k x^k - a_{k-1} x^{k-1} - \dots - a_2 x^2 - (a_1 - 1)x - (a_1 - 1), \end{aligned}$$

$$P_1(x) = x - a_1.$$

Subtracting $P_1(x)$ from $P_{N-1}(x)$ we get the companion polynomial $R_m^+(x)$ for this sequence.

$$\begin{aligned} R_m^+(x) =& P_{N+1}(x) - P_1(x) \\ =& x^{N+1} - a_1 x^N - a_2 x^{N-1} - \dots - a_k x^{N-k+1} \\ & - a_k x^k - a_{k-1} x^{k-1} - \dots - a_2 x^2 - a_1 x + 1. \end{aligned}$$

Recall that since $d_\theta(1)$ is finite the companion polynomial for this greedy expansion is given by $R(x) = x^k - a_1 x^{k-1} - \dots - a_{k-1} x - a_k$. So we have;

$$\begin{aligned} R_m^+(x) =& R(x)x^{N-k+1} + R^*(x) \\ =& M(x)Q(x)x^{N-k+1} + M^*(x)Q^*(x) \\ =& Q(x)(M(x)x^{N-k+1} + M^*(x)). \end{aligned}$$

The final equality holds since $Q(x)$ is reciprocal, so that $Q^*(x) = Q(x)$. We have $m = N - k + 1$ so $x^m M(x) + M^*(x) | R_m^+(x)$ as required. So the proposed sequence is a valid α_m^+-representation for 1.

We complete the proof by showing $\mathbf{a}_m^+$ is the greedy expansion for 1 using Theorem 1.1.7. That is, we show $\sigma^j(\mathbf{a}_m^+) <_{\text{lex}} \mathbf{a}_m^+$ for all $j = 1, \ldots, N$.

1. $\sigma^j(\mathbf{a}_m^+) <_{\text{lex}} \mathbf{a}_m^+$ for $j = 1, \ldots k$ since $d_\theta(1)$ is the greedy θ-expansion for 1.
2. $\sigma^j(\mathbf{a}_m^+) <_{\text{lex}} \mathbf{a}_m^+$ for $j = k+1, \ldots m$ since $\sigma^j(\mathbf{a}_m^+)$ starts with a 0 and $\mathbf{a}_m^+$ begins with $\lfloor \theta \rfloor > 0$.
3. $\sigma^j(\mathbf{a}_m^+) <_{\text{lex}} \mathbf{a}_m^+$ for $j = m+1, \ldots, N$ since $d_\theta(1)$ is finite reversibly greedy.

□

Note that this result holds for all finite reversibly greedy PV-numbers and sufficiently large m, in what follows, our results will be conditional on further properties to be discussed.

2.1.2 The Periodic Case

Let θ be a PV number with minimal polynomial $M(x)$. Suppose that the greedy θ-expansion for 1 is given by $a_1 \ldots a_k (a_{k+1} \ldots a_{k+\ell})^\omega$ and that it satisfies Condition 1.3.2. We aim to show a relationship between the greedy expansion of 1 with the base being a PV number θ, and the greedy expansions of 1 with the base being the Salem number α_m^+ associated to θ.

Example 2.1.4. *Consider the PV number θ with the minimal polynomial $M(x) = \chi(x) = x^4 - x^3 - 2x^2 + 1$, we know that it has the beta-expansion given by $11(10)^\omega$. We know $d_\theta(1) = a_1 \ldots a_k (a_{k+1} \ldots a_{k+\ell})^\omega = 11(10)^\omega$. In this case, we have $k = 2, \ell = 2$. Define $\kappa = a_2 \ldots a_k = 1$ and $\gamma = a_{k+1} \ldots a_{k+\ell} = 10$. For sufficiently large values of m we can define the polynomial $T_m(x) = M(x)x^m + M^*(x)$ which yields a Salem root α_m^+. We look at the greedy expansions of 1 with base α_m^+ for various values of m. See Table 2.1.1.*

We observe a pattern. The odd values of $m = 2n + 1$ give a greedy expansion of the form $1(\kappa\gamma^{n-1}0110(\gamma^*)^{n-1}\kappa^*00)^\omega$. The even values of $m = 2n + 2$ give a greedy expansion of the form $1(\kappa\gamma^{n-1}10001(\gamma^*)^{n-1}\kappa^*00)^\omega$

This is an example of a more general phenomenon that we will prove with the following theorem:

m	Greedy expansion	Reduced Expansion
3	$1(10110100)^\omega$	$1(\kappa 0110\kappa^*00)^\omega$
5	$1(110011001100)^\omega$	$1(\kappa\gamma 0110\gamma^*\kappa^*00)^\omega$
7	$1(1101001100101100)^\omega$	$1(\kappa\gamma\gamma 0110\gamma^*\gamma^*\kappa^*00)^\omega$
9	$1(11010100110010101100)^\omega$	$1(\kappa\gamma\gamma\gamma 0110\gamma^*\gamma^*\gamma^*\kappa^*00)^\omega$
4	$1(110001100)^\omega$	$1(\kappa 10001\kappa^*00)^\omega$
6	$1(1101000101100)^\omega$	$1(\kappa\gamma 10001\gamma^*\kappa^*00)^\omega$
8	$1(11010100010101100)^\omega$	$1(\kappa\gamma\gamma 10001\gamma^*\gamma^*\kappa^*00)^\omega$
10	$1(110101010001010101100)^\omega$	$1(\kappa\gamma\gamma\gamma 10001\gamma^*\gamma^*\gamma^*\kappa^*00)^\omega$

Table 2.1.1: Greedy expansions for 1 under base α_m^+ in Example 2.1.4

Theorem 2.1.5. *Let θ be a PV number such that the greedy θ-expansion of 1 is infinite and reversibly greedy. Let $d_\theta(1) = a_1 \dots a_k(a_{k+1} \dots a_{k+\ell})^\omega = a_1\kappa(\gamma)^\omega$. Let $M(x)$ be the minimal polynomial of θ and $\delta = a_1 - 1$. Let $R(x)$ be the companion polynomial and $Q(x)$ be the co-factor polynomial for $d_\theta(1)$. Let $j \in \{1, \dots, \ell\}$ and $\alpha_{n\ell+j}^+$ be the Salem number satisfying $T_{n\ell+j}^+(x) = M(x)x^{n\ell+j} + M^*(x)$.*

Assume $Q(x)$ is reciprocal, there exists a τ_j such that

$$d_{\alpha_{\ell+j}^+}(1) = a_1(\kappa\tau_j\kappa^*\delta\delta)^\omega$$

and the companion polynomial of $d_{\alpha_{\ell+j}^+}(1)$ is

$$R_{\ell+j}(x) = Q(x)(M(x)x^{\ell+j} + M^*(x))(1 + x^\ell).$$

Then

$$d_{\alpha_{n\ell+j}^+}(1) = a_1(\kappa \underbrace{\gamma \dots \gamma}_{n-1} \tau_j \underbrace{\gamma^* \dots \gamma^*}_{n-1} \kappa^*\delta\delta)^\omega \tag{2.1.1}$$

and the companion polynomial of $d_{\alpha_{n\ell+j}^+}(1)$ is

$$R_{n\ell+j}(x) = Q(x)(M(x)x^{n\ell+j} + M^*(x))(1 + x^\ell + \dots + x^{n\ell}).$$

Remark 2.1.6. *We can write Example 2.1.4 using the notation of Theorem 2.1.5. As the greedy expansion for the PV number θ is given by $11(10)^\omega$ we have $\kappa = 1$, $\gamma = 10$, $\delta = 0$, $\ell = 2$ and we find $\tau_1 = 0110$ and $\tau_2 = 10001$.*

We introduce the following notation that will be used in both the proof of Theorem 2.1.5 and Chapter 4.

Definition 2.1.7. *Let* 1 *have an infinite greedy expansions* $a_1a_2\dots a_k(a_{k+1}\dots a_{k+\ell})^\omega = a_1\kappa(\gamma)^\omega$ *with respect to some base, with period of length* ℓ *and pre-period of length* k*. Let* $R(x)$ *be the companion polynomial, as defined in Definition 1.1.13. Let* $R^*(x)$ *be the reciprocal polynomial of* $R(x)$*. Define* $S(x)$ *such that* $P_{k+\ell}(x) = x^\ell P_k(x) - S(x)$*. That is, the coefficients of* $S(x)$ *correspond to the word* γ *and* $S(x) = a_{k+1}x^{\ell-1}+a_{k+2}x^{\ell-2}+\dots+a_{k+\ell}$*. Define* $S^*(x) = a_{k+\ell}x^{\ell-1} + a_{k+\ell-1}x^{\ell-2} + \dots + a_{k+1}$*. This gives us*

$$R(x) = P_{k+\ell}(x) - P_k(x), \quad R^*(x) = P^*_{k+\ell}(x) - x^\ell P^*_k(x). \tag{2.1.2}$$

and,

$$P_{k+\ell}(x) = x^\ell P_k(x) - S(x), \quad P^*_{k+\ell}(x) = P^*_k(x) - x^{k+1}S^*(x). \tag{2.1.3}$$

Remark 2.1.8. *It is often convenient to think of* $S^*(x)$ *as the reciprocal polynomial of* $S(x)$*. If* $a_{k+1} = 0$*, then this isn't exactly true.*

Proof of Theorem 2.1.5. As $Q(x)$ is reciprocal, $Q(x)M^*(x) = R^*(x)$. Using this and the notation in Definition 2.1.7, we get

$$\begin{aligned}
R_{\ell+j}(x) &= Q(x)\left(M(x)x^{\ell+j} + M^*(x)\right)(1+x^\ell)\\
&= R(x)x^{2\ell+j} + R(x)x^{\ell+j} + R^*(x)x^\ell + R^*(x)\\
&= P_k(x)x^{3\ell+j} - P_k(x)x^{2\ell+j} + P_k(x)x^{2\ell+j} - P_k(x)x^{\ell+j} - S(x)x^{2\ell+j} - S(x)x^{\ell+j}\\
&\quad + P^*_k(x)x^\ell - P^*_k(x)x^{2\ell} + P^*_k(x) - P^*_k(x)x^\ell - S^*(x)x^{k+1+\ell} - S^*(x)x^{k+1}\\
&= P_k(x)x^{3\ell+j} - \Big(P_k(x)x^{j+\ell-k-1} + S(x)x^{2\ell+j-k-1} + S(x)x^{\ell+j-k-1}\\
&\quad + P^*_k(x)x^{2\ell-k-1} + S^*(x)x^\ell + S^*(x)\Big)x^{k+1} + P^*_k(x)\\
&= P_k(x)x^{3\ell+j} - Z^{(1)}_j(x)x^{k+1} + P^*_k(x)
\end{aligned}$$

Here, we define

$$\begin{aligned}
Z^{(1)}_j(x) = {} & P_k(x)x^{j+\ell-k-1} + S(x)x^{2\ell+j-k-1} + S(x)x^{\ell+j-k-1}\\
& + P^*_k(x)x^{2\ell-k-1} + S^*(x)x^\ell + S^*(x).
\end{aligned} \tag{2.1.4}$$

We can interpret $Z^{(1)}_j(x)$ as a polynomial of degree $3\ell + j - k - 2$, this is the degree of $S(x)x^{2\ell+j-k-1}$. It is possible (as we will see in Example 2.1.9) that the lead and tail coefficients of $Z^{(1)}_j(x)$ are 0. By assumption $R_{\ell+j}(x)$ is the companion polynomial of the

representation $a_1(\kappa\tau_j\kappa^*\delta\delta)^\omega$. Let $N = |a_1\kappa\tau_j\kappa^*\delta\delta|$. This shows that:

$$\begin{aligned} R_{\ell+j}(x) = & \Bigg(x^N - a_1x^{N-1} - \underbrace{(a_2x^{N-2} + \cdots + a_kx^{N-k})}_{\kappa} \\ & - \underbrace{(t_1x^{N-k-1} + \cdots + t_{N-2k-1}x^{k+1})}_{\tau_j} - \underbrace{(a_2x^k + \ldots a_kx^2)}_{\kappa^*} - \delta x - \delta \Bigg) \\ & - (x - a_1) \end{aligned}$$

By comparing degrees, we see that $N = 3\ell+k+j$, and hence $|\tau_j| = 3\ell-k+j-1$. Noticing that

$$x^N - a_1x^{N-1} - \underbrace{(a_2x^{N-2} + \cdots + a_kx^{N-k})}_{\kappa} = P_k(x)x^{3\ell+j}$$

and

$$- \underbrace{(a_2x^k + \ldots a_kx^2)}_{\kappa^*} - \delta x - \delta - x + a_1 = P_k^*(x)$$

we find that $Z_j^{(1)}(x)$ corresponds to the digits of τ_j.

To prove this result, we will prove that $R_{n\ell+j}(x)$ is the companion polynomial associated with the representation given on the right hand side of Equation (2.1.1). As $T^+_{n\ell+j}(x)$ divides this companion polynomial, we have that the expansion on the right hand side of Equation (2.1.1) is a $\alpha^+_{n\ell+j}(x)$-representation of 1. We then show that this representation is a greedy expansion by Theorem 1.1.7.

Let $R_{n\ell+j}(x) = Q(x)(M(x)x^{n\ell+j} + M^*(x))(1 + x^\ell + \cdots + x^{\ell n})$

Then

$$\begin{aligned} R_{n\ell+j}(x) &= (Q(x)M(x)x^{n\ell+j} + Q(x)M^*(x))(1 + x^\ell + \cdots + x^{\ell n}) \\ &= (R(x)x^{n\ell+j} + R^*(x))(1 + x^\ell + \cdots + x^{\ell n}) \\ &= R(x)\sum_{i=0}^{n} x^{j+\ell(n+i)} + R^*(x)\sum_{i=0}^{n} x^{\ell i}. \end{aligned}$$

Since $R(x)$ is the companion polynomial of the greedy expansion under the PV base, we can separate the summands into the components $P_{k+\ell}(x), P_k(x), P^*_{k+\ell}(x)$ and $P^*_k(x)$. Next, we use the telescoping properties of these sums to rewrite the summands in terms of $S(x)$ and $S^*(x)x^{k+1}$.

$$
\begin{aligned}
R_{n\ell+j}(x) = &\ (P_{k+\ell}(x) - P_k(x)) \sum_{i=0}^{n} x^{j+\ell(n+i)} + \left(P^*_{k+\ell}(x) - x^\ell P^*_k(x)\right) \sum_{i=0}^{n} x^{\ell i} \\
&= P_{k+\ell}(x)x^{2n\ell+j} + \left(P_{k+\ell}(x) - x^\ell P_k(x)\right) \sum_{i=0}^{n-1} x^{j+\ell(n+i)} - P_k(x)x^{n\ell+j} \\
&\quad + P^*_{k+\ell}(x) + \left(P^*_{k+\ell}(x) - P^*_k(x)\right) \sum_{i=1}^{n} x^{\ell i} - P^*_k(x)x^{\ell(n+1)} \\
&= P_{k+\ell}(x)x^{2n\ell+j} - S(x) \sum_{i=0}^{n-1} x^{j+\ell(n+i)} - P_k(x)x^{n\ell+j} \\
&\quad + P^*_{k+\ell}(x) - S^*(x)x^{k+1} \sum_{i=1}^{n} x^{\ell i} - P^*_k(x)x^{\ell(n+1)} \\
&= P_k(x)x^{2n\ell+\ell+j} - S(x) \sum_{i=0}^{n} x^{j+\ell(n+i)} - P_k(x)x^{n\ell+j} \\
&\quad + P^*_k(x) - S^*(x)x^{k+1} \sum_{i=0}^{n} x^{\ell i} - P^*_k(x)x^{\ell(n+1)}.
\end{aligned}
$$

We next rearrange the summation into three parts. The first part will correspond to the subword $a_1\kappa\gamma^{n-1}$ of $a_1\kappa\gamma^{n-1}\tau_j(\gamma^*)^{n-1}\kappa^*\delta\delta$. The second part, $Z_j^{(1)}(x)$, will correspond to the subword τ_j. The last part will correspond to the subword $(\gamma^*)^{n-1}\kappa^*\delta\delta$ with an additional $-(x - a_1)$. This $-(x - a_1)$ is needed because $R_{n\ell+j}(x)$ is a companion polynomial of a periodic representation with preperiod a_1.

$$
\begin{aligned}
R_{n\ell+j}(x) = &\left(P_k(x)x^{j+2n\ell+\ell} - S(x)\sum_{i=2}^{n} x^{j+\ell(n+i)}\right) \\
&- \Big(P_k(x)x^j + S(x)x^j + S(x)x^{j+\ell} \qquad (2.1.5)\\
&+ P_k^*(x)x^\ell + S^*(x)x^{k+1} + S^*(x)x^{k+1-\ell}\Big)x^{n\ell} \\
&+ \left(P_k^*(x) - S^*(x)\sum_{i=0}^{n-2} x^{\ell i+k+1}\right) \\
= &\left(P_k(x)x^{j+2n\ell+\ell} - S(x)\sum_{i=2}^{n} x^{j+\ell(n+i)}\right) - Z_j^{(1)}(x)x^{k+1+(n-1)\ell} \\
&+ \left(P_k^*(x) - S^*(x)\sum_{i=0}^{n-2} x^{\ell i+k+1}\right) \qquad (2.1.6)
\end{aligned}
$$

To see why this is the companion polynomial for the right hand side of Equation (2.1.1) consider the components separately. We note that

$$
P_k(x)x^{j+(2n+1)\ell} = x^{j+(2n+1)\ell}\left(x^k - a_1x^{k-1} - \underbrace{a_2x^{k-2} + \cdots + a_k}_{\kappa}\right).
$$

When considering $S(x)\sum_{i=0}^{n-2} x^{\ell i+k}$ we note that:

$$
S(x)\sum_{i=0}^{n-2} x^{\ell i} = \underbrace{\underbrace{a_{k+1}x^{\ell(n-1)-1} + \cdots + a_{k+\ell}x^{\ell(n-2)}}_{\gamma} + \cdots + \underbrace{a_{k+1}x^{\ell-1} + \cdots + a_{k+\ell}}_{\gamma}}_{n-1}.
$$

The next component, $Z_j^{(1)}(x)$, corresponds to τ_j. Similarly $S^*(x)\sum_{i=0}^{n-2} x^{\ell i+k+1}$ corresponds to $\gamma^*\ldots\gamma^*$. Lastly $P_k^*(x)$ corresponds to $\kappa^*\delta\delta$ and the factor $-(x-a_1)$.

Based on the above argument, we see that $R_{n\ell+j}(x)$ as defined is the companion polynomial of the right hand side of Equation (2.1.1). Noting that $M(x)x^{n\ell+j}+M^*(x)$ divides $R_{n\ell+j}(x)$, we conclude that this is indeed a valid representation for 1 in base $\alpha^+_{\ell n+j}$. It remains to show that this representation is the greedy expansion.

It suffices to apply Theorem 1.1.7 to complete the proof. Let $\mathbf{a_{(n,j)}}$ be the right hand side of Equation (2.1.1). We have $\sigma^i(\mathbf{a_{(n,j)}}) < \mathbf{a_{(n,j)}}$ for $i = 1,\ldots k+\ell(n-1)$ since $a_1\ldots a_k(a_{k+1}\ldots a_{k+\ell})^\omega$ is a greedy expansion of θ. Next, $\sigma^i(\mathbf{a_{(n,j)}}) < \mathbf{a_{(n,j)}}$ for $i = k+\ell(n-$

$1)+1\ldots k+\ell(n-1)+|\tau_j|$ since the relation holds for $\mathbf{a_{(1,j)}}$ by assumption. Finally, we see that $\sigma^j(\mathbf{a_{(n,j)}}) < \mathbf{a_{(n,j)}}$ for the remaining terms, as $a_1\ldots a_k(a_{k+1}\ldots a_{k+\ell})^\omega$ is reversibly greedy.

□

We show that the set of PV numbers for which the hypothesis of Theorem 2.1.5 holds is non-trivial. We will do this by giving an infinite family of PV numbers and a particular value of j for which the hypothesis holds.

Example 2.1.9. *Let θ be the PV root of $\Psi_{1,q}^{A-}(x)$ for $q \geq 3$, as given in Table 1.2.1. We see that $d_\theta(1) = 11(0^{q-2}10)^\omega$. Notice that $k = 2$ and $\ell = q$. For this example, we will take $j = q$.*

Using the notation of Definition 2.1.7 we get

$$\begin{aligned}
P_k(x) &= x^2 - x - 1\\
P_k^*(x) &= -x^2 - x + 1\\
S(x) &= 0\cdot x^{q-1} + 0\cdot x^{q-2} + \cdots + 0\cdot x^2 + x + 0\\
&= x\\
S^*(x) &= 0\cdot x^{q-1} + x^{q-2} + 0\dot{x}^{q-3} - \cdots + 0\cdot x + 0\\
&= x^{q-2}\\
Z_j^{(1)}(x) &= P_k(x)x^{j+\ell-k-1} + S(x)x^{2\ell+j-k-1} + S(x)x^{\ell+j-k-1}\\
&\quad + P_k^*(x)x^{2\ell-k-1} + S^*(x)x^\ell + S^*(x)\\
&= \big((x^2-x-1)x^{2q-3} + x^{3q-2} + x^{2q-2}\\
&\quad +(-x^2-x+1)x^{2q-3} + x^{2q-2} + x^{q-2}\big)\\
&= x^{3q-2} + x^{q-2}
\end{aligned}$$

Recall that $Z_q^{(1)}(x)$ is interpreted as a polynomial of degree $3\ell+j-k-2 = 4q-4$. Observing that $\gamma = 0^{q-2}10$ we get $\tau_j = 0^{q-2}10^{2q-1}10^{q-2} = \gamma 0^{2q-3}\gamma^*$.

This gives us an infinite family of Salem numbers satisfying the polynomial

$$\Psi_{1,q}^{A-}(x)x^{nq+q} + \left(\Psi_{1,q}^{A-}\right)^*(x)$$

with greedy expansion

$$1(1(0^{q-2}10)^n0^{2q-3}(010^{q-2})^n00)^\omega.$$

The process of finding $Z_j^{(1)}(x)$ in this example can be used with any PV number. Whether that PV number and a given value of j can be used in the theorem is based on the coefficients of the polynomial $Z_j^{(1)}(x)$. If its coefficients come from $\mathcal{A}_\theta$ this corresponds to a valid representation. In the case where the coefficients are greater than θ or negative, then the word τ_j represented by them is not valid. We also continue to require the fact that $d_\theta(1)$ is reversibly greedy and the corresponding companion polynomial $Q(x)$ is reciprocal.

2.2 The Negative case: $T_m^-(x)$

Let $M(x)$ be the minimal polynomial of a PV number θ and consider the Salem number α_m^- defined by the polynomial $T_m^-(x) = M(x)x^m - M^*(x)$. As before, in the positive case, α_m^+ we will divide this into two cases depending on whether the greedy θ-expansion of 1 is finite or eventually periodic.

