

Characterizing cofree representations of $\mathrm{SL}_n \times \mathrm{SL}_m$

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

The study, and in particular classification, of cofree representations has been an interest of research for over 70 years. The Chevalley-Shepard-Todd Theorem provides a beautiful intrinsic characterization for cofree representations of finite groups. Specifically, this theorem says that a representation V of a finite group G is cofree if and only if G is generated by pseudoreflections. Up until the late 1900s, with the exception of finite groups, all of the existing classifications of cofree representations of a particular group consist of an explicit list, as opposed to an intrinsic group-theoretic characterization. However, in 2019, Edidin, Satriano, and Whitehead formulated a conjecture which intrinsically characterizes stable irreducible cofree representations of connected reductive groups and verified their conjecture for simple Lie groups. The conjecture states that for a stable irreducible representation V of a connected reductive group G , V is cofree if and only if V is pure. In comparison to the classifications comprised of a list of cofree representations, this conjecture can be viewed as an analogue of the Chevalley-Shepard-Todd Theorem for actions of connected reductive groups. The aim of this thesis is to further expand upon the techniques formulated by Edidin, Satriano, and Whitehead as a means to work towards the verification of the conjecture for all connected semisimple Lie groups. The main result of this thesis is the verification of the conjecture for stable irreducible representations $V \otimes W$ of $\mathrm{SL}_n \times \mathrm{SL}_m$ satisfying $\dim V \geq n^2$ and $\dim W \geq m^2$.

As the main group under study in this thesis is $\mathrm{SL}_n \times \mathrm{SL}_m$, in Chapter 2 we provide a thorough analysis of the structure of irreducible representations of SL_n from the view point of them being in one-to-one correspondence with irreducible representations of the Lie algebra $\mathrm{Lie}(\mathrm{SL}_n) = \mathfrak{sl}_n$. The last section of Chapter 2 describes the general theory of irreducible representations of complex semisimple Lie algebras, with $\mathfrak{sl}_n$ as a toy example. In Chapter 3, we provide a brief introduction to Geometric Invariant Theory (GIT) and present the main results of the theory. We then discuss the history of GIT and the known characterization results for properties of representations that arise from GIT. In particular, we introduce cofree representations and the current classification results for cofree representations of certain classes of groups. We finish Chapter 3 by introducing pure representations and the conjecture formulated by Edidin, Satriano, and Whitehead. In Chapter 4, we verify that for all stable irreducible representations $V \otimes W$ of $\mathrm{SL}_n \times \mathrm{SL}_m$ satisfying $\dim V \geq n^2$ and $\dim W \geq m^2$, $V \otimes W$ is cofree if and only if $V \otimes W$ is pure. This involves proving an upper bound on the dimension of pure representations of $G_1 \times G_2$, with G_i connected reductive Lie groups. We also introduce two methods that can be used to show that a given representation is not pure. The last section in Chapter 4 discusses the difficulties and obstacles when trying to verify the conjecture for the remaining cases, namely when $\dim V < n^2$ or $\dim W < m^2$.

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Dedication

To Quannik and Francesca, my four-legged best friends, for always reminding me of what truly matters in life.

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Chapter 1

Introduction

This thesis focuses on a classical problem from *geometric invariant theory* (GIT) that aims to understand when quotients arising from representations of reductive groups are extremely well-behaved. In particular, given a finite-dimensional representation V of a reductive group G over an algebraically closed field k of characteristic 0, we consider the space $k[V]$ of **polynomial functions** $V \rightarrow k$ on V . The representation action induces an action of G on $k[V]$ via

$$(g \cdot f)(v) := f(g^{-1} \cdot v), \quad g \in G, \quad f \in k[V], \quad v \in V.$$

The main object under study in GIT is the algebra $k[V]^G$ of **G -invariant polynomials**, which are those in $k[V]$ that are fixed under the action of G . Nagata [17, pg. 372] showed that $k[V]^G$ is a finitely generated k -algebra when G is reductive and so the inclusion $\pi^* : k[V]^G \hookrightarrow k[V]$ induces a morphism

$$\pi : V \rightarrow \operatorname{Spec}(k[V]^G) =: V//G$$

of affine varieties over k . We call π the affine **GIT quotient** associated to π . Although the GIT quotient behaves in many of the same ways as the usual quotient, in general, it does not parametrize the set of orbits. In other words, $\pi : V \rightarrow V//G$ often identifies several orbits, namely those whose closures intersect. Nevertheless, one can get around this by restricting π to an open set of *stable points* V^s to get a true orbit space $V//G$ and quotient $\pi : V^s \rightarrow \pi(V^s) \subseteq V//G$. When $V^s \neq \emptyset$, we say that V is **stable**.

