

Asymptotics of the number of lattice points in the transportation polytope via optimization on Lorentzian polynomials

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

We formally extend the theory of polynomial capacity to power series and totally unimodular matrices. Using these results, we prove the log-asymptotic correctness of bounds by Brändén, Leake, and Pak developed through the use of Lorentzian polynomials ([BLP23]) under certain conditions, and provide a counterexample where these bounds are not log-asymptotically correct, even when symmetry exists.

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Dedication

Τῷ δὲ δυναμένῳ φυλάξαι αὐτοὺς ἀπταίστους καὶ στήσαι κατενώπιον τῆς δόξης αὐτοῦ ἀμώμους
ἐν ἀγαλλιάσει μόνῳ σοφῷ θεῷ σωτῇρι ἡμῶν δόξα καὶ μεγαλωσύνη κράτος καὶ ἐξουσία καὶ νῦν
καὶ εἰς πάντας τοὺς αἰῶνας ἀμήν

Ίούδας 24-25

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Chapter 1

Introduction

I conceive that these things, king Gelon, will appear incredible to the great majority of people who have not studied mathematics, but that to those who are conversant therewith and have given thought to the question of the distances and sizes of the earth the sun and moon and the whole universe the proof will carry conviction. And it was for this reason that I thought the subject would be not inappropriate for your consideration.

Archimedes ([AH97], p. 232)

So wrote Archimedes at the end of one of his lesser-known works, titled *The Sand-Reckoner*, where he both approximates the number of grains of sand that can fit in the universe and devises a number system for recording that number [AH97]. As far as counting problems go, it is largely a matter of setting up good bounds on the size of celestial objects as the ancient Greeks understood them. As we will see, the key that relates Archimedes' sand-reckoning problem to the upcoming problem is that both are approximations of what can be incredibly large numbers – neither counting problem has an efficiently computable formula, nor should one be expected.

As the above quote suggests, counting problems have been around since at least antiquity. There are many such problems that are considered $\#P$ -hard - the counting analogue of NP hard for decision problems ([Val79]). Examples include the permanent ([Val79]), counting the number of integer flows in a directed graph for fixed net flow vector ([LM24]), counting the number of contingency tables for fixed marginals ([BLP23]), and counting the number of bases of a matroid ([AGV18]).

The next few sections introduction will focus on a key approximate counting technique and scenarios where it has been applied, prior work counting particular objects known as contingency tables, and information about related objects. A particular focus is given on the intersection of these topics, mirroring the structure of our work.

1.1 Capacity

The following concept of “capacity” was isolated in the work of L. Gurvits:

Definition 1.1. Let $p : \mathbb{R}^n \rightarrow \mathbb{R}$ be a polynomial with nonnegative coefficients and $\gamma \in \mathbb{R}^n$. The γ -**capacity** of p , denoted $\text{Cap}_\gamma(p)$, is given by

$$\text{Cap}_\gamma(p) = \inf_{x>0} \frac{p(x)}{x^\gamma}$$

To the best of this author’s knowledge, Gurvits’ first iteration of capacity appeared in [Gur04]. In this paper, capacity is defined in a narrow sense for the product of linear functions and with $\gamma = \mathbf{1}$, and is identified as a key technique used in earlier work between Gurvits and A. Samorodnitsky on approximating the size of the permanent of a matrix ([GS02]). Gurvits expands on this concept in [Gur08] when he generalizes the definition to homogeneous polynomials with positive coefficients, as well as for any $\gamma \in \mathbb{R}_+$. In this paper, the expanded definition is used as an approach to approximating the number of $\{0, 1\}$ contingency table with fixed row and column sums, and contains many of the types of inequalities seen in later work such as [BLP23], [AGV18].

Why is the γ -capacity of a polynomial of interest? In brief, for certain classes of polynomials, the capacity can be used to provide bounds on the coefficients of these polynomials. Originally applied to the determinant polynomial, the list of polynomials has expanded to include real stable polynomials, Hurwitz stable polynomials, log-concave and completely log-concave polynomials, and Lorentzian polynomials. In particular, this technique can be extended to generating polynomials that fall into these classes of polynomials to bound the number of corresponding objects being counted.

One natural question that arises when considering capacity and classes of polynomials is how capacity changes as the polynomial changes. In [GL20], Gurvits and J. Leake look at how linear transformations of polynomials result. Building on the work of J. Borcea and P. Brändén in [BB09] that characterizes linear operators that preserves real stability, the authors develop a list of linear operators that limit how much the capacity of a polynomial can change upon applying the linear operator. They call these linear operators *capacity-preserving operators* ([GL20]). The concept of the symbol of a linear operator used by Borcea and Brändén in [BB09] is also highlighted by Gurvits and Leake as carrying the key to how capacity can change. They use these results on capacity-preserving operators to give another proof of a bound by Csikvári in [Csi17] on the number of k -matchings of a biregular bipartite graph ([GL20]).

Gurvits & Leake were also inspired by the work of N. Anari and S. Oveis Gharan in [AO21], who produced some of the linear operator bounds listed in [GL20]. More broadly, Anari and Oveis Gharan’s paper generalizes some of the inequalities involving capacity. The authors also provide algorithms for efficiently approximating values by applying another generalization of Gurvits’ original $\mathbf{1}$ -capacity involving the product of a pair of real stable polynomials. The authors also prove that a polynomial p with nonnegative coefficients has a positive γ -capacity if and only if γ is in the Newton polytope of p , a result we generalize to power series with nonnegative coefficients.

Anari and Oveis Gharan also worked together with C. Vinzant in [AGV18] to efficiently and deterministically compute bounds on the number of bases of any matroid, provided access to an independence oracle. They extend their main result to efficiently computing

bounds on the number of bases of a given size, as well as bounds on the number of bases in the intersection of any two matroids. To show this, they take advantage of results given by Adiprasito, Huh, and Katz in [AHK18] to show that the basis generating polynomial of a matroid is completely log-concave. They then derive bounds by considering the uniform probability distribution over all bases of the matroid and deriving upper and lower bounds on the entropy of said distribution. In particular, they link finding these bounds to a convex optimization problem to maximizing an entropy-based function over a polytope generated as the convex hull of the bases of the matroid. A similar convex program appears in our work over the transportation polytope.

J. Leake and A. Morales in [LM24] apply the methods discussed in [BLP23] to obtain improved bounds on the number of integral flows on the complete graph under different types of netflow vectors. They use the same capacity-style optimization problem on the generating polynomial to provide upper and lower bounds on the number of integer flows. They apply the contingency table results in [BLP23] to flows using the fact that a flow polytope can be represented as a face on a corresponding transportation polytope. Unlike [BLP23], they choose a point in the flow polytope to generate an explicit lower bound on the capacity. In this paper, they pick the average of the vertices of the flow polytope and provide some reasoning behind their heuristic. They also show that their capacity bound is log-asymptotically correct under polynomial growth in the flow vector. We draw inspiration from their techniques for proving log-asymptotics.

Originally, capacity was applied to homogeneous real stable polynomials. However, bounds on the coefficients of non-homogeneous real stable polynomials were obtained by Gurvits, N. Klein, and Leake in [GKL24]. They apply these bounds to the real stable generating polynomial of strongly Rayleigh probability distributions to generate bounds on certain probabilities. These bounds, in turn, are applied to the metric Travelling Salesman Problem, which the authors use to improve the approximation factor of the algorithm given by A. R. Karlin, Klein, and Oveis Gharan in [KKG21]. In particular, they show that the set of worst-case real stable polynomials can be restricted to “sparse products of affine linear forms” ([GKL24], p. 5), providing them better bounds than if they considered all real-stable polynomials.

A. Barvinok applies the capacity technique in the context of contingency tables in [Bar09a], but does not label the technique as Gurvits does. However, the authors of [BLP23] apply capacity explicitly to counting contingency tables, the details of which will be discussed in the following section.

1.2 Contingency Tables

Of particular interest to current readers is previous work on counting (nonnegative integral) contingency tables with fixed row and column sums. The formal definition will appear in Chapter 2, but contingency tables are $m \times n$ nonnegative matrices where the row and column sums are of particular interest. Contingency tables appear in a variety of contexts. In design theory, they appear in the counting of magic squares [DG95]. In group theory, they

appear when discussing the descents of permutations, double cosets [DG95], Young tableaux, and Kostka numbers [DG95]. In statistics, they are key in χ^2 tests for independence, and enumerating them for fixed row and column sums arises when finding the proportion of tables with given row and column sums above a certain threshold chi-squared value [DG95]; and are also the focus in the “table end security problem”, where a malicious actor wishes to extract information about the original table based on the public row and column sums [LK13]. In graph theory, they can be related to integral flows [LM24].

A. Barvinok in [Bar09a] provides upper and lower bounds on the number of (α, β) contingency tables in the general case. To do so, he first calculates approximations to the volume of the transportation polytope (see Section 1.4), then combines them with earlier results from [Bar07] and [Bar08] to relate the number of contingency tables to an integral of a scalar function ϕ over the volume of the standard simplex. The key ingredient, as Barvinok helpfully notes, is that the standard simplex can be partitioned into affine regions where ϕ is constant [Bar09a]. Barvinok also generalizes his approach to counting contingency tables where some entries are forced to be 0, applying this to the problem of counting integer flows on non-bipartite digraphs.

As mentioned above, Barvinok in [Bar09a] uses an equivalent version of capacity in his bounds on the number of contingency tables. While using the same upper bound as shown in [BLP23], Barvinok uses a different multiplicative factor for his lower capacity bound. In the same paper, Barvinok finds that his bound is log-asymptotically correct except in cases where the dimensions of the table are larger asymptotically than the marginals; our log-asymptotics complement his results by ensuring correct asymptotics within this case, among others. He uses this result to elucidate cases where the independence heuristic from [Goo76] fails log-asymptotically.

One well-known technique for approximating the number of contingency tables for given row and column sums is known as Good’s Heuristic, also known as the Independence heuristic, which approximates the number of contingency tables via a heuristic method [Goo76]. As H. Lyu and I. Pak explain in [LP22], this heuristic works well in cases where the row and column sums are mostly independent, performing less well as the row and columns become more dependent.

As a consequence of the need to not just count but enumerate all possible contingency tables for fixed row and column sums, researchers have worked on exact algorithms for exactly enumerating all such tables. As even counting the number of contingency tables for fixed row and column sums is $\#P$ -complete even when the number of rows is fixed to 2 ([DKM97]), such algorithms are not expected to have polynomial running time.

Another related topic of interest is generating a random contingency table for fixed row and column sums. An example of such work is given by M. Dyer, R. Kannan, and J. Mount in [DKM97], where they develop efficient algorithms for sampling using random walks (assuming sufficiently large row and column sums). A crucial step to their algorithm is reducing their sampling problem to sampling from the continuous transportation polytope. The authors highlight the work of Diaconis and Efron in [DE85], where they extend the χ^2 -statistic to additional tests for categorizing contingency tables into different classes that

lie between independent tables and uniformly sampled contingency tables. More recently, S. DeSalvo and J. Zhao applied the “probabilistic divide-and-conquer” (p. 837) method to randomly sampling contingency tables ([DZ20]). Markov chain Monte Carlo methods have also been commonly used ([DZ20]). For a more detailed discussion of sampling techniques, see [DZ20].

For a matrix $K = (K_{ij})$ where $K_{ij} \in \mathbb{Z}_+ \cup \{+\infty\}$, K -contingency tables are a generalization of contingency tables where every contingency table A must also satisfy $A \leq K$, where the inequality applies entrywise. Regular contingency tables can be thought of as K -contingency tables where $K_{ij} = +\infty$ for every entry (i, j) . In [BLP23], P. Brändén, J. Leake, and I. Pak develop general upper and lower bounds on the number of K -contingency tables using the concept of capacity and the capacity-preserving operators listed in [GL20] to obtain bounds on the coefficient of the generating polynomial P_K for (α, β) K -contingency tables. They obtain these bounds by showing that P_K can be transformed into a product of denormalized Lorentzian polynomial, which allows them to apply existing capacity-based bounds on the coefficients of P_K . They compare their results with Barvinok’s bound from [Bar09a] and A. Shapiro’s bound in [Sha10] from both theoretical and numerical comparisons.

1.3 Asymptotics of Contingency Tables

One way of assessing the accuracy of bounds on the number of contingency tables is to ask if they are correct asymptotically. More precisely, given a sequence of row and column sums that is getting progressively “larger” in some sense, what happens to the accuracy of our bounds as we take the limit of the sequence to infinity?

The study of the asymptotics of the number of contingency tables can be categorized into two different branches:

1. Precise asymptotics, and
2. Log-asymptotics.

1.3.1 Precise Asymptotics

Precise asymptotics for contingency tables are in general quite difficult – the highlighted work below are for special cases with stronger conditions than log-asymptotics.

E. R. Canfield and B. D. McKay in [RCM10] focus on analyzing the asymptotics of the number of contingency tables in the case where the marginals were uniform, but the dimensions could differ. As the authors note, only a few papers had been written rigorously proving asymptotics of any special cases of contingency tables before them. They find and prove precise asymptotics in this case provided the dimensions go to infinity and certain conditions concerning the dimension of the contingency tables and the average value of a cell in the contingency tables. They do this using a method unlike the capacity-based methods highlighted thus far – rather, they use Cauchy’s theorem to write the relevant coefficient of the generating polynomial as a multivariate integral, then apply complex analysis and

polyhedral theory on the transportation polytope to closely approximate the value of the integral.

Shortly after this paper, Barvinok and J. A. Hartigan in [BH10] develop precise asymptotics for the number of “smooth” contingency tables using capacity-like optimization problems. Here, smoothness is defined to ensure that the “typical table” (the expectation of the maximum-entropy distribution over a set of contingency table distributions defined by row α, β) of smooth marginals are bounded between two proportions of a constant and that the dimensions of the contingency table are of the same order. They note how this smoothness condition controls the optimal solution to the capacity optimization problem, and highlight some particular cases where smoothness fails and how this impacts the optimal solution. Notably, this also requires the average entry in the contingency table to stay away from zero asymptotically ([BH10]).

More recently, Lyu and Pak focus on the case of square tables where the vector of row sums equals the vector of column sums in [LP22]. Taking inspiration from a special case considered by Barvinok, they consider the case where margins are one of two possible values, where the ratio between the number of margins of each value is controlled for. Under this scenario, the authors provide asymptotics and identify an explicit threshold that determines whether or not the row and column sums are independent or not.

1.3.2 Log-Asymptotics

Far weaker than standard asymptotics, log-asymptotically correct bounds ensure that the logarithm of the number of contingency tables approach the logarithm of the bounds as the sequence of row and column sums goes to infinity. Log-asymptotically correct bounds tend to be more general than standard asymptotics as they only require that the dominant term is correct.

In [Bar09a], Barvinok uses his bounds on the number of contingency tables in the general case to state the main term log-asymptotically of the number of contingency tables, except in cases with marginals he calls “very sparse”, where all individual row and column sums are dominated by the dimensions of the rows and columns. Barvinok also notes that for his bounds, “the case of moderate to high margins seems to be the most difficult” ([Bar09a], p. 350). In this paper, we show that the bounds given by [BLP23] (where the growth of the bound is entirely due to the capacity term) are log-asymptotically correct for the very sparse marginals case outlined by Barvinok, and more generally where no subset of the row or column sums are a fixed proportion of the total sum.

Leake and Morales also evaluate the log-asymptotics of their bound on the number of integer flows in [LM24], which they show is equivalent to counting the number of K -contingency tables where certain entries are fixed to 0. We emulate their framework in the case of unrestricted contingency tables.

1.4 Transportation Polytopes

Contingency tables have an interesting relation to the **transportation problem**, defined as follows:

Suppose you have a set of warehouses $i = 1, \dots, m$ storing some material and a set of clients $j = 1, \dots, n$ that need the material, where each warehouse i has a supply α_i of the material and each client j has a demand β_j . Shipping from warehouse i to client j has an associated cost per unit c_{ij} . Fractions of materials can be sent, and the total supply equals the total demand. Find the cheapest way of ensuring all demand is satisfied.

Let (x_{ij}) be the decision variables corresponding to the amount of material shipped from warehouse i to client j . The feasible region of this problem is constructed from the following constraints:

$$\begin{aligned} \sum_{j=1}^n x_{ij} &= \alpha_i \quad \forall i = 1, \dots, m \\ \sum_{i=1}^m x_{ij} &= \beta_j \quad \forall j = 1, \dots, n \\ x &\geq 0 \quad \forall i = 1, \dots, m, j = 1, \dots, n \end{aligned}$$

The set of all feasible points is known as the **transportation polytope**. Any element of the transportation polytope can be mapped bijectively to a contingency table with row sums α and column sums β by identifying the (i, j) th cell of the table with x_{ij} . In particular, the set of all (nonnegative) integral (α, β) -contingency tables maps to precisely the lattice points of the (α, β) -transportation polytope, providing a nice geometric interpretation to the counting problem.

Various attributes of the transportation polytope (and its many generalizations) have been studied, with it being the feasible region of one of the oldest problems in linear programming ([LK13]). Classes of transportation polytopes of particular interest include

- The *Birkhoff polytope*, with identical unit row and column sums ([LK13]).
- The *central transportation polytope*, where the row sums are uniform and the column sums are uniform, but whose dimensions may differ ([LK13]).
- A *generic* transportation polytope can have arbitrary dimensions and entries, but must satisfy $\sum_{i \in S} \alpha_i \neq \sum_{j \in T} \beta_j$ for every subset $S \subseteq [m]$ and $T \subseteq [n]$ ([LK13]).

The study of transportation polytopes overlaps with the broader study of polyhedral theory, including the study of the faces of the transportation polytope. While the vertices and edges of the transportation polytope are known, questions remain about the relationships between them ([LK13]). In the case where (α, β) are integral, then the vertices of the corresponding

transportation polytope are integral also, and this follows from the fact that the constraint matrix is TU. All vertices of generic transportation polytopes have exactly $m+n-1$ nonzero entries ([LK13]), and all such polytopes are simple ([LK13]).

Dyer in [Dye83] proved that counting the number of vertices of a transportation polytope is #P-hard even when the transportation polytope is restricted to be simple. Nonetheless, there exist algorithms for constructing the vertices of the transportation polytope (see e.g. [HM02]), as well as algorithms for sampling from the vertices of transportation polytopes using random walks on the graph with vertex and edge sets equal to the vertices and edges of the transportation polytope ([CDMS08]).

Computing the vertex centroid of a general polyhedron is #P-hard ([ET12]), and this author was unable to find any result showing otherwise for general transportation polytopes.

More related to the problem of counting contingency tables is the problem of considering the volume of a transportation polytope. In many cases, the volume of a polytope approximates the number of lattice points contained in the polytope as the polytope is scaled to infinity, with the volume providing a bound on the total number of lattice points. As touched on earlier, Barvinok approximates the volume of a transportation polytope in the general case in [Bar09a] as a way to approximate the number of contingency tables. Applying this relationship in the reverse direction, Brändén, Leake, and Pak apply their bounds on the number of contingency tables to approximate the volume of the transportation polytope in [BLP23].

Another similar problem is calculating the diameter of the transportation polytope. G. Brightwell, J. Van den Heuvel, and L. Stougie showed in [BVdHS06] that the diameter of a transportation polytope with dimension $m \times n$ is at most $8(m+n-2)$. Their approach was to look at the bipartite graph corresponding to an instance of an extreme point of the transportation polytope and bound the number of “pivot” steps it takes to generate the bipartite graph corresponding to any other extreme point ([BVdHS06]). Multiple sources ([LK13], [DLKOS09]) also attribute an improved bound of $4(m+n-1)$ to C. Hurkens.

As highlighted by [DKM97], counting the number of lattice points of a contingency table is #P-hard, even when restricting the dimensions to $2 \times n$. Practically, this means that any efficient calculation of the number of contingency tables for given row and column sums is in general at best approximate.

For a survey of research on the transportation polytope and its generalizations, see [LK13].

1.5 This work

The main result of this work are conditions on the row and column sums that guarantee the bounds on the number of contingency tables given in [BLP23] are log-asymptotically correct. It builds off of the work in [BLP23] in various ways, including formalizing the extension of capacity to power series. It also applies the technique and language of infimal convolution to simplify proofs about capacity.

Chapter 2 begins by introducing notation and conventions used throughout the paper. It also introduces some basic results for key functions used later.