2.2.1 Finite Case

Example 2.2.1. *Consider the PV number θ with minimal polynomial $M(x) = \Phi_3(x) = x^4 - 2x^3 + x - 1$. We know $d_\theta(1) = a_1 \dots a_k = 111001$. In this case, we have $k = 6$, let $\kappa = a_2 \dots a_{k-1}(a_k - 1) = 11000$ and $\gamma = a_1 \dots a_{k-1}(a_k - 1) = 111000$. We look at the greedy α_m^--expansions of 1 where α_m^- is the Salem root of $T_m^-(x)$ for various values of m 2.2.1.*

We observe two different patterns. For $m = \ell n + j$ when $j = 1, 2, 5, 6$ we see that the greedy expansions look like $1(\kappa\gamma^{n-1}\tau_j(\gamma^*)^{n-1}\kappa^*00)^\omega$ for the appropriate choice of τ_j. Similarly for $j = 3$ the greedy expansions look like $1(\kappa\gamma^{n-1}\tau_j00)^\omega$ for $\tau_j = 11$.

As before, these are examples of more general patterns.

Theorem 2.2.2. *Let θ be a PV number such that the greedy θ-expansion of 1 is finite and reversibly greedy with $d_\theta(1) = a_1a_2 \dots a_k$. Let $\kappa = a_2 \dots a_{k-1}(a_k - 1)$, $\gamma = a_1a_2 \dots a_{k-1}(a_k - 1)$ and $\delta = a_1 - 1$. Let $M(x)$ be the minimal polynomial of θ. Let $R(x) = P_k$ be the companion polynomial and $Q(x)$ be the co-factor polynomial for $d_\theta(1)$. Let $j \in \{1, \dots, k\}$. Let α_{nk+j}^- be the Salem number that satisfies $T_{nk+j}^-(x) = M(x)x^{nk+j} - M^*(x)$.*

Assume $k = 2p$ is even. Assume $Q(x)$ is reciprocal, there exists a τ_j such that

$$d_{\alpha_{k+j}^-}(1) = a_1(\kappa\tau_j\kappa^*\delta\delta)^\omega$$

and the companion polynomial of $d_{\alpha_{k+j}^-}(1)$ is

$$R_{k+j}(x) = Q(x)(M(x)x^{k+j} - M^*(x))\left(\frac{x^{3p}+1}{x^{2p}-1}\right).$$

m	greedy expansion of 1
7	$1(\kappa 010\kappa^*00)^\omega$
13	$1(\kappa\gamma 010\gamma^*\kappa^*00)^\omega$
19	$1(\kappa\gamma\gamma 010\gamma^*\gamma^*\kappa^*00)^\omega$
25	$1(\kappa\gamma\gamma\gamma 010\gamma^*\gamma^*\gamma^*\kappa^*00)^\omega$
8	$1(\kappa 1001\kappa^*00)^\omega$
14	$1(\kappa\gamma 1001\gamma^*\kappa^*00)^\omega$
20	$1(\kappa\gamma\gamma 1001\gamma^*\gamma^*\kappa^*00)^\omega$
26	$1(\kappa\gamma\gamma\gamma 1001\gamma^*\gamma^*\gamma^*\kappa^*00)^\omega$
10	$1(\kappa 1100)^\omega$
16	$1(\kappa\gamma 1100)^\omega$
22	$1(\kappa\gamma\gamma 1100)^\omega$
28	$1(\kappa\gamma\gamma\gamma 1100)^\omega$
11	$1(\kappa 1101011\kappa^*00)^\omega$
17	$1(\kappa\gamma 1101011\gamma^*\kappa^*00)^\omega$
23	$1(\kappa\gamma\gamma 1101011\gamma^*\gamma^*\kappa^*00)^\omega$
29	$1(\kappa\gamma\gamma\gamma 1101011\gamma^*\gamma^*\gamma^*\kappa^*00)^\omega$
12	$1(\kappa 11011011\kappa^*00)^\omega$
18	$1(\kappa\gamma 11011011\gamma^*\kappa^*00)^\omega$
24	$1(\kappa\gamma\gamma 11011011\gamma^*\gamma^*\kappa^*00)^\omega$
30	$1(\kappa\gamma\gamma\gamma 11011011\gamma^*\gamma^*\gamma^*\kappa^*00)^\omega$

Table 2.2.1: Greedy expansions of 1 for base α_m^- in Example 2.2.1

Then

$$d_{\alpha^-_{nk+j}}(1) = a_1(\kappa \underbrace{\gamma \dots \gamma}_{n-1} \tau_j \underbrace{\gamma^* \dots \gamma^*}_{n-1} \kappa^* \delta\delta)^\omega \tag{2.2.1}$$

and the companion polynomial of $d_{\alpha^-_{nk+j}}(1)$ *is*

$$R_{nk+j}(x) = Q(x)(M(x)x^{nk+j} - M^*(x)) \left(\frac{x^{2np+p}+1}{x^{2p}-1}\right).$$

Proof. This proof is similar to the proof of Theorem 2.1.5. We have the following key

observations which are used in the algebraic manipulation;

$$\begin{aligned}\frac{x^{2np+p}+1}{x^{2p}-1} &= \frac{x^{2np+p}+x^{2np}}{(x^p-1)(x^p+1)} + \frac{1-x^{2np}}{x^{2p}-1} \\ &= \frac{x^{2np}}{x^p-1} - \left(\sum_{i=0}^{n-1} x^{2pi}\right) \qquad (2.2.2) \\ &= \frac{x^{2np+p}-x^p}{x^{2p}-1} + \frac{x^p+1}{x^{2p}-1} \\ &= x^p\left(\sum_{i=0}^{n-1} x^{2pi}\right) + \frac{1}{x^p-1}. \qquad (2.2.3)\end{aligned}$$

As $d_\theta(1)$ is finite, we have

$$R(x) = P_k(x) = x^k - \underbrace{(a_1x^{k-1} + \underbrace{a_2x^{k-2} + a_3x^{k-3} + \cdots + a_{k-1}x + (a_k-1)}_{\kappa})}_{\gamma} - 1$$

We similarly define $S(x) = a_1x^{k-1} + \cdots + a_{k-1}x + (a_k-1)$ to encode the digits of γ and $S^*(x) = (a_k-1)x^{k-1} + a_{k-1}x^{k-2} + \cdots + a_2x + a_1$ for the digits of γ^*. We note that $R(x) = x^k - S(x) - 1$ and $R^*(x) = -x^k - S^*(x) + 1$. The equivalent equation to Equation (2.1.6) is

$$\begin{aligned}R_{nk+j}(x) =& x^{2np+j+p}R(x)\left(\sum_{i=0}^{n-1} x^{2pi}\right) + \frac{R(x)x^j - R^*(x)}{x^p-1}x^{2pn} + R^*(x)\left(\sum_{i=0}^{n-1} x^{2pi}\right) \\ =& x^{4np+j-p}(P_k(x)+1) - S(x)\left(\sum_{i=0}^{n-2} x^{2p(n+i)+j+p}\right) \\ & - x^{2pn+j+p} + \frac{R(x)x^j - R^*(x)}{x^p-1}x^{2pn} - x^{2pn} \\ & - S^*(x)\left(\sum_{i=1}^{n-1} x^{2pi}\right) - x^{2p} + P_k^*(x) \\ =& x^{4np+j-p}(P_k(x)+1) - S(x)\left(\sum_{i=0}^{n-2} x^{2p(n+i)+j+p}\right) + Z^{(2)}(x)x^{2pn} \\ & - S^*(x)\left(\sum_{i=1}^{n-1} x^{2pi}\right) - x^{2p} + P_k^*(x)\end{aligned}$$

where

$$\begin{aligned}
Z_j^{(2)}(x) =& -x^{j+p} + \frac{R(x)x^j - R^*(x)}{x^p - 1} - 1 \\
=& \frac{x^{2p+j} - S(x)x^j - x^j - (-x^{2p} - S^*(x) + 1)}{x^p - 1} - x^{j+p} - 1 \\
=& x^j \frac{x^{2p} - 1}{x^p + 1} + \frac{-S(x)x^j - S^*(x)}{x^p - 1} + \frac{x^{2p} - 1}{x^p - 1} - x^{j+p} - 1 \\
=& x^j(x^p + 1) + \frac{-x^j S(x) + S^*(x)}{x^p - 1} + x^p + 1 - x^{j+p} - 1 \\
=& x^j + \frac{-x^j S(x) + S^*(x)}{x^p - 1} + x^p
\end{aligned}$$

It should be noted that the lead coefficient of $R(x)$ is 1 and the tail coefficient is $-a_k$. This means that the coefficient in front of $x^{2np+j+p+2pi}$ is $a_k - 1$ for $i = 1, \ldots, n-2$. This gives the first $n-1$ occurrences of γ in the representation. The contribution to the coefficient in front of $x^{2np+j+p}$ from $x^{2np+j+p}$ is -1. There is an additional contribution of 1 coming from $x^{2pn}Z^{(2)}(x)$. This corresponds to the final γ before the τ_j in the representation. Similarly for γ^*. □

Theorem 2.2.3. *Let θ be a PV number such that the greedy θ-expansion of 1 is finite and reversibly greedy with $d_\theta(1) = a_1 a_2 \ldots a_k$. Let $\kappa = a_2 \ldots a_{k-1}(a_k - 1)$, $\gamma = a_1 a_2 \ldots a_{k-1}(a_k - 1)$ and $\delta = a_1 - 1$. Let $M(x)$ be the minimal polynomial of θ. Let $R(x) = P_k$ be the companion polynomial and $Q(x)$ be the co-factor polynomial for $d_\theta(1)$. Let $j \in \{1, \ldots, k\}$. Let α^-_{nk+j} be the Salem number that satisfies $T^-_{nk+j}(x) = M(x)x^{nk+j} - M^*(x)$.*

Assume $Q(x)$ is reciprocal, there exists a τ_j such that

$$d_{\alpha^-_{k+j}}(1) = a_1(\kappa\tau_j\delta\delta)^\omega$$

and the companion polynomial of $d_{\alpha^-_{k+j}}(1)$ is

$$R_{k+j}(x) = Q(x)(M(x)x^{k+j} - M^*(x))\left(\frac{1}{x^k - 1}\right).$$

Then

$$d_{\alpha^-_{nk+j}}(1) = a_1(\kappa \underbrace{\gamma \ldots \gamma}_{n-1} \tau_j \delta\delta)^\omega \tag{2.2.4}$$

and the companion polynomial of $d_{\alpha^-_{nk+j}}(1)$ is

$$R_{nk+j}(x) = Q(x)(M(x)x^{nk+j} - M^*(x))\left(\frac{1}{x^k - 1}\right).$$

Proof. This proof is similar to the proof of Theorem 2.2.2. The equivalent equation to Equation (2.1.6) is

$$\begin{aligned}
R_{nk+j}(x) &= x^j R(x)\left(1 + x^k + x^{2k} + \cdots + x^{(n-1)k} + \frac{1}{x^k - 1}\right) - R^*/(x^k - 1) \\
&= x^{(n-1)k+j} R(x) + x^j R(x) \sum_{i=0}^{n-2} x^{ki} + \frac{R(x)x^j - R^*(x)}{x^k - 1} \\
&= x^{(n-1)k+j}(P_k(x) + 1 - 1) + x^j(x^k - S(x) + 1)\sum_{i=0}^{n-2} x^{ki} + \frac{R(x)x^j - R^*(x)}{x^k - 1} \\
&= x^{(n-1)k+j}(P_k(x) + 1) - x^j S(x)\sum_{i=0}^{n-2} x^{ki} - x^j + \frac{R(x)x^j - R^*(x)}{x^k - 1} \\
&= x^{(n-1)k+j}(P_k(x) + 1) - x^j S(x)\sum_{i=0}^{n-2} x^{ki} + Z^{(3)}(x) - (a_1 x - 1)
\end{aligned}$$

where

$$\begin{aligned}
Z_j^{(3)}(x) &= -x^j + \frac{R(x)x^j - R^*(x)}{x^k - 1} + a_1 x - 1 \\
&= \frac{-S(x)x^j + S^*(x)}{x^k - 1} + a_1 x
\end{aligned}$$

Although not obvious, the coefficient in front of x is 0 upon expansion. This is due to the assumption on the form of the greedy α_{k+j}-expansion for 1. The coefficients from x^2 to x^{j-1} correspond to the terms of τ_j. □

2.2.2 Periodic case

Example 2.2.4. *Consider the PV number θ with the minimal polynomial $M(x) = \Psi_{1,6}^{A-}(x) = x^8 - x^7 - x^6 - x^2 + 1$. We know $d_\theta(1) = a_1 \dots a_k(a_{k+1} \dots a_{k+\ell})^\omega = 11(000010)^\omega$. Here $k = 2$ and $\ell = 6$. Let $\kappa = a_2 \dots a_k = 1$ and $\gamma = a_{k+1} \dots a_{k+\ell} = 000010$. For sufficiently large values of m we can define the polynomial $T_m^-(x) = M(x)x^m + M^*(x)$ which yields a Salem number α_m^-. We look at the greedy expansions of 1 with base α_m^- for various m See Table 2.2.2.*

We observe two different patterns. For $m = \ell n + j$ with $j = 3$ we see that the greedy expansions look like $1(\kappa\gamma^n\tau_j(\gamma^*)^n\kappa^*00)^\omega$ for $\tau_j = 010$. Similarly for $j = 6$ the greedy expansions look like $1(\kappa\gamma^n\tau_j\kappa^*00)^\omega$ for $\tau_j = 000$.

As before, these are examples of a more general pattern.

m	greedy expansion of 1
9	$1(\kappa\gamma 010\gamma^*\kappa^* 00)^\omega$
15	$1(\kappa\gamma\gamma 010\gamma^*\gamma^*\kappa^* 00)^\omega$
21	$1(\kappa\gamma\gamma\gamma 010\gamma^*\gamma^*\gamma^*\kappa^* 00)^\omega$
27	$1(\kappa\gamma\gamma\gamma\gamma 010\gamma^*\gamma^*\gamma^*\gamma^*\kappa^* 00)^\omega$
12	$1(\kappa\gamma 000\kappa^* 00)^\omega$
18	$1(\kappa\gamma\gamma 000\kappa^* 00)^\omega$
24	$1(\kappa\gamma\gamma\gamma 000\kappa^* 00)^\omega$
30	$1(\kappa\gamma\gamma\gamma\gamma 000\kappa^* 00)^\omega$

Table 2.2.2: Greedy expansions of 1 for base α_m^- in Example 2.2.4

Theorem 2.2.5. *Let θ be a PV number such that the greedy expansion of 1 is periodic and reversibly greedy. Let $d_\theta(1) = a_1 \dots a_k(a_{k+1} \dots a_{k+\ell})^\omega = a_1\kappa(\gamma)^\omega$. Let $M(x)$ be the minimal polynomial of θ and $\delta = a_1 - 1$. Let $R(x)$ be the companion polynomial and $Q(x)$ be the co-factor polynomial for $d_\theta(1)$. Let $j \in \{1, \dots, \ell\}$. Let $\alpha^-_{n\ell+j}$ be the Salem number that satisfies $T^-_{n\ell+j}(x) = M(x)x^{n\ell+j} - M^*(x)$.*

Assume $\ell = 2p$ is even. Assume $Q(x)$ is reciprocal, there exists a τ_j such that

$$d_{\alpha^-_{\ell+j}}(1) = a_1(\kappa\gamma\tau_j\gamma^*\kappa^*\delta\delta)^\omega$$

and the companion polynomial of the greedy $\alpha^-_{\ell+j}$-expansion is

$$R_{\ell+j}(x) = Q(x)(M(x)x^{\ell+j} - M^*(x))\left(\frac{x^{3p}+1}{x^{2p}-1}\right).$$

Then

$$d_{\alpha^-_{n\ell+j}}(1) = a_1(\kappa \underbrace{\gamma \dots \gamma}_{n} \tau_j \underbrace{\gamma^* \dots \gamma^*}_{n} \kappa^*\delta\delta)^\omega \tag{2.2.5}$$

and the companion polynomial of this greedy $\alpha^-_{n\ell+j}$-expansion is

$$R_{n\ell+j}(x) = Q(x)(M(x)x^{n\ell+j} - M^*(x))\left(\frac{x^{2np+p}+1}{x^{2p}-1}\right).$$

Proof. This proof is similar to the proof of Theorem 2.2.2 with the observation that since $d_\theta(1)$ is periodic and $R(x) = P_{k+\ell}(x) - P_k(x)$.

Using the notation from Definition 2.1.7, the equivalent equation to Equation (2.1.6) is

$$\begin{aligned} R_{n\ell+j}(x) =& P_k(x)x^{4np+p+j} - S(x)\left(\sum_{i=0}^{n-1} x^{j+2np+2pi}\right) \\ &+ Z_j^{(3)}(x)x^{2np} - S^*(x)\left(\sum_{i=0}^{n-1} x^{2pi+k+1}\right) + P_k^*(x) \end{aligned}$$

where

$$Z_j^{(4)}(x) = \frac{R(x)x^j - R^*(x)}{x^p - 1} - P_k(x)x^{p+j} - P_k^*(x)$$

We note that the coefficients in front of $x^0, \ldots, x^{k-1}$ of $Z^{(4)}(x)$ are zero and are not included the calculation of τ_j. □

Theorem 2.2.6. *Let θ be a PV number such that the greedy expansion of 1 is periodic and reversibly greedy. Let $d_\theta(1) = a_1 \ldots a_k(a_{k+1} \ldots a_{k+\ell})^\omega = a_1\kappa(\gamma)^\omega$. Let $M(x)$ be the minimal polynomial of θ and $\delta = a_1 - 1$. Let $R(x)$ be the companion polynomial and $Q(x)$ be the co-factor polynomial for $d_\theta(1)$. Let $j \in \{1, \ldots, \ell\}$. Let $\alpha^-_{n\ell+j}$ be the Salem numbers that satisfy $T^-_{n\ell+j}(x) = M(x)x^{n\ell+j} - M^*(x)$.*

Assume $Q(x)$ is reciprocal, there exists a τ_j such that

$$d_{\alpha^-_{\ell+j}}(1) = a_1(\kappa\gamma\tau_j\kappa^*\delta\delta)^\omega$$

and the companion polynomial of this greedy $\alpha^-_{\ell+j}$-expansion is

$$R_{\ell+j}(x) = Q(x)(M(x)x^{\ell+j} - M^*(x))\left(\frac{1}{x^\ell - 1}\right).$$

Then

$$d_{\alpha^-_{n\ell+j}}(1) = a_1(\kappa\underbrace{\gamma\ldots\gamma}_{n}\tau_j\kappa^*\delta\delta)^\omega \tag{2.2.6}$$

and the companion polynomial of this greedy $\alpha^-_{n\ell+j}$-expansion is

$$R_{n\ell+j}(x) = Q(x)(M(x)x^{n\ell+j} - M^*(x))\left(\frac{1}{x^\ell - 1}\right).$$

Proof. This is very similar to the proof of Theorem 2.1.5. with the observation that since $d_\theta(1)$ is periodic, we have $R(x) = P_{k+\ell}(x) - P_k(x)$.

Using the notation of Definition 2.1.7, the equivalent equation to Equation (2.1.6) is

$$x^{j+n\ell}P_k(x) - S(x)\sum_{i=0}^{n-1} x^{\ell i+j} + Z^{(5)}(x) + P_k^*(x)$$

where

$$Z_j^{(5)}(x) = \frac{R(x)x^j - R^*(x)}{x^\ell - 1} - P_k(x)x^j - P_k^*(x).$$

□

2.2.3 Negative Case Examples

In this section, we wish to show that the sets of PV numbers where hypotheses of the Theorems in this section hold are non-trivial. We will do this by giving an infinite family of PV numbers and a particular value j for which each hypothesis holds.

Example 2.2.7. *We will consider four examples, each corresponding to one of Theorems 2.2.2, 2.2.3, 2.2.5, and 2.2.6.*

For Theorems 2.2.2 and 2.2.3 consider θ the root of $\Phi_q(x)$ as defined in Table 1.2.1. In this case, we have $d_\theta(1) = 1^q0^{q-1}1$. We have $\kappa = 1^{q-1}0^q$ and $\gamma = 1^q0^q$.

For $j = 1$ this satisfies the conditions of Theorem 2.2.2. In this case, we get $p = q$, and

$$\begin{aligned} Z_j^{(2)}(x) &= x + \frac{-xS(x) + S^*(x)}{x^q - 1} + x^q \\ &= x + \frac{-x^{q+1}\frac{x^q-1}{x-1} + x\frac{x^q-1}{x-1}}{x^q - 1} + x^q \\ &= x - \frac{-x^{q+1}}{x-1} + \frac{x}{x-1} + x^q \\ &= -x^{q-1} - x^{q-2} - \cdots - x^2 - x. \end{aligned}$$

From this we find that $\tau_1 = 01^{q-2}0$.

For $j = q+1$ this satisfies the conditions of Theorem 2.2.3. In this case, we get $Z_j^{(3)}(x) = -\sum_{i=2}^{2n} x^i$ and $\tau_j = 1^{q-1}$. (Recall that the coefficients in front of x^1 and x^0 do not contribute to τ_j in this case.)

For Theorems 2.2.5 and 2.2.6 consider θ the root of $\Psi_{1,q}^{A-}(x)$ for q even. In this case, we have $d_\theta(1) = 11(0^{q-2}10)^\omega$.

For $j = q/2$ this satisfies the conditions of Theorem 2.2.5. In this case, after simplification, we get $Z_j^{(4)}(x) = x^{q/2+1}$. This gives $\tau_j = 0^{q/2-1}10^{q/2-1}$. (Recall that the coefficients in front of $x^0, \ldots, x^{k-1} = x^0, x^1$ do not contribute to τ_j in this case.)

For $j = q$ this satisfies the conditions of Theorem 2.2.6. In this case, after simplification, we get $Z_j^{(5)}(x) = 0$ and and $\tau_j = 0^{q-3}$.

2.3 Frequency of Theorem Application

Recall that each of the four main theorems have the initial condition of first finding a Salem base α so that the greedy α-expansion for 1 has the required co-factor. Using this fact, we were then able to show that the pattern continued and write down the greedy expansions for 1 in other related Salem bases. In this section, we explore how often each theorem can be applied and provide some useful examples to show why they cannot be applied universally.

We consider a data set that includes all PV numbers in the interval $(1, 2)$ with a minimal polynomial of degree at most 15. There are 555 such polynomials for which the greedy expansion for 1 is finite, and a further 292 for which the greedy expansion is eventually periodic. Using computer calculations, we find that 518 of these are finite reversibly greedy and satisfy Condition 1.3.2 and a further 254 are infinite reversibly greedy. Hare and Tweedle [15], who explored the finite case, performed this analysis for PV numbers up to degree 30. They found from the 3704 such PV numbers. There were 2420 with finite greedy expansions and 2237 which satisfied Condition 1.3.2. We can see that PV numbers which could satisfy the conditions of each of our theorems are relatively common. It remains to explore how often we find an initial Salem base with a greedy expansion for 1 in the desired form.

The following example illustrates why, at least in the positive periodic case, not each value of $j \pmod \ell$ has a representation of the desired form.