Geometric invariant theory aims to characterize groups by various properties arising from the interactions between $k[V]$ and $k[V]^G$ that are deemed to be “nice” through an

invariant theory lens. The main property that this thesis is concerned with is called *cofree*. We say that a representation V of G is **cofree** if $k[V]$ is a free $k[V]^G$ -module. Schwarz proved that V is cofree if and only if $V//G$ is smooth and the fibres of $\pi : V \rightarrow V//G$ all have the same dimension [21, Proposition 17.29]. The Chevalley-Shepard-Todd Theorem is a beautiful intrinsic characterization of cofree representations V of finite groups G and says that V is cofree if and only if G is generated by pseudoreflections. In contrast, if G is a connected reductive group, then there does not exist a precise group-theoretic characterization of when V is cofree. Although we have yet to characterize all cofree representations, several classifications have been found for various classes of connected reductive groups. For example, the cofree representations of simple Lie groups were classified independently by Schwarz [20], and Adamovich and Golovina [1], while Littelmann [13] classified irreducible cofree representations of semisimple groups. Unlike the Chevalley-Shepard-Todd Theorem, these results for connected reductive groups consist only of a list of cofree representations, rather than a general group-theoretic characterization.

In 2019, Edidin, Satriano, and Whitehead stated a conjecture that intrinsically characterizes stable irreducible cofree representations of connected reductive groups [8]. Specifically, the conjecture states that if V is a stable irreducible representation of a connected reductive group G , then V is cofree if and only if V is *pure*; we say that a representation V is **pure** if $\pi(V \setminus V^s)$ is pure of codimension-one in $V//G$, where V^s is the set of stable points. Furthermore, Edidin, Satriano, and Whitehead verified their conjecture for all simple Lie groups [8]. As such, the geometric property *pure* is a candidate for an intrinsic condition that characterizes cofree. The aim of this thesis is to work towards the verification of this conjecture for all semisimple Lie groups by first studying cofree and pure representations of $\mathrm{SL}_n \times \mathrm{SL}_m$. The first main result of this thesis proves an upper bound on the dimension of a pure representation $V_1 \otimes V_2$ of $G_1 \times G_2$, when G_i are connected reductive Lie groups; see Theorem 4.5.4. This upper bound on pure representations is essential for turning the classification of pure representations into a finite problem. The second main result of this thesis proves that irreducible representations $V \otimes W$ of $\mathrm{SL}_n \times \mathrm{SL}_m$ satisfying $\dim V \geq n^2$ and $\dim W \geq m^2$ are not pure, which leads to the verification of the conjecture for this class of representations; see Theorem 4.8.1. It is expected that a proof of the conjecture that works for all of $\mathrm{SL}_n \times \mathrm{SL}_m$ will naturally generalize to any connected semisimple Lie group. Based off of this thesis work and the literature, there is reason to believe that the case $\mathrm{SL}_n \times \mathrm{SL}_m$ is the most difficult to verify the conjecture for. In fact, the difficulty of the case $\mathrm{SL}_n \times \mathrm{SL}_m$ is explicitly pointed out in [24], where a similar, yet different, GIT property of representations is studied. The work in this thesis, along with future work, will not only close a gap in the invariant theory literature, but may unveil the obstructions that make studying the invariant theory of $\mathrm{SL}_n \times \mathrm{SL}_m$ difficult.

Since the focus of this thesis is representations of $\mathrm{SL}_n \times \mathrm{SL}_m$, we start by providing an in-depth analysis of the representation theory of SL_n . In Chapter 2, we introduce *Lie groups* and their associated *Lie algebras*. We then show that representations of a simply connected Lie group G are in one-to-one correspondence with representations of its Lie algebra $\mathfrak{g}$. This bijection is particularly nice, as it linearizes the problem of classifying representations of a simply connected Lie group by converting it to a problem about maps between vector spaces. Since SL_n is simply connected, we apply this strategy to classify its representations. We start by classifying the representations of SL_2 , and then using the SL_2 case as a framework, extend the classification naturally to SL_n . We finish Chapter 2 by studying the structure of irreducible representations of semisimple Lie algebras.