Chapter 3 contains our extensions of existing results on capacity to power series, relations between nonzero capacity and totally unimodular matrices, and an alternate proof of the value of capacity in a symmetric case.

Chapter 4, develops the machinery leading to showing the bounds from [BLP23] are log-asymptotically correct under simple conditions involving the maximum value of the row and column sums as a fraction of the total sum. Using this machinery, we derive additional conditions for log-asymptotic correctness in a few different cases. We also prove the existence of scenarios where the log-asymptotics of [BLP23] fail, as well as weaker conditions for the log-asymptotic correctness of explicit lower bounds that does not rely on solving the capacity problem.

We conclude in Chapter 5 with a discussion of topics and questions for future research.

Chapter 2

Preliminaries

2.1 Foundational Notation

Throughout this paper, we use the following notation:

- $\mathbb{Z}_+$ refers to the set of nonnegative integers.
- For any $n \in \mathbb{Z}_+$, $[n] := \{1, 2, \dots, n\}$.
- $\mathbb{R}_+$ refers to the set of nonnegative real numbers and $\mathbb{R}_{++}$ refers to the set of (strictly) positive real numbers
- $\mathbb{R}_+^n$ refers to the set of all n -dimensional real vectors where every component is non-negative, and $\mathbb{R}_{++}^n$ refers to the elements of $\mathbb{R}_+^n$ with (strictly) positive components. $\mathbb{R}_{++}^n$ is also referred to as the *positive orthant*.
- $\mathbb{R}[[x_1, \dots, x_n]]$ is the set of all real n -variable power series, with $\mathbb{R}_+[[x_1, \dots, x_n]]$ being the set of all n -variable power series with nonnegative coefficients.
- Unless otherwise noted, $e_1, \dots, e_n$ is the set of standard basis vectors in $\mathbb{R}^n$.
- $\mathbb{R}^{m \times n}$ refers to the set of all $m \times n$ matrices.
- For any $A_{ij} \in \mathbb{R}^{m \times n}$, A_{ij} refers to the (i, j) th entry in the matrix A ; however, when the indices become challenging to keep track of, the common computer science notation $A[i, j]$ for the (i, j) th entry of the matrix is also used. In a similar style, $A[i, :]$ and $A[:, j]$ represent the i th row and j th column of A respectively.
- For any matrix $A \in \mathbb{R}^{m \times n}$, $\bar{A} = \frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n A_{ij}$ denotes the average value of A . This notation extends naturally to vectors.
- For any $x, y \in \mathbb{R}^n$, $x^y = \prod_{j=1}^n x_j^{y_j}$.
- For any $f : \mathbb{R}^n \rightarrow \mathbb{R}$, $\nabla^2 f$ is the Hessian of f (NOT the Laplacian of f).

- $\mathbf{1}$ represents the all-ones vector. The dimension will depend on context.
- I is the identity matrix.
- $J = \mathbf{1}\mathbf{1}^T$ is the square all-ones matrix.

Let $P \in \mathbb{R}_+[[x_1, \dots, x_n]]$, $P(x) = \sum_{\mu} a_{\mu} x^{\mu}$ be a nonnegative power series.

- For any $\mu \in \mathbb{Z}^n$, $[x^{\mu}]P(x)$ denotes the coefficient a_{μ} of the μ degree term. In this context, μ is referred to as the *degree vector*.
- The *Support* of P , denoted $\text{Supp}(P)$, is the set of all degree vectors $\mu \in \mathbb{R}^n$ such that $[x^{\mu}]P(x) \neq 0$. Note that $\text{Supp}(P) \subseteq \mathbb{Z}_+^n$.
- The *Newton polyhedron* of P , denoted $\text{Newt}(P) = \text{conv}(\text{Supp}(P))$, is the convex hull of the support of P .
- The γ -*capacity* of P , denoted $\text{Cap}_{\gamma}(P)$, is defined as

$$\text{Cap}_{\gamma}(P) = \inf_{x \in \mathbb{R}^n: x > 0} \frac{P(x)}{x^{\gamma}}$$

2.2 Framework and basic results for contingency tables

A contingency table A is a $m \times n$ matrix with nonnegative entries. The row sums and column sums of A form vectors $\alpha \in \mathbb{R}^m$ and $\beta \in \mathbb{R}^n$ respectively where each component satisfies

$$\sum_{j=1}^n A_{ij} = \alpha_i, \quad \sum_{i=1}^m A_{ij} = \beta_j$$

Let $\alpha \in \mathbb{R}^m, \beta \in \mathbb{R}_{++}^n$. Throughout this paper, these will be the prescribed row and column sums for a contingency table. An (α, β) -contingency table is a contingency table with row sums α and column sums β . The number of (α, β) -contingency tables is denoted by $\text{CT}(\alpha, \beta)$. When $\|\alpha\|_1 \neq \|\beta\|_1$, $\text{CT}(\alpha, \beta) = 0$ by necessity, so unless stated otherwise we assume $\|\alpha\|_1 = \|\beta\|_1 = N$, where N will conventionally denote the total sum of the entire contingency table.

Following the notation used in [BLP23], we define

$$P_{\infty}(x, y) := \prod_{i=1}^m \prod_{j=1}^n \sum_{\ell=1}^{\infty} x_i^{\ell} y_j^{\ell} = \prod_{i=1}^m \prod_{j=1}^n \left(\frac{1}{1 - x_i y_j} \right),$$

noting that $P_{\infty}(x)$ is the generating function for the number of contingency tables since the $[x^{\alpha} y^{\beta}]$ th coefficient of P_{∞} is $\text{CT}(\alpha, \beta)$.

We define $T_{\alpha\beta}$ to be the set of all (α, β) -contingency tables.

2.3 Results for key functions

We begin with the following fact that we use repeatedly when proving log-asymptotics:

Fact 2.1. *Let $a, b, c, d \in \mathbb{R}_+$ such that $c, d > 0$. Then*

$$\max \left\{ \frac{a}{c}, \frac{b}{d} \right\} \geq \frac{a+b}{c+d} \geq \min \left\{ \frac{a}{c}, \frac{b}{d} \right\}$$

Corollary 2.2. *Let $(a_k)_{k=1}^n, (b_k)_{k=1}^n$ be real sequences where $b_1, \dots, b_n \notin \{0\}$ and $\sum_{k=1}^n b_k \neq 0$. Then*

$$\max_{k \in [n]} \left\{ \frac{a_k}{b_k} \right\} \geq \frac{\sum_{k=1}^n a_k}{\sum_{k=1}^n b_k} \geq \min_{k \in [n]} \left\{ \frac{a_k}{b_k} \right\}$$

Proof. Apply mathematical induction using Fact 2.1 for the inductive step with $\frac{\sum_{k=1}^{n-1} a_k}{\sum_{k=1}^{n-1} b_k}$ and $\frac{a_n}{b_n}$, then use inductive hypothesis. $\square$

The following function notation comes from [LM24] and is frequently used in the results below (though it existed earlier at least to the work of [Bar09a] and [BH10]):

Definition 2.3. *Let $h : \mathbb{R}_+ \rightarrow \mathbb{R}, h(0) = 0$, and $h(x) := (x+1) \log(x+1) - x \log(x)$ for $x > 0$.*

Lemma 2.4. *h is continuous.*

Proof. By standard continuity results, h is continuous over the interval $(0, \infty)$. It remains to show h is continuous at 0. Indeed,

$$\begin{aligned} \lim_{h \rightarrow 0^+} h(x) &= \lim_{z \rightarrow 0^+} (x+1) \log(x+1) - \lim_{z \rightarrow 0^+} x \log(x) \\ &= 0 + \lim_{z \rightarrow 0^+} \frac{-\log(x)}{1/x} \\ &= \lim_{z \rightarrow 0^+} \frac{-1/x}{-1/x^2} && \text{(By L'Hôpital's Rule)} \\ &= \lim_{z \rightarrow 0^+} x \\ &= 0, \end{aligned}$$

as required. $\square$

Remarks 2.5.

$$1. \ h(x) \text{ can equivalently be written as } h(x) = \log \left(\frac{(x+1)^{x+1}}{x^x} \right)$$

$$2. \ h \text{ is infinitely differentiable on } \mathbb{R}_{++}, \text{ and in particular } h'(x) = \log(1 + \frac{1}{x}), h''(x) = \frac{-1}{x(x+1)}$$

Lemma 2.6. *h' is convex.*

Proof. For any $x > 0$,

$$h'''(x) = \frac{1}{x(x+1)^2} + \frac{1}{x^2(x+1)} > 0,$$

so by second-order sufficient conditions h' is convex. □

Lemma 2.7. *h is increasing.*

Proof. For any $x > 0$,

$$h'(x) = \log\left(1 + \frac{1}{x}\right) > 0$$

□

Lemma 2.8. *For any $x > 0$, $xh'(x) < h(x) < (x+1)h'(x)$*

Proof. Let $x > 0$. Then

$$\begin{aligned} xh'(x) &= \log\left(\left(\frac{x+1}{x}\right)^x\right) \\ &= h(x) - \log(x+1) \\ &< h(x) \\ &< h(x) + \log(x) \\ &= \log\left(\left(\frac{x+1}{x}\right)^{x+1}\right) \\ &= (x+1)h'(x) \end{aligned}$$

□

Lemma 2.9. *For any $x > 0$, $\log(x+1) < h(x) < 1 + \log(x+1)$*

Proof. Recall that $(1 + \frac{1}{x})^x \rightarrow 0$ as $x \rightarrow 0^+$, $(1 + \frac{1}{x})^x \rightarrow e$ as $x \rightarrow \infty$, and that $(1 + \frac{1}{x})^x$ increases as x increases. Then for any $x > 0$,

$$\begin{aligned} \log(x+1) &< \log\left(\left(1 + \frac{1}{x}\right)^x\right) + \log(x+1) \\ &= h(x) \\ &= \log\left(\left(1 + \frac{1}{x}\right)^x\right) + \log(x+1) \\ &< \log(e) + \log(x+1) \\ &= 1 + \log(x+1) \end{aligned}$$

□

Lemma 2.10.

$$\lim_{x \rightarrow \infty} h(x) = \infty$$

Proof. Let $x > 0$. Then

$$h(x) = \log\left(\frac{(x+1)^{x+1}}{x^x}\right) = \log\left(\left(1 + \frac{1}{x}\right)^x\right) + \log(x+1)$$

Since

$$\lim_{x \rightarrow \infty} \log\left(\left(1 + \frac{1}{x}\right)^x\right) = \log(e) = 1 \quad \text{and} \quad \lim_{x \rightarrow \infty} \log(x+1) = \infty,$$

we conclude that $\lim_{x \rightarrow \infty} h(x) = 0$. □

Lemma 2.11. *For any $x > 0$,*

$$xh\left(\frac{1}{x}\right) = h(x)$$

Proof.

$$\begin{aligned} xh\left(\frac{1}{x}\right) &= x \left(\left(\frac{1}{x} + 1\right) \log\left(\frac{1}{x} + 1\right) - \frac{1}{x} \log\left(\frac{1}{x}\right) \right) \\ &= (1+x) \log\left(\frac{x+1}{x}\right) - \log\left(\frac{1}{x}\right) \\ &= (1+x) \log(x+1) - (x+1) \log(x+1) + \log(x) \\ &= (x+1) \log(x+1) - x \log(x) \\ &= h(x) \end{aligned}$$

□

Lemma 2.12. *Let $F : \mathbb{R}_+^n \rightarrow \mathbb{R}$, $F(x) = \sum_{j=1}^n h(x_j)$. Then F is concave.*

Proof. Observe that for all $x > 0$, $\partial_i F(x) = h'(x_i)$, $\partial_i^2 F(x) = h''(x_i) < 0$, and $\partial_i \partial_j F(x) = 0$ for all $i \neq j$, so $\nabla^2 F(x) \prec 0$ for all x . By second-order sufficient conditions, F is concave. □

Lemma 2.13. *For all $x > 0$,*

$$\log(x) \leq \log(x+1) \leq \log(x) + \frac{1}{x}$$

Proof. Since $\log(x)$ is an increasing function, $\log(x) \leq \log(x+1)$ for all $x > 0$. Since $x = 0 = \log(x+1)$ and $\frac{d}{dx}(x) = 1 > \frac{d}{dx}(\log(x+1))$, $\log(x+1) < x$ for all $x > 0$. Thus

$$\log(x+1) - \left(\log(x) + \frac{1}{x}\right) = \log\left(1 + \frac{1}{x}\right) - \frac{1}{x} < \frac{1}{x} - \frac{1}{x} = 0$$

for all $x > 0$, showing that $\log(x+1) < \log(x) + \frac{1}{x}$, as required. □

Lemma 2.14. *For all $x \geq 0$,*

$$x - \frac{x^2}{2} \leq \log(x+1) \leq x$$

Proof. First, observe that equality holds when $x = 0$. Next, taking derivatives, we see that

$$1 - x < \frac{1}{x + 1} < 1$$

for all $x > 0$ (where the first inequality holds since $(1 - x^2) < 1$ for $x > 0$); thus by a calculus argument the inequalities hold. $\square$

Chapter 3

Capacity Results

3.1 Formalizing capacity reformulations for power series

Brändén, Leake, and Pak provide the following result in [BLP23]:

Proposition 3.1. *(Proposition 6.2 in [BLP23]) Let $p_1, \dots, p_m \in \mathbb{R}_+[x_1, \dots, x_r], \gamma \in \mathbb{R}$. Then*

$$\text{Cap}_\gamma \left(\prod_{k=1}^m p_k \right) = \sup_{\substack{\delta^1, \dots, \delta^m \in \mathbb{R}_+^r \\ \delta^1 + \dots + \delta^m = \gamma}} \prod_{k=1}^m \text{Cap}_{\delta^k}(p_k)$$

This inspired the following result, which generalizes to the case where $p_1, \dots, p_m$ are power series:

Proposition 3.2. *Let $p_1, \dots, p_m \in \mathbb{R}_+[[x_1, \dots, x_r]]$ such that for every $k = 1, \dots, m$, $p_k(z^k) \in (0, \infty)$ for some $z^k \in \mathbb{R}_+^r$, and let $\gamma \in \mathbb{R}^r$. Then*

$$\text{Cap}_\gamma \left(\prod_{k=1}^m p_k \right) = \sup_{\substack{\delta^1, \dots, \delta^m \in \mathbb{R}_+^r \\ \delta^1 + \dots + \delta^m = \gamma}} \prod_{k=1}^m \text{Cap}_{\delta^k}(p_k)$$

Unlike the original proof of Proposition 3.1, this theorem is proven thanks to the concept of **infimal convolution** and its application in a theorem from Rockafellar:

Definition 3.3. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$. The **convex conjugate** of f , denoted by f^* , is $f^*(x) = \sup_{y \in \mathbb{R}^n} \{\langle x, y \rangle - f(y)\}$. It is also known as the Fenchel conjugate.*

Definition 3.4. *Let $f_1, \dots, f_m$ be functions. The **infimal convolution of $f_1, \dots, f_m$ at x** is*

$$(f_1 \square \dots \square f_m)(x) := \inf_{x^{(1)} + \dots + x^{(m)} = x} \{f_1(x^{(1)}) + \dots + f_m(x^{(m)})\}$$

Definition 3.5. A function $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ is proper if $f(x) > -\infty$ for all $x \in \mathbb{R}^n$ and there exists a point $x \in \mathbb{R}^n$ such that $f(x) < \infty$.

Lemma 3.6. (Theorem 16.4 in [Roc70]) Let $f_1, \dots, f_m$ be proper convex functions in $\mathbb{R}^r$. Then

$$(f_1 \square \dots \square f_m)^* = f_1^* + \dots + f_m^*$$

To prove the following result, we will need Carathéodory's Theorem:

Proposition 3.7. (Carathéodory's Theorem) (Corollary 3.41 of [CCZ14]) If a point $v \in \mathbb{R}^n$ is a convex combination of points in some set $X \subseteq \mathbb{R}^n$, then it is a convex combination of at most $\dim(X) + 1$ affinely independent points in X .

Lemma 3.8. Let $S \subseteq \mathbb{R}^n$. For any $\gamma \in \text{conv}(S)$, let $T_\gamma \subseteq S$, $\gamma = \sum_{\mu \in T_\gamma} \lambda_\mu \mu$, $\sum_{\mu \in T_\gamma} \lambda_\mu = 1$ be written so that $|T_\gamma|$ is minimized. For any $y \in \mathbb{R}^n$ there exists $\mu \in T_\gamma$ such that $(\mu - \gamma) \cdot y \geq 0$.

Proof. First, note that by Proposition 3.7, $|T_\gamma|$ is finite, so T_γ is well-defined.

We will prove the result by induction on $|T_\gamma|$.

When $|T_\gamma| = 1$, $\gamma \in T_\gamma$ so taking $\mu = \gamma$ we get $(\mu - \gamma) \cdot y = 0$ for any y .

Now suppose $n > 1$, that the property holds for any $\gamma \in \text{conv}(S)$ where $|T_\gamma| = n - 1$, and that $\gamma = \sum_{\mu \in T_\gamma} \lambda_\mu \mu \in \text{conv}(S)$ where $|T_\gamma| = n$ and $\sum_{\mu \in T_\gamma} \lambda_\mu = 1$. Then we can write $T_\gamma = \{\mu_1, \dots, \mu_n\}$ and $\lambda_i = \lambda_{\mu_i}$ for each $i \in [n]$. Observe that we can define $\gamma' = \sum_{i=1}^{n-1} \frac{\lambda_i}{1-\lambda_n} \mu_i$, noting that $|T_{\gamma'}| \leq n - 1$ and $\gamma = \lambda_n \mu_n + (1 - \lambda_n) \gamma'$. Let $y \in \mathbb{R}^n$. Then by the inductive hypothesis, we can pick $\mu' \in T_{\gamma'}$ such that $(\mu' - \gamma') \cdot y \geq 0$. If $(\mu' - \gamma) \cdot y > 0$ then we are done, so suppose that $(\mu' - \gamma) \cdot y < 0$. Then

$$0 > (\mu' - \gamma) \cdot y = (\mu' - \lambda_n \mu_n - (1 - \lambda_n) \gamma') \cdot y = \lambda_n (\mu' - \mu_n) \cdot y + (1 - \lambda_n) (\mu' - \gamma') \cdot y,$$

so

$$\lambda_n (\mu_n - \mu') \cdot y = -\lambda_n (\mu' - \mu_n) \cdot y > (1 - \lambda_n) (\mu' - \gamma') \cdot y \geq 0;$$

since $\lambda_n, (1 - \lambda_n) > 0$, we conclude that $(\mu_n - \mu') \cdot y > 0$, $(\mu' - \gamma') \cdot y \geq 0$. Then

$$\begin{aligned} (\mu_n - \gamma) \cdot y &= (\mu_n - \lambda_n \mu_n - (1 - \lambda_n) \gamma') \cdot y \\ &= (1 - \lambda_n) ((\mu_n - \mu') \cdot y + (\mu' - \gamma') \cdot y) \\ &> 0 \end{aligned}$$

By induction, the property holds for any size of T_γ , so we are done. $\square$

Lemma 3.9. Let S be a set of points in $\mathbb{Z}^n$ with positive values a_μ associated to each $\mu \in S$. Let $P(x) = \sum_{\mu \in S} a_\mu x^\mu$ and define $\psi(y) = \log(P(e^y))$. Suppose $\psi(y)$ is proper. Then $\gamma \notin \text{conv}(S) = \text{Newt}(P) \iff \psi^*(\gamma) = \infty$

Proof.

($\implies$) : Suppose $\gamma \notin \text{conv}(S)$. Since $\{\gamma\}$ is a nonempty compact convex set and $\text{conv}(S)$ is a nonempty closed convex set, by the hyperplane separation theorem there exists $\bar{y} \in \mathbb{R}^n, c \in \mathbb{R}$ such that $\gamma \cdot \bar{y} > c$ and $\mu \cdot \bar{y} \leq c$ for all $\mu \in S$. Then $(\mu - \gamma) \cdot \bar{y} = \mu \cdot \bar{y} - \gamma \cdot \bar{y} < 0$ for all $\mu \in S$.

For all $k \geq 1$, let $y^k = k\bar{y}$, noting that $\mu \cdot y^k - \gamma \cdot y^k = k(\mu - \gamma) \cdot \bar{y} \rightarrow -\infty$ as $k \rightarrow \infty$. Since ψ is proper, there exists some point $y_0 \in \mathbb{R}^n$ such that $\psi(y_0) \in \mathbb{R}$. This means that $P(e^{y_0}) = \sum_{\mu} a_{\mu} e^{\mu \cdot y_0}$ converges.