Example 2.3.1. *The beta-expansion for 1 where the base is the PV root of $\Phi_{(r,\ell)}^{C-}(x)$ is given by $1^r0^{r-1}1(0^{\ell-2r}1^r0^r)^\omega$. We then calculate the constituent components of the companion*

polynomial and find the following;

$$P_k(x) = x^{2r} - x^{2r-1} - \cdots - x^r - 1 \tag{2.3.1}$$
$$P_k^*(x) = -x^{2r} - x^r - x^{r-1} - \cdots - x + 1 \tag{2.3.2}$$
$$S(x) = x^{2r-1} + x^{2r-2} + \cdots + x^r \tag{2.3.3}$$
$$S^*(x) = x^{\ell-1-r} + x^{\ell-1-(r+1)} + \cdots + x^{\ell-1-(2r-1)} \tag{2.3.4}$$

We then calculate the polynomial,

$$\begin{aligned} Z_j^{(1)}(x) = {} & P_k(x)x^{j+\ell-k-1} + S(x)x^{2\ell+j-k-1} + S(x)x^{\ell+j-k-1} \\ & + P_k^*(x)x^{2\ell-k-1} + S^*(x)x^\ell + S^*(x). \end{aligned} \tag{2.3.5}$$

The coefficients of this polynomial to exactly determine the string τ_j in Theorem 2.1.5.

For example, fix $r = 2$ and $\ell = 5$. Note that ℓ is the length of the periodic part of the PV expansion. Let $M(x) = \Phi_{(r,\ell)}^{C-}(x)$. Then we can calculate the greedy expansions for the Salem numbers with defining polynomials $M(x)x^{n\ell+j} + M^*(x)$ for each $1 \leq j \leq 5$ if the theorem were to hold true. We have

$$P_k(x) = x^4 - x^3 - x^2 - 1 \tag{2.3.6}$$
$$P_k^*(x) = -x^4 - x^2 - x + 1 \tag{2.3.7}$$
$$S(x) = x^3 + x^2 \tag{2.3.8}$$
$$S^*(x) = x^2 + x \tag{2.3.9}$$

We calculate the following polynomials,

$$Z_1^{(1)}(x) = x^8 + 2x^5 + x^2 \tag{2.3.10}$$
$$Z_2^{(1)}(x) = x^{10} + x^6 + x^5 + x^1 \tag{2.3.11}$$
$$Z_3^{(1)}(x) = x^{11} + x^{10} - x^9 + x^7 + x^5 - x^3 + x^2 + x^1 \tag{2.3.12}$$
$$Z_4^{(1)}(x) = x^{12} + x^{11} - x^9 + x^8 + x^5 - x^4 + x^2 + x^1 \tag{2.3.13}$$
$$Z_5^{(1)}(x) = x^{13} + x^{12} + x^2 + x^1 \tag{2.3.14}$$

We can then use these polynomials to determine the strings τ_j for each j, we must be careful about leading and trailing zeros due to the highest possible degree and the possible degree span of these polynomials. These strings are self inverse and their last digit should be the constant coefficient.

Here, the coefficients of this polynomial for $j = 1$ give the digits of a beta-expansion which includes a 2 which is not an allowable digit. For cases $j = 3$ and $j = 4$ we find negative

j	τ_j	Notes
1	00100200100	Invalid expansion
2	010^3110^310	Valid expansion
3	$011(-1)01010(-1)110$	Invalid expansion
4	$0110(-1)1001(-1)0110$	Invalid expansion
5	0110^9110	Valid expansion

Table 2.3.1: Value of τ_j for greedy expansions under Salem bases related to $\Phi^{C-}_{(2,5)}(x)$

Theorem	# PV which satisfy the hypothesis of the Theorem
Theorem 2.1.1	1000
Theorem 2.1.5	322
Theorem 2.2.2	110
Theorem 2.2.3	269
Theorem 2.2.5	45
Theorem 2.2.6	296

Table 2.3.2: PV numbers of degree at most 20 for which our results can be applied

coefficients which would result in negative digits in the beta-expansion. This shows us that $R_{n\ell+j}(x)$ *is not the companion polynomial for the beta-expansion of the Salem number induced by the polynomial* $M(x)x^{n\ell+j} + M^*(x)$ *for* $j = 1, 3, 4$*. In the other cases, all the coefficients of* $Z_j(x)$ *are in* $\{0, 1\}$*, so they induced string* τ_j *is valid.*

The remainder of this section provides some numerical analysis for our results. We take a look at a data set that includes all PV numbers, $\theta \in (1, 2)$, with minimal polynomial of degree at most 20.

There are 1731 PV numbers in the data set. Of these, 1000 have a finite reversible greedy expansions, 547 had an infinitely reversible greedy expansion, and the remaining 184 did not have a reversible greedy expansion. We have summarized in Table 2.3.2 for how many polynomials there exist a j such that the hypothesis of one of the Theorems in this paper holds.

2.4 Degree 4 Salem numbers

This section provides a complete characterization for greedy expansions of 1 under Salem bases with minimal polynomials of degree 4. This result is the work of Boyd, in Section 4.3 we will prove a similar result for lazy expansions of 1.

Theorem 2.4.1 (Boyd, [8]). *Let β be a Salem number of degree 4 with minimal polynomial*

$$T(x) = x^4 - ax^3 + bx^2 - ax + 1.$$

Then the greedy β-expansion for 1, (c_i), is eventually periodic and the expansions are given as follows;

1. *If $c_1 = a + 1$ and $b = -2a - 1$, then*
$$(c_i) = (a+1), [(a-1), (a+1), (a-1), 0, (a-1), (a+1), (a-1), a, a]^\omega$$
2. *If $c_1 = a + 1$ and $b > -2a - 1$, then*
$$(c_i) = (a+1), [(-b-a-1), (2a+b), (-b-a-1), a, a]^\omega$$
3. *If $c_1 = a$, so that $-a \leq b \leq 0$, $b \neq -a + 1$, then*
$$(c_i) = a, [(-b), (a-1), (a-1)]^\omega$$
4. *If $c_1 = a - 1$, so that $0 < b \leq a$, $b \neq 2$, then*
$$(c_i) = (a-1), [(a-b), (a-b), (a-2), (a-2)]^\omega$$
5. *If $c_1 = a-2$, so $a+1 < b < 2a-2$ let $b_k = (2a-2) - (a-3)/k$ for $k = 1, 2, \ldots, a-3$, so that $a + 1 = b_1 < b_2 < \cdots < b_{a-3} = 2a - 3$. If $b_{k-1} < b \leq b_k$ then*
$$\begin{aligned}
c_1 &= a - 2, \\
c_2 &= 2a - b - 3, \\
c_i &= (i-1)(2a-b-2), \qquad \text{if } 2 \leq i \leq k-1, \\
c_k &= 2(k-1)a - (k-1)b - (2k-3), \\
c_{k+1} &= (2k-1)a - kb - (2k-3), \\
c_i &= c_{(2k+3)-i}, \qquad \text{if } k+2 \leq i \leq 2k+1, \\
c_{2k+2} &= c_{2k+3} = a - 3.
\end{aligned}$$

We provide the exact details of this result for two reasons in particular.

First, notice in each of these cases a similar structure to the theorems in Sections 2.1 and 2.2. In each of the five forms above, if we let $c_1 - 1 = \delta$, then we always have $c_\ell = c_{\ell+1} = \delta$. Furthermore, the remaining digits of the periodic part, $c_2 \ldots c_{\ell-2}$, form a reciprocal sequence.

Second, we note that there are only finitely many forms that the greedy expansions for 1 under degree four Salem bases can take. We will find that these same structures appear when we look at the lazy expansions for 1 under the same bases. Using this, we will explore when degree four Salem numbers are univoque.

Chapter 3

Lazy Expansions

Recall that for a non-integer $\beta > 1$ the lazy β-expansion for x is the lexicographically smallest β-representation for the given real number x. We begin this chapter with some results related to lazy expansions for 1. Our main goal in Section 3.1 is to give a criterion for a sequence being a lazy expansion for 1 similar in form to Theorem 1.1.7. With this tool, we explore which algebraic integers can have finite or eventually periodic lazy expansions. We show shortly that no lazy β-expansion for 1 is ever finite, so we restrict ourselves to finding eventually periodic expansions.

In Section 3.2 we show that the bases, β_1 for which the lazy β_1-expansion for 1 is eventually periodic share many similar properties with the bases, β_2, where the greedy β_2-expansion for 1 is finite or eventually periodic. That is, these bases have properties similar to the Parry numbers. We will question if these two sets are equal, or otherwise intersect non trivially. This analysis leads us to considering the PV and Salem numbers as bases for lazy expansions of 1.

Recall that for θ a PV number, the greedy expansion $d_\theta(1)$ is always finite or eventually periodic. That is, θ is a Parry number. We will show that a similar result holds for lazy expansions of PV bases. In Section 3.3 we turn our attention back to regular PV numbers. Having shown that all of these induce eventually periodic lazy expansions for 1 we take the time to write them down. Recall for a regular PV number $\theta \in (1, 2)$, Amara [2] wrote down the greedy θ-expansion for 1 and the corresponding pseudo co-factor. We write down the lazy θ-expansion for 1 and the corresponding pseudo co-factor

3.1 Lazy Expansion Criteria

We begin by first showing that unlike the greedy expansion for 1, the lazy expansion in a given base β is never finite.

Proposition 3.1.1. *Suppose $\beta \in (M, M+1)$ and $(a_i)_{i\geq 0}$ is the lazy β-expansion for 1. Then there exists arbitrarily large n where $a_n \neq 0$. That is, (a_i) is not finite.*

Proof. Suppose towards a contradiction that (a_i) is a finite expansion. Then there exists $k \in \mathbb{N}$ such that $a_k \neq 0$ and $a_n = 0$ for all $n > k$. Since $a_k \neq 0$ the real number $x = a_k\beta^{-k}$ is non-zero. If we show that $(b_i) = \ell_\beta(x) \neq 0^{k-1}a_k$ then we are done. Since $x = a_k\beta^{-k}$, the sequence $0^{k-1}a_k$ is a valid β-representation of x. Therefore, since $(b_i) \leq_{\text{lex}} 0^{k-1}a_k$ we must have $b_i = 0$ for $i < k$ and $b_k \leq a_k$. Suppose towards a contradiction that $b_k = a_k$ then,

$$(a_k - 1)\beta^{-k} + \sum_{i=k+1}^{\infty} M\beta^{-i} < x = a_k\beta^{-k}.$$

Equivalently,

$$\sum_{i=k+1}^{\infty} M\beta^{-i} < \beta^{-k} \iff \sum_{i=1}^{\infty} M\beta^{-i} < 1.$$

This final inequality fails because $\sum_{i=1}^{\infty} M\beta^{-i} > \sum_{i=1}^{\infty} M \cdot (M+1)^{-i}$ since $\beta < M+1$. But we know since β is not an integer that $\sum_{i=1}^{\infty} M \cdot (M+1)^{\infty} > 1$. Therefore, $b_k \neq a_k$ and (a_i) is not the lazy β-expansion for 1.

□

Our main goal in this section is to prove a lazy analog to Theorem 1.1.7. We begin with the following result, which is limited to the case $\beta \in (1, 2)$. Using this result, we can determine whether a given sequence is the lazy β-expansion of 1 for some base β.

Theorem 3.1.2 (Farzaliyev, [13])**.** *A sequence $\overline{a} = a_1a_2\ldots$ is a lazy expansion of 1 in some base $\beta \in (1, 2)$. Provided for all $k \in \mathbb{N}$ such that $a_k = 1$ we have $(a_i) = a_1a_2\cdots >_{lex} (b_{k+i}) = b_{k+1}b_{k+2}\ldots$, where $b_n = 1 - a_n$. That is, if $\overline{b} = (b_{i+k})$ then $\sigma(\overline{b}) <_{\text{lex}} \sigma(\overline{a})$ for all k such that $a_k = 1$.*

The restriction here to $\beta \in (1, 2)$ limits the use of this result outside the interval. However, note that we have exact knowledge of the regular PV numbers that fall within this interval. This criterion can be used to show the lazy expansions for 1 under those bases. The following example illustrates how a given sequence can be verified as a lazy expansion for 1 in finite time.

Example 3.1.3. *The sequence* $\overline{a} = 1101(0110)^\omega$ *is a lazy expansion of* 1 *in some base* β*.*

$$\begin{aligned} k = 1, \quad 010(1001)^\omega &<_{\text{lex}} 1101(0110)^\omega \\ k = 2, \quad 10(1001)^\omega &<_{\text{lex}} 1101(0110)^\omega \\ k = 3, \quad 0(1001)^\omega &<_{\text{lex}} 1101(0110)^\omega \\ k = 4, \quad (1001)^\omega &<_{\text{lex}} 1101(0110)^\omega \\ k = 5, \quad (0011)^\omega &<_{\text{lex}} 1101(0110)^\omega \\ k = 6, \quad (0110)^\omega &<_{\text{lex}} 1101(0110)^\omega \\ k = 7, \quad (1100)^\omega &<_{\text{lex}} 1101(0110)^\omega \\ k = 8, \quad (1001)^\omega &<_{\text{lex}} 1101(0110)^\omega \end{aligned}$$

Notice how in the periodic case we need only to check finitely many sequences. Here we check all the values of k*, the values of* k *for which* $a_k = 0$ *are not required. This can be seen because the sequence for* $k - 1$ *will be larger. In this example, the values* $k = 3, 5, 8$ *correspond to* $a_k = 0$*, and we see that* $b_k b_{k+1} \cdots >_{\text{lex}} b_{k+1} b_{k+2} \ldots$ *as expected.*

To prove a generalized version of Theorem 3.1.2, we require some other criteria for both lazy and greedy expansions. The following criteria for greedy and lazy expansions is a modified result of Erdős, Joó, and Komornik [12]. We have modified the statements to generalize the original result. The original had the restriction of $\beta \in (1, 2)$, for all non-integer $\beta > 1$.

Proposition 3.1.4. *Let* (a_i) *be a* β*-representation for some* $x \in [0, \lfloor\beta\rfloor/(\beta - 1)]$*.*

1. *Then* (a_i) *is the greedy representation for* x *if and only if*

$$\sum_{i=1}^{\infty} a_{k+i}\beta^{-i} < 1, \quad \text{whenever } a_k \neq \lfloor\beta\rfloor. \tag{3.1.1}$$

2. *Further,* (a_i) *is the lazy representation for* x *if and only if*

$$\sum_{i=1}^{\infty} (\lfloor\beta\rfloor - a_{k+i})\beta^{-i} < 1, \quad \text{whenever } a_k \neq 0. \tag{3.1.2}$$

Proof. To prove the forward direction of 1, suppose that (a_i) does not satisfy equation (3.1.1) for some k with $a_k \neq \lfloor\beta\rfloor$. Then x has another representation

$$x = \sum_{i=1}^{\infty} a_i' \beta^{-i},$$

with $a_i = a'_i$ for all $i < k$ and $a'_k \geq a_k + 1$. We select one such representation with $a'_k = a_k + 1$. Clearly $(a_i) <_{\text{lex}} (a'_i)$ and so (a_i) is not the greedy representation.

On the other hand, for the backward direction, if (a_i) is not the greedy β-expansion, then there is another representation (a'_i) of x and a positive integer k such that $a_i = a'_i$ for all $i < k$ and $a_k < a'_k \leq \lfloor \beta \rfloor$. Then,

$$\sum_{i=k+1}^{\infty} a_i \beta^{-i} \geq \beta^{-k},$$

which contradicts equation (3.1.1). Multiplying both sides by β^k and re-indexing, we have

$$\sum_{i=1}^{\infty} a_{k+i} \beta^{-i} \geq 1$$

.

In order to prove 2, we take advantage of the following fact. (a_i) is the lazy β-expansion of x if and only if $(\lfloor \beta \rfloor - a_i)$ is the greedy β-expansion of $\lfloor \beta \rfloor / (\beta - 1) - x$. From the above case, we know that the sequence $(b_i) = (\lfloor \beta \rfloor - a_i)$ is a greedy β-expansion if and only if

$$\sum_{i=1}^{\infty} b_{k+i} \beta^{-i} < 1, \quad \text{whenever } b_k \neq \lfloor \beta \rfloor. \tag{3.1.3}$$

Since $b_k = \lfloor \beta \rfloor$ if and only if $a_k = 0$ and $b_i = \lfloor \beta \rfloor - a_i$, we can rewrite equation (3.1.3) as

$$\sum_{i=1}^{\infty} (\lfloor \beta \rfloor - a_{k+i}) \beta^{-i} < 1, \quad \text{whenever } a_k \neq 0. \tag{3.1.4}$$

□

The following result, due to Baiocchi and Komornik [3] in their exploration of quasi-greedy representations, is modified here slightly to align with our needs. In our case, we aim to prove that a sequence is the greedy expansion. Removing the possibility of equality of representations in the statement of Lemma 3.1.5 also removes the possibility that the sequence (b_i) is the quasi-greedy expansion. We will use Lemma 3.1.5 later to prove that a given sequence is lazy.

Lemma 3.1.5. *Suppose (a_i) is an arbitrary β-representation for 1 and (b_i) is a β-representation for $y \in [0, \lfloor \beta \rfloor / (\beta - 1)]$. Assume $(b_{k+i}) < (a_i)$ whenever $b_k \neq \lfloor \beta \rfloor$. Then (b_i) is the greedy β-expansion of y.*

Proof. We will show $\sum_{i=k+1}^{\infty} b_i\beta^{-i} < \beta^{-k}$ whenever $b_k \neq \lfloor\beta\rfloor$. This is equivalent to the condition of Proposition 3.1.4. To do this, let $k \in \mathbb{N}$ with $b_k \neq \lfloor\beta\rfloor$.

Let $k_0 = k$ and define a sequence of indices $k_0 < k_1 < \dots$ satisfying

$$b_{k_{j-1}+i} = a_i \text{ for } i = 1, \dots, k_j - k_{j-1} - 1, \text{ and } b_{k_j} < a_{k_j - k_{j-1}}.$$

There are two cases, if we can continue this process by always finding a new value k_i ,then the sequence of indices is infinite. If this is not the case, then there is some index N where $(b_{k_N+i}) = (a_i)$.

If we obtain an infinite sequence, then we have;

$$\sum_{i=k+1}^{\infty} b_i\beta^{-i} = \sum_{j=1}^{\infty}\sum_{i=1}^{k_j-k_{j-1}} b_{k_{j-1}+i}\beta^{-(k_{j-1}+i)} \tag{3.1.5}$$

$$< \sum_{j=1}^{\infty}\left(\left(\sum_{i=1}^{k_j-k_{j-1}} a_i\beta^{-(k_{j-1}+i)} - 1/\beta^{k_j}\right)\right) \tag{3.1.6}$$

$$\leq \sum_{j=1}^{\infty}\left(\frac{1}{\beta^{k_{j-1}}} - \frac{1}{\beta^{k_j}}\right) \tag{3.1.7}$$

$$= \frac{1}{\beta^{k_0}} = \frac{1}{\beta^k}. \tag{3.1.8}$$

If the sequence of indices was finite, then we have $(b_{k_N+i}) = (a_i)$ for some $N > 0$. In this case, we have;

$$\sum_{i=k+1}^{\infty} b_i\beta^{-i} = \sum_{j=1}^{N}\sum_{i=1}^{k_j-k_{j-1}} b_{k_{j-1}+i}\beta^{-(k_{j-1}+i)} + \sum_{i=1}^{\infty} b_{k_N+i}\beta^{-(k_N+i)} \tag{3.1.9}$$

$$< \sum_{j=1}^{N}\left(\left(\sum_{i=1}^{k_j-k_{j-1}} a_i\beta^{-(k_{j-1}+i)} - 1/\beta^{k_j}\right)\right)\sum_{i=1}^{\infty} a_i\beta^{-(k_N+i)} \tag{3.1.10}$$

$$\leq \sum_{j=1}^{N}\left(\frac{1}{\beta^{k_{j-1}}} - \frac{1}{\beta^{k_j}}\right) + \frac{1}{\beta^{k_N}} \tag{3.1.11}$$

$$= \frac{1}{\beta^{k_0}} = \frac{1}{\beta^k}. \tag{3.1.12}$$

In either case, we have $\sum_{i=k+1}^{\infty} b_i\beta^{-i} < \beta^{-k}$ whenever $b_k \neq \lfloor\beta\rfloor$. If we multiply by β^k on both sides of the inequality and re-index the summation, we have $\sum_{i=1}^{\infty} b_{k+i}\beta^{-i} < 1$. This is exactly the criteria for Proposition 3.1.4, so (b_i) is a greedy expansion. □

The following theorem will be our main tool for determining if a given expansion is the lazy expansion. This is the generalized version of Theorem 3.1.2 and the lazy analog of Theorem 1.1.7.

Theorem 3.1.6. *Let (a_i) be a β-representation of 1. If $(\lfloor\beta\rfloor - a_{k+i}) <_{\text{lex}} (a_i)$ whenever $a_k \neq 0$ then (a_i) is the lazy β-expansion.*

Proof. We apply Lemma 3.1.5. Set $(b_i) = (\lfloor\beta\rfloor - a_i)$ so that $\lfloor\beta\rfloor - a_k = b_k \neq \lfloor\beta\rfloor$ whenever $a_k \neq 0$. Then $(\lfloor\beta\rfloor - a_{k+i}) <_{\text{lex}} (a_i)$ is equivalent to $(b_{k+i}) <_{\text{lex}} (a_i)$. By the definition of (b_i) this lexicographic ordering is true whenever $a_k \neq 0$. Therefore, (b_i) is a greedy β-expansion and the sequence $(\lfloor\beta\rfloor - d_i) = (\lfloor\beta\rfloor - (\lfloor\beta\rfloor - a_i)) = (a_i)$ is a lazy β-expansion. We already know that (a_i) is a β-representation for 1 so it must be the lazy β-expansion for 1. □

With this criterion we can use to show that a given representation is a lazy expansion for 1 when we know that it is a valid β-expansion. Unlike Theorem 1.1.7 this result does not work in both directions. If we are given a lazy β-expansion for 1, we cannot say that it satisfies the premise of Theorem 3.1.6. This is largely due to the case for $\beta < \Phi$, where Φ is the golden ratio. When $\beta < \Phi$ the lazy β-expansion $(a_i) = d_\beta(1)$ has $a_1 = 0$ because $\sum_{i=2}^{\infty} \beta^{-i} > 1$. The following result describes the criteria in the case where $\beta \in (1, 2)$.

Theorem 3.1.7 (Farzaliyev, [13])**.** *Let (a_i) be a β-representation for 1, that is $1 = \sum_{i=1}^{\infty} a_i \beta^{-i}$.*

1. *If $\beta \in (1, (1+\sqrt{5})/2]$ and (a_i) is lazy, then $(1 - a_{k+i}) < (a_i)$ whenever $a_k = 1$.*
2. *If $\beta \in ((1+\sqrt{5})/2, 2)$ and (a_i) is lazy, then $(a_k) > (a_i)$ whenever $a_k = 0$.*

Notice that the first conclusion is the exact assumption in Theorem 3.1.6, recalling that $\beta < 2$. For $\beta \in (1, (1+\sqrt{5})/2]$ we have $\lfloor\beta\rfloor = 1$. The second result is significantly different due to the fact $a_1 = 0$.

3.2 Lazy Parry Numbers

In this section, we prove some results that were used in the literature regarding greedy expansions. We will show that they continue to hold when we examine lazy expansions. These results give evidence to look at the PV and Salem numbers in regard to finite or eventually periodic expansions. We are interested in finding families of real numbers that induce eventually periodic lazy expansions for 1. Recall that the lazy expansion for 1 is never finite, so this is the lazy analog of being a Parry number. To this end, we have the following definition.

Definition 3.2.1. *We say that a real algebraic number β is a* ***lazy Parry Number*** *if $\ell_\beta(1)$ is eventually periodic.*

We will also require the following transformation.