In Chapter 3, we introduce the basic theory and results from GIT. Many of these results are about properties of $\pi : V \rightarrow V//G$. We also define the notion of a *stable point* and prove some nice features of stable points. Given the conjecture above, representations that have stable points are of particular interest to us in this thesis. We further motivate the problem of intrinsically characterizing cofree representations by discussing the history of GIT and many of the classifications already proved. The motivation behind the definition of *pure* and its relationship with cofreeness is expanded upon from [8]. We finish Chapter 3 with an analysis of cofree representations, by extracting the complete list of all irreducible cofree representations of $\mathrm{SL}_n \times \mathrm{SL}_m$ from the classification done by Littelmann [13].

Chapter 4 is the bulk of the thesis and utilizes the previous chapters as background. Recall that the aim of this thesis is to work towards the verification of the conjecture that says that any stable irreducible representation $V \otimes W$ of $\mathrm{SL}_n \times \mathrm{SL}_m$ is cofree if and only if it is pure. Because we already know the complete list of all irreducible cofree representations $V \otimes W$ of $\mathrm{SL}_n \times \mathrm{SL}_m$ from Chapter 3, the general strategy to prove the conjecture is to first write down the complete list of all pure representations. Given this, we first introduce a necessary condition for a representation to be pure from [8]. This condition essentially says that a representation is pure if there exists a hyperplane with a relatively large number of *weights* on it; the weights of a representation are certain eigenvalues that completely describe the representation. This necessary condition allows us to translate the problem of finding which representations are pure into a combinatorial problem about counting weights on hyperplanes. Furthermore, we utilize this hyperplane condition to prove the first main result of this thesis, which gives an upper bound on the dimension of certain pure representations. In the case of irreducible representations $V \otimes W$ of $\mathrm{SL}_n \times \mathrm{SL}_m$ satisfying $\dim V \geq n^2$ and $\dim W \geq m^2$, this upper bound allows us to write down a finite list that contains all possible pure representations. We then show that all of the representations in this list are not pure via two methods. One of the methods, called the “toral method”, amounts to computing the dimension of the fixed points of the

representations under toral elements. The other method uses the necessary condition on pure representations by showing that for a given representation there is no hyperplane that contains enough weights. Since there are no cofree representations $V \otimes W$ satisfying $\dim V \geq n^2$ and $\dim W \geq m^2$, the conjecture is verified in this case.

Chapter 2

Representations of SL_n

2.1 Introduction

Representation theory is the study of the ways in which a given algebraic structure may act on vector spaces [9]. In particular, a *representation* of an algebraic structure allows us to represent its elements as linear transformations of vector spaces. There is a vast amount of choices we can make in regards to the type of algebraic object being represented, and the type of vector space on which the algebraic object is represented. As a consequence, representation theory is prevalent in mathematics and has many diverse applications. It is connected to geometry via invariant theory [10], impacts number theory via automorphic forms and the Langlands program, and has several applications to physics. For example, as first noted in the 1930s by Eugene Wigner, there is a strong connection between representation theory and particle physics [5]. This connection links the properties of elementary particles to the structure of Lie groups and Lie algebras. For instance, under this connection, the different quantum states of an elementary particle give rise to an *irreducible representation* of a well-known Lie group, called the Poincaré group.

There are several different approaches to studying the theory of representations. In this chapter, we focus our attention to finite-dimensional representations of simply connected Lie groups and semisimple Lie algebras, over $\mathbb{C}$. One of the main goals of this chapter is to understand the techniques used to classify the representations of the Lie algebra $\mathfrak{sl}_n$ corresponding to the Lie group $\mathrm{SL}_n(\mathbb{C})$. Several of the techniques and concepts that we introduce are helpful for understanding the representation theory of other Lie groups and Lie algebras.

In Chapter 2.2, we define Lie groups, Lie algebras, and their representations. We show that every Lie group has associated to it a Lie algebra which is equal to its tangent space at the identity. The main result of this chapter is that representations of a simply connected Lie group are in one-to-one correspondence with representations of its Lie algebra. To prove this, we introduce the adjoint representation of a Lie group and Lie algebra, and the exponential map of a Lie group. Lastly, we define semisimple Lie algebras and highlight some of their properties, which are fundamental in the classification of their representations. In Chapter 2.3, we classify the irreducible representations of the semisimple Lie algebra $\mathfrak{sl}_2$ corresponding to $\mathrm{SL}_2(\mathbb{C})$. Given the results from Chapter 2.2, this gives us a complete classification of the representations of $\mathrm{SL}_2(\mathbb{C})$. In Chapter 2.4, we illustrate through $\mathfrak{sl}_3$ how the procedure for classifying irreducible representations of $\mathfrak{sl}_2$ can be naturally extended to $\mathfrak{sl}_n$ for higher $n \in \mathbb{N}$. To do this, we use the $\mathfrak{sl}_2$ case as a framework and rely on explicit results from Chapter 2.3. In Chapter 2.5, we see how the process for classifying representations of $\mathfrak{sl}_3$ generalizes to a step-by-step procedure for classifying representations of any complex semisimple Lie algebra. We introduce objects that are fundamental in the study of representation theory such as Cartan subalgebras, roots and root spaces, the Weyl group, and the Weyl chamber. Throughout, we demonstrate the theory with the semisimple Lie algebra $\mathfrak{sl}_n$. At the end of the chapter, we arrive at a complete classification of the irreducible representations of $\mathfrak{sl}_n$.