Let $\varepsilon > 0$. Since for any $\mu \in S$ $\mu \cdot y^k - \gamma \cdot y^k \rightarrow -\infty$ as $k \rightarrow \infty$, we can pick sufficiently large k to guarantee that $\mu \cdot y^k - \gamma \cdot y^k < \gamma \cdot y_0 + \log(\varepsilon)$, which implies

$$\begin{aligned} \mu \cdot (y^k + y_0) - \gamma \cdot (y^k + y_0) &= (\mu \cdot y^k - \gamma \cdot y^k) + (\mu \cdot y_0 - \gamma \cdot y_0) \\ &< \gamma \cdot y_0 + \log(\varepsilon) + \mu \cdot y_0 - \gamma \cdot y_0 \\ &= \mu \cdot y_0 + \log(\varepsilon). \end{aligned}$$

Then for sufficiently large k ,

$$0 < e^{\mu \cdot (y^k + y_0) - \gamma \cdot (y^k + y_0)} < \varepsilon e^{\mu \cdot y_0},$$

so

$$0 < \sum_{\mu \in S} a_{\mu} e^{\mu \cdot (y^k + y_0) - \gamma \cdot (y^k + y_0)} < \varepsilon \sum_{\mu \in S} a_{\mu} e^{\mu \cdot y_0}.$$

Since this is true for any $\varepsilon > 0$, $\sum_{\mu \in S} a_{\mu} e^{\mu \cdot (y^k + y_0) - \gamma \cdot (y^k + y_0)} \rightarrow 0$ as $k \rightarrow \infty$.

Let $f_{\gamma} : \mathbb{R}^n \rightarrow \mathbb{R}, f_{\gamma}(y) := \gamma \cdot y - \psi(y)$. Then

$$\begin{aligned} \lim_{k \rightarrow \infty} f_{\gamma}(y^k + y_0) &= \lim_{k \rightarrow \infty} (\gamma \cdot (y^k + y_0) - \log(P(e^{(y^k + y_0)}))) \\ &= \lim_{k \rightarrow \infty} \left(\log \left(e^{\gamma \cdot (y^k + y_0)} \right) - \log \left(\sum_{\mu \in S} a_{\mu} e^{\mu \cdot (y^k + y_0)} \right) \right) \\ &= \lim_{k \rightarrow \infty} - \left(\log \left(\sum_{\mu \in S} a_{\mu} e^{\mu \cdot (y^k + y_0)} \right) - \log \left(e^{\gamma \cdot (y^k + y_0)} \right) \right) \\ &= \lim_{k \rightarrow \infty} - \left(\log \left(\sum_{\mu \in S} a_{\mu} e^{\mu \cdot (y^k + y_0) - \gamma \cdot (y^k + y_0)} \right) \right) \\ &= \infty, \end{aligned}$$

and so $\psi^*(\gamma) = \sup_{y \in \mathbb{R}^n} (\gamma \cdot y - \psi(y)) = \infty$.

($\impliedby$) : Let $\gamma \in \text{conv}(S)$, and let f_{γ} be defined as above. By Lemma 3.8, there exists $\mu' \in S$ such that $(\mu' - \gamma) \cdot y > 0$. Then

$$\sum_{\mu \in S} a_{\mu} e^{(\mu - \gamma) \cdot y} \geq a_{\mu'} e^{(\mu' - \gamma) \cdot y} \geq a_{\mu'},$$

so

$$\begin{aligned}
f_\gamma(y) &= \gamma \cdot y - \log(P(e^y)) \\
&= \left(\log(e^{\gamma \cdot y}) - \log \left(\sum_{\mu \in S} a_\mu e^{\mu \cdot y} \right) \right) \\
&= - \left(\log \left(\sum_{\mu \in S} a_\mu e^{\mu \cdot y} \right) - \log(e^{\gamma \cdot y}) \right) \\
&= - \log \left(\sum_{\mu \in S} a_\mu e^{(\mu - \gamma) \cdot y} \right) \\
&\leq -\log(a_{\mu'}) \\
&< \infty,
\end{aligned}$$

as required. □

Corollary 3.10. *Let $P \in \mathbb{R}_+[[x_1, \dots, x_n]]$, $P(x) = \sum_\mu a_\mu x^\mu$ be a power series that converges at a point $a \in \mathbb{R}_+$ to a value $0 < P(a) < \infty$. Then $\text{Cap}_\gamma(P) > 0 \iff \gamma \in \text{Newt}(P)$*

Proof. Let $\psi(y) = \log(P(e^y))$. Then

$$\begin{aligned}
\text{Cap}_\gamma(P) > 0 &\iff \log \text{Cap}_\gamma(P) > -\infty \\
&\iff \inf_{y \in \mathbb{R}^n} \log(P(e^y)) - \gamma \cdot y > -\infty \\
&\iff -\psi^*(\gamma) > -\infty \\
&\iff \psi^*(\gamma) < \infty \\
&\iff \gamma \in \text{conv}(S) \quad (\text{By Lemma 3.9}) \\
&\iff \gamma \in \text{Newt}(P)
\end{aligned}$$
□

Proof of Proposition 3.2: Let $\psi_k(y) = \log(p_k(e^y))$. The log of the L.S. of the equation of

Proposition 3.2 can be rewritten as

$$\begin{aligned}
\log \left(\text{Cap}_\gamma \left(\prod_{k=1}^m p_k \right) \right) &= \log \left(\inf_{x>0} \left(\frac{\prod_{k=1}^m p_k(x)}{x^\gamma} \right) \right) \\
&= \inf_{y \in \mathbb{R}^n} \left(\sum_{k=1}^m \log p_k(e^y) - \sum_{i=1}^n \gamma_i \log e^{y_i} \right) \\
&= - \sup_{y \in \mathbb{R}^n} \left(\gamma \cdot y - \sum_{k=1}^m \log p_k(e^y) \right) \\
&= - \sup_{y \in \mathbb{R}^n} \left(\gamma \cdot y - \sum_{k=1}^m \psi_k(y) \right) \\
&= - \left(\sum_{k=1}^m \psi_k \right)^* (\gamma)
\end{aligned}$$

The log of the R.S. of 6.2 can be rewritten as

$$\log \left(\sup_{\substack{\delta^1, \dots, \delta^m \in \mathbb{R}_+^n \\ \delta^1 + \dots + \delta^m = \gamma}} \prod_{k=1}^m \text{Cap}_{\delta^k}(p_k) \right) = \sup_{\substack{\delta^1, \dots, \delta^m \in \mathbb{R}_+^n \\ \delta^1 + \dots + \delta^m = \gamma}} \sum_{k=1}^m \log \text{Cap}_{\delta^k}(p_k) \quad (3.1)$$

Now

$$\begin{aligned}
\sum_{k=1}^m \log \text{Cap}_{\delta^k}(p_k) &= \sum_{k=1}^m \inf_{y \in \mathbb{R}^n} (\log(p_k(e^y)) - \delta^k \cdot y) \\
&= - \sum_{k=1}^m \sup_{y \in \mathbb{R}^n} (\delta^k \cdot y - \log(p_k(e^y))) \\
&= - \sum_{k=1}^m \sup_{y \in \mathbb{R}^n} (\delta^k \cdot y - \psi_k(y)) \\
&= - \sum_{k=1}^m \psi_k^*(\delta^k),
\end{aligned}$$

so equation (3.1) becomes

$$\begin{aligned} \sup_{\substack{\delta^1, \dots, \delta^m \in \mathbb{R}_+^n \\ \delta^1 + \dots + \delta^m = \gamma}} \sum_{k=1}^m \log \text{Cap}_{\delta^k}(p_k) &= \sup_{\substack{\delta^1, \dots, \delta^m \in \mathbb{R}_+^n \\ \delta^1 + \dots + \delta^m = \gamma}} - \sum_{k=1}^m \psi_k^*(\delta^k) \\ &= - \inf_{\substack{\delta^1, \dots, \delta^m \in \mathbb{R}_+^n \\ \delta^1 + \dots + \delta^m = \gamma}} \sum_{k=1}^m \psi_k^*(\delta^k) \end{aligned}$$

By Lemma 3.9, we know that $\psi_k^*(\delta^k) = \infty \iff \delta^{(k)} \in \text{Newt}(p_k) \subseteq \mathbb{R}_+^n$, so we can relax the nonnegativity constraints as follows:

$$\begin{aligned} - \inf_{\substack{\delta^1, \dots, \delta^m \in \mathbb{R}_+^n \\ \delta^1 + \dots + \delta^m = \gamma}} \sum_{k=1}^m \psi_k^*(\delta^k) &= - \inf_{\substack{\delta^1, \dots, \delta^m \in \mathbb{R}^n \\ \delta^1 + \dots + \delta^m = \gamma}} \sum_{k=1}^m \psi_k^*(\delta^k) \\ &= -(\psi_1^* \square \dots \square \psi_m^*)(\gamma) \end{aligned}$$

Since $\psi_1, \dots, \psi_m$ are proper convex functions, by Lemma 3.6,

$$-(\psi_1^* \square \dots \square \psi_m^*)(\gamma) = - \left(\sum_{k=1}^m \psi_k \right)^* (\gamma),$$

so the log of the L.S. equals the log of the R.S.; hence the L.S. equals the R.S., as required. $\square$

3.2 Capacity for products of geometric-style series

Now that we have shown that Proposition 3.2 is correct, we will next simplify the the right side of Proposition 3.2 in the case where the entire power series is built from a product of geometric-style series using the following sequence of results:

Lemma 3.11. *For any $x_1, \dots, x_n \geq 0$, $\lambda \geq 0$, $(x_1 + \dots + x_n)^\lambda \geq x_1^\lambda + \dots + x_n^\lambda$.*

Proof. Suppose first that $\lambda = \frac{p}{q}$ is rational, and suppose for a contradiction that $(x_1 + \dots + x_n)^\lambda < x_1^\lambda + \dots + x_n^\lambda$. Then since $0 \leq (x_1 + \dots + x_n)$, $(x_1 + \dots + x_n)^p < (x_1^{p/q} + \dots + x_n^{p/q})^q \leq x_1^p + \dots + x_n^p$, a contradiction. Thus $(x_1 + \dots + x_n)^\lambda \geq x_1^\lambda + \dots + x_n^\lambda$.

Now suppose λ is irrational. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, we can pick a sequence $z_p \in \mathbb{Q}$, $p \geq 1$ such that $z_p \rightarrow \lambda$. By what we proved above, $(x_1 + \dots + x_n)^{z_p} \geq x_1^{z_p} + \dots + x_n^{z_p}$ for all $p \geq 1$, so by convergence results, $(x_1 + \dots + x_n)^\lambda \geq x_1^\lambda + \dots + x_n^\lambda$. $\square$

Corollary 3.12. *For any $(x_k)_{k=1}^\infty, (y_k)_{k=1}^\infty \subseteq \mathbb{R}_+$, $\lambda \in (0, 1)$, it follows that*

$$\left(\sum_{k=1}^\infty x_k \right)^\lambda \left(\sum_{k=1}^\infty y_k \right)^{1-\lambda} \geq \sum_{k=1}^\infty x_k^\lambda y_k^{1-\lambda}$$

Proof. By Lemma 3.11, for any $n \geq 1$,

$$(x_1 + \dots + x_n)^\lambda (y_1 + \dots + y_n)^{1-\lambda} \geq (x_1^\lambda + \dots + x_n^\lambda)(y_1^{1-\lambda} + \dots + y_n^{1-\lambda}).$$

Distributing, we see that since all terms are nonnegative, $(x_1 + \dots + x_n)^\lambda (y_1 + \dots + y_n)^{1-\lambda} \geq x_1^\lambda y_1^{1-\lambda} + \dots + x_n^\lambda y_n^{1-\lambda}$. Since this holds for all $n \geq 1$, taking $n \rightarrow \infty$ gives

$$\left(\sum_{k=1}^{\infty} x_k \right)^\lambda \left(\sum_{k=1}^{\infty} y_k \right)^{1-\lambda} \geq \sum_{k=1}^{\infty} x_k^\lambda y_k^{1-\lambda}$$

□

Lemma 3.13. *Let $P(x) = \sum_{\mu} a_{\mu} x^{\mu} \in \mathbb{R}_+[[x_1, \dots, x_n]]$ be a power series. Then $\psi(y) = \log(P(e^y))$ is convex.*

Proof. Let $x, y \in \mathbb{R}^n, \lambda \in (0, 1)$. Observe that

$$\begin{aligned} e^{\lambda\psi(x) + (1-\lambda)\psi(y) - \psi(\lambda x + (1-\lambda)y)} &= \frac{P(e^x)^\lambda \cdot P(e^y)^{1-\lambda}}{P(e^{\lambda x + (1-\lambda)y})} \\ &= \frac{(\sum_{\mu} a_{\mu} e^{\mu \cdot x})^\lambda (\sum_{\mu} a_{\mu} e^{\mu \cdot y})^{1-\lambda}}{\sum_{\mu} a_{\mu} e^{\mu \cdot (\lambda x + (1-\lambda)y)}} \\ &\geq \frac{\sum_{\mu} a_{\mu}^\lambda (e^{\mu \cdot x})^\lambda a_{\mu}^{1-\lambda} (e^{\mu \cdot y})^{1-\lambda}}{\sum_{\mu} a_{\mu} e^{\mu \cdot (\lambda x + (1-\lambda)y)}} \quad (\text{by Corollary 3.12}) \\ &= \frac{\sum_{\mu} a_{\mu} e^{\lambda \mu \cdot x + (1-\lambda) \mu \cdot y}}{\sum_{\mu} a_{\mu} e^{\mu \cdot (\lambda x + (1-\lambda)y)}} \\ &= 1 \end{aligned}$$

Therefore, taking the log of both sides of the inequality and rearranging gives

$$\lambda\psi(x) + (1-\lambda)\psi(y) \geq \psi(\lambda x + (1-\lambda)y),$$

so ψ is convex. □

Remark 3.14. *Similar results for polynomials were known before (see Fact 2.12 of [GL20]), but the above result generalizes this to power series.*

Lemma 3.15. *Let $a \in \mathbb{R}^n$ and*

$$q(x) = \frac{1}{1 - x^a}$$

Then

$$\text{Cap}_{\gamma}(q) > 0 \iff \text{Cap}_{\gamma}(q) = \frac{(z+1)^{z+1}}{z^z}, \gamma = za, z \geq 0$$

Proof. First, note that by Corollary 3.10, $\text{Cap}_{\gamma}(q) > 0 \iff \gamma \in \text{Newt}(q) \iff \gamma = za$ for some $z > 0$, so it suffices to show that in this case when $\gamma = za$ for some $z > 0$ that $\text{Cap}_{\gamma}(q) = \frac{(z+1)^{z+1}}{z^z}$.

Observe that

$$\log(\text{Cap}_{\gamma}(q)) = \inf_{y \in \mathbb{R}^n} \left(\log \left(\frac{1}{1 - e^{a \cdot y}} \right) - \gamma \cdot y \right)$$

Since $\text{Cap}_\gamma(q)$ is finite, $1 \neq e^{a \cdot y}$; hence $a \cdot y \neq 0$. Note also that for $\tau(y) = \log\left(\frac{1}{1-e^{a \cdot y}}\right) - \gamma \cdot y$, $\partial_j \tau(y) = \frac{a_j e^{a \cdot y}}{1-e^{a \cdot y}} - \gamma_j$. By Lemma 3.13, $\psi(y) = \log\left(\frac{1}{1-e^{a \cdot y}}\right)$ is convex, so $\tau(y) = \psi(y) - \gamma \cdot y$ is also convex. Since τ is convex, any extreme point of τ is a global minimum of τ .

Setting $\nabla \tau(y) = 0$ and solving for y , we see that for any $j \in [n]$,

$$\begin{aligned} \partial_j \tau(y) &= 0 \\ \iff a_j e^{a \cdot y} &= \gamma_j (1 - e^{a \cdot y}) \\ \iff (a_j + \gamma_j) e^{a \cdot y} &= \gamma_j \\ \iff a \cdot y &= \log\left(\frac{\gamma_j}{a_j + \gamma_j}\right) \quad (\text{Since } \gamma = za) \\ \iff a \cdot y &= \log\left(\frac{z}{1+z}\right) \end{aligned}$$

This is a system of n variables and n identical equations. Since $a \cdot y \neq 0$, this system has a solution. Let $\bar{y}$ be an arbitrary solution to this system, recalling that $\bar{y}$ is an extreme point that minimizes τ . Plugging in $\bar{y}$ into τ , we get

$$\begin{aligned} \log(\text{Cap}_\gamma(q)) &= \tau(\bar{y}) \\ &= \log\left(\frac{1}{1 - \frac{z}{z+1}}\right) - z \log\left(\frac{z}{z+1}\right) \\ &= \log(z+1) - z \log(z) + z \log(z+1) \\ &= (z+1) \log(z+1) - z \log(z) \\ &= \log\left(\frac{(z+1)^{z+1}}{z^z}\right) \end{aligned}$$

Taking exponents of both sides returns the result. □

Proposition 3.16. *Let $P = \{x \in \mathbb{R}^n : Ax = b, x \geq 0\}$, A a real $m \times n$ matrix, and let*

$$Q(x) = \prod_{j=1}^m \frac{1}{1 - x^{A[:,j]}} ,$$

where $x^{A[:,j]} = \prod_{i=1}^m x_i^{A_{ij}}$. Then

$$\begin{aligned} \text{Cap}_b(Q) &= \max_{s.t. \quad z \in P} \prod_{j=1}^n \frac{(z_j+1)^{z_j+1}}{z_j^{z_j}} \end{aligned} \quad (\text{Cap-Sup})$$

Proof. Let $q_j(x) = \frac{1}{1-x^{A[:,j]}}$ so that $Q(x) = \prod_{j=1}^n q_j(x)$. By Proposition 3.2,

$$\text{Cap}_b(Q) = \sup_{\substack{\delta^1, \dots, \delta^n \in \mathbb{R}_+^m \\ \delta^1 + \dots + \delta^n = b}} \prod_{j=1}^n \text{Cap}_{\delta^j}(q_j)$$

From this equation, we see that $\text{Cap}_b(Q) > 0 \iff \text{Cap}_{\delta^j}(q_j) > 0$ for all $j \in [n]$. By Lemma 3.15, for any $j \in [n]$, $\text{Cap}_{\delta^j}(q_j) > 0 \iff \text{Cap}_{\delta^j}(q_j) = \frac{(z_j+1)^{z_j+1}}{z_j^{z_j}}, \delta^j = z_j A[:, j], z_j \geq 0$. Putting these two results together with the supremum constraints gives

$$\begin{aligned} \text{Cap}_b(Q) > 0 &\iff \begin{aligned} \text{Cap}_b(Q) &= \max && \prod_{j=1}^n \frac{(z_j+1)^{z_j+1}}{z_j^{z_j}} \\ \text{s.t.} &&& Az = b \\ &&& z \geq 0 \end{aligned} \\ &\iff \begin{aligned} \text{Cap}_b(Q) &= \max && \prod_{j=1}^n \frac{(z_j+1)^{z_j+1}}{z_j^{z_j}} \\ \text{s.t.} &&& z \in P \end{aligned}, \end{aligned}$$

as required. $\square$

To understand why Proposition 3.16 is important, consider the following result found in [Bar17]. For the next couple of results, we define $P = \{x \in \mathbb{R}^n : Ax = b, x \geq 0\}$, A a real $m \times n$ matrix, and

$$Q(x) = \prod_{j=1}^n \frac{1}{1 - x^{A[:,j]}} ,$$

where $x^{A[:,j]} = \prod_{i=1}^m x_i^{A_{ij}}$.

Lemma 3.17. ([Bar17], p. 199) *Let P, A, Q be defined as above. Then $|P \cap \mathbb{Z}^d| = [x^b]Q(x)$.*

Given that capacity can be used to bound the coefficients of a power series under certain conditions, Lemma 3.17 indicates that capacity can be used to bound the number of lattice points in a polyhedron P , provided the conditions for applying capacity are satisfied. One particular case that may prove more generally useful beyond the case of contingency tables is covered below.

Definition 3.18. *A $m \times n$ matrix A is Totally Unimodular (TU) if every square submatrix of A formed by deleting rows and/or columns has a determinant in the set $\{-1, 0, 1\}$.*

One key fact about any TU matrix A is that all the vertices of $P = \{Ax = b, x \geq 0\}$ are integral.