Definition 3.2.2. *Let $L_\beta(x)$ be the lazy β-transformation of x. That is, if $(\ell_j(x))_{j\geq 0}$ is the lazy expansion in base β for x we have $\beta^n\left(x-\sum_{k=0}^n \ell_k(x)\beta^{-k}\right) = L_\beta^n(x)$.*

This is a lazy version of the β-transformation originally used by Rényi [20].

Definition 3.2.3. *For a fixed $\beta > 1$. The β-transformation of the unit interval $T_\beta(x)$ is given by*

$$T_\beta(x) = \beta x \pmod 1, x \in [0,1).$$

If $(d_j(x))_{j\geq 0}$ is the greedy expansion in base β for x we have $\beta^n\left(x-\sum_{k=0}^n d_k(x)\beta^{-k}\right) = T_\beta^n(x)$.

The following result gives a condition for being a lazy Parry number. This result is due to Solomyak [24] in the greedy expansion case.

Theorem 3.2.4. *Suppose β has an eventually periodic lazy expansion for 1. Then if γ is a Galois conjugate of β, we have $|\gamma| < (1+\sqrt{5})/2$.*

Proof. Let $\mathcal{D}$ be the open unit disc and $\mathbb{D}$ its closure in the complex plane. Define $\mathcal{B} = \{f(z) = 1+\sum_{j=1}^\infty c_j z^j, 0 \leq c_j \leq 1\}$ and $\mathcal{G} = \{\lambda \in \mathcal{D}, f(\lambda) = 0 \text{ for some } f \in \mathcal{B}\}$.

We show that if Φ is the closure of the set of all conjugates of lazy Parry numbers, then we have $\Phi \setminus \mathbb{D} \subseteq \mathcal{G}^{-1} = \{\lambda^{-1}, \lambda \in \mathcal{G}\}$. $\mathcal{G} \cup \mathbb{T}$ and $\mathcal{G}^{-1} \cup \mathbb{T}$ are closed, where $\mathbb{T}$ is the unit torus. so it suffices to prove that if λ is a conjugate of a lazy Parry number and $|\lambda| > 1$ then $\lambda^{-1} \in \mathcal{G}$.

Let $1 = \sum_{j=1}^\infty a_j\beta^{-j}$ be such that $a_1 \dots a_k(a_{k+1}\dots a_{k+\ell})^\infty$ is the lazy β-expansion of 1, the conjugate of λ. It follows that β satisfies;

$$\left(1-\sum_{j=1}^{k+\ell} a_j z^{-j}\right) z^{k+\ell} = \left(1-\sum_{j=1}^{k} a_j z^{-j}\right) z^k. \tag{3.2.1}$$

which is a monic algebraic equation with integer coefficients. Therefore, β is an algebraic integer, and so its Galois conjugates also satisfy 3.2.1. Therefore, λ must also satisfy $1 = \sum_{j=1}^\infty a_j\lambda^{-j}$.

For any z, with $|z| > 1$ we have

$$1-\sum_{j-1}^{\infty} a_j z^{-j} = (1-\beta/z)\left(1+\sum_{j=1}^{\infty} b_j z^{-j}\right),$$

where $b_1 = \beta - a_1, b_2 = b_1\beta - a_2, \ldots, b_j = b_{j-1}\beta - a_j, \ldots$. Using the definition of the lazy β transformation we have $b_j = L_\beta^j(1)$ for each $j \geq 1$. It follows that for λ, a conjugate of β, its reciprocal λ^{-1} is a zero of $f(z) = 1 + \sum_{j=1}^{\infty}(L_\beta^j(1))z^j$. Since $L_\beta^j(1) \in [0, 1)$ for each $j \geq 1$, we know that $f \in \mathcal{B}$.

Let $\Omega := \{z \in \mathcal{D} : |1 - z|^{-1}|2 - z| \leq (1 - |z|)^{-1}|z|\}$. Then Lemma 2.2 of Solomyak [24] states $\mathcal{G} \subseteq \Omega$. They show that $\Omega \cap \mathbb{R} = (-1, -\chi]$ where $\chi = (-1 + \sqrt{5})/2$, which proves $\min\{|z|, z \in \Omega\} = \chi$. Then invoking our set inclusion result above, for any λ, the conjugate of a lazy Parry number, we must have $|\lambda| \leq (1 + \sqrt{5})/2$ as $\lambda^{-1} \geq \chi$. □

Remark 3.2.5. *Notice that the proof does not rely on the fact that (a_i) is the lazy expansion. We only require the existence of the transformation $L_\beta(x)$ that sends the interval $[0, 1)$ to itself and defines the representation. Any other representation which is defined similarly and is eventually periodic will provide the same conclusion. It follows if 1 has an eventually periodic representation in base β then all the Galois conjugates of β have modulus at most $(1 + \sqrt{5})/2$.*

This result being the same as the greedy case leads one to ask whether the set of Parry numbers coincides with the set of lazy Parry numbers. We show that this is not the case. In what follows, let (c_i) be an arbitrary β-representation for the real number x. Define

$$[(c_i)]_\beta = [c_1 c_2 \ldots]_\beta = \sum \frac{c_i}{\beta^i} = x,$$

and consider the shifts of (c_i).

$$[\sigma^k(c_i)]_\beta = [c_{k+1} c_{k+2} \ldots]_\beta = \beta^k \left(x - \sum_{i=1}^{k-1} \frac{c_i}{\beta^i} \right)$$

We see that when $\mathbf{c} = d_\beta(x)$ is the greedy expansion, then $[\sigma^k(c_i)]_\beta = T_\beta^k(x)$. When $\mathbf{c} = \ell_\beta(x)$ is the lazy expansion, then $[\sigma^k(c_i)]_\beta = L_\beta^k(x)$. We can view $[\sigma^k(\mathbf{c})]_\beta$ as a β-transformation with respect to the representation (c_i).

We learned in Theorem 3.2.4 that if the representation (c_i) is eventually periodic, then β has all its conjugates with modulus at most $(1 + \sqrt{5})/2$. Suppose $\beta > 1$ is not PV or Salem, so that we can select a conjugate $\gamma \neq \beta$ with $|\gamma| > 1$. Let x' be the image of x by the conjugate map $\mathbb{Q}(\beta) \to \mathbb{Q}(\gamma)$. The following result, due to Akiyama [1] when (c_i) is the greedy expansion, gives a criterion for when (c_i) is eventually periodic.

Theorem 3.2.6. *Let $\beta > 1$ have a conjugate γ with $|\gamma| > 1$ and let x' be the image of x by the conjugate map $\mathbb{Q}(\beta) \to \mathbb{Q}(\gamma)$. Suppose that (c_i) is a β-representation for 1.*

If there exists $k \in \mathbb{N}$ such that $|[(\sigma^k(\mathbf{c})]_\beta)'| > \lfloor \beta \rfloor / (|\gamma| - 1)$, then the representation (c_i) is not eventually periodic.

Proof. Let $x_m = [\sigma^m(\mathbf{c})]_\beta$, then we have $x_{m+1} = \beta x_m - c_{m+1}$. Since $|x'_k| > \lfloor\beta\rfloor/(|\gamma| - 1)$, we have

$$|x'_{m+1}| = |\gamma x'_m - c_{m+1}| > |x'_m|$$

for $m \geq k$. Therefore, the sequence $(|x'_i|)$ diverges, which contradicts (c_i) being eventually periodic. In such a case, the sequence (x_i) has finitely many unique terms. □

Akiyama showed this for the greedy expansion; the proof is adapted here for any similarly defined representation, including the lazy expansion. Another such representation could include one defined to use the greedy algorithm to find a_{2n+1} and the lazy algorithm to calculate a_{2p}. The following two examples show that having an eventually periodic greedy expansion for 1, or being a Parry number, does not imply being a lazy Parry number and vise versa.

Example 3.2.7. *Consider the polynomial,*

$$P(x) = x^{13} - x^{12} - x^2 - x + 1,$$

and let $\beta \approx 1.1935724$ be the real root greater than 1. This base has a greedy expansion for 1 given by $d_\beta(1) = 1(0000000001000)^\omega$. Using a computer, we are able to calculate the first few digits of the lazy expansion.

$$\ell_\beta(1) = \underbrace{0\ldots0}_{9}\underbrace{1\ldots1}_{7}0\underbrace{1\ldots1}_{20}0\underbrace{1\ldots1}_{9}0\underbrace{1\ldots1}_{20}0\underbrace{1\ldots1}_{18}0\ldots$$

Taking $k = 34$ suffices so that $|L^k_\beta(1)'| > \lfloor\beta\rfloor/(|\gamma|-1)$. Where γ is the root with the largest modulus and different from β.

Therefore, the expansion $\ell_\beta(1)$ is not eventually periodic and β is a Parry number, but not a lazy Parry number.

Example 3.2.8. *Consider the polynomial,*

$$P(x) = x^{13} - 2x^{12} - x^{11} - 2x^{10} + x^9 - 2x^8 + x^7 - 2x^6 + 2x^5 - 2x^4 + 2x^3 - 3x^2 + 2x - 2,$$

and let $\beta \approx 1.9928744$ be the real root greater than 1. This base has lazy expansion for 1 given by $d_\beta(1) = 1(1111110011011)^\omega$. Using a computer, we are able to calculate the first few digits of the greedy expansion,

$$d_\beta(1) = \underbrace{1\ldots1}_{7}000111\underbrace{0\ldots0}_{8}1110001001100110101\underbrace{0\ldots0}_{4}1100100\underbrace{1\ldots1}_{5}001000\ldots$$

Taking $k = 48$ suffices so that $|T^k_\beta(1)'| > \lfloor\beta\rfloor/(|\gamma|-1)$. Where γ is the root with the largest modulus and different from β.

Therefore, the expansion $d_\beta(1)$ is not eventually periodic and β is a lazy Parry number, but not a Parry number.

We can now say that the sets of Parry numbers and lazy Parry numbers do not intersect in a trivial manner. Therefore, we will continue to explore lazy Parry numbers. The remainder of this section considers known results for Parry numbers in the context of lazy Parry numbers. This brings our attention back to the PV and Salem numbers, two families of real numbers that satisfy the conditions of Theorem 3.2.4. The next result in the greedy case is due to Schmidt [22] who showed that if β is a PV-number then $\mathrm{Per}(\beta) = \mathbb{Q}(\beta) \cap [0, 1)$, where $\mathrm{Per}(\beta)$ is the set of points which eventually become periodic under the β-transformation.

We will show that this result holds in the case of the lazy β-transformation, and by extension we show that every PV-number is a lazy Parry number. Recall that a real number q is a Parry number provided that the greedy expansion for 1 in base q is finite or eventually periodic. Let $\mathrm{Per}_\ell(\beta)$ be the set of eventually periodic points of L_β in the interval $[0, 1)$.

Theorem 3.2.9. *Let θ be a PV-number. Then $\mathrm{Per}_\ell(\theta) = \mathbb{Q}(\theta) \cap [0, 1)$.*

In order to prove this theorem, we require some results from Schmidt [22]. To that end, let β be a PV-number of degree d with minimal polynomial $M(x) = x^d + \sum_{i=0}^{d-1} c_i x^i$. Fix an $x \in \mathbb{Q}(\beta)$ and write

$$x = q^{-1} \sum_{i=0}^{d-1} p_i \beta^i, \quad p_i, q \in \mathbb{Z}, \quad q > 0, \tag{3.2.2}$$

with q minimal such that this representation is unique. Then for each $n \geq 0$ write

$$\varrho_n(x) = \beta^n \left(x - \sum_{k=0}^{n} \ell_k(x) \beta^{-k} \right) = L_\beta^n(x).$$

Lemma 3.2.10 (Schmidt, [22] Lemma 2.1)**.** *For every $n \geq 0$ there exists a unique element $r_n = (r_{n,1}, \ldots, r_{n,d}) \in \mathbb{Z}^d$ such that*

$$\varrho_n(x) = q^{-1} \sum_{k=1}^{d} r_{n,k} \beta^{-k}.$$

The proof of Lemma 3.2.10 is unchanged when replacing the greedy expansion $d_\beta(x)$ with the lazy expansions $\ell_\beta(x)$ when we define the constants $\varrho_n(x)$.

Lemma 3.2.11 (Schmidt, [22] Lemma 2.3)**.** *Let $\beta = \beta_1, \ldots, \beta_d$ be the roots of $M(x)$ and, for every $n \geq 0$, $i = 1, \ldots d$, put*

$$\varrho_{n,i}(x) = q^{-1} \sum_{k=1}^{d} r_{n,k} \beta_i^{-k}.$$

By construction we have that $\varrho_{n,1}(x) = \varrho_n(x)$ for all n. Then the following conditions are equivalent;

1. $\mathrm{frac}(x) \in \mathrm{Per}_\ell(\beta)$,
2. $\max_{1\le i\le d} \sup_n |\varrho_{n,i}(x)| < \infty$,
3. $\sup_n \max_{1\le k\le d} |r_{n,k}| = \sup_n ||r_n|| < \infty$.

Note that this result originally concerned the set of periodic points of T_β, the β-transformation, which can be used to define the greedy expansion, which tells us about eventually periodic greedy expansions. The proof of this result is unchanged when we use the lazy expansion $\ell_\beta(x)$ in the construction of $\varrho_{n,i}(x)$ and r_n, so the proof is omitted.

Proof of Theorem 3.2.9. Let $x \in \mathbb{Q}(\beta) \cap [0,1)$ be arbitrary but fixed, write it in the form of 3.2.2. Let $\varrho_{n,i}(x)$ be as in Lemma 3.2.11 for each $n \ge 0$ and $i = 1, \dots, d$. We know $\varrho_n(x) = \varrho_{n,1}(x)$ so it is clear that $0 \le \varrho_{n,1}(x) < 1$ for each $n \ge 0$. Let $\beta_2, \dots, \beta_d$ be the Galois conjugates of β, then $\eta = \max_{i=2,\dots,d} |\beta_i| < 1$, since β is a PV-number. The following inequalities thus hold for every $n \ge 0$ and $i = 2, \dots, d$.;

$$\begin{aligned} |\varrho_{n,i}(x)| &\le \eta^n q^{-1} \sum_{j=0}^{d-1} |p_i| \eta^j + \eta^n \sum_{k=0}^{n} \ell_k(x) |\beta_j^{-k}| \\ &\le q^{-1} \sum_{j=0}^{d-1} |p_i| \eta^{j+n} + \lfloor \beta \rfloor \sum_{k=0}^{n} \eta^k < \infty \end{aligned}$$

The second inequality follows from the change of variables $k \to n - k$ over the second summation and the fact that $\ell_k(x)$ takes a maximum value of $\lfloor \beta \rfloor$. Noting that from the definition of $\varrho_n(x) = \varrho_{n,1}(x)$ we must have $0 \le \varrho_n(x) < 1$ for each $n \ge 0$. Then we conclude that $\max_{1\le i\le d} \sup_n |\varrho_{n,i}(x)| < \infty$.

Therefore, we are in a position to apply Lemma 3.2.11 and conclude that $(x) \in \mathrm{Per}_\ell(\beta)$. □

Corollary 3.2.12. *Let β be a PV-number, then 1 has an eventually periodic lazy expansion in the base β. In other words, β is a lazy Parry number.*

Proof. Consider $L_\beta(1) \in [0,1)$. We know $L_\beta(1) \in \mathbb{Q}(\beta) \cap [0,1) = \mathrm{Per}_\ell(\beta)$. Therefore, the set $\{L_\beta^k(1) : k \ge 0\}$ is finite, and we must have that the representation for 1 under the lazy transformation is eventually periodic. □

3.3 Regular PV-numbers

We recall from the introduction that greedy β-expansions for 1 where β is a regular PV number have been established. Towards applying similar results we have to those in Chapter 2 we first would like to establish what the lazy β-expansions for 1 are when β is a regular PV-numbers.

Tables A.0.2 and A.0.3 give the lazy expansions and pseudo co-factors along with restrictions of the families of regular PV-numbers. The pseudo co-factor is the polynomial $Q^*(x)$ such that the companion polynomial $R(x)$ can be written as $R(x) = M^*(x)Q^*(x)$. Here $M^*(x)$ is the defining polynomial given in Table 1.2.1 . $M^*(x)$ may not be the minimal polynomial $M(x)$. Proposition 3.3.1 below is an example of one of these results given in Tables A.0.2 and A.0.3. As many results have similar proofs, we will provide only a few illustrative examples.

Proposition 3.3.1. *Consider the case* $\Psi^{A-}_{(r,q)}(x) = \Psi^{A-}_{(r,n(r+1)+j)}(x)$ *then for* $r > 1$ *and* $q > r$ *the lazy expansion for 1 is given by;*

$$(1^r0)^{n-1}1^{r+1}[1^r0]^\omega, \text{ for } j = 0 \tag{3.3.1}$$
$$(1^r0)^{n-1}1^{r+1}[0^j(1^r0)^n]^\omega, \text{ for } j = 1, \dots, r-1 \tag{3.3.2}$$
$$(1^r0)^{n-1}1^{r+1}0^r(1^r0)^{n-1}1^{r-1}01^{r+1}[(1^r0)^{n-1}1^{2r}0]^\omega, \text{ for } j = r. \tag{3.3.3}$$

Proof. We prove this by first showing that $\Psi^{A-}_{(r,n(r+1)+j)}(x)$ is a factor of the companion polynomial for the given sequence. This will show that the sequence is a valid representation for 1 in the required base. Then we finish by showing that the expansion is lazy by using Theorem 3.1.2. We have

$$\Psi^{A-}_{(r,n(r+1)+j)}(x) = x^{n(r+1)+j}\left(x^{(r+1)} - x^r - \cdots - 1\right) - x^{(r+1)} + 1.$$

Let $R_{(r,q)}(x) = R_{(r,n(r+1)+j)}(x)$ be the companion polynomial for the sequence in the statement of the theorem. We will show that the pseudo co-factor in each case is as follows.

$$Q^*_{(r,n,j)}(x) = \begin{cases} 1 & \text{if } j = 0 \\ 1 + x^{r+1} + \cdots + x^{(n-1)(r+1)} & \text{if } j = 1, \dots, r-1 \\ 1 + x^{q+r}(1 + x^{r+1} + \cdots + x^{(n-1)(r+1)}) & \text{if } j = r. \end{cases} \tag{3.3.4}$$

That is $R_{(r,n(r+1)+j)}(x) = Q^*_{(r,n,j)}(x) \cdot \Psi^{A-}_{(r,n(r+1)+j)}(x)$.

For the case $j = 0$ we have;

$$R_{(r,n(r+1)+0)}(x) = x^{(n+1)(r+1)} - \sum_{i=0}^{n-2}\sum_{m=1}^{r} x^{(n+1)(r+1)-i(r+1)-m} - \sum_{i=1}^{2r+1} x^{2(r+1)-i}$$
$$- \left[x^{n(r+1)} - \sum_{i=0}^{n-2}\sum_{m=1}^{r} x^{n+(r+1)-i(r+1)-m}\right] \tag{3.3.5}$$
$$= x^{(n+1)(r+1)} - x^{(n+1)(r+1)-1} - \cdots - x^{n(r+1)} - x^{r+1} + 1 \tag{3.3.6}$$
$$= \Psi^{A-}_{(r,n(r+1)+0)}(x). \tag{3.3.7}$$

For the case $j = 1, \ldots, r-1$, we let $R(x) = R_{(r,n(r+1)+j)}(x)$;

$$R(x) = x^{2n(r+1)+j} - \sum_{i=0}^{n-2}\sum_{m=1}^{r} x^{2n(r+1)+j-i(r+1)-m}$$
$$- \sum_{i=1}^{r+1} x^{(n+1)(r+1)+j-i} - \sum_{i=0}^{n-1}\sum_{m=1}^{r} x^{n(r+1)-i(r+1)-m}$$
$$- \left[x^{n(r+1)} - \sum_{i=0}^{n-2}\sum_{m=1}^{r} x^{n(r+1)-i(r+1)-m} - \sum_{i=1}^{r+1} x^{(r+1)-i}\right] \tag{3.3.8}$$
$$= x^{(n+1)(r+1)+j}\left[x^{(r+1)} - x^r - \cdots - 1\right]\left(1 + x^{r+1} + \cdots + x^{(n-1)(r+1)}\right) - x^{n(r+1)} + 1 \tag{3.3.9}$$
$$= \left[x^{(n+1)(r+1)+j}\left[x^{(r+1)} - x^r - \cdots - 1\right] - x^{(r+1)} + 1\right]\left(1 + x^{r+1} + \cdots + x^{(n-1)(r+1)}\right) \tag{3.3.10}$$
$$= \Psi^{A-}_{(r,n(r+1)+j)}(x) \cdot Q^{*}_{(r,n,j)}(x) \tag{3.3.11}$$

For the case $j = r$, we find $\Psi^{A-}_{(r,n(r+1)+j)}(x) \cdot Q^{*}_{(r,n,j)}(x)$. It remains to show each that these representations given in the statement of the theorem have the lazy property. From the criteria given in Theorem 3.1.6, it suffices to notice that given a $k \in \mathbb{N}$ with $a_k = 1$ the sequence (a_{k+1}) is lexicographically larger than $(0^r1)^{n-1}0^{r+1}$, a prefix of $(1 - a_i)$. We see that for each 1 in the expansion, the sequence that follows is larger in every case, there is only a single instance where we find the required number of zero digits to require further checking. □

Some of the families of regular PV numbers appear to have more complicated lazy expansions for 1, which we were unable to directly prove via these methods. We discuss these families in the following subsections and give conjectures for their respective lazy expansions. The lazy expansions for these cases have been calculated, using a computer, up to $r, q = 50$ and align with our stated conjectures.

3.3.1 Case $\Psi^{B+}_{(r,q)}(x)$

To deal with this case, we first consider the following definition.

Definition 3.3.2. *We say that a word $a_1 \dots a_k(a_{k+1} \dots a_{k+\ell})^\omega$ is **balanced** if for any two sub-words of the same length, $a_i \dots a_{i+m} = b_1 \dots b_m$ and $a_j \dots a_{j+m} = c_1 \dots c_m$ we have $|\#\{b_i = 1\} - \#\{c_i = 1\}| \leq 1$.*

Example 3.3.3. *The sequence $(101010)^\omega$ is a balanced and fully periodic word since every sub-word of length $2k$ has k ones and every sub-word of length $2k+1$ has k or $k+1$ ones.*

Conjecture 3.3.4. *For the case $\Psi^{B+}_{(r,q)}(x) = \Psi^{B+}_{(r,n(r+1)+j)}(x)$ where $j = -1, \dots, r-1$. We let $\kappa = 1^r0$, $\sigma(\kappa) = \sigma = 1^{r-1}0$ and $\gamma_i = 1^{i-1}01^{r-(i-1)-1}0$ where we reduce $i \mod r$ so that $i \in \{1, 2, \dots, r\}$. Then the lazy expansion for 1 in this PV base looks like.*

$$[\kappa^{(n-1)+n_1}\gamma_{j+1}\kappa^{(n-1)+n_2}\gamma_{2(j+1)}\kappa^{(n-1)+n_3} \dots \kappa^{(n-1)+n_m}\gamma_{m(j+1)}\kappa^{(n-1)+n_{m+1}}\sigma]^\omega$$

where $m+1 = (r+1)/\gcd(r+1, j+1)$ and n_i is determined in the following way.