Notation and prerequisites

All vector spaces are finite-dimensional with base field $\mathbb{R}$ or $\mathbb{C}$. Given a vector space V , we denote its dual space by V^* . We denote the n th symmetric power of V by $\mathrm{Sym}^n V$, and the n th exterior power of V by $\bigwedge^n V$. Introductory manifold theory that is needed can be found in the first few chapters of [12]. We adopt standard manifold theory terminology and notation from [12].

2.2 Lie groups and Lie algebras

2.2.1 Lie groups

A Lie group is a set endowed with both the structure of a group and a manifold in a *compatible* way. Here, compatible means that the group operations agree with the structure of the manifold. For our purposes, we only consider Lie groups over $\mathbb{R}$ and $\mathbb{C}$. A **real Lie**

group is a group endowed with the structure of a smooth manifold such that the group operations of multiplication and inversion are smooth maps. Similarly, a **complex Lie group** is a group endowed with the structure of a complex manifold such that the group operations of multiplication and inversion are holomorphic maps. If G is a Lie group, we say that $H \subseteq G$ is a **Lie subgroup** if H is both a subgroup of G and an embedded submanifold that is closed. Given this definition, a Lie subgroup is itself a Lie group. Furthermore, we define an **immersed subgroup** to be the image of a Lie group H under an injective Lie group morphism (see below) to G .

Set F to be either $\mathbb{R}$ or $\mathbb{C}$, and let $M_n(F)$ denote the set of $n \times n$ matrices over F . The first example of a Lie group that we look at is the **general linear group**

$$\mathrm{GL}_n(F) := \{A \in M_n(F) : \det(A) \neq 0\}.$$

First consider the case when $F = \mathbb{R}$. Since $\mathrm{GL}_n(\mathbb{R})$ is an open subset of the smooth manifold $M_n(\mathbb{R})$ it is itself a smooth manifold. The group multiplication for $\mathrm{GL}_n(\mathbb{R})$ is smooth since if $A, B \in \mathrm{GL}_n(\mathbb{R})$ then the entries of AB are polynomials in the entries of A and B . Smoothness of group inversion for $\mathrm{GL}_n(\mathbb{R})$ follows by Cramers rule and the fact that cofactors are polynomials. If we replace the word smooth with holomorphic, then the exact same argument shows that $\mathrm{GL}_n(\mathbb{C})$ is a complex Lie group. Sometimes it will be helpful to view $\mathrm{GL}_n(F)$ as the group $\mathrm{GL}(V)$ of invertible linear operators $T : V \rightarrow V$, where $V \cong F^n$.

Other examples of Lie groups include the **special linear group**

$$\mathrm{SL}_n(F) := \{A \in \mathrm{GL}_n(F) : \det(A) = 1\},$$

the **orthogonal group**

$$\mathrm{O}_n(F) := \{A \in \mathrm{GL}_n(F) : A^T A = A A^T = I_n\},$$

the **special orthogonal group**

$$\mathrm{SO}_n(F) := \mathrm{O}_n(F) \cap \mathrm{SL}_n(F),$$

and the **symplectic group**

$$\mathrm{Sp}_{2n}(F) := \{A \in M_{2n}(F) : A^T \Omega A = \Omega\},$$

where

$$\Omega := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

and I_n is the $n \times n$ identity matrix.

If G and H are two real (resp. complex) Lie groups, then a **morphism of real (resp. complex) Lie groups** is a map $\varphi : G \rightarrow H$ that is smooth (resp. holomorphic) and a group homomorphism. When the Lie group structure is clear, we simply say that φ is a Lie group morphism. A Lie group morphism φ is a **Lie group isomorphism** if φ is bijective and its inverse φ^{-1} is a Lie group morphism. We say that a (finite-dimensional) **representation** of a Lie group G on a vector space $V \cong F^n$ is a Lie group morphism from G to $\text{GL}(V)$. If $\rho : G \rightarrow \text{GL}(V)$ is a representation of G , we denote the action of G on V by

$$g(v) := \rho(g)(v), \quad g \in G.$$

When the morphism ρ is clear from context, we often refer to V itself or (V, G) as the representation.