When A is TU, Proposition 3.16 and Lemma 3.17 can be used to justify the following result:

Proposition 3.19. *Suppose that A is a real $m \times n$ TU matrix and Q as defined above. Then $\text{Cap}_b(Q) > 0$ if and only if $[x^b]Q(x) > 0$.*

Proof. If $[x^b]Q(x) > 0$, then $b \in \text{Newt}(P)$, so by Corollary 3.10 $\text{Cap}_b(Q) > 0$.

Conversely, suppose $\text{Cap}_b(Q) > 0$. Then $b \in \text{Newt}(Q)$, so we can write $b = \sum_{\mu \in \text{Supp}(Q)} \lambda_\mu \mu$ for $\lambda \in [0, 1], \sum_\mu \lambda_\mu = 1$. For any $\gamma \in \mathbb{R}^m$, let $P_\gamma = \{x \in \mathbb{R}^n : Ax =$

$\gamma, x \geq 0\}$. Then for any $\mu \in \text{Supp}(Q)$, $|P_\mu \cap \mathbb{Z}^d| = [x^\mu]Q(x) > 0$, so for every $\mu \in \text{Supp}(Q)$ we can pick $x_\mu \in P_\mu$. Let $x^* = \sum_\mu \lambda_\mu x_\mu \geq 0$, and observe that

$$Ax^* = \sum_\mu \lambda_\mu Ax_\mu = \sum_\mu \lambda_\mu \mu = b,$$

so $P_b \neq \emptyset$. Since P_b does not contain a line (since $P_b \subseteq \{x \geq 0\}$), P_b must contain an extreme point. Since A is TU, every extreme point of P_b is integral; hence P_b contains an integral point. Thus $[x^b]Q(x) = |P_b \cap \mathbb{Z}^n| > 0$, as required. $\square$

Remark 3.20. *Proposition 3.19 suggests that counting problems where a TU matrix encodes the generating function are scenarios in which capacity bounds are possible – the full discussion is left until Section 5.3.*

Corollary 3.21. *Let $\alpha \in \mathbb{Z}_{++}^m, \beta \in \mathbb{Z}_{++}^n$ with $\|\alpha\|_1 = \|\beta\|_1 = N$. Then*

$$\text{Cap}_{\alpha\beta}(P_\infty) \geq \prod_{i=1}^m \prod_{j=1}^n \frac{\left(\frac{\alpha_i \beta_j}{N} + 1\right)^{\frac{\alpha_i \beta_j}{N} + 1}}{\frac{\alpha_i \beta_j}{N}^{\frac{\alpha_i \beta_j}{N}}}$$

Proof. It is easy to check that $\alpha\beta^T/N$ is feasible for (Cap-Sup). Then the inequality follows by Proposition 3.16. $\square$

In the following special case, symmetry allows us to find the optimal solution to (Cap-Sup). This result was known as far back as [Bar09a] from a symmetry argument, but the proof below uses KKT conditions.

Proposition 3.22. *Let $\alpha = \bar{\alpha} \mathbf{1} \in \mathbb{Z}_{++}^m, \beta \in \mathbb{Z}_{++}^n$ such that $\|\alpha\|_1 = N = \|\beta\|_1$. Then $(\alpha\beta^T/N)_{ij} = \alpha_i \beta_j / N = \beta_j / m$ is the optimal solution to (Cap-Sup).*

Proof. Finding an optimal solution to (Cap-Sup) is equivalent to finding an optimal solution to

$$-\min \quad H(z) = \sum_{i=1}^m \sum_{j=1}^n z_{ij} \log(z_{ij}) - (z_{ij} + 1) \log(z_{ij} + 1) \quad (3.2)$$

$$\begin{aligned} \text{s.t.} \quad & \sum_{j=1}^n z_{ij} = \alpha_i \quad \forall i \in [m] \\ & \sum_{i=1}^m z_{ij} = \beta_j \quad \forall j \in [n] \\ & z_{ij} \geq 0 \quad \forall i \in [m], j \in [n] \end{aligned}$$

The KKT constraints for (3.2) are:

$$\mu_i + \nu_j - \lambda_{ij} = -\partial_{ij}H(z) \quad \forall i \in [m], j \in [n] \quad (\text{Stationarity}) \quad (3.3)$$

$$\mu_i \left(\sum_{j=1}^m z_{ij} - \alpha_i \right) = 0 \quad \forall i \in [m] \quad (\text{CS conditions}) \quad (3.4)$$

$$\nu_j \left(\sum_{i=1}^m z_{ij} - \beta_j \right) = 0 \quad \forall j \in [n] \quad (3.5)$$

$$-\lambda_{ij} z_{ij} = 0 \quad \forall i \in [m], j \in [n] \quad (3.6)$$

$$\sum_{j=1}^n z_{ij} = \alpha_i \quad \forall i \in [m] \quad (\text{Primal Feasibility}) \quad (3.7)$$

$$\sum_{i=1}^m z_{ij} = \beta_j \quad \forall j \in [n] \quad (3.8)$$

$$-z_{ij} \leq 0 \quad \forall i \in [m], j \in [n] \quad (3.9)$$

$$\lambda_{ij} \geq 0 \quad (\text{Dual Feasibility}) \quad (3.10)$$

By Lemma 2.12, (3.2) is a convex program. Because all the constraints are affine, the linear constraint qualification holds and therefore the KKT conditions are sufficient.

Consider the KKT conditions at the point $\alpha\beta^T/N$ with dual variables $\lambda_{ij} = 0$ for all $i \in [m], j \in [n]$, $\mu_i = 0$ for all $i \in [m]$, $\nu_j = -\partial_{1j}H(\alpha\beta^T/N)$:

- **Stationarity:** First, note that for fixed j , (3.3) is the same for all $i \in [m]$ (since the rows are all identical); hence there are only $[n]$ distinct Stationarity constraints. Also for fixed j , $\mu_i + \nu_j - \lambda_{ij} = -\partial_{1j}H(\alpha\beta^T/N) = -\partial_{ij}H(\alpha\beta^T/N)$, so the Stationarity constraints are satisfied.
- **CS conditions:** Since $\alpha\beta^T/N$ is feasible, conditions (3.4) and (3.5) are satisfied automatically. Since $\lambda_{ij} = 0$ for all $i \in [m], j \in [m]$, conditions (3.6) are satisfied.
- **Feasibility:** It is clear that both primal and dual variables are feasible.

Since all KKT conditions are satisfied, $\alpha\beta^T/N$ is an optimal solution to (3.2) (and therefore also to eq. (Cap-Sup)). $\square$

In this particular case, the optimal solution also has a straightforward geometric interpretation:

Proposition 3.23. *Let $\alpha = \bar{\alpha}\mathbf{1} \in \mathbb{R}_+^m, \beta \in \mathbb{R}_+^n$. Then $\alpha\beta^T/N$ is the average of all the vertices of $T_{\alpha\beta}$*

In order to prove this, we will need to develop some notation.

Let $\bar{\alpha} \in \mathbb{R}_+, \alpha = \bar{\alpha}\mathbf{1} \in \mathbb{R}_+^m, \beta \in \mathbb{R}_+^n$ so that $\|\alpha\|_1 = N = \|\beta\|_1$. Let $T_{\alpha\beta}$ be the set of all contingency tables whose rows sum to α and columns sum to β , and let $E(T_{\alpha\beta})$ be the set of extreme points of $T_{\alpha\beta}$.

Note that for any $A \in T_{\alpha\beta}$, any contingency table A' formed by permuting the rows of A is also a contingency table, as permuting the rows does not change the row sums (they each sum to $\bar{\alpha}$) nor the column sums, and all entries remain nonnegative. In particular, for any extreme point $A \in E(T_{\alpha\beta})$, any contingency table A' formed by permuting the rows of A is also an extreme point, as permuting rows of A is the same as a change of coordinates in the vector space of matrices, which is a linear transformation.

We define the group action of the symmetric group S_m on $E(T_{\alpha\beta})$ so that for $\sigma \in S_m$ and $A \in E(T_{\alpha\beta})$, σA is the vertex of $T_{\alpha\beta}$ formed by permuting the rows of A according to σ . By the previous note, this group action is well-defined. For any $A \in T_{\alpha\beta}$, define $S_m^A = \{\sigma \in S_m : \sigma A = A\}$ to be the stabilizer of A and $S_m A = \{\sigma A : \sigma \in S_m\}$ to be the orbit of A .

Lemma 3.24. *Let $A \in E(T_{\alpha\beta})$, $B \in S_m A$. The number of permutations mapping A to B is $|S_m^A|$.*

Proof. Since $B \in S_m A$, we can pick $\sigma \in S_m$ such that $\sigma A = B$. Let $U = \{\tau \in S_m : \tau A = B\}$. For all $\tau \in U$, $\sigma^{-1}\tau A = \sigma^{-1}B = A$, so $\sigma^{-1}\tau \in S_m^A$. Moreover, for any $\tau_1, \tau_2 \in U$, $\sigma^{-1}\tau_1 = \sigma^{-1}\tau_2 \iff \tau_1 = \tau_2$, so $|U| \leq |S_m^A|$. Now let $v \in S_m^A$. Observe that $\sigma v A = \sigma A = B$, so $\sigma v \in U$, and for any $v_1, v_2 \in S_m^A$, $\sigma v_1 = \sigma v_2 \iff v_1 = v_2$; hence $|S_m^A| \leq |U|$. Thus $|S_m^A| = |U|$, proving the proposition. $\square$

Proof of Proposition 3.23. Observe that by Lemma 3.24, for any given $A \in E(T_{\alpha\beta})$,

$$\sum_{\sigma \in S_m} \sigma A = \sum_{B: B \in S_m A} |S_m^A| B = |S_m^A| \sum_{B: B \in S_m A} B.$$

By orbit-stabilizer theorem, $|S_m^A| = |S_m|/|S_m A| = m!/|S_m A|$, so

$$\sum_{\sigma \in S_m} \sigma A = |S_m^A| \sum_{B: B \in S_m A} B = \frac{m!}{|S_m A|} \sum_{B: B \in S_m A} B$$

Dividing by $m!$, we get that

$$\frac{1}{m!} \sum_{\sigma \in S_m} \sigma A = \frac{1}{|S_m A|} \sum_{B: B \in S_m A} B,$$

where the right side of the equation is precisely the formula for the average of all extreme points in the orbit of A .

Moreover, since for each pair of rows (i, j) there are exactly $(m-1)!$ permutations that send row i to row j ,

$$\frac{1}{m!} \sum_{\sigma \in S_m} \sigma A = \frac{1}{m!} \begin{bmatrix} \sum_{i=1}^m (m-1)! A[i, :] \\ \sum_{i=1}^m (m-1)! A[i, :] \\ \vdots \\ \sum_{i=1}^m (m-1)! A[i, :] \end{bmatrix} = \frac{1}{m} \begin{bmatrix} \sum_{i=1}^m A[i, :] \\ \sum_{i=1}^m A[i, :] \\ \vdots \\ \sum_{i=1}^m A[i, :] \end{bmatrix} = \frac{1}{m} \begin{bmatrix} \beta^T \\ \beta^T \\ \vdots \\ \beta^T \end{bmatrix},$$

hence

$$\frac{1}{|S_m A|} \sum_{B: B \in S_m A} B = \frac{1}{m} \begin{bmatrix} \beta^T \\ \beta^T \\ \vdots \\ \beta^T \end{bmatrix}$$

Now since the orbits of $E(T_{\alpha\beta})$ partition $E(T_{\alpha\beta})$, we get that the average of all extreme points of $T_{\alpha\beta}$ is

$$\begin{aligned} \frac{1}{|E(T_{\alpha\beta})|} \sum_{A: A \in E(T_{\alpha\beta})} A &= \frac{1}{|E(T_{\alpha\beta})|} \sum_{R \in E(T_{\alpha\beta})/S_m} \sum_{B: B \in R} B \\ &= \frac{1}{|E(T_{\alpha\beta})|} \sum_{R \in E(T_{\alpha\beta})/S_m} \frac{|R|}{m} \begin{bmatrix} \beta^T \\ \beta^T \\ \vdots \\ \beta^T \end{bmatrix} \\ &= \frac{1}{m} \frac{1}{|E(T_{\alpha\beta})|} \left(\sum_{R \in E(T_{\alpha\beta})/S_m} |R| \right) \begin{bmatrix} \beta^T \\ \beta^T \\ \vdots \\ \beta^T \end{bmatrix} \\ &= \frac{1}{m} \frac{1}{|E(T_{\alpha\beta})|} |E(T_{\alpha\beta})| \begin{bmatrix} \beta^T \\ \beta^T \\ \vdots \\ \beta^T \end{bmatrix} \\ &= \frac{1}{m} \begin{bmatrix} \beta^T \\ \beta^T \\ \vdots \\ \beta^T \end{bmatrix} \end{aligned}$$

□

Remark 3.25. *It is not true in general that the average of the vertices of $T_{\alpha\beta}$ is $\alpha\beta^T/N$. As a counterexample, consider $\alpha = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \beta = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$. The extreme points of $T_{\alpha\beta}$ are $\begin{bmatrix} 2 & 1 \\ 0 & 4 \end{bmatrix}$ and $\begin{bmatrix} 0 & 3 \\ 2 & 2 \end{bmatrix}$ with average $\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$, but $\alpha\beta^T/N = \frac{1}{7} \begin{bmatrix} 6 & 15 \\ 8 & 20 \end{bmatrix}$.*

Chapter 4

Log Asymptotics

4.1 Conditions for log-asymptotically correct bounds

In this section, the goal is to prove that the capacity bounds from [BLP23] are correct log-asymptotically under certain conditions. To do so, we use the main result of [BLP23], which we list here:

Theorem 4.1. (*Theorem 2.1 in [BLP23]*) Let $\alpha \in \mathbb{Z}_{++}^m, \beta \in \mathbb{Z}_{++}^n$ such that $\sum_{i=1}^m \alpha_i = \sum_{j=1}^n \beta_j$. Then:

$$\text{Cap}_{\alpha\beta}(P_\infty) \geq \text{CT}(\alpha, \beta) \geq \left[\prod_{i=2}^m \frac{\alpha_i^{\alpha_i}}{(\alpha_i + 1)^{\alpha_i+1}} \prod_{j=1}^n \frac{\beta_j^{\beta_j}}{(\beta_j + 1)^{\beta_j+1}} \right] \text{Cap}_{\alpha\beta}(P_\infty), \quad (4.1)$$

Remark 4.2. *The full result applies more generally to contingency tables where the cell entries are bounded, but we only consider the unrestricted case.*

4.1.1 Properties of the ratio of h and h'

Recall that $h(x) = (x+1)\log(x+1)$ and $h'(x) = \log(1+1/x)$. We will develop mathematical machinery to see how the ratio between h and h' changes.

Lemma 4.3. Let $\lambda_1, \dots, \lambda_n > 0$ such that $\lambda_1 + \dots + \lambda_n = 1$, $F : \mathbb{R}_+ \rightarrow \mathbb{R}, F(x) = \frac{h(x)}{\sum_{j=1}^n h(\lambda_j x)}$. Then $F(1) \geq F(x)$ for all $x \geq 1$.

To show this, we prove a couple of lemmas:

Lemma 4.4. For any $z \in \mathbb{R}_{++}$,

$$h'(z) \log(z+1) < \frac{h(z)}{z+1}.$$

Proof. Let $f : \mathbb{R}_+ \rightarrow \mathbb{R}$, $f(0) = 0$, and $f(z) = h'(z) \log(z+1) - \frac{h(z)}{z+1}$ for $z > 0$, noting that f is continuous everywhere and differentiable on $\mathbb{R}_{++}$. Observe that

$$\begin{aligned}
\lim_{z \rightarrow 0^+} f(z) &= \lim_{z \rightarrow 0^+} h'(z) \log(z+1) - \lim_{z \rightarrow 0^+} \frac{h(z)}{z+1} \\
&= \lim_{z \rightarrow 0^+} \log(z+1)^2 - \lim_{z \rightarrow 0^+} \log(z) \log(z+1) - 0 \quad (\text{Since } h \text{ is continuous by 2.4}) \\
&= 0 - \lim_{z \rightarrow 0^+} \frac{\log(z+1)}{\frac{1}{\log(z)}} \\
&= \lim_{z \rightarrow 0^+} \frac{\frac{1}{z+1}}{\frac{-1}{\log(z)^2 z}} \quad (\text{By L'Hôpital's Rule}) \\
&= \lim_{z \rightarrow 0^+} \frac{-\log(z)^2}{\frac{1}{z}} \quad (\text{Evaluating } \lim_{z \rightarrow 0^+} z+1 = 1) \\
&= \lim_{z \rightarrow 0^+} \frac{-2\log(z) \frac{1}{z}}{\frac{-1}{z^2}} \quad (\text{By L'Hôpital's Rule}) \\
&= 2 \lim_{z \rightarrow 0^+} \frac{\log(z)}{\frac{1}{z}} \\
&= 2 \lim_{z \rightarrow 0^+} \frac{\frac{1}{z}}{-\frac{1}{z^2}} \quad (\text{By L'Hôpital's Rule}) \\
&= -2 \lim_{z \rightarrow 0^+} z \\
&= 0,
\end{aligned}$$

$$f(1) = h'(1) \log(2) - \frac{h(1)}{2} = \log(2)^2 - \log(2) = -\log(2)(1 - \log(2)) < 0,$$

and

$$\begin{aligned}
\lim_{z \rightarrow \infty} f(z) &= \lim_{z \rightarrow \infty} h'(z) \log(z+1) - \lim_{z \rightarrow \infty} \frac{h(z)}{z+1} \\
&= \lim_{z \rightarrow \infty} \frac{h'(z)}{\frac{1}{\log(z+1)}} - \lim_{z \rightarrow \infty} \frac{h(z)}{z+1} \\
&= \lim_{z \rightarrow \infty} \frac{h''(z)}{\frac{-1}{\log(z+1)^2(z+1)}} - \lim_{z \rightarrow \infty} h'(z) && \text{(By L'Hôpital's Rule)} \\
&= \lim_{z \rightarrow \infty} \frac{\log(z+1)^2}{z} - 0 \\
&= \lim_{z \rightarrow \infty} 2 \log(z+1) \frac{1}{z+1} && \text{(By L'Hôpital's Rule)} \\
&= 2 \lim_{z \rightarrow \infty} \frac{1}{z+1} && \text{(By L'Hôpital's Rule)} \\
&= 0,
\end{aligned}$$

Next, note that $f'(z) = \frac{z \log(1 + \frac{1}{z}) - \frac{1}{z} \log(1 + z)}{(z+1)^2}$, and let $q(z) = z \log(1 + \frac{1}{z})$. Observe that $q'(z) = \log(1 + \frac{1}{z}) + \frac{1}{1+z} > 0$, and therefore:

- For $z \in (0, 1)$, $q(z) < q(\frac{1}{z})$, so $f'(z) = \frac{q(z) - q(\frac{1}{z})}{(z+1)^2} < 0$
- For $z > 1$, $q(z) > q(\frac{1}{z})$, so $f'(z) = \frac{q(z) - q(\frac{1}{z})}{(z+1)^2} > 0$
- $f'(1) = \frac{q(1) - q(1)}{(z+1)^2} = 0$

Putting everything together, we conclude from the continuity and differentiability of f that $f(z) < 0$ for all $z > 0$, which completes the proof. $\square$

Lemma 4.5. *Let $x > 0$. Then for any $\lambda \in (0, 1)$,*

$$\frac{\lambda h'(\lambda x)}{h(\lambda x)} > \frac{h'(x)}{h(x)}.$$

Proof. Let $g : \mathbb{R}_{++} \rightarrow \mathbb{R}, g(\lambda) = \frac{\lambda h'(\lambda x)}{h(\lambda x)}$, noting that

$$\begin{aligned}
g'(\lambda) &= \frac{(\lambda x h''(\lambda x) + h'(\lambda x))h(\lambda x) - \lambda x h'(\lambda x)^2}{h(\lambda x)^2} \\
&= \frac{h'(\lambda x)(h(\lambda x) - h'(\lambda x)\lambda x) - \frac{\lambda x h(\lambda x)}{\lambda x(\lambda x+1)}}{h(\lambda x)^2} \\
&= \frac{h'(\lambda x)((\lambda x + 1) \log(\lambda x + 1) - \lambda x \log(\lambda x) - \lambda x(\log(\lambda x + 1) - \log(\lambda x)) - \frac{\lambda x h(\lambda x)}{\lambda x(\lambda x+1)})}{h(\lambda x)^2} \\
&= \frac{h'(\lambda x) \log(\lambda x + 1) - \frac{\lambda x h(\lambda x)}{\lambda x(\lambda x+1)}}{h(\lambda x)^2} \\
&< 0,
\end{aligned}$$

where the last line holds by Lemma 4.4. Then g is strictly decreasing on $\mathbb{R}_{++}$, and in particular for any $\lambda \in (0, 1)$,

$$\frac{\lambda h'(\lambda x)}{h(\lambda x)} = g(\lambda) > g(1) = \frac{h'(x)}{h(x)}$$

