

Robust Decision-Making in Finance and Insurance

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

Traditional finance models assume a decision maker (DM) has a single view on the stochastic dynamics governing the price process but, in practice, the decision maker (DM) may be uncertain about the true probabilistic model that governs the occurrence of different states. If only risk is present, that is, the DM fully relies on a single probabilistic model $\mathbb{P}$. When the DM is ambiguous, he holds different views on the precise distributions of the price dynamics. This type of model uncertainty due to the multiple probabilistic views is called *ambiguity*.

In the presence of model ambiguity, [Maccheroni et al. \(2013\)](#) propose a novel robust mean-variance model which is referred to as the mean-variance-variance (M-V-V) criterion in my thesis. The M-V-V model is an analog of the Arrow-Pratt approximation to the well-known smooth ambiguity model, but it offers a more tractable structure and meanwhile separates the modeling of ambiguity, ambiguity aversion, and risk aversion.

In Chapters [3](#) and [4](#), we study the dynamic portfolio optimization and the dynamic reinsurance problem under the M-V-V criterion and derive the equilibrium strategies in light of the issue of time inconsistency. We find the equilibrium strategies share many properties with the ones from smooth ambiguity, but the time horizon appears inconsistently in the objective function of the M-V-V criterion, in turn causing the equilibrium strategies to be non-monotonic with respect to the risk aversion. To resolve this issue, we further propose a mean-variance-standard deviation (M-V-SD) criterion. The corresponding equilibrium investment strategy exhibits the appealing feature of limited stock market participation, a well-documented stylized fact in empirical studies. The corresponding equilibrium reinsurance strategy also displays the property of restricted insurance retention.

Chapter [5](#) analyzes optimal longevity risk transfers, focusing on differing buyer and seller risk aversions using a Stackelberg game framework. We compare static contracts, which offer long-term protection with fixed terms, to dynamic contracts, which provide short-term coverage with variable terms. Our numerical analysis with real-life mortality data shows that risk-averse buyers prefer static contracts, leading to higher welfare gains and flexible market conditions, while less risk-averse buyers favor dynamic contracts. Ambiguity, modeled as information asymmetry, reduces welfare gains and market flexibility but does not change contract preferences. These findings explain key empirical facts and offer insights into the longevity-linked capital market.

In the rest of the chapters, Chapter [1](#) introduces the background literature and main motivations of this thesis. Chapter [2](#) covers the mathematical preliminaries for the subsequent chapters. The core analysis and findings are presented in the following chapters. Finally, Chapter [6](#) concludes the thesis and suggests potential directions for future research.

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Dedication

To my parents, Shiping Zhang and Kaiying Xiao.

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Chapter 1

Introduction

1.1 Background and Motivation

This thesis explores the interplay between these concepts and their implications for portfolio optimization, reinsurance strategies, and the transfer of longevity risk. We aim to contribute to the literature by addressing gaps in dynamic portfolio optimization under ambiguity, investigating optimal reinsurance strategies considering model uncertainty, and examining the optimal transaction of longevity risk using a game-theoretic approach. This introduction outlines the key motivations and existing literature that inform our research objectives.

1.1.1 Ambiguity Model

Ambiguity describes situations where the probabilities of outcomes are unknown. In this context, the decision maker considers multiple perspectives, each associated with a different probability measure. This contrasts with risk, where a single, known probability measure is available to guide choices. The distinction between risk and ambiguity is behaviorally significant. Risk involves making decisions based on known probabilities. Ambiguity, on the other hand, occurs when the decision maker is uncertain about the probabilities due to cognitive or informational constraints. Essentially, risk involves certainty about probabilities, while ambiguity involves uncertainty about which probabilities to use.

Ambiguity is a richly studied phenomenon with abundant empirical evidence and wide applications in finance and insurance. The well-known Ellsberg paradox ([Ellsberg \(1961\)](#))

highlights individuals' preference for known risks over unknown probabilities and underscores the importance of ambiguity models as foundational tools for guiding decision-making in the presence of ambiguity. This paradox demonstrates that people tend to prefer a gamble with known probabilities over one with unknown probabilities, even when the expected outcomes are identical. [Easley and O'Hara \(2009\)](#) investigate the implications of ambiguity aversion for the performance and regulation of markets and [Ju and Miao \(2012\)](#) propose a generalized recursive smooth ambiguity model that permits a three-way separation among risk aversion, ambiguity aversion, and intertemporal substitution. Beyond theoretical developments, there is substantial empirical research on portfolio choices and asset pricing under ambiguity, including works by [Dimmock et al. \(2016\)](#), [Brenner and Izhakian \(2018\)](#), and [Kostopoulos et al. \(2022\)](#).

Ambiguity model acts as the foundation to guide decision making in the presence of ambiguity modeling. A number of ambiguity models have been proposed and studied in the literature. Among them, the *maxmin expected utility* (MEU) introduced by [Gilboa and Schmeidler \(1989\)](#) has drawn considerable attention. For the MEU model, the standard setup consists of

$$\min_{\mathbb{Q} \in \Delta} \mathbb{E}^{\mathbb{Q}}[U(X)], \quad (1.1)$$

where Δ is a set of prior probability measures (i.e., the collection of plausible probability measures in decision making), U is a standard utility function capturing the decision maker (DM)'s risk preference, and X is a random payoff. The MEU model considers the expected payoff under the worst-case scenario.¹ That is to say, DMs with MEU preferences are extremely ambiguity-averse. Hence, the MEU criterion does not allow DMs to have different ambiguity attitudes, as only the worst-case scenario matters.²

To address this issue, there are two widely used ambiguity models that allow for the disentangling of *risk*, *ambiguity level*, and *degree of ambiguity aversion*. One is the *α -maxmin expected utility* (α -MEU) criterion, introduced and axiomatically characterized by [Marinacci \(2002\)](#), [Ghirardato et al. \(2004\)](#), and [Maccheroni et al. \(2006\)](#), which is defined as

$$\alpha \min_{\mathbb{Q} \in \Delta} \mathbb{E}^{\mathbb{Q}}[U(X)] + (1 - \alpha) \max_{\mathbb{Q} \in \Delta} \mathbb{E}^{\mathbb{Q}}[U(X)], \quad (1.2)$$

where the ambiguity attitude is characterized by the parameter $\alpha \in [0, 1]$, which weights the worst and best perspectives. However, α -MEU inherits a similar limitation from the

¹The decision maker (DM)'s objective is to find an optimal control to the random payoff under the worst-case perspective, and thus it leads to a maxmin form.

²There have been many empirical/experimental works on eliciting DMs' degree of ambiguity aversion (see, e.g., [Dimmock et al. \(2015\)](#), [Baillon et al. \(2018\)](#), and [Cavatorta and Schröder \(2019\)](#)).

MEU criterion—only the two extreme priors play a role while all other priors are ignored in the decision making.

The other model is the *smooth ambiguity* model, introduced by [Klibanoff et al. \(2005\)](#), which aggregates all priors in a weighted form:

$$\int_{\Delta} \varphi (\mathbb{E}^{\mathbb{Q}} [U(X)]) d\nu(\mathbb{Q}), \quad (1.3)$$

where each $\mathbb{Q} \in \Delta$ represents a prior probability measure; $\nu(\cdot)$ assigns a probability measure on Δ , i.e. probabilistic beliefs for all priors; and φ captures the ambiguity attitude by smoothly aggregating the expected values under each $\mathbb{Q}$. A concave (resp. convex) φ represents ambiguity aversion (resp. ambiguity loving) attitudes. The smooth ambiguity criterion (1.3) can be re-expressed as

$$\mathbb{E}_{\nu} [\varphi (\mathbb{E}^{\mathbb{Q}} [U(X)])],$$

where $\mathbb{E}_{\nu}$ denotes the expectation under the probability measure $\nu(\cdot)$, which is often referred to as the *second-order distribution*. Admittedly, smooth ambiguity offers the desired separation of risk, ambiguity, and ambiguity aversion, but its structure is usually complex and thus its application in dynamic portfolio optimization is challenging. Analytic (not necessarily fully explicit) solutions can only be obtained under very specific choices for φ , U , and $\nu(\cdot)$; see, e.g., [Kellner \(2017\)](#), [Balter et al. \(2021\)](#), and [Guan and Li \(2022\)](#).

[Maccheroni et al. \(2013\)](#) introduces a robust mean-variance (M-V) model of the form

$$\mathbb{E}^{\overline{\mathbb{Q}}}[X] - \frac{\gamma}{2} \text{Var}^{\overline{\mathbb{Q}}}[X] - \frac{\beta}{2} \text{Var}_{\nu} [\mathbb{E}^{\mathbb{Q}}[X]] \quad (1.4)$$

as an analog of the classic Arrow-Pratt approximation to the smooth ambiguity model. Here, $\overline{\mathbb{Q}}$ is called the reduction measure, satisfying

$$\overline{\mathbb{Q}}(A) = \mathbb{E}_{\nu} [\mathbb{Q}(A)] = \int_{\Delta} \mathbb{Q}(A) d\nu(\mathbb{Q})$$

for any measurable set A , and Var_{ν} denotes the variance under the probability measure $\nu(\cdot)$, i.e.,

$$\text{Var}_{\nu} [\mathbb{E}^{\mathbb{Q}}[X]] = \mathbb{E}_{\nu} [(\mathbb{E}^{\mathbb{Q}}[X])^2] - (\mathbb{E}_{\nu} [\mathbb{E}^{\mathbb{Q}}[X]])^2 = \int_{\Delta} (\mathbb{E}^{\mathbb{Q}}[X])^2 d\nu(\mathbb{Q}) - \left(\int_{\Delta} \mathbb{E}^{\mathbb{Q}}[X] d\nu(\mathbb{Q}) \right)^2.$$

Heuristically, the measure $\overline{\mathbb{Q}}$ is a weighted average of all priors. In this thesis, we consider a dynamic portfolio optimization problem under a criterion that we call the mean-variance-variance (M-V-V) model:

$$\mathbb{E}^{\mathbb{P}}[X] - \frac{\gamma}{2}\text{Var}^{\mathbb{P}}[X] - \frac{\beta}{2}\text{Var}_{\nu}[\mathbb{E}^{\mathbb{Q}}[X]], \quad (1.5)$$

where $\mathbb{P}$ is a pre-specified probability measure called the *reference measure*. Note that the main difference between criteria (1.4) and (1.5) is that $\mathbb{P}$ is pre-specified while $\overline{\mathbb{Q}}$ needs to be derived based on the set of priors Δ and associated measure $\nu(\cdot)$. The reason to replace the measure $\overline{\mathbb{Q}}$ by $\mathbb{P}$ will be postponed to the introduction section in Chapter 3. In (1.5), $\mathbb{E}^{\mathbb{P}}[X] - \frac{\gamma}{2}\text{Var}^{\mathbb{P}}[X]$ is of the M-V form to represent a DM's risk aversion (with risk aversion parameter γ); $\text{Var}_{\nu}(\mathbb{E}^{\mathbb{Q}}[X])$ measures the variance of the expected payoff under different priors; and β is the *ambiguity aversion parameter*. The robust M-V model (1.5) provides a *simpler but more intuitive* structure than smooth ambiguity. The risk and ambiguity attitudes reduce to two parameters (γ and β , respectively), compared to the two functions U and φ in smooth ambiguity. Despite the nice features, applications of the robust M-V criterion (1.5) are rather scarce in the literature, with the exception of Section 6 of [Maccheroni et al. \(2013\)](#) which considers a static portfolio optimization problem. Therefore, **the first objective of this thesis is to contribute to this line of research by analyzing a dynamic portfolio optimization problem under the mean-variance-variance (M- V-V) criterion.**

1.1.2 Time Inconsistency

The issue of time inconsistency arises in the dynamic optimization problem of maximizing (1.5) from the two variance terms, $\text{Var}^{\mathbb{P}}[\cdot]$ and $\text{Var}_{\nu}[\cdot]$. The first variance is inherited from the original M-V criterion, which leads to time inconsistency as a result of the quadratic (nonlinear) structure of the expected terminal payoff under $\mathbb{P}$. On the other hand, the second variance $\text{Var}_{\nu}[\cdot]$ yields time inconsistency as it seeks to *coordinate and integrate* various values of the expected terminal payoff under different prior probability measures.

There are two major approaches in the literature for addressing time inconsistency, known as the precommitted strategy and the equilibrium strategy. The precommitted strategy focuses solely on determining the optimal strategy at the initial time and obligates the DM's future selves to adhere to the strategy established at the initial time; see, e.g., [Bajeux-Besnainou and Portait \(1998\)](#), [Li and Ng \(2000\)](#), [Zhou and Li \(2000\)](#), [Bielecki et al. \(2005\)](#), [Cvitanic et al. \(2008\)](#), and [Dai et al. \(2010\)](#). Alternatively, the equilibrium approach addresses time inconsistency by considering the DM's future selves as different

players and adopting a game-theoretic perspective to seek subgame perfect Nash equilibria (SPNE). For M-V problems, [Basak and Chabakauri \(2010\)](#) is the first to derive an explicit equilibrium strategy for the M-V criterion for terminal wealth optimization. Their work has been extended by many subsequent studies, including [Czichowsky \(2013\)](#), [Björk et al. \(2014\)](#), [Li et al. \(2016\)](#), [Pedersen and Peskir \(2017\)](#), and [He and Jiang \(2022\)](#). Recently, equilibrium strategies for M-V problems have also been applied to robo-advising ([Dai et al. \(2021\)](#)), insurance risk sharing ([Chen et al. \(2021\)](#)), and reinforcement learning ([Dai et al. \(2023\)](#)). In this thesis, we adopt the equilibrium approach to derive equilibrium strategies through a formal verification argument. The basic idea is that when deciding on a control action at time $-t$, we must consider that our objectives will change over time. Therefore, we can frame the entire problem as a non-cooperative game, with a different player representing our future self at each time $-t$. Each player (or future self) has their own preferences and strategies. The time $-t$ DM controls her strategy within a short period, selecting an optimal strategy for that time, while future selves choose their own optimal strategies. Thus, an equilibrium strategy is time-consistent.

1.1.3 Optimal Reinsurance

Naturally, the M-V-V criterion (1.5) can also be examined in actuarial science. As an important part of the insurance industry, reinsurance plays a vital role in the development of insurance and the stability of the global financial system. In recent years, various types of unpredictable risks, such as those pertaining to public health events, climate change, financial risk, etc., have made reinsurance under a model uncertainty setting a particularly relevant research topic with much-needed guidance for risk management purposes. The early studies on optimal reinsurance can be traced back to [Borch \(1960\)](#), who focused on risk minimization, and [Arrow \(1978\)](#), who concentrated on expected utility maximization. The dynamic optimal reinsurance problem has also received considerable attention, with various optimization criteria being studied. [David and Young \(2005\)](#) identified the optimal proportional reinsurance strategy by minimizing the probability of ruin. [Bai and Guo \(2008\)](#) examined the proportional reinsurance strategy for both minimizing the probability of ruin and maximizing the expected exponential utility of terminal wealth. The mean-variance criterion is closely related to maximizing the expected utility of terminal wealth. [Li et al. \(2017a\)](#) study an insurer's excess-loss reinsurance-investment problem under a mean-variance criterion. They show that excess-loss is the unique equilibrium reinsurance strategy under a spectrally negative Lévy insurance model when the reinsurance premium is computed according to the expected value premium principle.

Although the problem of optimal reinsurance has been widely investigated, the rein-

insurance decision under ambiguity requires further exploration. Ambiguity is also present in insurance risk modeling, and an ambiguity-averse insurer aims to quantitatively address this uncertainty. This uncertainty can significantly impact the effectiveness and robustness of reinsurance strategies. Moreover, ambiguity can lead to more conservative decision-making, affecting the pricing, structuring, and selection of reinsurance contracts. Understanding how to manage and mitigate this ambiguity is crucial for developing more resilient and adaptable reinsurance models. [Zheng et al. \(2016\)](#) investigate a robust optimal portfolio and reinsurance problem under the *maxmin expected* utility (1.1). [Li et al. \(2016\)](#) analyze the robust reinsurance-investment problem under the α -maxmin mean-variance criterion, similar to (1.2), which differentiates between the level of ambiguity aversion and ambiguity. [Guan and Li \(2022\)](#) study an optimal investment and reinsurance problem under the smooth ambiguity model proposed. Even though smooth ambiguity can generate richer forms of optimal strategies than other ambiguity models, analytic (not necessarily fully explicit) solutions can only be obtained under very specific choices for φ , U , and ν in (1.3). **The second objective of this thesis is to study the optimal reinsurance-investment strategy under the mean-variance-variance (M-V-V) criterion and determine whether the excess-loss strategy is the equilibrium strategy under the model uncertainty.**

1.1.4 Longevity Risk Transfer

Reinsurance has become increasingly necessary to effectively manage and transfer exposure to longevity risk, which is the risk of individuals living longer than expected, due to the rising demand for defined benefit pension plans and life insurance. Most longevity risk transactions have occurred within the reinsurance sector, where reinsurers offer protection for a premium. However, [Blake and Morales \(2017\)](#) state that despite steady development over the past two decades, the reinsurance sector cannot absorb the global longevity risk exposure, which exceeds its capacity. Therefore, attracting investment from external capital market investors into longevity risk transactions is crucial for creating a market capable of managing global longevity risk exposure sustainably. [Kessler \(2021\)](#) highlights that entities seeking risk transfer can access more capital, while investors benefit from diversification, as most insurance risks, including catastrophic and mortality risks, show little correlation with other financial assets. However, transactions involving mortality or longevity risks in the capital market remain rare. Several reasons have been proposed for the scarcity of capital market transactions involving longevity risk, including the tension between the long-term nature of such risks and investors' preference for short-term investments, as well as information asymmetry by [Blake et al. \(2019\)](#) and [Kiff \(2022\)](#).

Furthermore, a detailed elucidation of the aforementioned reasons is structured as follows. First, due to the long-term nature of pension and annuity portfolios, longevity reinsurance contracts are long-term and remain effective until the covered portfolio is depleted. This kind of contract can be modeled by the static contract, which sets a constant hedge ratio and fixed periodic payments at inception. However, capital market investors, primarily concerned with returns and liquidity, generally prefer short-term investments in insurance risks and are reluctant to commit to long-term contracts with limited liquidity. The short-term reinsurance contract can be modeled by the dynamic contract, which adjusts these parameters periodically based on the latest available information. In our study, we adopt the equilibrium approach in the context of dynamic contracting. Given the disparity in capital market investors' preferences and those of traditional reinsurers, the preferred contract structures between the reinsurance sector and the capital market should be further investigated. Second, longevity risk is a new and lesser-known risk to general capital market investors. Such information asymmetries could make longevity risk less attractive to capital market investors and possibly prevent them from entering the market. This information asymmetry is addressed in longevity risk transfers by introducing ambiguity.

Furthermore, in reinsurance, the seller acts as the principal, offering a variety of swap contracts to the buyer, each distinguished by a specific longevity risk loading. The buyer, acting as the follower, then selects an optimal hedge ratio—either constant or time-varying, depending on the contract type—for each given risk loading. Therefore, it is natural to employ a Stackelberg (principal-agent) game framework to explore optimal transactions of longevity risk, as suggested by [Chen and Shen \(2019\)](#). However, despite their investigation into the optimal reinsurance contract in the Stackelberg differential game, [Chen and Shen \(2019\)](#) fail to address a crucial issue: the potential information asymmetry between the insurer and reinsurer regarding the claim process. Hence, **The third objective of this thesis is to develop a Stackelberg (principal-agent) game framework to explore the optimal transaction of longevity risk. This model aims to identify factors, including contract structure and ambiguity, that hinder longevity risk transfer in the capital market, and to analyze contract structure preferences across different markets.**

1.2 Structure of the Thesis

Motivated by the aforementioned objectives, this thesis comprises three research projects and is organized as follows: Chapter [2](#) presents the core mathematical preliminaries for the

later chapters. Chapter 3 proposes a robust dynamic portfolio optimization problem under the mean-variance-variance (M-V-V) criterion. Chapter 4 considers a robust dynamic optimal reinsurance problem under the mean-variance-variance (M-V-V) criterion. Chapter 5 develops a theoretical model to explore optimal transactions of longevity risk using a Stackelberg game framework. Chapter 6 summarizes the main results of each chapter and outlines potential future work streams.

Chapter 3 studies a dynamic portfolio optimization problem under the mean-variance-variance (M-V-V) criterion proposed by [Maccheroni et al. \(2013\)](#). It is an analog of the Arrow-Pratt approximation to the well-known smooth ambiguity model. Under the standard Black-Scholes framework, we derive fully explicit equilibrium investment strategies in which a decision maker's level of ambiguity, risk aversion, and ambiguity aversion are transparently captured. We find that the time horizon appears inconsistently in the objective function of the M-V-V criterion, in turn causing the equilibrium strategies to be non-monotonic with respect to risk aversion. In response, we introduce a new mean-variance-standard deviation (M-V-SD) criterion to address this issue. Equilibrium strategies under the M-V-SD criterion exhibit an appealing feature of limited stock market participation, which provides a theoretical justification to this phenomenon.

Chapter 4 examines a dynamic optimal reinsurance problem under the mean-variance-variance (M-V-V) criterion within the spectrally negative Lévy framework. The closed form of the equilibrium strategy under the M-V-V criterion is derived. However, a horizon-scale inconsistency exists between the risk variance and the ambiguity variance in the optimal reinsurance problem. To address this inconsistency, we introduce the M-V-SD criterion and derive the corresponding explicit equilibrium strategy. The M-V-SD criterion results in a non-retention zone for the equilibrium strategy.

Chapter 5 develops an economic framework to analyze optimal longevity risk transfers, focusing on the differing risk aversions of buyers and sellers of longevity risk transfer contracts. Utilizing a Stackelberg game framework, we compare static longevity swap contracts, which offer long-term protection with constant hedge ratios and predetermined hedging costs, against dynamic contracts that provide short-term coverage with variable terms. Our numerical analysis using real-life mortality data reveals that static contracts are preferred when buyers are more risk-averse, leading to larger welfare gains for both parties and more flexible market conditions. Conversely, dynamic contracts are favored when buyers are less risk-averse. Additionally, we incorporate information asymmetry in the form of ambiguity. While ambiguity reduces welfare gains for both parties and imposes stricter market conditions, it does not alter contract preferences. Finally, we discuss the implications of our analysis on optimal contract forms and market dynamics in both the traditional reinsurance market and the emerging longevity-linked capital market. Our

study offers theoretical insights into key empirical observations in the current longevity risk transfer market and provides new perspectives on the evolution of the longevity-linked capital market.

It is important to highlight that the models and methodologies vary across these three research problems. In Chapters 3 and 4, both of dynamic optimization problems are analyzed under the mean-variance-variance (M-V-V) criterion, while in Chapter 5, we use the modified max-min principle as the ambiguity criterion. The underlying asset process in Chapter 3 follows the Black-Scholes framework, whereas the surplus process in Chapter 4 follows a spectrally negative Lévy process to model claim occurrences. In Chapter 5, the surplus process becomes a discrete model since the random count of surviving policyholders follows a binomial distribution. In Chapters 3 and 4, the goal is to solve for the equilibrium strategy using extended Hamilton-Jacobi-Bellman (HJB) equations. In contrast, in Chapter 5, solving the equilibrium strategy is not the final goal but rather using it for empirical study.

Each of Chapters 3 through 5 represents an independent research project. While efforts have been made to maintain consistent notation, some inconsistencies may remain. Readers are encouraged to consider each chapter separately regarding notation.

Chapter 2

Mathematical Preliminaries

2.1 Itô's formula and Stochastic Differential Equations

Let us consider a stochastic process $X = \{X_t\}_{t \geq 0}$ defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ satisfying the usual conditions of completeness and right continuity. Throughout the chapter, we assume the state space of X is $\mathbb{R}$.

We first state the famous Itô's formula, which is defined for semimartingales, the broadest class of stochastic processes.

Definition 2.1. (*Semimartingale*). A semimartingale is a càdlàg adapted process X having a decomposition in the form:

$$X_t = X_0 + M_t + A_t$$

where X_0 is finite and $\mathcal{F}_0$ -measurable, $M_0 = A_0 = 0$, M is a càdlàg local martingale, and A is an adapted process with finite variation.

Theorem 2.1. (*Itô's Formula*) Let $X = \{X_t\}_{t \geq 0}$ be a semimartingale with state space $\mathbb{R}$ and f be a C^2 function on $\mathbb{R}$. Then $f(X)$ is also a semimartingale, and

$$f(X_t) = f(X_0) + \int_{0+}^t f_x(X_{s-}) dX_s + \frac{1}{2} \int_{0+}^t f_{xx}(X_{s-}) d[X, X]_s^c \quad (2.1)$$

$$+ \sum_{0 < s \leq t} \{f(X_s) - f(X_{s-}) - f_x(X_{s-}) \Delta X_s\} \quad (2.2)$$

where f_x (f_{xx} resp.) is the first (second resp.) order derivative of f with respect to X , $[X, X]^c$ is the continuous part of the quadratic variation process of X , and $\Delta X_t = X_t - X_{t-}$.

Note that Itô's formula (2.1) is applied in the following chapters to prove verification theorems and derive partial differential equations. In the rest of this chapter, we explore the solution of a stochastic differential equation (SDE) driven by Lévy processes.

Theorem 2.2. (*Existence and Uniqueness of Solutions of Lévy SDEs*) Consider the following Lévy SDE in

$$\begin{cases} dX_t = \alpha(t, X_t)dt + \sigma(t, X_t)dB_t + \int_{\mathbb{R}} \gamma(t, X_{t-}, z) \tilde{N}(dt, dz), t \in [0, T] \\ X_t = x, \end{cases} \quad (2.3)$$

where $\alpha : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$, $\sigma : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ and $\gamma : [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$. Here,

$$\tilde{N}(dt, dz) = N(dt, dz) - \nu(dz)dt$$

is the compensated Poisson random measure of X_t , and B_t is a Brownian motion independent of $\tilde{N}(dt, dz)$. Coefficients of the SDE satisfy the following conditions

- (At most linear growth) There exists a constant $C_1 < \infty$ such that

$$|\sigma(t, x)|^2 + |\alpha(t, x)|^2 + \int_{\mathbb{R}} |\gamma(t, x, z)|^2 \nu(dz) \leq C_1 (1 + |x|^2)$$

for all $x \in \mathbb{R}$,

- (Lipschitz continuity) There exists a constant $C_2 < \infty$ such that

$$|\sigma(t, x) - \sigma(t, y)|^2 + |\alpha(t, x) - \alpha(t, y)|^2 + \int_{\mathbb{R}} |\gamma(t, x, z) - \gamma(t, y, z)|^2 \nu(dz) \leq C_2 |x - y|^2$$

for all $x, y \in \mathbb{R}$.

Then there exists a unique càdlàg adapted solution X_t such that

$$E[|X_t|^2] < \infty \text{ for all } t.$$

2.2 Formulation of Some Stochastic Control Problems

2.2.1 Standard Stochastic Control Problems

A controlled jump diffusion process $X = \{X_t\}_{t \in [0, T]}$ can be described by the following SDE:

$$\begin{cases} dX_t = \alpha(t, X_t, u_t)dt + \sigma(t, X_t, u_t)dB_t + \int_{\mathbb{R}} \gamma(t, X_{t-}, u_{t-}, z) \tilde{N}(dt, dz), t > 0 \\ X_t = x, \end{cases} \quad (2.4)$$

where

$$b : [0, T] \times \mathbb{R} \times U \rightarrow \mathbb{R}, \quad \sigma : [0, T] \times \mathbb{R} \times U \rightarrow \mathbb{R}, \quad \text{and } \gamma : [0, T] \times \mathbb{R} \times U \times \mathbb{R} \rightarrow \mathbb{R}.$$

We denote by $\{X_s^{t,x,u}\}_{s \in [t, T]}$ the wealth process under strategy $\mathbf{u}$ with initial value $X_t^{\mathbf{u}} = x$. The process $u(t) = u(t, \omega) : [0, \infty) \times \Omega \rightarrow U$ is our control process, assumed to be càdlàg and adapted. We consider a performance criterion $J = J^{(u)}(x)$ of the form

$$J(t, x, u) = \mathbb{E} \left[\int_t^T f(s, X_s^{t,x,u}, u_s) ds + g(X_T^{t,x,u}) \right], \quad (2.5)$$

where f is a function from $[0, T] \times \mathbb{R} \times U$ to $\mathbb{R}$ and g is a function from $\mathbb{R}$ to $\mathbb{R}$. We say that the control process u is *admissible* and write $u \in \mathcal{A}$ if (2.4) has a unique, strong solution X_t and further assume

- a quadratic growth condition, $|g(x)| \leq C(1 + |x|^2)$ for $\forall x \in \mathbb{R}$ and a constant C independent of x ;
- the set of controls $\{u_t\}_{t \geq 0}$ is restricted to

$$\mathcal{A}(t, x) = \left\{ u \in \mathcal{A} : E \left[\int_t^T |f(s, X_s^{t,x,u}, u_s)| ds \right] < \infty \right\} \text{ for } (t, x) \in [0, T] \times \mathbb{R}.$$

Definition 2.2. (*Value function*). For all $(t, x) \in [0, T] \times \mathbb{R}$ and $u \in \mathcal{A}(t, x)$, the performance criterion is defined in (2.5) and the associated value function is

$$v(t, x) = \sup_{u \in \mathcal{A}(t, x)} J(t, x; u).$$

Given an initial condition $(t, x) \in [0, T] \times \mathbb{R}$, we say that $u^* \in \mathcal{A}(t, x)$ is an optimal control if $v(t, x) = J(t, x; u^*)$. It suffices to consider Markov controls, which has the form of $u_s = u(s, X_s^{t,x,u})$, in this thesis.

Definition 2.3. (*Infinitesimal generator*) For a constant control u and a function $f \in \mathcal{C}^{1,2}([0, T] \times \mathbb{R})$, define the infinitesimal generator of the controlled diffusion process by

$$\begin{aligned} \mathcal{A}^u f(t, x) &= f_t(t, x) + \alpha(x, u) f_x(t, x) + \frac{1}{2} \sigma^2(x, u) f_{xx}(t, x) \\ &\quad + \int_{\mathbb{R}} f(t, x + \gamma(x, u, z)) - f(t, x) - f_x(t, x) \gamma(x, u, z) \nu(dz). \end{aligned} \quad (2.6)$$

In the following, we briefly discuss how the solutions to the Hamilton-Jacobi-Bellman (HJB) equation can aid in determining optimal control strategies.

Theorem 2.3. (*Verification Theorem*) Suppose $V(t, x) \in \mathcal{C}^{1,2}([0, T] \times \mathbb{R})$ satisfies the following:

1. $\sup_{u \in U} \{\mathcal{A}^u V(t, x) + f(t, x, u)\} = 0, \quad (t, x) \in [0, T] \times \mathbb{R},$
2. $V(T, x) = g(x), \quad x \in \mathbb{R},$
3. $\mathbb{E} \left[|V(t, X_T^{t,x,u})| + \int_t^T |\mathcal{A}^u V(s, X_T^{s,x,u})| ds \right] < \infty, \text{ for all } u \in \mathcal{A},$
4. Suppose that for each $x \in \mathbb{R}$, there exists a $u^* = u^*(t, x) \in \mathcal{A}$ such that

$$\mathcal{A}^{u^*} V(t, x) + f(t, x, u^*) = 0$$

5. $\{V(s, X_T^{s,x,u})\}_{s \in [t, T]}$ and $\{V(s, X_T^{s,x,u^*})\}_{s \in [t, T]}$ are uniformly integrable.

Then, u^* is an optimal control and

$$V(t, x) = J(t, x; u^*) = \sup_{u \in \mathcal{A}(t, x)} J(t, x; u) \text{ for all } x \in \mathbb{R}.$$

2.2.2 Time Inconsistent Control Problems

In this section, we consider a class of stochastic control problems in continuous time which have the property of being time-inconsistent in the sense that they do not allow a Bellman optimality principle. We still consider a stochastic process $X = \{X_t\}_{t \in [0, T]}$ defined by SDE (2.4). A control process u is considered *admissible* and is denoted $u \in \mathcal{A}$ if u ensures that the equation (2.4) has a unique, strong solution X_t .

Definition 2.4. (*Reward Functional*) For a fixed $(t, x) \in [0, T] \times \mathbb{R}$ and a fixed admissible control law $\mathbf{u}$, the corresponding reward functional J is defined by

$$J(t, x, u) = E [F (X_T^{t,x,u})] + G (E [X_T^{t,x,u}]) .$$

We assume for all $u \in \mathcal{A}$ that

$$E [|F (X_T^{t,x,u})|] < \infty, \quad E [|X_T^{t,x,u}|] < \infty$$

and hence

$$G (E [X_T^{t,x,u}]) < \infty$$

Since we cannot apply a Bellman optimality principle due to a nonlinear function G operating on the expected value $E [X_T^{t,x,u}]$, the definition of “optimal” is unclear, as the concept of optimality can vary with different initial times t and different initial states x . This paper adopts a game-theoretic perspective, seeking equilibrium strategies that achieve subgame perfect Nash equilibrium. A more general reward functional form is considered in Chapter 3 and Chapter 4.

Definition 2.5. (*Equilibrium Strategy*) Consider an admissible control law $\hat{u}$. Choose an arbitrary admissible control law $u \in U$ and a fixed real number $h > 0$. Also fix an arbitrarily chosen initial point (t, x) . Define the control law u_h by

$$u_h(s, x) = \begin{cases} u(s, x) & \text{for } t \leq s < t + h, x \in \mathbb{R}, \\ \hat{u}(s, x) & \text{for } t + h \leq s \leq T, x \in \mathbb{R} \end{cases}$$

If

$$\liminf_{h \rightarrow 0} \frac{J(t, x, \hat{u}) - J(t, x, u_h)}{h} \geq 0$$

for all $u \in U$, we say that $\hat{u}$ is an equilibrium control law. Corresponding to the equilibrium law $\hat{u}$, we define the equilibrium value function V by

$$V(t, x) = J(t, x, \hat{u}).$$

Definition 2.6. Consider an arbitrary admissible control $u \in U$. We say that a function $h : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ belongs to the space $L_T^2 (X^u)$ if it satisfies the condition

$$E \left[\int_t^T |h_x (s, X_s^{t,x,u}) \sigma (s, X_s^{t,x,u})|^2 ds \right] < \infty$$

for every (t, x) . In this expression, h_x denotes the first order derivative of h in the x -variable.

We can now present the verification theorem in the following.

Theorem 2.4. (*Verification theorem*) Assume that the functions $V(t, x)$, $g(t, x)$ are in $C^{1,2}$ and $u^*(t, x) \in \mathcal{A}$ have the following properties:

1. V and g solve the extended HJB system:

$$\sup_{u \in U} \left\{ \mathcal{A}^u V(t, x) + (G_x \circ g)(t, x) \mathcal{A}^u g(t, x) - \mathcal{A}^u (G \circ g)(t, x) \right\} = 0, \quad (2.7)$$

$$V(T, x) = F(x) + G(x), \quad (2.8)$$

for any $(t, x) \in [0, T) \times \mathbb{R}$.

2. There is a measurable function $u^*(t, x) \in \mathcal{A}$, satisfying

$$\mathcal{A}^{u^*} V(t, x) + (G_x \circ g)(t, x) \mathcal{A}^{u^*} g(t, x) - \mathcal{A}^{u^*} (G \circ g)(t, x) = 0, \quad (2.9)$$

and, for any $(t, x, \lambda) \in [0, T) \times \mathbb{R} \times \mathbb{R}$,

$$\mathcal{A}^{u^*} g(t, x) = 0. \quad (2.10)$$

3. V, g and $G \circ g$ all belong to the space $L_T^2(X^{t,x,u^*})$.

Then u^* is an equilibrium law, and V is the corresponding equilibrium value function. Furthermore,

$$g(t, x) = E \left[X_T^{t,x,u^*} \right].$$

We refer interested readers to Björk et al. (2017) for formulations of the time-inconsistent control problem and further examples. In Chapter 3 and Chapter 4, we establish the verification theorem for a broader class of reward functionals. This approach allows us to derive the equilibrium strategy under the M-V-V criterion.

Chapter 3

Robust Mean-Variance Investment Optimization

3.1 Introduction

Ambiguity is a richly studied phenomenon with abundant empirical evidence and wide applications in finance; see, e.g., [Easley and O’Hara \(2009\)](#), [Bossaerts et al. \(2010\)](#), [Ju and Miao \(2012\)](#), [Dimmock et al. \(2016\)](#), [Brenner and Izhakian \(2018\)](#), and [Kostopoulos et al. \(2022\)](#). The well-known Ellsberg paradox ([Ellsberg \(1961\)](#)) stresses the importance of *ambiguity models* as the foundation to guide decision making in the presence of ambiguity.

A number of criteria (also known as ambiguity models) have been proposed and studied in the literature. Among them, the *maxmin expected utility* (MEU) introduced by [Gilboa and Schmeidler \(1989\)](#) has drawn considerable attention. For the MEU model, the standard setup consists of

$$\min_{\mathbb{Q} \in \Delta} \mathbb{E}^{\mathbb{Q}} [U(X)], \quad (3.1)$$

where Δ is a set of prior probability measures (i.e., the collection of plausible probability measures in decision making), U is a standard utility function capturing the decision maker (DM)’s risk preference, and X is a random payoff. The MEU model considers the expected payoff under the worst-case scenario.¹ That is to say, DMs with MEU preferences

¹The decision maker (DM)’s objective is to find an optimal control to the random payoff under the worst-case perspective, and thus it leads to a maxmin form.

are extremely ambiguity-averse. Hence, the MEU criterion does not allow DMs to have different ambiguity attitudes, as only the worst-case scenario matters.²

To address this issue, there are two widely used ambiguity models that allow for the disentangling of *risk*, *ambiguity level*, and *degree of ambiguity aversion*. One is the α -*maxmin expected utility* (α -MEU) criterion, introduced and axiomatically characterized by [Marinacci \(2002\)](#), [Ghirardato et al. \(2004\)](#), and [Maccheroni et al. \(2006\)](#), which is defined as

$$\alpha \min_{\mathbb{Q} \in \Delta} \mathbb{E}^{\mathbb{Q}}[U(X)] + (1 - \alpha) \max_{\mathbb{Q} \in \Delta} \mathbb{E}^{\mathbb{Q}}[U(X)], \quad (3.2)$$

where the ambiguity attitude is characterized by the parameter $\alpha \in [0, 1]$, which weights the worst and best perspectives. However, α -MEU inherits a similar limitation from the MEU criterion—only the two extreme priors play a role while all other priors are ignored in the decision making.

The other model is the *smooth ambiguity* model, introduced by [Klibanoff et al. \(2005\)](#), which aggregates all priors in a weighted form:

$$\int_{\Delta} \varphi(\mathbb{E}^{\mathbb{Q}}[U(X)]) d\nu(\mathbb{Q}), \quad (3.3)$$

where each $\mathbb{Q} \in \Delta$ represents a prior probability measure; $\nu(\cdot)$ assigns a probability measure on Δ , i.e. probabilistic beliefs for all priors; and φ captures the ambiguity attitude by smoothly aggregating the expected values under each $\mathbb{Q}$. A concave (resp. convex) φ represents ambiguity aversion (resp. ambiguity loving) attitudes. The smooth ambiguity criterion (3.3) can be re-expressed as

$$\mathbb{E}_{\nu}[\varphi(\mathbb{E}^{\mathbb{Q}}[U(X)])],$$

where $\mathbb{E}_{\nu}$ denotes the expectation under the probability measure $\nu(\cdot)$, which is often referred to as the *second-order distribution*. Admittedly, smooth ambiguity offers the desired separation of risk, ambiguity, and ambiguity aversion, but its structure is usually complex and thus its application in dynamic portfolio optimization is challenging. Analytic (not necessarily fully explicit) solutions can only be obtained under very specific choices for φ , U , and ν ; see, e.g., [Kellner \(2017\)](#), [Balter et al. \(2021\)](#), and [Guan and Li \(2022\)](#).

[Maccheroni et al. \(2013\)](#) introduces a robust mean-variance (M-V) model of the form

$$\mathbb{E}^{\bar{\mathbb{Q}}}[X] - \frac{\gamma}{2} \text{Var}^{\bar{\mathbb{Q}}}[X] - \frac{\beta}{2} \text{Var}_{\nu}[\mathbb{E}^{\mathbb{Q}}[X]] \quad (3.4)$$

²There have been many empirical/experimental works on eliciting DMs' degree of ambiguity aversion (see, e.g., [Dimmock et al. \(2015\)](#), [Baillon et al. \(2018\)](#), and [Cavatorta and Schröder \(2019\)](#)).

as an analog of the classic Arrow-Pratt approximation to the smooth ambiguity model. Here, $\overline{\mathbb{Q}}$ is called the reduction measure, satisfying

$$\overline{\mathbb{Q}}(A) = \mathbb{E}_\nu [\mathbb{Q}(A)] = \int_{\Delta} \mathbb{Q}(A) d\nu(\mathbb{Q})$$

for any measurable set A , and Var_ν denotes the variance under the probability measure $\nu(\cdot)$, i.e.,

$$\text{Var}_\nu (\mathbb{E}^\mathbb{Q}[X]) = \mathbb{E}_\nu \left[(\mathbb{E}^\mathbb{Q}[X])^2 \right] - (\mathbb{E}_\nu [\mathbb{E}^\mathbb{Q}[X]])^2 = \int_{\Delta} (\mathbb{E}^\mathbb{Q}[X])^2 d\nu(\mathbb{Q}) - \left(\int_{\Delta} \mathbb{E}^\mathbb{Q}[X] d\nu(\mathbb{Q}) \right)^2.$$

Heuristically, the measure $\overline{\mathbb{Q}}$ is a weighted average of all priors. In (3.4), $\mathbb{E}^{\overline{\mathbb{Q}}}[X] - \frac{\gamma}{2} \text{Var}^{\overline{\mathbb{Q}}}[X]$ is of the M-V form to represent a DM's risk aversion (with risk aversion parameter γ); $\text{Var}_\nu (\mathbb{E}^\mathbb{Q}[X])$ measures the variance of the expected payoff under different priors; and β is the *ambiguity aversion parameter*. The robust M-V model (3.4) provides a *simpler but more intuitive* structure than smooth ambiguity. The risk and ambiguity attitudes reduce to two parameters (γ and β , respectively), compared to the two functions U and φ in smooth ambiguity. Despite the nice features, applications of the robust M-V criterion (3.4) are rather scarce in the literature, with the exception of Section 6 of [Maccheroni et al. \(2013\)](#) which considers a static portfolio optimization problem.

This chapter considers a dynamic portfolio optimization problem under a criterion that we call the *mean-variance-variance (M-V-V)* model:

$$\mathbb{E}^\mathbb{P}[X] - \frac{\gamma}{2} \text{Var}^\mathbb{P}[X] - \frac{\beta}{2} \text{Var}_\nu [\mathbb{E}^\mathbb{Q}[X]], \quad (3.5)$$

where $\mathbb{P}$ is a pre-specified probability measure called the *reference measure*. Note that the main difference between criteria (3.4) and (3.5) is that $\mathbb{P}$ is pre-specified while $\overline{\mathbb{Q}}$ needs to be derived based on the set of priors Δ and associated measure $\nu(\cdot)$. In this chapter, we adopt the M-V-V model (3.5) for the following reasons. First, $\overline{\mathbb{Q}}$ is not necessarily a prior probability measure. For example, if we only have two priors ($\Delta = \{\mathbb{Q}_1, \mathbb{Q}_2\}$) with some binary distribution, it follows that $\overline{\mathbb{Q}} \notin \Delta$ and the distribution of the payoff X is unknown under $\overline{\mathbb{Q}}$. Second, from a modeling perspective, it is natural for a DM to set up a criterion based on a reference measure (e.g., it may be the prior with the highest likelihood) and then penalize for the variance of the expected payoff under different priors. Note that [Maccheroni et al. \(2013\)](#) also assumes $\overline{\mathbb{Q}} = \mathbb{P}$ in its application to a static robust portfolio optimization problem.