2.2.2 Lie algebras

In this section we define Lie algebras and morphisms between them. We show that every Lie group has associated with it a corresponding Lie algebra. In fact, if G and H are Lie groups, with G simply connected, the study of Lie group morphisms from G to H is reduced to the study of morphisms between their corresponding Lie algebras. If we take H to be $\text{GL}(V)$, then this says that representations of G can be understood solely through the study of morphisms between Lie algebras. Since Lie algebras do not have a topology, classifying morphisms between them is a much simpler problem.

Lemma 2.2.1. *Let G be a Lie group. For any $g \in G$, the map $m_g : G \rightarrow G$ defined by $m_g(h) = gh$ is a diffeomorphism.*

Proof. Note that m_g is bijective with inverse $(m_g)^{-1} = m_{g^{-1}}$. Let $m : G \times G \rightarrow G$ be group multiplication, and define $\varphi_g : G \rightarrow G \times G$ by $\varphi_g(h) = (g, h)$. Then the diagram

$$\begin{array}{ccccc} & & m_g & & \\ & \nearrow & & \searrow & \\ G & \xrightarrow{\varphi_g} & G \times G & \xrightarrow{m} & G \end{array}$$

commutes. Since G is a Lie group, the multiplication operation m is continuous. Define $c_g : G \rightarrow G$ by $c_g(h) = g$, and let $p_i : G \times G \rightarrow G$ for $i = 1, 2$ be projection onto the i th coordinate. Then the diagrams

$$\begin{array}{ccc}
& G \times G & \\
\varphi_g \nearrow & \downarrow p_1 & \\
G & \xrightarrow{c_g} & G
\end{array}
\quad
\begin{array}{ccc}
& G \times G & \\
\varphi_g \nearrow & \downarrow p_2 & \\
G & \xrightarrow{\text{id}_G} & G
\end{array}$$

commute. Since c_g and id_G are continuous, by the universal property of product spaces, the map φ_g is continuous. So, m_g is continuous for all $g \in G$. In particular, m_g and $m_{g^{-1}}$ are continuous and so m_g is a diffeomorphism. $\square$

Proposition 2.2.2. [9, Exercise 8.1] *Let G be a connected Lie group with identity e . If $U \subset G$ is any open neighborhood of e , then U generates G as a group, i.e., $\langle U \rangle = G$.*

Proof. Since G is connected, its only clopen sets are the empty set and G itself. Note that $\langle U \rangle \neq \emptyset$ since $e \in U$. Thus, it suffices to show that $\langle U \rangle$ is clopen.

We first show that $\langle U \rangle$ is open. Let $g \in G$, and consider

$$gU := \{gu : u \in U\} \subseteq \langle U \rangle.$$

By Lemma 2.2.1, the map m_g is a diffeomorphism and hence open. Thus $m_g(U) = gU$ is open for all $g \in G$. Taking $g \in \langle U \rangle$, we find that $g = ge \in gU$ with gU open.

To prove that $\langle U \rangle$ is closed, we must show that $\overline{\langle U \rangle} \subseteq \langle U \rangle$. Let $\partial\langle U \rangle$ denote the boundary of $\langle U \rangle$. Since $\overline{\langle U \rangle} = \partial\langle U \rangle \cup \langle U \rangle$ it suffices to show that $\partial\langle U \rangle \subseteq \langle U \rangle$. If $g \in \partial\langle U \rangle$, then every open neighborhood of g contains at least one element of $\langle U \rangle$. Since gU is an open neighborhood of g , there exists $h \in gU \cap \langle U \rangle$. Then $h = gu$ for some $u \in U$ and $h = u_1 \dots u_k$ for some $u_i \in U \cup U^{-1}$. So $gu = u_1 \dots u_k$ and hence $g = u_1 \dots u_k u^{-1} \in \langle U \rangle$. $\square$

Remark 2.2.3. Proposition 2.2.2 implies that any morphism $\rho : G \rightarrow H$ between connected Lie groups is determined by its germ at the identity $e \in G$. In other words, we can understand ρ by just looking at what it does locally around $e \in G$.

We further extend this idea of local data with the following proposition, whose proof we defer to Section 2.2.4.