□

Proof of 4.3: Observe that

$$F'(x) = \frac{h'(x)(\sum_{j=1}^n h(\lambda_j x)) - h(x)(\sum_{j=1}^n \lambda_j h'(\lambda_j x))}{(\sum_{j=1}^n h(\lambda_j x))^2}$$

and in particular,

$$\begin{aligned}
F'(x) \leq 0 &\iff h'(x) \left(\sum_{j=1}^n h(\lambda_j x) \right) \leq h(x) \left(\sum_{j=1}^n \lambda_j h'(\lambda_j x) \right) \\
&\iff \frac{h'(x)}{h(x)} \leq \frac{\sum_{j=1}^n \lambda_j h'(\lambda_j x)}{\sum_{j=1}^n h(\lambda_j x)}.
\end{aligned}$$

But by Corollary 2.2 and Lemma 4.5

$$\frac{h'(x)}{h(x)} \leq \min_{j \in [n]} \left\{ \frac{\lambda_j h'(\lambda_j x)}{h(\lambda_j x)} \right\} \leq \frac{\sum_{j=1}^n \lambda_j h'(\lambda_j x)}{\sum_{j=1}^n h(\lambda_j x)},$$

so $F'(x) \leq 0$ and therefore $F(1) \geq F(x)$ for all $x \geq 1$. □

4.1.2 Result and consequences

Having developed a sufficient amount of mathematical machinery, we now state the main asymptotic result:

Theorem 4.6. Let $(\alpha^k, \beta^k)_{k=1}^\infty$ be a sequence of positive integral vectors whose dimensions may change but where for all $k \geq 1$, $\|\alpha^k\|_1 = N_k = \|\beta^k\|_1$. If $\|\frac{\alpha^k}{N_k}\|_\infty \rightarrow 0$ and $\|\frac{\beta^k}{N_k}\|_\infty \rightarrow 0$ as $k \rightarrow \infty$, then

$$\frac{\log(\text{CT}(\alpha^k, \beta^k))}{\log(\text{Cap}_{\alpha^k \beta^k}(P_\infty))} \rightarrow 1 \quad \text{as } k \rightarrow \infty$$

Proof. Recall that

$$h(x) = \log \left(\frac{(x+1)^{x+1}}{x^x} \right).$$

Taking the log of both sides of (4.1) and dividing by $\log(\text{Cap}_{\alpha^k \beta^k}(P_K))$ gives

$$1 \geq \frac{\log(\text{CT}_K(\alpha^k \beta^k))}{\log(\text{Cap}_{\alpha^k \beta^k}(P_K))} \geq 1 - \frac{\sum_{i=2}^{m_k} h(\alpha_i^k) + \sum_{j=1}^{n_k} h(\beta_j^k)}{\log(\text{Cap}_{\alpha^k \beta^k}(P_K))}$$

Now using Proposition 3.16 and plugging in $\alpha^k \beta^{k^T} / N_k$ to lower bound $\text{Cap}_{\alpha^k \beta^k}(P_K)$, we get

$$1 \geq \frac{\log(\text{CT}_K(\alpha^k \beta^k))}{\log(\text{Cap}_{\alpha^k \beta^k}(P_K))} \geq 1 - \frac{\sum_{i=2}^{m_k} h(\alpha_i^k) + \sum_{j=1}^{n_k} h(\beta_j^k)}{\sum_{i=1}^{m_k} \sum_{j=1}^{n_k} h\left(\frac{\alpha_i^k \beta_j^k}{N_k}\right)} \quad (4.2)$$

Observe that by Corollary 2.2 and Lemma 4.3,

$$\begin{aligned} \frac{\sum_{i=2}^{m_k} h(\alpha_i^k)}{\sum_{i=1}^{m_k} \sum_{j=1}^{n_k} h\left(\frac{\alpha_i^k \beta_j^k}{N_k}\right)} &\leq \max_{i \in [m_k]} \left\{ \frac{h(\alpha_i^k)}{\sum_{j=1}^{n_k} h\left(\frac{\alpha_i^k \beta_j^k}{N_k}\right)} \right\} && \text{(By Corollary 2.2)} \\ &\leq \frac{h(1)}{\sum_{j=1}^{n_k} h\left(\frac{\beta_j^k}{N_k}\right)} && \text{(By Lemma 4.3)} \\ &\leq \frac{h(1)}{\sum_{j=1}^{n_k} \frac{\beta_j^k}{N_k} h'\left(\frac{\beta_j^k}{N_k}\right)} && \text{(By Lemma 2.8)} \\ &\leq \frac{h(1)}{h'\left(\sum_{j=1}^{n_k} \left(\frac{\beta_j^k}{N_k}\right)^2\right)} && \text{(By Lemma 2.6)} \\ &= \frac{h(1)}{h'\left(\left\|\frac{\beta^k}{N_k}\right\|_2^2\right)} \end{aligned}$$

Now by Hölder's inequality, $\left\|\frac{\beta^k}{N_k}\right\|_2^2 \leq \left\|\frac{\beta^k}{N_k}\right\|_1 \left\|\frac{\beta^k}{N_k}\right\|_\infty = 1 \left\|\frac{\beta^k}{N_k}\right\|_\infty$. By hypothesis, $\left\|\frac{\beta^k}{N_k}\right\|_\infty \rightarrow 0$ as $k \rightarrow \infty$, so taking the limit as $k \rightarrow \infty$ and applying the squeeze theorem gives $\left\|\frac{\beta^k}{N_k}\right\|_2^2 \rightarrow 0$

0. Since $h'(x) \rightarrow \infty$ as $x \rightarrow 0^+$,

$$0 \leq \lim_{k \rightarrow \infty} \frac{\sum_{i=2}^{m_k} h(\alpha_i^k)}{\sum_{i=1}^{m_k} \sum_{j=1}^{n_k} h\left(\frac{\alpha_i^k \beta_j^k}{N_k}\right)} \leq \lim_{k \rightarrow \infty} \frac{h(1)}{h'\left(\left\|\frac{\beta^k}{N_k}\right\|_2^2\right)} = 0,$$

so this fraction goes to 0 as $k \rightarrow \infty$.

Applying the same approach but swapping α with β shows

$$\lim_{k \rightarrow \infty} \frac{\sum_{j=1}^{n_k} h(\beta_j^k)}{\sum_{i=1}^{m_k} \sum_{j=1}^{n_k} h\left(\frac{\alpha_i^k \beta_j^k}{N_k}\right)} = 0,$$

which completes the proof. $\square$

Remark 4.7. Colloquially, $\|\beta_j^k/N_k\|_\infty \rightarrow 0$ provided the dimensions go to infinity and no finite subset of components dominates the remaining terms as $k \rightarrow \infty$. Therefore, Theorem 4.6 shows how, as long as the marginals are dominated by the total sum, the bound provided in Theorem 4.1 is log-asymptotically correct.

Remark 4.8. Note that the condition give in Theorem 4.6 is sufficient but not necessary. For example, it can be shown using integral approximations that if $\alpha^k, \beta^k \in \mathbb{R}^k, \alpha_i^k = \beta_i^k = 2^i$, then the log-asymptotics remain correct.

Remark 4.9. The proof of Theorem 4.6 uses a lower bound on the capacity to show the log-asymptotics is correct for the actual optimal value of the capacity. It does not show that this lower bound on the capacity, when used in a lower bound on $\text{CT}(\alpha^k, \beta^k)$, is log-asymptotically correct. This will be discussed later in Section 4.3.

Using this result, we can derive another condition that may be more useful depending on the type of table.

Corollary 4.10. Let $(\alpha^k, \beta^k)_{k=1}^\infty$ be a sequence of positive integral vectors where $\|\alpha^k\|_1 = N_k = \|\beta^k\|_1$, and let $\delta^k = \overline{\alpha^k} \mathbf{1} - \beta^k$. Suppose that $\|\alpha^k/N_k\|_\infty \rightarrow 0$ and $\|\delta^k/N_k\|_\infty \rightarrow 0$ as $k \rightarrow \infty$. Then

$$\frac{\log(\text{CT}(\alpha^k, \beta^k))}{\log(\text{Cap}_{\alpha^k \beta^k}(P_\infty))} \rightarrow 1 \quad \text{as } k \rightarrow \infty$$

Proof. Observe that as $k \rightarrow \infty$,

$$\left\|\frac{\beta^k}{N_k}\right\|_\infty = \left\|\frac{\overline{\alpha^k} \mathbf{1} + (\beta^k - \overline{\alpha^k} \mathbf{1})}{N_k}\right\|_\infty \leq \left\|\frac{\overline{\alpha^k} \mathbf{1}}{N_k}\right\|_\infty + \left\|\frac{\delta^k}{N_k}\right\|_\infty \leq \frac{\overline{\alpha^k}}{N_k} + \frac{\|\delta^k\|_\infty}{N_k} \rightarrow 0$$

Therefore, by Theorem 4.6,

$$\frac{\log(\text{CT}(\alpha^k, \beta^k))}{\log(\text{Cap}_{\alpha^k \beta^k}(P_\infty))} \rightarrow 1 \quad \text{as } k \rightarrow \infty$$

$\square$

If we restrict the case where we have uniform rows, then we have a second set of conditions for correct log-asymptotics.

Proposition 4.11. *Let $(\alpha^k, \beta^k)_{k=1}^\infty$ be a sequence of positive integral vectors where $\alpha^k = \bar{\alpha} \mathbf{1} \in \mathbb{Z}^{m_k}$, $\beta^k \in \mathbb{Z}^{n_k}$, and suppose that*

$$\frac{h(\bar{\alpha}^k)}{\sum_{j=1}^{n_k} h\left(\frac{\beta_j^k}{m_k}\right)} \rightarrow 0, m_k \rightarrow \infty \quad \text{as } k \rightarrow \infty.$$

Then

$$\frac{\log(\text{CT}(\alpha^k, \beta^k))}{\log(\text{Cap}_{\alpha^k \beta^k}(P_\infty))} \rightarrow 1 \quad \text{as } k \rightarrow \infty$$

Proof. First, note that by Proposition 3.22,

$$\log(\text{Cap}_{\alpha^k \beta^k}(P_\infty)) = \sum_{i=1}^{m_k} \sum_{j=1}^{n_k} h\left(\frac{\alpha_i^k \beta_j^k}{N_k}\right) = m_k \sum_{j=1}^{n_k} h\left(\frac{\beta_j^k}{m_k}\right). \quad (4.3)$$

As in the proof of Theorem 4.6, taking the log of both sides of (4.1) and dividing by $\log(\text{Cap}_{\alpha^k \beta^k}(P_\infty))$ gives

$$1 \geq \frac{\log(\text{CT}_K(\alpha^k \beta^k))}{\log(\text{Cap}_{\alpha^k \beta^k}(P_\infty))} \geq 1 - \frac{\sum_{i=2}^{m_k} h(\alpha_i^k) + \sum_{j=1}^{n_k} h(\beta_j^k)}{\log(\text{Cap}_{\alpha^k \beta^k}(P_\infty))}. \quad (4.4)$$

After plugging in for $\log(\text{Cap}_{\alpha^k \beta^k}(P_\infty))$ using 4.3, there are two fractions to consider:

$$\frac{\sum_{i=2}^{m_k} h(\alpha_i^k)}{m_k \sum_{j=1}^{n_k} h\left(\frac{\beta_j^k}{m_k}\right)}, \quad \text{and} \quad (i)$$

$$\frac{\sum_{j=1}^{n_k} h(\beta_j^k)}{m_k \sum_{j=1}^{n_k} h\left(\frac{\beta_j^k}{m_k}\right)} \quad (ii)$$

Simplifying (i) gives

$$\frac{m_k h(\bar{\alpha}^k)}{m_k \sum_{j=1}^{n_k} h\left(\frac{\beta_j^k}{m_k}\right)} = \frac{h(\bar{\alpha}^k)}{\sum_{j=1}^{n_k} h\left(\frac{\beta_j^k}{m_k}\right)}$$

which limits to 0 by assumption.

Simplifying (ii) using Fact 2.1, Lemma 4.3, and Lemma 2.11,

$$0 \leq \frac{\sum_{j=1}^{n_k} h(\beta_j^k)}{m_k \sum_{j=1}^{n_k} h\left(\frac{\beta_j^k}{m_k}\right)} \leq \max_{j \in [n_k]} \left\{ \frac{h(\beta_j)}{m_k h\left(\frac{\beta_j^k}{m_k}\right)} \right\} \leq \frac{h(1)}{m_k h\left(\frac{1}{m_k}\right)} = \frac{h(1)}{h(m_k)}.$$

Since $m_k \rightarrow \infty$, h is continuous (Lemma 2.4), and $\lim_{x \rightarrow \infty} h(x) = \infty$ (Lemma 2.10), this expression limits to 0 as $k \rightarrow \infty$. $\square$

While the condition given in Proposition 4.11 is not the simplest condition, it allows for a short proof of multiple simpler conditions, as shown below:

Corollary 4.12. *Let $(\alpha^k, \beta^k)_{k=1}^\infty$ be a sequence of positive integral vectors where $\|\alpha^k\|_1 = N_k = \|\beta^k\|_1$, $\alpha^k = \bar{\alpha}^k \mathbf{1}$, and $m_k \rightarrow \infty$. For any $\varepsilon > 0$, let $S_\varepsilon^k = \{j \in [n_k] : \beta_j^k > \varepsilon m_k\}$. If any of the following conditions hold, then*

$$\frac{\log(\text{CT}(\alpha^k, \beta^k))}{\log(\text{Cap}_{\alpha^k \beta^k}(P_\infty))} \rightarrow 1 \quad \text{as } k \rightarrow \infty :$$

- (i) *There exists $M > 0$ such that $\bar{\alpha}^k < M$ for all $k \geq 1$ and $\|\beta^k/m_k\|_\infty \rightarrow \infty$.*
- (ii) *There exists $M > 0$ such that $\bar{\alpha}^k < M$ for all $k \geq 1$ and there exists some $\varepsilon > 0$ such that $|S_\varepsilon^k| \rightarrow \infty$.*
- (iii) *There exists $\varepsilon > 0$ such that $\bar{\alpha}^k = o(|S_\varepsilon^k|)$ with respect to k .*

Proof.

- (i) Suppose there exists $M > 0$ such that $\bar{\alpha}^k < M$ for all $k \geq 1$ and $\|\beta^k/m_k\|_\infty \rightarrow \infty$. Then by Lemma 2.7 and Lemma 2.10, as $k \rightarrow \infty$,

$$\frac{h(\bar{\alpha}^k)}{\sum_{i=1}^{n_k} h\left(\frac{\beta_i^k}{m_k}\right)} \leq \frac{h(M)}{h\left(\left\|\frac{\beta^k}{m_k}\right\|_\infty\right)} \rightarrow 0,$$

so by Proposition 4.11, the result follows.

- (ii) Suppose there exists $M > 0$ such that $\bar{\alpha}^k < M$ for all $k \geq 1$ and there exists some $\varepsilon > 0$ such that $|S_\varepsilon^k| \rightarrow \infty$. Then by Lemma 2.7 and Lemma 2.10, as $k \rightarrow \infty$,

$$\frac{h(\bar{\alpha}^k)}{\sum_{i=1}^{n_k} h\left(\frac{\beta_i^k}{m_k}\right)} \leq \frac{h(M)}{\sum_{j \in S_\varepsilon^k} h\left(\frac{\beta_j^k}{m_k}\right)} < \frac{h(M)}{|S_\varepsilon^k| h(\varepsilon)} \rightarrow 0,$$

so by Proposition 4.11, the result follows.

- (iii) Suppose there exists $\varepsilon > 0$ such that $\bar{\alpha}^k = o(|S_\varepsilon^k|)$ with respect to k . Then by Lemma 2.7 and Lemma 2.9, as $k \rightarrow \infty$,

$$\frac{h(\bar{\alpha}^k)}{\sum_{i=1}^{n_k} h\left(\frac{\beta_i^k}{m_k}\right)} \leq \frac{h(\bar{\alpha}^k)}{\sum_{j \in S_\varepsilon^k} h\left(\frac{\beta_j^k}{m_k}\right)} < \frac{h(\bar{\alpha}^k)}{|S_\varepsilon^k| h(\varepsilon)} \rightarrow 0,$$

and so by Proposition 4.11 the result follows.

□

Combining conditions in Corollary 4.10 and Proposition 4.11 leads to the following result:

Corollary 4.13. *Let $(\alpha^k, \beta^k)_{k=1}^\infty$ be a sequence of positive integral vectors where $\|\alpha^k\|_1 = N_k = \|\beta^k\|_1$ and $\alpha^k = \overline{\alpha^k} \mathbf{1}$, and let $\delta^k = \overline{\alpha^k} \mathbf{1} - \beta^k$. Suppose that $\|\delta^k\|_\infty / N_k \rightarrow 0$ as $k \rightarrow \infty$ and $m_k \rightarrow \infty$. Then*

$$\frac{\log(\text{CT}(\alpha^k, \beta^k))}{\log(\text{Cap}_{\alpha^k \beta^k}(P_\infty))} \rightarrow 1 \quad \text{as } k \rightarrow \infty.$$

Proof. Since $m_k \rightarrow \infty$ and $N_k = \overline{\alpha^k} m_k$,

$$\left\| \frac{\alpha^k}{N_k} \right\|_\infty = \frac{\overline{\alpha^k}}{N_k} = \frac{1}{m_k} \rightarrow 0 \quad \text{as } k \rightarrow \infty$$

Since $\|\delta^k\|_\infty / N_k \rightarrow 0$ as $k \rightarrow \infty$, by Corollary 4.10

$$\frac{\log(\text{CT}(\alpha^k, \beta^k))}{\log(\text{Cap}_{\alpha^k \beta^k}(P_\infty))} \rightarrow 1 \quad \text{as } k \rightarrow \infty.$$

□

The Independence Heuristic, first introduced by Good in [Goo76], approximates the number of (α, β) -contingency tables by

$$I(\alpha, \beta) = \binom{N + mn - 1}{mn - 1}^{-1} \prod_{i=1}^m \binom{\alpha_i + n - 1}{n - 1} \prod_{j=1}^n \binom{c_j + m - 1}{m - 1}$$

Let $I(\alpha^k, \beta^k)$ be the approximation of $\text{CT}(\alpha^k, \beta^k)$ according to the independence heuristic.