The issue of time inconsistency arises in the dynamic optimization problem of maximizing (3.5) from the two variance terms, $\text{Var}^\mathbb{P}[\cdot]$ and $\text{Var}_\nu[\cdot]$. A problem is called time

inconsistent if it does not admit the Bellman optimality principle: “An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision” (p.83 on [Bellman \(1957\)](#)). More precisely, if a control policy is optimal on the time interval $[t, T]$, it should continue to be optimal for any subinterval $[s, T]$, where $s > t$. Known examples of time-inconsistent problems include the M-V criterion ([Basak and Chabakauri, 2010](#)), non-exponential discounting ([Ebert et al., 2020](#)), state-dependent risk aversion ([Björk et al., 2014](#)), S-shaped utility maximization ([Jin and Yu Zhou, 2008](#)), endogenous habit formation ([Chapman, 1998](#)), α -MEU ([Marinacci, 2002](#)), and smooth ambiguity ([Klibanoff et al., 2005](#)). It is noteworthy that the two variance terms induce time inconsistency in two *different* ways. The first variance is inherited from the original M-V criterion, which leads to time inconsistency as a result of the quadratic (nonlinear) structure of the expected terminal payoff under $\mathbb{P}$. On the other hand, the second variance $\text{Var}_\nu[\cdot]$ yields time inconsistency as it seeks to *coordinate and integrate* various values of the expected terminal payoff under different prior probability measures.

There are two major approaches in the literature for addressing time inconsistency, known as the precommitted strategy and the equilibrium strategy. The precommitted strategy focuses solely on determining the optimal strategy at the initial time and obligates the DM’s future selves to adhere to the strategy established at the initial time. This strategy, which disregards future suboptimality and results in time-inconsistent decisions, is commonly solved using the Lagrange dual method. For existing works on precommitted strategies for the M-V problems, see, for instance, [Bajeux-Besnainou and Portait \(1998\)](#), [Li and Ng \(2000\)](#), [Zhou and Li \(2000\)](#), [Bielecki et al. \(2005\)](#), [Cvitanic et al. \(2008\)](#), and [Dai et al. \(2010\)](#).

Alternatively, the equilibrium approach addresses time inconsistency by considering the DM’s future selves as different players and adopting a game-theoretic perspective to seek subgame perfect Nash equilibria (SPNE). Heuristically, the time- t DM can control her strategy only within a small time period and selects an optimal strategy that can be implemented during this period, given that future selves determine their own optimal strategies. As such, an equilibrium strategy is a time-consistent control policy. This idea can be traced back to [Strotz \(1956\)](#), but significant breakthroughs have been made more recently. For M-V problems, [Basak and Chabakauri \(2010\)](#) is the first to derive an explicit equilibrium strategy for the M-V criterion for terminal wealth optimization. Their work has been extended by many subsequent studies, including [Czichowsky \(2013\)](#), [Björk et al. \(2014\)](#), [Li et al. \(2016\)](#), [Pedersen and Peskir \(2017\)](#), and [He and Jiang \(2022\)](#). Recently, equilibrium strategies for M-V problems have also been applied to robo-advising ([Dai et al. \(2021\)](#)), insurance risk sharing ([Chen et al. \(2021\)](#)), and reinforcement learning ([Dai et al.](#)

(2023)).

In this chapter, we adopt the equilibrium approach to derive equilibrium investment strategies under the standard Black-Scholes financial market model through a formal verification argument. Our main findings are as follows: (1) The equilibrium strategies under the M-V-V criterion are fully explicit (for any second-order distribution $\nu(\cdot)$) and exhibit a nice separation of key modeling factors (risk aversion, ambiguity aversion, and ambiguity level). This contrasts with α -MEU and smooth ambiguity, where explicit or semi-explicit solutions can only be obtained under very limited specifications of utility functions, ambiguity functions, and second-order distributions (e.g., Li et al. (2019a), Balter et al. (2021), and Guan and Li (2022)). (2) The impact of ambiguity and ambiguity attitudes on the equilibrium strategy decays with time. This time decay effect also holds for smooth ambiguity, but not α -MEU or MEU (see Guan and Li (2022)). (3) The most intriguing finding is that the equilibrium strategy is not always monotone with respect to the risk aversion parameter γ . We attribute this to *horizon-scale inconsistency*—more specifically, that $\mathbb{E}^{\mathbb{P}}[X_T] - \frac{\gamma}{2}\text{Var}^{\mathbb{P}}[X_T]$ is linear in T but $\text{Var}_{\nu}[\mathbb{E}^{\mathbb{Q}}[X_T]]$ is quadratic in T . This leads DMs to postpone risky investment to reduce the heavy penalty from the ambiguity variance $\text{Var}_{\nu}[\mathbb{E}^{\mathbb{Q}}[X_T]]$ when the investment’s time horizon is long.

To address the aforementioned issue of horizon-scale inconsistency, we then introduce a new ambiguity model, referred to as the *mean-variance-standard deviation (M-V-SD)* criterion, given by

$$\mathbb{E}^{\mathbb{P}}[X] - \frac{\gamma}{2}\text{Var}^{\mathbb{P}}[X] - \frac{\beta}{2}\sigma_{\nu}[\mathbb{E}^{\mathbb{Q}}[X]], \quad (3.6)$$

where $\sigma_{\nu}[\cdot] := \sqrt{\text{Var}_{\nu}[\cdot]}$ is the standard deviation operator under ν . We find that the equilibrium strategies under the M-V-SD criterion share many properties with the M-V-V criterion, but with two notable differences. First, the equilibrium strategies become more conservative with the DM’s degree of risk aversion, which aligns better with behavioral intuition. Second, the most appealing feature of the M-V-SD criterion is that the associated equilibrium investment strategy exhibits *limited stock market participation*, given that DMs choose to abstain from investing in stocks unless a stock’s risk premium under the reference measure significantly differs from 0. This result provides theoretical justification to the empirical evidence that a significant portion of households does not invest in the stock market (see, e.g., Antoniou et al. (2015) and Dimmock et al. (2016)).

The remainder of this paper is organized as follows. Section 3.2 introduces the ambiguity model and formulates the portfolio optimization problem under a general robust M-V criterion, called the mean-variance-X (M-V-X) criterion, which includes M-V-V and M-V-SD as special cases. Section 3.3 provides a verification theorem for the equilibrium

strategies under the general M-V-X criterion. Sections 3.4 and 3.5 present our main results under the M-V-V criterion and the M-V-SD criterion, respectively. All proofs are relegated to the Appendix.

3.2 Problem Formulation

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathbb{P}$ is called the *reference measure*. Let $L^2(\Omega, \mathbb{P})$ be the set of square-integrable random variables under $\mathbb{P}$. We use $\mathbb{E}^\mathbb{P}[\cdot]$ and $\text{Var}^\mathbb{P}[\cdot]$ to denote the expectation and variance under $\mathbb{P}$, respectively. We equip the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t \in [0, T]}$, where $T > 0$ is a fixed time horizon. The filtered complete probability space is assumed to satisfy the usual conditions.

We consider the standard Black-Scholes financial market model, consisting of one risk-free bond and one risky stock. The value of the risk-free bond $\{B_t\}_{t \in [0, T]}$ has a constant interest rate $r \geq 0$ with

$$dB_t = rB_t dt.$$

The stock price process $\{S_t\}_{t \in [0, T]}$ is assumed to follow a geometric Brownian motion whose dynamics are governed by

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

where $\mu \in \mathbb{R}$ is the instantaneous expected rate of return, $\sigma > 0$ is the constant volatility, and $W = \{W_t\}_{t \in [0, T]}$ is a standard $\mathbb{F}$ -adapted $\mathbb{P}$ -Brownian motion.

Due to the time inconsistency of the M-V-V criterion, which will be elaborated upon later, we restrict the investment strategies to feedback controls, a.k.a., Markov controls. A measurable deterministic function $u : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is called an *admissible* feedback control if it satisfies the linear growth condition

$$|u(t, x)| \leq C_1(1 + |x|), \quad \text{for all } (t, x) \in [0, T] \times \mathbb{R}, \quad (3.7)$$

and the Lipschitz condition

$$|u(t, x) - u(t, y)| \leq C_2|x - y|, \quad \text{for all } (t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}, \quad (3.8)$$

where C_1 and C_2 are two positive constants. The set of all admissible investment strategies is denoted by $\mathbb{U}$. Under an investment strategy $\mathbf{u} = \{u(t, x)\}_{(t, x) \in [0, T] \times \mathbb{R}} \in \mathbb{U}$, the corresponding portfolio value process $X^\mathbf{u} = \{X_t^\mathbf{u}\}_{t \in [0, T]}$, with initial portfolio value $x_0 \in \mathbb{R}$, under $\mathbb{P}$ follows the dynamics

$$dX_t^\mathbf{u} = (rX_t^\mathbf{u} + (\mu - r)u(t, X_t^\mathbf{u})) dt + \sigma u(t, X_t^\mathbf{u}) dW_t, \quad (3.9)$$

where $u(t, X_t^{\mathbf{u}})$ is the dollar amount of wealth invested in the stock at time t . Under any investment strategy $\mathbf{u} \in \mathbb{U}$, the stochastic differential equation (3.9) admits a unique càdlàg solution $X^{\mathbf{u}}$ such that

$$\mathbb{E}^{\mathbb{P}} \left[\sup_{t \in [0, T]} (X_t^{\mathbf{u}})^2 \right] < \infty. \quad (3.10)$$

See, e.g., Theorem 1.3.16 of Pham (2009) or Theorem 2.9 in Chapter 5 of Karatzas and Shreve (2014).

It is worth noting that under any feedback control $\mathbf{u} \in \mathbb{U}$, the controlled portfolio value $X^{\mathbf{u}}$ is a Markov diffusion process (see Section III.5 of Fleming and Soner (2006)). The restriction to feedback controls is a standard assumption for the equilibrium approach to ensure the *Markovian* structure of the controlled processes (see Björk and Murgoci (2014) and Björk et al. (2017)). This assumption plays a crucial role in the proof of the verification theorem; see Step 3 of Appendix A.3. Intuitively, this stems from the backward induction principle inherent in the equilibrium approach, where an optimal decision at time t can be made only if the decisions after time t are also optimal. To illustrate the necessity, one may consider a discrete-time model. Even as the first step of backward induction, the DM in the last period is unable to determine her optimal strategy if the controls are not necessarily feedback ones, meaning that they may depend on the states at earlier times.

We then introduce ambiguity under the Black-Scholes framework. For any given $\lambda \in \mathbb{R}$, by Girsanov's Theorem, there exists a unique prior probability measure $\mathbb{Q}^{\lambda}$, equivalent to $\mathbb{P}$, such that the Radon-Nikodym derivative is given by

$$q_T^{\lambda} := \frac{d\mathbb{Q}^{\lambda}}{d\mathbb{P}} \Big|_{\mathcal{F}_T} = e^{-\frac{1}{2}\lambda^2 T - \lambda W_T}. \quad (3.11)$$

Under $\mathbb{Q}^{\lambda}$, the portfolio value follows the dynamics

$$dX_t^{\mathbf{u}} = (rX_t^{\mathbf{u}} + \alpha(\lambda)u(t, X_t^{\mathbf{u}})) dt + \sigma u(t, X_t^{\mathbf{u}}) dW_t^{\lambda}, \quad (3.12)$$

where $\{W_t^{\lambda}\}_{t \in [0, T]}$, with $dW_t^{\lambda} = dW_t + \lambda dt$, is a $\mathbb{Q}^{\lambda}$ -Brownian motion and

$$\alpha(\lambda) := \mu - r - \sigma \lambda$$

is the *risk premium* of the stock under $\mathbb{Q}^{\lambda}$. Since the drift and volatility terms in the stochastic differential equation (3.12) still satisfy the linear growth condition and the Lipschitz continuity, we know that (3.12) also admits a unique càdlàg solution $X^{\mathbf{u}}$ such that, under any prior $\mathbb{Q}^{\lambda}$,

$$\mathbb{E}^{\mathbb{Q}^{\lambda}} \left[\sup_{t \in [0, T]} (X_t^{\mathbf{u}})^2 \right] < \infty. \quad (3.13)$$

We denote by $\{X_s^{t,x,\mathbf{u}}\}_{s \in [t,T]}$ the wealth process under strategy $\mathbf{u}$ with initial value $X_t^{\mathbf{u}} = x$. It is worth noting that $\lambda \rightarrow \mathbb{E}^{\mathbb{Q}^\lambda} [X_T^{t,x,\mathbf{u}}]$ is a *continuous* function on $\mathbb{R}$. Indeed, for any $\lambda_1, \lambda_2 \in \mathbb{R}$,

$$\begin{aligned} \left| \mathbb{E}^{\mathbb{Q}^{\lambda_1}} [X_T^{t,x,\mathbf{u}}] - \mathbb{E}^{\mathbb{Q}^{\lambda_2}} [X_T^{t,x,\mathbf{u}}] \right| &= \left| \int_{\Omega} X_T^{t,x,\mathbf{u}} (q_T^{\lambda_1} - q_T^{\lambda_2}) d\mathbb{P} \right| \\ &\leq \|X_T^{t,x,\mathbf{u}}\|_{L^2(\Omega, \mathbb{P})} \|q_T^{\lambda_1} - q_T^{\lambda_2}\|_{L^2(\Omega, \mathbb{P})} \\ &= \|X_T^{t,x,\mathbf{u}}\|_{L^2(\Omega, \mathbb{P})} \left\| e^{-\frac{1}{2}\lambda_1^2 T - \lambda_1 W_T} - e^{-\frac{1}{2}\lambda_2^2 T - \lambda_2 W_T} \right\|_{L^2(\Omega, \mathbb{P})} \\ &= \|X_T^{t,x,\mathbf{u}}\|_{L^2(\Omega, \mathbb{P})} \left| e^{\lambda_1 \sqrt{T}} - e^{\lambda_2 \sqrt{T}} \right|. \end{aligned}$$

The ambiguity in a DM's perspective is characterized by a probability measure ν on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$, where $\mathcal{B}_{\mathbb{R}}$ denotes the Borel sigma-algebra on $\mathbb{R}$. Note that ν is also referred to as the *second-order distribution*, introducing an additional layer of uncertainty to decision making, reflecting various possible perspectives. For example, ν can be modeled by a discrete measure with finite mass points $\{\lambda_1, \dots, \lambda_n\}$ when the DM holds only finite prospective views. Alternatively, ν can assume a uniform distribution if the DM maintains equal confidence across a continuous set of priors, indicating no strong preference for the selection of the reference measure. It reduces to the case without ambiguity when ν is a Dirac measure. The mean and variance of $\lambda \rightarrow \mathbb{E}^{\mathbb{Q}^\lambda} [X_T^{t,x,\mathbf{u}}]$ with respect to ν are defined as

$$\mathbb{E}_\nu [\mathbb{E}^{\mathbb{Q}} [X_T^{t,x,\mathbf{u}}]] := \int_{\mathbb{R}} \mathbb{E}^{\mathbb{Q}^\lambda} [X_T^{t,x,\mathbf{u}}] d\nu(\lambda) \quad (3.14)$$

and

$$\begin{aligned} \text{Var}_\nu [\mathbb{E}^{\mathbb{Q}} [X_T^{t,x,\mathbf{u}}]] &:= \mathbb{E}_\nu \left[\left(\mathbb{E}^{\mathbb{Q}} [X_T^{t,x,\mathbf{u}}] \right)^2 \right] - \left(\mathbb{E}_\nu [\mathbb{E}^{\mathbb{Q}} [X_T^{t,x,\mathbf{u}}]] \right)^2 \\ &= \int_{\mathbb{R}} \left(\mathbb{E}^{\mathbb{Q}^\lambda} [X_T^{t,x,\mathbf{u}}] \right)^2 d\nu(\lambda) - \left(\int_{\mathbb{R}} \mathbb{E}^{\mathbb{Q}^\lambda} [X_T^{t,x,\mathbf{u}}] d\nu(\lambda) \right)^2, \end{aligned} \quad (3.15)$$

respectively. It is worth noting that since $\mathbb{E}_\nu$ and Var_ν operate on the real-valued function $\lambda \rightarrow \mathbb{E}^{\mathbb{Q}^\lambda} [X_T^{t,x,\mathbf{u}}]$, we use a dot to abbreviate the variable λ in this function.

The reward function under the M-V-V criterion is given by

$$\mathbb{E}^{\mathbb{P}} [X_T^{t,x,\mathbf{u}}] - \frac{\gamma}{2} \text{Var}^{\mathbb{P}} [X_T^{t,x,\mathbf{u}}] - \frac{\beta}{2} \text{Var}_\nu [\mathbb{E}^{\mathbb{Q}} [X_T^{t,x,\mathbf{u}}]], \quad (3.16)$$

where $\gamma > 0$ is the risk aversion parameter and $\beta > 0$ is the ambiguity aversion parameter. In later analysis, we will find out some issues with the M-V-V criterion in dynamic portfolio

optimization. Therefore, in this paper, we also propose and study the *Mean-Variance-Standard Deviation (M-V-SD) criterion* of the form

$$\mathbb{E}^{\mathbb{P}} [X_T^{t,x,\mathbf{u}}] - \frac{\gamma}{2} \text{Var}^{\mathbb{P}} [X_T^{t,x,\mathbf{u}}] - \frac{\beta}{2} \sigma_{\nu} [\mathbb{E}^{\mathbb{Q}} [X_T^{t,x,\mathbf{u}}]] , \quad (3.17)$$

where

$$\sigma_{\nu} [\mathbb{E}^{\mathbb{Q}} [X_T^{t,x,\mathbf{u}}]] := \sqrt{\text{Var}_{\nu} [\mathbb{E}^{\mathbb{Q}} [X_T^{t,x,\mathbf{u}}]]}.$$

Since our priors have been parametrized by $\lambda \in \mathbb{R}$, our second-order distribution ν is defined on the space $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$. While [Maccheroni et al. \(2006\)](#) defines the second-order distribution on the space of Radon-Nikodym derivatives $q = \frac{d\mathbb{Q}}{d\mathbb{P}}$, we will show in [Appendix A.1](#) that both definitions are consistent via a *pushforward measure*.

To unify the above two models, we consider the following more general *Mean-Variance-X (M-V-X) criterion*, namely,

$$J(t, x, \mathbf{u}) = \mathbb{E}^{\mathbb{P}} [X_T^{t,x,\mathbf{u}}] - \frac{\gamma}{2} \text{Var}^{\mathbb{P}} [X_T^{t,x,\mathbf{u}}] - \frac{\beta}{2} \mathbb{X}_{\nu} [\mathbb{E}^{\mathbb{Q}} [X_T^{t,x,\mathbf{u}}]] . \quad (3.18)$$

Here, $\mathbb{X}_{\nu} : L^2(\mathbb{R}, \nu) \rightarrow [0, \infty)$ is a non-negative functional, where $L^2(\mathbb{R}, \nu)$ denotes the set of square-integrable real-valued functions with respect to ν , and it satisfies

$$\mathbb{X}_{\nu} [c(\cdot)] = 0, \quad (3.19)$$

when $c(\lambda) \equiv c$ is a constant function for any $\lambda \in \mathbb{R}$. Clearly, Var_{ν} and σ_{ν} are two special cases satisfying (3.19). The non-negativity of $\mathbb{X}_{\nu}$ implies that the term $\mathbb{X}_{\nu} [\mathbb{E}^{\mathbb{Q}} [X_T^{t,x,\mathbf{u}}]]$ serves to penalize DM's ambiguity in the reward function (3.18). The condition (3.19) ensures that the M-V-X criterion reduces to the classical M-V criterion if the expected payoff remains unchanged under different priors. To simplify the notation, we rewrite the reward function as

$$\begin{aligned} J(t, x, \mathbf{u}) &= \mathbb{E}^{\mathbb{P}} \left[X_T^{t,x,\mathbf{u}} - \frac{\gamma}{2} (X_T^{t,x,\mathbf{u}})^2 \right] + \frac{\gamma}{2} (\mathbb{E}^{\mathbb{P}} [X_T^{t,x,\mathbf{u}}])^2 - \frac{\beta}{2} \mathbb{X}_{\nu} [\mathbb{E}^{\mathbb{Q}} [X_T^{t,x,\mathbf{u}}]] \\ &= \mathbb{E}^{\mathbb{P}} [F(X_T^{t,x,\mathbf{u}})] + G(\mathbb{E}^{\mathbb{P}} [X_T^{t,x,\mathbf{u}}]) + \mathbb{H}_{\nu} [\mathbb{E}^{\mathbb{Q}} [X_T^{t,x,\mathbf{u}}]] , \end{aligned} \quad (3.20)$$

where $F(x) := x - \frac{\gamma}{2}x^2$, $G(x) := \frac{\gamma}{2}x^2$ and $\mathbb{H}_{\nu}[\cdot] := -\frac{\beta}{2}\mathbb{X}_{\nu}[\cdot]$.

The objective of the investor is to maximize the reward function over the admissible set of investment strategies $\mathbb{U}$, that is

$$\max_{\mathbf{u} \in \mathbb{U}} J(t, x, \mathbf{u}). \quad (3.21)$$

By (3.13) and the non-negativity of $\mathbb{X}_\nu[\cdot]$, we know that $X_T^{t,x,\mathbf{u}} \in L^2(\Omega, \mathbb{P})$ and, furthermore,

$$J(t, x, \mathbf{u}) \in [-\infty, \infty)$$

for any investment strategy $\mathbf{u} \in \mathbb{U}$. This implies that the optimization problem (3.21) is well-posed, meaning the value function is finite. However, for the purpose of defining equilibrium investment strategies, it is necessary to propose the additional condition that

$$J(t, x, \mathbf{u}) > -\infty. \quad (3.22)$$

Condition (3.22) is reasonable because any strategy $\mathbf{u}$ with a reward $J(t, x, \mathbf{u}) = -\infty$ shall *not* be optimal in any sense. Moreover, since $\mathbb{X}_\nu[\cdot]$ is a functional on $L^2(\mathbb{R}, \nu)$, condition (3.22) is equivalent to

$$\lambda \rightarrow \mathbb{E}^{\mathbb{Q}^\lambda} [X_T^{t,x,\mathbf{u}}] \in L^2(\mathbb{R}, \nu). \quad (3.23)$$

Due to the continuity of the mapping $\lambda \rightarrow \mathbb{E}^{\mathbb{Q}^\lambda} [X_T^{t,x,\mathbf{u}}]$, we know that (3.23) clearly holds when the second-order distribution ν has compact support on $\mathbb{R}$. More generally, the following lemma ensures that (3.23) (and equivalently (3.22)) holds for unbounded distributions satisfying the exponential integrability condition (3.24).

In the rest of the paper, we assume that ν satisfies the integrability condition (3.24). In addition to all bounded distributions, condition (3.24) also includes unbounded distributions such as the normal distribution. The proof of Lemma 3.1 is relegated to Appendix A.2.

Lemma 3.1. *Suppose that the Borel probability measure ν satisfies*

$$\int_{\mathbb{R}} e^{C|\lambda|} d\nu(\lambda) < \infty, \quad (3.24)$$

for some sufficiently large positive constant C depending on r, μ, σ, C_1 , and T . For any fixed $(t, x) \in [0, T] \times \mathbb{R}$ and $\mathbf{u} \in \mathbb{U}$, the mapping $\lambda \rightarrow \mathbb{E}^{\mathbb{Q}^\lambda} [X_T^{t,x,\mathbf{u}}]$ belongs to $L^2(\mathbb{R}, \nu)$ and the reward function $J(t, x, \mathbf{u}) \in (-\infty, \infty)$.

The optimization problem (3.21) with reward function (3.20) has the issue of time inconsistency, stemming from both the nonlinear function $G(\cdot)$ and the functional $\mathbb{H}_\nu[\cdot]$. It is noteworthy that they induce time inconsistency in two different ways. The quadratic function $G(\cdot)$ is inherited from the original M-V criterion, with time inconsistency resulting from its nonlinear structure. On the other hand, the functional $\mathbb{H}_\nu[\cdot]$, which encompasses $\text{Var}_\nu[\cdot]$ and $\sigma_\nu[\cdot]$ operators, yields time inconsistency as it seeks to coordinate and integrate various values of the expected terminal payoff under different prior probability measures. There

are two major approaches in the literature for addressing time inconsistency, known as the precommitted strategy and the equilibrium strategy. The precommitted strategy focuses solely on determining the optimal strategy at the initial time and disregards future suboptimality. Thus, this paper adopts the equilibrium approach by considering the DM's future selves as different players and adopting a game-theoretic perspective to solve for subgame perfect Nash equilibria (SPNE).

We adopt the following well-established definition of equilibrium strategies (see, e.g., Björk and Murgoci (2010), Björk et al. (2014), and Björk et al. (2017)). The intuition behind Definition 3.1 is that the time- t DM can only control her strategy within a small time period. Therefore, she aims to solve an optimal strategy that can be executed during this period, under the assumption that her future selves have already determined their own optimal strategies.

Definition 3.1. *Given $(t, x) \in [0, T] \times \mathbb{R}$ and an investment strategy $\mathbf{u}^* = \{u^*(s, y)\}_{(s, y) \in [t, T] \times \mathbb{R}} \in \mathbb{U}$, we define a perturbed strategy $\mathbf{u}^\varepsilon$ as*

$$u^\varepsilon(s, y) = \begin{cases} \hat{u}, & s \in [t, t + \varepsilon), \\ u^*(s, y), & s \in [t + \varepsilon, T], \end{cases}$$

where $\varepsilon > 0$ and $\hat{u} \in \mathbb{R}$. Suppose that, for any $(t, x) \in [0, T] \times \mathbb{R}$ and any such perturbed strategy $\mathbf{u}^\varepsilon$, we have

$$\limsup_{\varepsilon \downarrow 0} \frac{J(t, x, \mathbf{u}^\varepsilon) - J(t, x, \mathbf{u}^*)}{\varepsilon} \leq 0.$$

Then, $\mathbf{u}^*$ is an equilibrium investment strategy to Problem (3.21), and $J(t, x, \mathbf{u}^*)$ is the corresponding equilibrium value function.

3.3 Verification Theorem

In this section, we present a verification theorem for the equilibrium strategies of the dynamic portfolio optimization problem (3.21).

We say a function $\phi(t, x, \lambda) \in C^{1,2,0}([0, T] \times \mathbb{R} \times \mathbb{R})$ if it is (jointly) continuous on $[0, T] \times \mathbb{R} \times \mathbb{R}$, and the partial derivatives $\phi_t := \frac{\partial \phi}{\partial t}$, $\phi_x := \frac{\partial \phi}{\partial x}$, $\phi_{xx} := \frac{\partial^2 \phi}{\partial x^2}$ are continuous on $[0, T] \times \mathbb{R} \times \mathbb{R}$. For any $u \in \mathbb{R}$ and $\lambda \in \mathbb{R}$, we define a differential operator

$$\mathcal{A}^{u, \lambda} \phi(t, x, \lambda) := \phi_t(t, x, \lambda) + (rx + (\mu - r - \lambda\sigma)u) \phi_x(t, x, \lambda) + \frac{1}{2} \sigma^2 u^2 \phi_{xx}(t, x, \lambda), \quad (3.25)$$

where $\phi(t, x, \lambda) \in C^{1,2,0}([0, T] \times \mathbb{R} \times \mathbb{R})$. Besides, we define

$$\mathcal{A}^u \phi(t, x) := \phi_t(t, x) + (rx + (\mu - r)u) \phi_x(t, x) + \frac{1}{2} \sigma^2 u^2 \phi_{xx}(t, x),$$

where $\phi(t, x) \in C^{1,2}([0, T] \times \mathbb{R})$.

Definition 3.2. Consider an admissible control $\mathbf{u} = \{u(t, x)\}_{(t,x) \in [0,T] \times \mathbb{R}} \in \mathbb{U}$. We say that a function $\phi(t, x, \lambda) : [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ belongs to the space $L_T^2(X^{\mathbf{u}})$ if

$$\mathbb{E}^{\mathbb{Q}^\lambda} \left[\int_t^T \phi_x(s, X_s^{t,x,\mathbf{u}}, \lambda)^2 u(s, X_s^{t,x,\mathbf{u}})^2 ds \right] < \infty,$$

for every $(t, x, \lambda) \in [0, T] \times \mathbb{R} \times \mathbb{R}$. For brevity of notation, we also say that a function $\phi(t, x) : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ belongs to the space $L_T^2(X^{\mathbf{u}})$ if

$$\mathbb{E}^{\mathbb{P}} \left[\int_t^T \phi_x(s, X_s^{t,x,\mathbf{u}})^2 u(s, X_s^{t,x,\mathbf{u}})^2 ds \right] < \infty,$$

for every $(t, x) \in [0, T] \times \mathbb{R}$.

A sufficient condition for $\phi \in L_T^2(X^{\mathbf{u}})$ is that ϕ_x has at most quadratic growth in x , and the feedback control $\mathbf{u}$ is independent of x . The latter holds for our equilibrium strategies in Theorems 3.2 and 3.3, as well as all constant strategies.

The following definition of the Gâteaux derivative of the functional $\mathbb{H}_\nu[\cdot]$ will be used in the verification theorem.

Definition 3.3. We say the functional $\mathbb{H}_\nu : L^2(\mathbb{R}, v) \rightarrow [0, \infty)$ is Gâteaux differentiable at $f \in L^2(\mathbb{R}, v)$ if there exists a linear continuous operator denoted by $\frac{\delta \mathbb{H}_\nu[f(\cdot)]}{\delta f(\cdot)} : L^2(\mathbb{R}, v) \rightarrow \mathbb{R}$ such that

$$\left\langle \frac{\delta \mathbb{H}_\nu[f(\cdot)]}{\delta f(\cdot)}, \varphi(\cdot) \right\rangle := \lim_{\varepsilon \rightarrow 0} \frac{\mathbb{H}_\nu[f(\cdot) + \varepsilon \varphi(\cdot)] - \mathbb{H}_\nu[f(\cdot)]}{\varepsilon} = \frac{d}{d\varepsilon} \mathbb{H}_\nu[f(\cdot) + \varepsilon \varphi(\cdot)] \Big|_{\varepsilon=0}, \quad (3.26)$$

for any $\varphi \in L^2(\mathbb{R}, v)$.

When $\mathbb{H}_\nu$ is Gâteaux differentiable at f , by Taylor expansion,

$$\mathbb{H}_\nu[f + \varepsilon \varphi] - \mathbb{H}_\nu[f] = \varepsilon \frac{d}{d\varepsilon} \mathbb{H}_\nu[f + \varepsilon \varphi] \Big|_{\varepsilon=0} + o(\varepsilon) = \varepsilon \left\langle \frac{\delta \mathbb{H}_\nu[f]}{\delta f}, \varphi \right\rangle + o(\varepsilon). \quad (3.27)$$

We now present a verification theorem for the M-V-X maximization problem (3.21). The proof of this verification theorem is relegated to Appendix A.3. Note that we use $g \circ f$ to denote the composition of the functions g and f .

Theorem 3.1. *Suppose that there exist $V(t, x) \in C^{1,2}([0, T] \times \mathbb{R})$ and $g(t, x, \lambda) \in C^{1,2,0}([0, T] \times \mathbb{R} \times \mathbb{R})$ satisfying the following conditions:*

1. *For any $(t, x) \in [0, T] \times \mathbb{R}$, the functional $\mathbb{H}_\nu$ is Gâteaux differentiable at $g(t, x, \cdot)$, and moreover, $\mathbb{H}_\nu[g(t, x, \cdot)] \in C^{1,2}([0, T] \times \mathbb{R})$.*
2. *For any $(t, x) \in [0, T] \times \mathbb{R}$,*

$$\sup_{u \in \mathbb{R}} \left\{ \mathcal{A}^u V(t, x) + (G_x \circ g)(t, x) \mathcal{A}^u g(t, x) - \mathcal{A}^u (G \circ g)(t, x) + \left\langle \frac{\delta \mathbb{H}_\nu[g(t, x, \cdot)]}{\delta g(t, x, \cdot)}, \mathcal{A}^{u, \cdot} g(t, x, \cdot) \right\rangle - \mathcal{A}^u \mathbb{H}_\nu[g(t, x, \cdot)] \right\} = 0, \quad (3.28)$$

where $g(t, x) := g(t, x, 0)$, and $\mathcal{A}^{u, \cdot} g(t, x, \cdot) \in L^2(\mathbb{R}, \nu)$ for any $u \in \mathbb{R}$.

3. *There is a measurable function $u^*(t, x)$, such that $\mathbf{u}^* = \{u^*(t, x)\}_{(t, x) \in [0, T] \times \mathbb{R}} \in \mathbb{U}$, satisfying*

$$\mathcal{A}^{u^*} V(t, x) + (G_x \circ g)(t, x) \mathcal{A}^{u^*} g(t, x) - \mathcal{A}^{u^*} (G \circ g)(t, x) + \left\langle \frac{\delta \mathbb{H}_\nu[g(t, x, \cdot)]}{\delta g(t, x, \cdot)}, \mathcal{A}^{u^*, \cdot} g(t, x, \cdot) \right\rangle - \mathcal{A}^{u^*} \mathbb{H}_\nu[g(t, x, \cdot)] = 0, \quad (3.29)$$

and, for any $(t, x, \lambda) \in [0, T] \times \mathbb{R} \times \mathbb{R}$,

$$\mathcal{A}^{u^*, \lambda} g(t, x, \lambda) = 0. \quad (3.30)$$

4. *For any $(x, \lambda) \in \mathbb{R} \times \mathbb{R}$,*

$$V(T, x) = x \text{ and } g(T, x, \lambda) = x. \quad (3.31)$$

5. *The functions $g(t, x, \lambda)$, $V(t, x)$, $(G \circ g)(t, x)$, and $\mathbb{H}_\nu[g(t, x, \cdot)]$ all belong to $L_T^2(X^{\mathbf{u}^*}) \cap L_T^2(X^{\hat{\mathbf{u}}})$, where $\hat{\mathbf{u}} = \{\hat{u}(t, x)\}_{(t, x) \in [0, T] \times \mathbb{R}}$ with $\hat{u}(t, x) \equiv \hat{u}$ for all $(t, x) \in [0, T] \times \mathbb{R}$ is a constant control.*

Then, $\mathbf{u}^* = \{u^*(t, x)\}_{(t, x) \in [0, T] \times \mathbb{R}}$ is an equilibrium investment strategy to the M-V-X maximization problem (3.21). Moreover,

$$V(t, x) = J(t, x, \mathbf{u}^*), \quad g(t, x) = \mathbb{E}^\mathbb{P}[X_T^{t, x, \mathbf{u}^*}], \quad \text{and} \quad g(t, x, \lambda) = \mathbb{E}^{\mathbb{Q}^\lambda}[X_T^{t, x, \mathbf{u}^*}].$$

Since the Hamilton-Jacobi-Bellman type of equation (3.28) arises from a time-inconsistent problem, we refer to it as an extended Hamilton-Jacobi-Bellman (EHJB) equation following the seminal work by Björk and Murgoci (2010). In the following two examples, we consider the M-V-V and the M-V-SD criterion for which the EHJB equation (3.28) can be expressed more explicitly. Furthermore, we show that the two functionals, Var_v and σ_v , are actually Fréchet differentiable, which is stronger than Gâteaux differentiability (e.g., Kesavan (2004)).

Example 3.1 (M-V-V criterion). *Consider the M-V-V criterion in which $\mathbb{X}_v = \text{Var}_v$. For any $f, \varphi \in L^2(\mathbb{R}, v)$, by (3.26),*

$$\begin{aligned} & \left\langle \frac{\delta \text{Var}_v[f(\cdot)]}{\delta f(\cdot)}, \varphi(\cdot) \right\rangle \\ &= \frac{d}{d\varepsilon} \text{Var}_v[f(\cdot) + \varepsilon \varphi(\cdot)] \Big|_{\varepsilon=0} \\ &= \frac{d}{d\varepsilon} \left\{ \text{Var}_v[f(\cdot)] + 2\varepsilon \text{Cov}_v[f(\cdot), \varphi(\cdot)] + \varepsilon^2 \text{Var}_v[\varphi(\cdot)] \right\} \Big|_{\varepsilon=0} \\ &= 2\text{Cov}_v[f(\cdot), \varphi(\cdot)], \end{aligned}$$

where Cov_v is the covariance operator under v . Moreover, the Gâteaux derivative $\frac{\delta \text{Var}_v[f]}{\delta f} = 2\text{Cov}_v[f, \cdot] : L^2(\mathbb{R}, v) \rightarrow \mathbb{R}$ is a linear and continuous³ operator. Then, the EHJB equation (3.28) becomes

$$\begin{aligned} & \sup_{u \in \mathbb{R}} \left\{ \mathcal{A}^u V(t, x) + (G_x \circ g)(t, x) \mathcal{A}^u g(t, x) - \mathcal{A}^u (G \circ g)(t, x) \right. \\ & \quad \left. - \frac{\beta}{2} (2\text{Cov}_v[g(t, x, \cdot), \mathcal{A}^{u, \cdot} g(t, x, \cdot)] - \mathcal{A}^u \text{Var}_v[g(t, x, \cdot)]) \right\} = 0. \end{aligned} \quad (3.32)$$

Furthermore, we show that Var_v is Fréchet differentiable at any $f \in L^2(\mathbb{R}, v)$. For any $\varphi \in L^2(\mathbb{R}, v)$, by

$$\text{Var}_v[\varphi(\cdot)] = \int_{\mathbb{R}} \varphi^2(\lambda) d\nu(\lambda) - \left(\int_{\mathbb{R}} \varphi(\lambda) d\nu(\lambda) \right)^2 \leq \int_{\mathbb{R}} \varphi^2(\lambda) d\nu(\lambda) = \|\varphi\|_{L^2(\mathbb{R}, v)}^2,$$

³The continuity is in terms of the topology induced by $L^2(\mathbb{R}, v)$.

we have

$$\begin{aligned}
& \lim_{\|\varphi\|_{L^2(\mathbb{R},v)} \rightarrow 0} \frac{|\text{Var}_\nu[f(\cdot) + \varphi(\cdot)] - \text{Var}_\nu[f(\cdot)] - 2\text{Cov}_\nu[f(\cdot), \varphi(\cdot)]|}{\|\varphi\|_{L^2(\mathbb{R},v)}} \\
&= \lim_{\|\varphi\|_{L^2(\mathbb{R},v)} \rightarrow 0} \frac{\text{Var}_\nu[\varphi(\cdot)]}{\|\varphi\|_{L^2(\mathbb{R},v)}} \\
&\leq \lim_{\|\varphi\|_{L^2(\mathbb{R},v)} \rightarrow 0} \|\varphi\|_{L^2(\mathbb{R},v)} \\
&= 0.
\end{aligned}$$

This implies the Fréchet derivative of Var_ν at f is equal to $2\text{Cov}_\nu[f, \cdot]$.

Example 3.2 (M-V-SD criterion). Consider the M-V-SD criterion in which $\mathbb{X}_\nu[\cdot] = \sigma_\nu[\cdot] = \sqrt{\text{Var}_\nu[\cdot]}$. For any $f, \varphi \in L^2(\mathbb{R}, v)$ with $\sigma_\nu[f] > 0$, by (3.26) and Example 3.1,

$$\begin{aligned}
\left\langle \frac{\delta \sigma_\nu[f(\cdot)]}{\delta f(\cdot)}, \varphi(\cdot) \right\rangle &= \frac{d}{d\varepsilon} \sigma_\nu[f(\cdot) + \varepsilon \varphi(\cdot)] \Big|_{\varepsilon=0} \\
&= \frac{\frac{d}{d\varepsilon} \text{Var}_\nu[f(\cdot) + \varepsilon \varphi(\cdot)]}{2\sigma_\nu[f(\cdot) + \varepsilon \varphi(\cdot)]} \Big|_{\varepsilon=0} \\
&= \frac{\text{Cov}_\nu[f(\cdot), \varphi(\cdot)]}{\sigma_\nu[f(\cdot)]}.
\end{aligned}$$

Moreover, the Gâteaux derivative $\frac{\delta \sigma_\nu[f]}{\delta f} = \frac{\text{Cov}_\nu[f, \cdot]}{\sigma_\nu[f]} : L^2(\mathbb{R}, v) \rightarrow \mathbb{R}$ is a linear and continuous operator. Then, the EHJB equation (3.28) becomes

$$\begin{aligned}
& \sup_{u \in \mathbb{R}} \left\{ \mathcal{A}^u V(t, x) + (G_x \circ g)(t, x) \mathcal{A}^u g(t, x) - \mathcal{A}^u (G \circ g)(t, x) \right. \\
& \quad \left. - \frac{\beta}{2} \left[\frac{\text{Cov}_\nu[g(t, x, \cdot), \mathcal{A}^{u, \cdot} g(t, x, \cdot)]}{\sigma_\nu[g(t, x, \cdot)]} - \mathcal{A}^u \sigma_\nu[g(t, x, \cdot)] \right] \right\} = 0. \quad (3.33)
\end{aligned}$$

Furthermore, we show that σ_ν is Fréchet differentiable at any f with $\sigma_\nu[f] > 0$. For any $\varphi \in L^2(\mathbb{R}, v)$, by Taylor expansion that $(x + h)^{\frac{1}{2}} = x^{\frac{1}{2}} + \frac{1}{2}x^{-\frac{1}{2}}h + o(h)$ for $x > 0$, we have

$$\begin{aligned}
\sigma_\nu[f(\cdot) + \varphi(\cdot)] &= (\text{Var}_\nu[f(\cdot)] + 2\text{Cov}_\nu[f(\cdot), \varphi(\cdot)] + \text{Var}_\nu[\varphi(\cdot)])^{\frac{1}{2}} \\
&= \sigma_\nu[f(\cdot)] + \frac{2\text{Cov}_\nu[f(\cdot), \varphi(\cdot)] + \text{Var}_\nu[\varphi(\cdot)]}{2\sigma_\nu[f(\cdot)]} + o(2\text{Cov}_\nu[f(\cdot), \varphi(\cdot)] + \text{Var}_\nu[\varphi(\cdot)]) \\
&= \sigma_\nu[f(\cdot)] + \frac{2\text{Cov}_\nu[f(\cdot), \varphi(\cdot)] + \text{Var}_\nu[\varphi(\cdot)]}{2\sigma_\nu[f(\cdot)]} + o(\|\varphi\|_{L^2(\mathbb{R},v)}),
\end{aligned}$$

where the last step is by

$$\begin{aligned}
|2\text{Cov}_\nu[f(\cdot), \varphi(\cdot)] + \text{Var}_\nu[\varphi(\cdot)]| &\leq 2\sigma_\nu[f(\cdot)]\sigma_\nu[\varphi(\cdot)] + \text{Var}_\nu[\varphi(\cdot)] \\
&\leq 2\sigma_\nu[f(\cdot)]\|\varphi\|_{L^2(\mathbb{R},v)} + \|\varphi\|_{L^2(\mathbb{R},v)}^2 \\
&= \|\varphi\|_{L^2(\mathbb{R},v)} (2\sigma_\nu[f(\cdot)] + \|\varphi\|_{L^2(\mathbb{R},v)}) .
\end{aligned}$$

Then,

$$\begin{aligned}
&\lim_{\|\varphi\|_{L^2(\mathbb{R},v)} \rightarrow 0} \frac{\left| \sigma_\nu[f(\cdot) + \varphi(\cdot)] - \sigma_\nu[f(\cdot)] - \frac{\text{Cov}_\nu[f(\cdot), \varphi(\cdot)]}{\sigma_\nu[f(\cdot)]} \right|}{\|\varphi\|_{L^2(\mathbb{R},v)}} \\
&= \lim_{\|\varphi\|_{L^2(\mathbb{R},v)} \rightarrow 0} \frac{\left| \frac{\text{Var}_\nu[\varphi(\cdot)]}{2\sigma_\nu[f(\cdot)]} + o(\|\varphi\|_{L^2(\mathbb{R},v)}) \right|}{\|\varphi\|_{L^2(\mathbb{R},v)}} \\
&\leq \frac{1}{2\sigma_\nu[f(\cdot)]} \lim_{\|\varphi\|_{L^2(\mathbb{R},v)} \rightarrow 0} \frac{\|\varphi\|_{L^2(\mathbb{R},v)}^2}{\|\varphi\|_{L^2(\mathbb{R},v)}} \\
&= 0.
\end{aligned}$$

This implies the Fréchet derivative of σ_ν at f is equal to $\frac{\text{Cov}_\nu[f, \cdot]}{\sigma_\nu[f]}$.