Proposition 2.2.4 (First Principle). [9, Section 8.1] *Let G and H be Lie groups, with G connected. A morphism $\rho : G \rightarrow H$ is uniquely determined by its differential $(d\rho)_e : T_e G \rightarrow T_e H$ at the identity.*

Given Proposition 2.2.4, it is natural to ask which maps between $T_e G$ and $T_e H$ arise as differentials of Lie group morphisms. Before we answer this, it will be helpful to understand how the group homomorphism property of a Lie group morphism is reflected in its differential. For any Lie group G and any $g \in G$, we define the Lie group automorphism

$$\begin{aligned}\Psi_g : G &\rightarrow G \\ h &\mapsto ghg^{-1}.\end{aligned}$$

If $\rho : G \rightarrow H$ is a Lie group morphism, then the group homomorphism property

$$\rho(gh) = \rho(g)\rho(h), \quad \forall g, h \in G,$$

ensures that the diagram

$$\begin{array}{ccc} G & \xrightarrow{\rho} & H \\ \Psi_g \downarrow & & \downarrow \Psi_{\rho(g)} \\ G & \xrightarrow{\rho} & H \end{array}$$

commutes. This commutative diagram is equivalent to saying that ρ respects the action of G on itself by conjugation. Since Ψ_g preserves the identity $e \in G$, its differential at e is

$$\text{Ad}(g) := (d\Psi_g)_e : T_e G \rightarrow T_e G,$$

where $T_e G$ denotes the tangent space of G at e . Then

$$\begin{aligned}\text{Ad} : G &\rightarrow \text{GL}(T_e G) \\ g &\mapsto \text{Ad}(g)\end{aligned}$$

is a representation of G on $T_e G$, called the **adjoint representation** of G . We show that any Lie group morphism $\rho : G \rightarrow H$ respects the adjoint action of G on $T_e G$.

Proposition 2.2.5. [9, Section 8.1] *Let $\rho : G \rightarrow H$ be a Lie group morphism. Then for any $g \in G$ the diagram*

$$\begin{array}{ccc} T_e G & \xrightarrow{(d\rho)_e} & T_e H \\ \text{Ad}(g) \downarrow & & \downarrow \text{Ad}(\rho(g)) \\ T_e G & \xrightarrow{(d\rho)_e} & T_e H \end{array}$$

commutes.

Proof. Given $X \in T_e G$, we can find a smooth path $\gamma : (-\varepsilon, \varepsilon) \rightarrow G$ such that $\gamma(0) = e$ and $\gamma'(0) = X$. Then

$$(\text{Ad}(\rho(g)) \circ (d\rho)_e)(X) = ((d\Psi_{\rho(g)})_e \circ (d\rho)_e)(X) = (\Psi_{\rho(g)} \circ \rho \circ \gamma)'(0)$$

and

$$((d\rho)_e \circ \text{Ad}(g))(X) = ((d\rho)_e \circ (d\Psi_g)_e)(X) = (\rho \circ \Psi_g \circ \gamma)'(0).$$

Since ρ is a group homomorphism, we have that $\Psi_{\rho(g)} \circ \rho = \rho \circ \Psi_g$ and so

$$(\Psi_{\rho(g)} \circ \rho \circ \gamma)'(0) = (\rho \circ \Psi_g \circ \gamma)'(0). \quad \square$$

Preservation of the adjoint representation $\text{Ad} : G \rightarrow \text{GL}(T_e G)$ by a Lie group morphism $\rho : G \rightarrow H$ is not purely a condition on the differential $(d\rho)_e$; it still explicitly evokes the group G through $\rho : G \rightarrow H$. To get a condition that is purely on the differential $(d\rho)_e$, we take the differential of Ad at the identity $e \in G$. Since $\text{GL}(T_e G)$ is an open subset of the vector space $\text{End}(T_e G)$, its tangent space at e is naturally identified with $\text{End}(T_e G)$. So, the differential of Ad at e is

$$\text{ad} := (d\text{Ad})_e : T_e G \rightarrow \text{End}(T_e G).$$

It will be helpful to interpret ad as the bilinear map

$$[\cdot, \cdot] : T_e G \times T_e G \rightarrow T_e G,$$

where

$$[X, Y] := \text{ad}(X)(Y).$$

We refer to this bilinear map as the **Lie bracket** associated to $T_e G$ (or G). Sometimes we write $[\cdot, \cdot]_G$ to emphasize the group G . We show that the differential $(d\rho)_e$ of any Lie group morphism $\rho : G \rightarrow H$ respects the adjoint action of $T_e G$ on itself.