Combining Theorem 4.6 with the analysis of a sequence of contingency tables computed by Barvinok in [Bar09a] shows that there are cases where the Independence Heuristic fails:

Lemma 4.14. *(From section 2.3 in [Bar09a]) Let $r \in \mathbb{R}_+^m, c \in \mathbb{R}_+^n$ be nonuniform marginals with the same sum N and suppose without loss of generality that the entries of r, c are in increasing order. Consider the following sequence of row sum vectors $\alpha^k \in \mathbb{R}^{km}$ and column sum vectors $\beta^k \in \mathbb{R}^{kn}$:*

$$\begin{aligned} \alpha^k &= (kr_1, \dots, kr_1, kr_2, \dots, kr_2, \dots, kr_m, \dots, kr_m) = (kr_1 \mathbf{1}, kr_2 \mathbf{1}, \dots, kr_n \mathbf{1}), \\ \beta^k &= (kc_1, \dots, kc_1, kc_2, \dots, kc_2, \dots, kc_m, \dots, kc_m) = (kc_1 \mathbf{1}, kc_2 \mathbf{1}, \dots, kc_n \mathbf{1}) \end{aligned}$$

Then

$$\lim_{k \rightarrow \infty} \frac{1}{k^2} \log(\text{CT}(\alpha^k, \beta^k)) > \lim_{k \rightarrow \infty} \frac{1}{k^2} \log(I(\alpha^k, \beta^k))$$

Proposition 4.15. *There exist sequences of row and column sums for which the bound of [BLP23] is log-asymptotically correct and for which the independence heuristic is not log-asymptotically correct.*

Proof. Let $r \in \mathbb{R}_+^m, c \in \mathbb{R}_+^n$ with the same sum N where r, c are both non-uniform. Consider the following sequence of row sum vectors $\alpha^k \in \mathbb{R}^{km}$ and column sum vectors $\beta^k \in \mathbb{R}^{kn}$:

$$\begin{aligned}\alpha^k &= (kr_1, \dots, kr_1, kr_2, \dots, kr_2, \dots, kr_m, \dots, kr_m) = (kr_1 \mathbf{1}, kr_2 \mathbf{1}, \dots, kr_m \mathbf{1}), \\ \beta^k &= (kc_1, \dots, kc_1, kc_2, \dots, kc_2, \dots, kc_m, \dots, kc_m) = (kc_1 \mathbf{1}, kc_2 \mathbf{1}, \dots, kc_n \mathbf{1})\end{aligned}$$

Let $I(\alpha^k, \beta^k)$ be the approximation of $\text{CT}(\alpha^k, \beta^k)$ according to the independence heuristic. By Lemma 4.14,

$$\frac{\log(\text{CT}(\alpha^k, \beta^k))}{\log(I(\alpha^k, \beta^k))} \rightarrow L > 1 \quad \text{as } k \rightarrow \infty.$$

However,

$$\left\| \frac{\alpha^k}{N_k} \right\|_\infty \leq \frac{\alpha_m^k}{N_k} \leq \frac{kr_m}{k^2 N} = \frac{r_m}{kN} \rightarrow 0 \quad \text{and} \quad \left\| \frac{\beta^k}{N_k} \right\|_\infty \leq \frac{\beta_n^k}{N_k} \leq \frac{kc_n}{k^2 N} = \frac{c_n}{kN} \rightarrow 0 \quad \text{as } k \rightarrow \infty,$$

so by Theorem 4.6

$$\frac{\log(\text{CT}(\alpha^k, \beta^k))}{\log(\text{Cap}_{\alpha^k \beta^k}(P_K))} \rightarrow 1 \quad \text{as } k \rightarrow \infty$$

□

However, there are scenarios where the independence heuristic is log-asymptotically correct, but our bounds fail.

4.2 Cases where log-asymptotics fail

First, note the following result thanks to [Rob55]:

Lemma 4.16. *Let $n \in \mathbb{Z}_+$. Then*

$$\sqrt{2\pi n} \left(\frac{n}{e}\right)^n e^{\frac{1}{12n+1}} < n! < \sqrt{2\pi n} \left(\frac{n}{e}\right)^n e^{\frac{1}{12n}}$$

Using this result, we can show that there exist cases where our log-asymptotics fail, even when we restrict to the cases where the row sums are uniform:

Proposition 4.17. *There exist sequences of row and column sums such that the bound Theorem 4.1 fails, even when restricting to sequences where the dimensions go to ∞ and the row sums are uniform.*

Proof. Let $\alpha^k = \mathbf{1}$, $\alpha^k \in \mathbb{R}^{2^{k+1}-2}$, and $\beta^k \in \mathbb{R}^k, \beta_j^k = 2^j$ for all $j \in [k]$. This sequence of marginals does not lead to the same log asymptotics between $\text{CT}(\alpha^k, \beta^k)$ and $\text{Cap}_{\alpha^k, \beta^k}(P_\infty)$. To show this, note first that there is a bijection between contingency tables satisfying

marginals α^k, β^k and an assignment of each row $i = 1, \dots, 2^{k+1} - 2$ to one of k containers, where the container S_j can fit 2^j elements, $j = 1, \dots, n$. As a result, the multinomial expression gives

$$\text{CT}(\alpha^k, \beta^k) = \frac{(2^{k+1} - 2)!}{\prod_{j=1}^k (2^j)!}.$$

Taking the logarithm of both sides and applying Lemma 4.16 provides

$$\begin{aligned} \log(\text{CT}(\alpha^k, \beta^k)) &\leq \log \left(\sqrt{2\pi(2^{k+1} - 2)} \left(\frac{(2^{k+1} - 2)}{e} \right)^{(2^{k+1} - 2)} e^{\frac{1}{12(2^{k+1} - 2)}} \right) \\ &\quad - \sum_{j=1}^k \log \left(\sqrt{2\pi 2^j} \left(\frac{2^j}{e} \right)^{2^j} e^{\frac{1}{12 \cdot 2^j + 1}} \right) \\ &= \frac{1}{2} \log(2\pi) + \left((2^{k+1} - 2) + \frac{1}{2} \right) \log(2^{k+1} - 2) - (2^{k+1} - 2) \\ &\quad + \frac{1}{12(2^{k+1} - 2)} \\ &\quad - \sum_{j=1}^k \left[\frac{1}{2} \log(2\pi) + \left(2^j + \frac{1}{2} \right) \log(2^j) - 2^j + \frac{1}{12 \cdot 2^j + 1} \right] \\ &= -\frac{1}{2}(k-1) \log(2\pi) + \left(2^{k+1} - \frac{3}{2} \right) \log(2^{k+1} - 2) \\ &\quad - \sum_{j=1}^k \left(2^j + \frac{1}{2} \right) j \log(2) + \frac{1}{12(2^{k+1} - 2)} - \sum_{j=1}^k \frac{1}{12 \cdot 2^j + 1} \\ &< -\frac{1}{2}(k-1) \log(2\pi) + \left(2^{k+1} - \frac{3}{2} \right) (k+1) \log(2) \\ &\quad - \log(2) \sum_{j=1}^k 2^j j - \frac{1}{2} \log(2) \sum_{j=1}^k j. \end{aligned}$$

Now

$$\sum_{j=1}^k j = \frac{1}{2} k(k+1),$$

$$\sum_{j=1}^k 2^j j = \sum_{j=1}^k \sum_{\ell=j}^n 2^j = \sum_{j=1}^k (2^{k+1} - 2^j) = k 2^{k+1} - (2^{k+1} - 2) = (k-1) 2^{k+1} + 2,$$

so simplifying the last two terms gives

$$\begin{aligned}
\log(\text{CT}(\alpha^k, \beta^k)) &< -\frac{1}{2}(k-1)\log(2\pi) + \left(2^{k+1} - \frac{3}{2}\right)(k+1)\log(2) \\
&\quad - \log(2)(k-1)2^{k+1} + 2\log(2) - \frac{1}{4}\log(2)k(k+1) \\
&= -\frac{1}{2}(k-1)\log(2\pi) + 2 \cdot 2^{k+1}\log(2) - \frac{3}{2}(k+1)\log(2) \\
&\quad + 2\log(2) - \frac{1}{4}\log(2)k(k+1) \\
&= 2^{k+1}[-2^{-k-2}(k-1)\log(2\pi) + 2\log(2) - 3 \cdot 2^{-k-2}(k+1)\log(2) \\
&\quad + 2^{-k}\log(2) - 2^{-k-3}\log(2)k(k+1)].
\end{aligned}$$

Taking the limit as $k \rightarrow \infty$ of the expression inside the square brackets gives $2\log(2)$.

On the other side, using the fact that $\int \log(x) = x\log(x) - x + C$, we calculate that for any constant c ,

$$\begin{aligned}
\int_{j=0}^k h(c2^j) dj &= \int_{j=1}^k c2^j \log(c2^j + 1) - c2^j \log(c2^j) + \log(c2^j + 1) dj \\
&= \frac{1}{\log(2)} \left((c2^j + 1) \log(c2^j + 1) - (c2^j + 1) \right) \Big|_{j=0}^k \\
&\quad + \frac{1}{\log(2)} \left(c2^j \log(c2^j) - c2^j \right) \Big|_{j=0}^k + \int_{j=0}^k \log(c2^j + 1) dj \\
&= \frac{1}{\log(2)} \left((c2^k + 1) \log(c2^k + 1) - (c + 1) \log(c + 1) - c2^k \log(c2^k) + c \log(c) \right) \\
&\quad + \int_{j=0}^k \log(c2^j + 1) dj.
\end{aligned}$$

Next, note that for any $x \in [0, 1]$, $2\log(x+1) > x$ since $2\log(x+1) = x$ when $x = 0$ and for $x \in [0, 1]$, $\frac{d}{dx}(2\log(x+1)) = \frac{2}{x+1} \geq 1 = \frac{d}{dx}(x)$. Then for any $j \in [0, k]$, $\log(2^{j-k-1} + 1) \geq 2^{j-k-2}$; hence

$$\int_{j=0}^k \log((2^{k-1})2^j + 1) dj > \int_{j=0}^k 2^{j-k-2} dj = \frac{2^{-k-2}}{\log(2)} (2^k - 2^0) = \frac{1}{\log(2)} \left(\frac{1}{4} - 2^{-k-2} \right)$$

Applying these two expressions and setting $c = 2^{-k-1}$, we get

$$\begin{aligned}
\log(\text{Cap}_{\alpha^k \beta^k}(P_\infty)) &= (2^{k+1} - 2) \sum_{j=1}^k h\left(\frac{2^j}{2^{k+1} - 2}\right) \\
&\geq (2^{k+1} - 2) \int_0^k h(2^{j-k-1}) dj \\
&> \frac{(2^{k+1} - 2)}{\log(2)} \left[\frac{3}{2} \log\left(\frac{3}{2}\right) - (2^{-k-1} + 1) \log(2^{-k-1} + 1) - \frac{1}{2} \log\left(\frac{1}{2}\right) \right. \\
&\quad \left. + 2^{-k-1} \log(2^{-k-1}) + \frac{1}{4} - 2^{-k-1} \right], \\
&> \frac{(2^{k+1} - 2)}{\log(2)} \left[h\left(\frac{1}{2}\right) - h(2^{-k-1}) + \frac{1}{4} - 2^{-k-1} \right],
\end{aligned}$$

where taking the limit as $k \rightarrow \infty$ of the expression inside the square brackets gives $h\left(\frac{1}{2}\right) + \frac{1}{4}$.

Putting this together, we get

$$\lim_{k \rightarrow \infty} \frac{\log(\text{CT}(\alpha^k, \beta^k))}{\log(\text{Cap}_{\alpha^k \beta^k}(P_K))} \leq \frac{2 \log(2)}{h\left(\frac{1}{2}\right) + \frac{1}{4}} \cdot \lim_{k \rightarrow \infty} \frac{2^{k+1} \log(2)}{2^{k+1} - 2} = \frac{2 \log(2)^2}{h\left(\frac{1}{2}\right) + \frac{1}{4}} \approx 0.79758 < 1,$$

as required. $\square$

4.3 Log-asymptotics for explicit lower bounds

Theorem 4.6 provides a simple condition under which the bounds of Theorem 4.1 are guaranteed to be log-asymptotically correct. However, this does not address whether or not the point $\alpha\beta^T/N$, used in the proof of Theorem 4.6 as a feasible point for (Cap-Sup) whose objective value provides a lower bound on $\text{Cap}_{\alpha\beta}(P_\infty)$, could also be used to derive log-asymptotically correct lower bounds on $\text{CT}(\alpha, \beta)$. If this explicit lower bound is also log-asymptotically correct (even only in certain cases), then this would allow for further analysis into what drives the growth of $\text{Cap}_{\alpha\beta}(P_\infty)$ without having to solve a nonlinear program. We already know $\alpha\beta^T/N$ is the optimizer of (Cap-Sup) when the row sums are uniform, so we only consider the general case.

Let's return to the inequality (4.1) after replacing a_i, b_j with α_i, β_j respectively (see Remark 4.2):

$$\text{Cap}_{\alpha\beta}(P_K) \geq \text{CT}_K(\alpha, \beta) \geq \left[\prod_{i=2}^m \frac{\alpha_i^{\alpha_i}}{(\alpha_i + 1)^{\alpha_i+1}} \prod_{j=1}^n \frac{\beta_j^{\beta_j}}{(\beta_j + 1)^{\beta_j+1}} \right] \text{Cap}_{\alpha\beta}(P_K), \quad (4.1)$$

As usual, we set $P_K = P_\infty$. Let $\hat{X}^k$ be the optimal solution to (Cap-Sup). Then $\text{Cap}_{\alpha^k \beta^k}(P_K) = e^{H(\hat{X}^k)}$. This time, however, we let $X^k \in \mathbb{R}_+^{m \times n}$ be an arbitrary feasible point for (Cap-Sup) so that our lower bound becomes

$$e^{H(\hat{X}^k)} \geq \text{CT}_K(\alpha, \beta) \geq \left[\prod_{i=2}^m \frac{\alpha_i^{\alpha_i}}{(\alpha_i + 1)^{\alpha_i + 1}} \prod_{j=1}^n \frac{\beta_j^{\beta_j}}{(\beta_j + 1)^{\beta_j + 1}} \right] e^{H(X^k)}, \quad (4.5)$$

Taking the logarithm of the inequalities and dividing by $H(X^k)$ we get

$$\frac{H(\hat{X}^k)}{H(X^k)} \geq \frac{\log(\text{CT}_K(\alpha^k, \beta^k))}{H(X^k)} \geq 1 - \frac{\sum_{i=2}^m h(\alpha_i^k) + \sum_{j=1}^n h(\beta_j^k)}{H(X^k)}$$

We will show that this lower bound is log-asymptotically correct under certain conditions. To do this, we provide a characterization for $\hat{X}^k$ being optimal.

Recall the first order concavity condition:

Proposition 4.18. (Section 3.1.3 of [BV04] modified for concavity) Suppose f is differentiable. Then f is concave if and only if $\mathbf{dom} f$ is convex and $f(y) \leq f(x) + \nabla f(x)^T(y - x)$ for all $x, y \in \mathbf{dom} f$.

This leads to the following result:

Lemma 4.19. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a concave differentiable function, $P \subseteq \mathbb{R}^n$ be a convex set, $x^* \in P$, and $S \supseteq P$ be a subspace containing $P - \{x^*\}$. x^* maximizes f if $\text{proj}_S(\nabla f(x^*)) = 0$.

Proof. By Proposition 4.18, for any $x \in P$,

$$f(x) \leq f(x^*) + \nabla f(x^*)^T(x - x^*) = f(x^*) + \text{proj}_S(\nabla f(x^*))^T(x - x^*) = f(x^*),$$

so by definition x^* maximizes f . □

Proposition 4.20. Let A be the coefficient matrix corresponding to a transportation polytope in standard equality form (i.e. $T_{\alpha\beta} = \{x \in \mathbb{R}^{m \times n} : Ax = [\alpha \ \beta]^T\}$) and let $S = \text{Null}(A)$. Then for any $Y \in \mathbb{R}^{m \times n}$,

$$\text{proj}_S(Y) = \left(I - \frac{1}{m} J \right)^T Y \left(I - \frac{1}{n} J \right)$$

Proof. Let $S = \text{Null}(A)$. By the Fundamental Theorem of Linear Algebra, $\text{Row}(A)^\perp = \text{Null}(A) = S$, so for any $Y \in \mathbb{R}^{m \times n}$,

$$\text{proj}_S(Y) = Y - \text{proj}_{S^\perp}(Y)$$

Let $R'_i, C'_j \in \mathbb{R}^{m \times n}$ correspond to the i th row and j th column constraint coefficient vector in A , noting that for standard basis vectors $e_1, \dots, e_n$, $R'_i = [e_i \ \cdots \ e_i]$, $C'_j = [e_j \ \cdots \ e_j]^T$.

Since R'_i, C'_j form a basis of $\text{Row}(A)$, we can apply the Gram-Schmidt orthogonalization process to get an orthogonal basis $R_1, \dots, R_m, C_1, \dots, C_{n-1}$, defined as

$$\begin{aligned} R_i &= e_i \mathbf{1}^T, & C_j &= \mathbf{1} \begin{bmatrix} 0 & \cdots & 0 & \frac{(n-j)}{n-(j-1)} & -\frac{1}{n-(j-1)} & \cdots & -\frac{1}{n-(j-1)} \end{bmatrix} \\ \|R_i\|_F^2 &= \frac{1}{n} & \|C_j\|_F^2 &= m \frac{(n-j)^2}{(n-(j-1))^2} + m(n-(j-1)) \frac{1}{(n-(j-1))^2} = \frac{m(n-j)}{n-(j-1)} \end{aligned}$$

Then for any $Y \in \mathbb{R}^{m \times n}$,

$$\begin{aligned} \text{proj}_{R_s}(Y) &= \left(\frac{1}{n} \sum_{\ell=1}^n Y[s, \ell] \right) R_s = \overline{Y[s, :]} R_s \\ \text{proj}_{C_t}(Y) &= \left(\frac{1}{m} \sum_{k=1}^m Y[k, t] - \frac{1}{m(n-j)} \sum_{\ell=t+1}^n \sum_{k=1}^m Y[k, \ell] \right) C_t \\ &= \left(\overline{Y[:, t]} - \overline{Y[:, (t+1) : n]} \right) C_t \end{aligned}$$

$$\begin{aligned} \left[\sum_{s=1}^m \text{proj}_{R_s}(Y) \right]_{ij} &= \overline{Y[i, :]} \\ \left[\sum_{t=1}^{n-1} \text{proj}_{C_t}(Y) \right]_{ij} &= \sum_{\ell=1}^{j-1} \frac{-1}{n-(\ell-1)} \left(\overline{Y[:, \ell]} - \overline{Y[:, (\ell+1) : n]} \right) \\ &\quad + \left(\frac{n-j}{n-(j-1)} \right) \left(\overline{Y[:, j]} - \overline{Y[:, (j+1) : n]} \right) \\ &= \sum_{\ell=1}^{j-1} \frac{-1}{n-(\ell-1)} \left(\overline{Y[:, \ell]} - \frac{1}{n-\ell} \sum_{v=\ell+1}^n \overline{Y[:, v]} \right) \\ &\quad + \left(\frac{n-j}{n-(j-1)} \right) \left(\overline{Y[:, j]} - \frac{1}{n-j} \sum_{v=j+1}^n \overline{Y[:, v]} \right) \\ &= \sum_{\ell=1}^{j-1} \overline{Y[:, \ell]} \left(\frac{-1}{n-(\ell-1)} + \sum_{u=1}^{\ell-1} \frac{1}{(n-u)(n-(u-1))} \right) \\ &\quad + \overline{Y[:, j]} \left(\sum_{u=1}^{j-1} \frac{1}{(n-u)(n-(u-1))} - \frac{n-j}{n-(j-1)} \right) \\ &\quad + \sum_{\ell=j+1}^n \overline{Y[:, \ell]} \left(\sum_{u=1}^{j-1} \frac{1}{(n-u)(n-(u-1))} - \frac{n-j}{(n-(j-1))(n-j)} \right) \end{aligned}$$

Using the telescoping sum

$$\sum_{p=1}^k \frac{1}{(n-p)(n-(p-1))} = \sum_{p=1}^k \left(\frac{1}{n-p} - \frac{1}{n-(p-1)} \right) = \frac{1}{n-k} - \frac{1}{n} = \frac{k}{n(n-k)},$$

we deduce that

$$\begin{aligned}
\left[\sum_{t=1}^{n-1} \text{proj}_{C_t}(Y) \right]_{ij} &= \sum_{\ell=1}^{j-1} \overline{Y[:, \ell]} \left(\frac{-1}{n - (\ell - 1)} + \sum_{u=1}^{\ell-1} \frac{1}{(n - u)(n - (u - 1))} \right) \\
&\quad + \overline{Y[:, j]} \left(\sum_{u=1}^{j-1} \frac{1}{(n - u)(n - (u - 1))} - \frac{n - j}{n - (j - 1)} \right) \\
&\quad + \sum_{\ell=j+1}^n \overline{Y[:, \ell]} \left(\sum_{u=1}^{j-1} \frac{1}{(n - u)(n - (u - 1))} - \frac{n - j}{(n - (j - 1))(n - j)} \right) \\
&= \sum_{\ell=1}^{j-1} \overline{Y[:, \ell]} \left(\frac{-1}{n - (\ell - 1)} + \frac{\ell - 1}{n(n - (\ell - 1))} \right) \\
&\quad + \overline{Y[:, j]} \left(\frac{j - 1}{n(n - (j - 1))} - \frac{n - j}{n - (j - 1)} \right) \\
&\quad + \sum_{\ell=j+1}^n \overline{Y[:, \ell]} \left(\frac{j - 1}{n(n - (j - 1))} - \frac{1}{(n - (j - 1))} \right) \\
&= \sum_{\ell=1}^{j-1} \overline{Y[:, \ell]} \left(\frac{-1}{n} \right) + \overline{Y[:, j]} \left(\frac{n - 1}{n} \right) + \sum_{\ell=j+1}^n \overline{Y[:, \ell]} \left(\frac{-1}{n} \right) \\
&= \frac{(n - 1)}{n} \overline{Y[:, j]} - \frac{1}{n} \sum_{\substack{\ell=1 \\ \ell \neq j}}^n \overline{Y[:, \ell]} \\
&= \overline{Y[:, j]} - \overline{Y}
\end{aligned}$$

Then

$$\begin{aligned}
[\text{proj}_S(Y)]_{ij} &= Y_{ij} - \left(\overline{Y[i, :]} \right) - \left(\overline{Y[:, j]} - \overline{Y} \right) \\
&= \left(e_i - \frac{1}{m} \mathbf{1} \right)^T Y \left(e_j - \frac{1}{n} \mathbf{1} \right) \\
&= \frac{(n - 1)}{n} \left(Y[i, j] - \overline{Y[:, j]} \right) - \frac{1}{n} \sum_{\substack{\ell=1 \\ \ell \neq j}}^n \left(Y[i, \ell] - \overline{Y[:, \ell]} \right)
\end{aligned}$$

so

$$\text{proj}_S(Y) = \left(I - \frac{1}{m} J \right)^T Y \left(I - \frac{1}{n} J \right)$$

□

Note that

$$\begin{aligned}
\|\text{proj}_S(Y)\|_F^2 &= \text{tr} \left(\left(I - \frac{1}{m} J \right)^T Y \left(I - \frac{1}{n} J \right) \left(I - \frac{1}{m} J \right)^T Y^T \left(I - \frac{1}{n} J \right) \right) \\
&= \text{tr} \left(Y \left(I - \frac{1}{n} J \right) Y^T \left(I - \frac{1}{m} J \right) \right) \\
&\leq \left\| Y \left(I - \frac{1}{n} J \right) \right\|_F \left\| Y^T \left(I - \frac{1}{m} J \right) \right\|_F \\
&= \sqrt{\sum_{i=1}^m \sum_{j=1}^n \left(Y[i, j] - \overline{Y[i, :]} \right)^2} \sqrt{\sum_{i=1}^m \sum_{j=1}^n \left(Y[i, j] - \overline{Y[:, j]} \right)^2}
\end{aligned}$$

Since $\hat{X}^k$ lies in the relative interior of the transportation polytope, $\text{proj}_S(\nabla H(\hat{X}^k)) = 0$.