3.4 Equilibrium Strategies Under the M-V-V Criterion

In this section, we derive and analyze the equilibrium investment strategies to maximize the M-V-V reward function, namely

$$J(t, x, \mathbf{u}) = \mathbb{E}^\mathbb{P} [X_T^{t,x,\mathbf{u}}] - \frac{\gamma}{2} \text{Var}^\mathbb{P} [X_T^{t,x,\mathbf{u}}] - \frac{\beta}{2} \text{Var}_\nu [\mathbb{E}^\mathbb{Q} [X_T^{t,x,\mathbf{u}}]] . \quad (3.34)$$

For ease of notation, we denote the variance of the second-order distribution ν by

$$\delta_\nu^2 := \int_{\mathbb{R}} \lambda^2 d\nu(\lambda) - \left(\int_{\mathbb{R}} \lambda d\nu(\lambda) \right)^2 .$$

Theorem 3.2. *An equilibrium investment strategy to maximize the M-V-V reward function (3.34) is given by*

$$u_t^* = \frac{\mu - r}{\gamma\sigma^2} e^{-\left(\frac{\beta\delta_\nu^2}{\gamma} + r\right)(T-t)} . \quad (3.35)$$

Moreover, the associated equilibrium value functions are given by

$$V(t, x) = J(t, x, \mathbf{u}^*) = e^{r(T-t)}x + \frac{(\mu - r)^2}{4\beta\sigma^2\delta_\nu^2}(1 - e^{-\frac{2\beta\delta_\nu^2}{\gamma}(T-t)})$$

and

$$g(t, x, \lambda) = \mathbb{E}^{\mathbb{Q}^\lambda}[X_T^{t,x,\mathbf{u}^*}] = e^{r(T-t)}x + \frac{\alpha(\lambda)(\mu - r)}{\beta\sigma^2\delta_\nu^2}(1 - e^{-\frac{\beta\delta_\nu^2}{\gamma}(T-t)}),$$

where we recall $\alpha(\lambda) = \mu - r - \sigma\lambda$.

We now examine the properties of the equilibrium investment strategy (3.35) in more detail. First, note that in the absence of ambiguity regarding the stock return (i.e., ν is a degenerate probability distribution), then $\delta_\nu^2 = 0$ and (3.35) reduces to

$$u_t^{\text{NR}} = \frac{\mu - r}{\gamma\sigma^2}e^{-r(T-t)}, \quad (3.36)$$

which corresponds to the equilibrium strategy in the standard M-V criterion; see Proposition 7.1 of Björk and Murgoci (2010). The superscript NR stands for non-robust.

Second, the sign of u_t^* is identical to that of $\mu - r$, the stock's risk premium under $\mathbb{P}$. It follows that the equilibrium investment strategy is to take a long (resp. short) position on the stock when its risk premium under the reference measure $\mathbb{P}$ is positive (resp. negative).

Third, the equilibrium investment strategy is more conservative if the investor faces more ambiguity or is more ambiguity-averse. This can be seen as $|u_t^*|$ decreases with respect to δ_ν^2 and β . This result is intuitive and also aligns with other ambiguity models; see Proposition 3.3 and Remark 3.5 of Li et al. (2016) for α -MEU, and Proposition 3.1 of Guan and Li (2022) for smooth ambiguity. In particular, when the investor is ambiguity-neutral, i.e., $\beta \downarrow 0$, the equilibrium strategy becomes

$$u_t^{\text{AN}} = \frac{\mu - r}{\gamma\sigma^2}e^{-r(T-t)} = u_t^{\text{NR}},$$

where the superscript AN stands for ambiguity-neutral. Intuitively, the ambiguity-neutral strategy is identical to the non-robust strategy in the absence of ambiguity.

Fourth, $|u_t^*|$ decreases with the remaining time to maturity $T - t$, and further, for all values of β and δ_ν^2 , they converge to the same limit; that is,

$$|u_t^*| \uparrow |u_T^{\text{AN}}|, \quad \text{when } t \rightarrow T.$$

This implies that the impact of ambiguity and ambiguity attitudes decays with time and all equilibrium strategies (for different levels of ambiguity and ambiguity aversion) converge to the ambiguity-neutral strategy. As illustrated in Guan and Li (2022), this time decay effect is a special feature of smooth ambiguity which is not observed in the MEU or α -MEU models.

Fifth, we shall stress that the M-V-V criterion has the merit of generating a *fully explicit* equilibrium strategy in comparison with other ambiguity models. All key factors (perceived level of ambiguity δ_ν^2 , ambiguity aversion β , and risk aversion γ) appear explicitly in the solution. Maccheroni et al. (2013) demonstrates that the M-V-V criterion is an approximation to the general smooth ambiguity model by offering a much more tractable structure under milder modeling assumptions. Our results in continuous time further stress this key merit of the M-V-V criterion. Indeed, as a comparison, closed-form equilibrium strategies have been obtained for α -MEU and smooth ambiguity only under very specific assumptions: for example, α -MEU with M-V or log utility and entropy penalty (see Li et al. (2016) and Li et al. (2019a)); and smooth ambiguity with M-V or exponential utility, power ambiguity function, and normally distributed second-order distribution (see Guan et al. (2018) and Balter et al. (2021)).

Last, it is most intriguing to see that the equilibrium strategy (3.35) is *not always monotone* with respect to the risk aversion parameter γ . A “more natural” conjecture would be that $|u_t^*|$ decreases with γ , meaning that the equilibrium strategy becomes more conservative if the investor is more risk averse. This would be the case if the ambiguity aversion parameter β is linear or of a higher order than the risk aversion parameter γ . That is,

$$\beta = c\gamma^k, \quad \text{for } k \geq 1, \quad (3.37)$$

for some constant $c > 0$. The condition (3.37) aligns with the empirical studies showing evidence of a positive correlation between risk aversion and ambiguity aversion (see Bossaerts et al. (2010)). Nevertheless, in general, when β and γ are two “independent” parameters, we see from (3.35) that $|u_t^*|$ *increases* with γ when

$$T - t > \frac{\gamma}{\beta\delta_\nu^2}. \quad (3.38)$$

The key issue leading to the non-monotonicity of γ is that the M-V-V criterion does not have a consistent scale of time horizon. We use a simple example to illustrate this. Recall that

$$W_t^\lambda = W_t + \lambda t, \quad t \in [0, T]$$

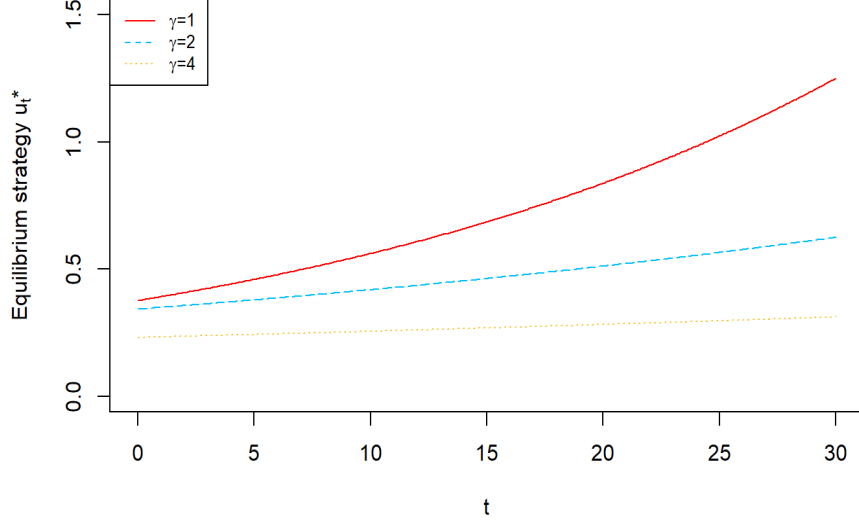


Figure 3.1: Equilibrium strategies under the M-V-V criterion for different levels of risk aversion ($T = 30$).

is a $\mathbb{Q}^\lambda$ -Brownian motion. It follows that

$$\begin{aligned}
& \mathbb{E}^{\mathbb{P}} [W_T] - \frac{\gamma}{2} \text{Var}^{\mathbb{P}} [W_T] - \frac{\beta}{2} \text{Var}_{\nu} [\mathbb{E}^{\mathbb{Q}} [W_T]] \\
&= 0 - \frac{\gamma}{2} T - \frac{\beta}{2} \left[\int_{\mathbb{R}} \left(\mathbb{E}^{\mathbb{Q}^\lambda} [W_T^\lambda - \lambda T] \right)^2 d\nu(\lambda) - \left(\int_{\mathbb{R}} \mathbb{E}^{\mathbb{Q}^\lambda} [W_T^\lambda - \lambda T] d\nu(\lambda) \right)^2 \right] \\
&= 0 - \frac{\gamma}{2} T - \frac{\beta}{2} T^2 \left[\int_{\mathbb{R}} \lambda^2 d\nu(\lambda) - \left(\int_{\mathbb{R}} \lambda d\nu(\lambda) \right)^2 \right] \\
&= 0 - \frac{\gamma}{2} T - \frac{\beta \delta_{\nu}^2}{2} T^2.
\end{aligned} \tag{3.39}$$

It is seen that $\text{Var}^{\mathbb{P}} [W_T]$ is *linear* in T , while $\text{Var}_{\nu} [\mathbb{E}^{\mathbb{Q}} [W_T]]$ is *quadratic*. This implies that the penalty from the ambiguity-variance term is of higher order than the first two M-V terms with respect to the time horizon. Note that such a “*linear–linear–quadratic relation in T* ” in (3.39) holds for much more general underlying processes such as Lévy processes. We refer to this issue as the *horizon-scale inconsistency* of the M-V-V criterion.

To visualize this issue, Figures 3.1 and 3.2 plot the equilibrium strategy u_t^* under the

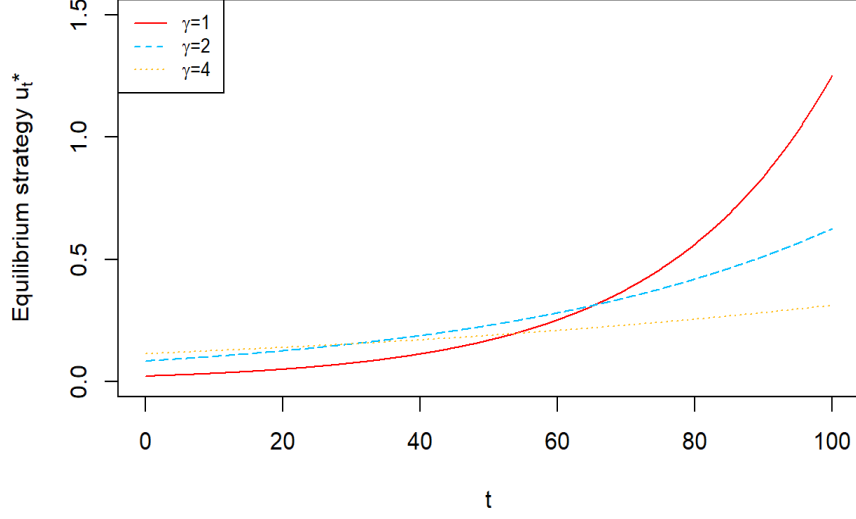


Figure 3.2: Equilibrium strategies under the M-V-V criterion for different levels of risk aversion ($T = 100$).

M-V-V criterion for different risk aversion levels. The other parameters are assumed to be $\mu = 0.05$, $\sigma = 0.2$, $\beta = 2$, $\delta_\nu^2 = 0.02$, and $r = 0$. Figure 3.1 uses a time horizon of $T = 30$ while Figure 3.2 considers $T = 100$. In Figure 3.1, we see that u_t^* is monotonically decreasing in γ . However, Figure 3.2 shows that the monotonicity with respect to γ may fail for a sufficiently long time horizon.

The economic intuition is as follows: Given that the ambiguity variance term $\text{Var}_\nu[\cdot]$ heavily penalizes the investment horizon, a DM prefers to *postpone risky investment* to mitigate the penalty stemming from the ambiguity variance term. As an extreme case, letting $\gamma \downarrow 0$ in (3.35), we have

$$u_t^* = \begin{cases} 0, & t < T, \\ \infty, & t = T. \end{cases}$$

This means that risk-neutral DMs would choose to postpone all risky investments to the last moment. Risk-averse DMs would still invest some amount in stock prior to maturity for the purpose of risk diversification. By comparing the investment time schedules of DMs with different levels of risk aversion in Figure 3.2, we observe that a more risk-averse DM might invest slightly more aggressively when the time horizon is long, but will adopt a

significantly more conservative approach as the maturity approaches. Notably, disparities in u_t^* (for different values of γ) are more significant when $t \approx T$ than $t \approx 0$.

We shall point out that the non-monotonicity of u_t^* in relation to risk aversion is a *new finding* of the M-V-V criterion in *multi-period* models. While [Maccheroni et al. \(2013\)](#) studies some single-period portfolio optimization problems under M-V-V, their strategies are always monotone with respect to their risk aversion parameter γ ; see their equation (25). This is because DMs in multi-period models have the option to postpone their investment to reduce ambiguity variance, while this option is not available in static decision models.

3.5 Equilibrium Strategies under the M-V-SD Criterion

To address the aforementioned issue of the M-V-V criterion's horizon-scale inconsistency, we now study the equilibrium strategies under the Mean-Variance-Standard Deviation (M-V-SD) criterion by defining the reward function as

$$J(t, x, \mathbf{u}) = \mathbb{E}^{\mathbb{P}} [X_T^{t,x,\mathbf{u}}] - \frac{\gamma}{2} \text{Var}^{\mathbb{P}} [X_T^{t,x,\mathbf{u}}] - \frac{\beta}{2} \sigma_{\nu} [\mathbb{E}^{\mathbb{Q}} [X_T^{t,x,\mathbf{u}}]] . \quad (3.40)$$

One can see from (3.39) that the rationale for this modification is to have a consistent linear scale of the investment horizon in the reward function.

The following theorem presents the equilibrium investment strategies under the M-V-SD criterion. Recall that

$$\delta_{\nu} = \sqrt{\int_{\mathbb{R}} \lambda^2 d\nu(\lambda) - \left(\int_{\mathbb{R}} \lambda d\nu(\lambda) \right)^2}$$

is the standard deviation of the second-order distribution ν .

Theorem 3.3. *An equilibrium investment strategy that maximizes the M-V-SD reward function (3.40) is given by*

$$u_t^* = \begin{cases} \frac{\mu - r - \frac{1}{2}\beta\sigma\delta_{\nu}}{\gamma\sigma^2} e^{-r(T-t)}, & \text{if } \mu - r > \frac{1}{2}\beta\sigma\delta_{\nu}, \\ 0, & \text{if } \mu - r \in \left[-\frac{1}{2}\beta\sigma\delta_{\nu}, \frac{1}{2}\beta\sigma\delta_{\nu}\right], \\ \frac{\mu - r + \frac{1}{2}\beta\sigma\delta_{\nu}}{\gamma\sigma^2} e^{-r(T-t)}, & \text{if } \mu - r < -\frac{1}{2}\beta\sigma\delta_{\nu}. \end{cases} \quad (3.41)$$

Moreover, the associated equilibrium value functions are given by

$$V(t, x) = J(t, x, \mathbf{u}^*) = e^{r(T-t)}x + \frac{(\mu - r - \text{sgn}(\mu - r)\frac{\beta\sigma}{2}\delta_\nu) u_t^* e^{r(T-t)}}{2}(T - t) \quad (3.42)$$

and

$$g(t, x, \lambda) = \mathbb{E}^{\mathbb{Q}^\lambda}[X_T^{t,x,\mathbf{u}^*}] = e^{r(T-t)}x + \alpha(\lambda)u_t^* e^{r(T-t)}(T - t). \quad (3.43)$$

We find that the M-V-SD strategy (3.41) shares a lot of properties with the M-V-V strategy: 1) it reduces to the non-robust strategy (3.36) in the absence of ambiguity ($\delta_\nu = 0$); 2) $|u_t^*|$ decreases with β and δ_ν ; 3) $|u_t^*|$ decreases with $T - t$ and converges to the ambiguity-neutral strategy as $t \rightarrow T$.

However, contrary to the M-V-V strategy, the M-V-SD strategy $|u_t^*|$ is monotonically decreasing in risk aversion γ , which better fits behavioral intuition. It is seen that, when $r = 0$, the M-V-SD value functions (3.42) and (3.43) are both linear in time to maturity, $T - t$. As such, the DM has no incentive to postpone risky investment.

The most appealing feature of the M-V-SD strategy (3.41) is that it yields a notable “no-trading zone” of the stock,

$$\mu - r \in \left[-\frac{1}{2}\beta\sigma\delta_\nu, \frac{1}{2}\beta\sigma\delta_\nu \right].$$

That is, the DM takes a long (resp. short) position in the stock only if its risk premium under the reference measure, $\mu - r$, is significantly positive (resp. negative). Otherwise, the DM refrains from participating in risky investments. Moreover, the no-trading zone $[-\frac{1}{2}\beta\sigma\delta_\nu, \frac{1}{2}\beta\sigma\delta_\nu]$ expands with the DM’s level of ambiguity δ_ν as well as her level of ambiguity aversion β .

It is a well-known stylized fact that a significant portion of households does not invest in stock markets. To the best of our knowledge, Cao et al. (2005) is the first to justify that *limited participation in stock markets* is possible in the presence of ambiguity, using the MEU model. This finding is also supported by empirical work. For instance, Antoniou et al. (2015) finds that increases in ambiguity are associated with outflows from equity funds. Dimmock et al. (2016) investigates the negative relationship between ambiguity aversion and stock market participation, but only for subjects who perceive stock returns to be highly ambiguous. There are a few theoretical extensions that adopt a similar worst-case approach to demonstrate the phenomenon of limited stock market participation in the presence of ambiguity, for example, Biagini and Pinar (2017) and Lin and Riedel (2021).

Our M-V-SD model provides further theoretical justification for limited stock market participation but differs from existing literature (e.g., [Cao et al. \(2005\)](#), [Biagini and Pinar \(2017\)](#), and [Lin and Riedel \(2021\)](#)) in two major respects. First, previous works rely on the assumption of bounded priors (i.e., $\nu(\cdot)$ has compact support) to derive a no-trading parameter set. However, the M-V-SD model does not rely on this assumption, and the support of the second-order distribution $\nu(\cdot)$ can be unbounded. Second, previous works are based on the MEU model, and thus the no-trading zone depends only on extreme priors, whereas our M-V-SD model integrates perspectives under all priors.

Chapter 4

Robust Mean-Variance Reinsurance Optimization

4.1 Introduction

Reinsurance is crucial to the insurance industry, contributing significantly to its development and the stability of the global financial system. Recently, unpredictable risks, including public health events, climate change, and financial instability, have heightened the importance of reinsurance under model uncertainty. This has made it a critical research topic, providing essential guidance for risk management. The study of an insurer's optimal reinsurance strategy has garnered significant attention in actuarial science literature under various criteria. Early studies on optimal reinsurance date back to [Borch \(1960\)](#), who focused on risk minimization, and [Arrow \(1978\)](#), who concentrated on expected utility maximization. [David and Young \(2005\)](#) identified the optimal proportional reinsurance strategy by minimizing the probability of ruin, while [Bai and Guo \(2008\)](#) examined proportional reinsurance strategies for both minimizing the probability of ruin and maximizing the expected exponential utility of terminal wealth. Additionally, the mean-variance criterion is closely related to maximizing the expected utility of terminal wealth. [Li et al. \(2017a\)](#) study an insurer's excess-loss reinsurance-investment problem under a mean-variance criterion. They show that excess-loss is the unique equilibrium reinsurance strategy under a spectrally negative Lévy insurance model when the reinsurance premium is computed according to the expected value premium principle.

Although optimal reinsurance has been extensively studied, reinsurance decisions under ambiguity still require further exploration. Ambiguity is also present in insurance risk

modeling, and ambiguity-averse insurers seek to quantitatively address this uncertainty. [Zheng et al. \(2016\)](#) investigate a robust optimal portfolio and reinsurance problem using the *maxmin expected* utility. [Li et al. \(2016\)](#) analyze the robust reinsurance-investment problem under the α -maxmin mean-variance criterion, which differentiates between levels of ambiguity aversion and ambiguity. [Guan and Li \(2022\)](#) study an optimal investment and reinsurance problem using the smooth ambiguity model. While the smooth ambiguity model can generate richer optimal strategies than other ambiguity models, analytic solutions can only be obtained under very specific conditions for φ , U , and $\nu(\cdot)$ in (1.3).

We model the insurer's basic surplus process by a spectrally negative Lévy process. A spectrally negative Lévy process is a more general stochastic process that includes continuous components and jump components but only allows for negative jumps. The spectrally negative Lévy process can incorporate a variety of jump behaviors and does not restrict the inter-arrival times to be exponentially distributed. It can model more complex behaviors, such as heavy-tailed distributions and varying jump sizes. This model is widely used in risk theory within the actuarial literature; see, e.g., [Chiu and Yin \(2005\)](#) and [Landriault et al. \(2011\)](#).

In this chapter, we study a dynamic optimal reinsurance problem under two criteria: the first one is the mean-variance-variance (MVV) criterion,

$$\mathbb{E}^{\mathbb{P}}[X] - \frac{\gamma}{2}\text{Var}^{\mathbb{P}}[X] - \frac{\beta}{2}\text{Var}_{\nu}[\mathbb{E}^{\mathbb{Q}}[X]], \quad (4.1)$$

the second one is the mean-variance-standard deviation (M-V-SD) criterion,

$$\mathbb{E}^{\mathbb{P}}[X] - \frac{\gamma}{2}\text{Var}^{\mathbb{P}}[X] - \frac{\beta}{2}\sigma_{\nu}[\mathbb{E}^{\mathbb{Q}}[X]]. \quad (4.2)$$

Our work is as follows: We find the closed form of the equilibrium strategy under the M-V-V criterion. But there is the horizon-scale inconsistency between the risk measure $\text{Var}^{\mathbb{P}}[X]$ and the ambiguity measure $\text{Var}_{\nu}[\mathbb{E}^{\mathbb{Q}}[X]]$ existing in the M-V-V criterion. To fix the inconsistency between the two measures, we introduce the M-V-SD criterion and find the corresponding equilibrium strategy.

Our findings are summarized as follows. The closed form of the equilibrium strategy under both criteria is obtained without specific assumptions, such as the second-order distribution $\nu(\cdot)$ being normal. For an ambiguity-neutral insurer or in the absence of ambiguity, the equilibrium strategy reduces to the non-robust case. The ambiguity-averse reinsurance strategy is more conservative than the ambiguity-neutral reinsurance strategy. The equilibrium strategy under the M-V-V criterion is not always monotonic with respect

to the risk aversion parameter γ because the ambiguity variance $\text{Var}_\nu[\mathbb{E}^\mathbb{Q}[X]]$ is a quadratic function of the time horizon scale, while the risk variance $\text{Var}^\mathbb{P}[X]$ is a linear function of the time horizon scale. This horizon-scale inconsistency is resolved by introducing the M-V-SD criterion. The equilibrium strategy under the M-V-SD criterion decreases with increasing risk aversion. Additionally, the M-V-SD criterion produces a non-retention zone: even though the reinsurance loading is positive, the insurer chooses to purchase reinsurance policies to cover all of their insurance policies. The insurer begins to retain insurance policies only when the reinsurance loading is significantly large.

This chapter is organized as follows. Section 4.2 defines the controlled surplus process and Section 4.3 defines the ambiguity model. Section 4.4 provides the corresponding verification theorem. Section 4.5 presents the equilibrium strategy under the M-V-V criterion and Section 4.6 gives the equilibrium strategy under the M-V-SD criterion. Section 4.7 summarizes all the main results. All proofs are postponed to the Appendix.

4.2 Controlled Surplus Process

Let $(\Omega, \mathcal{F}, \mathbf{F} = \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a filtered, complete probability space satisfying the usual conditions, and let $T > 0$ be a finite time horizon. We consider a classical risk model: an insurer's basic surplus process is modeled by a spectrally negative Lévy process defined on this probability space with dynamics,

$$dU_t = cdt - \int_0^{+\infty} zN(dt, dz), \quad (4.3)$$

where $c \geq 0$ is the insurance premium rate and $N(dt, dz)$ is a Poisson random measure on $[0, T] \times (0, \infty)$ representing the number of insurance claims of size $(z, z + dz)$ within the time period $(t, t + dt)$. The corresponding compensated measure of $N(dt, dz)$ is denoted by $\tilde{N}(dt, dz) = N(dt, dz) - v(dz)dt$, in which v is a Lévy measure that represents the expected number of insurance claims with size range of $(z, z + dz)$ within a unit time interval. We assume that $\int_0^1 zv(dz) < \infty$ and $\int_1^\infty z^2v(dz) < \infty$ to ensure the second moment of U_t exists (see Theorem 25.3 of Sato [Ken-Iti \(1999\)](#)). The insurer's premium c is determined under the expected value principle, that is,

$$c = \frac{(1 + \theta)}{t} \mathbb{E} \left[\int_0^t \int_0^\infty zN(ds, dz) \right] = (1 + \theta) \int_0^\infty zv(dz), \quad (4.4)$$

where $\theta > 0$ is the proportional risk loading of the insurer. Spectrally negative Lévy processes (4.3) include the compound Poisson process, known as the Cramér-Lundberg

model in actuarial science, as a special case with $v(dz) = \rho F(dz)$, where ρ is the Poisson intensity and $F(dz)$ is the claim size distribution function.

The insurer can manage its insurance liabilities by purchasing a reinsurance policy with retained claim ℓ . The retained claim $\ell(t, z) : [0, T] \times (0, \infty) \rightarrow [0, z]$ is a deterministic function with the overinsurance restriction $0 \leq \ell(t, z) \leq z$, where $z = U_{t-} - U_t$ represents the size of a claim occurring at time t . We call the retention claim $\ell(s, z)$ which satisfies the overinsurance restriction as the *admissible* retention strategy. The reinsurer is responsible to cover the excess loss $z - \ell(t, z)$ so the time t premium rate of the reinsurance policy is given by

$$(1 + \eta) \int_0^\infty (z - \ell(t, z))v(dz), \quad (4.5)$$

in which η is the reinsurer's proportional safety loading ($\eta \geq 0$). We assume that $\eta > \theta$ since the reinsurance policy is generally more expensive than the primary insurance. Furthermore, suppose the insurer invests in a financial market consisting of a risk-free asset with a constant interest rate $r \geq 0$,

$$dB_t = rB_t dt.$$

Let $\{X_t^\ell\}_{t \in [0, T]}$ denote the corresponding insurance surplus process under a reinsurance strategy. Under the retention $\ell(t, z)$, the dynamics of the surplus process X_t is governed by

$$\begin{aligned} dX_t^\ell &= rX_t^\ell dt + dU_t + \int_0^\infty (z - \ell(t, z))N(dt, dz) - (1 + \eta) \int_0^\infty (z - \ell(t, z))v(dz)dt \\ &= \left[rX_t^\ell + \int_0^\infty ((\theta - \eta)z + \eta\ell(t, z))v(dz) \right] dt - \int_0^\infty \ell(t, z)\tilde{N}(dt, dz). \end{aligned} \quad (4.6)$$

Under any admissible retention strategies, the stochastic differential equation (4.6) admits a unique càdlàg solution X^ℓ such that

$$\mathbb{E}^\mathbb{P} [|X_t^\ell|^2] < \infty, \quad \text{for all } t \in [0, T]. \quad (4.7)$$

See, e.g., Theorem 1.19 of [Øksendal and Sulem \(2005\)](#).

4.3 Ambiguity Model

Insurers often discern ambiguity towards the underlying distribution of insurance. We adopt the analog of the mean-variance approximation model introduced by [Maccheroni](#)

et al. (2013) to determine the reinsurance strategy an insurer should make under the ambiguity on the insurance.

Consider a probability measure $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathbb{P}$ is called the *reference measure*. Let $L^2(\Omega, \mathbb{P})$ be the set of square-integrable random variables under $\mathbb{P}$. In the following, the ambiguity in a DM's perspective is characterized by a probability measure ν on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$, where $\mathcal{B}_{\mathbb{R}}$ denotes the Borel sigma-algebra on $\mathbb{R}$. Note that ν is also referred to as the *second-order distribution*, introducing an additional layer of uncertainty to decision making, reflecting various possible perspectives. We can define the ambiguity model under the spectrally negative Lévy process. For any given $\lambda \in \mathbb{R}$, by Girsanov's Theorem, there exists a unique prior probability measure $\mathbb{Q}^\lambda$, the set of Radon-Nikodym derivatives is given by

$$q_t^\lambda := \frac{d\mathbb{Q}^\lambda}{d\mathbb{P}} \Big|_{\mathcal{F}_t} = \exp \left\{ \int_0^t \int_0^\infty \lambda \tilde{N}(ds, dz) + \int_0^t \int_0^\infty (\lambda + 1 - e^\lambda) v(dz) ds \right\}.$$

By Girsanov's Theorem,

$$\tilde{N}^\lambda(dt, dz) = \tilde{N}(dt, dz) + (1 - e^\lambda) v(dz) dt \quad (4.8)$$

is a $\mathbb{Q}^\lambda$ -compensated Poisson random measure with compensator $e^\lambda v(dz)$. Under the prior measure $\mathbb{Q}^\lambda$, the dynamics of the surplus process X_t becomes

$$dX_t^\ell = \left[rX_t^\ell + \int_0^\infty ((\theta - \eta)z + (1 + \eta - e^\lambda)\ell(t, z)) v(dz) \right] dt - \int_0^\infty \ell(t, z) \tilde{N}^\lambda(dt, dz). \quad (4.9)$$

Under any admissible retention strategies $\ell \in [0, z]$, the stochastic differential equation (4.9) still admits a unique càdlàg solution X^ℓ such that

$$\mathbb{E}^{\mathbb{Q}^\lambda} [|X_t^\ell|^2] < \infty, \quad \text{for all } t \in [0, T]. \quad (4.10)$$

It is worth noting the continuity of $\mathbb{E}^{\mathbb{Q}^\lambda} [X_T^{t,x,\ell}]$ in λ . The proof is postponed to the Appendix.

Proposition 4.1. $\lambda \rightarrow \mathbb{E}^{\mathbb{Q}^\lambda} [X_T^{t,x,\ell}]$ is a continuous function on $\mathbb{R}$.

The mean and variance of $\lambda \rightarrow \mathbb{E}^{\mathbb{Q}^\lambda} [X_T^{t,x,\ell}]$ with respect to ν are defined as

$$\mathbb{E}_\nu \left[\mathbb{E}^{\mathbb{Q}^\lambda} [X_T^{t,x,\ell}] \right] := \int_{\mathbb{R}} \mathbb{E}^{\mathbb{Q}^\lambda} [X_T^{t,x,\ell}] d\nu(\lambda), \quad (4.11)$$

and

$$\begin{aligned}\text{Var}_\nu \left[\mathbb{E}^\mathbb{Q} \left[X_T^{t,x,\ell} \right] \right] &:= \mathbb{E}_\nu \left[\left(\mathbb{E}^\mathbb{Q} \left[X_T^{t,x,\ell} \right] \right)^2 \right] - \left(\mathbb{E}_\nu \left[\mathbb{E}^\mathbb{Q} \left[X_T^{t,x,\ell} \right] \right] \right)^2 \\ &= \int_{\mathbb{R}} \left(\mathbb{E}^{\mathbb{Q}^\lambda} \left[X_T^{t,x,\ell} \right] \right)^2 d\nu(\lambda) - \left(\int_{\mathbb{R}} \mathbb{E}^{\mathbb{Q}^\lambda} \left[X_T^{t,x,\ell} \right] d\nu(\lambda) \right)^2, \end{aligned} \quad (4.12)$$

respectively.

Maccheroni et al. (2013) derive the mean-variance-variance approximation model for the smooth ambiguity equivalent certainty that helps us formulate investor's risk and ambiguity preference. They extend the standard mean-variance model to a mean-variance-variance model:

$$J(t, x, \ell) = \mathbb{E}^\mathbb{P} \left[X_T^{t,x,\ell} \right] - \frac{\gamma}{2} \text{Var}^\mathbb{P} \left[X_T^{t,x,\ell} \right] - \frac{\beta}{2} \text{Var}_\nu \left[\mathbb{E}^\mathbb{Q} \left[X_T^{t,x,\ell} \right] \right], \quad (4.13)$$

where γ, β , which represent negative attitudes toward risk and ambiguity, respectively. In this paper, we also propose and study the *Mean-Variance-Standard Deviation (M-V-SD) criterion* of the form

$$\mathbb{E}^\mathbb{P} \left[X_T^{t,x,\ell} \right] - \frac{\gamma}{2} \text{Var}^\mathbb{P} \left[X_T^{t,x,\ell} \right] - \frac{\beta}{2} \sigma_\nu \left[\mathbb{E}^\mathbb{Q} \left[X_T^{t,x,\ell} \right] \right], \quad (4.14)$$

where

$$\sigma_\nu \left[\mathbb{E}^\mathbb{Q} \left[X_T^{t,x,\ell} \right] \right] := \sqrt{\text{Var}_\nu \left[\mathbb{E}^\mathbb{Q} \left[X_T^{t,x,\ell} \right] \right]}.$$

To unify the above two models, we consider the following more general *Mean-Variance-X (M-V-X) criterion*, namely,

$$J(t, x, \ell) = \mathbb{E}^\mathbb{P} \left[X_T^{t,x,\ell} \right] - \frac{\gamma}{2} \text{Var}^\mathbb{P} \left[X_T^{t,x,\ell} \right] - \frac{\beta}{2} \mathbb{X}_\nu \left[\mathbb{E}^\mathbb{Q} \left[X_T^{t,x,\ell} \right] \right]. \quad (4.15)$$

Here $\mathbb{X}_\nu : L^2(\mathbb{R}, v) \rightarrow [0, \infty)$ is a non-negative functional, where $L^2(\mathbb{R}, v)$ denotes the set of square-integrable real-valued functions with respect to v , and it satisfies

$$\mathbb{X}_\nu [c(\cdot)] = 0, \quad (4.16)$$

when $c(\lambda) \equiv c$ for any $\lambda \in \mathbb{R}$ is a constant function. Clearly, Var_ν and σ_ν are two special cases satisfying (4.16). The non-negativity of $\mathbb{X}_\nu$ implies that the term $\mathbb{X}_\nu \left[\mathbb{E}^\mathbb{Q} \left[X_T^{t,x,\ell} \right] \right]$ serves to penalize DM's ambiguity in the reward function (4.15). The condition (4.16) ensures that the M-V-X criterion reduces to the classical M-V criterion if the expected

payoff remains unchanged under different priors. To simplify the notation, we rewrite the reward function as

$$\begin{aligned} J(t, x, \ell) &= \mathbb{E}^{\mathbb{P}} \left[X_T^{t,x,\ell} - \frac{\gamma}{2} (X_T^{t,x,\ell})^2 \right] + \frac{\gamma}{2} \left(\mathbb{E}^{\mathbb{P}} [X_T^{t,x,\ell}] \right)^2 - \frac{\beta}{2} \mathbb{X}_{\nu} \left[\mathbb{E}^{\mathbb{Q}} [X_T^{t,x,\ell}] \right] \\ &= \mathbb{E}^{\mathbb{P}} \left[F(X_T^{t,x,\ell}) \right] + G \left(\mathbb{E}^{\mathbb{P}} [X_T^{t,x,\ell}] \right) + \mathbb{H}_{\nu} \left[\mathbb{E}^{\mathbb{Q}} [X_T^{t,x,\ell}] \right], \end{aligned} \quad (4.17)$$

where $F(x) := x - \frac{\gamma}{2}x^2$, $G(x) := \frac{\gamma}{2}x^2$ and $\mathbb{H}_{\nu}[\cdot] := -\frac{\beta}{2}\mathbb{X}_{\nu}[\cdot]$. The objective of the insurer is to maximize the reward function with the overinsurance restriction,

$$\sup_{\ell} J(t, x, \ell). \quad (4.18)$$

By (4.7) and the non-negativity of $\mathbb{X}_{\nu}[\cdot]$, we know that $X_T^{t,x,\ell} \in L^2(\Omega, \mathbb{P})$ and furthermore,

$$J(t, x, \ell) \in [-\infty, \infty),$$

for any retention strategy $\ell \in [0, z]$. This implies that the optimization problem (4.18) is well-posed, meaning the value function is finite. However, for the purpose of defining equilibrium retention strategies, it is necessary to propose the additional condition that

$$J(t, x, \ell) > -\infty. \quad (4.19)$$

The condition (4.19) is reasonable because any strategy ℓ with a reward $J(t, x, \ell) = -\infty$ shall *not* be optimal in any sense. Moreover, since $\mathbb{X}_{\nu}[\cdot]$ is a functional on $L^2(\mathbb{R}, \nu)$, condition (4.19) is equivalent to

$$\lambda \rightarrow \mathbb{E}^{\mathbb{Q}^{\lambda}} [X_T^{t,x,\ell}] \in L^2(\mathbb{R}, \nu). \quad (4.20)$$

Due to the continuity of the mapping $\lambda \rightarrow \mathbb{E}^{\mathbb{Q}^{\lambda}} [X_T^{t,x,\ell}]$, we know that (4.20) clearly holds when the second-order distribution ν has compact support on $\mathbb{R}$. More generally, the following lemma ensures that (4.20), equivalently (4.19), holds for unbounded distributions satisfying the exponential integrability condition (4.21).

In the rest of the paper, we assume that ν satisfies the integrability condition (4.21). In addition to all bounded distributions, condition (4.21) also includes unbounded distributions. The proof of Lemma 4.1 is postponed to Appendix B.2.

Lemma 4.1. *Suppose that the Borel probability measure ν satisfies*

$$\int_{\mathbb{R}} \exp \{ C e^{\lambda} \} d\nu(\lambda) < \infty, \quad (4.21)$$

for some sufficiently large positive constant C depending on $r, \eta, \theta, \int_0^\infty zv(dz), \int_0^\infty z^2v(dz)$, and T . For any fixed $(t, x) \in [0, T] \times \mathbb{R}$ and $\ell \in [0, z]$, the mapping $\lambda \rightarrow \mathbb{E}^{\mathbb{Q}^\lambda} [X_T^{t,x,\ell}]$ belongs to $L^2(\mathbb{R}, \nu)$ and the reward function $J(t, x, \ell) \in (-\infty, \infty)$.

Due to time inconsistency induced by two variance terms, we cannot maximize $J(t, x, \ell)$ directly in a time-consistent manner. Instead, we seek an equilibrium feedback strategy for this M-V-X reinsurance problem.

Definition 4.1. Given $(t, x) \in [0, T] \times \mathbb{R}$ and a retention strategy $\ell^* = \{\ell^*(s, z)\}_{(s,z) \in [t,T] \times (0,+\infty)} \in [0, z]$, we define an admissible perturbed strategy ℓ^ε as

$$\ell^\varepsilon(s, z) = \begin{cases} \hat{\ell}(z), & s \in [t, t + \varepsilon), \\ \ell^*(s, z), & s \in [t + \varepsilon, T], \end{cases}$$

where $\varepsilon > 0$ and $\hat{\ell} : (0, \infty) \rightarrow [0, z]$. Suppose that, for any $(t, x) \in [0, T] \times \mathbb{R}$ and any such admissible perturbed strategy ℓ^ε , we have

$$\limsup_{\varepsilon \downarrow 0} \frac{J(t, x, \ell^\varepsilon) - J(t, x, \ell^*)}{\varepsilon} \leq 0.$$

Then, ℓ^* is an equilibrium retention strategy to Problem (4.18), and $J(t, x, \ell^*)$ is the corresponding equilibrium value function.

4.4 Verification Theorem

In this section, we provide a verification theorem for the problem, an equilibrium feedback strategy and the corresponding value function.

We say a function $f(t, x, \lambda) \in C^{1,1,0}([0, T] \times \mathbb{R} \times \mathbb{R})$ if it is jointly continuous on $[0, T] \times \mathbb{R} \times \mathbb{R}$, and the partial derivatives $f_t := \frac{\partial f}{\partial t}$ and $f_x := \frac{\partial f}{\partial x}$ are continuous on $[0, T] \times \mathbb{R} \times \mathbb{R}$. For any $\ell \in [0, z]$ and $\lambda \in \mathbb{R}$, we define a differential operator:

$$\begin{aligned} \mathcal{A}^{\ell,\lambda} f(t, x, \lambda) &:= \lim_{\varepsilon \downarrow 0} \frac{\mathbb{E}^{\mathbb{Q}^\lambda} \left[f\left(t + \varepsilon, X_{t+\varepsilon}^{t,x,\ell}, \lambda\right) \right] - f(t, x, \lambda)}{\varepsilon} \\ &= f_t(t, x, \lambda) + \left[rx + \int_0^\infty ((\theta - \eta)z + (1 + \eta)\ell(t, z))v(dz) \right] f_x(t, x, \lambda) \\ &\quad + \int_0^\infty (f(t, x - \ell(t, z), \lambda) - f(t, x, \lambda)) e^\lambda v(dz), \end{aligned} \tag{4.22}$$

in which $f \in \mathcal{C}^{1,1}([0, T] \times \mathbb{R})$. In particular, we obtain the integro-differential operator $\mathcal{A}^{\ell,0}$ under $\mathbb{P}$ when $\lambda = 0$. We use the following notations

$$\mathcal{A}^\ell f(t, x) := \mathcal{A}^{\ell,0} f(t, x) \text{ and } f(t, x) := f(t, x, 0). \quad (4.23)$$

Next, we give a verification theorem that we will use to determine an equilibrium feedback strategy and corresponding equilibrium value function.

Definition 4.2. Consider an arbitrary admissible retention strategy ℓ . We say that a function $f(t, x, \lambda) : [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ belongs to the space $L_T^2(X^\ell)$ if

$$\mathbb{E}^{\mathbb{Q}^\lambda} \left[\int_t^T \int_0^\infty (f(s, X_s^{t,x,\ell} - \ell(s, z), \lambda) - f(s, X_s^{t,x,\ell}, \lambda))^2 e^{\lambda v}(\mathrm{d}z) \mathrm{d}s \right] < \infty, \quad (4.24)$$

for every $(t, x, \lambda) \in [0, T] \times \mathbb{R} \times \mathbb{R}$. For brevity of notation, we also say that a function $f(t, x) : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ belongs to the space $L_T^2(X^\ell)$ if

$$\mathbb{E}^{\mathbb{P}} \left[\int_t^T \int_0^\infty (f(s, X_s^{t,x,\ell} - \ell(s, z)) - f(s, X_s^{t,x,\ell}))^2 v(\mathrm{d}z) \mathrm{d}s \right] < \infty, \quad (4.25)$$

for every $(t, x) \in [0, T] \times \mathbb{R}$.