Proposition 2.2.6. [9, Section 8.1] *Let $\rho : G \rightarrow H$ be a morphism of Lie groups. Then for any $X \in T_e G$ the diagram*

$$\begin{array}{ccc} T_e G & \xrightarrow{(d\rho)_e} & T_e H \\ \text{ad}(X) \downarrow & & \downarrow \text{ad}((d\rho)_e(X)) \\ T_e G & \xrightarrow{(d\rho)_e} & T_e H \end{array}$$

commutes.

Proof. For any $X, Y \in T_e G$, we show that

$$(d\rho)_e([X, Y]) = [(d\rho)_e(X), (d\rho)_e(Y)].$$

Let $\gamma : (-\varepsilon, \varepsilon) \rightarrow G$ be smooth path such that $\gamma(0) = e$ and $\gamma'(0) = X$. Since a derivative commutes with a linear transformation, we may write

$$\begin{aligned} (d\rho)_e([X, Y]) &= (d\rho)_e(\text{ad}(X)(Y)) \\ &= (d\rho)_e\left(\left.\frac{d}{dt}\right|_{t=0} \text{Ad}(\gamma(t))(Y)\right) \\ &= \left.\frac{d}{dt}\right|_{t=0} ((d\rho)_e \circ \text{Ad}(\gamma(t)))(Y). \end{aligned}$$

Then by Proposition 2.2.5, we get that

$$\begin{aligned} (d\rho)_e([X, Y]) &= \left.\frac{d}{dt}\right|_{t=0} (\text{Ad}(\rho(\gamma(t))) \circ (d\rho)_e)(Y) \\ &= d(\text{Ad} \circ \rho)_e(X)((d\rho)_e(Y)) \\ &= (d\text{Ad})_e((d\rho)_e(X))((d\rho)_e(Y)) \\ &= [(d\rho)_e(X), (d\rho)_e(Y)]. \end{aligned} \quad \square$$

Remark 2.2.7. By Proposition 2.2.6, if a linear map $\phi : T_e G \rightarrow T_e H$ arises as the differential of a Lie group morphism $G \rightarrow H$, then it necessarily preserves the Lie bracket in the sense that

$$\phi([X, Y]) = [\phi(X), \phi(Y)] \quad (2.1)$$

for all $X, Y \in T_e G$. The following proposition, whose proof we defer to Section 2.2.4, tells us that this condition is sufficient for G simply connected.

Proposition 2.2.8 (Second Principle). [9, Section 8.1] *Let G and H be Lie groups, with G simply connected. Then a linear map $\phi : T_e G \rightarrow T_e H$ is the differential of a Lie group morphism $G \rightarrow H$ if and only if ϕ respects the Lie bracket operation in the sense of Equation (2.1) above.*

Example 2.2.9. [9, Section 8.1] Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$, and consider $G = \text{GL}(V)$. We find an explicit formula for the Lie bracket associated to $T_e G$. Let's compute the differential

$$\text{Ad}(g) = (d\Psi_g)_e : \text{End}(V) \rightarrow \text{End}(V),$$

where $e \in G$ is the identity. Given $X \in \text{End}(V)$, we can find a smooth path $\gamma : (-\varepsilon, \varepsilon) \rightarrow G$ such that $\gamma(0) = e$ and $\gamma'(0) = X$. Then

$$(d\Psi_g)_e(X) = \left. \frac{d}{dt} \right|_{t=0} \Psi_g(\gamma(t)) = \left. \frac{d}{dt} \right|_{t=0} g\gamma(t)g^{-1} = g\gamma'(0)g^{-1} = gXg^{-1}.$$

Extending Ψ_g to a linear map on $\text{End}(V)$, this computation shows that $(d\Psi_g)_e = \Psi_g$. So, the adjoint representation of G is given by

$$\begin{aligned} \text{Ad} : \text{GL}(V) &\rightarrow \text{GL}(\text{End}(V)) \\ g &\mapsto \Psi_g. \end{aligned}$$

Next, we compute the differential

$$\text{ad} = (d\text{Ad})_e : \text{End}(V) \rightarrow \text{End}(\text{End}(V)).$$

Let X and γ be as above. Then for any $Y \in \text{End}(V)$, we compute that

$$\text{ad}(X)(Y) = \left. \frac{d}{dt} \right|_{t=0} \text{Ad}(\gamma(t))Y = \left. \frac{d}{dt} \right|_{t=0} \Psi_{\gamma(t)}(Y) = \left. \frac{d}{dt} \right|_{t=0} \gamma(t)Y\gamma^{-1}(t).$$