Theorem 4.21. *Let (α^k, β^k) be a sequence of row and column marginals satisfying $\|\alpha^k\|_1 = \|\beta^k\|_1 = N_k$. Let $\hat{X}^k$ be the optimizer of (Cap-Sup). Suppose that $\|\hat{X}^k\|_\infty$ is bounded above by M as $k \rightarrow \infty$ and*

$$\frac{\|\alpha^k\|_\infty \|\beta^k\|_\infty}{N_k} \rightarrow 0, \text{ as } k \rightarrow \infty.$$

Then the bounds of (4.5) are log-asymptotically correct for $X^k = \alpha^k \beta^{kT} / N_k$.

In order to prove this result, we use the following results on μ -strongly-concave functions.

Definition 4.22. *Let $D \subseteq \mathbb{R}^n$ be a convex set and $f : D \rightarrow \mathbb{R}$ be a differentiable function. f is μ -strongly concave if $\nabla^2 f(x) \preceq \mu I$.*

Remark 4.23. *Here, we assume $\mu \leq 0$.*

The following lemma follows directly from applying the definition of μ -strongly convex to the second-order Taylor's Theorem.

Lemma 4.24. *[Section 9.1.2, page 459, of [BV04]] Let $D \subseteq \mathbb{R}^n$ be a convex set and $f : D \rightarrow \mathbb{R}$ be a μ -strongly concave differentiable function. Then for any $x, y \in D$,*

$$f(y) \leq f(x) + \nabla f(x)^T (y - x) + \frac{\mu}{2} \|y - x\|_2^2$$

Using this lemma, we can show the following result.

Lemma 4.25. *[derived from Section 9.1.2, page 460, of [BV04]] Let $D \subseteq \mathbb{R}^n$ be a convex set and $f : D \rightarrow \mathbb{R}$ be a μ -strongly concave differentiable function. Let $x \in D$, let $S \subseteq \mathbb{R}^n$ be a subspace containing $D - \{x\}$, and suppose that $\hat{x}$ is a global maximizer of f on D . Then*

$$f(\hat{x}) \leq f(x) - \frac{1}{2\mu} \|\text{proj}_S(\nabla f(x))\|_2^2$$

Proof. (Derived from Section 9.1.2, page 460, of [BV04]) Let $y \in D$. By Lemma 4.24,

$$\begin{aligned} f(y) &\leq f(x) + \nabla f(x)^T(y - x) + \frac{\mu}{2} \|y - x\|_2^2 \\ &= f(x) + \text{proj}_S(\nabla f(x))^T(y - x) + \frac{\mu}{2} \|y - x\|_2^2 \end{aligned}$$

Observe that the expression on the right side is a concave function with respect to y for fixed x whose gradient is $\text{proj}_S(\nabla f(x)) + \mu(y - x)$. Setting the gradient equal to zero and solving for y , we find the of the right side expression is maximized by $y^* = x - \frac{1}{\mu} \text{proj}_S(\nabla f(x))$. Then

$$\begin{aligned} f(y) &\leq f(x) + \text{proj}_S(\nabla f(x))^T(y - x) + \frac{\mu}{2} \|y - x\|_2^2 \\ &\leq f(x) + \text{proj}_S(\nabla f(x))^T(y^* - x) + \frac{\mu}{2} \|y^* - x\|_2^2 \\ &= f(x) - \frac{1}{\mu} \|\text{proj}_S(\nabla f(x))\|_2^2 + \frac{1}{2\mu} \|\text{proj}_S(\nabla f(x))\|_2^2 \\ &= f(x) - \frac{1}{2\mu} \|\text{proj}_S(\nabla f(x))\|_2^2, \end{aligned}$$

as required. □

Lemma 4.26. *Let $U > 0$. H is $-\frac{1}{U(U+1)}$ -strongly concave on $D = (0, U]^{m \times n}$.*

Proof. Observe that for any $X \in D$, $[\nabla H(X)]_{ij} = h'(X_{ij})$, so

$$[\nabla^2 H(X)]_{ij, k\ell} = \begin{cases} h''(X_{ij}) = -\frac{1}{X_{ij}(X_{ij}+1)} & \text{if } (i, j) = (k, \ell) \\ 0 & \text{otherwise} \end{cases},$$

showing that $\nabla^2 H(X)$ is diagonal and negative definite for any $X \in D$. In particular,

$$\nabla^2 H(X) \preceq -\frac{1}{U(U+1)} I,$$

as required. □

We now have enough mathematical machinery to prove the main result of this section.

Proof of Theorem 4.21. First, note that

$$0 \leq \left\| \frac{\alpha^k}{n_k} \right\|_\infty \leq \frac{\|\alpha^k\|_\infty \|\beta^k\|_\infty}{n_k \beta^k} = \frac{\|\alpha^k\|_\infty \|\beta^k\|_\infty}{N_k} \rightarrow 0$$

and

$$0 \leq \left\| \frac{\beta^k}{m_k} \right\|_\infty \leq \frac{\|\alpha^k\|_\infty \|\beta^k\|_\infty}{\alpha^k m_k} = \frac{\|\alpha^k\|_\infty \|\beta^k\|_\infty}{N_k} \rightarrow 0$$

as $k \rightarrow \infty$, so $\|\alpha^k/n_k\| \rightarrow 0$ and $\|\beta^k/m_k\|_\infty \rightarrow 0$ as $k \rightarrow \infty$.

Observe that for $Y^k = \nabla H(X^k)$, applying Lemma 2.13 gives us

$$\begin{aligned}
Y^k[i, j] - \overline{Y^k[i, :]} &= \log \left(1 + \frac{N_k}{\alpha_i^k \beta_j^k} \right) - \frac{1}{n_k} \sum_{t=1}^{n_k} \log \left(1 + \frac{N_k}{\alpha_i^k \beta_t^k} \right) \\
&\geq \log \left(\frac{N_k}{\alpha_i^k \beta_j^k} \right) - \frac{1}{n_k} \sum_{t=1}^{n_k} \left(\log \left(\frac{N_k}{\alpha_i^k \beta_t^k} \right) + \frac{\alpha_i^k \beta_t^k}{N_k} \right) \\
&= \log(N_k) - \log(\alpha_i^k) - \log(\beta_j^k) - \log(N_k) + \log(\alpha_i^k) + \frac{1}{n_k} \sum_{t=1}^{n_k} \log(\beta_t^k) - \frac{\alpha_i^k}{n_k} \\
&= \frac{1}{n_k} \sum_{t=1}^{n_k} \log(\beta_t^k) - \log(\beta_j^k) - \frac{\alpha_i^k}{n_k},
\end{aligned}$$

$$\begin{aligned}
\overline{Y^k} - \overline{Y^k[:, j]} &= \frac{1}{m_k n_k} \sum_{s=1}^{m_k} \sum_{t=1}^{n_k} \log \left(1 + \frac{N_k}{\alpha_s^k \beta_t^k} \right) - \frac{1}{m_k} \sum_{s=1}^{m_k} \log \left(1 + \frac{N_k}{\alpha_s^k \beta_j^k} \right) \\
&\geq \frac{1}{m_k} \sum_{s=1}^{m_k} \left(\frac{1}{n_k} \sum_{t=1}^{n_k} \log \left(\frac{N_k}{\alpha_s^k \beta_t^k} \right) - \log \left(\frac{N_k}{\alpha_s^k \beta_j^k} \right) - \frac{\alpha_s^k \beta_j^k}{N_k} \right) \\
&= \frac{1}{m_k} \sum_{s=1}^{m_k} \left(\log(N_k) - \log(\alpha_s^k) - \frac{1}{n_k} \sum_{t=1}^{n_k} \log(\beta_t^k) - \log(N_k) + \log(\alpha_s^k) \right. \\
&\quad \left. + \log(\beta_j^k) - \frac{\beta_j^k}{m_k} \right) \\
&= -\frac{1}{n_k} \sum_{t=1}^{n_k} \log(\beta_t^k) + \log(\beta_j^k) - \frac{\beta_j^k}{m_k},
\end{aligned}$$

so

$$\begin{aligned}
[\text{proj}_S(Y^k)]_{ij} &= Y^k[i, j] - \overline{Y^k[i, :]} - \overline{Y^k[:, j]} + \overline{Y^k} \\
&\geq \frac{1}{n_k} \sum_{t=1}^{n_k} \log(\beta_t^k) - \log(\beta_j^k) - \frac{\alpha_i^k}{n_k} - \frac{1}{n_k} \sum_{t=1}^{n_k} \log(\beta_t^k) + \log(\beta_j^k) - \frac{\beta_j^k}{m_k} \\
&= -\left(\frac{\alpha_i^k}{n_k} + \frac{\beta_j^k}{m_k} \right).
\end{aligned}$$

Note that $\alpha_i^k/n_k + \beta_j^k/m_k \rightarrow 0$ since

$$0 \leq \frac{\alpha_i^k}{n_k} + \frac{\beta_j^k}{m_k} \leq \left\| \frac{\alpha^k}{n_k} \right\|_\infty + \left\| \frac{\beta_j^k}{m_k} \right\|_\infty \rightarrow 0.$$

Similarly,

$$\begin{aligned}
Y^k[i, j] - \overline{Y^k[i, :]} &= \log \left(1 + \frac{N_k}{\alpha_i^k \beta_j^k} \right) - \frac{1}{n_k} \sum_{t=1}^{n_k} \log \left(1 + \frac{N_k}{\alpha_i^k \beta_t^k} \right) \\
&\leq \log \left(\frac{N_k}{\alpha_i^k \beta_j^k} \right) + \frac{\alpha_i^k \beta_j^k}{N_k} - \frac{1}{n_k} \sum_{t=1}^{n_k} \log \left(\frac{N_k}{\alpha_i^k \beta_t^k} \right) \\
&= \log(N_k) - \log(\alpha_i^k) - \log(\beta_j^k) + \frac{\alpha_i^k \beta_j^k}{N_k} - \log(N_k) + \log(\alpha_i^k) + \frac{1}{n_k} \sum_{t=1}^{n_k} \log(\beta_t^k) \\
&= \frac{1}{n_k} \sum_{t=1}^{n_k} \log(\beta_t^k) + \frac{\alpha_i^k \beta_j^k}{N_k} - \log(\beta_j^k),
\end{aligned}$$

$$\begin{aligned}
\overline{Y^k} - \overline{Y^k[:, j]} &= \frac{1}{m_k n_k} \sum_{s=1}^{m_k} \sum_{t=1}^{n_k} \log \left(1 + \frac{N_k}{\alpha_s^k \beta_t^k} \right) - \frac{1}{m_k} \sum_{s=1}^{m_k} \log \left(1 + \frac{N_k}{\alpha_s^k \beta_j^k} \right) \\
&\leq \frac{1}{m_k} \sum_{s=1}^{m_k} \left(\frac{1}{n_k} \sum_{t=1}^{n_k} \left(\log \left(\frac{N_k}{\alpha_s^k \beta_t^k} \right) + \frac{\alpha_s^k \beta_t^k}{N_k} \right) - \log \left(\frac{N_k}{\alpha_s^k \beta_j^k} \right) \right) \\
&= \frac{1}{m_k} \sum_{s=1}^{m_k} \left(\log(N_k) - \log(\alpha_s^k) - \frac{1}{n_k} \sum_{t=1}^{n_k} \log(\beta_t^k) + \frac{\alpha_s^k}{n_k} - \log(N_k) \right. \\
&\quad \left. + \log(\alpha_s^k) + \log(\beta_j^k) \right) \\
&= -\frac{1}{n_k} \sum_{t=1}^{n_k} \log(\beta_t^k) + \log(\beta_j^k) + \frac{N_k}{m_k n_k},
\end{aligned}$$

so by Proposition 4.20,

$$\begin{aligned}
[\text{proj}_S(Y^k)]_{ij} &= Y^k[i, j] - \overline{Y^k[i, :]} - \overline{Y^k[:, j]} + \overline{Y^k} \\
&\leq -\frac{1}{n_k} \sum_{t=1}^{n_k} \log(\beta_t^k) + \log(\beta_j^k) + \frac{N_k}{m_k n_k} + \frac{1}{n_k} \sum_{t=1}^{n_k} \log(\beta_t^k) + \frac{\alpha_i^k \beta_j^k}{N_k} - \log(\beta_j^k) \\
&= \frac{N_k}{m_k n_k} + \frac{\alpha_i^k \beta_j^k}{N_k}.
\end{aligned}$$

Since

$$\begin{aligned}
\left\| \frac{\alpha^k}{n_k} \right\|_\infty &\rightarrow 0, \left\| \frac{\beta^k}{m_k} \right\|_\infty \rightarrow 0, \text{ and } \frac{\|\alpha^k\|_\infty \|\beta^k\|_\infty}{N_k} \rightarrow 0 \text{ as } k \rightarrow \infty, \\
0 &\leq \frac{N_k}{m_k n_k} + \frac{\alpha_i^k \beta_j^k}{N_k} \leq \frac{\sum_{s=1}^{m_k} \alpha_s^k}{m_k n_k} + \frac{\|\alpha^k\|_\infty \|\beta^k\|_\infty}{N_k} \leq \frac{\|\alpha^k\|_\infty}{n_k} + \frac{\|\alpha^k\|_\infty \|\beta^k\|_\infty}{N_k} \rightarrow 0;
\end{aligned}$$

thus $N_k/m_k n_k + \alpha_i^k \beta_j^k/N_k \rightarrow 0$.

Since

$$\frac{\alpha_i^k}{n_k} + \frac{\beta_j^k}{m_k} > 0, \frac{N_k}{m_k n_k} + \frac{\alpha_i^k \beta_j^k}{N_k} > 0,$$

it follows that

$$|[\text{proj}_S(Y^k)]_{ij}| \leq \frac{N_k}{m_k n_k} + \frac{\alpha_i^k \beta_j^k}{N_k} + \frac{\alpha_i^k}{n_k} + \frac{\beta_j^k}{m_k}.$$

Then

$$\begin{aligned} \|\text{proj}_S(Y^k)\|_1 &\leq \sum_{i=1}^{m_k} \sum_{j=1}^{n_k} \frac{N_k}{m_k n_k} + \frac{\alpha_i^k \beta_j^k}{N_k} + \frac{\alpha_i^k}{n_k} + \frac{\beta_j^k}{m_k} \\ &= N_k + N_k + N_k + N_k \\ &= 4N_k \end{aligned}$$

and

$$\|\text{proj}_S(Y^k)\|_\infty \leq \frac{N_k}{m_k n_k} + \frac{\|\alpha^k\|_\infty \|\beta^k\|_\infty}{N_k} + \frac{\|\alpha^k\|_\infty}{n_k} + \frac{\|\beta^k\|_\infty}{m_k},$$

and so by Hölder's inequality

$$\begin{aligned} \|\text{proj}_S(Y^k)\|_2^2 &\leq \|\text{proj}_S(Y^k)\|_1 \|\text{proj}_S(Y^k)\|_\infty \\ &= 4N_k \left(\frac{N_k}{m_k n_k} + \frac{\|\alpha^k\|_\infty \|\beta^k\|_\infty}{N_k} + \frac{\|\alpha^k\|_\infty}{n_k} + \frac{\|\beta^k\|_\infty}{m_k} \right) \end{aligned}$$

Next, observe that

$$\begin{aligned} H(X^k) &= \sum_{i=1}^{m_k} \sum_{j=1}^{n_k} h(X_{ij}^k) \\ &\geq \sum_{i=1}^{m_k} \sum_{j=1}^{n_k} \log(X_{ij}^k + 1) \\ &\geq \sum_{i=1}^{m_k} \sum_{j=1}^{n_k} \left(X_{ij}^k - \frac{(X_{ij}^k)^2}{2} \right) && \text{(by Lemma 2.14)} \\ &= \sum_{i=1}^{m_k} \sum_{j=1}^{n_k} \frac{\alpha_i^k \beta_j^k}{N_k} - \frac{1}{2} \left\| \frac{\alpha^k \beta^{kT}}{N_k} \right\|_2^2 \\ &= N_k - \frac{1}{2} \left\| \frac{\alpha^k \beta^{kT}}{N_k} \right\|_2^2 \\ &\geq N_k - \frac{1}{2} N_k \left\| \frac{\alpha^k \beta^{kT}}{N_k} \right\|_\infty && \text{(by Hölder's inequality)} \\ &= N_k \left(1 - \frac{1}{2} \left\| \frac{\alpha^k \beta^{kT}}{N_k} \right\|_\infty \right). \end{aligned}$$

Since $\|\alpha^k\|_\infty \|\beta^k\|_\infty / N_k \rightarrow 0$, for sufficiently large k we can guarantee that $\left\| \frac{\alpha^k \beta^{kT}}{N_k} \right\|_\infty \in (0, 1)$ and thus that $\left(1 - \frac{1}{2} \left\| \frac{\alpha^k \beta^{kT}}{N_k} \right\|_\infty \right) \in (\frac{1}{2}, 1)$.