The following definition of Gâteaux derivative of the functional $\mathbb{H}_\nu[\cdot]$ will be used in the verification theorem.

Definition 4.3. We say the functional $\mathbb{H}_\nu : L^2(\mathbb{R}, v) \rightarrow [0, \infty)$ is Gâteaux differentiable at $f \in L^2(\mathbb{R}, v)$ if there exists a linear continuous operator denoted by $\frac{\delta \mathbb{H}_\nu[f(\cdot)]}{\delta f(\cdot)} : L^2(\mathbb{R}, v) \rightarrow \mathbb{R}$ such that

$$\left\langle \frac{\delta \mathbb{H}_\nu[f(\cdot)]}{\delta f(\cdot)}, \varphi(\cdot) \right\rangle := \lim_{\varepsilon \rightarrow 0} \frac{\mathbb{H}_\nu[f(\cdot) + \varepsilon \varphi(\cdot)] - \mathbb{H}_\nu[f(\cdot)]}{\varepsilon} = \left. \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \mathbb{H}_\nu[f(\cdot) + \varepsilon \varphi(\cdot)] \right|_{\varepsilon=0}, \quad (4.26)$$

for any $\varphi \in L^2(\mathbb{R}, v)$.

When $\mathbb{H}_\nu$ is Gâteaux differentiable at f , by Taylor expansion,

$$\mathbb{H}_\nu[f + \varepsilon \varphi] - \mathbb{H}_\nu[f] = \varepsilon \left. \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \mathbb{H}_\nu[f + \varepsilon \varphi] \right|_{\varepsilon=0} + o(\varepsilon) = \varepsilon \left\langle \frac{\delta \mathbb{H}_\nu[f]}{\delta f}, \varphi \right\rangle + o(\varepsilon). \quad (4.27)$$

We now present a verification theorem for the M-V-X maximization problem (4.15). The proof of this verification theorem is relegated to Appendix B.3. Note that we use $g \circ f$ to denote the composition of two functions g and f .

Theorem 4.1. Suppose that there exist $V(t, x) \in C^{1,1}([0, T] \times \mathbb{R})$ and $g(t, x, \lambda) \in C^{1,1,0}([0, T] \times \mathbb{R} \times \mathbb{R})$ satisfying the following conditions:

1. For any $(t, x) \in [0, T] \times \mathbb{R}$, the functional $\mathbb{H}_\nu$ is Gâteaux differentiable at $g(t, x, \cdot)$, and moreover, $\mathbb{H}_\nu[g(t, x, \cdot)] \in C^{1,1}([0, T] \times \mathbb{R})$.
2. For any $(t, x) \in [0, T] \times \mathbb{R}$,

$$\sup_{\ell \in \mathbb{R}} \left\{ \mathcal{A}^\ell V(t, x) + (G_x \circ g)(t, x) \mathcal{A}^\ell g(t, x) - \mathcal{A}^\ell (G \circ g)(t, x) + \left\langle \frac{\delta \mathbb{H}_\nu[g(t, x, \cdot)]}{\delta g(t, x, \cdot)}, \mathcal{A}^{\ell, \cdot} g(t, x, \cdot) \right\rangle - \mathcal{A}^\ell \mathbb{H}_\nu[g(t, x, \cdot)] \right\} = 0, \quad (4.28)$$

where $g(t, x) := g(t, x, 0)$, and $\mathcal{A}^{\ell, \cdot} g(t, x, \cdot) \in L^2(\mathbb{R}, \nu)$ for any $\ell \in \mathbb{R}$.

3. There is a measurable function $\ell^*(t, x)$ such that $\ell^* = \{\ell^*(t, x)\}_{(t,x) \in [0,T] \times \mathbb{R}} \in [0, z]$ for $z \in (0, \infty)$ satisfying

$$\mathcal{A}^{\ell^*} V(t, x) + (G_x \circ g)(t, x) \mathcal{A}^{\ell^*} g(t, x) - \mathcal{A}^{\ell^*} (G \circ g)(t, x) + \left\langle \frac{\delta \mathbb{H}_\nu[g(t, x, \cdot)]}{\delta g(t, x, \cdot)}, \mathcal{A}^{\ell^*, \cdot} g(t, x, \cdot) \right\rangle - \mathcal{A}^{\ell^*} \mathbb{H}_\nu[g(t, x, \cdot)] = 0, \quad (4.29)$$

and, for any $(t, x, \lambda) \in [0, T] \times \mathbb{R} \times \mathbb{R}$,

$$\mathcal{A}^{\ell^*, \lambda} g(t, x, \lambda) = 0. \quad (4.30)$$

4. For any $(x, \lambda) \in \mathbb{R} \times \mathbb{R}$,

$$V(T, x) = x \text{ and } g(T, x, \lambda) = x. \quad (4.31)$$

5. The functions $g(t, x, \lambda)$, $V(t, x)$, $(G \circ g)(t, x)$, and $\mathbb{H}_\nu[g(t, x, \cdot)]$ all belong to $L_T^2(X^{\ell^*}) \cap L_T^2(X^{\hat{\ell}})$, where $\hat{\ell} = \{\hat{\ell}(t, z)\}_{(t,z) \in [0,T] \times [0,\infty)}$ with $\hat{\ell}(t, z) \equiv \hat{\ell}$ for all $(t, z) \in [0, T] \times (0, \infty)$ is a constant control.

Then $\ell^* = \{\ell^*(t, z)\}_{(t,z) \in [0,T] \times (0,\infty)}$ is an equilibrium retention strategy to the M-V-X maximization problem (4.15). Moreover,

$$V(t, x) = J(t, x, \ell^*), \quad g(t, x) = \mathbb{E}^\mathbb{P}[X_T^{t,x,\ell^*}], \quad \text{and} \quad g(t, x, \lambda) = \mathbb{E}^{\mathbb{Q}^\lambda}[X_T^{t,x,\ell^*}].$$

Example 4.1 (M-V-V criterion). Consider the M-V-V criterion in which $\mathbb{X}_\nu = \text{Var}_\nu$. For any $f, \varphi \in L^2(\mathbb{R}, \nu)$, by (4.26),

$$\left\langle \frac{\delta \text{Var}_\nu[f(\cdot)]}{\delta f(\cdot)}, \varphi(\cdot) \right\rangle = 2 \text{Cov}_\nu[f(\cdot), \varphi(\cdot)],$$

where Cov_ν is the covariance operator under ν . Moreover, the Gâteaux derivative $\frac{\delta \text{Var}_\nu[f]}{\delta f} = 2 \text{Cov}_\nu[f, \cdot] : L^2(\mathbb{R}, \nu) \rightarrow \mathbb{R}$ is a linear and continuous¹ operator. Then the EHJB equation (4.28) becomes

$$\sup_{\ell \in \mathbb{R}} \left\{ \mathcal{A}^\ell V(t, x) + (G_x \circ g)(t, x) \mathcal{A}^\ell g(t, x) - \mathcal{A}^\ell (G \circ g)(t, x) - \frac{\beta}{2} (2 \text{Cov}_\nu[g(t, x, \cdot), \mathcal{A}^\ell g(t, x, \cdot)] - \mathcal{A}^\ell \text{Var}_\nu[g(t, x, \cdot)]) \right\} = 0. \quad (4.32)$$

Example 4.2 (M-V-SD criterion). Consider the M-V-SD criterion in which $\mathbb{X}_\nu[\cdot] = \sigma_\nu[\cdot] = \sqrt{\text{Var}_\nu[\cdot]}$. For any $f, \varphi \in L^2(\mathbb{R}, \nu)$ with $\sigma_\nu[f] > 0$, by (4.26) and Example 4.1,

$$\left\langle \frac{\delta \sigma_\nu[f(\cdot)]}{\delta f(\cdot)}, \varphi(\cdot) \right\rangle = \frac{\text{Cov}_\nu[f(\cdot), \varphi(\cdot)]}{\sigma_\nu[f(\cdot)]}.$$

Moreover, the Gâteaux derivative $\frac{\delta \sigma_\nu[f]}{\delta f} = \frac{\text{Cov}_\nu[f, \cdot]}{\sigma_\nu[f]} : L^2(\mathbb{R}, \nu) \rightarrow \mathbb{R}$ is a linear and continuous operator. Then the EHJB equation (4.28) becomes

$$\sup_{\ell \in \mathbb{R}} \left\{ \mathcal{A}^\ell V(t, x) + (G_x \circ g)(t, x) \mathcal{A}^\ell g(t, x) - \mathcal{A}^\ell (G \circ g)(t, x) - \frac{\beta}{2} \left[\frac{\text{Cov}_\nu[g(t, x, \cdot), \mathcal{A}^\ell g(t, x, \cdot)]}{\sigma_\nu[g(t, x, \cdot)]} - \mathcal{A}^\ell \sigma_\nu[g(t, x, \cdot)] \right] \right\} = 0. \quad (4.33)$$

4.5 Equilibrium Reinsurance Strategy under the Mean-Variance-Variance Criterion

In this section, we derive and analyze the equilibrium reinsurance strategies to maximize the M-V-V reward function, i.e.,

$$J(t, x, \ell) = \mathbb{E}^\mathbb{P} \left[X_T^{t, x, \ell} \right] - \frac{\gamma}{2} \text{Var}^\mathbb{P} \left[X_T^{t, x, \ell} \right] - \frac{\beta}{2} \text{Var}_\nu \left[\mathbb{E}^\mathbb{Q} \left[X_T^{t, x, \ell} \right] \right], \quad (4.34)$$

¹The continuity is in terms of the topology induced by $L^2(\mathbb{R}, \nu)$.

where we recall from (4.12) that

$$\text{Var}_\nu \left[\mathbb{E}^\mathbb{Q} \left[X_T^{t,x,\ell} \right] \right] = \mathbb{E}_\nu [(\mathbb{E}^\mathbb{Q} [X_T^{t,x,\ell}])^2] - (\mathbb{E}_\nu [\mathbb{E}^\mathbb{Q} [X_T^{t,x,\ell}]])^2.$$

We denote by the variance of the second-order distribution ν by

$$\delta_\nu^2 := \int_{\mathbb{R}} (e^\lambda)^2 d\nu(\lambda) - \left(\int_{\mathbb{R}} e^\lambda d\nu(\lambda) \right)^2.$$

To obtain the equilibrium strategy, we need to establish the differentiability and monotonicity of the function $D(t)$ firstly.

Lemma 4.2. *For any $t \in (0, \infty)$, we consider the following ordinary differential equation (ODE),*

$$\begin{cases} D'(t) = e^{r(T-t)} \int_0^{+\infty} \left(0 \vee \frac{\eta - \beta \delta_\nu^2 D(t)}{\gamma e^{r(T-t)}} \wedge z \right) v(dz), & t \geq 0 \\ D(0) = 0. \end{cases} \quad (4.35)$$

Then, there exists a unique solution $D(t)$ that solves the above ODE (4.35) and has the following properties when $\eta > 0$,

1. $0 < D(t) < \frac{\eta}{\beta \delta_\nu^2}$,
2. $D(t)$ is strictly increasing with respect to t .

When $\eta = 0$, $D(t) = 0$ for all $t \in [0, \infty)$.

Then, we obtain the following equilibrium retention strategy.

Theorem 4.2. *An equilibrium reinsurance strategy for the mean–variance–variance criterion is given by*

$$\ell^*(t, z) = \frac{\eta - \beta \delta_\nu^2 D(T-t)}{\gamma e^{r(T-t)}} \wedge z, \quad (4.36)$$

where $D(T-t) = D(t)$ is the unique solution for the ODE (4.35).

We next discuss the behavior of the equilibrium strategy given in (4.36). First, when the insurer is ambiguity-neutral ($\beta = 0$) or the ambiguity towards the claim is absent ($\delta_\nu^2 = 0$), the equilibrium reinsurance strategy reduces to

$$\ell^*(t, z) = \frac{\eta}{\gamma e^{r(T-t)}} \wedge z, \quad (4.37)$$

which becomes the equilibrium reinsurance strategy under the standard M-V criterion; see Theorem 3.2 of Li et al. (2017a). This also means that the ambiguity-neutral strategy is identical to the non-robust strategy in the absence of ambiguity. Second, an insurer retains insurance policies only when the reinsurance loading is positive. Conversely, the insurer purchases full reinsurance if the reinsurance loading is zero.

For the impact of ambiguity attitude β and ambiguity δ_ν^2 on the equilibrium retention strategy, we have the following proposition.

Proposition 4.2. *For any $t \in [0, T]$, $\ell^*(t, z)$ is decreasing in β and δ_ν^2 when $\eta > 0$.*

The proposition (4.2) implies that the insurer is willing to take less insurance risk by purchasing reinsurance policies when the ambiguity or ambiguity attitude increases. In comparison with (4.37) and (4.36), the insurer takes less (resp. more) risk if she is ambiguity averse (resp. loving).

In the end, the most surprising property of the equilibrium strategy (4.36) is the non-monotonicity with respect to the risk aversion parameter γ . Even though the analytical form of the equilibrium strategy prevents a deeper investigation for the non-monotonicity in γ , this fact can be found clearly in the case with no interest rate.

Corollary 4.1. *When $r = 0$, an equilibrium reinsurance strategy has a closed form,*

$$\ell^*(t, z) = \frac{\eta - \beta\delta_\nu^2\Psi^{-1}(T - t)}{\gamma} \wedge z. \quad (4.38)$$

where

$$\Psi(x) = \int_0^x \frac{1}{\psi(s)} ds, \quad x \in [0, \frac{\eta}{\beta\delta_\nu^2}), \quad (4.39)$$

and

$$\psi(x) = \int_0^\infty \frac{\eta - \beta\delta_\nu^2 x}{\gamma} \wedge zv(dz). \quad (4.40)$$

Example 4.3. *The Lévy measure is selected to be*

$$v(dz) = \mu e^{-\mu z} dz, \quad z > 0,$$

which means that the aggregate loss $\int_0^t \int_0^\infty zN(ds, dz)$ is a compound Poisson process with a Poisson intensity of 1, and each loss follows an exponential distribution with parameter μ . Then, we obtain

$$\ell^*(t, z) = \frac{1}{\mu} \log(\exp(-\frac{\beta\delta_\nu^2}{\gamma}(T - t) + \log(e^{\frac{\mu\eta}{\gamma}} - 1)) + 1) \wedge z. \quad (4.41)$$

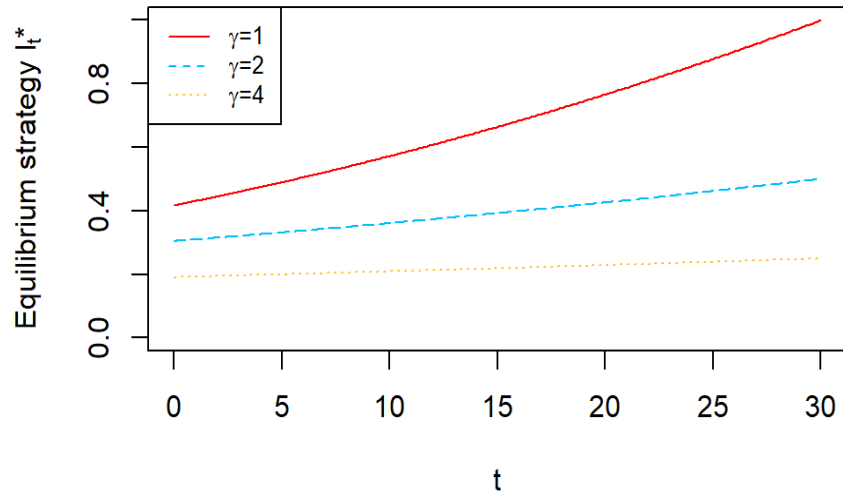


Figure 4.1: Equilibrium reinsurance strategies under the M-V-V criterion for different levels of risk aversion ($T = 30$)

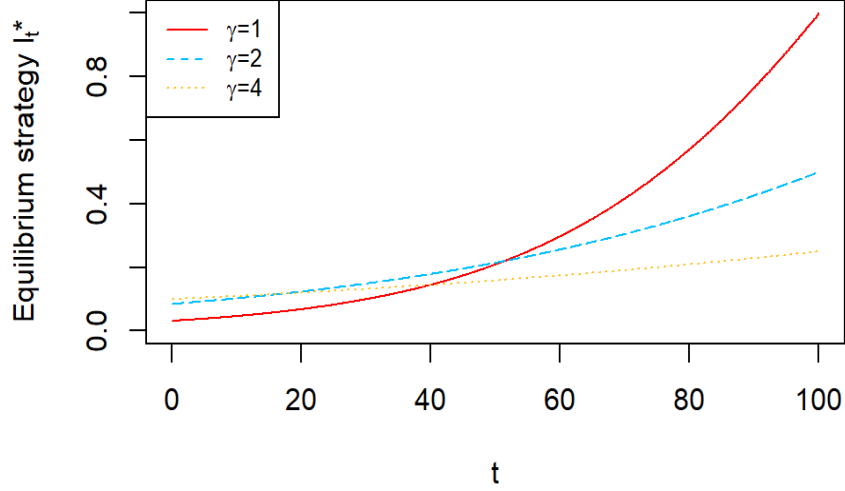


Figure 4.2: Equilibrium reinsurance strategies under the M-V-V criterion for different levels of risk aversion ($T = 100$)

In the formula of the equilibrium strategy (4.41), we discover that $\ell^*(t, z)$ is not always monotone with respect to the risk aversion parameter γ . However, our intuition suggests that $|\ell^*(t, z)|$ should decrease with γ , indicating that the reinsurance strategy becomes more conservative if the insurer is more risk-averse. If we assume that the ambiguity aversion parameter β is a *linear or higher order* function of the risk aversion parameter γ , that is,

$$\beta = c\gamma^k, \quad \text{for } k \geq 1 \quad \text{and} \quad c \geq 0,$$

in this case, our equilibrium strategy (4.41) is decreasing with respect to γ . This condition aligns with empirical studies showing evidence of a positive correlation between risk aversion and ambiguity aversion (see [Bossaerts et al. \(2010\)](#)). However, we still need to explain the more general case when β and γ are independent. A similar property of the equilibrium strategy is also found in the investment problem. The *horizon-scale inconsistency* in the M-V-V criterion also holds for much more general underlying processes such as Lévy processes. The risk variance $\text{Var}^{\mathbb{P}}[\cdot]$ is a linear function of the terminal time T , while the ambiguity variance $\text{Var}_{\nu}[\mathbb{E}^{\mathbb{Q}}[\cdot]]$ is a quadratic function of T .

Figure 4.1 and Figure 4.2 plot the equilibrium strategy ℓ^* under the M-V-V criterion for different risk aversion levels. The parameters are set to $\eta = 1$, $\beta = 2$, $\delta_{\nu}^2 = 0.02$, $r = 0$, and

$\mu = 1$. We see that ℓ^* decreases with γ when the time to maturity is short. However, when the time to maturity is long, the equilibrium strategy is no longer monotone with respect to risk aversion γ . The economic intuition is as follows: since the ambiguity variance term $\text{Var}_\nu [\mathbb{E}^\mathbb{Q} [\cdot]]$ penalizes quadratically with the time horizon, a less risk-averse insurer prefers to *buy the reinsurance policy at the beginning* in order to lower the exposure to insurance risk, which reduces the heavy penalty from the ambiguity variance for long-term risky insurance claims.

4.6 Equilibrium Reinsurance Strategy under the Mean-Variance-Standard Deviation Criterion

In this section, we present a corrected ambiguity criterion: the M-V-SD criterion, which prevents the horizon-scale inconsistency in the criterion. From the previous discussion, the ambiguity measure overpenalizes on horizon scale compared to the risk measure. To unify the scale of time horizon in both uncertainty measures, we modify the extended mean-variance criterion in [Maccheroni et al. \(2013\)](#),

$$J(t, x, \ell) = \mathbb{E}^\mathbb{P} [X_T^{t,x,\ell}] - \frac{\gamma}{2} \text{Var}^\mathbb{P} [X_T^{t,x,\ell}] - \frac{\beta}{2} \sigma_\nu [\mathbb{E}^\mathbb{Q} [X_T^{t,x,\ell}]], \quad (4.42)$$

where σ_ν denotes the standard deviation under the measure ν ,

$$\sigma_\nu[\cdot] = \sqrt{\text{Var}_\nu[\cdot]}.$$

We also note that

$$\delta_\nu = \sqrt{\int_{\mathbb{R}} (e^\lambda)^2 d\nu(\lambda) - \left(\int_{\mathbb{R}} e^\lambda d\nu(\lambda) \right)^2}$$

is the standard deviation of the second-order distribution ν . We replace the variance of the expectation under each prior measure $\mathbb{Q}^\lambda$ with its standard deviation so that the two ambiguity measures offer the same penalties in continuous case.

Theorem 4.3. *An equilibrium reinsurance strategy for the M-V-SD criterion is given by*

$$\ell^*(t, z) = \begin{cases} 0, & \text{if } \eta \leq \frac{\beta}{2} \delta_\nu, \\ \frac{\eta - \frac{\beta}{2} \delta_\nu}{\gamma e^{r(T-t)}} \wedge z, & \text{if } \eta > \frac{\beta}{2} \delta_\nu, \end{cases} \quad (4.43)$$

We find that the M-V-SD strategy (4.43) shares many properties with the M-V-V strategy: (1) the equilibrium strategy becomes the non-robust equilibrium strategy under the standard M-V criterion in the absence of an ambiguity measure. (2) $\ell^*(t, z)$ is also decreasing in β and δ_ν .

The more tractable structure of the M-V-SD strategy (4.43) also provides other properties. First, $\ell^*(t, z)$ is decreasing and convex with respect to the risk-free rate r . When large claims occur, the insurer can borrow money from the risk-free bond in exchange for solvency. Therefore, if the cost of borrowing money increases then the insurer retains less risky insurance. But this change in the equilibrium strategy with respect to r diminishes as r increases since ℓ^* is convex in r .

Second, the equilibrium strategy (4.43) is always decreasing in the risk aversion γ because we fix the horizon-inconsistency between the risk measure and the ambiguity measure by replacing the variance with the standard deviation. The equilibrium strategy is a decreasing and convex function with respect to γ . As the insurer becomes more risk-averse, he wants to retain fewer insurance policies. The absolute magnitude of the rate of change of ℓ^* in risk aversion γ is decreasing when γ is increasing.

Third, The retained claim ℓ^* increases as the reinsurance risk-loading η increases when $\eta > \frac{\beta}{2}\delta_\nu$. In other words, as the reinsurance policy becomes more expensive, the insurer retains more insurance risk.

Fourth, the most interesting property of the M-V-SD strategy is the non-retention zone for reinsurance loading η :

$$\eta \in [0, \frac{\beta}{2}\delta_\nu].$$

for which the insurer does not retain any insurance policy even though the reinsurance loading is positive. In other words, the insurer cedes all his insurance policies when $\eta \leq \frac{\beta}{2}\delta_\nu$. The insurer begins to retain insurance policies only when the reinsurance loading is significantly large. Also, this non-retention set enlarges when the ambiguity attitude β or the ambiguity δ_ν increases. To the best of our knowledge, this limited retention result is the first finding in the equilibrium reinsurance strategy. Other equilibrium reinsurance strategies under models with ambiguity do not have such property. See, e.g., theorem 3.3 of (Guan and Li (2022)).

4.7 Conclusion

In this chapter, we study the equilibrium reinsurance strategy under the M-V-V criterion and the M-V-SD criterion. The closed form of equilibrium strategies are derived by solving

the EHJB equation. The equilibrium strategy under the M-V-V criterion is determined by a corresponding ODE so the comparison principle is used when proving the monotonicity of the equilibrium strategy with respect to the insurer's ambiguity and ambiguity attitude.

We have two findings about the M-V-V strategy: (1) The optimal exposure to retention insurance decreases as the ambiguity aversion increases. (2) The most surprising feature of the equilibrium strategy is that the M-V-V equilibrium strategy is non-monotonic with respect to the risk aversion parameter due to the inconsistent scale of the time horizon between the risk measure and the ambiguity measure. To resolve the issue with the horizon-scale inconsistency of the M-V-V criterion, we introduce the M-V-SD criterion to evaluate the insurer's ambiguity belief. Under the M-V-SD criterion, the equilibrium strategy is always decreasing in the risk aversion parameter and has the appealing of property: the non-retention region. Finally, we note that the analytical tractability of the M-V-SD criterion renders our model especially suitable for the study of reinsurance behaviors under ambiguity.

Chapter 5

Contract Structure and Risk Aversion in Longevity Risk Transfers

5.1 Introduction

The global aging population has increased the need for defined benefit pension plans, life insurers, and reinsurers to effectively manage and transfer their exposure to longevity risk—the risk of individuals living longer than expected. This paper introduces an economic framework for analyzing optimal longevity risk transfers, offering theoretical explanations for key observations in the current longevity risk transfer market and new insights into market development.

To date, most existing longevity risk transactions have occurred within the reinsurance sector, where reinsurers offer longevity risk protection in exchange for a reinsurance premium. Although the reinsurance sector has steadily developed over the past two decades, it alone cannot absorb the global longevity risk exposure, which far exceeds its capacity ([Blake and Morales, 2017](#)). Therefore, attracting investment from external capital market investors into longevity risk transactions is crucial for creating a market capable of managing global longevity risk exposure sustainably. Insurance risk capital markets are argued to be a valuable complement to the reinsurance sector. On one hand, entities seeking risk transfer can access more capital; on the other hand, investors gain diversification benefits since most insurance risks, including catastrophic and mortality risks, exhibit little correlation with other financial assets ([Kessler, 2021](#)). The capital market of insurance risks has been evolving steadily for the past two decades, with catastrophe (CAT) bonds as the predominant instrument ([Li and Su, 2024](#); [Herrmann and Hibbeln, 2023](#)).

However, most existing catastrophe bond transactions are based on natural catastrophic risks, while transactions involving mortality or longevity risks remain rare. Several reasons have been proposed for the scarcity of capital market transactions involving longevity risk, including the tension between the long-term nature of such risks and investors’ preference for short-term investments, as well as information asymmetry (Blake et al., 2019; Kiff, 2022).¹

First, pension and annuity portfolios, inherently long-term in nature and often spanning several decades, require sustained hedging against longevity risk. Reinsurers, traditionally absorbing such risks, are willing to provide extended coverage. In contrast to most property and casualty policies, longevity reinsurance contracts are long-term and remain effective until the covered portfolio is depleted. However, capital market investors, primarily concerned with returns and liquidity, generally prefer short-term investments on insurance risks, and are reluctant to commit to long-term contracts with limited liquidity. This hesitance is particularly evident in the longevity risk transfer market, which has seen rather limited activity, making it challenging for investors to exit long-term longevity-linked investments. A notable example is the European Investment Bank’s failed attempt in 2004 to issue a 25-year longevity bond, which was withdrawn in 2005 due to lack of investor interest (Blake et al., 2006a). Additionally, the only series of CAT bonds targeting mortality risk to date, the Vita Capital series issued by Swiss Re starting in December 2003, includes eight bonds with the most recent, Vita Capital VI Limited, issued in July 2021. These bonds have terms ranging from three to five years. Recently, longevity sidecars have also emerged as capital market solutions with similar maturities of three to five years (Bugler et al., 2021).

Second, longevity risk is a new and lesser-known risk to general capital market investors. Consequently, investors will be wary of longevity-linked assets because they are worried they will be sold a ‘lemon’. Such information asymmetries could make longevity risk less attractive to capital market investors, and possibly prevent them from entering the market.

In response to these issues, this paper develops a theoretical model to explore optimal transactions of longevity risk using a Stackelberg game framework, involving a principal (seller) and a follower (buyer). The model encompasses a seller contemplating a range of longevity risk transfer contracts, each specified by a distinct risk loading (price). Upon being offered a contract, the buyer—who holds a life annuity portfolio and is exposed to longevity risk—responds by selecting an optimal hedge ratio based on the risk loading. Subsequently, considering the buyer’s optimal response based on her utility, the seller

¹Other rationales include product design, pricing, credit risk, and basis risk. See Blake et al. (2006a), Lin and Cox (2008), and MacMinn and Brockett (2017) for more discussions.

determines the optimal contract to offer in order to maximize her own utility. Both parties operate within a mean-variance expected utility framework. This analysis is demonstrated through the use of indemnity longevity swaps, a prevalent hedging instrument in the current market.² In a longevity swap, the buyer makes a fixed periodic payment to the seller in exchange for a floating payment that is linked to the actual mortality experience of her annuity portfolio.

We first investigate the preference for contract structure. In particular, we evaluate two contract types: the *static contract*, which sets a constant hedge ratio and fixed periodic payments at inception (time 0), and the *dynamic contract*, which adjusts these parameters periodically based on the latest available information. The static contract provides long-term protection with known hedging costs and reflects existing long-term reinsurance products, offering *high risk coverage*. The dynamic contract, resembling the rolling of short-dated instruments such as mortality CAT bonds or longevity sidecar, provides *low risk coverage* due to variable fixed payments. Our quantitative analysis uses the Age-Period-Cohort-Improvement model (CMI Mortality Projections Committee, 2016), fitted to UK mortality data, to simulate future mortality experiences in the buyer’s portfolio. We find that contract preference depends on the longevity risk aversion of both parties. With a more risk-averse buyer, the static contract is favored, yielding higher welfare gains for both parties. Conversely, with a more risk-averse seller, the dynamic contract is preferred as it avoids long-term commitments.

We next address information asymmetry in longevity risk transfers by introducing ambiguity. Information asymmetry, where one party has superior knowledge of the underlying risk, is a well-recognized issue in insurance markets (Akerlof, 1978; Rothschild and Stiglitz, 1976) and has been extensively explored in the context of longevity risk (see, e.g., Biffis and Blake, 2010, 2013, 2014; Chen et al., 2023). Policyholder mortality experiences, which can vary significantly across portfolios, often leave capital market investors less informed about the specific longevity risks facing the buyer’s portfolio. Employing the commonly used maxmin expected utility (Gilboa and Schmeidler, 1989), the seller’s objective is to make a robust decision based on the worst-case estimate of the uncertain probability distribution of the mortality experience in the buyer’s portfolio. Our findings reveal that while ambiguity reduces welfare gains for both parties in all scenarios, it does not alter our primary conclusions. Specifically, the static contract continues to be preferred by buyers with greater aversion to longevity risk, whereas sellers with higher risk aversion find dynamic contracts more appealing. Additionally, the presence of ambiguity narrows the range of longevity risk loadings leading to market collapse for both contract types. In particular,

²For detailed discussions of the longevity swap, see Dowd et al. (2006); Dawson et al. (2010).

with higher seller risk aversion, even moderate ambiguity discourages offering static contracts at any price, thus potentially excluding a market for such contracts. Conversely, the market for longevity risk transactions demonstrates greater resilience to ambiguity when the buyer exhibits stronger risk aversion.

While our findings concerning contract structure preferences and market viability are relevant to any risk transfer market, these findings offer insights into the observed dynamics within the current longevity risk transfer market and shed lights on market development. Existing literature suggests that an agent’s risk aversion can be influenced by several factors, such as familiarity with the risk, regulatory frameworks governing risk management, and the availability of hedging or transfer mechanisms (see, for example [Khaw et al., 2021](#); [Laeven and Levine, 2009](#); [Guiso and Paiella, 2008](#)).

In reinsurance transactions, buyers of the reinsurance contract, such as defined benefit pension plans and life insurers, face stricter solvency regulations compared to large-scale global reinsurers, who are typically the sellers. Additionally, reinsurers have a greater capacity to diversify longevity risks across multiple buyers and can hedge these risks by pooling liabilities with opposite risk exposures, such life insurance portfolios, or sharing risks with other reinsurers. Furthermore, pension managers, life insurance portfolio managers, or fund managers may also exhibit risk aversion as financial losses or insolvency within their institutions could jeopardize their job security ([Ge and Weisbach, 2021](#)). As a result, sellers in these scenarios are typically *less risk-averse* than the buyers. Conversely, in the capital market setting, sellers—often general institutional investors like investment banks, mutual funds, or endowments—exhibit *greater risk aversion* due to their lesser familiarity with longevity risks and reduced capacity to manage these risks compared to large reinsurers, the typical buyers in these transactions. Arguably, existing studies suggest that longevity-linked securities are considered low-beta investments ([Blake et al., 2006a](#)). This would imply that general capital market investors should be less risk-averse about taking on small portions of the risk. Such claim may be true in a complete and liquid market, where capital market investor could transfer any volume of their longevity risk exposure at low transaction costs. However, in the current incomplete and illiquid longevity risk transfer capital market, it is costly and challenging for investors to divest their longevity risk exposure once assumed. Such market frictions could substantially increase the risk aversion of capital market investors to longevity risk. Lastly, our findings indicate that while short-term solutions in the capital market for longevity risk are suboptimal in terms of providing the long-term coverage desired by risk holders, they represent a promising initial step towards developing an active longevity-linked capital market.

Finally, it is worth noting that, while static contracts have found widespread use in both the literature and the real-world insurance industry, dynamic contracts are becoming

increasingly relevant in insurance research (see, e.g., [Wong et al., 2017](#); [Zhou and Li, 2017](#); [Wang et al., 2023](#)). Nonetheless, the full range of advantages and disadvantages of these contract types for buyers and sellers remains largely unexplored. To our knowledge, this chapter is the first to elucidate the relationships between contract preferences and the risk aversions of the parties involved. We posit that the patterns observed in managing longevity risk—specifically, the preference for static contracts by less risk-averse sellers and for dynamic contracts by more risk-averse sellers—may apply to other economic contracting scenarios as well.

Relationship to the Literature and Organization of the Chapter

The importance of an active longevity-linked capital market on protecting the long-term viability of the global retirement system has been argued by [Blake et al. \(2013a\)](#). Regarding its development, [MacMinn and Brockett \(2017\)](#) point to moral hazard as a key contributing factor to the failure of the longevity-linked capital market, while [Börger et al. \(2023\)](#) investigate the effects of market saturation on longevity risk transactions, proposing that capital market deals may prove more profitable when the reinsurance sector reaches its capacity. Additionally, discussions exist on the link between capital market solutions and traditional reinsurance products. For example, [Biffis et al. \(2016\)](#) analyze the cost of collateralization in longevity swaps, while [Lakdawalla and Zanjani \(2012\)](#) investigate optimal collateral strategies involving reinsurance companies and catastrophe bonds, a prominent type of risk transfer instrument traded in the capital market. Furthermore, [Finken and Laux \(2009\)](#) discuss the potential of catastrophe bonds with parametric or index triggers as an alternative to reinsurance contracts in the presence of information asymmetry. Comprehensive overviews of the development of the longevity-linked capital market, as well as existing (successful and failed) capital market transactions can be found in [Blake et al. \(2013b\)](#), [Blake and Morales \(2017\)](#), and [Blake et al. \(2023\)](#). Our paper examines the development of the longevity risk transfer market from an alternative perspective of relative risk aversions of the participating parties.

The topic of managing longevity risk amidst ambiguity has been a focus in existing literature. Works by [Cairns \(2013\)](#) and [Li et al. \(2017b\)](#), for example, explore this area. However, unlike our approach where ambiguity exists solely from the seller’s viewpoint due to information asymmetry, these studies address ambiguity arising from estimation inaccuracies. In their context, the exact probability distribution of future longevity-linked cash flows is unknown to all parties, and estimations of this distribution are subject to various estimation errors. Information asymmetry underlying longevity risk transactions have also been widely discussed in the literature, see, for example, [Biffis and Blake \(2010\)](#),

2013, 2014); [Chen et al. \(2023\)](#). While many studies discuss information asymmetry from the seller’s perspective, [Biffis and Blake \(2013\)](#) examines pension buyout transactions and the securitization of longevity risk in scenarios where only the buyer, or both parties are uncertain about the true distribution of longevity risk. Such situations are interesting extensions of the proposed model in the current chapter.

In terms of methodology, this chapter is also connected to the emerging field of time-inconsistent dynamic optimization, particularly in the case of multi-period mean-variance problems. A common, yet simplistic method is to overlook the time inconsistency and optimize the objective for each period separately. This approach, however, tends to be sub-optimal due to its myopic nature. [Basak and Chabakauri \(2010\)](#) pioneered a more sophisticated approach by conceptualizing the time-inconsistent problem as a strategic game involving the individual’s future selves, subsequently seeking a sub-game perfect Nash equilibrium. This method not only addresses time inconsistency more effectively but has also become a predominant methodology in recent studies, as evidenced in [Björk et al. \(2014, 2017\)](#), and [Dai et al. \(2021\)](#). In our study, we adopt this equilibrium approach in the context of dynamic contracting. For more details about the intuition and definition of the equilibrium strategy, we refer interested readers to the introduction chapter 1.

The remainder of the chapter is organized as follows. Section 5.2 introduces the notation and the Stackelberg game framework. Section 5.3 presents the Age-Period-Cohort-Improvement model. Sections 5.4 and 5.5 discuss the optimal longevity swap contracts and their welfare implications in the reinsurance and the capital market setting, respectively. Finally, Section 5.7 concludes. Technical details are gathered in the Appendix.

5.2 Model Setting

This section outlines our model structure. We begin in Section 5.2.1 with an introduction to the notation and surplus dynamics of buyers and sellers. Section 5.2.2 defines the two types of contracts under consideration. Section 5.2.3 then presents the optimization problem for the two parties under complete information, framed within a Stackelberg game context. Finally, Section 5.2.4 introduces the Stackelberg game in the presence of ambiguity.

5.2.1 Surplus Dynamics

We commence by setting the groundwork for a contract arrangement concerning an indemnity longevity swap, involving two participating parties: the seller and the buyer. We

assume, for ease of understanding, that the buyer possesses a life annuity portfolio consisting of a single cohort of policyholders aged x at time 0. This is mathematically parallel to a defined-benefit pension scheme, characterized by a sequence of lifetime unitary payments made annually to the policyholders. It is also assumed that these policyholders are homogeneous, sharing identical future survival probabilities. The buyer's primary concern is longevity risk stemming from policyholders outliving expected lifespans. The objective is to either eliminate or reduce this risk through indemnity longevity swaps.³

Throughout this paper, we concentrate exclusively on longevity risk, setting aside financial risk. The market for longevity risk is modeled within the measurable space $(\Omega, \mathcal{F})$. Our analysis unfolds in a discrete-time framework, represented by the set of time points $\tau = \{0, 1, \dots, T\}$, T being the maximal survival horizon of the policyholders. Within this time frame, we construct a finite and discrete-time filtration, $\mathbb{F} = \{\mathcal{F}_t\}_{t \in \tau}$. Each σ -algebra $\mathcal{F}_t$ at time (year) t , $t \in \tau$, encompasses all information available up to and including that moment. We define all random variables and stochastic processes relevant to our study within this filtered probability space. Key notation used in our analysis include:

- $\mathbb{P}$: The reference measure for estimating the expected count of policyholders alive during contract formulation. Note that in the presence of seller's ambiguity (see Section 5.2.4), it serves as the most plausible prior probability measure.
- ${}_t p_x$: The true t -year survival probability for the policyholders aged x at time 0.

Our focus is on a single cohort aged x at time 0, and thus survival probabilities discussed in this paper pertain solely to this cohort. Unless noted otherwise, ${}_{t-s} p_{x+s}$ represents the $(t-s)$ -year survival probability for policyholders aged $x+s$ at time s for any $s, t \in \tau$ and $s < t$. While the true survival probabilities are known to the buyer, the seller may be *unsure* about these probabilities in the presence of ambiguity (see Section 5.2.4).

- ${}_t \hat{p}_x \equiv \mathbb{E}^{\mathbb{P}}[{}_t p_x]$: The estimated value of ${}_t p_x$ under the reference measure $\mathbb{P}$.⁴

³According to Artemis, longevity swaps and longevity risk reinsurance represent the most common form of longevity risk transfer in the current market. Although both mechanisms facilitate the transfer of longevity risk, longevity swaps are cash-settled (discussed later), whereas longevity reinsurance typically requires an upfront premium payment. The inclusion of longevity reinsurance within our analysis can be achieved straightforwardly. Source: <https://www.artemis.bm/library/what-is-longevity-risk-transfer/>.

⁴In this paper, we assume $\mathbb{E}^{\mathbb{P}}[{}_{t-s} p_{x+s}] = \mathbb{E}_h^{\mathbb{P}}[{}_{t-s} p_{x+s}]$ for any $h, s, t \in \tau$ and $0 < h < s < t$. The possibility of $\mathbb{E}_0^{\mathbb{P}}[{}_{t-s} p_{x+s}] \neq \mathbb{E}_h^{\mathbb{P}}[{}_{t-s} p_{x+s}]$ is referred to as recalibration risk (Cairns, 2013). Although not explicitly formulated, recalibration risk is implicitly incorporated into the optimal contracting problem in the presence of ambiguity, which will be presented in Section 5.2.4.

- l_0 : Initial count of policyholders aged x at time 0 within the annuity portfolio.
- l_t : The random count of survived policyholders in year t . Note that l_t is $\mathcal{F}_t$ -measurable and remains unobserved before t . Given information up to any s , where $s < t$, we assume that l_t follows a binomial distribution $B(l_s, {}_{t-s}p_{x+s})$.
- $\hat{l}_t$: The expected count of policyholders alive in year t under the reference measure $\mathbb{P}$. Note that this expected value takes a *different form* in static contracts and dynamic contracts, depending on whether the information of the realized count of survived policyholders is used (see Section 5.2.2).

The surplus of the buyer and the seller at time t is denoted by B_t and S_t , respectively. We assume the buyer's initial surplus, B_0 , is known. It could represent, for example, the total premium income from annuities or the assets of a pension plan in the payout phase. Our analysis exclusively focuses on longevity risk, assuming a financial market with a single, risk-free asset and a constant interest rate r . The buyer's surplus at time t is defined as:

$$B_t = B_{t-1}(1 + r) - l_t, \quad (5.1)$$

which is the surplus from the preceding period, B_{t-1} , adjusted for interest and reduced by the annuity payments at time t . Longevity risk originates from the uncertainty in future policyholder counts, l_t .

We explore longevity risk transfer through indemnity longevity swaps — agreements to exchange future cash flows based on the counts of survived policyholders within the buyer's portfolio. With an indemnity longevity swap, the seller offers either partial or complete protection against the uncertainties of l_t in exchange of a series of premiums. The longevity swap is a T -year contract and is cash-settled, meaning that no cash exchange is required at time 0. In year t , the seller pays a floating payment of $u_{t-1}l_t$ to the buyer and, in return, receives a fixed amount $u_{t-1}(1 + \eta)\hat{l}_t$, where $u_{t-1} \in [0, 1]$ is the *hedge ratio* at time t , η is the longevity risk loading, and recall that $\hat{l}_t$ is the expected count of policyholders alive in year t under the reference measure $\mathbb{P}$. In each year t , if $l_t > (1 + \eta)\hat{l}_t$, i.e., the realized number of policyholders alive is larger than the expected value plus the risk premium, then the buyer receives a net payment of $u_{t-1}(l_t - (1 + \eta)\hat{l}_t)$. In contrast, if $l_t < (1 + \eta)\hat{l}_t$, she makes a net payment of $u_{t-1}((1 + \eta)\hat{l}_t - l_t)$ to the seller.