Let w_j be the jth smallest non-zero balanced fully periodic word with period length $r+1$. Then we let n_i is the ith digit of w_{j+1} where for $j = -1$ we use the word $w_{r+1} = 1^\omega$.

Example 3.3.5. *For $r = 5$ we have $\kappa = 111110$, $\gamma_i = 1^{i-1}01^{5-(i-1)}0$, and $\sigma = 11110$.*

$$\gamma_1 = 011110,\ \gamma_2 = 101110,\ \gamma_3 = 110110,\ \gamma_4 = 111010,\ \gamma_5 = 111100.$$

The non-zero balanced fully periodic words of length $r+1 = 6$ are the following;

$$w_1 = 100000,\ w_2 = 100100,\ w_3 = 101010,\ w_4 = 110110,\ w_5 = 111110,\ w_6 = 111111.$$

Let $j = 1$ and $n = 3$ be such that $q = 19$. $\gcd(1+1, 5+1) = 2$ hence, the expansion will have $(5+1)/2 = 3$ terms of the form $\kappa^{(n-1)+n_i}$. We have $w_{j+1} = w_2 = 100100$ so that $n_1 = 1, n_2 = 0$ and $n_3 = 0$. Then the lazy expansion for 1 in base $\Psi^{B+}_{(5,19)}(x) = \Psi^{B+}_{(5,3(5+1)+1)}(x)$ is given by

$$\kappa^3\gamma_2\kappa^2\gamma_4\kappa^2\sigma = (111110)^3101110(111110)^2111010(111110)^211110.$$

3.3.2 Case $\Phi^{B-}_{(r,q)}(x)$

We consider $q = 2nr + r + j$ for $j = 1, \dots, 2r$. The cases where $j = 1$, $j = r, \dots, 2r-1$, and $j = 2r$ are relatively straightforward. They are given in Tables A.0.2 and A.0.3. In this section, we explore the remaining cases.

Conjecture 3.3.6. *Let θ be the PV root of $\Phi^{B-}_{(r,q)}(x)$. For $q = 2nr + r + j$ let $j = r - i$, $\lambda = 0^{r-(i+2)}10^i1, \gamma = 1^{r-(i+2)}01^i0$ then for $r \geq \min(2i, i+2)$ the lazy expansion for 1 in the PV base θ is given by;*

$$(1^r0^r)^n1^r0[(\lambda\gamma)^n\lambda 1^{r-(i+2)}01^{i-1}01^{r-2i}0^{i-1}10^{r-(i+2)}(\gamma^*\lambda^*)^n01^{r-1}01^{(2n+1)r+1}]^\omega \tag{3.3.12}$$

$$= (1^r0^r)^n1^r0[(0^{r-(i+2)}10^i11^{r-(i+2)}01^i0)^n0^{r-(i+2)}10^i1 \\ 1^{r-(i+2)}01^{i-1}01^{r-2i}0^{i-1}10^{r-(i+2)}(01^i01^{r-(i+2)}10^i10^{r-(i+2)})^n01^{r-1}01^{(2n+1)r+1}]^\omega \tag{3.3.13}$$

Example 3.3.7. *Let $r = 7$ and $j = r-2$ be so that $i = 2$. We have $7 \geq \min(2\cdot 2, 2+2)$, so we can apply the result. Let $q = 54 = 2(3)(7) + 7 + (7-2)$ be such that $n = 3$. Numerical calculations find that the expansion is given by;*

$$(1^70^7)^31^70[(0^310011^30110)^30^310011^30101^3010^3(01101^310010^3)^301^601^{50}]^\omega$$

This is exactly the expansion we expect from the above conjecture.

It remains to detail the expansions for $i + 2 \leq r < \max(2i, i + 2)$ in this case. This is only possible for $i \geq 3$.

Computationally, we find that this case has a similar form, that is;

$$(1^r0^r)^n1^r0[(\lambda\gamma)^n\lambda\tau_{(r,i,n)}(\gamma^*\lambda^*)^n01^{r-1}01^{(2n+1)r+1}]^\omega.$$

Here λ and γ are the same as above. The difference occurs in the fact that τ is now dependent on n whereas in the other cases in this section it depends only on r and i. Table 3.3.1 gives the first few values of r and q for which this case applies, we are yet to find a general form for expansions in this case.

(r,i,n)	τ
$(5,3,1)$	$(01)^9 10 (01)^7 0^3 1$
$(5,3,2)$	$(01)^{14} 10 (01)^{12} 0^3 1$
$(5,3,3)$	$(01)^{19} 10 (01)^{17} 0^3 1$
$(6,4,1)$	$(01)^2((10)^3(01)^3)(10)^3 1 (10)^3(01)^3(10)^3 0^3 1$
$(6,4,2)$	$(01)^2((10)^3(01)^3)^2(10)^3 1((10)^3(01)^3)^2(10)^3 0^3 1$
$(6,4,3)$	$(01)^2((10)^3(01)^3)^3(10)^3 1((10)^3(01)^3)^3(10)^3 0^3 1$
$(7,4,1)$	$10110(10010010110110)1001001100(01101101001001)011011000010$
$(7,4,2)$	$10110(10010010110110)^2 1001001100(01101101001001)^2 011011000010$
$(7,4,3)$	$10110(10010010110110)^3 1001001100(01101101001001)^3 011011000010$

Table 3.3.1: Examples for special cases related to $\Phi^{B-}_{(r,q)}(x)$

Chapter 4

Lazy Salem Representations

In Chapter 2 we related the greedy expansions for 1 under Salem bases to the underlying PV number and its greedy expansion for 1. In this chapter, we will provide some similar results for lazy expansions. As in the greedy case, we deal with both a positive case $T_m^+(x) = x^m M(x) + M^*(x)$ and the negative case $T_m^-(x) = x^m M(x) - M^*(x)$.

We begin by exploring results similar to those found in Chapter 2. Recall that in the greedy expansion case we had to consider the difference when the PV number had a finite versus eventually periodic greedy expansion for 1. We showed earlier that the lazy expansion for 1 is never finite, so we do not need to consider different cases. In Chapter 3 we found many PV numbers θ with lazy expansion for 1, $\ell_\theta(1)$ of the following form;

$$\ell_\theta(1) = (a_1 \dots a_k)^\omega.$$

We call these expansions **fully periodic**. PV numbers with fully periodic expansions will be dealt with separately from eventually periodic expansions. Recall in the results from Chapter 2, we often needed the object $a_2 \dots a_k$. In the fully periodic case, this object is not well defined.

We end this chapter with a complete description of lazy expansions for 1 under Salem bases, where the Salem number is degree 4. Recall that Boyd [8] showed that all such Salem numbers are Parry numbers. We will show that they are also lazy Parry numbers, and we will exactly describe the underlying expansions.

In Chapter 2 we used the notion of reversibly greedy PV numbers. The purpose was to prove that the sequences in the statements of our results were greedy expansions. In this chapter, we introduce the concept of a reversibly lazy expansion. We will require this concept to prove that sequences are lazy expansions under Salem bases.

Condition 4.0.1. *We say an algebraic integer θ with $\ell_\theta(1) = a_1 \dots a_k(a_{k+1} \dots a_{k+\ell})^\omega$ has a **reversibly lazy** θ-expansion if it is lazy and;*

$$a_1 a_2 \dots a_k(a_{k+1} \dots a_{k+\ell})^\omega >_{lex} (\lfloor \beta \rfloor - a_{i-1})(\lfloor \beta \rfloor - a_{i-2}) \dots (\lfloor \beta \rfloor - a_2) \text{ for all } i,$$
$$\text{with } 0 \leq i \leq \max(k + 2\ell + 1, 3k + 1).$$

This condition is the direct analog to Condition 1.3.2 for θ having a reversibly greedy expansion. We use the criteria for (a_i) being a lazy expansion found in Chapter 3 on the inverted sequence $a_{k+\ell} \dots a_2$.

4.1 The Positive case: $T_m^+(x)$

We begin with two examples, the first of which is a PV number θ with $\ell_\theta(1)$ that is eventually periodic. This case is distinguished from the second one by the fact that the preperiodic part is never empty.

Example 4.1.1. *Consider the PV number θ with the minimal polynomial $M(x) = \Phi_{3,6}^{A-}(x) = x^{10} - 2x^9 + x^7 - x^6 - x^3 + x^2 - 1$. We know $\ell_\theta(1) = a_1 \dots a_k(a_{k+1} \dots a_{k+\ell})^\omega = 11100(1011111)^\omega$. Here $k = 5$ and $\ell = 7$. Let $\kappa = a_2 \dots a_k = 1100$ and $\gamma = a_{k+1} \dots a_{k+\ell} = 1011111$. For sufficiently large values of m we can define the polynomial $T_m^+(x) = M(x)x^m + M^*(x)$ which yields a Salem number α_m^+. We look at the lazy expansions of 1 with base α_m^+ for various m. See Table 4.1.1.*

m	greedy expansion of 1
8	$1(\kappa 111101111\kappa^*00)^\omega$
15	$1(\kappa\gamma 111101111\gamma^*\kappa^*00)^\omega$
22	$1(\kappa\gamma\gamma 1111001111\gamma^*\gamma^*\kappa^*00)^\omega$
29	$1(\kappa\gamma\gamma\gamma 1111001111\gamma^*\gamma^*\gamma^*\kappa^*00)^\omega$
9	$1(\kappa 1101111011\kappa^*00)^\omega$
16	$1(\kappa\gamma 1101111011\gamma^*\kappa^*00)^\omega$
23	$1(\kappa\gamma\gamma 1101111011\gamma^*\gamma^*\kappa^*00)^\omega$
30	$1(\kappa\gamma\gamma\gamma 1101111011\gamma^*\gamma^*\gamma^*\kappa^*00)^\omega$

Table 4.1.1: Lazy expansions of 1 for base α_m^+ in Example 4.1.1

For $m = \ell n + j$ with $j = 1$ we see that the lazy expansions look like $1(\kappa\gamma^{n-1}\tau_j(\gamma^*)^{n-1}\kappa^*00)^\omega$ with $\tau_1 = 111101111$. Similarly for $j = 2$ the lazy expansions look like $1(\kappa\gamma^{n-1}\tau_j(\gamma^*)^{n-1}\kappa^*00)^\omega$ with $\tau_2 = 1101111011$.

As before, these are examples of a more general pattern.

The second example is a PV number θ such that $\ell_\theta(1)$ is fully periodic. That is, the minimal preperiodic part of this expansion is empty.

Example 4.1.2. *Consider the PV number θ with the minimal polynomial $M(x) = \Psi_2(x) = x^3 - x^2 - x - 1$. We know $\ell_\theta(1) = (a_1 \dots a_\ell)^\omega = (110)^\omega$. Here $\ell = 3$, let and $\gamma = a_2 \dots a_\ell a_1 = 101$. For sufficiently large values of m we can define the polynomial $T_m^+(x) = M(x)x^m + M^*(x)$ which yields a Salem number α_m^+. We look at the lazy expansions of 1 with base α_m^+ for various m See Table 4.1.2.*

m	greedy expansion of 1
4	$1(111100)^\omega$
7	$1(\gamma 1111\gamma^* 00)^\omega$
10	$1(\gamma\gamma 1111\gamma^*\gamma^* 00)^\omega$
13	$1(\gamma\gamma\gamma 1111\gamma^*\gamma^*\gamma^* 00)^\omega$
6	$1(1100)^\omega = 1(11001100)^\omega$
9	$1(\gamma 110011\gamma^* 00)^\omega$
12	$1(\gamma\gamma 110011\gamma^*\gamma^* 00)^\omega$
15	$1(\gamma\gamma\gamma 110011\gamma^*\gamma^*\gamma^* 00)^\omega$

Table 4.1.2: Lazy expansions of 1 for base α_m^+ in Example 4.1.2

For $m = \ell n + j$ with $j = 1$ the lazy expansions look like $1(\gamma^{n-1}\tau_j(\gamma^*)^{n-1}00)^\omega$ with $\tau_1 = 1111$. Similarly for $j = 3$ we see that the lazy expansions look like $1(\gamma^{n-1}\tau_j(\gamma^*)^{n-1}00)^\omega$ with $\tau_3 = 110011$.

As before, these are examples of a more general pattern.

We have the following two similar results, similar to those from Chapter 2. Here, the distinction occurs for when $k = 0$ where k is the length of the pre-periodic part of the lazy PV-expansion. That is, the case where the PV-representation is fully periodic.

Theorem 4.1.3. *Let θ be a PV number such that the lazy θ-expansion of 1 is infinite and reversibly lazy. Let $\ell_\theta(1) = a_1 \dots a_k(a_{k+1} \dots a_{k+\ell})^\omega = a_1\kappa(\gamma)^\omega$. Let $M(x)$ be the minimal polynomial of θ and $\delta = a_1 - 1$. Let $R(x)$ be the companion polynomial and $Q(x)$ be the co-factor polynomial for $\ell_\theta(1)$. Let $j \in \{1, \dots, \ell\}$ and $\alpha_{n\ell+j}^+$ be the Salem number satisfying $T_{n\ell+j}^+(x) = M(x)x^{n\ell+j} + M^*(x)$.*

Assume $Q(x)$ is reciprocal, there exists a τ_j such that

$$\ell_{\alpha_{\ell+j}^+}(1) = a_1(\kappa\tau_j\kappa^*\delta\delta)^\omega$$

and the companion polynomial of $\ell_{\alpha_{\ell+j}^+}(1)$ is

$$R_{\ell+j}(x) = Q(x)(M(x)x^{\ell+j} + M^*(x)).$$

Then

$$\ell_{\alpha^+_{n\ell+j}}(1) = a_1(\kappa \underbrace{\gamma \dots \gamma}_{n-1} \tau_j \underbrace{\gamma^* \dots \gamma^*}_{n-1} \kappa^* \delta\delta)^\omega \tag{4.1.1}$$

and the companion polynomial of $\ell_{\alpha^+_{n\ell+j}}(1)$ *is*

$$R_{n\ell+j}(x) = Q(x)(M(x)x^{n\ell+j} + M^*(x))(1 + x^\ell + \cdots + x^{(n-1)\ell}).$$

Proof. This proof is similar to the proof of Theorem 2.1.5 with the observation that the extra factor in $R_{\ell+j}(x)$ differs.

Using the notation from Definition 2.1.7, the equivalent equation to Equation (2.1.6) is

$$\begin{aligned}
R_{n\ell+j}(x) = &\left(P_k(x)x^{j+2n\ell} - S(x)\sum_{i=1}^{n-1} x^{j+\ell(n+i)}\right) \\
&- \left(P_k(x)x^{n\ell+j} + S(x)x^{n\ell+j} + P_k^*(x)x^{n\ell} + S^*(x)x^{(n-1)\ell+k+1}\right) \\
&+ \left(P_k^*(x) - S^*(x)\sum_{i=0}^{n-2} x^{\ell i+k+1}\right) \\
= &\left(P_k(x)x^{j+2n\ell} - S(x)\sum_{i=1}^{n-1} x^{j+\ell(n+i)}\right) - Z_j^{(6)}(x)x^{(n-1)\ell} \\
&+ \left(P_k^*(x)x^{n\ell} + S^*(x)x^{(n-1)\ell+k+1}\right)
\end{aligned}$$

where we recall

$$Z_j^{(6)}(x) = x^{\ell+j}(P_k(x) + S(x)) + x^\ell P_k^*(x) + x^{k+1}S^*(x)$$

In the greedy case, we used Theorem 1.1.7 to prove that the representation was the greedy expansion. Here, it suffices to apply Theorem 3.1.6 to complete the proof. Let (a_i) be the right hand side of Equation (4.1.1). We have $(\lfloor \alpha^+_{n\ell+j} \rfloor - a_{i+r}) < (a_i)$ whenever $a_r \neq 0$ for $r = 1, \dots k + \ell(n-1)$ since $a_1 \dots a_k(a_{k+1} \dots a_{k+\ell})^\omega$ is a lazy expansion of θ. Next, $(\lfloor \alpha^+_{n\ell+j} \rfloor - a_{i+r}) < (a_i)$ whenever $a_r \neq 0$ for $r = k + \ell(n-1) + 1 \dots k + \ell(n-1) + |\tau_j|$ since the relation holds for the case $n = 1$ by assumption. Finally, we see that $(\lfloor \alpha^+_{n\ell+j} \rfloor - a_{i+r}) < (a_i)$ whenever $a_r \neq 0$ for the remaining terms, as $a_1 \dots a_k(a_{k+1} \dots a_{k+\ell})^\omega$ is reversibly greedy. □

Theorem 4.1.4. *Let* θ *be a PV number such that the lazy* θ*-expansion of* 1 *is infinite and reversibly greedy. Let* $\ell_\theta(1) = (a_1 \dots a_\ell)^\omega$ *be fully periodic. Let* $M(x)$ *be the minimal polynomial of* θ*,* $\gamma = a_2 \dots a_\ell a_1$*, and* $\delta = a_1 - 1$*. Let* $R(x)$ *be the companion polynomial*

and $Q(x)$ be the co-factor polynomial for $\ell_\theta(1)$. Let $j \in \{1, \ldots, \ell\}$ and $\alpha^+_{n\ell+j}$ be the Salem number satisfying $T^+_{n\ell+j}(x) = M(x)x^{n\ell+j} + M^*(x)$.

Assume $Q(x)$ is reciprocal, there exists a τ_j such that

$$\ell_{\alpha^+_{\ell+j}}(1) = a_1(\tau_j \delta\delta)^\omega$$

and the companion polynomial of $\ell_{\alpha^+_{\ell+j}}(1)$ is

$$R_{\ell+j}(x) = Q(x)(M(x)x^{\ell+j} + M^*(x)).$$

Then

$$\ell_{\alpha^+_{n\ell+j}}(1) = a_1(\underbrace{\gamma \ldots \gamma}_{n-1} \tau_j \underbrace{\gamma^* \ldots \gamma^*}_{n-1} \delta\delta)^\omega \tag{4.1.2}$$

and the companion polynomial of $\ell_{\alpha^+_{n\ell+j}}(1)$ is

$$R_{n\ell+j}(x) = Q(x)(M(x)x^{n\ell+j} + M^*(x))(1 + x^\ell + \cdots + x^{(n-1)\ell}).$$

Proof. This proof is similar to the proof of Theorem 4.1.3 with the observation that $\ell_\theta(1)$ is periodic and $R(x) = P_\ell(x) - 1 = x^\ell - S(x) - 1$.

Using the notation of Definition 2.1.7, the equivalent equation to Equation (2.1.6) is

$$\begin{aligned} R_{n\ell+j}(x) =& P_\ell(x)x^{(2n-1)\ell+j} - S(x)\left(\sum_{i=1}^{n-2} x^{(n+i)\ell+j}\right) \\ &+ Z^{(7)}_j(x) - S^*(x)\left(\sum_{i=1}^{n-2} x^{i\ell}\right) + P^*_\ell(x) \end{aligned}$$

where

$$\begin{aligned} Z^{(7)}_j(x) &= P_\ell(x)x^{n\ell+j} + P^*_\ell(x)x^{(n-1)\ell} - (x^{(n+1)\ell+j} + x^{n\ell+j} + x^{n\ell} + x^{(n-1)\ell}) \\ &= (x^\ell - S(x))x^{n\ell+j} + (1 - S^*(x))x^{(n-1)\ell} - (x^{(n+1)\ell+j} + x^{n\ell+j} + x^{n\ell} + x^{(n-1)\ell}) \\ &= -S(x)x^{n\ell+j} - S^*(x)x^{(n-1)\ell} - x^{n\ell+j} - x^{n\ell} \end{aligned}$$

We note that the coefficients in front of $Z^{(7)}(x)$ included in the calculation of τ_j are those between $x^{(n+1)\ell+j-1}$ and $x^{(n-1)\ell+1}$ non inclusive. □

4.2 The Negative case: $T_m^-(x)$

Example 4.2.1. *Consider the PV number θ with the minimal polynomial $M(x) = \Phi_{2,4}^{C-}(x) = x^7-2x^6+x^5-x^4-x^3+x^2-x+1$. We know $\ell_\theta(1) = a_1 \dots a_k(a_{k+1} \dots a_{k+\ell})^\omega = 1101(1100)^\omega$. Here $k = 4$ and $\ell = 4$. Let $\kappa = a_2 \dots a_k = 101$ and $\gamma = a_{k+1} \dots a_{k+\ell} = 1100$. For sufficiently large values of m we can define the polynomial $T_m^-(x) = M(x)x^m - M^*(x)$ which yields a Salem number α_m^-. We look at the lazy expansions of 1 with base α_m^- for various m. See Table 4.2.1.*

m	greedy expansion of 1
6	$1(\kappa 1111111\kappa^*00)^\omega$
10	$1(\kappa\gamma 1111111\gamma^*\kappa^*00)^\omega$
14	$1(\kappa\gamma\gamma 1111111\gamma^*\gamma^*\kappa^*00)^\omega$
18	$1(\kappa\gamma\gamma\gamma 1111111\gamma^*\gamma^*\gamma^*\kappa^*00)^\omega$
7	$1(\kappa\gamma 11\kappa^*00)^\omega$
11	$1(\kappa\gamma\gamma 11\kappa^*00)^\omega$
15	$1(\kappa\gamma\gamma\gamma 11\kappa^*00)^\omega$
19	$1(\kappa\gamma\gamma\gamma\gamma 11\kappa^*00)^\omega$

Table 4.2.1: Lazy expansions of 1 for base α_m^- in Example 4.2.1

We observe two different patterns. For $m = \ell n + j$ with $j = 2$ we see that the lazy expansions look like $1(\kappa\gamma^{n-1}\tau_j(\gamma^*)^{n-1}\kappa^*00)^\omega$ with $\tau_2 = 1111111$. Similarly for $j = 3$ the lazy expansions look like $1(\kappa\gamma^n\tau_j\kappa^*00)^\omega$ with $\tau_3 = 11$.

As before, these are examples of a more general pattern.

Theorem 4.2.2. *Let θ be a PV number such that the lazy expansion of 1 is periodic and reversibly lazy. Let $\ell_\theta(1) = a_1 \dots a_k(a_{k+1} \dots a_{k+\ell})^\omega = a_1\kappa(\gamma)^\omega$. Let $M(x)$ be the minimal polynomial of θ and $\delta = a_1 - 1$. Let $R(x)$ be the companion polynomial and $Q(x)$ be the co-factor polynomial for $\ell_\theta(1)$. Let $j \in \{1, \dots, \ell\}$. Let $\alpha_{n\ell+j}^-$ be the Salem number that satisfies $T_{n\ell+j}^-(x) = M(x)x^{n\ell+j} - M^*(x)$.*

Assume $\ell = 2p$ is even. Assume $Q(x)$ is reciprocal, there exists a τ_j such that

$$\ell_{\alpha_{\ell+j}^-}(1) = a_1(\kappa\tau_j\kappa^*\delta\delta)^\omega$$

and the companion polynomial of $\ell_{\alpha_{\ell+j}^-}(1)$ is

$$R_{\ell+j}(x) = Q(x)(M(x)x^{\ell+j} - M^*(x))\left(\frac{x^{3p}+1}{x^{2p}-1}\right).$$

Then

$$\ell_{\alpha^-_{n\ell+j}}(1) = a_1(\kappa \underbrace{\gamma \dots \gamma}_{n-1} \tau_j \underbrace{\gamma^* \dots \gamma^*}_{n-1} \kappa^* \delta\delta)^\omega \tag{4.2.1}$$

and the companion polynomial of $\ell_{\alpha^-_{n\ell+j}}(1)$ *is*

$$R_{n\ell+j}(x) = Q(x)(M(x)x^{n\ell+j} - M^*(x)) \left(\frac{x^{2np+p}+1}{x^{2p}-1} \right).$$

Proof. This proof is similar to the proof of Theorem 2.2.2 with the observation that since $\ell_\theta(1)$ is periodic and $R(x) = P_{k+\ell}(x) - P_k(x)$.