Let $U_k = \max\{\|\alpha^k\|_\infty \|\beta^k\|_\infty / N_k, M\}$, $\mu_k = -\frac{1}{U_k(U_k+1)}$, noting that $0 < -\frac{1}{\mu_k} = U_k(U_k + 1) = M(M + 1)$ for sufficiently large k since $\|\alpha^k\|_\infty \|\beta^k\|_\infty / N_k \rightarrow 0$. By Lemma 4.25 and Lemma 4.26, since $X^k, \hat{X}^k$ lies in the hypercube $(0, U_k]^{m \times n}$,

$$\begin{aligned} H(\hat{X}^k) &\leq H(X^k) - \frac{1}{2\mu_k} \|\text{proj}_S(\nabla H(X^k))\|_2^2 \\ &= H(X^k) - \frac{1}{2\mu_k} \|\text{proj}_S(Y^k)\|_2^2 \end{aligned}$$

Dividing by $H(X^k)$, for sufficiently large k we get

$$\begin{aligned} \frac{H(\hat{X}^k)}{H(X^k)} &\leq 1 - \frac{1}{2\mu_k} \frac{\|\text{proj}_S(Y^k)\|_2^2}{H(X^k)} \\ &\leq 1 - \frac{1}{2\mu_k} \frac{4N_k \left(\frac{N_k}{m_k n_k} + \frac{\|\alpha^k\|_\infty \|\beta^k\|_\infty}{N_k} + \frac{\|\alpha^k\|_\infty}{n_k} + \frac{\|\beta^k\|_\infty}{m_k} \right)}{N_k \left(1 - \frac{1}{2} \left\| \frac{\alpha^k \beta^{kT}}{N_k} \right\|_\infty \right)} \quad \left(\text{since } \frac{1}{2\mu_k} < 0 \right) \\ &\rightarrow 1 \end{aligned}$$

since

$$\frac{N_k}{m_k n_k} \rightarrow 0, \quad \frac{\|\alpha^k\|_\infty \|\beta^k\|_\infty}{N_k} \rightarrow 0, \quad \frac{\|\alpha^k\|_\infty}{n_k} \rightarrow 0, \quad \frac{\|\beta^k\|_\infty}{m_k} \rightarrow 0, \quad 1 - \frac{1}{2} \left\| \frac{\alpha^k \beta^{kT}}{N_k} \right\|_\infty \rightarrow 1,$$

and for sufficiently large k ,

$$-\frac{1}{2\mu_k} = \frac{1}{2} U_k(U_k + 1) = \frac{1}{2} M(M + 1).$$

Then since

$$0 \leq \left\| \frac{\alpha^k}{N_k} \right\|_\infty \leq \left\| \frac{\alpha^k}{n_k} \right\|_\infty \rightarrow 0, \quad 0 \leq \left\| \frac{\beta^k}{N_k} \right\|_\infty \leq \left\| \frac{\beta^k}{m_k} \right\|_\infty \rightarrow 0,$$

the conditions for Theorem 4.6 are satisfied, and so we conclude that

$$\frac{\log(\text{CT}(\alpha^k, \beta^k))}{H(X^k)} \rightarrow \frac{\log(\text{CT}(\alpha^k, \beta^k))}{\log(\text{Cap}_{\alpha^k \beta^k}(P_K))} \rightarrow 1 \quad \text{as } k \rightarrow \infty.$$

□

The main challenge with applying Theorem 4.21 comes from showing that the sequence of optimizers $\hat{X}^k$ have bounded entries. We suspect, as the following conjecture formalizes, that the other condition on the marginals guarantees boundedness.

Conjecture 4.27. Let (α^k, β^k) be a sequence of row and column marginals satisfying $\|\alpha^k\|_1 = \|\beta^k\|_1 = N_k$. Let $\hat{X}^k$ be the optimizer of (Cap-Sup). Suppose that

$$\frac{\|\alpha^k\|_\infty \|\beta^k\|_\infty}{N_k} \rightarrow 0, \text{ as } k \rightarrow \infty.$$

Then $\|\hat{X}^k\|_\infty$ is bounded.

The following proposition provides evidence towards this conjecture:

Proposition 4.28. Let $\alpha^k = (\alpha_{k+}, 1, \dots, 1) \in \mathbb{Z}_{++}^m, \beta^k = (\beta_{k+}, 1, \dots, 1) \in \mathbb{Z}_{++}^n$ so that $\|\alpha^k\|_1 = \|\beta^k\|_1 = N_k$ and $\alpha_{k+}\beta_{k+}/N_k \rightarrow \infty$ as $k \rightarrow \infty$, and let $\hat{X}^k$ be the optimizer of (Cap-Sup). Then $\|\hat{X}^k\|_\infty$ is bounded.

In order to show this, we use a probabilistic interpretation of $\hat{X}^k$ provided by [Bar09b]:

Lemma 4.29. [Theorem 1.7 in [Bar09b]] Let $\alpha \in \mathbb{Z}_{++}^m, \beta \in \mathbb{Z}_{++}^n$ with $\|\alpha\|_1 = \|\beta\|_1 = N$, let $\hat{X}$ be the optimizer of (Cap-Sup), and let Φ be a random $m \times n$ matrix where each entry is an independent geometric random variable with mean $\mathbb{E}[\Phi_{ij}] = \hat{X}_{ij}$ for all $i \in [m], j \in [n]$. Then the probability mass function of Φ is constant on $T_{\alpha\beta} \cap \mathbb{Z}$.

Proof of Proposition 4.28. Let Φ_{ij}^k be a random $m_k \times n_k$ matrix where each entry is an independent geometric random variable with mean $\mathbb{E}[\Phi_{ij}^k] = \hat{X}_{ij}^k$. Let $U_k = \min\{\alpha_{k+}, \beta_{k+}\}$. For each $\eta = 0, 1, \dots, U_k$, let C_η^k be the number of integral (α^k, β^k) -contingency tables where the $(1, 1)$ -entry is η . We will first derive a formula for C_η^k , then apply it to prove $\hat{X}_{ij}^k = \mathbb{E}[\Phi_{ij}^k] \rightarrow 0$.

Let k be sufficiently large so that $N - \alpha_{k+} - \beta_{k+} > 0$. Note that for any $\eta = 0, 1, \dots, U_k$,

- There are $\binom{n_k - 1}{\alpha_{k+} - \eta}$ ways of distributing the remaining $\alpha_{k+} - \eta$ ones remaining in the first row,
- There are $\binom{m_k - 1}{\beta_{k+} - \eta}$ ways of distributing the remaining $\beta_{k+} - \eta$ ones remaining in the first column, and
- After distributing $\alpha_{k+} - \eta$ across the first row and distributing $\beta_{k+} - \eta$ across the first column, since

$$(m_k - 1) - (\beta_{k+} - \eta) = N_k - \alpha_{k+} - \beta_{k+} + \eta = (n_k - 1) - (\alpha_{k+} - \eta),$$

there remains a square $(N_k - \alpha_{k+} - \beta_{k+} + \eta) \times (N_k - \alpha_{k+} - \beta_{k+} + \eta)$ sized integral contingency table where all row and column sums are all ones. Therefore, this table is a permutation of the identity matrix, of which there are $(N_k - \alpha_{k+} - \beta_{k+} + \eta)!$.

Putting this together, we derive a formula for C_η^k :

$$\begin{aligned} C_\eta^k &= \binom{n_k - 1}{\alpha_{k+} - \eta} \binom{m_k - 1}{\beta_{k+} - \eta} (N - \alpha_{k+} - \beta_{k+} + \eta)! \\ &= \frac{(m_k - 1)!(n_k - 1)!}{(\alpha_{k+} - \eta)!(\beta_{k+} - \eta)!(N - \alpha_{k+} - \beta_{k+} + \eta)!} \end{aligned}$$

Now, applying Corollary 2.2, we calculate that for any $\zeta = 1, \dots, U_k$

$$\mathbb{P}(\Phi_{11}^k = \zeta) = \frac{C_\zeta^k}{\sum_{\eta=0}^{U_k} C_\eta^k} \leq \frac{C_\zeta^k}{\sum_{\eta=0}^{\zeta-1} C_\eta^k} \leq \max_{\eta \in [\zeta]} \left\{ \frac{\frac{1}{\zeta} C_\zeta^k}{C_{\eta-1}^k} \right\}$$

Observe that for any $\eta = 1, \dots, U_k$,

$$\begin{aligned} \frac{C_\eta^k}{C_{\eta-1}^k} &= \frac{(\alpha_{k+} - \eta + 1)(\beta_{k+} - \eta + 1)}{N_k - \alpha_{k+} - \beta_{k+} + \eta} \\ &= \frac{\alpha_{k+}\beta_{k+}}{N_k} \left(\frac{(1 - \frac{\eta-1}{\alpha_{k+}})(1 - \frac{\eta-1}{\beta_{k+}})}{1 - \frac{\alpha_{k+} + \beta_{k+} - \eta}{N_k}} \right) \\ &\leq \frac{\alpha_{k+}\beta_{k+}}{N_k} \left(\frac{1}{1 - \frac{\alpha_{k+} + \beta_{k+}}{N_k}} \right). \end{aligned}$$

Since $\alpha_{k+}\beta_{k+}/N_k \rightarrow 0$, we can guarantee that for sufficiently large k , $\alpha_{k+}\beta_{k+}/N_k < 1/3$ and $(\alpha_{k+} + \beta_{k+})/N_k < 1/3$. Then for sufficiently large k ,

$$\frac{C_\eta^k}{C_{\eta-1}^k} \leq \frac{\alpha_{k+}\beta_{k+}}{N_k} \left(\frac{1}{1 - \frac{\alpha_{k+} + \beta_{k+}}{N_k}} \right) < \frac{1}{3} \left(\frac{1}{\frac{2}{3}} \right) = \frac{1}{2},$$

so for any $0 \leq \eta < \zeta$,

$$\frac{C_\zeta^k}{C_\eta^k} = \prod_{\tau=\eta+1}^{\zeta} \frac{C_\tau^k}{C_{\tau-1}^k} \leq \left(\frac{\alpha_{k+}\beta_{k+}}{N_k} \left(\frac{1}{1 - \frac{\alpha_{k+} + \beta_{k+}}{N_k}} \right) \right)^{\zeta-\eta} \leq \frac{\alpha_{k+}\beta_{k+}}{N_k} \left(\frac{1}{1 - \frac{\alpha_{k+} + \beta_{k+}}{N_k}} \right)$$

Then it follows that for sufficiently large k ,

$$0 \leq \mathbb{P}(\Phi_{11}^k = \zeta) \leq \max_{\eta \in [\zeta]} \left\{ \frac{\frac{1}{\zeta} C_\zeta^k}{C_{\eta-1}^k} \right\} \leq \frac{1}{\zeta} \frac{\alpha_{k+}\beta_{k+}}{N_k} \left(\frac{1}{1 - \frac{\alpha_{k+} + \beta_{k+}}{N_k}} \right)$$

In particular, this means that

$$0 \leq \mathbb{E}[\Phi_{11}^k] = \sum_{\zeta=1}^{U_k} \zeta \cdot \mathbb{P}(\Phi_{11}^k = \zeta) \leq \frac{\alpha_{k+}\beta_{k+}}{N_k} \left(\frac{1}{1 - \frac{\alpha_{k+} + \beta_{k+}}{N_k}} \right) \rightarrow 0$$

as $k \rightarrow \infty$ since

$$\frac{\alpha_{k+}\beta_{k+}}{N_k} \rightarrow 0, \quad \frac{\alpha_{k+} + \beta_{k+}}{N_k} \rightarrow 0.$$

By Lemma 4.29, $\hat{X}_{11}^k = \mathbb{E}[\Phi_{11}^k] \rightarrow 0$ as $k \rightarrow \infty$. Moreover, all other entries of $\hat{X}^k$ are bounded by row sum 1 or column sum 1, so $\left\|\hat{X}^k\right\|_{\infty}$ is bounded, as required. $\square$

Chapter 5

Further Discussion

The original idea behind this thesis was to derive explicit lower bounds on the number of (α, β) -contingency tables and compare these bounds with other existing bounds using a well-defined (α, β) -contingency table to explicitly lower bound (Cap-Sup).

However, while we could easily identify the matrix $\alpha\beta^T/N$ as a feasible point in the formulation (Cap-Sup), it proved surprisingly difficult to prove anything about it outside of the case where all row sums (resp. all column sums) were identical and the column sums (resp. row sums) were free.

The following is a list of results that didn't lead to anything significant in this thesis, but might prove useful at some point to others.

5.1 Improved points in supremum formulation of capacity

The point $\alpha\beta^T/N$ was used in the proof of Theorem 4.6 to lower bound the capacity since it was both a feasible contingency table and had a simple closed-form. To a lesser extent, it was also used to generate an explicit lower bound without relying on an optimization problem in Section 4.3. However, it is likely that there exists a general contingency table that is closer to the value of the capacity. Some high-level considerations when deciding on such a contingency table Z are:

- Z should be “better” (i.e. have a greater objective value) than $\alpha\beta^T/N$ in some way:
 - Z should show the bounds of (4.1) are log-asymptotically correct in cases not covered by $\alpha\beta^T/N$, or
 - Plugging in the objective value of Z in the place of capacity in (4.1) should result in bounds that are log-asymptotically correct in more cases than $\alpha\beta^T$
- Z must have a form that allows for a theoretical analysis of log-asymptotics to be performed.

Let's return to the inequality (4.1) after replacing a_i, b_j with α_i, β_j respectively (see Remark 4.2):

$$\text{Cap}_{\alpha\beta}(P_K) \geq \text{CT}_K(\alpha, \beta) \geq \left[\prod_{i=2}^m \frac{\alpha_i^{\alpha_i}}{(\alpha_i + 1)^{\alpha_i+1}} \prod_{j=1}^n \frac{\beta_j^{\beta_j}}{(\beta_j + 1)^{\beta_j+1}} \right] \text{Cap}_{\alpha\beta}(P_K), \quad (4.1)$$

As usual, we set $P_K = P_\infty$. Let $\hat{X}^k$ be the optimal solution to (Cap-Sup). Then $\text{Cap}_{\alpha^k\beta^k}(P_K) = e^{H(\hat{X}^k)}$. This time, however, we let $Z^k \in \mathbb{R}_+^{m \times n}$ be an arbitrary feasible point for (Cap-Sup) so that our lower bound becomes

$$e^{H(\hat{X}^k)} \geq \text{CT}_K(\alpha, \beta) \geq \left[\prod_{i=2}^m \frac{\alpha_i^{\alpha_i}}{(\alpha_i + 1)^{\alpha_i+1}} \prod_{j=1}^n \frac{\beta_j^{\beta_j}}{(\beta_j + 1)^{\beta_j+1}} \right] e^{H(Z^k)},$$

Taking the logarithm of the inequalities and dividing by $H(Z^k)$ we get

$$\frac{H(\hat{X}^k)}{H(Z^k)} \geq \frac{\log(\text{CT}_K(\alpha^k\beta^k))}{H(Z^k)} \geq 1 - \frac{\sum_{i=2}^m h(\alpha_i^k) + \sum_{j=1}^n h(\beta_j^k)}{H(Z^k)}$$

Suppose that $\|\alpha^k/N_k\|_\infty \rightarrow 0, \|\beta^k/N_k\|_\infty \rightarrow 0$. Then the right side of the inequality will go to 1 by the proof of Theorem 4.6, so in order to show our new lower bound is log-asymptotically correct, it remains to show that

$$\frac{\log(\text{Cap}_{\alpha^k\beta^k}(P_\infty))}{H(Z^k)} = \frac{H(\hat{X}^k)}{H(Z^k)} \rightarrow 1$$

Then by Proposition 4.18,

$$\begin{aligned} H(\hat{X}^k) &\leq H(Z^k) + \nabla H(Z^k)^T (\hat{X}^k - Z^k) \\ &= H(Z^k) + \text{proj}_S(\nabla H(Z^k))^T (\hat{X}^k - Z^k), \\ &\leq H(Z^k) + \|\text{proj}_S(\nabla H(Z^k))\|_F \|\hat{X}^k - Z^k\|_F, \end{aligned}$$

where S is a subspace containing $d := \hat{X}^k - Z^k$. Then

$$\frac{H(\hat{X}^k)}{H(Z^k)} \leq 1 + \frac{\|\text{proj}_S(\nabla H(Z^k))\|_F \|\hat{X}^k - Z^k\|_F}{H(Z^k)},$$

so it suffices to show

$$\frac{\|\text{proj}_S(\nabla H(Z^k))\|_F \|\hat{X}^k - Z^k\|_F}{H(Z^k)} \rightarrow 0$$

Now we know that since $\hat{X}^k, Z^k$ lies in the affine space defined by $AZ = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ that characterizes the set of all $\alpha\beta$ contingency tables, $d = \hat{X}^k - Z^k$ must satisfy $Ad = 0$. Then we can set $S = \text{Null}(A)$ and apply Proposition 4.20 to get an explicit formula for $\text{proj}_S(\nabla H(Z^k))$.

Note that by a result from Hurkens attested to by [DLKOS09, LK13], the diameter of the transportation polytope with $m \times n$ components is at most $4(m+n-1)$. This provides a reasonable bound on $\|\hat{X}^k - Z^k\|_F$ even without attempting any further analysis on $\hat{X}^k - Z^k$.

Also note that by Proposition 4.20 we have for any $Y \in \mathbb{R}^{m \times n}$,

$$\begin{aligned} \|\text{proj}_S(Y)\|_F^2 &= \text{tr} \left(\left(I - \frac{1}{m} J \right)^T Y \left(I - \frac{1}{n} J \right) \left(I - \frac{1}{m} J \right)^T Y^T \left(I - \frac{1}{n} J \right) \right) \\ &= \text{tr} \left(Y \left(I - \frac{1}{n} J \right) Y^T \left(I - \frac{1}{m} J \right) \right) \\ &\leq \left\| Y \left(I - \frac{1}{n} J \right) \right\|_F \left\| Y^T \left(I - \frac{1}{m} J \right) \right\|_F \\ &= \sqrt{\sum_{i=1}^m \sum_{j=1}^n \left(Y[i, j] - \overline{Y[i, :]} \right)^2} \sqrt{\sum_{i=1}^m \sum_{j=1}^n \left(Y[i, j] - \overline{Y[:, j]} \right)^2}, \end{aligned}$$

which provides another condition for optimality. However, this is likely not a good condition for optimality in general, as even in the proof of Theorem 4.21, it wasn't true that $Y^k[i, j] - \overline{Y^k[i, :]} \rightarrow 0$ as $k \rightarrow \infty$.

Altogether, this provides some technical criteria to consider when choosing an improved contingency table Z better than $\alpha\beta^T/N$.

5.2 Bounds for dominating cases

This work, along with much of the previous work, focuses either on bounds for special cases that have some symmetry or approximations to symmetry (e.g. the central transportation polytope ([RCM10]) or for the general case under certain “smoothness” conditions ([BH10])). In particular, this work shows how the capacity bounds of [BLP23] are log-asymptotically correct provided none of the marginals dominate the others. Notably, we found a specific counterexample where these capacity bounds fail to be log-asymptotically correct, even with uniform row sums (see Proposition 4.17 and proof). These “strongly asymmetric” cases, then, represent situations where even log-asymptotically correct bounds would be helpful.

5.3 Capacity bounds with TU matrices

Proposition 3.19 shows that the power series

$$Q(x) = \prod_{j=1}^n \frac{1}{1 - x^{A[:,j]}}$$

generated from a $m \times n$ TU matrix A has positive γ -capacity if and only if the γ -coefficient of the power series is also positive. This is not true for polynomials in general, let alone power series. Such a result is necessary for any class of power series for which we can hope to establish general capacity-based bounds on its coefficients, and it is for this reason we conjecture that we can develop capacity-based bounds on power series with the same form as $Q(x)$.

Current examples of TU matrices that we already have capacity-based bounds for include the transportation polytope's constraint matrix (see e.g. [BLP23]) and the integer flow problem constraint matrix (see e.g. [LM24]). Both of these examples have exactly two nonzero entries per row – a property that proves critical in the approach in [BLP23] (and by extension [LM24]) to relate these power series to Lorentzian polynomials that share the same coefficients. However, there exist TU matrices with more than two nonzero entries per row. Developing a technique for bounding the coefficients in such situations will require new mathematical tools.

One sub-problem of the above is to restrict to the $\{0, 1\}$ -TU matrices where in every row, the nonzero entries appear consecutively.

Another simplification would be to consider the polynomials formed by limiting the degree of each column in the power series. For example, if we restrict the degree of each column to at most 1, then $Q(x)$ becomes

$$Q(x) = \prod_{j=1}^n (1 + x^{A[:,j]}).$$

(In the case of contingency tables, this is equivalent to restricting all entries of the table to be binary). Having a bound on the degree of a coefficient may assist in the capacity analysis.

5.4 Log-asymptotics for random marginals

In the chapter on log-asymptotics, we only worked with deterministic sequences of marginals (α^k, β^k) . However, motivated especially by the condition given in Corollary 4.10 and Corollary 4.13, we began wondering how the log-asymptotics analysis would change based on a random sequence of marginals picked according to some distribution. Intuitively, if our distribution is symmetric about the mean, then δ would naturally correspond to standard deviation in some fashion. Such analysis might prove useful in cases where the goal is to sample from the space of all contingency tables.

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