Incorporating longevity swap into the buyer's portfolio modifies her hedged surplus, denoted by $\{B_t(u_{t-1})\}_{t=1, \dots, T}$, to the following:

$$B_t(u_{t-1}) = B_{t-1}(u_{t-2}) \cdot (1 + r) - l_t + u_{t-1}(l_t - (1 + \eta)\hat{l}_t). \quad (5.2)$$

Correspondingly, for the seller, assuming no preexisting assets or liabilities, the surplus follows the dynamics:

$$S_t = S_{t-1}(1 + r) - u_{t-1}(l_t - (1 + \eta)\hat{l}_t). \quad (5.3)$$

5.2.2 Static and Dynamic Contracts

In this study, we define two distinct types of longevity swap contracts: **static** and **dynamic** contracts. These contracts are differentiated based on the way how the hedge ratio and fixed payments are determined:

1. **Static Contract:** The static contract maintains a *constant hedge ratio*, $u_{t-1} \equiv u$, for all periods $t = 1, \dots, T$. In addition, $\hat{l}_t$ in the fixed payment formula for each t are determined at the *beginning of the contract (time 0)*, i.e., for $t = 1, \dots, T$

$$\hat{l}_t = \mathbb{E}^{\mathbb{P}} [l_t] = {}_t\hat{p}_x l_0. \quad (5.4)$$

2. **Dynamic Contract:** The dynamic contract allows for *changes* in the the hedge ratio over time. Specifically, the time- t hedge ratio, u_{t-1} , is determined at time $t - 1$ and is measurable with respect to $\mathcal{F}_{t-1}$. For the payment determination, $\hat{l}_t$ is also updated based on $\mathcal{F}_{t-1}$ at each $t - 1$. In other words,

$$\hat{l}_t = \mathbb{E}_{t-1}^{\mathbb{P}} [l_t] = \hat{p}_{x+t-1} l_{t-1}, \quad (5.5)$$

with $\hat{p}_{x+t-1}$ being the estimated one-year survival probability for a policyholder aged $x + t - 1$ in year $t - 1$, and l_{t-1} the observed number of policyholders alive at time $t - 1$.

It is worth noting that both contract types share the same maturity date, T . The terms ‘static’ and ‘dynamic’ are used to describe the structural difference in the scope of risk coverage. Static contracts are characterized by the constant hedge ratio and predetermined fixed payments, which make hedging costs completely known from the outset. This provides *high-risk coverage* to the buyer, delivering long-term coverage throughout the contract duration. In contrast, at each time t , dynamic contracts address longevity risk only for the upcoming year rather than the entire remaining contract duration. In this sense, like a series of one-year forwards, dynamic contracts offer *low-risk coverage* since the hedge ratio and fixed payment are periodically updated.

In real-world reinsurance markets for longevity risks, static contracts currently *dominate* in indemnity longevity swaps and reinsurance transactions, whereas dynamic contracts are less frequently utilized. However, dynamic contracts have garnered considerable attention in the academic literature on longevity risk management (see, for example, [Wong et al., 2017](#); [Wang et al., 2023](#)). Furthermore, a significant body of literature exists on dynamic contracts within the broader context of optimal insurance; see, for example, ([Chen et al., 2021](#); [Ghili et al., 2024](#); [Baradel, 2024](#)) for some recent works on dynamic contracts in risk sharing, health insurance, and catastrophe bonds. Therefore, one of the main objectives of this paper is to compare these two contract types. For a more comprehensive understanding, we analyze the welfare implications for both buyers and sellers.

We find that the performance of these contracts varies markedly with the risk aversion of the two parties. Specifically, when the buyer exhibits higher risk aversion than the seller, static contracts are preferred due to their alignment with the long-term hedging needs inherent in longevity risk. Conversely, when the seller displays higher risk aversion, dynamic contracts emerge as the more favorable option. This is because when the seller has high risk aversion, she is reluctant to offer long-term, higher-coverage contracts. Meanwhile, demand is low from less risk-averse buyers.

5.2.3 Stackelberg Game with Complete Information

To identify optimal contracts, we adopt a Stackelberg (principal-agent) game framework, recognizing the seller as the price maker and the buyer as the price taker. More specifically, the seller, acting as the principal, offers a range of swap contracts to the buyer, each distinguished by a specific longevity risk loading η . The buyer, as the follower, responds by selecting an optimal hedge ratio – either constant or time-varying depending on the contract type – for each given η . With knowledge of the buyer’s optimal choice of hedging ratio, the seller then determines the optimal value of risk loading η that best serves her own objective. Note that the seller presents only one type of contract at a time. Therefore, the optimization problem for static and dynamic contracts is conducted through two separate Stackelberg game analyses.

Our choice to employ the Stackelberg game framework is based on several reasons. First, the Stackelberg game framework can be seen as a middle ground between the classical monopoly and competitive models ([Rothschild and Stiglitz, 1976](#); [Stiglitz, 1977](#)), which are two extremes. It is widely acknowledged that real insurance markets lie between perfect competition and monopoly. Even though the seller is the principal and leads the game, it is clearly not optimal to set the risk loading extremely high as the buyer’s demand would

diminish. Second, the buyer's demand is *endogenously* determined according to the seller's pricing strategy (η). Due to the lack of empirical data, it is generally difficult to accurately estimate the buyers' demand curve in response to changes in insurance prices. In the Stackelberg game, buyers' demand is well incorporated by solving for the buyer's optimal hedge ratio in the first step (the agent's problem). Third, to the best of our knowledge, we are among the first to apply the Stackelberg game framework to longevity risk transfer in the presence of information asymmetry, despite its adoption in general optimal insurance studies (e.g., [Chen and Shen, 2018](#); [Cao et al., 2022](#); [Boonen and Ghossoub, 2023](#)).⁵ Furthermore, unlike existing works, we focus on investigating the relationship between contract type preferences (static or dynamic) and the risk preferences of the players.

In the Stackelberg game, we select the mean-variance (MV) criterion as the objective function for both players. This choice is based on the following reasons. First, stemming from Markowitz's modern portfolio theory, the MV criterion has been widely used as a common performance measure in asset management for firms, applicable to both single-period and multi-period problems (see, e.g., [Aït-Sahali and Brandt, 2001](#); [Acharya and Pedersen, 2005](#); [Basak and Chabakauri, 2010](#); [Björk et al., 2014](#); [Dai et al., 2021](#)). Second, the MV criterion can be viewed as an approximation to the standard von Neumann–Morgenstern expected utility maximization, as per the Arrow–Pratt approximation of certainty equivalence ([Maccheroni et al., 2013](#)). This implies that the findings based on the MV criterion should remain similar when general utility functions are used. Third, the MV criterion simply reduces to the risk-neutral case when the risk aversion parameter becomes zero.

Formally, for the *static contract*, the buyer selects the optimal constant hedging ratio u given a risk loading $\eta \geq 0$. This is achieved by solving the following optimization problem:

$$\sup_u \left\{ \mathbb{E}^{\mathbb{P}} [B_T(u, \eta)] - \frac{\gamma_b}{2} \text{Var}^{\mathbb{P}} [B_T(u, \eta)] \right\}, \quad (5.6)$$

where γ_b represents the buyer's degree of risk aversion, and $B_T(u, \eta)$ denotes the buyer's surplus at time T , given hedge ratio u and risk loading η . The seller, considering the buyer's optimal response, $u^*(\eta)$, then selects the optimal risk loading η to maximize the mean-variance criterion:

$$\sup_{\eta \geq 0} \left\{ \mathbb{E}^{\mathbb{P}} [S_T(u^*(\eta))] - \frac{\gamma_s}{2} \text{Var}^{\mathbb{P}} [S_T(u^*(\eta))] \right\}, \quad (5.7)$$

where $S_T(u^*(\eta))$ is the seller's surplus at time T given hedge ratio $u^*(\eta)$ and γ_s denotes the seller's degree of risk aversion.

⁵[Chen et al. \(2023\)](#) have also applied the Stackelberg game framework to optimize longevity risk transfer contract designs when buyers exhibit two distinct risk types. Their study involves offering a single longevity swap with a fixed η to both risk types, each determining their respective optimal hedge ratio.

In the context of a *dynamic contract*, the buyer adjusts her strategy to maximize the mean-variance criterion with a *time-varying* hedge ratio:

$$\sup_{\mathbf{u}} \left\{ \mathbb{E}^{\mathbb{P}} [B_T(\mathbf{u}, \eta)] - \frac{\gamma_b}{2} \text{Var}^{\mathbb{P}} [B_T(\mathbf{u}, \eta)] \right\}, \quad (5.8)$$

with $\mathbf{u} = (u_t)_{t \in \{0,1,2,\dots,T-1\}}$. The seller's corresponding optimization problem is then defined as:

$$\sup_{\eta \geq 0} \left\{ \mathbb{E}^{\mathbb{P}} [S_T(\mathbf{u}^*(\eta))] - \frac{\gamma_s}{2} \text{Var}^{\mathbb{P}} [S_T(\mathbf{u}^*(\eta))] \right\}, \quad (5.9)$$

where $\mathbf{u}^* = (u_t^*(\eta))_{t \in \{0,1,2,\dots,T-1\}}$. The optimization problems above represent the process of optimal contracting in the case of *complete information*, where the seller has perfect knowledge of the true survival probability of the policyholders. In this case, it is natural to assume that the reference measure coincides with the true probabilities, ${}_{t-s}\mathbf{p}_{x+s} = {}_{t-s}\hat{\mathbf{p}}_{x+s}$ for any $s, t \in \tau$, and $s < t$.

We remark that, throughout the formulation of the Stackelberg game, we assume that the buyer and the seller exhibit risk aversion $\gamma_b > 0$ and $\gamma_s \geq 0$, respectively. The buyer's risk aversion is crucial because it determines her willingness to engage in longevity risk transfer. If the buyer were not risk-averse, she would lack the motivation to transfer longevity risk considering the risk loading imposed by the seller in addition to the expected loss. This additional cost is a deterrent unless mitigated by the benefits of risk transfer. In contrast, the seller could be either risk averse or risk neutral. Throughout the analysis, we consider a general situation of a risk-averse seller. As a specific scenario, we will discuss the results with a risk-neutral seller ($\gamma_s = 0$) in Section 5.4.3.

5.2.4 Stackelberg Game under Ambiguity

We now introduce the optimal longevity risk transfer problem under information asymmetry. In this scenario, the seller does not know the true probability measure of the buyer's portfolio. Therefore, the seller considers a collection of probability measures, $\mathcal{Q}$, which she believes includes all potential candidates for the true probability measure. Each probability measure in $\mathcal{Q}$, denoted as $\mathbb{Q}^\lambda$, is referred to as a *prior probability measures* and is formally defined as follows.

Under a given prior probability measure $\mathbb{Q}^\lambda$, the $(t - s)$ -year survival probability for the policyholder aged $x + s$ is given by:

$$\mathbb{E}^{\mathbb{Q}^\lambda} [{}_{t-s}\mathbf{p}_{x+s}] := ({}_{t-s}\hat{\mathbf{p}}_{x+s})^\lambda, \quad (5.10)$$

where we recall that ${}_{t-s}\hat{p}_{x+s} = \mathbb{E}^{\mathbb{P}}[{}_{t-s}p_{x+s}]$, and we denote by $\lambda > 0$ the parameter labeling the prior probability measures. In particular, the reference measure $\mathbb{P}$ corresponds to the case that $\lambda = 1$. The formulation in Equation (5.10) ensures that the survival probabilities are confined within the interval of $[0, 1]$ under all priors. With this construction, a λ value *greater* than 1 implies *lower* survival probabilities and, consequently, a reduced life expectancy compared to what is predicted by the reference measure. It is important to note that this definition is merely a convenient way to represent the expected values of true survival probabilities under various priors within our framework. Alternative definitions are also feasible and can be integrated as needed.

In response to the uncertainty regarding the true probability measure, we adopt the maxmin expected utility framework (Gilboa and Schmeidler, 1989) to find the best strategy under the worst-case scenario. This is arguably the most common approach in economic decision-making under ambiguity (see, e.g., Bossaerts et al., 2010; Ju and Miao, 2012; Dimmock et al., 2016; Kostopoulos et al., 2022). In our context, the worst-case scenario corresponds to the prior probability measure $\mathbb{Q}^\lambda$ that leads to the largest survival probabilities of the policyholders. The optimal contracting process in the presence of ambiguity is detailed below.

For the *static contract*, given a prior probability measure $\mathbb{Q}^\lambda$, the buyer selects the optimal constant hedging ratio u for a given risk loading $\eta \geq 0$ by solving:

$$\sup_u \left\{ \mathbb{E}^{\mathbb{Q}^\lambda} [B_T(u, \eta)] - \frac{\gamma_b}{2} \text{Var}^{\mathbb{Q}^\lambda} [B_T(u, \eta)] \right\}. \quad (5.11)$$

The seller, considering the buyer's optimal response $u^*(\eta)$ and following the maxmin principle, then calculates the optimal η under the worst-case scenario within $\mathcal{Q}$, the set of prior probability measures:

$$\sup_{\eta \geq 0} \inf_{\mathbb{Q}^\lambda \in \mathcal{Q}} \left\{ \mathbb{E}^{\mathbb{Q}^\lambda} [S_T(u^*(\eta))] - \frac{\gamma_s}{2} \text{Var}^{\mathbb{Q}^\lambda} [S_T(u^*(\eta))] \right\}. \quad (5.12)$$

In the context of a *dynamic contract*, the buyer maximizes the mean-variance criterion with a time-varying hedge ratio:

$$\sup_{\mathbf{u}} \left\{ \mathbb{E}^{\mathbb{Q}^\lambda} [B_T(\mathbf{u}, \eta)] - \frac{\gamma_b}{2} \text{Var}^{\mathbb{Q}^\lambda} [B_T(\mathbf{u}, \eta)] \right\}, \quad (5.13)$$

with $\mathbf{u} = (u_t)_{t \in \{0, 1, 2, \dots, T-1\}}$. The seller's corresponding optimization problem is then defined as:

$$\sup_{\eta \geq 0} \inf_{\mathbb{Q}^\lambda \in \mathcal{Q}} \left\{ \mathbb{E}^{\mathbb{Q}^\lambda} [S_T(\mathbf{u}^*(\eta))] - \frac{\gamma_s}{2} \text{Var}^{\mathbb{Q}^\lambda} [S_T(\mathbf{u}^*(\eta))] \right\}, \quad (5.14)$$

where $\mathbf{u}^* = (u_t^*(\eta))_{t \in \{0,1,2,\dots,T-1\}}$.

The derivation of the corresponding hedging ratios for the static contract is detailed in Appendix C.1. A notable challenge in the dynamic contract scenario is the time inconsistency introduced by the time-varying hedging ratio, a complex aspect often encountered in stochastic mean-variance optimization problems (see, e.g., Björk and Murgoci, 2010; Björk et al., 2014). To address this, we apply the mainstream game-theoretic approach as proposed in Björk and Murgoci (2010), formulating the optimal hedge ratio as an equilibrium strategy by solving a system of extended Hamilton-Jacobi-Bellman (EHJB) equations. Comprehensive explanations of the equilibrium strategy and associated methodologies are provided in Appendix C.2.

5.3 Mortality Model

To model the randomness of annuitants, we first need to estimate the survival probability. This section introduces the Age-Period-Cohort-Improvement (APCI) mortality model, which will be used to produce the survival probability estimates under different measures. Section 5.3.1 introduces the model, and Section 5.3.2 discusses the construction of the set of prior probability measures, $\mathcal{Q}$.

5.3.1 The APCI model

The Age-Period-Cohort-Improvement (APCI) model, as used by the CMI Mortality Projections Committee,⁶ generates life tables for UK life insurers and pension funds. The model articulates the annual death rates at age x and year h , $m_{x,h}$, logarithmically:

$$\ln(m_{x,h}) = \beta_x^{(1)} + \beta_x^{(2)}(h - \bar{h}) + \kappa_h + \gamma_c + \sigma_{\omega_x}\omega_{x,h}, \quad (5.15)$$

where $\beta_x^{(1)}$ and $\beta_x^{(2)}$ represent the age and period effects, respectively, $\bar{h}$ is a normalization parameter, κ_h is a time-varying mortality improvement factor, γ_c denotes the cohort effect, σ_{ω_x} is the standard deviation of error terms for age x , and $\omega_{x,h}$ are independent standard normal distributed error terms.

Forecasting future survival probabilities involves projecting the mortality improvement factor, κ_h . We model κ_h as a random walk following an ARIMA(0,1,0) process:

$$\kappa_h = \kappa_{h-1} + \sigma_\kappa \varepsilon_h, \quad (5.16)$$

⁶Source: CMI Mortality Projections Committee (2016).

where σ_κ is the standard deviation of the error terms, and ε_h are independent standard normal distributed error terms.

The APCI model is calibrated to UK uni-sex death rates sourced from the Human Mortality Database,⁷ using one-age, one-year death rates for ages 20 to 100 and years 1956 to 2020. The model parameters in Equation (5.15) are estimated via maximum likelihood (Richards et al., 2019). Subsequently, the ARIMA(0,1,0) model (5.16) is fitted to the estimated mortality improvement parameters. After estimating the model, the estimated future survival probabilities under the reference measure are obtained by standard simulation practice. For the sake of completeness, we provide a brief summary of the simulation process in Appendix C.3.

5.3.2 Construction of $\mathcal{Q}$

This subsection discusses how the set of prior probability measures, $\mathcal{Q}$, is constructed. According to the definition, $\mathcal{Q}$ is fully characterized by specifying a set of values for the parameter λ . However, the challenge is that the value of λ is somewhat abstract for a decision maker to specify. To address this, we construct the range of λ by linking it with the projected *remaining life expectancy* at age x .

More specifically, by (5.10), we know that the remaining life expectancy at age x under $\mathbb{Q}^\lambda$ is given by

$$\hat{e}_x(\lambda) := \mathbb{E}^{\mathbb{Q}^\lambda} \left[\sum_{k=1}^T {}_k p_x \right] = \sum_{k=1}^T {}_k \hat{p}_x^\lambda.$$

In particular, $\hat{e}_x := \hat{e}_x(1)$ represents the projected life expectancy under the reference measure $\mathbb{P}$. We then define the set of prior probability measures as follows:

$$\mathcal{Q} = \{\mathbb{Q}^\lambda : \text{the value of } \lambda \text{ satisfies } \hat{e}_x(\lambda) \in [(1 - \alpha)\hat{e}_x, (1 + \alpha)\hat{e}_x]\}.$$

Here, we use α as a parameter to represent the *degree of ambiguity*. It reduces to the scenario without ambiguity if $\alpha = 0$. As an illustrative example, suppose that the initial age of policyholders is $x = 65$ and the degree of ambiguity equals $\alpha = 0.05$, meaning that the decision maker speculates that the expected life expectancy can be 5% higher or lower than predicted under the reference measure $\mathbb{P}$. Then we have

$$\mathcal{Q} = \{\mathbb{Q}^\lambda : \lambda \in [0.86, 1.15]\}$$

because $\hat{e}_{65}(0.86) = 1.05\hat{e}_{65}$ and $\hat{e}_{65}(1.15) = 0.95\hat{e}_{65}$. Table 5.1 outlines more ranges of λ for by varying the degree of ambiguity α .

⁷Source: <https://www.mortality.org/>.

Table 5.1: Interval of λ with different degree of ambiguity α

α	Interval of λ
0.05	[0.86, 1.15]
0.1	[0.74, 1.33]
0.15	[0.64, 1.53]
0.2	[0.54, 1.76]
0.4	[0.24, 3.18]

5.4 Buyer with Stronger Risk Aversion ($\gamma_b > \gamma_s$)

This section evaluates the welfare implications of static versus dynamic contracts when the buyer exhibits greater risk aversion than the seller. In our numerical analysis, we set the risk aversion parameters for the buyer (γ_b) at 0.3 and for the seller (γ_s) at 0.1. The base scenario parameters are an interest rate (r) of 0.02, an initial age (x) of 65, an initial annuitant count (l_0) of 10,000, and a contract duration (T) of 35 years. Without loss of generality, the initial wealth of both the buyer (B_0) and the seller (S_0) is assumed to be zero.

The principal finding from the numerical analysis is that *static contracts consistently outperform dynamic contracts* in terms of providing greater welfare gains to both parties. This outcome results from the buyer's higher risk aversion, which intensifies the demand for long-term longevity risk hedging. Additionally, the introduction of ambiguity reduces welfare gain for both parties, yet does not alter the superior performance of static contracts. This scenario also reveals that ambiguity narrows the range of longevity risk loadings that benefit both parties. More detailed analysis of these results is presented in the sections addressing scenarios without ambiguity in Section 5.4.1 and with ambiguity in Section 5.4.2. Moreover, we consider the case that the seller is risk neutral ($\gamma_s = 0$) in Section 5.4.3. As the difference in risk aversion is now larger, we find that both parties exhibit an even stronger preference for static contracts.

5.4.1 Without Ambiguity

We initiate our analysis with the benchmark scenario in the absence of ambiguity, where the seller possesses complete knowledge of the true probability measure, which is assumed to be the reference measure $\mathbb{P}$. In Figure 5.1(a), we illustrate the welfare gains for the seller with both static and dynamic contracts. For a specific η , the welfare gain from

a contract is defined as the difference between the seller's optimal expected utility from selling the contract at equilibrium volume at that η and the baseline utility in the absence of any transaction. Thus, a positive welfare gain indicates that the seller benefits from the transaction at that particular η value.

From Figure 5.1(a), several key insights emerge. First, the range of η values yielding a positive welfare gain is significantly *broader* for the static contract compared to the dynamic one, suggesting that static contracts facilitate market existence more readily. Second, the η values that sustain a functional market are notably *higher* for static contracts, reflecting the greater longevity risk the seller assumes for the duration of these contracts. Third, the maximum welfare gain achievable with static contracts considerably surpasses that with dynamic contracts, due to the buyer's stronger preference for the more comprehensive risk coverage offered by static contracts. This preference leads buyers to accept higher η values for static contracts than for dynamic ones.

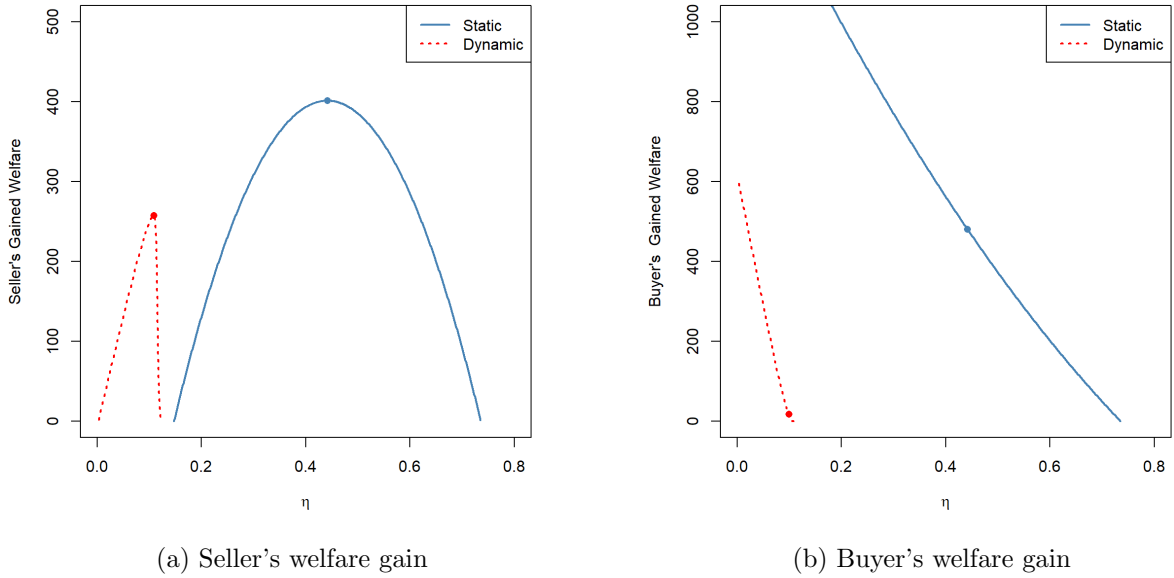


Figure 5.1: Welfare gain of the seller (left panel) and the buyer (right panel) when trading the static and the dynamic contract.

Figure 5.1(b) depicts the welfare gains for the buyer across both contract types, with the bold dot on each line indicating the buyer's welfare gain at the optimal η^* . Consistently, the equilibrium static contract yields a substantially higher welfare gain for the buyer.

5.4.2 With Ambiguity

We now shift our focus to scenarios involving ambiguity, where the true survival probabilities of the policyholders remain unknown to the seller. Our analysis considers two specifications of $\mathcal{Q}$ representing different degrees of ambiguity. The first specification, characterized by $\alpha = 0.1$, considers all probability measures leading to $\hat{e}_{65}(\lambda) \in [0.9\hat{e}_{65}, 1.1\hat{e}_{65}]$. The second specification represents a higher degree of ambiguity with $\alpha = 0.2$ and thus $\hat{e}_{65}(\lambda) \in [0.8\hat{e}_{65}, 1.2\hat{e}_{65}]$.

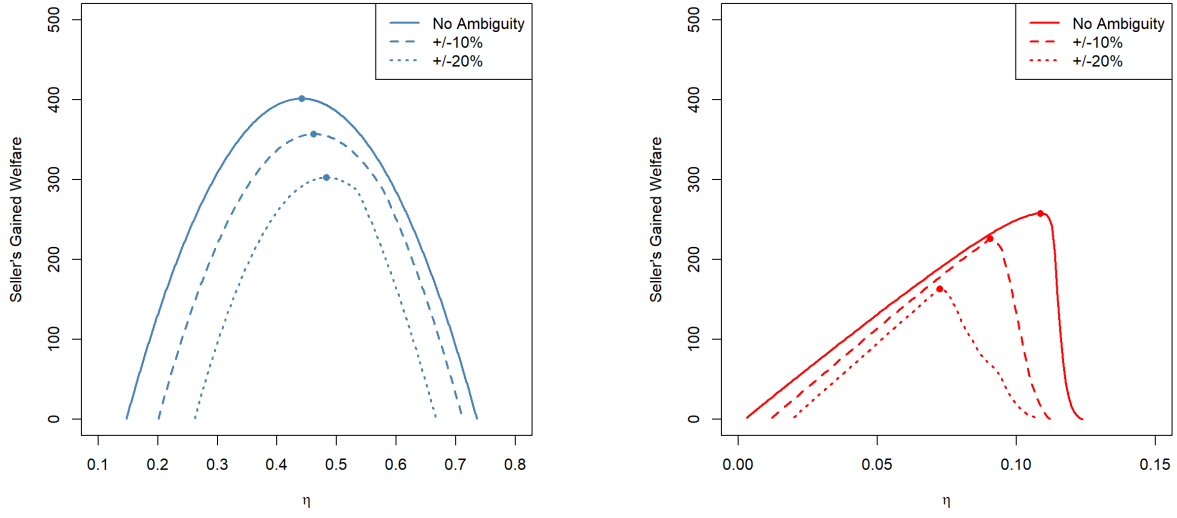
Welfare Gain for Both Parties

We begin by assessing the seller's welfare. Figures 5.2(a) and 5.2(b) illustrate the seller's welfare with both contract types under three ambiguity scenarios: no ambiguity, and with ambiguity sets characterized by $\alpha = 0.1$ and 0.2 , respectively. The $\pm\%$ notation indicates the degree of ambiguity, e.g., $\pm 10\%$ means $\alpha = 0.1$.

Several observations can be made from Figures 5.2. First, under all ambiguity levels, the seller's welfare diminishes across all η values. This reduction stems from the seller's maxmin expected utility approach, where the worst-case scenario is assumed—where the buyer's true probability measure leads to the highest expected annuity payouts, thereby increasing the seller's payment obligations and decreasing welfare gains. Additionally, ambiguity narrows the range of η values that yield favorable outcomes for the seller, reducing her inclination to offer longevity swaps and making it more difficult for the longevity transfer market to exist.

Crucially, despite the presence of ambiguity, the static contract remains more advantageous, providing positive welfare gains over broader η ranges and higher gains at the optimal η^* in all scenarios. Optimal η^* values and corresponding welfare gains in all scenarios are summarized in Table 5.2. Notably, the static contract maintains more *stable* optimal η^* values across different degrees of ambiguity compared to the dynamic contract. Moreover, the dynamic contract's maximum welfare gain decreases significantly with increasing ambiguity, which suggests that the dynamic contract is less robust to ambiguity.

We then turn to the buyer's welfare gain. Figures 5.3(a) and 5.3(b) depict the buyer's welfare gain across all prior probability measures in the ambiguity sets with $\alpha = 0.1$ and 0.2 , respectively. For each set, welfare gains at each λ measure the buyer's welfare gain from engaging in transactions with η^* and $u_t^*(\eta^*)$ when her true probability measure is $\mathbb{Q}^\lambda$. Evidently, the static contract yields higher welfare for the buyer across all prior probability measures, making it the optimal choice for the buyer irrespective of the actual survival probabilities in her portfolio.



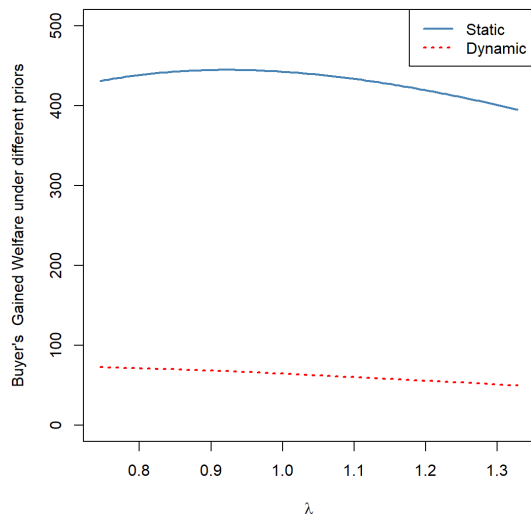
(a) Static Contract

(b) Dynamic Contract

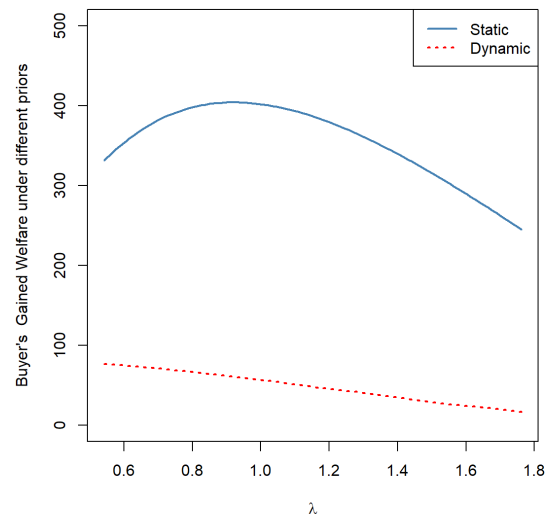
Figure 5.2: Seller's welfare gain when trading the static (left panel) and the dynamic contract (right panel) across different degrees of ambiguity.

Table 5.2: Seller's optimal risk loading and welfare gain.

	Static Contract		Dynamic Contract	
	η^*	Welfare Gain	η^*	Welfare Gain
No ambiguity	0.4417	401.0935	0.1087	257.1561
$\alpha = 0.1$	0.4618	356.7751	0.0906	225.5776
$\alpha = 0.2$	0.4836	302.4461	0.0725	163.1762



(a) $\alpha = 0.1$



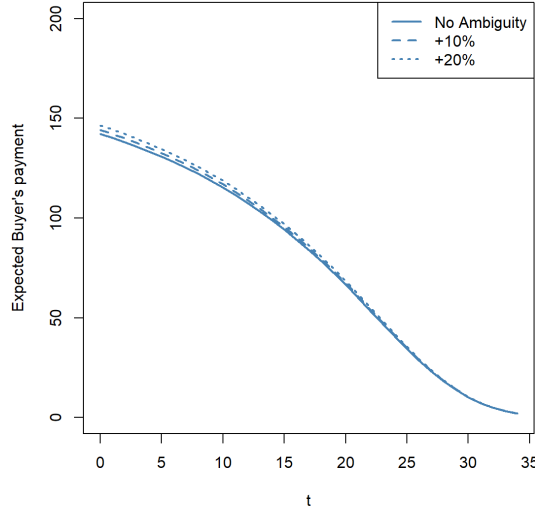
(b) $\alpha = 0.2$

Figure 5.3: Buyer's welfare gain across all prior probability measures with $\alpha = 0.1$ (left panel) and 0.2 (right panel).

Fixed Payments and Optimal Hedge Ratios

In order to get a better idea of the risk coverage provided by both contracts, we examine the time patterns of fixed payments and the optimal hedge ratios selected by the buyer for both contract types.

Figure 5.4 illustrates the fixed payments for the static contract under full hedge, denoted as $(1 + \eta^*)\hat{l}_t$ across all ambiguity scenarios. Notably, there is a consistent downward trend in fixed payments over time due to the natural decline in the expected number of surviving policyholders. Additionally, the fixed payments are very similar across scenarios. This is expected, as the fixed payments are predetermined at time 0 and not affected by the mortality experience of the policyholders after contract inception. Furthermore, the fact that optimal η^* values for the static contract are rather similar across different levels of ambiguity further contributes to the consistent fixed payments (see Table 5.2).



(a) Static Contract

Figure 5.4: The fixed payments with full hedge, $(1 + \eta^*)\hat{l}_t$, for the static contract.

Conversely, Figure 5.5 presents the dynamic contract's fixed payments. Here, the fixed payments in each year t are determined based on the most recently observed count of survived policyholders, l_{t-1} . Consequently, there are uncertainties associated with the future fixed payments because of the randomness of l_{t-1} . To account for this uncertainty, Figure 5.5 includes the 95% confidence interval for these payments. This interval is represented

both for scenarios without ambiguity (Figure 5.5(a)) and for the case with ambiguity at an α level of 0.2 (Figure 5.5(b)). Even without ambiguity, the variability in fixed payments is pronounced, contrasting with the fixed payments with the static contracts. Moreover, the variability of fixed payments is markedly amplified when ambiguity is present. This further indicates the less predictable nature of the hedging costs with dynamic contracts compared to their static counterparts.

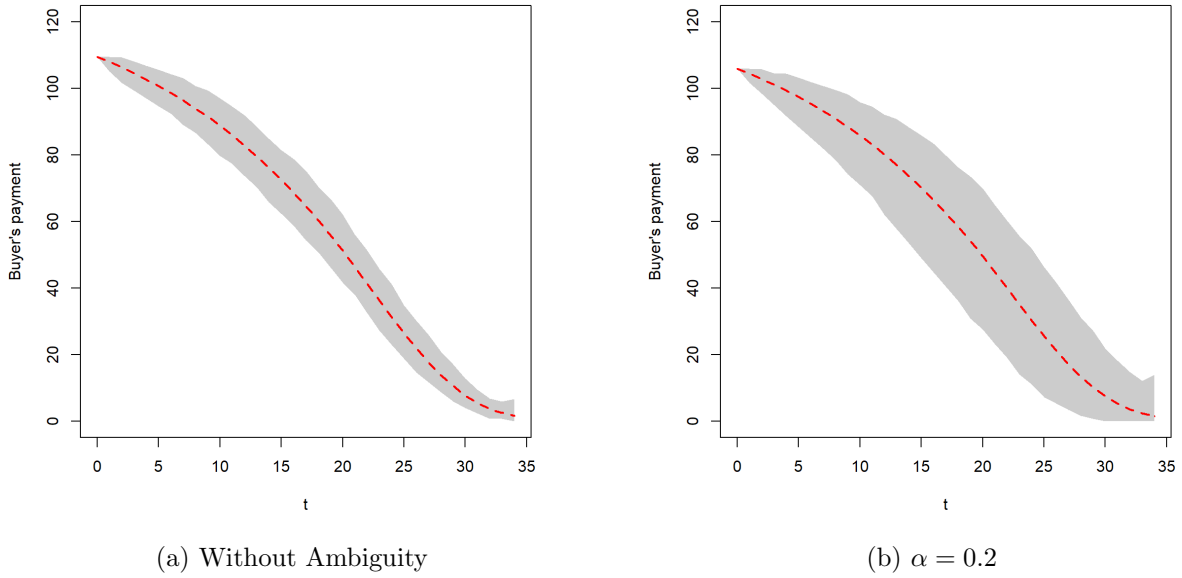


Figure 5.5: The 95% confidence intervals of the fixed payments with full hedge, $(1 + \eta^*)\hat{l}_t$, with the dynamic contract in the absence of ambiguity (left panel) and in the presence of ambiguity with $\alpha = 0.2$ (right panel).

Turning to the optimal hedging ratios, Figure 5.6 compares these ratios for both contract types across all levels of ambiguity, assuming a hypothetical buyer with the worst-case prior measure at each level. For both contracts, the optimal hedge ratio increases with ambiguity level due to heightened hedging demand from the ‘worst-case’ buyer, who anticipates higher future annuity payouts. Notably, the optimal hedge ratio for static contracts is more robust compared to dynamic contracts, as it is considerably more stable across ambiguity scenarios.

In the dynamic contract, the optimal hedge ratio exhibits a declining trend over time. In particular, in the absence of ambiguity, the buyer opts for a full hedge for the first 25 years,

then gradually reduces the hedge ratio thereafter. This decline is driven by the buyer's trade-off between the impact of hedging on the mean and variance of her liability. Initially, the reduction in variance outweighs the increase in mean, motivating higher hedge ratios. However, maintaining a full hedge results in a disproportionate rise in expected liability compared to the reduction in variance over time, leading to lower hedge ratios near the end of the contract. When the ambiguity level increases, the buyer maintains full hedge for a longer period of time.

Lastly, we observe an interesting pattern of optimal hedge ratios in the presence of ambiguity for the dynamic contract. Specifically, when $\alpha = 0.2$, the optimal hedge ratio starts below full hedge and gradually increases to full hedge over a few years. This outcome results from the recursive nature of determining the optimal hedge ratio. When the ambiguity level is high, the buyer anticipates maintaining a higher hedge ratio for an extended period, which results in higher hedging costs. Consequently, to mitigate the overall cost of hedging, the buyer strategically opts for a slightly lower hedge ratio in the early years when uncertainty regarding the number of surviving policyholders is lower.

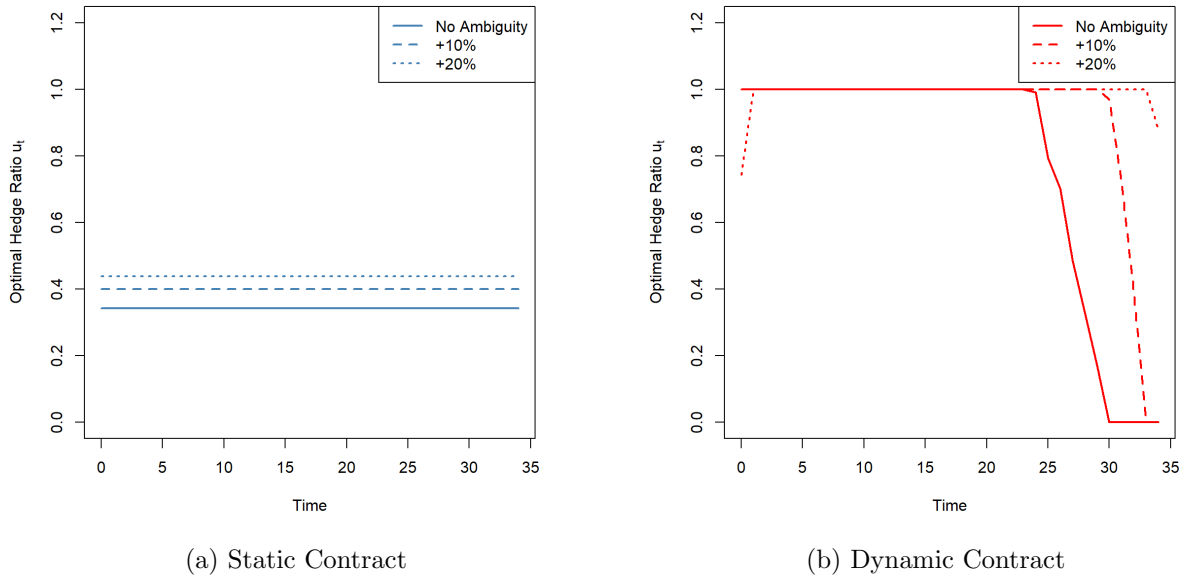


Figure 5.6: Buyer's optimal hedge ratio with both contract types in the reinsurance market at different ambiguity levels.

5.4.3 Risk-Neutral Seller

This subsection examines the robustness of our findings regarding risk aversion parameters by considering the case where the seller is risk-neutral ($\gamma_s = 0$), while buyer's risk aversion γ_b remains at 0.3. This setup indicates a larger difference in risk aversion between the two parties.

Figure 5.7 displays the welfare gains for both parties under this configuration. Similar to observations from Figure 5.1, trading the static contract yields significantly greater welfare gains for both parties compared to the dynamic contract. Notably, when the seller is risk-neutral, these gains are even more pronounced than in scenarios where the seller is risk-averse.

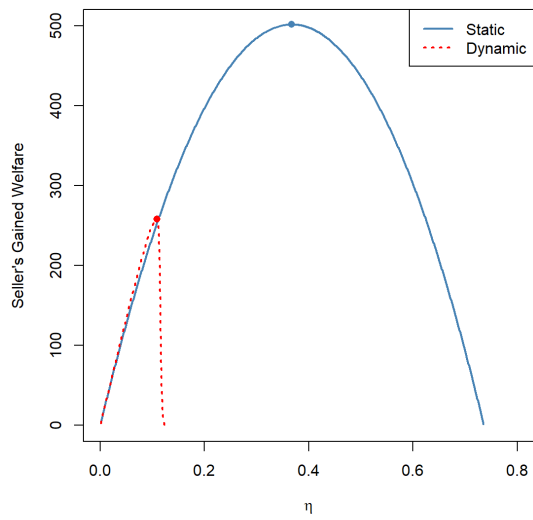
Additionally, Figure 5.8 depicts the seller's welfare for both contract types under the three ambiguity scenarios discussed in Section 5.4.2. Consistent with outcomes with a risk-averse seller, we observe that the seller's welfare decreases across all values of η with the introduction of ambiguity, and the viable η intervals for transactions narrow. However, in each ambiguity scenario, these intervals are broader for a risk-neutral seller than for a risk-averse seller, particularly for the static contract. This suggests that the existence of a longevity risk transfer market is more probable when the seller is risk-neutral.

In summary, this section assesses the welfare implications of the two contract types of indemnity longevity swap for both parties in the transaction when the buyer is more risk averse. We find that the static contract consistently outperforms the dynamic counterpart, providing more pronounced benefits to both parties. This is because the static contract provides a higher risk coverage to the buyer over the entire lifespan of her annuity portfolio. This conclusion holds true in scenarios both with and without ambiguity.

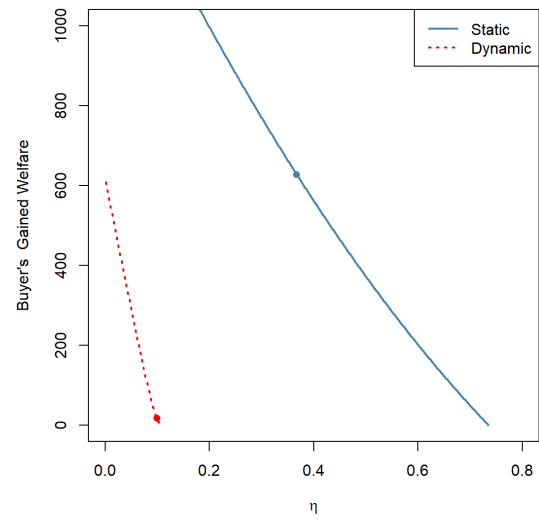
5.5 Seller with Stronger Risk Aversion ($\gamma_b < \gamma_s$)

In this section, we examine indemnity longevity swap transactions where the seller is more risk-averse. We set the risk aversion parameters at $\gamma_b = 0.1$ for the buyer and $\gamma_s = 0.3$ for the seller, with all other base parameters remaining constant.