Using the notation of Definition 2.1.7, the equivalent equation to Equation (2.1.6) is

$$\begin{aligned} R_{n\ell+j}(x) =& x^{2np+j+p} R(x) \left(\sum_{i=0}^{n-1} x^{2pi} \right) + \frac{R(x)x^j - R^*(x)}{x^p - 1} x^{2pn} + R^*(x) \left(\sum_{i=0}^{n-1} x^{2pi} \right) \\ =& P_k(x) x^{4np+p+j} - S(x) \left(\sum_{i=1}^{n-1} x^{j+2np+2pi} \right) \\ &+ Z_j^{(8)}(x) - S^*(x) \left(\sum_{i=0}^{n-2} x^{2pi+k+1} \right) + P_k^*(x) \end{aligned}$$

where

$$\begin{aligned} Z_j^{(8)}(x) =& -(P_k(x) + S(x))x^{2pn+j+p} + \left(\frac{R(x)x^j - R^*(x)}{x^p-1} - P_k^*(x) \right) x^{2pn} - S^*(x)x^{2p(n-1)+k+1} \\ =& -S(x)x^{2pn+j+p} + P_k(x)x^{2pn+j} + \frac{S^*(x)x^{k+1} - S(x)x^j}{x^p-1} \cdot x^{2pn} \\ &+ P_k^*(x)x^{2pn+p} - S^*(x)x^{2p(n-1)+k+1}. \end{aligned}$$

We note that the coefficients in front of $Z^{(8)}(x)$ included the calculation of τ_j beginning at $x^{2p(n-1)+k+1}$. □

Theorem 4.2.3. *Let* θ *be a PV number such that the lazy expansion of* 1 *is periodic and reversibly lazy. Let* $\ell_\theta(1) = a_1 \dots a_k (a_{k+1} \dots a_{k+\ell})^\omega = a_1 \kappa(\gamma)^\omega$. *Let* $M(x)$ *be the minimal polynomial of* θ *and* $\delta = a_1 - 1$. *Let* $R(x)$ *be the companion polynomial and* $Q(x)$ *be the co-factor polynomial for* $\ell_\theta(1)$. *Let* $j \in \{1, \dots, \ell\}$. *Let* $\alpha^-_{n\ell+j}$ *be the Salem numbers that satisfy* $T^-_{n\ell+j}(x) = M(x)x^{n\ell+j} - M^*(x)$.

Assume $Q(x)$ *is reciprocal, there exists a* τ_j *such that*

$$\ell_{\alpha^-_{\ell+j}}(1) = a_1(\kappa\gamma\tau_j\kappa^*\delta\delta)^\omega$$

and the companion polynomial of $\ell_{\alpha^-_{\ell+j}}(1)$ *is*

$$R_{\ell+j}(x) = Q(x)(M(x)x^{\ell+j} - M^*(x))\left(\frac{1}{x^\ell - 1}\right).$$

Then

$$\ell_{\alpha^-_{n\ell+j}}(1) = a_1(\kappa\underbrace{\gamma\ldots\gamma}_{n}\tau_j\kappa^*\delta\delta)^\omega \tag{4.2.2}$$

and the companion polynomial of $\ell_{\alpha^-_{n\ell+j}}(1)$ *is*

$$R_{n\ell+j}(x) = Q(x)(M(x)x^{n\ell+j} - M^*(x))\left(\frac{1}{x^\ell - 1}\right).$$

Proof. This is very similar to the proof of Theorem 2.1.5. with the observation that since $d_\theta(1)$ is periodic, we have $R(x) = P_{k+\ell}(x) - P_k(x)$.

Using the notation of Definition 2.1.7, the equivalent equation to Equation (2.1.6) is

$$x^{j+n\ell}P_k(x) - S(x)\sum_{i=0}^{n-1} x^{\ell i+j} + Z^{(5)}(x) + P_k^*(x)$$

where

$$Z_j^{(5)}(x) = \frac{R(x)x^j - R^*(x)}{x^\ell - 1} - P_k(x)x^j - P_k^*(x).$$

□

This result is exactly the same as Theorem 2.2.6 with the only change being the use of the lazy expansion instead of the greedy expansion. This is to say that greedy and lazy expansions for α_m^- can both appear to have this form. Note that this does not imply that α_m^- is univoque, as we do not know $d_\theta(1) = \ell_\theta(1)$. These expansions under the base θ are used to construct the expansions under α_m^-. Furthermore, there is no guaranty that the same value of j works in both greedy and lazy cases.

Remark 4.2.4. *We can apply Theorem 4.2.3 in the case where* θ *has a fully periodic lazy expansion for* 1*, say* $\ell_\theta(1) = (a_1\ldots,a_\ell)^\omega$*. Suppose that the companion polynomial of* $\ell_\theta(1)$ *is* $R(x)$*. Then the sequence* $a_1(a_2\ldots a_\ell a_1)^\omega = a_1(\gamma)^\omega$ *has companion polynomial* $R_1(x) = xR(x)$*. If we apply Theorem 4.2.3 with this form of the lazy expansion we find*

$$\ell_{\alpha_{n\ell+j}}(1) = a_1(\kappa\underbrace{\gamma\ldots\gamma}_{n}\tau_j\kappa^*\delta\delta)^\omega$$

and the companion polynomial of $\ell_{\alpha^-_{n\ell+j}}(1)$ *is*

$$\begin{aligned} R_{n\ell+j}(x) &= Q(x)(M(x)x^{n\ell+j} - M^*(x))\left(\frac{1}{x^\ell - 1}\right) \\ &= (R(x)x^{n\ell+j} - R^*(x))\left(\frac{1}{x^\ell - 1}\right) \\ &= (R_1(x)x^{n\ell+j} - R_1^*(x))\left(\frac{1}{x^{\ell+1} - x}\right). \end{aligned}$$

4.3 Degree 4 Salem Numbers

In this section, we show that all Salem numbers of degree 4 are lazy Parry numbers. Through computational experiments, we determined the seven forms the expansions take, with a finite number of exceptional cases. We will show that every degree 4 Salem number falls into exactly one of these cases. Furthermore, we prove that the lazy expansion for 1 in that base is the expected result. For each case, we show two things; first, that the expansion is a lazy expansion for 1 in some base β and second, that the expansion is an expansion in the desired base. Combined these facts will prove the expansion is the lazy expansion in the desired base. There are 13 Salem numbers of degree four which would otherwise fall into one of these cases but do not have the expected lazy expansions for 1. We prove these are nevertheless lazy Parry numbers and we show why they occur, we also show that no other exceptions exist in degree four.

Let $T(x) = x^4 - ax^3 + bx^2 - ax + 1$ be a minimal Salem polynomial. Then we know $-2a - 2 < b < 2a - 2$, $b \neq \pm a + 1$ and $b \neq 2$ due to Boyd [8]. Let $(a_i) = \ell_\beta(1)$ be the lazy expansion for 1 in base β, where β is the Salem root of $T(x)$

We use the lazy expansion criteria of Proposition 3.1.6 to prove that the proposed sequences are, in fact, lazy expansions. To do this, we require the use of $\lfloor\beta\rfloor$ when considering the sequence $(\lfloor\beta\rfloor - a_{k+i})$. Unlike the greedy case where we know $a_1 = \lfloor\beta\rfloor$, it is possible that $a_1 < \lfloor\beta\rfloor$. Then we cannot simply read the value of $\lfloor\beta\rfloor$ from the lazy expansion. The following lemma gives the exact value of $\lfloor\beta\rfloor$ depending on a and b.

Lemma 4.3.1 (Boyd, [8])**.** *Let* $T(x) = x^4 - ax^3 + bx^2 - ax + 1$ *be the minimal polynomial of a Salem number* β*, then* $\lfloor\beta\rfloor$ *takes the following values;*

$$\lfloor\beta\rfloor = \begin{cases} a+1 & \text{if } -2a-2 < b < -a, \\ a & \text{if } -a \leq b \leq 0, b \neq -a+1, \\ a-1 & \text{if } 0 < b \leq a, b \neq 2, \\ a-2 & \text{if } a+1 < b < 2a-2. \end{cases} \tag{4.3.1}$$

Recall that $T(x)$ is not the minimal polynomial of a Salem number if $b = 2$ or $b = \pm a + 1$. This is why these values of b do not appear in the above cases.

Theorem 4.3.2. *Every Salem number of degree 4 is a lazy Parry number, and their expansions are given by the following cases. Here we let k be the length of the preperiod and ℓ the length of the periodic part. In all cases, except for the finitely many exceptional cases, we have $k = 1$. We let $T(x) = x^4 - ax^3 + bx^2 - ax + 1$ be the minimal polynomial for a Salem number, then;*

1. *For $b = -2a-1$ we have $\ell = 9$ and the expansion is $a+1(a-1, a+1, a-1, 0, a-1, a+1, a-1, a, a)^\omega$. We have exceptional cases at $(a, b) = (1, -3)$ and $(a, b) = (2, -5)$.*

2. *For $-2a - 1 < b < -a - 1$ we have $\ell = 5$ and the expansion is $a + 1(-a - b - 1, 2a + b, -a - b - 1, a, a)^\omega$. Exceptional cases exist at the points $(a, b) = (2, -4)$ and $(a, b) = (3, -6)$.*

3. *For $-a-1 \leq b < 0$ we have $\ell = 3$ and the expansion is $a(-b, a-1, a-1)$. Exceptional cases exist at the points $(a, b) = (1, -1)$ and $(a, b) = (1, -2)$.*

4. *For $0 \leq b < a$ we have $\ell = 4$ and the expansion is $a - 1(a - b, a - b, a - 2, a - 2)^\omega$. There is a unique exceptional case at the point $(a, b) = (2, 0)$.*

5. *For $a \leq b < \lfloor 3a/2 \rfloor$ we have $\ell = 6$ and the expansion is $a - 2(2a - b - 2, 3a - 2b - 1, 3a - 2b - 1, 2a - b - 2, a - 3, a - 3)^\omega$. There is a unique exceptional case at the point $(a, b) = (3, 3)$.*

6. *For $\lfloor 3a/2 \rfloor \leq b < 2a - 3$ let $b_k = (2a - 2) - (a - 3)/k$ for $k = 1, 2, \ldots, a - 3$, so that $a + 1 = b_1 < b_2 < \cdots < b_{a-3} = 2a - 3$. Then if $b_{k-1} \leq b < b_k$ we have $\ell = 2k + 2$ and the expansion is given by $a_1(a_2, \ldots, a_{2k+3})^\omega$, where*

$$\begin{aligned}
a_1 &= a - 2, \\
a_2 &= 2a - b - 3, \\
a_i &= (i - 1)(2a - b - 2), \quad \text{if } 2 \leq i \leq k - 1, \\
a_k &= 2(k - 1)a - (k - 1)b - (2k - 3), \\
a_{k+1} &= (2k - 1)a - kb - (2k - 3), \\
a_i &= a_{(2k+3)-i}, \quad \text{if } k + 2 \leq i \leq 2k + 1, \\
a_{2k+2} &= a_{2k+3} = a - 3.
\end{aligned}$$

7. *For $b = 2a - 3$ we have $\ell = 13 + 2(p - 4)$ where $a = 2p$ or $2p + 1$ and the expansion is $a - 3(a - 2, 2, 5, 7, \ldots, 2p - 5, 2p - 2, \tau, 2p - 2, 2p - 5, \ldots 7, 5, 2, a - 2, a - 4, a - 4)$ where $\tau = 2, 4, 4, 2$ if a is even and $\tau = 1, 3, 3, 1$ if a is odd. This gives exceptional points $(a, b) = (5, 7), (6, 9), (7, 11), (8, 13)$ and $(9, 15)$.*

Proof. We first use Proposition 3.1.6 to show that the sequences in the statement of the theorem are lazy expansions for 1 under some base. To this end, let (a_i) be the sequence from the statement of the theorem. We use Lemma 4.3.1 to determine the sequence $(d_i) = (\lfloor\beta\rfloor - a_{m+i})$. We use (d_i) in the application of Proposition 3.1.6. In the case where $(d_i) \geq (a_i)$ Proposition 3.1.6 fails, this indicates that (a_i) is not a lazy expansion and is one of the exceptional cases.

1. For $b = -2a - 1$ we know $\lfloor\beta\rfloor = a + 1$. The sequence $(d_i) \geq_{\text{lex}} (a_i)$ exactly when $k = 4$ and $a + 1 - (a - 1) = 2 > a - 1$ corresponding to the exceptional points $(a, b) = (1, -3)$ and $(a, b) = (2, -5)$.

2. For $-2a-1 < b < -a-1$ we know $\lfloor\beta\rfloor = a+1$. The sequence (d_i) is lexicographically larger than (a_i) only when $2a+b = 0$ and $a+1-(-a-b-1) = 2a+b+2 > -a-b-1$ corresponding to the exceptional points $(a, b) = (2, -4)$ and $(a, b) = (3, -6)$.

3. For $-a - 1 \leq b < 0$ we know $\lfloor\beta\rfloor = a + 1$ if $b = -a - 1$ and $\lfloor\beta\rfloor = a$ otherwise by Lemma 4.3.1. This fails the lazy criteria when $\lfloor\beta\rfloor - (a - 1) \geq a$ corresponding to the exceptional points $(a, b) = (1, -2)$ and $(a, b) = (1, -1)$.

4. For $0 \leq b < a$ we know $\lfloor\beta\rfloor = a$ if $b = 0$ and $\lfloor\beta\rfloor = a - 1$ otherwise. The sequence is lazy provided $\lfloor\beta\rfloor - (a - 2) < a - 1$, the only integer point for which this does not occur is $(a, b) = (2, 0)$.

5. For $a \leq b < \lfloor 3a/2\rfloor$ we know $\lfloor\beta\rfloor = a - 1$ if $b = a$ and $\lfloor\beta\rfloor = a - 2$ otherwise. The provided sequence is lazy in all cases except $(a, b) = (3, 3)$, as we have $d_1 \leq a_1$ in all other valid cases.

6. For $\lfloor 3a/2\rfloor \leq b < 2a - 3$ we always have $\lfloor\beta\rfloor = a - 2$, so it suffices to check $a_i \neq 0$ for $i \geq 2$. This is the case provided $b \neq b_k$ which is as defined.

7. For $b = 2a - 3$ we have $\lfloor\beta\rfloor = a - 2$. Since $a_i \neq 0$ for large enough a the given sequence is lazy whenever $2p - 5 > 0$ which corresponds to $a \geq 10$. This restriction causes the exception points $(a, b) = (5, 7), (6, 9), (7, 11), (8, 13)$ and $(9, 15)$.

Then, except in the 13 exceptional cases we have enumerated above, we have shown that (a_i) is the lazy expansion for 1 in base. Next, we show that these expansions are representations in the desired Salem base by showing $T(x)|R(x)$, where $R(x)$ is the companion polynomial for the sequence in question. To show this, we enumerate the following cofactors $Q(x)$ such that $T(x)Q(x) = R(x)$. Recall $T(x) = x^4 - ax^3 + bx^2 - ax + 1$.

1. For $b = -2a - 1$ we claim $Q(x) = x^6 - x^5 + 2x^4 - 2x^3 + 2x^2 - x + 1$. As $b = -2a - 1$ we consider $T(x) = x^4 - ax^3 + (-2a - 1)x^2 - ax + 1$ and verify $T(x)Q(x) = R(x)$ as required by expanding $R(x) = P_{10}(x) - P_1(x)$. Here $P_n(x)$ is as defined in Definition 1.1.13.

2. For $-2a - 1 < b < -a - 1$ we claim $Q(x) = x^2 - x + 1$. Expanding the companion polynomial, we have $R(x) = (x^6 - x^5 + x^4 + x^2 - x + 1) + a(-x^5 + x^4 - 2x^3 + x^2 - x) + b(x^4 - x^3 + x^2) = (x^2 - x + 1)(x^4 + 1) + a(x^2 - x + 1)(-x^3 - x) + b(x^2 - x + 1)(x^2)$, which is exactly the required form.

3. For $-a-1 \leq b < 0$ we claim $Q(x) = 1$. Expanding the companion polynomial $R(x) = P_4(x) - T(x) = (x^4 - ax^3 + bx^2 - (a-1)x - (a-1)) - (x - a) = x^4 - ax^3 + bx^2 - ax + 1$ as expected.

4. For $0 \leq b < a$ we claim $Q(x) = x + 1$. $R(x) = P_5(x) - P_1(x) = (x^5 - (a-1)x^4 - (a-b)x^3 - (a-b)x^2 - (a-2)x - (a-2)) - (x - (a-1)) = (x+1)(x^4+1) + a(x+1)(-x^3 - x) + b(x+1)(x^2)$ as required.

5. For $a \leq b < \lfloor 3a/2 \rfloor$ we claim $Q(x) = x^3 + 2x^2 + 2x + 1$. Upon expanding $R(x)$, we find $T(x)Q(x) = R(x)$ as required.

6. For $\lfloor 3a/2 \rfloor \leq b < 2a - 3$ we claim $Q(x) = (x^k - 1)(x^{k+1} - 1)/(x - 1)^2 = x^{2k-1} + 2x^{2k-2} + \cdots + (k-1)x^{k+1} + kx^k + kx^{k-1} + (k-1)x^{k-2} + \cdots + 2x + 1$. For this case, we individually assess the 3 polynomial components. $Q(x) \cdot (x^4 + 1) = 1 + 2x + 3x^2 + 4x^3 + 6x^4 + 8x^5 + \cdots + (k - 1 + k - 3)x^{k-2} + (2k - 3)x^{k-1} + (2k - 3)x^k + (2k - 4)x^{k+} + \cdots + 6x^{2k} + 3x^{2k+1} + 2x^{2k+2} + x^{2k+3}$. This is exactly the component of $R(x)$ whose coefficients do not include a or b. Similarly, $a \cdot Q(x) \cdot (-x^3 - x)$ is the component of $R(x)$ whose coefficients include only a and $b \cdot Q(x)(x^2)$ is the component with coefficients that include exactly b. Combining these facts, we have $R(x) = Q(x)(x^4 - ax^3 + bx^2 - ax + 1)$ as required.

7. For $b = 2a - 3$ we claim $Q(x) = x^{2p+1} + 3x^{2p} + 5x^{2p-1} + \cdots + (2p-1)x^{p+2} + 2px^{p+1} + 2px^p + (2p-1)x^{p-1} + \cdots + 5x^2 + 3x + 1$, where $a = 2p$ or $2p + 1$. As $b = 2a - 3$ we consider $T(x) = x^4 - ax^3 + (2a - 3)x^2 - ax + 1$ and verify $T(x)Q(x) = R(x)$ as required.

Finally, we use a computer program to calculate the lazy expansions for 1 in the finitely many exceptional cases encountered in the previous analysis. We verify these Salem numbers are lazy Parry Numbers. The entire list of exceptional points is provided in Table 4.3.1, along with their respective lazy expansions for 1.

□

a	b	Expansion
1	-3	$12(21)^\omega$
1	-2	$1112(12221)^\omega$
1	-1	$101(10)^\omega$
2	-5	$3(13033303122)^\omega$
2	-4	$303(22332)^\omega$
2	0	$12(121)^\omega$
3	-6	$4(1444133)^\omega$
3	3	$112(1221)^\omega$
5	7	$303(112332)^\omega$
6	9	$3(4324423422)^\omega$
7	11	$4(5313313533)^\omega$
8	13	$5(626244262644)^\omega$
9	15	$6(726133162755)^\omega$

Table 4.3.1: Exceptional Degree 4 Salem Lazy Expansions

Figures 4.3.1 and 4.3.2 show when each of the degree four Salem numbers of lazy or greedy expansions in each of the seven forms described in Theorem 4.3.2. We notice significant overlap in the types of expansions, as these sequences rely only on the values of a and b and we can conclude that any overlap indicates a univoque Salem number. With the important distinction that Salem numbers with expansions of Type 6 may not agree.

Note that all greedy expansions covered by Theorem 2.4.1 are found in lazy expansions for 1 under Salem numbers of degree 4. Only in the boundary areas where the expansions shift from one type to another do we find a difference in expansion. We reference Theorem 4.3.2 and the result for greedy expansions from Boyd, Theorem 2.4.1 and conclude the following.

Theorem 4.3.3. *Let α be a degree four Salem number. Let $T(x) = x^4 - ax^3 + bx^2 - ax + 1$ be the minimal polynomial for α. Then α is univoque if any of the following holds;*

1. $-2a - 1 \leq b < -a - 1$ *and* $(a, b) \neq (1, -3), (2, -5), (2, -4), (3, -6)$,
2. $-a \leq b < 0$ *and* $(a, b) \neq (1, -2), (1, -1)$,
3. $1 \leq b < a$ *and* $(a, b) \neq (2, 0)$,
4. $a + 1 \leq b < \lfloor 3a/2 \rfloor$,
5. $\lfloor 3a/2 \rfloor < b < 2a - 3$ *and* $b \neq b_i$ *where* $b_k = (2a - 2) - (a - 3)/k$.

Proof. Let $T(x) = x^4 - ax^3 + bx^2 - ax + 1$ be a minimal Salem polynomial with Salem root α. Then lazy expansion $\ell_\alpha(1)$ from Theorem 4.3.2 and greedy expansion $d_\alpha(1)$ from Theorem 2.4.1 agree whenever the pair (a, b) satisfies any of the above conditions. □

Figures 4.3.1 and 4.3.2 indicate which of the seven types from Theorem 4.3 the degree four Salem number with minimal polynomial $T(x) = x^4 - ax^3 + bx^2 - ax + 1$ falls under. The horizontal axis indicates the value of $-b$ and the vertical axis indicates the value of a. The colours are as follows;

Red: Type 1, Purple: Type 2, Blue: Type 3, Green: Type 4,

Yellow: Type 5, Orange: Type 6, Pink: Type 7, Cyan: Exceptional Cases

Note that only the lazy case, Figure 4.3.1 contains exceptional cases. That is, every degree four Salem number has a greedy expansion for 1 given by one of the seven types. In fact, Type 7 does not occur in the greedy case.