Our quantitative analysis indicates a notable shift in outcomes due to the altered risk aversion levels. Specifically, the dynamic contract becomes more favorable for the seller because it involves a lesser extent of longevity risk transfer. This allows the seller to offer the dynamic contract at a lower longevity risk loading, increasing its attractiveness to the buyer. Furthermore, we observe that increased risk aversion of the seller heightens the

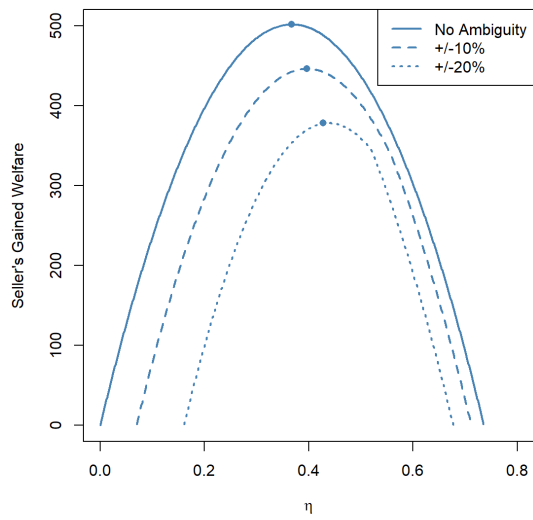


(a) Seller's welfare gain

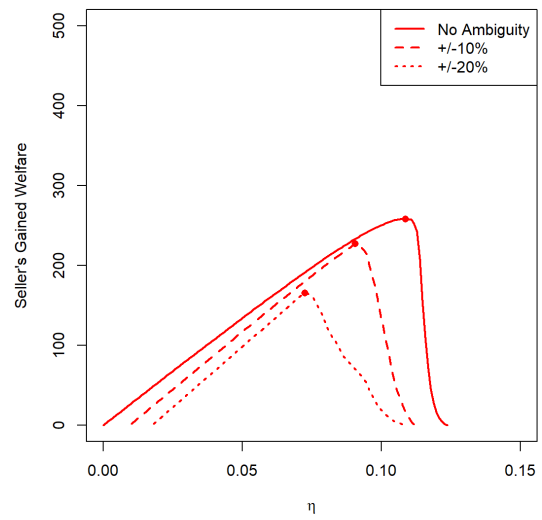


(b) Buyer's welfare gain

Figure 5.7: Welfare gain of the seller (left panel) and the buyer (right panel) when trading the static and the dynamic contract when seller is risk-neutral



(a) Static Contract



(b) Dynamic Contract

Figure 5.8: Welfare gain of the seller when trading the static (left panel) and the dynamic contract (right panel) across different degrees of ambiguity when seller is risk neutral.

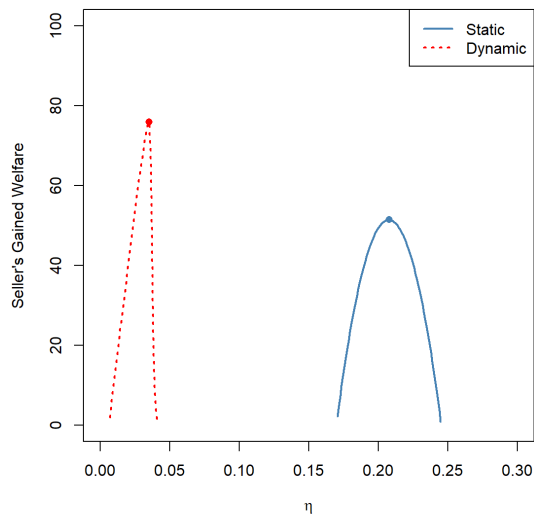
potential for market collapse, a scenario in which issuing any contract could leave one or both parties worse off than if no transaction were made. Detailed numerical results including scenarios both with and without ambiguity, are presented in Sections 5.5.1 and 5.5.2, respectively.

5.5.1 Without Ambiguity

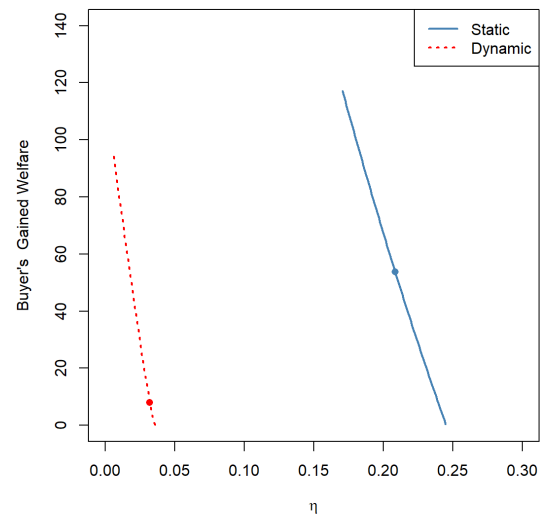
Figure 5.9 illustrates the welfare gains for both parties across different η values for each contract type in the absence of ambiguity. From the seller's perspective (Figure 5.9(a)), although the range of η values favoring transactions remains broader for the static contract, it has narrowed significantly compared to scenarios involving a more risk-averse buyer (see Figure 5.1(a)). Notably, the dynamic contract now yields a maximal welfare gain that exceeds that of the static contract, suggesting a preference shift by the seller towards the dynamic contract due to their increased risk aversion, which discourages them from committing to long-term longevity risk protection. Conversely, Figure 5.9(b) shows that the buyer could still achieve higher welfare gains with a static contract, if available. This preference aligns with the buyer's ongoing need for long-term longevity risk protection, even when her risk aversion is comparatively lower. Nonetheless, a static contract is unlikely to be offered by the seller, who is the price maker, as it leads to a lower maximum welfare gain than the dynamic contract.

5.5.2 With Ambiguity

Moving to scenarios with ambiguity, Figures 5.10(a) and 5.10(b) display the welfare gains for both contract types across three different degrees of ambiguity for the seller: no ambiguity, $\alpha = 0.05$, and $\alpha = 0.1$. In contrast to findings in Section 5.4, welfare gains from static contracts become a lot more *sensitive* to changes in ambiguity levels compared to dynamic contracts. Specifically, at an ambiguity level of $\alpha = 0.1$, static contracts no longer yield beneficial outcomes for the seller at any η , leading to a *cessation of the market for static contracts*. Conversely, the welfare gain for dynamic contracts does not decrease as sharply with increased ambiguity, demonstrating their stronger robustness to ambiguity when the seller is more risk averse. Even at $\alpha = 0.1$, the range of η ensuring positive welfare gain remains substantial. Notably, the ambiguity levels addressed here are lower than those considered in Section 5.4, as the unfavorable outcomes for static contracts at $\alpha = 0.1$ discourage exploration into higher levels of ambiguity.

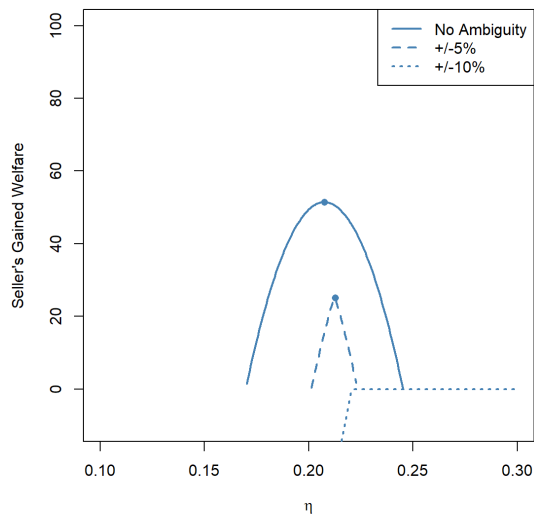


(a) Seller's Welfare Gain

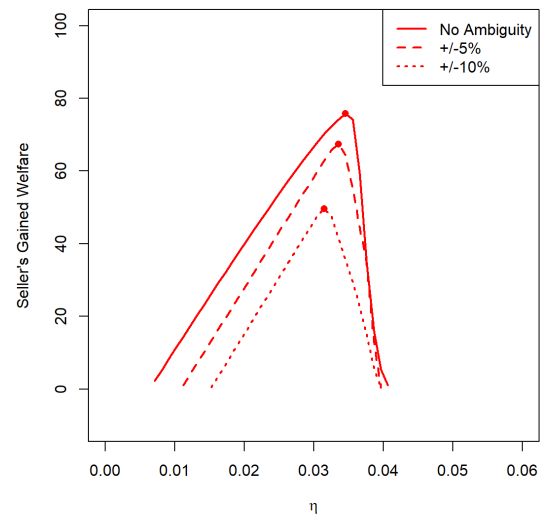


(b) Buyer's Welfare Gain

Figure 5.9: Welfare gain of the seller (left panel) and the buyer (right panel) when trading the static and the dynamic contract in the absence of ambiguity.



(a) Static Contract



(b) Dynamic Contract

Figure 5.10: Welfare gain of the seller when trading the static (left panel) and the dynamic contract (right panel) across different ambiguity scenarios.

Figure 5.11 then examines the buyer’s welfare gain with both static and dynamic contracts at an ambiguity level of $\alpha = 0.05$. In contrast the findings in Section 5.4.2 (see Figure 5.3), the dynamic contract offers higher welfare gains for the buyer across almost all prior probability measures. This shift occurs as the seller, now more risk averse, would require significantly higher risk loadings to offer the static contract. Such increased loadings would drastically reduce the buyer’s welfare gains, making the dynamic contract a more cost-effective choice despite its lower risk coverage. Specifically, for λ values above 1.06, the buyer avoids purchasing any static contract. Lastly, with $\alpha = 0.1$, the static contract ceases to be offered, resulting in a welfare gain of zero for the static contract, irrespective of the buyer’s true probability measure.

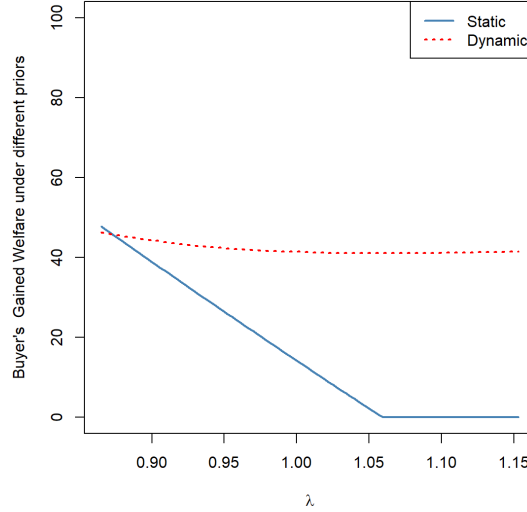


Figure 5.11: The buyer’s welfare gain across all prior probability measures with both contract types with the ambiguity level of $\alpha = 0.05$ when the seller is more risk averse.

5.6 Implications for Longevity Risk Transfer Market

This section discusses the implications of our findings for the current longevity risk transfer market. We begin with an overview of the market landscape in Section 5.6.1, followed by an exploration of the differing levels of risk aversion between parties in both the reinsurance sector and the emerging longevity-linked capital market in Sections 5.6.2 and 5.6.3. Our

analysis aims to illuminate several key observed facts within the existing market and provide insights into potential development pathways for the longevity-linked capital market.

5.6.1 Overview of the Longevity Risk Transfer Market

Since its inception in 2006, the modern longevity risk transfer market has predominantly involved transactions within the reinsurance sector, where the typical seller is a reinsurance company. These transactions are generally long-term, with the contract term covering the lifetime of the underlying annuity or pension liability, and the reinsured liability defined at inception. For example, in December 2023, Nationale-Nederlanden (NN) Group transferred €13 billion of longevity risk related to pension liabilities in the Netherlands to Prudential Financial, Inc. and Swiss Re, covering approximately 300,000 Netherlands pension policies until the portfolio runs off.⁸

Although the global reinsurance sector has effectively absorbed longevity risks, the growing volume of risk carried by pension plans and funds, due to increasing life expectancy, suggests a need for the capital markets to become conduits for longevity risk transfer, as the risk will eventually exceed the reinsurance sector’s capacity (Blake et al., 2019; Chen et al., 2023). Longevity swaps and other longevity-linked assets, with their low correlation to conventional asset classes like equities, bonds, and real estate, are considered attractive to general capital market investors for diversifying asset portfolios (Blake and Kearns, 2022).

Despite these theoretical advantages, capital market transactions involving longevity risk remain rare. Investment banks, including major players like J.P. Morgan, initially engaged alongside insurers since market inception but have gradually receded. In addition, banks with insurance subsidiaries, such as Goldman Sachs (owner of Rothesay Life) and Deutsche Bank (owner of Abbey Life), participated briefly before divesting from their life insurance businesses (Blake et al., 2023). We refer to (Kessler, 2021; Blake et al., 2023) for a comprehensive review of the capital market’s role in longevity risk transfer and the analysis of existing transactions.

A key factor in the scarcity of capital market transactions and the withdrawal of capital market investors is likely their limited understanding of longevity risk and the challenge of forecasting longevity-linked cash flows. This knowledge gap could increase risk aversion among capital market investors toward longevity risk. Moreover, concerns about adverse

⁸NN Group has successfully transferred €13 billion in Dutch pension longevity risk to Prudential Financial, Inc.’s insurance subsidiary and Swiss Re: <https://www.artemis.bm/news/nn-group-taps-prudential-swiss-re-for-e13bn-longevity-risk-transfer/>.

selection, where the counterparty might have deeper insights into the risk, could deter participation (Blake and Kearns, 2022). The next subsections explore how risk aversion influences contract preference and market existence in longevity risk transfers, offering theoretical explanations for the current market dynamics and insights for cultivating the longevity-linked capital market.

5.6.2 Reinsurance Sector

In the reinsurance market, it is likely that buyers, particularly small or medium-sized life insurers and pension plans, are more risk-averse than sellers for several reasons. First, these buyers often manage relatively modest pools of annuitants or pensioners, leading to less stable and less predictable longevity-linked financial commitments. Additionally, many pension plans cater to homogeneous groups of pensioners who share similar socioeconomic characteristics such as occupation, income, and geographic location. This homogeneity tends to amplify systematic longevity risks. Furthermore, the financial stability of life insurers or pension plans is crucial for the well-being of retirees, as a failure to meet financial obligations could significantly reduce pension payments, adversely affecting the financial security and quality of life of pensioners. Finally, the direct accountability of buyers to policyholders results in stricter solvency regulations. Collectively, these factors heighten the risk aversion of buyers towards failing to meet their financial commitments.

On the other hand, reinsurers could exhibit lower risk aversion compared to buyers. First, reinsurers possess superior expertise in modeling and managing longevity-linked cash flows. Second, their ability to manage longevity risk across a diverse array of life insurers or pension funds allows them to diversify their risk exposure and enhance overall stability. Third, reinsurers have a greater capacity to hedge assumed longevity risks. For example, they can create internal natural hedging portfolios by combining life insurance portfolios, which face mortality risk, with annuity or pension portfolios, exposed to longevity risk. Furthermore, reinsurers enjoy more flexibility in sharing longevity risk, for example, with other reinsurers. Finally, reinsurers are subject to less stringent regulatory requirements than life insurers and pension plans. Therefore, as discussed in Section 5.4, the preference for static contracts in longevity reinsurance, driven by the higher risk aversions of buyers, aligns with the empirical observation that most longevity reinsurance transactions are *static and long-term* in nature.

5.6.3 Capital Market

In contrast, for capital market transactions, we believe that it is more common that the seller is more risk-averse than the buyer. This is because the seller for such transactions are general institutional capital market investors, such as investment banks, mutual funds, or endowments, while the buyers are reinsurers or large life insurers. This differs from reinsurance market transactions because the seller now possesses less expertise in modeling longevity-linked cash flows and a reduced capability to hedge longevity risk compared to the buyer. In this case, our analysis in Section 5.5 provides a possible explanation to the *rarity* of existing longevity risk transfer transactions in the capital market. Specifically, while buyers demand a static, long-term hedging instrument, sellers are unwilling to offer such contracts. In addition, in the presence of ambiguity, it is very easy for the market of static contract to collapse.

Finally, Table 5.3 reports the maximum ambiguity levels α at which markets will sustain existence for both contract types in scenarios with both a more risk-averse buyer (akin to the reinsurance sector) and a more risk-averse seller (similar to the capital market). First, we observe that the maximum ambiguity levels are significantly higher for a more risk-averse buyer, indicating a more robust environment for maintaining longevity risk transactions in the reinsurance sector. This finding corroborates the reality that most existing longevity risk transactions occur within the reinsurance sector. Second, with a more risk-averse seller, dynamic contracts exhibit a far greater tolerance to ambiguity compared to static contracts. This suggests that initiating transactions with dynamic contracts may be more advantageous for developing the longevity-linked capital market.

Table 5.3: The maximum ambiguity level at which the market will exist.

	Static	Dynamic
More risk averse buyer	$\alpha = 0.46$	$\alpha = 0.44$
More risk averse seller	$\alpha = 0.09$	$\alpha = 0.22$

The preference for dynamic contracts in the capital markets is reflected in the Vita Capital CAT bonds issued by Swiss Re, as discussed earlier, and the recent emergence of longevity sidecars in the capital market. Longevity sidecars started with late 2017 transactions by Athene Holding Ltd and Voya Financial, involving a \$19 billion in annuities sidecar transaction. Athene subsequently launched the Athene Co-Invest Reinsurance Affiliate (ACRA) sidecar program, which included a \$27 billion fixed annuity block reinsurance transaction. By the end of 2020, ACRA’s sidecar-related invested assets had grown

to \$25 billion. In February 2023, Athene launched ACRA 2 aiming to raise \$2 billion, with commitments later increased to \$4 billion by September 2023. These recent sidecar transactions have demonstrated the feasibility of transferring longevity risk using short-dated instruments in the capital markets. While such short-term instruments may not represent the ultimate solution given the long-term nature of longevity risk, they serve as valuable initial attempts towards an efficient and active capital market for longevity risk transfer. For further discussions of longevity sidecars, we refer to [Bugler et al. \(2021\)](#), [Blake and Kearns \(2022\)](#), and [Blake et al. \(2023\)](#).

5.7 Conclusion

In this study, we develop an economic framework to analyze optimal longevity risk transfer under ambiguity using a Stackelberg game approach. The framework involves a buyer, seeking to offload longevity risk, and a seller, providing hedge services. We focus on two types of indemnity longevity swap contracts: a static contract, characterized by a constant hedge ratio and fixed payments predetermined at time 0, and a dynamic contract, where both hedge ratios and fixed payments are periodically adjusted based on the latest available mortality information. The static contract offers higher risk coverage with known hedging costs at its onset, whereas the dynamic contract, which is similar to a rolling strategy of short-dated contracts over time, presents uncertain hedging costs and lower coverage due to its reliance on continually updated data. Ambiguity arises from information asymmetry, as the seller lacks complete information about the true probability distribution of the buyer's future mortality.

Our analysis assesses the welfare implications of these contracts for both parties under two scenarios: one where the buyer is more risk-averse, and another where the seller is more risk-averse. The findings indicate a preference for the static (resp. dynamic) contract when the buyer (resp. seller) is more risk-averse. These preferences persist regardless of the presence of ambiguity. Additionally, ambiguity tends to decrease welfare gains for both parties and reduces the likelihood of successful longevity risk transfers. This effect is especially pronounced when the seller is more risk averse, in which case even a moderate degree of ambiguity can deter the offering of static contracts at any price.

These insights offer theoretical explanations for several observed facts in the current longevity risk transfer market. First, economic theories and the current market status suggest that the buyer is commonly more risk averse in the reinsurance sector, while the opposite holds in capital market transactions. Our model explains why most longevity risk transactions in the reinsurance sector are static, why transactions in the capital market

are rare, and why there is a growing trend towards short-dated instruments like mortality CAT bonds and longevity sidecars in the capital market. Moreover, reducing information asymmetry and thus ambiguity could enhance the feasibility and benefits of longevity risk transfers across both market settings and contract types.

Future research could explore several promising directions. First, examining scenarios where information asymmetry occurs in the opposite direction—where the buyer lacks complete information—could provide valuable insights, particularly in the reinsurance market where reinsurers often have superior modeling expertise than small- and medium-size pension plans ([Biffis and Blake, 2013](#)). We conjecture that our main findings would hold if buyers perceive low degrees of ambiguity, but markets might cease to exist under high ambiguity levels for buyers. Furthermore, while this paper focuses on indemnity longevity swaps, exploring alternative hedging instruments, such as index-based longevity derivatives ([Cairns, 2013](#)), could provide valuable insights for capital market transactions.

Chapter 6

Conclusion and Future Works

The main objective of this thesis is to investigate the dynamic optimization problem in finance and insurance within the framework of the ambiguity model, from both theoretical and empirical perspectives. We hope this thesis will contribute to a better understanding of decision makers' behavior under ambiguity.

More specifically, the main contribution of chapter 3 and chapter 4 is that it has employed the ambiguity model proposed by [Maccheroni et al. \(2013\)](#) in the stochastic control problem. The M-V-V model is an analog of the Arrow-Pratt approximation to the well-known smooth ambiguity model, but it offers a more tractable structure and meanwhile separates the modeling of ambiguity, ambiguity aversion, and risk aversion. To fix the horizon inconsistency in the M-V-V criterion, we provide a new M-V-SD criterion. In chapter 3, we study a portfolio optimization problem under two criteria (M-V-V and M-V-SD) which models the DM's ambiguity belief to the financial risks. In chapter 4, we study the equilibrium reinsurance strategy under the M-V-V criterion and the M-V-SD criterion. The major challenge in our analysis is the time inconsistency due to the functional $\mathbb{X}_\nu[\cdot]$ of the expectation under prior probability measures, which is solved by introducing the Gâteaux derivative in the EHJB.

We find that the M-V-V equilibrium strategy has two noteworthy properties: The DM rebalances his investment portfolio in response to ambiguity depending on risk premium under $\mathbb{P}$. An increase in ambiguity aversion decreases the optimal exposure to the risky asset. One surprising finding is that the M-V-V equilibrium strategy is non-monotonic with respect to the risk aversion parameter due to the non-consistent scale of the time horizon between the risk measure and the ambiguity measure. To resolve the issue with the horizon-scale inconsistency of the M-V-V criterion, we introduce the M-V-SD criterion

to evaluate DM’s ambiguity belief. The M-V-SD strategy shares common properties with the M-V-V strategy but further also exhibits other appealing features: the monotonicity of risk aversion and the non-trading (non-retention) parameter zone. Finally, we note that the analytical tractability of the M-V-SD criterion renders our model specially fit for the study of investment and reinsurance behaviors under ambiguity.

Chapter 5 develops an economic framework to analyze optimal longevity risk transfer under ambiguity using a Stackelberg game approach, involving a buyer offloading longevity risk and a seller providing hedge services. We compare static contracts, with constant hedge ratios and fixed payments, to dynamic contracts, with periodically adjusted terms. Static contracts offer higher risk coverage but higher costs, while dynamic contracts have uncertain costs and lower coverage. Ambiguity arises from the seller’s incomplete information on the buyer’s future mortality probabilities. Our analysis shows that more risk-averse buyers prefer static contracts, while more risk-averse sellers prefer dynamic contracts. Ambiguity reduces welfare gains and the likelihood of successful risk transfers, especially for risk-averse sellers. These findings explain why most longevity risk transactions in reinsurance are static, why capital market transactions are rare, and why short-dated instruments are becoming more popular. Reducing information asymmetry could enhance longevity risk transfer feasibility and benefits across markets.

We highlight some of the most relevant avenues for future research below:

1. In chapter 3, it would be interesting to consider the dynamic investment optimization problem under the M-V-V criterion in the precommitted approach; see, e.g., [Li and Ng \(2000\)](#) and [Zhou and Li \(2000\)](#). The precommitted strategy focuses solely on determining the optimal strategy at the initial time and obligates the DM’s future selves to adhere to the strategy established at the initial time. This strategy, which disregards future suboptimality and results in time-inconsistent decisions, is commonly solved using the Lagrange dual method.
2. In Chapter 4, it is worth investigating the dynamic reinsurance problem under the M-V-V criterion within the framework of a Stackelberg game. [Chen and Shen \(2019\)](#) proposes a new continuous-time framework to analyze optimal reinsurance, where an insurer and a reinsurer are the two players in a stochastic Stackelberg differential game. The reinsurer moves first, followed by the insurer, to achieve a Stackelberg equilibrium for optimal reinsurance arrangements. The optimization objective in their model can be replaced by the M-V-V criterion in future work.
3. In chapter 5, future research could explore several promising directions. First, examining scenarios where the buyer lacks complete information could provide valuable

insights, particularly in the reinsurance market where reinsurers have superior modeling expertise compared to small- and medium-sized pension plans ([Biffis and Blake, 2013](#)). We conjecture that our findings would hold if buyers perceive low ambiguity, but markets might cease under high ambiguity. Second, incorporating recalibration risk ([Cairns, 2013](#)), which involves updating survival probabilities based on observed survival counts, is another important avenue. Finally, exploring alternative hedging instruments, such as index-based longevity derivatives, could provide valuable insights for capital market transactions.

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APPENDICES

Appendix A

Proofs for chapter 3

A.1 Equivalence with the setup in [Maccheroni et al. \(2013\)](#)

We first review the general setup presented in [Maccheroni et al. \(2013\)](#) for the ambiguity model. Consider a probability measure $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathbb{P}$ is the reference measure. Each prior probability measure $\mathbb{Q}^q$ on $(\Omega, \mathcal{F})$ is defined via the Radon-Nikodym derivative $q = \frac{d\mathbb{Q}^q}{d\mathbb{P}}$, where q belongs to the set

$$\Delta = \{q \in L^2(\Omega, \mathbb{P}) : q \geq 0 \text{ } \mathbb{P}\text{-a.s. and } \mathbb{E}^{\mathbb{P}}[q] = 1\}. \quad (\text{A.1})$$

We equip Δ with the Borel sigma-algebra $\mathcal{B}_\Delta$ induced by the L^2 -metric. A second-order distribution ν_Δ is a probability measure defined on $(\Delta, \mathcal{B}_\Delta)$ that characterizes the ambiguity of DMs. Given any $X \in L^2(\Omega, \mathbb{P})$, one can define a mapping $e_X : \Delta \rightarrow \mathbb{R}$ as

$$e_X(q) := \int_{\Omega} X q d\mathbb{P} = \mathbb{E}^{\mathbb{Q}^q}[X],$$

where $\mathbb{Q}^q$ is the prior probability measure generated by q . By the continuity of the mapping $q \rightarrow e_X(q)$, ensuring its measurability, we know that e_X defines a random variable on the probability space $(\Delta, \mathcal{B}_\Delta, \nu_\Delta)$. The mean and variance of e_X with respect to ν_Δ are denoted, respectively, by

$$\mathbb{E}_{\nu_\Delta}[e_X] = \int_{\Delta} e_X(q) d\nu_\Delta(q)$$

and

$$\begin{aligned}\text{Var}_{\nu_\Delta}[e_X] &= \mathbb{E}_{\nu_\Delta}[(e_X)^2] - (\mathbb{E}_{\nu_\Delta}[e_X])^2 \\ &= \int_{\Delta} e_X(q)^2 d\nu_\Delta(q) - \left(\int_{\Delta} e_X(q) d\nu_\Delta(q) \right)^2.\end{aligned}\tag{A.2}$$

The M-V-V criterion in [Maccheroni et al. \(2013\)](#) is defined according to (A.2).

Following the above general setup, one can define the ambiguity model under the Black-Scholes framework. By Girsanov's Theorem, the set of Radon-Nikodym derivatives is given by

$$\Delta = \left\{ q_T^\lambda : q_T^\lambda = e^{-\frac{1}{2}\lambda^2 T - \lambda W_T} \text{ for } \lambda \in \mathbb{R} \right\},$$

equipped with the Borel sigma-algebra $\mathcal{B}_\Delta$ induced by the L^2 -metric. We define a bijection $f : (\mathbb{R}, \mathcal{B}_\mathbb{R}) \rightarrow (\Delta, \mathcal{B}_\Delta)$ as

$$f(\lambda) = q_T^\lambda, \quad \text{for } \lambda \in \mathbb{R}.$$

Note that f is continuous as

$$\begin{aligned}\|q_T^{\lambda_1} - q_T^{\lambda_2}\|_{L^2(\Omega, \mathbb{P})} &= \mathbb{E} \left(\left| e^{-\frac{1}{2}\lambda_1^2 T - \lambda_1 W_T} - e^{-\frac{1}{2}\lambda_2^2 T - \lambda_2 W_T} \right|^2 \right)^{1/2} \\ &= \left| e^{\lambda_1 \sqrt{T}} - e^{\lambda_2 \sqrt{T}} \right|.\end{aligned}$$

For any given probability measure ν on $(\mathbb{R}, \mathcal{B}_\mathbb{R})$, also referred to as the second-order distribution, one can define a *pushforward probability measure* on $(\Delta, \mathcal{B}_\Delta)$ as

$$\nu_\Delta(B) := \nu(f^{-1}(B)), \quad \text{for any } B \in \mathcal{B}_\Delta.$$

By the property of the pushforward measure (e.g., Theorem 3.6.1 of [Bogachev and Ruas \(2007\)](#)), we have

$$\int_{\Delta} g(q_T^\lambda) d\nu_\Delta(q_T^\lambda) = \int_{\mathbb{R}} (g \circ f)(\lambda) d\nu(\lambda),$$

which implies

$$\mathbb{E}_{\nu_\Delta}[g] = \int_{\Delta} g(q_T^\lambda) d\nu_\Delta(q_T^\lambda) = \int_{\mathbb{R}} (g \circ f)(\lambda) d\nu(\lambda) = \int_{\mathbb{R}} g(q_T^\lambda) d\nu(\lambda),$$

for any integrable function g . By letting $g = e_X$ or $(e_X)^2$ for a fixed $X \in L^2(\Omega, \mathbb{P})$, we have

$$\mathbb{E}_{\nu_\Delta}[e_X] = \int_{\mathbb{R}} e_X(q_T^\lambda) d\nu(\lambda) = \int_{\mathbb{R}} \mathbb{E}^{\mathbb{Q}^\lambda}[X] d\nu(\lambda) = \mathbb{E}_\nu[\mathbb{E}^{\mathbb{Q}^\cdot}[X]], \tag{A.3}$$

where the last step is by (4.11), and

$$\begin{aligned}
\text{Var}_{\nu_\Delta}[e_X] &= \mathbb{E}_{\nu_\Delta}[(e_X)^2] - (\mathbb{E}_{\nu_\Delta}[e_X])^2 \\
&= \int_{\mathbb{R}} (e_X(q_T^\lambda))^2 d\nu(\lambda) - \left(\int_{\mathbb{R}} e_X(q_T^\lambda) d\nu(\lambda) \right)^2 \\
&= \int_{\mathbb{R}} \left| \mathbb{E}^{\mathbb{Q}^\lambda}[X] \right|^2 d\nu(\lambda) - \left(\int_{\mathbb{R}} \mathbb{E}^{\mathbb{Q}^\lambda}[X] d\nu(\lambda) \right)^2 \\
&= \text{Var}_\nu \left[\mathbb{E}^{\mathbb{Q}}[X] \right], \tag{A.4}
\end{aligned}$$

where the last step is by (4.12). Therefore, (A.3) and (A.4) imply that the two ways to define the second-order distribution, i.e. ν_Δ on Δ or ν on $\mathbb{R}$, are equivalent for the robust M-V model.

A.2 Proof of Lemma 3.1

For a given $(t, x) \in [0, T] \times \mathbb{R}$ and $\mathbf{u} \in \mathbb{U}$, we define a sequence of stopping times:

$$\tau_n = \inf \left\{ s \geq t : |X_s^{t,x,\mathbf{u}} - x| \geq n \right\},$$

for $n \in \mathbb{N}$. Let $f \in C_b^2(\mathbb{R})$, i.e., second-order continuously differentiable and with compact support on $\mathbb{R}$, such that $f(x) = x^2$ on the interval $[x - n, x + n]$. By Itô's formula and (3.12), we have

$$\begin{aligned}
& (X_{T \wedge \tau_n}^{t,x,\mathbf{u}})^2 - x^2 \\
&= f(X_{T \wedge \tau_n}^{t,x,\mathbf{u}}) - f(x) \\
&= \int_t^{T \wedge \tau_n} \left[f_x(X_s^{t,x,\mathbf{u}}) (rX_s^{t,x,\mathbf{u}} + \alpha(\lambda)u(s, X_s^{t,x,\mathbf{u}})) + \frac{1}{2}f_{xx}(X_s^{t,x,\mathbf{u}})\sigma^2 u(s, X_s^{t,x,\mathbf{u}})^2 \right] ds \\
&\quad + \int_t^{T \wedge \tau_n} \sigma u(s, X_s^{t,x,\mathbf{u}}) f_x(X_s^{t,x,\mathbf{u}}) dW_s^\lambda \\
&= \int_t^{T \wedge \tau_n} \left[2X_s^{t,x,\mathbf{u}} (rX_s^{t,x,\mathbf{u}} + \alpha(\lambda)u(s, X_s^{t,x,\mathbf{u}})) + \sigma^2 u(s, X_s^{t,x,\mathbf{u}})^2 \right] ds \\
&\quad + \int_t^{T \wedge \tau_n} \sigma u(s, X_s^{t,x,\mathbf{u}}) f_x(X_s^{t,x,\mathbf{u}}) dW_s^\lambda.
\end{aligned}$$

By the linear growth condition (3.7) and $\alpha(\lambda) = \mu - r - \sigma\lambda$, it follows that

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}^\lambda} \left[(X_{T \wedge \tau_n}^{t,x,\mathbf{u}})^2 \right] &= x^2 + \mathbb{E}^{\mathbb{Q}^\lambda} \left[\int_t^{T \wedge \tau_n} [2X_s^{t,x,\mathbf{u}} (rX_s^{t,x,\mathbf{u}} + \alpha(\lambda)u(s, X_s^{t,x,\mathbf{u}})) + \sigma^2 u(s, X_s^{t,x,\mathbf{u}})^2] ds \right] \\ &\leq x^2 + C(1 + |\lambda|) \mathbb{E}^{\mathbb{Q}^\lambda} \left[\int_t^{T \wedge \tau_n} (1 + (X_s^{t,x,\mathbf{u}})^2) ds \right] \\ &= x^2 + C(1 + |\lambda|) \int_t^T \left(1 + \mathbb{E}^{\mathbb{Q}^\lambda} [(X_{s \wedge \tau_n}^{t,x,\mathbf{u}})^2] \right) ds,\end{aligned}$$

for some positive constant C depending on r, μ, σ , and C_1 . By Grönwall's inequality (e.g., Lemma 1.3.1 of Pham (2009)), we have

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}^\lambda} \left[(X_{T \wedge \tau_n}^{t,x,\mathbf{u}})^2 \right] &\leq x^2 + C(1 + |\lambda|)(x^2 + 1) \int_0^{T-t} e^{C(1+|\lambda|)(T-t-s)} ds \\ &= (x^2 + 1)e^{C(1+|\lambda|)(T-t)} - 1.\end{aligned}$$

By letting $n \rightarrow \infty$ and using the dominated convergence theorem,

$$\mathbb{E}^{\mathbb{Q}^\lambda} \left[(X_T^{t,x,\mathbf{u}})^2 \right] \leq (x^2 + 1)e^{C(1+|\lambda|)(T-t)} - 1. \quad (\text{A.5})$$

It follows from Jensen's inequality (A.5) and the condition $\int_{\mathbb{R}} e^{C|\lambda|} d\nu(\lambda) < \infty$ that

$$\int_{\mathbb{R}} \left(\mathbb{E}^{\mathbb{Q}^\lambda} [X_T^{t,x,\mathbf{u}}] \right)^2 d\nu(\lambda) \leq \int_{\mathbb{R}} \mathbb{E}^{\mathbb{Q}^\lambda} [(X_T^{t,x,\mathbf{u}})^2] d\nu(\lambda) < \infty,$$

which confirms that the mapping $\lambda \rightarrow \mathbb{E}^{\mathbb{Q}^\lambda} [X_T^{t,x,\mathbf{u}}]$ belongs to $L^2(\mathbb{R}, \nu)$.

A.3 Proof of Theorem 3.1

Step 1. We show that $g(t, x, \lambda) = \mathbb{E}^{\mathbb{Q}^\lambda} [X_T^{t,x,\mathbf{u}^*}]$. By Itô's formula and the assumption that $g(t, x, \lambda) \in C^{1,2,0}([0, T] \times \mathbb{R} \times \mathbb{R})$, for any $(t, x, \lambda) \in [0, T] \times \mathbb{R} \times \mathbb{R}$, we have

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}^\lambda} [g(T, X_T^{t,x,\mathbf{u}^*}, \lambda)] &= g(t, x, \lambda) + \mathbb{E}^{\mathbb{Q}^\lambda} \left[\int_t^T \mathcal{A}^{u^*, \lambda} g(s, X_s^{t,x,\mathbf{u}^*}, \lambda) ds \right] \\ &\quad + \mathbb{E}^{\mathbb{Q}^\lambda} \left[\int_t^T \sigma g_x(s, X_s^{t,x,\mathbf{u}^*}, \lambda) u^*(s, X_s^{t,x,\mathbf{u}^*}) dW_s \right] \\ &= g(t, x, \lambda),\end{aligned}$$

where the last equality is due to (3.30) and the condition $g(t, x, \lambda) \in L_T^2(X^{\mathbf{u}^*})$, which ensures that the stochastic integral part is a martingale. Further, by $g(T, x, \lambda) = x$ as in (3.31), it follows that

$$\mathbb{E}^{\mathbb{Q}^\lambda}[X_T^{t,x,\mathbf{u}^*}] = \mathbb{E}^{\mathbb{Q}^\lambda}[g(T, X_T^{t,x,\mathbf{u}^*}, \lambda)] = g(t, x, \lambda). \quad (\text{A.6})$$

In particular, by letting $\lambda = 0$ and recalling $g(t, x) := g(t, x, 0)$, we have

$$\mathbb{E}^{\mathbb{P}}[X_T^{t,x,\mathbf{u}^*}] = \mathbb{E}^{\mathbb{P}}[g(T, X_T^{t,x,\mathbf{u}^*})] = g(t, x). \quad (\text{A.7})$$

Step 2. We show that $V(t, x) = J(t, x, \mathbf{u}^*)$. Substituting (3.30) into (3.29) yields

$$\mathcal{A}^{u^*} V(t, x) = \mathcal{A}^{u^*} (G \circ g)(t, x) + \mathcal{A}^{u^*} \mathbb{H}_\nu[g(t, x, \cdot)]. \quad (\text{A.8})$$

By Itô's formula, condition 5 that $V(t, x), (G \circ g)(t, x), \mathbb{H}_\nu[g(t, x, \cdot)] \in L_T^2(X^{\mathbf{u}^*})$, and (A.8), we have

$$\begin{aligned} \mathbb{E}^{\mathbb{P}}[V(T, X_T^{t,x,\mathbf{u}^*})] &= V(t, x) + \mathbb{E}^{\mathbb{P}} \left[\int_t^T \mathcal{A}^{u^*} V(s, X_s^{t,x,\mathbf{u}^*}) ds \right] \\ &= V(t, x) + \mathbb{E}^{\mathbb{P}} \left[\int_t^T \mathcal{A}^{u^*} (G \circ g)(s, X_s^{t,x,\mathbf{u}^*}) ds \right] \\ &\quad + \mathbb{E}^{\mathbb{P}} \left[\int_t^T \mathcal{A}^{u^*} \mathbb{H}_\nu[g(s, X_s^{t,x,\mathbf{u}^*}, \cdot)] ds \right], \end{aligned} \quad (\text{A.9})$$

$$\mathbb{E}^{\mathbb{P}}[(G \circ g)(T, X_T^{t,x,\mathbf{u}^*})] = (G \circ g)(t, x) + \mathbb{E}^{\mathbb{P}} \left[\int_t^T \mathcal{A}^{u^*} (G \circ g)(s, X_s^{t,x,\mathbf{u}^*}) ds \right], \quad (\text{A.10})$$

and

$$\mathbb{E}^{\mathbb{P}}[\mathbb{H}_\nu[g(T, X_T^{t,x,\mathbf{u}^*}, \cdot)]] = \mathbb{H}_\nu[g(t, x, \cdot)] + \mathbb{E}^{\mathbb{P}} \left[\int_t^T \mathcal{A}^{u^*} \mathbb{H}_\nu[g(s, X_s^{t,x,\mathbf{u}^*}, \cdot)] ds \right]. \quad (\text{A.11})$$

Substituting (A.10) and (A.11) into (A.9) yields

$$\begin{aligned} &\mathbb{E}^{\mathbb{P}}[V(T, X_T^{t,x,\mathbf{u}^*})] \\ &= V(t, x) + \mathbb{E}^{\mathbb{P}} \left[(G \circ g)(T, X_T^{t,x,\mathbf{u}^*}) \right] - (G \circ g)(t, x) + \mathbb{E}^{\mathbb{P}} \left[\mathbb{H}_\nu[g(T, X_T^{t,x,\mathbf{u}^*}, \cdot)] \right] - \mathbb{H}_\nu[g(t, x, \cdot)] \\ &= V(t, x) + \mathbb{E}^{\mathbb{P}} \left[G(X_T^{t,x,\mathbf{u}^*}) \right] - G \left(\mathbb{E}^{\mathbb{P}}[X_T^{t,x,\mathbf{u}^*}] \right) + \mathbb{E}^{\mathbb{P}} \left[\mathbb{H}_\nu[X_T^{t,x,\mathbf{u}^*}] \right] - \mathbb{H}_\nu \left[\mathbb{E}^{\mathbb{Q}}[X_T^{t,x,\mathbf{u}^*}] \right] \\ &= V(t, x) + \mathbb{E}^{\mathbb{P}} \left[G(X_T^{t,x,\mathbf{u}^*}) \right] - G \left(\mathbb{E}^{\mathbb{P}}[X_T^{t,x,\mathbf{u}^*}] \right) - \mathbb{H}_\nu \left[\mathbb{E}^{\mathbb{Q}}[X_T^{t,x,\mathbf{u}^*}] \right], \end{aligned}$$

where we use (3.31) and (A.7) in the second equality. In the last equality, since $X_T^{t,x,\mathbf{u}^*}$ is not a function of λ , by (3.19), we have $\mathbb{H}_\nu[X_T^{t,x,\mathbf{u}^*}] = 0$ a.s. Recall from (3.31) that

$$V(T, x) = x = F(x) + G(x).$$

It follows that

$$\begin{aligned} V(t, x) &= \mathbb{E}^\mathbb{P} \left[F(X_T^{t,x,\mathbf{u}^*}) \right] + \mathbb{E}^\mathbb{P} \left[G(X_T^{t,x,\mathbf{u}^*}) \right] - \mathbb{E}^\mathbb{P} \left[G(X_T^{t,x,\mathbf{u}^*}) \right] + G \left(\mathbb{E}^\mathbb{P} [X_T^{t,x,\mathbf{u}^*}] \right) \\ &\quad + \mathbb{H}_\nu \left[\mathbb{E}^\mathbb{Q} [X_T^{t,x,\mathbf{u}^*}] \right] \\ &= \mathbb{E}^\mathbb{P} \left[F(X_T^{t,x,\mathbf{u}^*}) \right] + G \left(\mathbb{E}^\mathbb{P} [X_T^{t,x,\mathbf{u}^*}] \right) + \mathbb{H}_\nu \left[\mathbb{E}^\mathbb{Q} [X_T^{t,x,\mathbf{u}^*}] \right] \\ &= J(t, x, \mathbf{u}^*), \end{aligned}$$

where the last step is by (3.20).