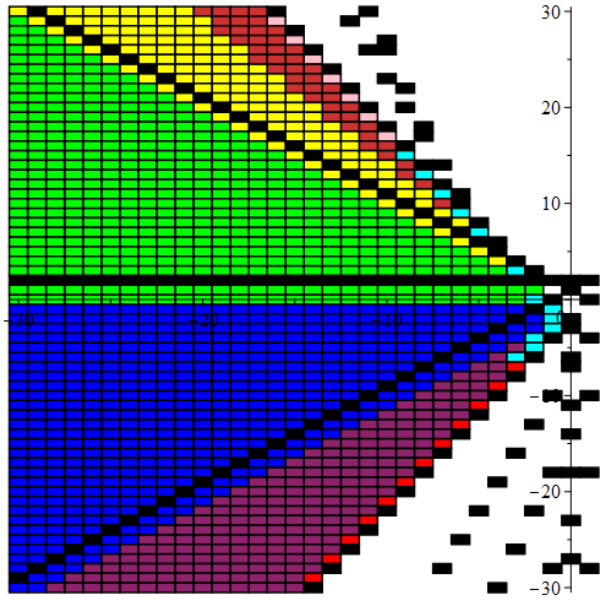

Figure 4.3.1: Types of Lazy expansion for 1 under degree 4 Salem bases

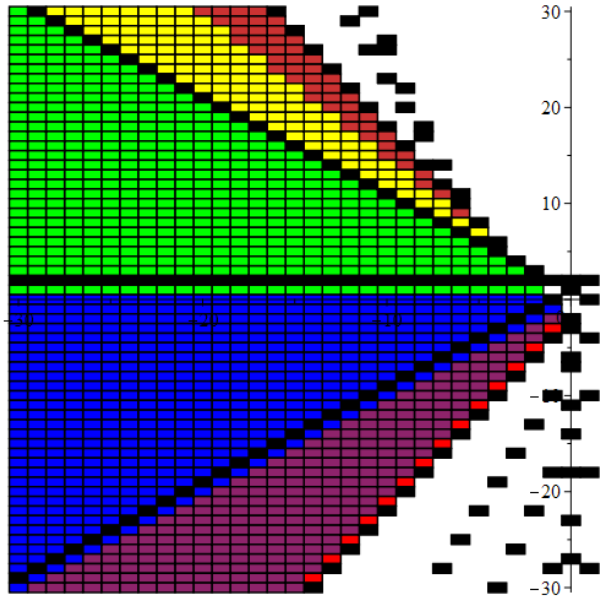

Figure 4.3.2: Types of Greedy expansion for 1 under degree 4 Salem bases

Chapter 5

Conclusion and Further Questions

This thesis provided generalizations of the work of Hare and Tweedle [15]. We provided criteria for Salem numbers to be Parry numbers and related the induced greedy expansion for 1 to the greedy expansions under PV numbers. While this does not prove all Salem numbers are Parry numbers we have found many infinite families for which this is the case.

Introducing lazy Parry numbers we expanded the knowledge about lazy expansions. First proving that all PV numbers are lazy Parry numbers and finding an easily applicable criteria for verifying a representation is the lazy expansion.

We continued by developing criteria for Salem numbers being lazy Parry numbers. Expanding our results of Parry numbers to lazy Parry numbers. We completed our analysis by focusing on degree four Salem numbers. We prove that they are all lazy Parry numbers and give the exact expansions that they induce.

The remainder of this chapter provides questions and conjectures about future results related to the study of this thesis. Section 5.1 explores the similarities of the results for the finite and eventually periodic cases in Section 2.2

Section 5.2 describes the fact that the restriction to PV and Salem numbers in our results may be premature. We ask questions about applying the results of Chapters 2 and 4 in the cases where $M(x)$ is not a PV polynomial.

Section 5.3 provides a few more results that give the lazy expansions for Salem numbers that appear as the roots of $T_m^{\pm}(x)$. These conjectures are yet unproven in general, but we verify them for certain families of PV numbers.

5.1 Quasi-Greedy Expansions

In this section, we would like to highlight the similarities between the theorems in Section 2.2. We observe that the companion polynomials in Theorem 2.2.2 and Theorem 2.2.5 have similar forms. The same can be said for Theorems 2.2.3 and 2.2.6. We can define the extra factor $F_n(x)$ where $R_{n\ell+j}(x) = Q(x)(M(x)x^{n\ell+j} - M^*(x))F_n(x)$ or $R_{nk+j}(x) = Q(x)(M(x)x^{nk+j} - M^*(x))F_n(x)$. In both cases, these extra factors are exactly the same.

The question is whether we can view the results for the finite case in terms of the periodic case. Recall that if $\beta > 1$ is an algebraic integer with $d_\beta(1) = a_1 \dots a_k$ then we can write the quasi-greedy expansion as $d^*_\beta(1) = (a_1 \dots a_{k-1}(a_k - 1))^\omega$.

We proved that if the greedy expansion has the reversibly greedy property then so does the quasi-greedy expansion. Then it would be logical to apply our results using this expansion. We keep in mind that the companion polynomial of $d^*_\beta(1)$ is not the same as its finite counterpart.

Consider Theorem 2.2.2. We can use a preperiod of any positive length in our results and apply Theorem 2.2.5. To that end, consider the expansion with preperiod of length one. $a_1(a_2 \dots a_{k-1}(a_k - 1)a_1)^\omega$ In this case we have $\gamma = a_2 \dots a_{k-1}(a_k - 1)a_1$ where as we expect $\gamma = a_1 \dots a_{k-1}(a_k - 1)$, the sequence is shifted. Once we account for κ in the sequence given by Theorem 2.2.2 we find that the sequence in this case has an extra digit.

We may be able to account for this digit by including it in the string τ_j, recalling that this digit a_1 is constant for all j. Then we expect to see that all valid strings τ_j should have the same starting digit. Example 2.2.1 contradicts this expectation. Here we have

$$\tau_1 = 010, \quad \tau_2 = 1001, \quad \tau_5 = 1101011, \quad \tau_6 = 11011011.$$

For $j = 2, 5, 6$ we see the expected value of $a_1 = 1$ as the first digit of τ_j. For the case $j = 1$ we appear to have a problem. We can do a similar analysis for Theorems 2.2.3 and 2.2.6.

With this evidence in mind, we pose the following questions.

Question 5.1.1. *Is Theorem 2.2.2 a restriction/extension of Theorem 2.2.5 when quasi-greedy expansions?*

Question 5.1.2. *Is Theorem 2.2.3 a restriction/extension of Theorem 2.2.6 when quasi-greedy expansions?*

At face value, we should expect a negative result. However, it is possible that the manipulation of companion polynomials proves that this is nonetheless the case. We can imagine a scenario where the polynomials $Z_j(x)$ defined in the respective theorems account for the difference that we noted above.

5.2 Results outside PV Numbers

Recall that PV and Salem numbers are not the only algebraic integers to be Parry and lazy Parry numbers. Each of the theorems in Chapters 2 and 4 use a PV polynomial $M(x)$ to construct Salem polynomials $T_m^{\pm}(x) = M(x)x^m \pm M^*(x)$. This construction is not restricted to PV numbers.

Let β be an algebraic integer with minimal polynomial $M(x)$ for which the greedy or lazy expansions, $d_\beta(1)$ or $\ell_\beta(1)$, are either finite or eventually periodic.

Question 5.2.1. *Is there such a β that is not a PV number which satisfies the conditions of our theorems? In such a case if $T_m^{\pm}(x)$ has a real algebraic integer root, β_m, do the expansions $d_{\beta_m}(1)$ and $\ell_{\beta_m}(1)$ ever have the form described by that theorem?*

Nothing in the proofs of our results require the use of PV and Salem except for the structure it imposes on the roots of the polynomials $T_m^{\pm}(x)$. By using a PV polynomial $M(x)$ we know that $T_m^{\pm}(x)$ will be the product of a Salem polynomial and a cyclotomic polynomial. This gives us a better chance of finding periodic expansions. However, we see no reason to restrict the use of these theorems to PV polynomials. We believe the existence of isolated algebraic integers that give a positive answer to this question to be likely.

Restricting ourselves to the greedy case for illustrative purposes. We cannot exclude the possibility of counter-examples to generalized versions of our theorems. Let β be an algebraic integer with minimal polynomial $M(x)$ and greedy expansion given by $d_\beta(1) = a_1 \dots a_k(a_{k+1} \dots a_{k+\ell})^\omega$. We can imagine there exists such that a real positive root $T_m^{\pm}(x)$ has a greedy or lazy expansion given by the statement of our theorems but $T_{m+\ell}^{\pm}(x)$ having non trivial factors. In such a case, the real root of this polynomial need not have greedy expansion as in our result. We found no examples after searching thousands of of degree four polynomials. With the lack of a single example, we find that a positive answer to the following question is unlikely.

Question 5.2.2. *Is there a family of algebraic integers, outside of the PV numbers, where the greedy or lazy expansions for* 1 *have the form describe in any of the theorems in this thesis?*

5.3 Lazy Salem: Case $\ell = 1$

Unlike the greedy case, when looking at lazy expansion for 1 it is possible to find bases β such that $\ell_\beta(1)$ has a period of length one. That is $\ell_\beta(1) = a_1 \dots a_k(a_{k+1})^\omega$.

Example 5.3.1. *Let β be the PV root of $\Phi_r(x) = x^{r+1} - 2x^r + x - 1$. Then $\ell_\beta(1) = 1^{r-1}0(1)^\omega$.*

While the theory behind results from Chapter 4 does not break down in these cases we find no examples of PV numbers that actually satisfy the premise of the of the theorems.

Example 5.3.2. *Consider the PV number θ with the minimal polynomial $M(x) = \Phi_4(x) = x^5 - 2x^4 + x - 1$. We know $\ell_\theta(1) = a_1 \dots a_k(a_{k+1})^\omega = 1110(1)^\omega$. Here $\ell = 1$, let $\kappa_0 = 0000, \kappa_1 = 1111$ and $\gamma = a_{k+1} = 1$. For sufficiently large values of m we can define the polynomial $T_m^+(x) = M(x)x^m + M^*(x)$ which yields a Salem number α_m^+. We look at the lazy expansions of 1 with base α_m^+ for various m See Table 5.3.1.*

m	Lazy Expansion of 1
5	$1(111100)^\omega = 1(111100111100)^\omega$
6	$1(1110101011100)^\omega$
7	$1(11100110011100)^\omega$
9	$1(1110001100011100)^\omega$
10	$1(11100010100011100)^\omega$
11	$1(11100010100011100)^\omega$
12	$1(111000100010011100)^\omega$
13	$\kappa_1\kappa_0 1(11110011110\kappa_1\kappa_1 0)^\omega$
14	$\kappa_1\kappa_0 1(111010101110\kappa_1\kappa_1 0)^\omega$
15	$\kappa_1\kappa_0 1(1110011001110\kappa_1\kappa_1 0)^\omega$
17	$\kappa_1\kappa_0 1(11100011000111 0\kappa_1\kappa_1 0)^\omega$
18	$\kappa_1\kappa_0 1(1110001010001110\kappa_1\kappa_1 0)^\omega$
19	$\kappa_1\kappa_0 1(11100010010001110\kappa_1\kappa_1 0)^\omega$
20	$\kappa_1\kappa_0 1(111000100010001110\kappa_1\kappa_1 0)^\omega$
21	$\kappa_1\kappa\kappa_1\kappa_0 1(11110011110\kappa_1\kappa_1\kappa_1\kappa_1 0)^\omega$
22	$\kappa_1\kappa\kappa_1\kappa_0 1(111010101110\kappa_1\kappa_1\kappa_1\kappa_1 0)^\omega$
23	$\kappa_1\kappa\kappa_1\kappa_0 1(1110011001110\kappa_1\kappa_1\kappa_1\kappa_1 0)^\omega$
25	$\kappa_1\kappa\kappa_1\kappa_0 1(11100011000111 0\kappa_1\kappa_1\kappa_1\kappa_1 0)^\omega$
26	$\kappa_1\kappa\kappa_1\kappa_0 1(1110001010001110\kappa_1\kappa_1\kappa_1\kappa_1 0)^\omega$
27	$\kappa_1\kappa\kappa_1\kappa_0 1(11100010010001110\kappa_1\kappa_1\kappa_1\kappa_1 0)^\omega$
28	$\kappa_1\kappa\kappa_1\kappa_0 1(111000100010001110\kappa_1\kappa_1\kappa_1\kappa_1 0)^\omega$

Table 5.3.1: Lazy expansions of 1 for base α_m^+ in Example 5.3.2

It appears that if $\ell_{\alpha^+_{k+j}}(1) = \ell_{\alpha^+_{4+j}}(1) = 1(\tau_j 0)^\omega$ for $j \in \{1 \dots 2k = 8\} \setminus \{k = 4\}$ then $\ell_{\alpha^+_{k+j+2nk}}(1) = \ell_{\alpha^+_{4+j+8n}}(1) = (\kappa_1\kappa_0)^n 1(\tau_j \kappa_1^{2n} 0)^\omega$. Where τ_j are the following sequences;

$\tau_1 = 11110011110, \quad \tau_2 = 111010101110, \quad \tau_3 = 1110011001110, \quad \tau_5 = 111000110001110,$

$$\tau_6 = 1110001010001110, \quad \tau_7 = 11100010010001110, \quad \tau_8 = 11110001000010001110.$$

We have the following conjecture, which we have verified for small values of n using the PV roots of $\Phi_r(x)$ for all $r \leq 20$.

Conjecture 5.3.3. *Let θ be a PV number such that the lazy expansion of 1 is periodic and reversibly lazy. Let $\ell_\theta(1) = a_1 \dots a_k (a_{k+1})^\omega$, $\kappa_0 = 0^k$ and $\kappa_1 = 1^k$. Let $M(x)$ be the minimal polynomial of θ and $\delta = a_1 - 1$. Let $R(x)$ be the companion polynomial and $Q(x)$ be the co-factor polynomial for $\ell_\theta(1)$. Let $j \in \{1, \dots, 2k\} \setminus \{k\}$. Let α^+_{2kn+j} be the Salem number that satisfies $T^+_{2kn+j}(x) = M(x)x^{2kn+j} + M^*(x)$.*

Assume $Q(x)$ is reciprocal, there exists a τ_j such that

$$\ell_{\alpha^+_{k+j}}(1) = 1(\tau_j 0)^\omega$$

and the companion polynomial of $\ell_{\alpha^+_{k+j}}(1)$ is

$$R_{k+j}(x) = Q(x)(M(x)x^{k+j} - M^*(x)) \left(\frac{x^k - 1}{x - 1} \right).$$

Then

$$\ell_{\alpha^+_{2kn+k+j}}(1) = (\kappa_1 \kappa_0)^n 1(\tau_j \kappa_1^{2n} 0)^\omega. \tag{5.3.1}$$

Furthermore the companion polynomial of $\ell_{\alpha^+_{2nk+k+j}}(1)$ is

$$R_{2nk+k+j}(x) = Q(x)(M(x)x^{2nk+k+j} - M^*(x)) \left(\frac{x^k - 1}{x - 1} \right) (1 + x^{2k} + \dots + x^{2nk}).$$

Unlike the theorems in Chapter 4 the preperiodic part changes with the value of n. We expect results to differ for PV numbers $\theta > 2$ where $\mathcal{A}_\theta \neq \{0, 1\}$.

Example 5.3.4. *Consider the PV number θ with the minimal polynomial $M(x) = \Phi_4(x) = x^5 - 2x^4 + x - 1$. We know $\ell_\theta(1) = a_1 \dots a_k (a_{k+1})^\omega = 1110(1)^\omega$. Here $\ell = 1$, let $\kappa = a_2 \dots a_k = 1110$ and $\gamma = a_{k+1} = 1$. For sufficiently large values of m we can define the polynomial $T^-_m(x) = M(x)x^m + M^*(x)$ which yields a Salem number α^-_m. We look at the lazy expansions of 1 with base α^-_m for various m See Table 5.3.2.*

It appears that $\ell_{\alpha^-_m}(1) = 1(\kappa \gamma^{m-6} \kappa^* \delta \delta)^\omega$.

We have the following conjecture, which we have verified all $n \leq 20$ using the PV roots of $\Phi_r(x)$ for all $r \leq 20$.

m	greedy expansion of 1
6	$1(\kappa 0)^\omega = 1(\kappa\kappa^*00)^\omega$
7	$1(\kappa\gamma\kappa^*00)^\omega$
8	$1(\kappa\gamma\gamma\kappa^*00)^\omega$
9	$1(\kappa\gamma\gamma\gamma\kappa^*00)^\omega$
10	$1(\kappa\gamma\gamma\gamma\gamma\kappa^*00)^\omega$
11	$1(\kappa\gamma\gamma\gamma\gamma\gamma\kappa^*00)^\omega$
12	$1(\kappa\gamma\gamma\gamma\gamma\gamma\gamma\kappa^*00)^\omega$
13	$1(\kappa\gamma\gamma\gamma\gamma\gamma\gamma\gamma\kappa^*00)^\omega$
14	$1(\kappa\gamma\gamma\gamma\gamma\gamma\gamma\gamma\gamma\kappa^*00)^\omega$

Table 5.3.2: Lazy expansions of 1 for base α_m^- in Example 5.3.4

Conjecture 5.3.5. *Let θ be a PV number such that the lazy expansion of 1 is periodic and reversibly lazy. Let $\ell_\theta(1) = a_1 \dots a_k(a_{k+1})^\omega$, $\kappa_0 = 0^k$ and $\kappa_1 = 1^k$. Let $M(x)$ be the minimal polynomial of θ and $\delta = a_1 - 1$. Let $R(x)$ be the companion polynomial and $Q(x)$ be the co-factor polynomial for $\ell_\theta(1)$. Let α_m^+ be the Salem number that satisfies $T_m^-(x) = M(x)x^m - M^*(x)$.*

Assume $Q(x)$ is reciprocal. Then

$$\ell_{\alpha_{k+n}^+}(1) = 1(\kappa\gamma^{n-1}\kappa^*00)^\omega$$

for all $n \geq 1$ and the companion polynomial of $\ell_{\alpha_{k+n}^+}(1)$ is

$$R_{k+n}(x) = Q(x)(M(x)x^{k+n} - M^*(x))\left(\frac{1}{x-1}\right).$$

We see a different structures from the theorems discussed in Chapters 2 and 4. There is no central string τ_j or mirrored copies of γ. At each iteration we are adding a single copy of γ instead of a copy of γ and $\gamma^* = \gamma$. The result of this is that the companion polynomial does not have an extra term. It is unclear how this result could change for $\theta > 2$.

If we look to double the period of the representation in Conjecture 5.3.5 the result is similar to that in Chapter 4.

$$\ell_{\alpha_{k+n}^-}(1) = 1(\kappa\gamma^{n-1}\tau\gamma^{n-1}\kappa^*00)^\omega, \quad \text{where } \tau = \kappa^*00\kappa.$$

We observe that this does not agree exactly with previous theorems since the expansions $\ell_{\alpha_m^-}(1)$ for $m \leq k+1$ are not properly covered.

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APPENDICES

Appendix A

Tables

The following tables contain the greedy and lazy expansions for 1 under regular PV bases in the interval $(1, 2)$. Each table is separated by family. Each family is divided into sub cases as needed. We give restrictions on r and q for which these expansions are valid. We also note the pseudo co factors for the companion polynomials of these expansions.

Recall that if $R(x)$ is the companion polynomial for the expansion and $\overline{M}(x)$ is the polynomial in Table 1.2.1 with the desired PV root, then $\overline{Q}(x)$ is the pseudo co factor provided $R(x) = \overline{M}(x)\overline{Q}(x)$.

Table A.0.1 provides the greedy expansions and the pseudo co factors of the same. These are the results compiled by Panju [18]. Table A.0.2 provides the lazy expansions for these regular PV numbers and Table A.0.3 contains the pseudo co factors. These two tables combine form the lazy analog of Table A.0.1 and are some results of this thesis.

Polynomial	Expansion	Restriction	Property	Pseudo Co-factor
$\Phi_r(x)$	$1^r0^{r-1}1$	$r \geq 1$	FRG	$(x^r-1)/(x-1)$
$\Psi_r(x)$	1^{r+1}	$r \geq 1$	FRG	1
$\chi(x)$	$11(10)^\omega$	NA	RG	1
$\Phi^{A-}_{(r,q)}(x)$	$1^r0^{r-1}10^{q-2r}10^r1^{r-1}$	$q \geq 2r$	FRG	$(x^r-1)/(x-1)$
$\Phi^{B-}_{(r,q)}(x)$	$1^r0^{r-1}10^{q-2r}1^{r-1}0^r1$	$q \geq 2r$	FRG	$(x^r-1)/(x-1)$
$\Phi^{C-}_{(r,q)}(x)$	$1^r0^{r-1}1[0^{q-2r}1^r0^r]^\omega$	$q \geq 2r+1$	RG	$(x^r-1)/(x-1)$
$\Phi^{A+}_{(r,q)}(x)$	$(1^r0^r)^s1^{r-1}$	$q = 2rs+r-1$	FRG	$1/(x^r+1)(x-1)$
$\Phi^{B+}_{(r,q)}(x)$	$(1^r0^r)^s1$	$q = 2rs+1$	FRG	$1/(x^r+1)(x-1)$
$\Phi^{C+}_{(r,q)}(x)$	$(1^r0^r)^{s-1}0^{q-1}1$	$q = 2rs$	FRG	$(x^q-1)/(x^r+1)(x-1)$
$\Psi^{A-}_{(r,q)}(x)$	$1^{r+1}[0^{q-r-1}1^r0]^\omega$	$q \geq r$	RG	1
$\Psi^{B-}_{(r,q)}(x)$	$1^{r+1}0^{q-r}1^r$	$q \geq r-1$	FRG	1
$\Psi^{A+}_{(r,q)}(x)$	$(1^r0)^s0^{q-1}1$	$q = (r+1)s$	FRG	$(1+x^{r+1}+\cdots+x^{(r+1)(s-1)})$
$\Psi^{B+}_{(r,q)}(x)$	$(1^r0)^{s-1}1^r$	$q = (r+1)s+r$	FRG	$1/(x^{r+1}-1)$
$\chi^-_A(x)$	$11(10)^{s-2}11011\ldots$ $\ldots[(10)^{s-2}0111(01)^{s-2}1000]^\omega$	$q = 2s$ for $s \geq 2$	RG	$(x^q-1)(x^{q+1}+1)/(x^2-1)$
$\chi^-_A(x)$	$11(10)^{s-2}11[00011(10)^{s-2}00]^\omega$	$q = 2s+1$ for $s \geq 2$	RG	$(x^q-1)/(x^2-1)$
$\chi^+_A(x)$	$11(10)^{s-2}011100(10)^{s-3}\ldots$ $\ldots000010(00)^{s-2}11$	$q = 2s$ for $s \geq 3$	FRG	$(x^{q-2}+1)(x^{q+2}-1)/(x^2-1)$
$\chi^+_A(x)$	$11(10)^{s-1}01000(10)^{s-1}0(00)^s11$	$q = 2s+1$ for $s \geq 1$	FRG	$(x^q+1)(x^{q+1}-1)/(x^2-1)$
$\chi^-_B(x)$	$11(10)^{s-3}1100000(10)^{s-3}001$	$q = 2s$ for $s \geq 3$	FRG	$(x^{q-2}-1)/(x^2-1)$
$\chi^-_B(x)$	$11(10)^{s-2}1101000(10)^{s-3}011\ldots$ $\ldots[1(00)^{s-1}10]^\omega$	$q = 2s+1$ for $s \geq 3$	RG	$(x^{2q-2}-x^{q-1}-x^{q-3}+1)/(x^2-1)$
$\chi^+_B(x)$	$11(10)^{s-2}0101\ldots$ $\ldots[1(10)^{s-3}(011)^2(10)^{s-3}010^4100]^\omega$	$q = 2s$ for $s \geq 3$	RG	$(x^4(x^{q-1}+1)(x^2-x+1)+x^{2q+4}-1)/(x^2-1)$
$\chi^+_B(x)$	$11(10)^{s-1}001$	$q = 2s+1$ for $s \geq 1$	FRG	$1/(x^2-1)$

Table A.0.1: Greedy Expansions for 1 under Regular PV bases, compiled from the work of Panju [18]