Step 3. We show that $\mathbf{u}^*$ is an equilibrium strategy. Recall that $\mathbf{u}^\varepsilon$ is the perturbed strategy in Definition 3.1 and $\hat{\mathbf{u}} = \{\hat{u}(t, x)\}_{(t,x) \in [0,T] \times \mathbb{R}}$ with $\hat{u}(t, x) \equiv \hat{u}$ for all $(t, x) \in [0, T] \times \mathbb{R}$ is a constant control. By (3.20), (A.6), (A.7), and the Markov property at time $t + \varepsilon$, as the controlled portfolio value under any admissible feedback control is a Markov diffusion process, we have

$$\begin{aligned} &\mathbb{E}^\mathbb{P} [V(t + \varepsilon, X_{t+\varepsilon}^{t,x,\hat{\mathbf{u}}})] \\ &= \mathbb{E}^\mathbb{P} [J(t + \varepsilon, X_{t+\varepsilon}^{t,x,\hat{\mathbf{u}}}, \mathbf{u}^*)] \\ &= \mathbb{E}^\mathbb{P} \left[\mathbb{E}^\mathbb{P} \left[F(X_T^{t+\varepsilon, X_{t+\varepsilon}^{t,x,\hat{\mathbf{u}}}, \mathbf{u}^*}) \right] \right] + \mathbb{E}^\mathbb{P} \left[G \left(\mathbb{E}^\mathbb{P} \left[X_T^{t+\varepsilon, X_{t+\varepsilon}^{t,x,\hat{\mathbf{u}}}, \mathbf{u}^*} \right] \right) \right] + \mathbb{E}^\mathbb{P} \left[\mathbb{H}_\nu \left[\mathbb{E}^\mathbb{Q} \left[X_T^{t+\varepsilon, X_{t+\varepsilon}^{t,x,\hat{\mathbf{u}}}, \mathbf{u}^*} \right] \right] \right] \\ &= \mathbb{E}^\mathbb{P} \left[F(X_T^{t,x,\mathbf{u}^\varepsilon}) \right] + \mathbb{E}^\mathbb{P} \left[(G \circ g) \left(t + \varepsilon, X_{t+\varepsilon}^{t,x,\hat{\mathbf{u}}} \right) \right] + \mathbb{E}^\mathbb{P} \left[\mathbb{H}_\nu \left[g(t + \varepsilon, X_{t+\varepsilon}^{t,x,\hat{\mathbf{u}}}, \cdot) \right] \right] \end{aligned}$$

and

$$\begin{aligned} J(t, x, \mathbf{u}^\varepsilon) &= \mathbb{E}^\mathbb{P} \left[F(X_T^{t,x,\mathbf{u}^\varepsilon}) \right] + G \left(\mathbb{E}^\mathbb{P} \left[X_T^{t,x,\mathbf{u}^\varepsilon} \right] \right) + \mathbb{H}_\nu \left[\mathbb{E}^\mathbb{Q} \left[X_T^{t,x,\mathbf{u}^\varepsilon} \right] \right] \\ &= \mathbb{E}^\mathbb{P} \left[F(X_T^{t,x,\mathbf{u}^\varepsilon}) \right] + G \left(\mathbb{E}^\mathbb{P} \left[\mathbb{E}^\mathbb{P} [X_T^{t+\varepsilon, X_{t+\varepsilon}^{t,x,\hat{\mathbf{u}}}, \mathbf{u}^*}] \right] \right) + \mathbb{H}_\nu \left[\mathbb{E}^\mathbb{Q} \left[\mathbb{E}^\mathbb{Q} [X_T^{t+\varepsilon, X_{t+\varepsilon}^{t,x,\hat{\mathbf{u}}}, \mathbf{u}^*}] \right] \right] \\ &= \mathbb{E}^\mathbb{P} \left[F(X_T^{t,x,\mathbf{u}^\varepsilon}) \right] + G \left(\mathbb{E}^\mathbb{P} \left[g(t + \varepsilon, X_{t+\varepsilon}^{t,x,\hat{\mathbf{u}}}) \right] \right) + \mathbb{H}_\nu \left[\mathbb{E}^\mathbb{Q} \left[g(t + \varepsilon, X_{t+\varepsilon}^{t,x,\hat{\mathbf{u}}}, \cdot) \right] \right]. \end{aligned}$$

It follows that

$$\begin{aligned}
J(t, x, \mathbf{u}^\varepsilon) - \mathbb{E}^\mathbb{P}[V(t + \varepsilon, X_{t+\varepsilon}^{t,x,\hat{\mathbf{u}}})] &= G\left(\mathbb{E}^\mathbb{P}\left[g(t + \varepsilon, X_{t+\varepsilon}^{t,x,\hat{\mathbf{u}}})\right]\right) - \mathbb{E}^\mathbb{P}\left[(G \circ g)\left(t + \varepsilon, X_{t+\varepsilon}^{t,x,\hat{\mathbf{u}}}\right)\right] \\
&\quad + \mathbb{H}_\nu\left[\mathbb{E}^\mathbb{Q}\left[g(t + \varepsilon, X_{t+\varepsilon}^{t,x,\hat{\mathbf{u}}}, \cdot)\right]\right] - \mathbb{E}^\mathbb{P}\left[\mathbb{H}_\nu\left[g(t + \varepsilon, X_{t+\varepsilon}^{t,x,\hat{\mathbf{u}}}, \cdot)\right]\right] \\
&= G\left(\mathbb{E}^\mathbb{P}\left[g(t + \varepsilon, X_{t+\varepsilon}^{t,x,\hat{\mathbf{u}}})\right]\right) - (G \circ g)(t, x) \\
&\quad - \left\{\mathbb{E}^\mathbb{P}\left[(G \circ g)\left(t + \varepsilon, X_{t+\varepsilon}^{t,x,\hat{\mathbf{u}}}\right)\right] - (G \circ g)(t, x)\right\} \\
&\quad + \mathbb{H}_\nu\left[\mathbb{E}^\mathbb{Q}\left[g(t + \varepsilon, X_{t+\varepsilon}^{t,x,\hat{\mathbf{u}}}, \cdot)\right]\right] - \mathbb{H}_\nu[g(t, x, \cdot)] \\
&\quad - \left\{\mathbb{E}^\mathbb{P}\left[\mathbb{H}_\nu\left[g(t + \varepsilon, X_{t+\varepsilon}^{t,x,\hat{\mathbf{u}}}, \cdot)\right]\right] - \mathbb{H}_\nu[g(t, x, \cdot)]\right\} \\
&:= D_1 - D_2 + D_3 - D_4, \tag{A.12}
\end{aligned}$$

where in the second equation, we insert two extra terms: $(G \circ g)(t, x)$ and $\mathbb{H}_\nu[g(t, x, \cdot)]$. We then analyze the four residual terms D_1 , D_2 , D_3 , and D_4 . First, we have

$$\begin{aligned}
D_1 &= G\left(\mathbb{E}^\mathbb{P}\left[g(t + \varepsilon, X_{t+\varepsilon}^{t,x,\hat{\mathbf{u}}})\right]\right) - (G \circ g)(t, x) \\
&= G_x(g(t, x))\left(\mathbb{E}^\mathbb{P}\left[g(t + \varepsilon, X_{t+\varepsilon}^{t,x,\hat{\mathbf{u}}})\right] - g(t, x)\right) + o(\varepsilon) \\
&= (G_x \circ g)(t, x)\mathbb{E}^\mathbb{P}\left[\int_t^{t+\varepsilon} \mathcal{A}^{\hat{\mathbf{u}}}g(s, X_s^{t,x,\hat{\mathbf{u}}})ds\right] + o(\varepsilon) \\
&= \varepsilon(G_x \circ g)(t, x)\mathcal{A}^{\hat{\mathbf{u}}}g(t, x) + o(\varepsilon)
\end{aligned}$$

and

$$\begin{aligned}
D_2 &= \mathbb{E}^\mathbb{P}\left[(G \circ g)(t + \varepsilon, X_{t+\varepsilon}^{t,x,\hat{\mathbf{u}}})\right] - (G \circ g)(t, x) \\
&= \mathbb{E}^\mathbb{P}\left[\int_t^{t+\varepsilon} \mathcal{A}^{\hat{\mathbf{u}}}(G \circ g)(s, X_s^{t,x,\hat{\mathbf{u}}})ds\right] \\
&= \varepsilon\mathcal{A}^{\hat{\mathbf{u}}}(G \circ g)(t, x) + o(\varepsilon).
\end{aligned}$$

For D_3 , since $\mathbb{H}_\nu[\cdot]$ is Gâteaux differentiable at $g(t, x, \cdot)$, by Taylor expansion (3.27) and

$$\mathbb{E}^{\mathbb{Q}^\lambda}\left[g(t + \varepsilon, X_{t+\varepsilon}^{t,x,\hat{\mathbf{u}}}, \lambda)\right] - g(t, x, \lambda) = \varepsilon\mathcal{A}^{\hat{\mathbf{u}},\lambda}g(t, x, \lambda) + o(\varepsilon),$$

it follows that

$$\begin{aligned}
D_3 &= \mathbb{H}_\nu \left[\mathbb{E}^\mathbb{Q} \left[g(t + \varepsilon, X_{t+\varepsilon}^{t,x,\hat{\mathbf{u}}}, \cdot) \right] \right] - \mathbb{H}_\nu[g(t, x, \cdot)] \\
&= \left\langle \frac{\delta \mathbb{H}_\nu[g(t, x, \cdot)]}{\delta g(t, x, \cdot)}, \mathbb{E}^\mathbb{Q} \left[g(t + \varepsilon, X_{t+\varepsilon}^{t,x,\hat{\mathbf{u}}}, \cdot) \right] - g(t, x, \cdot) \right\rangle + o(\varepsilon) \\
&= \left\langle \frac{\delta \mathbb{H}_\nu[g(t, x, \cdot)]}{\delta g(t, x, \cdot)}, \varepsilon \mathcal{A}^{\hat{\mathbf{u}}, \cdot} g(t, x, \cdot) + o(\varepsilon) \right\rangle + o(\varepsilon) \\
&= \varepsilon \left\langle \frac{\delta \mathbb{H}_\nu[g(t, x, \lambda)]}{\delta g(t, x, \lambda)}, \mathcal{A}^{\hat{\mathbf{u}}, \cdot} g(t, x, \cdot) \right\rangle + o(\varepsilon).
\end{aligned}$$

Note that we have used the linearity and continuity of $\frac{\delta \mathbb{H}_\nu[g(t, x, \lambda)]}{\delta g(t, x, \lambda)}$ in the last step because

$$\lim_{\varepsilon \rightarrow 0} \left\langle \frac{\delta \mathbb{H}_\nu[g(t, x, \cdot)]}{\delta g(t, x, \cdot)}, \frac{o(\varepsilon)}{\varepsilon} \right\rangle = \left\langle \frac{\delta \mathbb{H}_\nu[g(t, x, \cdot)]}{\delta g(t, x, \cdot)}, \lim_{\varepsilon \rightarrow 0} \frac{o(\varepsilon)}{\varepsilon} \right\rangle = \left\langle \frac{\delta \mathbb{H}_\nu[g(t, x, \cdot)]}{\delta g(t, x, \cdot)}, 0 \right\rangle = 0,$$

which implies

$$\left\langle \frac{\delta \mathbb{H}_\nu[g(t, x, \cdot)]}{\delta g(t, x, \cdot)}, o(\varepsilon) \right\rangle = o(\varepsilon).$$

For D_4 ,

$$\begin{aligned}
D_4 &= \mathbb{E}^\mathbb{P} \left[\mathbb{H}_\nu \left[g(t + \varepsilon, X_{t+\varepsilon}^{t,x,\hat{\mathbf{u}}}, \cdot) \right] \right] - \mathbb{H}_\nu[g(t, x, \cdot)] \\
&= \mathbb{E}^\mathbb{P} \left[\int_t^{t+\varepsilon} \mathcal{A}^{\hat{\mathbf{u}}} \mathbb{H}_\nu[g(s + \varepsilon, X_{s+\varepsilon}^{t,x,\hat{\mathbf{u}}}, \cdot)] ds \right] \\
&= \varepsilon \mathcal{A}^{\hat{\mathbf{u}}} \mathbb{H}_\nu[g(t, x, \cdot)] + o(\varepsilon).
\end{aligned}$$

Substituting the above expressions of D_1 , D_2 , D_3 , and D_4 into (A.12) and also using

$$\begin{aligned}
\mathbb{E}^\mathbb{P}[V(t + \varepsilon, X_{t+\varepsilon}^{t,x,\hat{\mathbf{u}}})] - V(t, x) &= \mathbb{E}^\mathbb{P} \left[\int_t^{t+\varepsilon} \mathcal{A}^{\hat{\mathbf{u}}} V(s, X_s^{t,x,\hat{\mathbf{u}}}) ds \right] \\
&= \varepsilon \mathcal{A}^{\hat{\mathbf{u}}} V(t, x) + o(\varepsilon),
\end{aligned}$$

we find that

$$\begin{aligned}
&J(t, x, \mathbf{u}^\varepsilon) - V(t, x) \\
&= J(t, x, \mathbf{u}^\varepsilon) - \mathbb{E}^\mathbb{P}[V(t + \varepsilon, X_{t+\varepsilon}^{t,x,\hat{\mathbf{u}}})] + \varepsilon \mathcal{A}^{\hat{\mathbf{u}}} V(t, x) + o(\varepsilon) \\
&= \varepsilon \left\{ \mathcal{A}^{\hat{\mathbf{u}}} V(t, x) + (G_x \circ g)(t, x) \mathcal{A}^{\hat{\mathbf{u}}} g(t, x) - \mathcal{A}^{\hat{\mathbf{u}}}(G \circ g)(t, x) \right. \\
&\quad \left. + \left\langle \frac{\delta \mathbb{H}_\nu[g(t, x, \cdot)]}{\delta g(t, x, \cdot)}, \mathcal{A}^{\hat{\mathbf{u}}, \cdot} g(t, x, \cdot) \right\rangle - \mathcal{A}^{\hat{\mathbf{u}}} \mathbb{H}_\nu[g(t, x, \cdot)] \right\} + o(\varepsilon).
\end{aligned}$$

By (C.7), we have

$$\limsup_{\varepsilon \downarrow 0} \frac{J(t, x, \mathbf{u}^\varepsilon) - J(t, x, \mathbf{u}^*)}{\varepsilon} = \limsup_{\varepsilon \downarrow 0} \frac{J(t, x, \mathbf{u}^\varepsilon) - V(t, x)}{\varepsilon} \leq 0,$$

which implies that $\mathbf{u}^*$ is an equilibrium strategy.

A.4 Proof of Theorem 3.2

Consider the following ansatz:

$$V(t, x) = e^{r(T-t)}x + B(t) \quad \text{and} \quad g(t, x, \lambda) = e^{r(T-t)}x + \alpha(\lambda)C(t), \quad (\text{A.13})$$

where $\alpha(\lambda) = \mu - r - \sigma\lambda$, and functions $B(t)$ and $C(t)$ are to be determined satisfying the boundary condition $B(T) = C(T) = 0$. It follows that

$$\text{Var}_\nu[g(t, x, \cdot)] = \text{Var}_\nu[\alpha(\cdot)]C(t)^2 = \sigma^2\delta_\nu^2C(t)^2 \quad (\text{A.14})$$

and

$$\begin{aligned} \text{Cov}_\nu[g(t, x, \cdot), \mathcal{A}^{u, \cdot}g(t, x, \cdot)] &= \text{Cov}_\nu[e^{r(T-t)}x + \alpha(\cdot)C(t), \alpha(\cdot)(C'(t) + ue^{r(T-t)})] \\ &= C(t)(C'(t) + ue^{r(T-t)})\text{Var}_\nu[\alpha(\cdot)] \\ &= C(t)(C'(t) + ue^{r(T-t)})\sigma^2\delta_\nu^2. \end{aligned} \quad (\text{A.15})$$

Substituting (A.13), (A.14), (A.15), and $G(x) = \frac{\gamma}{2}x^2$ into the EHJB equation (3.32) yields

$$B'(t) + \sup_{u \in \mathbb{R}} \left\{ (\mu - r)ue^{r(T-t)} - \beta\sigma^2\delta_\nu^2C(t)ue^{r(T-t)} - \frac{\gamma}{2}\sigma^2u^2e^{2r(T-t)} \right\} = 0. \quad (\text{A.16})$$

From the first-order condition, the equilibrium strategy u_t^* is given by

$$u_t^* = \frac{\mu - r - \beta\sigma^2\delta_\nu^2C(t)}{\gamma\sigma^2}e^{-r(T-t)}. \quad (\text{A.17})$$

Substituting (A.17) into (3.30) yields the following differential equation for $C(t)$,

$$C'(t) - \frac{\beta\delta_\nu^2}{\gamma}C(t) = -\frac{\mu - r}{\gamma\sigma^2}.$$

With the boundary condition $C(T) = 0$, we have

$$C(t) = \frac{\mu - r}{\beta\sigma^2\delta_\nu^2} \left(1 - e^{-\frac{\beta\delta_\nu^2}{\gamma}(T-t)} \right). \quad (\text{A.18})$$

Substituting (A.18) into (A.17) leads to the expression for u^* given in (3.35). By (3.35) and (A.16), together with $B(T) = 0$, we have

$$B(t) = \frac{(\mu - r)^2}{4\beta\sigma^2\delta_\nu^2} \left(1 - e^{-\frac{2\beta\delta_\nu^2}{\gamma}(T-t)} \right).$$

The equilibrium value functions are then given by

$$V(t, x) = e^{r(T-t)}x + \frac{(\mu - r)^2}{4\beta\sigma^2\delta_\nu^2} \left(1 - e^{-\frac{2\beta\delta_\nu^2}{\gamma}(T-t)} \right) \quad (\text{A.19})$$

and

$$g(t, x, \lambda) = e^{r(T-t)}x + \frac{\alpha(\lambda)(\mu - r)}{\beta\sigma^2\delta_\nu^2} \left(1 - e^{-\frac{\beta\delta_\nu^2}{\gamma}(T-t)} \right). \quad (\text{A.20})$$

Conditions 1 and 5 remain to be verified in Theorem 3.1. For condition 1, it is seen from (A.20) that $g(t, x, \lambda)$ is linear in λ , and thus $g(t, x, \cdot) \in L^2(\mathbb{R}, \nu)$ by the integrability condition (3.24). Then, it follows from Example 3.1 that $\text{Var}_\nu[\cdot]$ is Gâteaux differentiable at $g(t, x, \cdot)$ and, by (A.20),

$$\begin{aligned} \text{Var}_\nu[g(t, x, \cdot)] &= \frac{\mu - r}{\beta\sigma^2\delta_\nu^2} \left(1 - e^{-\frac{\beta\delta_\nu^2}{\gamma}(T-t)} \right) \text{Var}_\nu[\alpha(\cdot)] \\ &= \frac{\mu - r}{\beta} \left(1 - e^{-\frac{\beta\delta_\nu^2}{\gamma}(T-t)} \right), \end{aligned} \quad (\text{A.21})$$

which belongs to $C^{1,2}([0, T] \times \mathbb{R})$.

We then verify condition 5 that $g(t, x, \lambda), V(t, x), g^2(t, x), \text{Var}_\nu[g(t, x, \cdot)] \in L_T^2(X^{\mathbf{u}^*}) \cap L_T^2(X^{\hat{\mathbf{u}}})$. Note that both the equilibrium strategy $\mathbf{u}^*$ and the constant strategy $\hat{\mathbf{u}}$ are independent of x and are thus uniformly bounded. It suffices to show that

$$\mathbb{E}^{\mathbb{Q}^\lambda} \left[\int_t^T g_x(s, X_s^{t,x,\mathbf{u}}, \lambda)^2 ds \right] < \infty$$

and

$$\mathbb{E}^{\mathbb{P}} \left[\int_t^T f_x(s, X_s^{t,x,\mathbf{u}})^2 ds \right] < \infty, \quad (\text{A.22})$$

for $f(t, x) = V(t, x), g^2(t, x), \text{Var}_\nu[g(t, x, \cdot)]$ and $\mathbf{u} = \mathbf{u}^*, \hat{\mathbf{u}}$. From (A.19), (A.20), and (A.21), it is seen that $V_x(t, x) = g_x(t, x, \lambda) = e^{r(T-t)}$ and $\text{Var}_\nu[g(t, x, \cdot)]$ is independent of x . It remains to show that A.22 holds for $f = g^2$. Indeed,

$$\begin{aligned} \mathbb{E}^\mathbb{P} \left[\int_t^T \left(\frac{\partial g^2(s, X_s^{t,x,\mathbf{u}})}{\partial x} \right)^2 ds \right] &= \mathbb{E}^\mathbb{P} \left[\int_t^T (2e^{r(T-s)} (e^{r(T-s)} X_s^{t,x,\mathbf{u}} + (\mu - r)C(t)))^2 ds \right] \\ &\leq K_1 \int_t^T \mathbb{E}^\mathbb{P} [(X_s^{t,x,\mathbf{u}})^2] ds + K_2 \\ &\leq K_1(T-t) \mathbb{E}^\mathbb{P} \left[\sup_{s \in [t,T]} (X_s^{t,x,\mathbf{u}})^2 \right] + K_2 \\ &< \infty, \end{aligned}$$

where the last step is due to (3.13).

A.5 Proof of Theorem 3.3

We split our proof into two cases: $\mu - r \in (-\infty, -\frac{1}{2}\beta\sigma\delta_\nu) \cup (\frac{1}{2}\beta\sigma\delta_\nu, \infty)$ and $\mu - r \in [-\frac{1}{2}\beta\sigma\delta_\nu, \frac{1}{2}\beta\sigma\delta_\nu]$.

(1) For the case $\mu - r \in (-\infty, -\frac{1}{2}\beta\sigma\delta_\nu) \cup (\frac{1}{2}\beta\sigma\delta_\nu, \infty)$, we consider the following ansatz:

$$V(t, x) = e^{r(T-t)}x + D(t) \quad \text{and} \quad g(t, x, \lambda) = e^{r(T-t)}x + \alpha(\lambda)(\mu - r)E(t), \quad (\text{A.23})$$

where $D(t)$ and $E(t)$ are two functions to be determined satisfying $D(T) = E(T) = 0$ and $E(t) > 0$ for all $t \in [0, T)$. We then simplify the extended HJB equation (3.33). Similar to (A.14) and (A.15), we have

$$\sigma_\nu[g(t, x, \cdot)] = \sigma_\nu[\alpha(\cdot)]|\mu - r|E(t) = \sigma|\mu - r|\delta_\nu E(t)$$

and

$$\begin{aligned} \text{Cov}_\nu[g(t, x, \cdot), \mathcal{A}^{u,\cdot}g(t, x, \cdot)] &= \text{Cov}_\nu[e^{r(T-t)}x + \alpha(\cdot)(\mu - r)E(t), \alpha(\cdot)((\mu - r)E'(t) + ue^{r(T-t)})] \\ &= (\mu - r)E(t)((\mu - r)E'(t) + ue^{r(T-t)})\text{Var}_\nu[\alpha(\cdot)] \\ &= (\mu - r)E(t)((\mu - r)E'(t) + ue^{r(T-t)})\sigma^2\delta_\nu^2. \end{aligned}$$

It follows that

$$\begin{aligned}
& \frac{\text{Cov}_\nu [g(t, x, \cdot), \mathcal{A}^{u, \cdot} g(t, x, \cdot)]}{\sigma_\nu [g(t, x, \cdot)]} - \mathcal{A}^u \sigma_\nu [g(t, x, \cdot)] \\
&= \frac{(\mu - r)E(t) \left((\mu - r)E'(t) + ue^{r(T-t)} \right) \sigma^2 \delta_\nu^2}{\sigma |\mu - r| \delta_\nu E(t)} - \sigma |\mu - r| \delta_\nu E'(t) \\
&= \frac{(\mu - r)E(t) ue^{r(T-t)} \sigma^2 \delta_\nu^2}{\sigma |\mu - r| \delta_\nu E(t)} \\
&= \text{sgn}(\mu - r) \sigma \delta_\nu ue^{r(T-t)}.
\end{aligned}$$

Then, the EHJB equation (3.33) becomes

$$D'(t) + \sup_{u \in \mathbb{R}} \left\{ (\mu - r)ue^{r(T-t)} - \frac{\gamma}{2} \sigma^2 u^2 e^{2r(T-t)} - \frac{\beta}{2} \text{sgn}(\mu - r) \sigma \delta_\nu ue^{r(T-t)} \right\} = 0.$$

It follows from the first-order condition that the optimal control is

$$u_t^* = \frac{\mu - r - \frac{\beta}{2} \text{sgn}(\mu - r) \sigma \delta_\nu}{\gamma \sigma^2 e^{r(T-t)}}.$$

On the other hand, equation (3.30) implies that

$$E'(t) = -\frac{u_t^* e^{r(T-t)}}{\mu - r} = -\frac{\mu - r - \frac{\beta}{2} \text{sgn}(\mu - r) \sigma \delta_\nu}{(\mu - r) \gamma \sigma^2}.$$

This, together with $E(T) = 0$, yields

$$E(t) = \frac{\mu - r - \frac{\beta}{2} \text{sgn}(\mu - r) \sigma \delta_\nu}{(\mu - r) \gamma \sigma^2} (T - t).$$

It is easy to verify that $E(t) > 0$ for all $t < T$ when $\mu - r \in (-\infty, -\frac{1}{2}\beta\sigma\delta_\nu) \cup (\frac{1}{2}\beta\sigma\delta_\nu, \infty)$.

Lastly, by noting that the value functions $g(t, x, \lambda)$, $V(t, x)$, and the equilibrium strategy $\mathbf{u}^*$ all have the same structure as in Theorem 3.2, it is straightforward to verify that conditions 1 and 5 of Theorem 3.1 hold.

(2) When $\mu - r \in [-\frac{1}{2}\beta\sigma\delta_\nu, \frac{1}{2}\beta\sigma\delta_\nu]$, we show directly¹ that $u_t^* \equiv 0$ is an equilibrium strategy based on Definition 3.1. We consider the perturbed strategy $\mathbf{u}^\varepsilon$ defined by

$$u^\varepsilon(s, y) = \begin{cases} \hat{u}, & s \in [t, t + \varepsilon), \\ 0, & s \in [t + \varepsilon, T], \end{cases}$$

¹In fact, in this case, $\sigma_\nu [g(t, x, \cdot)] = 0$, and this term appears in the denominator of the EHJB (3.33). As such, one cannot solve equilibrium strategies from the EHJB.

where $\varepsilon > 0$ and $\hat{u} \in \mathbb{R}$. The corresponding objective function (3.40) is given by

$$\begin{aligned} J(t, x, \mathbf{u}^\varepsilon) &= \mathbb{E}^\mathbb{P} \left[X_T^{t,x,\mathbf{u}^\varepsilon} + \frac{\gamma}{2} (X_T^{t,x,\mathbf{u}^\varepsilon})^2 \right] - \frac{\gamma}{2} \left(\mathbb{E}^\mathbb{P} \left[X_T^{t,x,\mathbf{u}^\varepsilon} \right] \right)^2 - \frac{\beta}{2} \sigma_\nu \left(\mathbb{E}^{\mathbb{Q}^\cdot} \left[X_T^{t,x,\mathbf{u}^\varepsilon} \right] \right) \\ &= \mathbb{E}^\mathbb{P} \left[\mathbb{E}^\mathbb{P} \left[X_T^{t+\varepsilon, X_{t+\varepsilon}^{t,x,\hat{\mathbf{u}}}, 0} - \frac{\gamma}{2} (X_T^{t+\varepsilon, X_{t+\varepsilon}^{t,x,\hat{\mathbf{u}}}, 0})^2 \right] \right] + \frac{\gamma}{2} \left(\mathbb{E}^\mathbb{P} \left[\mathbb{E}^\mathbb{P} [X_T^{t+\varepsilon, X_{t+\varepsilon}^{t,x,\hat{\mathbf{u}}}, 0}] \right] \right)^2 \\ &\quad - \frac{\beta}{2} \sigma_\nu \left[\mathbb{E}^{\mathbb{Q}^\cdot} \left[\mathbb{E}^{\mathbb{Q}^\cdot} [X_T^{t+\varepsilon, X_{t+\varepsilon}^{t,x,\hat{\mathbf{u}}}, 0}] \right] \right]. \end{aligned}$$

Given that $X_T^{t+\varepsilon, X_{t+\varepsilon}^{t,x,\hat{\mathbf{u}}}, 0} = e^{r(T-t-\varepsilon)} X_{t+\varepsilon}^{t,x,\hat{\mathbf{u}}}$ almost surely, it follows that

$$\begin{aligned} J(t, x, \mathbf{u}^\varepsilon) &= \mathbb{E}^\mathbb{P} \left[e^{r(T-t-\varepsilon)} X_{t+\varepsilon}^{t,x,\hat{\mathbf{u}}} - \frac{\gamma}{2} (e^{r(T-t-\varepsilon)} X_{t+\varepsilon}^{t,x,\hat{\mathbf{u}}})^2 \right] + \frac{\gamma}{2} \left(\mathbb{E}^\mathbb{P} \left[e^{r(T-t-\varepsilon)} X_{t+\varepsilon}^{t,x,\hat{\mathbf{u}}} \right] \right)^2 \\ &\quad - \frac{\beta}{2} \sigma_\nu \left[\mathbb{E}^{\mathbb{Q}^\cdot} \left[e^{r(T-t-\varepsilon)} X_{t+\varepsilon}^{t,x,\hat{\mathbf{u}}} \right] \right]. \end{aligned}$$

By (3.12) and Itô's formula, we have

$$d(e^{r(T-s)} X_s^{t,x,\hat{\mathbf{u}}}) = \alpha(\lambda) \hat{u} e^{r(T-s)} ds + \sigma \hat{u} e^{r(T-s)} dW_s^\lambda$$

and

$$\begin{aligned} d(e^{r(T-s)} X_s^{t,x,\hat{\mathbf{u}}})^2 &= 2e^{r(T-s)} X_s^{t,x,\hat{\mathbf{u}}} (\alpha(\lambda) \hat{u} e^{r(T-s)} ds + \sigma \hat{u} e^{r(T-s)} dW_s^\lambda) + (\sigma \hat{u} e^{r(T-s)})^2 ds \\ &= (2\alpha(\lambda) \hat{u} X_s^{t,x,\hat{\mathbf{u}}} + \sigma^2 \hat{u}^2) e^{2r(T-s)} ds + 2\sigma \hat{u} e^{2r(T-s)} X_s^{t,x,\hat{\mathbf{u}}} dW_s^\lambda. \end{aligned}$$

It follows that

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}^\lambda} \left[e^{r(T-t-\varepsilon)} X_{t+\varepsilon}^{t,x,\hat{\mathbf{u}}} \right] &= e^{r(T-t)} x + \alpha(\lambda) \hat{u} \int_t^{t+\varepsilon} e^{r(T-s)} ds, \\ \mathbb{E}^\mathbb{P} \left[e^{r(T-t-\varepsilon)} X_{t+\varepsilon}^{t,x,\hat{\mathbf{u}}} \right] &= e^{r(T-t)} x + (\mu - r) \hat{u} \int_t^{t+\varepsilon} e^{r(T-s)} ds, \\ \mathbb{E}^\mathbb{P} \left[\left(e^{r(T-t-\varepsilon)} X_{t+\varepsilon}^{t,x,\hat{\mathbf{u}}} \right)^2 \right] &= e^{2r(T-t)} x^2 + \int_t^{t+\varepsilon} (2(\mu - r) \hat{u} X_s^{t,x,\hat{\mathbf{u}}} + \sigma^2 \hat{u}^2) e^{2r(T-s)} ds. \end{aligned}$$

Thus, we have

$$\begin{aligned} J(t, x, \mathbf{u}^\varepsilon) &= e^{r(T-t)} x + (\mu - r) \hat{u} \int_t^{t+\varepsilon} e^{r(T-s)} ds \\ &\quad - \frac{\gamma}{2} \left[e^{2r(T-t)} x^2 + \int_t^{t+\varepsilon} (2(\mu - r) \hat{u} X_s^{t,x,\hat{\mathbf{u}}} + \sigma^2 \hat{u}^2) e^{2r(T-s)} ds \right] \\ &\quad + \frac{\gamma}{2} \left(e^{r(T-t)} x + (\mu - r) \hat{u} \int_t^{t+\varepsilon} e^{r(T-s)} ds \right)^2 - \frac{\beta}{2} \sigma |\hat{u}| \delta_\nu \int_t^{t+\varepsilon} e^{r(T-s)} ds. \end{aligned}$$

On the other hand, under the strategy $u_t^* \equiv 0$,

$$J(t, x, \mathbf{u}^*) = e^{r(T-t)}x.$$

Then,

$$\begin{aligned} & \limsup_{\varepsilon \downarrow 0} \frac{J(t, x, \mathbf{u}^\varepsilon) - J(t, x, \mathbf{u}^*)}{\varepsilon} \\ &= (\mu - r)\hat{u}e^{r(T-t)} - \frac{\gamma}{2} (2(\mu - r)\hat{u}x + \sigma^2\hat{u}^2) e^{2r(T-t)} + \gamma e^{r(T-t)}x(\mu - r)\hat{u}e^{r(T-t)} - \frac{\beta}{2}\sigma|\hat{u}|\delta_\nu e^{r(T-t)} \\ &= (\mu - r)e^{r(T-t)}\hat{u} - \frac{\beta}{2}\sigma\delta_\nu e^{r(T-t)}|\hat{u}| - \frac{\gamma}{2}\sigma^2 e^{2r(T-t)}\hat{u}^2 \\ &\leq 0, \end{aligned}$$

where the last inequality is by $\mu - r \in [-\frac{1}{2}\beta\sigma\delta_\nu, \frac{1}{2}\beta\sigma\delta_\nu]$. Therefore, $u_t^* \equiv 0$ is an equilibrium strategy in this case.

Appendix B

Proofs for chapter 4

B.1 Proof of Proposition 4.1

Indeed, for any $\lambda_1, \lambda_2 \in \mathbb{R}$, we have

$$\begin{aligned} & \left| \mathbb{E}^{\mathbb{Q}^{\lambda_1}} \left[X_T^{t,x,\ell} \right] - \mathbb{E}^{\mathbb{Q}^{\lambda_2}} \left[X_T^{t,x,\ell} \right] \right| \\ &= \left| \int_{\Omega} X_T^{t,x,\ell} (q_T^{\lambda_1} - q_T^{\lambda_2}) d\mathbb{P} \right| \\ &\leq \left\| X_T^{t,x,\ell} \right\|_{L^2(\Omega, \mathbb{P})} \left\| q_T^{\lambda_1} - q_T^{\lambda_2} \right\|_{L^2} \\ &= \left\| X_T^{t,x,\ell} \right\|_{L^2(\Omega, \mathbb{P})} \left(\int_{\Omega} (q_T^{\lambda_1})^2 - 2q_T^{\lambda_1} q_T^{\lambda_2} + (q_T^{\lambda_2})^2 d\mathbb{P} \right)^{1/2} \end{aligned}$$

The expectation is calculated by

$$\begin{aligned} \int_{\Omega} q_T^{\lambda_1} q_T^{\lambda_2} d\mathbb{P} &= \int_{\Omega} \exp \left\{ \int_0^T \int_0^{\infty} (\lambda_1 + \lambda_2) \tilde{N}(ds, dz) + \int_0^T \int_0^{\infty} (\lambda_1 + \lambda_2 + 2 - e^{\lambda_1} - e^{\lambda_2}) v(dz) ds \right\} d\mathbb{P} \\ &= \exp \left\{ \int_0^T \int_0^{\infty} (2 - e^{\lambda_1} - e^{\lambda_2}) v(dz) ds \right\} \int_{\Omega} \exp \left\{ \int_0^T \int_0^{\infty} (\lambda_1 + \lambda_2) N(ds, dz) \right\} d\mathbb{P} \\ &= \exp \left\{ \int_0^T \int_0^{\infty} (2 - e^{\lambda_1} - e^{\lambda_2}) v(dz) ds \right\} \exp \left\{ \int_0^T \int_0^{\infty} (e^{\lambda_1 + \lambda_2} - 1) v(dz) ds \right\} \\ &= \exp \left\{ T \int_0^{\infty} (1 - e^{\lambda_1} - e^{\lambda_2} + e^{\lambda_1 + \lambda_2}) v(dz) \right\}. \end{aligned} \tag{B.1}$$

Similarly,

$$\int_{\Omega} q_T^{\lambda_1} q_T^{\lambda_1} d\mathbb{P} = \exp \left\{ T \int_0^{\infty} (1 - 2e^{\lambda_1} + e^{2\lambda_1}) v(dz) \right\}. \quad (\text{B.2})$$

By (B.1) and (B.2), The absolute difference between $\mathbb{E}^{\mathbb{Q}^{\lambda_1}} [X_T^{t,x,\ell}]$ and $\mathbb{E}^{\mathbb{Q}^{\lambda_2}} [X_T^{t,x,\ell}]$ is bounded by $\|X_T^{t,x,\ell}\|_{L^2(\Omega, \mathbb{P})}$ and a continuous function in λ_1 and λ_2 so $\lambda \rightarrow \mathbb{E}^{\mathbb{Q}^{\lambda}} [X_T^{t,x,\ell}]$ is a *continuous* function on $\mathbb{R}$.

B.2 Proof of Lemma 4.1

For given $(t, x) \in [0, T] \times \mathbb{R}$ and $\ell \in [0, z]$, we define a sequence of stopping times:

$$\tau_n = \inf \{s \geq t : |X_s^{t,x,\ell} - x| \geq n\},$$

for $n \in \mathbb{N}$. Let $f \in C_b^2(\mathbb{R})$, i.e., second-order continuously differentiable and with compact support on $\mathbb{R}$, such that $f(x) = x^2$ on the interval $[x - n, x + n]$. By Itô's formula and (4.6), we have

$$\begin{aligned} & (X_{T \wedge \tau_n}^{t,x,\ell})^2 - x^2 \\ &= f(X_{T \wedge \tau_n}^{t,x,\ell}) - f(x) \\ &= \int_t^{T \wedge \tau_n} \left[r X_s^{t,x,\ell} + \int_0^{\infty} ((\theta - \eta)z + (1 + \eta)\ell(s, z))v(dz) \right] f_x(X_s^{t,x,\ell}) ds \\ &+ \int_t^{T \wedge \tau_n} \int_0^{\infty} (f(X_s^{t,x,\ell} - \ell(s, z)) - f(X_s^{t,x,\ell})) e^{\lambda} v(dz) ds \\ &+ \int_t^{T \wedge \tau_n} \int_0^{\infty} (f(X_s^{t,x,\ell} - \ell(s, z)) - f(X_s^{t,x,\ell})) \tilde{N}^{\lambda}(ds, dz) \\ &= \int_t^{T \wedge \tau_n} 2X_s^{t,x,\ell} \left[r X_s^{t,x,\ell} + \int_0^{\infty} ((\theta - \eta)z + (1 + \eta)\ell(s, z))v(dz) \right] ds \\ &+ \int_t^{T \wedge \tau_n} \int_0^{\infty} ((X_s^{t,x,\ell} - \ell(s, z))^2 - (X_s^{t,x,\ell})^2) e^{\lambda} v(dz) ds \\ &+ \int_t^{T \wedge \tau_n} \int_0^{\infty} ((X_s^{t,x,\ell} - \ell(s, z))^2 - (X_s^{t,x,\ell})^2) \tilde{N}^{\lambda}(ds, dz). \end{aligned}$$

By the overinsurance restriction $0 \leq \ell(t, z) \leq z$, it follows that

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}^\lambda} \left[\left(X_{T \wedge \tau_n}^{t,x,\ell} \right)^2 \right] &= x^2 + \mathbb{E}^{\mathbb{Q}^\lambda} \left[\int_t^{T \wedge \tau_n} 2X_s^{t,x,\ell} \left[rX_s^{t,x,\ell} + \int_0^\infty ((\theta - \eta)z + (1 + \eta)\ell(s, z))v(dz) \right] ds \right] \\
&+ \mathbb{E}^{\mathbb{Q}^\lambda} \left[\int_t^{T \wedge \tau_n} \int_0^\infty ((X_s^{t,x,\ell} - \ell(s, z))^2 - (X_s^{t,x,\ell})^2) e^\lambda v(dz) ds \right] \\
&\leq x^2 + C(1 + e^\lambda) \mathbb{E}^{\mathbb{Q}^\lambda} \left[\int_t^{T \wedge \tau_n} (1 + (X_s^{t,x,\ell})^2) ds \right] \\
&= x^2 + C(1 + e^\lambda) \int_t^T \left(1 + \mathbb{E}^{\mathbb{Q}^\lambda} [(X_{s \wedge \tau_n}^{t,x,\ell})^2] \right) ds,
\end{aligned}$$

for some positive constant C depending on $r, \eta, \theta, \int_0^\infty zv(dz), \int_0^\infty z^2v(dz)$, and T . By Grönwall's inequality (e.g., Lemma 1.3.1 of [Pham \(2009\)](#)), we have

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}^\lambda} \left[\left(X_{T \wedge \tau_n}^{t,x,\ell} \right)^2 \right] &\leq x^2 + C(1 + e^\lambda)(x^2 + 1) \int_0^{T-t} e^{C(1+e^\lambda)(T-t-s)} ds \\
&= (x^2 + 1) \exp \{ C(1 + e^\lambda)(T - t) \} - 1.
\end{aligned}$$

By letting $n \rightarrow \infty$ and using the dominated convergence theorem,

$$\mathbb{E}^{\mathbb{Q}^\lambda} \left[\left(X_T^{t,x,\ell} \right)^2 \right] \leq (x^2 + 1) \exp \{ C(1 + e^\lambda)(T - t) \} - 1. \quad (\text{B.3})$$

It follows from Jensen's inequality, (B.3), and the condition $\int_{\mathbb{R}} \exp \{ Ce^\lambda \} d\nu(\lambda) < \infty$ that

$$\int_{\mathbb{R}} \left(\mathbb{E}^{\mathbb{Q}^\lambda} \left[X_T^{t,x,\ell} \right] \right)^2 d\nu(\lambda) \leq \int_{\mathbb{R}} \mathbb{E}^{\mathbb{Q}^\lambda} \left[(X_T^{t,x,\ell})^2 \right] d\nu(\lambda) < \infty,$$

which confirms that the mapping $\lambda \rightarrow \mathbb{E}^{\mathbb{Q}^\lambda} [X_T^{t,x,\ell}]$ belongs to $L^2(\mathbb{R}, \nu)$.

B.3 Proof of Theorem 4.1

Step 1. We show that $g(t, x, \lambda) = \mathbb{E}^{\mathbb{Q}^\lambda} [X_T^{t,x,\ell^*}]$. By Itô's formula and the assumption that $g(t, x, \lambda) \in C^{1,1,0}([0, T] \times \mathbb{R} \times \mathbb{R})$, for any $(t, x, \lambda) \in [0, T] \times \mathbb{R} \times \mathbb{R}$, we have

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}^\lambda} [g(T, X_T^{t,x,\ell^*}, \lambda)] &= g(t, x, \lambda) + \mathbb{E}^{\mathbb{Q}^\lambda} \left[\int_t^T \mathcal{A}^{\ell^*, \lambda} g(s, X_s^{t,x,\ell^*}, \lambda) ds \right] \\
&+ \mathbb{E}^{\mathbb{Q}^\lambda} \left[\int_t^T \int_0^\infty (g(s, X_s^{t,x,\ell^*} - \ell(s, z), \lambda) - g(s, X_s^{t,x,\ell^*}, \lambda)) \tilde{N}^\lambda(ds, dz) \right] \\
&= g(t, x, \lambda),
\end{aligned}$$

where the last equality is due to (4.30) and the condition $g(t, x, \lambda) \in L_T^2(X^{\ell*})$, which ensures the stochastic integral part is a martingale. Further, by $g(T, x, \lambda) = x$ as in (4.31), it follows that

$$\mathbb{E}^{\mathbb{Q}^\lambda}[X_T^{t,x,\ell*}] = \mathbb{E}^{\mathbb{Q}^\lambda}[g(T, X_T^{t,x,\ell*}, \lambda)] = g(t, x, \lambda). \quad (\text{B.4})$$

In particular, by letting $\lambda = 0$ and recalling $g(t, x) := g(t, x, 0)$, we have

$$\mathbb{E}^{\mathbb{P}}[X_T^{t,x,\ell*}] = \mathbb{E}^{\mathbb{P}}[g(T, X_T^{t,x,\ell*})] = g(t, x). \quad (\text{B.5})$$

For the following proof, we only need to replace the operator $\mathcal{A}^{u,\lambda}$ and $\mathcal{A}^u$ in the proof of verification theorem in Chapter 3 with the operator $\mathcal{A}^{\ell,\lambda}$ (4.22) and $\mathcal{A}^\ell$ (4.23) respectively.

B.4 Proof of Lemma 4.2

Recall the ODE,

$$\begin{cases} D'(t) = e^{r(T-t)} \int_0^{+\infty} \left(0 \vee \frac{\eta - \beta \delta_\nu^2 D(t)}{\gamma e^{r(T-t)}} \wedge z \right) v(dz), & t \geq 0 \\ D(0) = 0. \end{cases} \quad (\text{B.6})$$

First, the uniqueness and existence of the solution $D(t)$ are proved by checking the Lipchitz condition for the function $f(x)$,

$$\begin{aligned} f(x) &= e^{r(T-t)} \int_0^\infty \left(0 \vee \frac{\eta - \beta \delta_\nu^2 x}{\gamma e^{r(T-t)}} \wedge z \right) v(dz) \\ &= e^{r(T-t)} \int_0^{0 \vee \frac{\eta - \beta \delta_\nu^2 x}{\gamma e^{r(T-t)}}} S(z) dz, \end{aligned} \quad (\text{B.7})$$

where $S(z)$ is the survival function of claim size z . Without of loss generality, we suppose that $x_2 > x_1$,

$$\begin{aligned}
|f(x_1) - f(x_2)| &= e^{r(T-t)} \left| \int_{0 \vee \frac{\eta - \beta \delta_\nu^2 x_1}{\gamma e^{r(T-t)}}}^{0 \vee \frac{\eta - \beta \delta_\nu^2 x_2}{\gamma e^{r(T-t)}}} S(z) dz \right| \\
&\leq e^{r(T-t)} \left| \int_{0 \vee \frac{\eta - \beta \delta_\nu^2 x_1}{\gamma e^{r(T-t)}}}^{0 \vee \frac{\eta - \beta \delta_\nu^2 x_2}{\gamma e^{r(T-t)}}} 1 dz \right| \\
&= e^{r(T-t)} \left| 0 \vee \frac{\eta - \beta \delta_\nu^2 x_2}{\gamma e^{r(T-t)}} - 0 \vee \frac{\eta - \beta \delta_\nu^2 x_1}{\gamma e^{r(T-t)}} \right| \\
&= \frac{1}{2\gamma} |\beta \delta_\nu^2 (x_2 - x_1) + |\eta - \beta \delta_\nu^2 x_2| - |\eta - \beta \delta_\nu^2 x_1|| \\
&\leq \frac{\beta \delta_\nu^2}{\gamma} |x_2 - x_1|.
\end{aligned}$$

Next, we show that

$$0 < D(t) < \frac{\eta}{\beta \delta_\nu^2}$$

when $t \in (0, T)$. For the lower bound, we prove that $D(t) \neq 0$ for $t \in (0, T]$ by contradiction. When $D(t) = 0$, the ODE (B.6) becomes,

$$D'(t) = e^{r(T-t)} \int_0^{+\infty} \left(\frac{\eta}{\gamma e^{r(T-t)}} \wedge z \right) v(dz) > 0.$$

However, with the initial condition $D(0) = 0$, it implies that $D(t) > 0$ with $t \in (0, T]$. Then, it is easy to have $D(t) > 0$ because $D'(t) \geq 0$. For the upper bound, we suppose that, on the contrary, there exists $\tilde{t}$ such that $\tilde{t} = \inf \left\{ t \in (0, T] : D(t) \geq \frac{\eta}{\beta \delta_\nu^2} \right\}$. So we have $D'(\tilde{t}) = 0$ since

$$D'(\tilde{t}) = e^{r(T-\tilde{t})} \int_0^{+\infty} 0 \vee \frac{\eta - \beta \delta_\nu^2 D(\tilde{t})}{\gamma e^{r(T-t)}} \wedge z v(dz) = 0.$$

But we simply observe that

$$D^- D(\tilde{t}) = \lim_{t \rightarrow \tilde{t}^-} \frac{D(\tilde{t}) - D(t)}{\tilde{t} - t} > 0 \tag{B.8}$$

due to $D'(t) \geq 0$ and the definition of $\tilde{t}$. But the inequality (B.8) contradicts $D'(\tilde{t}) = 0$ so we obtain

$$0 < D(t) < \frac{\eta}{\beta\delta_\nu^2}.$$

The upper bound $\frac{\eta}{\beta\delta_\nu^2}$ for $D(t)$ implies that

$$D'(t) = e^{r(T-t)} \int_0^{+\infty} 0 \vee \frac{\eta - \beta\delta_\nu^2 D(\tilde{t})}{\gamma e^{r(T-t)}} \wedge zv(dz) > 0.$$

When $\eta = 0$, the ODE (4.35) affords $D'(0) = 0$ so $D(t) \equiv 0$ for $t \in [0, T]$.

We summarize properties of $D(t)$ when $\eta > 0$ as follows

1. $D(t)$ is the unique solution of

$$\begin{cases} D'(t) = e^{r(T-t)} \int_0^{+\infty} 0 \vee \frac{\eta - \beta\delta_\nu^2 D(t)}{\gamma e^{r(T-t)}} \wedge zv(dz), & t \geq 0 \\ D(0) = 0, \end{cases}$$

2. $0 < D(t) < \frac{\eta}{\beta\delta_\nu^2}$ when $t > 0$,
3. $D(t)$ is strictly increasing with respect to t .

B.5 Proof of Theorem 4.2

First, we need to simplify the extended HJB equation (4.28):

$$\sup_\ell \left\{ \mathcal{A}^\ell V(t, x) + (G_x \circ g)(t, x) \mathcal{A}^\ell g(t, x) - \mathcal{A}^\ell (G \circ g)(t, x) - \frac{\beta}{2} [2\text{Cov}_\nu(g(t, x, \lambda), \mathcal{A}^{\ell, \lambda} g(t, x, \lambda)) - \mathcal{A}^\ell \text{Var}_\nu[g(t, x, \lambda)]] \right\} = 0. \quad (\text{B.9})$$

Consider the following ansatz for $V(t, x)$ and $g(t, x, \lambda)$,

$$V(t, x) = e^{r(T-t)}x + B(t) \text{ and } g(t, x, \lambda) = e^{r(T-t)}x + C(t) + (1 + \eta - e^\lambda)D(t), \quad (\text{B.10})$$

with $B(T) = C(T) = D(T) = 0$. Then we find $\text{Cov}_\nu(g(t, x, \lambda), \mathcal{A}^{\ell, \lambda} g(t, x, \lambda))$ and $\mathcal{A}^\ell \text{Var}_\nu[g(t, x, \lambda)]$.

$$\begin{aligned}
& \text{Cov}_\nu(g(t, x, \lambda), \mathcal{A}^{\ell, \lambda} g(t, x, \lambda)) \\
&= \text{Cov}_\nu \left((1 + \eta - e^\lambda) D(t), (1 + \eta - e^\lambda) \left(D'(t) + \int_0^\infty e^{r(T-t)} \ell(t, z) v(dz) \right) \right) \\
&= D(t) D'(t) \delta_\nu^2 + D(t) \delta_\nu^2 \int_0^\infty e^{r(T-t)} \ell(t, z) v(dz)
\end{aligned} \tag{B.11}$$

and

$$\begin{aligned}
\mathcal{A}^\ell \text{Var}_\nu[g(t, x, \lambda)] &= \mathcal{A}^\ell \delta_\nu^2 (D(t))^2 \\
&= 2 \delta_\nu^2 D(t) D'(t).
\end{aligned} \tag{B.12}$$

Substituting above expressions (B.11) and (B.12) and the ansatz (B.10) into the EHJB (4.32),

$$B_t(t) + \sup_\ell \left\{ e^{r(T-t)} \int_0^{+\infty} (\theta - \eta) z + (\eta - \beta D(t) \delta_\nu^2) \ell(t, z) - \frac{\gamma}{2} e^{r(T-t)} \ell^2(t, z) v(dz) \right\} = 0 \tag{B.13}$$

The expression in (B.13) is concave with respect to ℓ ; thus, we obtain the optimal value of ℓ from the first-order condition. Specifically,

$$\ell^*(t, z) = 0 \vee \frac{\eta - \beta \delta_\nu^2 D(t)}{\gamma e^{r(T-t)}} \wedge z.$$

We insert $\ell^*(t, z)$ into (4.30). It follows that

$$C'(t) = -e^{r(T-t)} \int_0^\infty (\theta - \eta) z v(dz)$$

and

$$D'(t) + e^{r(T-t)} \int_0^{+\infty} 0 \vee \frac{\eta - \beta \delta_\nu^2 D(t)}{\gamma e^{r(T-t)}} \wedge z v(dz) = 0 \tag{B.14}$$

with $C(T) = D(T) = 0$. Then, we find

$$C(t) = \int_t^T \int_0^\infty e^{r(T-s)} (\theta - \eta) z v(dz) ds.$$

The explicit analytical solution cannot be obtained for (B.14). But we can simply set $D(T-t) = D(t)$ then transfer the (B.14) into the ODE in lemma (4.2). The uniqueness and existence of solution for the ordinary differential equation (B.14) is proved in lemma (4.2). According to the lemma (4.2), we obtain

$$l^*(t, z) = \frac{\eta - \beta\delta_\nu^2 D(T-t)}{\gamma e^{r(T-t)}} \wedge z.$$

Then, we plug the equilibrium strategy into (B.13). Using the boundary condition $B(T) = 0$, we obtain the corresponding equilibrium value function

$$V(t, x) = e^{r(T-t)}x + B(t), \quad (\text{B.15})$$

in which

$$B(t) = \int_t^T \int_0^{+\infty} e^{r(T-s)} \left[(\theta - \eta)z + (\eta - \beta D(t)\delta_\nu^2) \ell^*(t, z) - \frac{\gamma}{2} e^{r(T-s)} (\ell^*(t, z))^2 \right] v(dz) ds.$$

The expected value of the optimal portfolio is given by

$$g(t, x, \lambda) = e^{r(T-t)}x + \int_t^T \int_0^\infty e^{r(T-s)} (\theta - \eta)z v(dz) ds + (1 + \eta - e^\lambda)D(t). \quad (\text{B.16})$$

It is seen from (B.16) that $g(t, x, \lambda)$ is linear in e^λ , and thus $g(t, x, \cdot) \in L^2(\mathbb{R}, \nu)$ by the integrability condition (4.21). Then, it follows from Example 4.1 that $\text{Var}_\nu[\cdot]$ is Gâteaux differentiable at $g(t, x, \cdot)$ and, by (B.16),

$$\text{Var}_\nu[g(t, x, \cdot)] = (D(t))^2 \delta_\nu^2, \quad (\text{B.17})$$

which belongs to $C^1([0, T])$. It remains to show that $V, g, g^2 \in L_T^2(X^{\hat{\ell}}) \cap L_T^2(X^{\ell^*})$, where $\hat{\ell}(z)$ is a time-invariant strategy. By (4.24) and (4.25), it suffices to show that

$$\mathbb{E}^{\mathbb{Q}^\lambda} \left[\int_t^T \int_0^\infty (f(s, X_s^{t,x,\ell} - \ell(s, z), \lambda) - f(s, X_s^{t,x,\ell}, \lambda))^2 e^\lambda v(dz) ds \right] < \infty,$$

and

$$\mathbb{E}^{\mathbb{P}} \left[\int_t^T \int_0^\infty (f(s, X_s^{t,x,\ell} - \ell(s, z)) - f(s, X_s^{t,x,\ell}))^2 v(dz) ds \right] < \infty,$$

for $f = V, g, g^2$ and $\ell = \hat{\ell}, \ell^*$. From (B.10) and (B.17), it is seen that $V_x(t, x) = g_x(t, x, \lambda) = e^{r(T-t)}$ and $\text{Var}_\nu[g(t, x, \cdot)]$ is independent of x . By (B.10) and $\hat{\ell}(z), \ell^*(t, z) \in [0, z]$, it is only nontrivial to consider $f = g^2$. For $\ell = \hat{\ell}, \ell^*$, by (B.16) and (4.7),

$$\begin{aligned}
& \mathbb{E}^\mathbb{P} \left[\int_t^T \int_0^\infty (g^2(t, X_s^{t,x,\ell} - \ell(t, z)) - g^2(t, X_s^\ell))^2 v(dz) ds \right] \\
&= \mathbb{E}^\mathbb{P} \left[\int_t^T \int_0^\infty \ell(s, z)^2 e^{2r(T-s)} (2X_s^{t,x,\ell} e^{r(T-s)} - e^{r(T-s)} \ell(s, z) + 2h(s))^2 v(dz) ds \right] \\
&\leq \mathbb{E}^\mathbb{P} \left[\int_t^T (K_1(X_s^{t,x,\ell})^2 + K_2) ds \right] \\
&= K_1 \int_t^T \mathbb{E}^\mathbb{P} [(X_s^{t,x,\ell})^2] ds + K_2 \\
&\leq K_1(T-t) \mathbb{E}^\mathbb{P} \left[\sup_{s \in [t, T]} (X_s^{t,x,\ell})^2 \right] + K_2 \\
&< \infty,
\end{aligned}$$

where $h(t) = \int_t^T \int_0^\infty e^{r(T-s)} (\theta - \eta) z v(dz) ds + \eta D(t)$ is obtained from (B.10) and the boundary condition $h(T) = 0$. The bound of $D(t)$ is shown in proposition (4.2).

B.6 Proof of Proposition 4.2

When $\eta > 0$, we let $n(t) = \beta \delta_\nu^2 D(T-t)$ with $n(0) = 0$. The equilibrium strategy becomes

$$\ell^*(t, z) = \frac{\eta - n(t)}{\gamma e^{r(T-t)}} \wedge z. \quad (\text{B.18})$$

So it suffices to show that $n(\cdot)$ increases in β because it immediately implies ℓ_t^* is decreasing in β .

From the ODE (B.14), $n(t)$ solves the following ODE

$$\begin{cases} n'(t) = \beta \delta_\nu^2 K(n(t)), & t \geq 0, \\ n(0) = 0, \end{cases} \quad (\text{B.19})$$

where $K(x) = e^{r(T-t)} \int_0^{+\infty} \frac{\eta - x}{\gamma e^{r(T-t)}} \wedge z v(dz) \geq 0$. Let $\beta_2 \geq \beta_1 \geq 0$. We denote by $n_1(t)$ and $n_2(t)$ the solutions to (B.19). Specifically,

$$\begin{aligned} n_1'(t) &= \beta_1 \delta_\nu^2 K(n_1(t)) \\ n_2'(t) &= \beta_2 \delta_\nu^2 K(n_2(t)). \end{aligned}$$

Since $\beta_2 \geq \beta_1 \geq 0$, we have

$$\begin{aligned} n_1'(t) &= \beta_1 \delta_\nu^2 K(n_1(t)) \\ &\leq \beta_2 \delta_\nu^2 K(n_1(t)). \end{aligned}$$

So, $n_1(t)$ is a subsolution of the ODE (B.19) with $\beta = \beta_2$. In light of the comparison principle, we conclude that

$$n_1(t) \leq n_2(t), \text{ for all } t \geq 0. \quad (\text{B.20})$$

Therefore, $n(t)$ is increasing in β , which implies that ℓ^* is decreasing in β . We can also apply a similar method to show that ℓ^* is also decreasing in δ_ν^2 .

B.7 Proof of Corollary 4.1

When $r = 0$, we can solve the ODE (4.35)

$$\begin{cases} D'(t) = \int_0^{+\infty} 0 \vee \frac{\eta - \beta \delta_\nu^2 D(t)}{\gamma} \wedge zv(dz), & t \geq 0 \\ D(0) = 0. \end{cases} \quad (\text{B.21})$$

Recall that $D(t)$ is strictly increasing with respect to t , using separation of variables, one obtains

$$D(t) = \Psi^{-1}(t), \quad t \geq 0, \quad (\text{B.22})$$

where

$$\Psi(x) = \int_0^x \frac{1}{\psi(s)} ds, \quad x \in [0, \frac{\eta}{\beta \delta_\nu^2}), \quad (\text{B.23})$$

and

$$\psi(x) = \int_0^\infty \frac{\eta - \beta \delta_\nu^2 x}{\gamma} \wedge zv(dz). \quad (\text{B.24})$$

By (4.36), we get

$$\ell^*(t, z) = \frac{\eta - \beta \delta_\nu^2 \Psi^{-1}(T - t)}{\gamma} \wedge z.$$

B.8 Proof of Theorem 4.3

Under the M-V-SD criterion, we split our discussion into two cases $\eta > \frac{\beta}{2}\delta_\nu$ and $\eta \leq \frac{\beta}{2}\delta_\nu$.

Case 1 ($\eta > \frac{\beta}{2}\delta_\nu$): The EHJB equation under the M-V-SD criterion becomes

$$\sup_{\ell} \left\{ \mathcal{A}^\ell V(t, x) + (G_x \circ g)(t, x) \mathcal{A}^\ell g(t, x) - \mathcal{A}^\ell (G \circ g)(t, x) - \frac{\beta}{2} \left[\frac{\text{Cov}_\nu(g(t, x, \lambda), \mathcal{A}^{\ell, \lambda} g(t, x, \lambda))}{\sigma_\nu(g(t, x, \lambda))} - \mathcal{A}^\ell \sigma_\nu[g(t, x, \lambda)] \right] \right\} = 0. \quad (\text{B.25})$$

We consider the following ansatz,

$$V(t, x) = e^{r(T-t)}x + E(t) \text{ and } g(t, x, \lambda) = e^{r(T-t)}x + F(t) + (1 + \eta - e^\lambda)\eta H(t), \quad (\text{B.26})$$

with $E(T) = F(T) = H(T) = 0$ and $H(t) > 0$ for $t \neq T$. By (B.26), we have

$$\sigma_\nu(g(t, x, \lambda)) = \delta_\nu \eta H(t). \quad (\text{B.27})$$

Then, it follows that

$$\mathcal{A}^\ell \sigma_\nu(g(t, x, \lambda)) = \delta_\nu \eta H_t(t), \quad (\text{B.28})$$

and

$$\begin{aligned} & \text{Cov}_\nu(g(t, x, \lambda), \mathcal{A}^{\ell, \lambda} g(t, x, \lambda)) \\ &= |\eta|^2 H(t) H_t(t) \delta_\nu^2 + \eta H(t) \delta_\nu^2 \int_0^\infty e^{r(T-t)} \ell(t, z) v(dz). \end{aligned} \quad (\text{B.29})$$

Plugging the above ansatz into the EHJB equation (B.8),

$$E_t + \sup_{\ell} \left\{ e^{r(T-t)} \int_0^{+\infty} (\theta - \eta) z + (\eta - \frac{\beta}{2}\delta_\nu) \ell(t, z) - \frac{\gamma}{2} e^{r(T-t)} \ell^2(t, z) v(dz) \right\} = 0. \quad (\text{B.30})$$

The above expression is also concave with respect to ℓ so, by the first order condition, we obtain that

$$\ell^*(t, z) = \frac{\eta - \frac{\beta}{2}\delta_\nu}{\gamma e^{r(T-t)}} \wedge z.$$

We plug $\ell^*(t, z)$ into (4.30) and obtain

$$F(t) = \int_t^T \int_0^\infty e^{r(T-s)} (\theta - \eta) z v(dz) ds.$$

The equation (4.30) also implies a relationship between $H(t)$ and $\ell^*(t, z)$

$$H(t) = \frac{1}{\eta} \int_t^T \int_0^\infty \ell^*(s, z) e^{r(T-s)} v(dz) ds > 0.$$

Then, we plug the equilibrium strategy into (B.30). Using the boundary condition $E(T) = 0$, the corresponding equilibrium value function is given by

$$V(t, x) = e^{r(T-t)} x + E(t), \quad (\text{B.31})$$

in which

$$E(t) = \int_t^T \int_0^{+\infty} e^{r(T-s)} \left[(\theta - \eta) z + \left(\eta - \frac{\beta}{2} \delta_\nu \right) \ell^*(t, z) - \frac{\gamma}{2} e^{r(T-s)} (\ell^*(t, z))^2 \right] v(dz) ds.$$

The expected value of the optimal portfolio is given by

$$g(t, x, \lambda) = e^{r(T-t)} x + \int_t^T \int_0^{+\infty} e^{r(T-s)} \left[(\theta - \eta) z + (\eta + 1 - e^\lambda) \eta \ell^*(t, z) \right] v(dz) ds. \quad (\text{B.32})$$

Case 2 ($\eta \leq \frac{\beta}{2} \delta_\nu$): We want to show that $\ell^*(t, z) = 0$. The EHJB equation (B.8) cannot be used to find the equilibrium strategy in this case since $\sigma_\nu(g(t, x, \lambda)) = 0$ when $\ell^*(t, z) = 0$. We will prove that $\ell^*(t, z) = 0$ by definition (4.1). Recall the definition of an equilibrium strategy (4.1), we define a perturbed strategy ℓ^ε as

$$\ell_s^\varepsilon = \begin{cases} \hat{\ell}, & s \in [t, t + \varepsilon), \\ 0, & s \in [t + \varepsilon, T], \end{cases}$$

where $\varepsilon > 0$, $\hat{\ell} \in [0, z]$. On the one hand, the equilibrium strategy $\ell^* = 0$ implies that

$$J(t, x, \ell^*) = e^{r(T-t)} x + \int_t^T \int_0^\infty e^{r(T-s)} (\theta - \eta) z v(dz) ds. \quad (\text{B.33})$$

On the other hand, we rewrite the objective function (4.15),

$$\begin{aligned}
J(t, x, \ell^\varepsilon) &= \mathbb{E}^\mathbb{P} \left[X_T^{t,x,\ell^\varepsilon} + \frac{\gamma}{2} (X_T^{t,x,\ell^\varepsilon})^2 \right] - \frac{\gamma}{2} \left(\mathbb{E}^\mathbb{P} \left[X_T^{t,x,\ell^\varepsilon} \right] \right)^2 - \frac{\beta}{2} \sigma_\nu \left(\mathbb{E}^\mathbb{Q} \left[X_T^{t,x,\ell^\varepsilon} \right] \right) \\
&= \mathbb{E}^\mathbb{P} \left[\mathbb{E}^\mathbb{P} \left[X_T^{t+\varepsilon, X_{t+\varepsilon}^{t,x,\hat{\ell}}, 0} - \frac{\gamma}{2} (X_T^{t+\varepsilon, X_{t+\varepsilon}^{t,x,\hat{\ell}}, 0})^2 \right] \right] + \frac{\gamma}{2} \left(\mathbb{E}^\mathbb{P} \left[\mathbb{E}^\mathbb{P} [X_T^{t+\varepsilon, X_{t+\varepsilon}^{t,x,\hat{\ell}}, 0}] \right] \right)^2 \\
&\quad - \frac{\beta}{2} \sigma_\nu \left[\mathbb{E}^\mathbb{Q} \left[\mathbb{E}^\mathbb{Q} [X_T^{t+\varepsilon, X_{t+\varepsilon}^{t,x,\hat{\ell}}, 0}] \right] \right]. \tag{B.34}
\end{aligned}$$

When $t \in [t + \varepsilon, T]$, the dynamics of the surplus process X_t with $\ell = 0$ (4.6) yields

$$d(e^{r(T-t)} X_t^\ell) = e^{r(T-t)} \int_0^\infty (\theta - \eta) z v(dz) dt. \tag{B.35}$$

Then, we have the following equality holds almost surely,

$$X_T^{t+\varepsilon, X_{t+\varepsilon}^{t,x,\hat{\ell}}, 0} = e^{r(T-(t+\varepsilon))} X_{t+\varepsilon}^{t,x,\hat{\ell}} + \int_{t+\varepsilon}^T \int_0^\infty e^{r(T-s)} (\theta - \eta) z v(dz) ds.$$

By (4.9) and Itô's formula, we have

$$d(e^{r(T-s)} X_s^{t,x,\hat{\ell}}) = e^{r(T-s)} \int_0^\infty ((\theta - \eta) z + (1 + \eta - e^\lambda) \ell(s, z)) v(dz) ds - e^{r(T-s)} \int_0^\infty \ell(s, z) \tilde{N}^\lambda(ds, dz).$$

We compute the expectation under $\mathbb{P}$,

$$\begin{aligned}
&\mathbb{E}^\mathbb{P} \left[\mathbb{E}^\mathbb{P} \left[X_T^{t+\varepsilon, X_{t+\varepsilon}^{t,x,\hat{\ell}}, 0} \right] \right] \\
&= e^{r(T-t)} x + \int_t^T \int_0^\infty e^{r(T-s)} (\theta - \eta) z v(dz) ds + \int_t^{t+\varepsilon} \int_0^\infty e^{r(T-s)} \eta \hat{\ell}(s, z) v(dz) ds. \tag{B.36}
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}^\lambda} \left[\mathbb{E}^{\mathbb{Q}^\lambda} \left[X_T^{t+\varepsilon, X_{t+\varepsilon}^{t,x,\hat{\ell}}, 0} \right] \right] \\
&= e^{r(T-t)} x + \int_t^T \int_0^\infty e^{r(T-s)} (\theta - \eta) z v(dz) ds + \int_t^{t+\varepsilon} \int_0^\infty e^{r(T-s)} (\eta + 1 - e^\lambda) \hat{\ell}(s, z) v(dz) ds. \tag{B.37}
\end{aligned}$$

Then, it yields

$$\begin{aligned}\sigma_\nu \left[\mathbb{E}^{\mathbb{Q}} \left[\mathbb{E}^{\mathbb{Q}} \left[X_T^{t+\varepsilon, X_{t+\varepsilon}^{t,x,\hat{\ell}}, 0} \right] \right] \right] &= \delta_\nu \left| \int_t^{t+\varepsilon} \int_0^\infty e^{r(T-s)} \hat{\ell}(s, z) v(dz) ds \right| \\ &= \delta_\nu \int_t^{t+\varepsilon} \int_0^\infty e^{r(T-s)} \left| \hat{\ell}(s, z) \right| v(dz) ds.\end{aligned}\quad (\text{B.38})$$

By Ito's formula, we have that

$$\begin{aligned}d \left(e^{r(T-t)} X_t^\ell \right)^2 &= 2e^{2r(T-t)} X_t^\ell \left\{ \int_0^\infty ((\theta - \eta)z + \eta \ell(t, z)) v(dz) dt - \int_0^{+\infty} \ell(t, z) \tilde{N}(dt, dz) \right\} \\ &\quad + e^{2r(T-t)} \int_0^\infty \ell^2(t, z) \left(\tilde{N}(dt, dz) + v(dz) dt \right).\end{aligned}$$

It follows that

$$\begin{aligned}\mathbb{E}^\mathbb{P} \left[\mathbb{E}^\mathbb{P} \left[\left(X_T^{t+\varepsilon, X_{t+\varepsilon}^{t,x,\hat{\ell}}, 0} \right)^2 \right] \right] &= \mathbb{E}^\mathbb{P} \left[\left(e^{r(T-(t+\varepsilon))} X_{t+\varepsilon}^{t,x,\hat{\ell}} + \int_{t+\varepsilon}^T \int_0^\infty e^{r(T-s)} (\theta - \eta) z v(dz) ds \right)^2 \right] \\ &= e^{2r(T-t)} x^2 + e^{2r(T-t)} \int_t^{t+\varepsilon} \int_0^{+\infty} \hat{\ell}^2(s, z) v(dz) ds \\ &\quad + 2e^{r(t-t)} x \int_t^T \int_0^{+\infty} e^{r(T-s)} (\theta - \eta) z v(dz) ds \\ &\quad + 2e^{r(t-t)} x \int_t^{t+\varepsilon} \int_0^{+\infty} e^{r(T-s)} \eta \hat{\ell}(s, z) v(dz) ds \\ &\quad + 2 \left(\int_t^T \int_0^{+\infty} e^{r(T-s)} (\theta - \eta) z v(dz) ds \right) \left(\int_t^{t+\varepsilon} \int_0^{+\infty} e^{r(T-s)} \eta \hat{\ell}(s, z) v(dz) ds \right) \\ &\quad + \left(\int_t^T \int_0^{+\infty} e^{r(T-s)} (\theta - \eta) z v(dz) ds \right)^2 + o(\varepsilon).\end{aligned}\quad (\text{B.39})$$

Substituting (B.36), (B.38), and (B.39) into the reward function (B.34), we obtain

$$\begin{aligned}\limsup_{\varepsilon \downarrow 0} \frac{J(t, x, \ell^\varepsilon) - J(t, x, \ell^*)}{\varepsilon} &= \int_0^\infty e^{r(T-t)} \left[\eta \hat{\ell}(t, z) - \frac{\beta}{2} \delta_\nu \left| \hat{\ell}(t, z) \right| - \frac{\gamma}{2} e^{r(T-t)} (\hat{\ell}(t, z))^2 \right] v(dz) \\ &\leq 0.\end{aligned}$$

The equilibrium strategy is given by

$$\ell^*(t, z) = \begin{cases} 0, & \text{if } \eta \leq \frac{\beta}{2} \delta_\nu, \\ \frac{\eta - \frac{\beta}{2} \delta_\nu}{\gamma e^{r(T-t)}} \wedge z, & \text{if } \eta > \frac{\beta}{2} \delta_\nu. \end{cases}$$

Finally, we check the assumption 5 of the theorem (4.1). Note that the ansatz for $V(t, x)$, $g(t, x)$ and $g(t, x, \lambda)$ has the same form with the ansatz in the proof of theorem (4.2) so the proof in the theorem (4.3) also implies that $V(t, x), g(t, x, \lambda), g^2(t, x), \sigma_\nu(g(t, x, \lambda)) \in L_T^2(X^{\ell*}) \cap L_T^2(X^\ell)$.

Appendix C

Proofs for chapter 5

C.1 Static Contract

Theorem C.1. *Suppose we are given the seller's premium loading η , the buyer's optimal hedge ratio with the **static contract** equals:*

$$u^*(\cdot; \eta, \lambda) = 0 \vee \left(1 - \frac{C_1^\lambda}{\gamma_b D_1^\lambda}\right) \wedge 1, \quad (\text{C.1})$$

where

$$C_1^\lambda := l_0 \sum_{i=1}^T (1+r)^{T-i} \left((1+\eta)_i \hat{p}_x - {}_i \hat{p}_x^\lambda \right), \quad (\text{C.2})$$

$$D_1^\lambda := \sum_{i=1}^T \sum_{j=1}^T (1+r)^{2T-(i+j)} \text{Cov}^\lambda(l_i, l_j). \quad (\text{C.3})$$

Proof. We consider a constant hedge ratio over the term of the contract. For the **static contract**, the fixed payments at time t , $\hat{l}_t$ is equal to ${}_t \hat{p}_x l_0$ and the insurer's surplus with a constant hedging strategy $u_t = u \in [0, 1]$. In this case, the wealth process $\{B_t(u)\}_{t \in \{1, 2, \dots, T\}}$ in Equation (5.2) leads to the following terminal wealth $B_T(u)$:

$$B_T(u) = B_0(1+r)^T - \sum_{i=1}^T \left(l_i + u \left((1+\eta)_i \hat{p}_x l_0 - l_i \right) \right) (1+r)^{T-i}. \quad (\text{C.4})$$

We then calculate the expectation and variance of the terminal wealth $B_T(u)$ respectively as:

$$\mathbb{E}^{\mathbb{Q}^\lambda}[B_T(u)] = B_0(1+r)^T - \sum_{i=1}^T (l_{0i}\hat{p}_x^\lambda + ul_0((1+\eta)_i\hat{p}_x - {}_i\hat{p}_x^\lambda)) (1+r)^{T-i}, \quad (\text{C.5})$$

$$\text{Var}^{\mathbb{Q}^\lambda}[B_T(u)] = (1-u)^2 \sum_{i=1}^T \sum_{j=1}^T (1+r)^{2T-i-j} \text{Cov}^\lambda(l_i, l_j). \quad (\text{C.6})$$

We then plug the expectation (C.5) and the variance (C.6) into the insurer's objective (5.11), which gives:

$$\begin{aligned} \sup_{u \in [0,1]} \left\{ B_0(1+r)^T - \sum_{i=1}^T l_{0i}\hat{p}_x^\lambda (1+r)^{T-i} - u \sum_{i=1}^T l_0((1+\eta)_i\hat{p}_x - {}_i\hat{p}_x^\lambda) (1+r)^{T-i} \right. \\ \left. - \frac{\gamma_b}{2} (1-u)^2 \sum_{i=1}^T \sum_{j=1}^T (1+r)^{2T-i-j} \text{Cov}^\lambda(l_i, l_j) \right\} \end{aligned}$$

From the first-order condition, we find the optimal strategy u^* (C.1). □

C.2 Dynamic Contract

Definition C.1. We consider a fixed control law u and make the following construction

- Fix an arbitrary point (t, b) , where $t < T$, and choose an arbitrary control value $u \in [0, 1]$
- Now define the control law $\mathbf{u}^{u,t}$ on the time set $\{t, t+1, \dots, T-1\}$ by setting, for any $b \in \mathbb{R}$,

$$\mathbf{u}_k^{u,t}(b) = \begin{cases} \mathbf{u}_k^*(b) & \text{for } k = t+1, \dots, T-1 \\ u & \text{for } k = t \end{cases}$$

We say that $\mathbf{u}^*$ is a subgame perfect Nash equilibrium strategy if for every fixed (t, b) , we have

$$\sup_{u \in [0,1]} J_t(b, \mathbf{u}^{u,t}; \eta, \lambda) = J_t(b, \mathbf{u}^*; \eta, \lambda),$$

where $J_t(b, \mathbf{u}^*; \eta, \lambda) = \mathbb{E}_t^{\mathbb{Q}^\lambda} [B_T(\mathbf{u}^*)] - \frac{\gamma_b}{2} \text{Var}_t^{\mathbb{Q}^\lambda} [B_T(\mathbf{u}^*)]$. If an equilibrium control $\mathbf{u}^*$ exists, we define the equilibrium value function V by

$$V_t(b; \eta, \lambda) := J_t(b, \mathbf{u}^*; \eta, \lambda).$$

Theorem C.2. We define the function sequence $(g_t; \eta, \lambda)_{t=0}^T$, where $g_t : \mathbb{R} \rightarrow \mathbb{R}$, by:

$$g_t(b; \eta, \lambda) := \mathbb{E}_{t,b}^{\mathbb{Q}^\lambda} [B_T(\mathbf{u}^*)].$$

The value function V_t of the insurer's problem satisfies the recursion:

$$\begin{aligned} V_t(b; \eta, \lambda) &= \sup_{u \in [0,1]} \left\{ \mathbb{E}_{t,b}^{\mathbb{Q}^\lambda} [V_{t+1}(B_{t+1}(u); \eta, \lambda)] + \frac{\gamma_b}{2} \left(\mathbb{E}_{t,b}^{\mathbb{Q}^\lambda} [g_{t+1}(B_{t+1}(u); \eta, \lambda)] \right)^2 \right. \\ &\quad \left. - \frac{\gamma_b}{2} \mathbb{E}_{t,b}^{\mathbb{Q}^\lambda} [g_{t+1}^2(B_{t+1}(u); \eta, \lambda)] \right\}, \\ V_T(b; \eta, \lambda) &= b \end{aligned} \quad (\text{C.7})$$

where the recursion for g is given by:

$$g_t(b; \eta, \lambda) = \mathbb{E}_{t,b}^{\mathbb{Q}^\lambda} [g_{t+1}(B_{t+1}(\mathbf{u}^*); \eta, \lambda)], \quad g_T(b; \eta, \lambda) = b \quad (\text{C.8})$$

Proof. We rewrite the insurer's objective (5.13) as:

$$\mathbb{E}_{t,b}^{\mathbb{Q}^\lambda} [G(B_T(\mathbf{u}))] + H \left(\mathbb{E}_{t,b}^{\mathbb{Q}^\lambda} [B_T(\mathbf{u})] \right), \quad (\text{C.9})$$

where $G(x) = x - \frac{\gamma_b}{2} x^2$, $H(x) = \frac{\gamma_b}{2} x^2$. We can now apply the Theorem 3.5 in (Björk and Murgoci (2014)). After straightforward calculations, the extended Bellman equation reduces to the equation (C.7). $\square$

Theorem C.3. For $t = 0, 1, \dots, T-1$, the equilibrium reinsurance strategy $u^*(\cdot; \eta, \lambda)$ for (5.13) is given by:

$$u_t^*(\cdot; \eta, \lambda) = 0 \vee \left(1 - \frac{(1+\eta)\hat{\mathbf{p}}_{x+t} - \hat{\mathbf{p}}_{x+t}^\lambda}{\gamma_b(1+r)^{T-(t+1)}(1 - \hat{\mathbf{p}}_{x+t}^\lambda)\hat{\mathbf{p}}_{x+t}^\lambda} - \frac{f_{t+1}^\lambda}{(1+r)^{T-(t+1)}} \right) \wedge 1. \quad (\text{C.10})$$

The corresponding value functions are given by

$$\begin{cases} V_t(b; \eta, \lambda) = (1+r)^{T-t}b + F_t^\lambda l_t \\ g_t(b; \eta, \lambda) = (1+r)^{T-t}b + f_t^\lambda l_t \end{cases} \quad (\text{C.11})$$

where the coefficients F_t^λ and f_t^λ are determined recursively by

$$\begin{aligned} F_t^\lambda &= \hat{p}_{x+t}^\lambda f_{t+1}^\lambda - (1+r)^{T-(t+1)} (\hat{p}_{x+t}^\lambda + u_t^* ((1+\eta)\hat{p}_{x+t} - \hat{p}_{x+t}^\lambda)) \\ &\quad - \frac{\gamma_b}{2} ((u_t^* - 1)(1+r)^{T-(t+1)} + f_{t+1}^\lambda)^2 (1 - \hat{p}_{x+t}^\lambda) \hat{p}_{x+t}^\lambda, \\ f_t^\lambda &= \hat{p}_{x+t}^\lambda f_{t+1}^\lambda - (1+r)^{T-(t+1)} (\hat{p}_{x+t}^\lambda + u_t^* ((1+\eta)\hat{p}_{x+t} - \hat{p}_{x+t}^\lambda)), \end{aligned}$$

with boundary conditions

$$F_T^\lambda = 0 \quad \text{and} \quad f_T^\lambda = 0.$$

Proof. We assume the following ansatz,

$$V_t(b; \eta, \lambda) = A_t b + F_t^\lambda l_t, \tag{C.12}$$

$$g_t(b; \eta, \lambda) = a_t b + f_t^\lambda l_t. \tag{C.13}$$

Using this ansatz and the dynamics for B_T (5.2), the extended Bellman equation (C.7) reduces considerably, and we obtain

$$\begin{aligned} A_t b + F_t^\lambda l_t &= \sup_{u \in [0,1]} \left\{ A_{t+1} ((1+r)b - l_t \hat{p}_{x+t}^\lambda - u l_t ((1+\eta)\hat{p}_{x+t} - \hat{p}_{x+t}^\lambda)) + F_{t+1}^\lambda \hat{p}_{x+t}^\lambda l_t \right. \\ &\quad \left. - \frac{\gamma_b}{2} ((u-1)a_{t+1} + f_{t+1}^\lambda)^2 l_t (1 - \hat{p}_{x+t}^\lambda) \hat{p}_{x+t}^\lambda \right\}. \end{aligned}$$

The above expression is concave with respect to u . Thus, we obtain the optimal value of u from the first-order condition. Specifically,

$$u_t^*(\cdot; \eta, \lambda) = 0 \vee \left(1 - \frac{A_{t+1} ((1+\eta)\hat{p}_{x+t} - \hat{p}_{x+t}^\lambda)}{\gamma_b a_{t+1}^2 (1 - \hat{p}_{x+t}^\lambda) \hat{p}_{x+t}^\lambda} - \frac{f_{t+1}^\lambda}{a_{t+1}} \right) \wedge 1,$$

and inserting this into the equation above, we obtain A_t and the recursion formula for B_t ,

$$\begin{aligned} A_t &= (1+r)^{T-t}, \\ F_t^\lambda &= -(1+r)^{T-(t+1)} (\hat{p}_{x+t}^\lambda + u_t^* ((1+\eta)\hat{p}_{x+t} - \hat{p}_{x+t}^\lambda)) + F_{t+1}^\lambda \hat{p}_{x+t}^\lambda \\ &\quad - \frac{\gamma_b}{2} ((u_t^* - 1)a_{t+1} + f_{t+1}^\lambda)^2 (1 - \hat{p}_{x+t}^\lambda) \hat{p}_{x+t}^\lambda. \end{aligned} \tag{C.14}$$

We plug the ansatz (C.13) and the previously derived expression for u^* into the recursion (C.8),

$$a_t b + f_t^\lambda l_t = a_{t+1} ((1+r)b - \hat{p}_{x+t}^\lambda l_t - u_t^* l_t ((1+\eta)\hat{p}_{x+t} - \hat{p}_{x+t}^\lambda)) + f_{t+1}^\lambda \hat{p}_{x+t}^\lambda l_t.$$

After identifying coefficients, this gives us the recursions,

$$\begin{aligned} a_t &= (1+r)^{T-t}, \\ f_t^\lambda &= \hat{p}_{x+t}^\lambda f_{t+1}^\lambda - (1+r)^{T-(t+1)} \left(\hat{p}_{x+t}^\lambda + u_t^* \left((1+\eta) \hat{p}_{x+t} - \hat{p}_{x+t}^\lambda \right) \right). \end{aligned} \tag{C.15}$$

□

C.3 Forecasting Future Survival Probabilities

The procedure of forecasting future survival probabilities under the reference measure comprises the following steps:

1. Fit Equations (5.15) and (5.16) to historical mortality data to establish baseline parameters.
2. For a given simulation horizon T , simulate K paths of future κ values, denoted as $\kappa_t^{(k)}$ for $t = 1, 2, \dots, T$ and $k = 1, 2, \dots, K$, by:

$$\kappa_t^{(k)} = \kappa_{t-1}^{(k)} + \hat{\sigma}_\kappa \varepsilon_t^{(k)},$$

where $\varepsilon_t^{(k)}$ is the simulated shock to the mortality improvement factor in year t on the k -th path.

3. Utilizing the simulated κ paths, generate S paths of future death rates by:

$$m_{x,t}^{(k)} = \exp\left(\hat{\beta}_x^{(1)} + \hat{\beta}_x^{(2)}(t - \bar{h}) + \kappa_t^{(k)} + \hat{\gamma}_c + \hat{\sigma}_{\omega_x} \omega_{x,t}^{(k)}\right),$$

where $\omega_{x,t}^{(k)}$ is the simulated shock for age x in year t on the k -th path.

4. The $(t-s)$ -year survival probability for a policyholder aged $x+s$ at time s on the k -th path is approximated using (Pitacco, 2009):

$${}_{t-s}p_{x+s}^{(k)} \approx \exp\left(-\sum_{\mu=0}^{t-s} m_{x+s+\mu, s+\mu}^{(k)}\right)$$

for any $s, t \in \tau$ and $s < t$.

5. The estimate of ${}_{t-s}p_{x+s}$ is obtained by averaging the K simulated values:

$${}_{t-s}\hat{p}_{x+s} = \frac{1}{K} \sum_{k=1}^K {}_{t-s}p_{x+s}^{(k)}.$$