Polynomial	Expansion	Restriction
$\Phi_r(x)$	$1^{r-1}0[1]^\omega$	No Restriction
$\Psi_r(x)$	$[1^r0]^\omega$	$r \geq 2$
$\chi(x)$	$11(10)^\omega$	Not Applicable
$\Psi^{A+}_{(r,q)}(x)$	$(1^r0)^{n-1}1^{r-1}0[(1^r0)^n0(1^r0)^{n-1}1^r]^\omega$	$r \geq 2, q = n(r+1)$
	$(1^r0)^n1^{j-1}0[1^{r-j}01^j]^\omega$	$r \geq 2, q = n(r+1)+j$, for $j = 1, \ldots, r$
$\Psi^{B+}_{(r,q)}(x)$	Section 2.1	Section 2.1
$\Psi^{A-}_{(r,q)}(x)$	$(1^r0)^{n-1}1^{r+1}[1^r0]^\omega$	$q = n(r+1), n \geq 1$
	$(1^r0)^{n-1}1^{r+1}[0^j(1^r0)^{n-1}]^\omega$	$q = n(r+1)+j$, for $j = 1, \ldots, r-1, n \geq 1$
	$(1^r0)^{n-1}1^{r+1}0^r(1^r0)^{n-1}1^{r-1}01^{r+1}[(1^r0)^{n-1}1^{2r}0]^\omega$	$q = n(r+1)+r, n \geq 1$
$\Psi^{B-}_{(r,q)}(x)$	$1[(1^{r-1}01)^{n-1}1^r0^{j+1}(1^r0)^{n-1}1^{r-1}01]^\omega$	$r \geq 2, q = n(r+1)+j$, for $j = -1, \ldots, r-1$
$\Phi^{A+}_{(r,q)}(x)$	$0[1^{q-3}01]^\omega$	$r = 1, q \geq 5$
	$10[1^{q-3}011]^\omega$	$r = 2, q \geq 4$
	$1[1^{r-2}01^{q-r}01^{r-2}01]^\omega$	$r \geq 3, q \geq r+1$
$\Phi^{B+}_{(r,q)}(x)$	$1^{r-1}0[1^{q-r-1}01^r]^\omega$	$r \geq 1, q \geq \min(4, r+2)$
$\Phi^{C+}_{(r,q)}(x)$	$01^{2n-3}01^{2n+1}(01)^{n-1}1101^{2n-2}0[1]^\omega$	$r = 1, q = 2n, n \geq 2$
	$01^{2n-2}0[1^{2n+2}(01)^{n-1}10101^{2n-2}]^\omega$	$r = 1, q = 2n+1, n \geq 2$
	$1^{r-1}01^{q-r-1}01^{r-1}0[1]^\omega$	$r \geq 2, q \geq r+2$
$\Phi^{A-}_{(r,q)}(x)$	$(10)^{n-1}[1101^{2n-2}]^\omega$	$r = 1, q = 2n$
	$(10)^n[1]^\omega$	$r = 1, q = 2n+1$
	$(1^r0^r)^{n-1}1^r0^{r-1}[1]^\omega$	$r \geq 2, q = 2nr-1$
	$(1^r0^r)^{n-1}1^r0^{r-1}[10^{j+1}1^{2r-1}(1^{2r})^{n-1}]^\omega$	$r \geq 2, q = 2nr+j$, for , $j = 0, \ldots, r-1$
	$(1^r0^r)^{n-1}1^r0^{r-1}[01^r0^{j-r}10^{(2r-2)-j}10^{j-(r-1)}\ldots$	$r \geq 2, q = 2nr+j$, for , $j = r, \ldots, 2r-2$
	$\ldots 1^{r-1}01^{2r-1}(1^{2r})^{n-1}]^\omega$	
$\Phi^{B-}_{(r,q)}(x)$	$(1^r0^r)^n1^r0[1]^\omega$	$r \geq 1, q = 2rn+r+1$
	Section 2.2	$r \geq 1, q = 2rn+r+j$, for $j = 2, \ldots, r-1$
	$(1^r0^r)^n1^r0[0^{r-2}10^{j-r}1^{r-2}01^{r+1}(1^{2r})^n]^\omega$	$r \geq 1, q = 2nr+r+j$, for $j = r, \ldots, 2r-1$
	$(1^r0^r)^n1^r0[0^{r-1}1^{2r}0^{r-1}1^{r-1}01^{r+1}(1^{2r})^n]^\omega$	$r \geq 1, q = 2nr+r+2r$
$\Phi^{C-}_{(r,q)}(x)$	$(1^r0^r)^n1^r0^{j-1}1[0^{r-j}1^r0^j]^\omega$	$r \geq 1, q = 2nr+r+j$, for $j = 1, \ldots, r, n \geq 1$

continued ...

. . . continued

Polynomial	Expansion	Restriction
	$(1^r0^r)^n1^r0^{r-1}10^{j-r}1^{r-1}0(1^{2r})^n\ldots$ $\ldots[1^r0^{j-(r+1)}10^{2r-1-j}10^{j-r}1^{r-1}0(1^r)^{2n-1}1^{r-1}0]^\omega$ $(1^r0^r)^n1^r0^{r-1}10^r1^{r-1}0(1^{2r})^n1^r0^{r-1}1[(1^{2r})^{n+1}0^r]^\omega$	$r \geq 1, q = 2nr + r + j,$ for $j = r+1, \ldots, 2r-1, n \geq 1$ $r \geq 1, q = 2nr + r + 2r, n \geq 1$
$\chi_q^{A+}(x)$	$11(10)^{n-2}01101^{q-3}00[1]^\omega$ $11(10)^{n-2}01^{n-1}00[1]^\omega$	$q \geq 4, q = 2n$ $q \geq 3, q = 2n-1$
$\chi_q^{B+}(x)$	$1110[(10)^{n-3}01111]^\omega$ $1110[(10)^{n-3}01011(10)^{n-3}011011(10)^{n-3}\ldots$ $\ldots 00111(10)^{n-3}01111]^\omega$	$q \geq 7, q = 2n+1$ $q \geq 6, q = 2n$
$\chi_q^{A-}(x)$	$11(10)^{n-2}11011[(10)^{n-2}011(10)^{n-2}11000]^\omega$ $11(10)^{n-2}1100011(10)^{n-3}0111011[(10)^{n-3}111111000]^\omega$	$q \geq 4, q = 2n$ $q \geq 7, q = 2n+1$
$\chi_q^{B-}(x)$	$11(10)^{n-2}1110[1]^\omega$ $11(10)^{n-2}1100[1^{q-2}01]^\omega$	$q \geq 4, q = 2n$ $q \geq 5, q = 2n+1$

Table A.0.2: Lazy Expansions for 1 under Regular PV bases

Polynomial	Restriction	Pseudo Co-factor
$\Phi_r(x)$	No Restriction	1
$\Psi_r(x)$	$r > 1$	1
$\chi(x)$	NA	1
$\Psi^{A+}_{(r,q)}(x)$	$r \geq 2, q = n(r+1)$	$(1 + x^{r+1} + \cdots + x^{(n-1)(r+1)})(1 + x^{(n-1)(r+1)+r})$
	$r \geq 2, q = n(r+1) + j$, for $j = 1, \ldots, r$	1
$\Psi^{B+}_{(r,q)}(x)$	Section 3.3.1	Section 3.3.1
$\Psi^{A-}_{(r,q)}(x)$	$q = n(r+1), n \geq 1$	1
	$q = n(r+1) + j$, for $j = 1, \ldots, r-1, n \geq 1$	$1 + x^{r+1} + \cdots + x^{(n-1)(r+1)}$
	$q = n(r+1) + r, n \geq 1$	$1 + x^{q+r}(1 + x^{r+1} + \cdots + x^{(n-1)(r+1)})$
$\Psi^{B-}_{(r,q)}(x)$	$r \geq 2, q = n(r+1) + j$, for $j = -1, \ldots, r-1$	$x(1 + x^{r+1} + \cdots + x^{(n-1)(r+1)}$
$\Phi^{A+}_{(r,q)}(x)$	$r = 1, q \geq 5$	x
	$r = 2, q \geq 4$	$1/(x-1)$
	$r \geq 3, q \geq r+1$	$x/(x-1)$
$\Phi^{B+}_{(r,q)}(x)$	$r \geq 1, q \geq \min(4, r+2)$	$1/(x-1)$
$\Phi^{C+}_{(r,q)}(x)$	$r = 1, q = 2n, n \geq 2$	x
	$r = 1, q = 2n+1, n \geq 2$	x
	$r \geq 2, q \geq r+2$	1
$\Phi^{A-}_{(r,q)}(x)$	$r = 1, q = 2n$	x
	$r = 1, q = 2n+1$	x
	$r \geq 2, q = 2nr - 1$	$1/(x^r + 1)$
	$r \geq 2, q = 2nr + j$, for , $j = 0, \ldots, r-1$	$(1 + x + \cdots + x^{r-1})(1 + x^{2r} + \cdots + x^{(n-1)2r})$
	$r \geq 2, q = 2nr + j$, for , $j = r, \ldots, 2r-2$	$(1 + x + \cdots + x^{r-1})(1 + x^{2r} + \cdots + x^{n2r})$
$\Phi^{B-}_{(r,q)}(x)$	$r \geq 1, q = 2rn + r + 1$	$1/(x^r + 1)$
	$r \geq 1, q = 2rn + r + j$, for $j = 2, \ldots, r-1$	Section 3.3.2
	$r \geq 1, q = 2nr + r + j$, for $j = r, \ldots, 2r-1$	$(x^{2rn} + x^{2r(n-1)} + \cdots + 1)(x^{r-1} + x^{r-2} + \cdots + x + 1)$
	$r \geq 1, q = 2nr + r + 2r$	$(x^{2r(n+1)} + x^{2rn} + \cdots + 1)(x^{r-1} + x^{r-2} + \cdots + x + 1)$
$\Phi^{C-}_{(r,q)}(x)$	$r \geq 1, q = 2nr + r + j$, for $j = 1, \ldots, r, n \geq 1$	$x^{r-1} + x^{r-2} + \cdots + x + 1$

continued ...

...continued

Polynomial	Restriction	Pseudo Co-factor
	$r \geq 1, q = 2nr + r + j, \text{for } j = r+1, \ldots, 2r-1, n \geq 1$	$(x^{2rn} + x^{2r(n-1)} + \cdots + 1)(x^{r-1} + x^{r-2} + \cdots + x + 1)(1 + x^{2r(n+1)+j-r})$
	$r \geq 1, q = 2nr + r + 2r, n \geq 1$	$(x^{2rn} + x^{2r(n-1)} + \cdots + 1)(x^{r-1} + x^{r-2} + \cdots + x + 1)(1 + x^{2r(n+2)})$
$\chi_q^{A+}(x)$	$q \geq 4, q = 2n$	$(x^{2n} - x^{2n-1} + x - 1)/(x^2 - 1)$
	$q \geq 3, q = 2n - 1$	$(x^{2n} - x^{2n-1} + x - 1)/(x^2 - 1)$
$\chi_q^{B+}(x)$	$q \geq 7, q = 2n + 1$	$1/(x^2 - 1)$
	$q \geq 6, q = 2n$	$(x^{4q-2} - x^{3q-2} + x^{3q-3} - x^{q+1} + x^q - 1)/(x^{q+1} - x^{q-1} - x^2 + 1)$
$\chi_q^{A-}(x)$	$q \geq 4, q = 2n$	$(x^{2q+1} - x^{q+1} + x^q - 1)/(x^2 - 1)$
	$q \geq 7, q = 2n + 1$	$(x^{2q+4} - x^{q+5} + x^6 - 1)/(x^2 - 1)$
$\chi_q^{B-}(x)$	$q \geq 4, q = 2n$	$1/(x + 1)$
	$q \geq 5, q = 2n + 1$	$(x^{q-1} - 1)/(x^2 - 1)$

Table A.0.3: Pseudo Co-factors for Lazy Expansions of 1 under Regular PV bases

Appendix B

MAPLE Code

This appendix contains the MAPLE procedure used in the numerical analysis of this thesis.

Algorithm 1, labeled `Representation`, calculates the requested representation. It uses a combination of greedy and lazy algorithms as desired to calculate each digit. For example, if you want even digits calculated with the greedy algorithm and odd digits calculated with the lazy algorithm, we need only set $n = [0, 1]$. $n = [1]$ will calculate the greedy expansion and $n = [0]$ will calculate the lazy expansion. The output depends on whether we have found a periodic representation in the allotted iterations. In such a case, the output is a finite sequence of digits followed by a positive integer, which represents the point where the periodic part of the representations starts. If no such representation is found, then the output is simply a sequence of the requested length.

Here we give some sample outputs of this procedure, consider β with minimal polynomial $M(x) = x^3 - x - 1$.

$$\texttt{Representation}([1], 1, t^3 - t - 1, 50) \to [1, 0, 0, 0, 1, 0], 6.$$

$$\texttt{Representation}([0], 1, t^3 - t - 1, 50) \to [0, 0, 0, 0, 1], 5.$$

$$\texttt{Representation}([0, 1], 1, t^3 - t - 1, 50) \to [0, 1, 0, 1, 0, 0, 0, 1, 0, 0], 9.$$

$$\texttt{Representation}([1, 0, 0, 0, 0, 0], 1, t^3 - t - 1, 50) \to$$
$$[1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1], 13.$$

These outputs indicate that the greedy β-expansion for 1 is given by $10001[0]^\omega$ and the lazy β-expansion for 1 is given by $0000[1]^\omega$. The final two outputs indicate the $01010001[0]^\omega$

Algorithm 1 A procedure to calculate representations

procedure REPRESENTATION(n, y, p, k)

▷ n is a finite binary sequence (this decides at each step to use the greedy or lazy algorithm). y is a polynomial in t which represents the real number you want to find a representation for (generally 1). p is the minimal polynomial of the base with indeterminate t. k is number of iterations, the program will stop early if the representation is finite or eventually periodic

local $m,\ c,\ i,\ d,\ e,\ j,\ r,\ x,\ b,\ i2,\ B,\ q,\ l,\ s$;
$m := \mathbf{nops}(n);\ c := -1;\ i := 1;\ d := 0; j := 1;$
$b := \mathbf{fsolve}(p,\ t,\ 1.1..\ 100);\ r[0] := y;$
$\mathbf{gcdex}((t-1)/\mathbf{floor}(b),\ p,\ t,\ 'g',\ 'gg');$
while $i < k+1$ **and** $c \neq 0$ **do**
$i2 := ((i-1) \mod m) + 1;$
▷ $i2$ is used to track which entry in the sequence n is used at each step
$q := 0;\ \mathbf{Digits} := 1000;\ l := 0;$
while $q < 10^{-\mathbf{Digits}/2}$ **and** $l < 10$ **and not** $(r[i] = 0$ **or** $r[i] = g$ **or** $s[i] = 0)$ **do**
$b := \mathbf{fsolve}(p,\ t,\ 1.1..\ 100);\ B := \mathbf{subs}(t = b,\ g);$
if $n[i2] = 1$ **then** ▷ Using the greedy algorithm
$x[i] := \mathbf{convert}(\mathbf{simplify}(\mathbf{max}(0, \mathbf{min}(\mathbf{floor}(b), \mathbf{subs}(t = b,\ \mathbf{floor}(\mathbf{rem}(t * r[i-1],\ p,\ t))))))),\ \mathbf{integer});$ ▷ x_i are the digits of the representation
$r[i] := \mathbf{rem}(\mathbf{simplify}(t * r[i-1] - x[i]),\ p,\ t);$
$q := \mathbf{abs}(\mathbf{subs}(t = b,\ t * r[i-1]) - 1);$
if $r[i] \neq 0$ **then**
$s[i] := \mathbf{rem}(p,\ r[i],\ t);$
end if
end if
if $n[i2] = 0$ **then** ▷ Using the lazy algorithm
$x[i] := \mathbf{convert}(\mathbf{simplify}(\mathbf{min}(\mathbf{floor}(b), \mathbf{max}(0, \mathbf{subs}(t = b,\ \mathbf{ceil}(\mathbf{rem}(t * r[i-1] - B,\ p,\ t)))))),\ \mathbf{integer});$ ▷ x_i are the digits of the representation
$r[i] := \mathbf{rem}(\mathbf{simplify}(t * r[i-1] - x[i]), p, t);$
$q := \mathbf{abs}(\mathbf{subs}(t = b, t * r[i-1] - B));$
end if
$\mathbf{Digits} := \mathbf{Digits} + 10; l := l + 1;$
end while

Algorithm 1 continued from previous page

```
        if 2 ≤ i and c < 0 then
            for j to i − 1 do
                if r[i] = r[j] and i mod m = j mod m then
                    ▷ When the remainder terms match we have a periodic representation
                    c := i − j + 1;
                    d := i − j;                          ▷ d is the length of the period if it exists
                    e := j;    ▷ e is the length of the preperiodic part of the representation
                end if
            end for
        end if
        i := i + 1; c := c − 1; ▷ If no period is found then this variable remains negative,
    if this variable has possible value then the calculations end
    end while
    if d ≠ 0 then                          ▷ The representation is eventually periodic (or finite)
        return [seq(x[l], l = 1.. d + e)], e + 1;          ▷ Returns the representation in a
    sequence followed by the point where the periodic part begins
    else
        return seq(x[l], l = 1.. k);          ▷ Returns the requested amount of digits if no
    period is found sooner
    end if
end procedure
```

and $100000100000[1]^\omega$ are two other β-representations of 1. Notice that these two other representations are lexicographically between the greedy and lazy expansions.

Algorithm 2 which was originally written by Kevin Hare is used in later procedures, it reduces a given polynomial. In general we use it when constructing PV and Salem polynomials which way not be minimal. Recall that regular PV polynomials constructed from Table 1.2.1 may contain cyclotomic terms.

Algorithm 2 A procedure that reduces a polynomial to the minimal polynomial for its smallest real positive root larger than 1.1.

```
procedure CLEAN(poly)   ▷ poly is the polynomial with indeterminate t to be reduced
    local P, t;
    if type(poly,constant) then
        return poly;
    end if
    if nops(indets(poly)) ≠ 1 then
        returnpoly;
    end if
    t := indets(poly)[1];
    P := factor(poly);
    if type(P,'+') then                                   ▷ poly is irreducible
        return P;
    end if
    P := [op(P)];
    P := select(y→ {fsolve(y, t = 1.1 ..infinity)} ≠ {}, P);     ▷ Find the factor with the
desired root (generally PV or Salem)
    if P = [ ] then                                       ▷ No roots in the desired range
        return 1;
    end if
    return P[1]
end procedure
```

The following example outputs can be used in other procedures without the worry of including cyclotomic factors.

$$\texttt{Clean}(t^2 - t + 1) \ \to t^2 - t + 1.$$

$$\texttt{Clean}(t^3 - 2t + 1) \ \to 1.$$

$$\texttt{Clean}((t^5 - t^4 - t^3 + t^2 - 1) \cdot t^8 - (-t^5 + t^3 - t^2 - t + 1)) \ \to t^{12} - t^{10} - t^7 - t^6 - t^5 - t^2 + 1.$$

This last example is a Salem polynomial of the form $T_m^-(x)$ which we know always has a factor of $(x-1)$.

Algorithm 3 is used to construct the polynomial $T_m^+(x)$ using the input $M(x)$ and a positive integer m. It first constructs the polynomial $M^*(x)$ using the coefficients of $M(x)$. Combining $M(x)$ and $M^*(x)$ it then uses the `Clean` procedure to remove any cyclotomic factors. A simple modification can be used to construct the polynomial $T_m^-(x)$. We did not make separate procedures as each of our theorems are only concerned with one type at any given time.

Algorithm 3 A procedure to to construct Salem polynomials from a given PV polynomial

procedure SALEMM(p, m)
▷ p is a PV polynomial with indeterminate t and m is a positive integer
local r, q, v;
with(*PolynomialTools*):
$v :=$ *CoefficientList*(p, t);
$r := \sum_{i=0}^{degree(p,t)} v[i+1] \cdot t^{degree(p,t)-i}$; ▷ r is the reciprocal polynomial of p
$q := p \cdot t^m + r$; ▷ q is the requested Salem polynomial
return *Clean*(q); ▷ Returns the minimal polynomial with the same Salem root
end procedure

Example B.0.1. *Let $M(t) = t^5 - t^4 - t^3 + t^2 - 1$ and θ be the PV root of this polynomial. We know that the polynomials $T_m^{\pm}(t) = M(t)x^m \pm M^*(t)$ have Salem roots. For example*

$$T_8^+(t) = (t^5-t^4-t^3+t^2-1)t^8+(-t^5+t^3-t^2-t+1) = t^{13}-t^{12}-t^{11}+t^{10}-t^8-t^5+t^3-t^2-t+1.$$

We have,

$$\texttt{SalemM}(t^5 - t^4 - t^3 + t^2 - 1) \ \rightarrow t^{12} - 2t^{11} + t^{10} - t^7 + t^6 - t^5 + t^2 - 2t + 1.$$

This illustrates the fact that,

$$T_8^+(t) = (t+1)(t^{12} - 2t^{11} + t^{10} - t^7 + t^6 - t^5 + t^2 - 2t + 1).$$

The procedure removed the cyclotomic factor $(x+1)$.

Recall that Salem numbers appear only in even degrees, but the construction $T_m^{\pm}(x)$ can be of arbitrary degree. We use this procedure to remove the extra terms.

Algorithm 4 takes as input a representation, the input should be in the form given by the `Representation` procedure. This procedure constructs the companion polynomial

$R(x) = P_{k+\ell}(x) - P_k(x)$ using the digits of the representation. Note that we do not `Clean` this polynomial, we are interested in later factoring this into the minimal polynomial of the base and the co factor, $R(x) = M(x)Q(x)$. Here $M(x)$ is the polynomial used in the `Representation` procedure. This procedure works only when the representation input is eventually periodic, for finite representations it will output a multiple of the companion polynomial.

Algorithm 4 A procedure to calculate the companion polynomial of a given representation

procedure COMPANION(r) ▷ r is a representation in two parts, normally the output from `Representation`. The first is the sequence of digits, the second is where the periodic part begins
  local k, $p1$, $p2$
  $s := r[1]$ ▷ s is the sequence of digits that defines the representation
  $k := r[2] - 1$; ▷ k is the preperiod length
  $p1 := t^{nops(s)} - \sum_{i=1}^{nops(s)} s[i] \cdot t^{nops(s)-i}$ ▷ $p1$ is $P_{k+\ell}$
  $p2 := t^k - \sum_{i=1}^{k} s[i] \cdot t^{k-i}$ ▷ $p1$ is P_k
  return $p1 - p2$
end procedure

Example B.0.2. *Let $M(t) = t^5 - t^4 - t^3 + t^2 - 1$ and θ be the PV root of this polynomial. Then $\ell_\theta(1) = 00(1111011)^\infty$. Then we can write the companion polynomial $R(t) = P_{k+\ell}(t) - P_k(t) = t^9 - t^6 - t^5 - t^4 - t^3 - t - 1 - (t^2)$.*

$$\texttt{Companion(Representation(}[0], 1, t^5 - t^4 - t^3 + t^2 - 1, 50\texttt{))}$$
$$\to t^9 - t^6 - t^5 - t^4 - t^3 - t^2 - t - 1.$$

We see that the procedure outputs the desired companion polynomial.