Applying the product rule, we get that

$$\text{ad}(X)(Y) = \gamma'(0)Y\gamma^{-1}(0) + \gamma(0)Y(\gamma^{-1}(0))' = XY + Y(\gamma^{-1}(0))'.$$

To determine $(\gamma^{-1}(0))'$, we differentiate the relation $\gamma(t)\gamma^{-1}(t) = I$ at 0 to get that

$$\gamma'(0)\gamma^{-1}(0) + \gamma(0)(\gamma^{-1}(0))' = 0.$$

Then

$$(\gamma^{-1}(0))' = -\gamma^{-1}(0)\gamma'(0)\gamma^{-1}(0) = -X$$

and hence

$$\text{ad}(X)(Y) = XY - YX.$$

We conclude that the Lie bracket associated to $T_e G$ is the commutator

$$[X, Y] = XY - YX, \quad X, Y \in T_e G.$$

A similar proof shows that for any subgroup $H \subseteq \text{GL}(V)$, the Lie bracket associated to $T_e H$ is the commutator. $\diamond$

We prove two properties of the Lie bracket.

Proposition 2.2.10. [9, Section 8.1] *Let G be a Lie group. Then the Lie bracket associated to $T_e G$ is **skew-symmetric** in the sense that*

$$[X, Y] = -[Y, X]$$

for all $X, Y \in T_e G$.

Proof. First note that the Lie bracket being skew-symmetric is equivalent to the assertion that $[X, X] = 0$ for all $X \in T_e G$. So, it suffices to show that $[X, X] = 0$ for all $X \in T_e G$. If $X \in T_e G$, then there exists a homomorphism $\varphi_X : \mathbb{R} \rightarrow G$ such that $X = (d\varphi_X)_0(1)$. Existence of such a homomorphism will be proven in Section 2.2.4. Then, by Proposition 2.2.6, we get that

$$[X, X]_G = [(d\varphi_X)_0(1), (d\varphi_X)_0(1)]_G = (d\varphi_X)_0[1, 1]_{\mathbb{R}}.$$

Since $\mathbb{R}$ is abelian, its adjoint representation is constant and therefore $\text{ad} = 0$. We conclude that $[1, 1]_{\mathbb{R}} = 0$ and so $[X, X]_G = 0$. $\square$

Proposition 2.2.11. [9, Section 8.1] *Let G be a Lie group. Then the Lie bracket associated to $T_e G$ satisfies the **Jacobi identity**: for any $X, Y, Z \in T_e G$ we have that*

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

Moreover, the Jacobi identity is equivalent to the assertion that ad preserves the Lie bracket.

Proof. By Proposition 2.2.10, the Jacobi identity is equivalent to the relation

$$[[X, Y], Z] = [X, [Y, Z]] - [Y, [X, Z]].$$

Notice that

$$\text{ad}([X, Y])(Z) = [[X, Y], Z].$$

By Example 2.2.9, the bracket $[\text{ad}(X), \text{ad}(Y)]$ is the commutator of the operators. It follows that

$$\begin{aligned} [\text{ad}(X), \text{ad}(Y)](Z) &= (\text{ad}(X) \circ \text{ad}(Y) - \text{ad}(Y) \circ \text{ad}(X))(Z) \\ &= [X, [Y, Z]] - [Y, [X, Z]]. \end{aligned}$$

So, the Jacobi identity holds if and only if

$$\mathrm{ad}([X, Y])(Z) = [\mathrm{ad}(X), \mathrm{ad}(Y)](Z),$$

which proves the second statement. Since $\mathrm{Ad} : G \rightarrow \mathrm{GL}(T_e G)$ is a Lie group morphism, by Proposition 2.2.6, we get that

$$\mathrm{ad}([X, Y]) = [\mathrm{ad}(X), \mathrm{ad}(Y)]$$

for all $X, Y \in T_e G$. It follows from the second statement that the Lie bracket satisfies the Jacobi identity, which proves the first statement. $\square$

The above analysis motivates the following definition. A **Lie algebra** $\mathfrak{g}$ is a vector space together with a skew-symmetric bilinear map

$$[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$$

satisfying the Jacobi identity. To emphasize the extra Lie bracket structure, we write $\mathfrak{gl}(V)$ instead of $\mathrm{End}(V)$ when referring to the endomorphisms of a vector space as a Lie algebra. We refer to this bilinear map as the **Lie bracket** associated to $\mathfrak{g}$. If $\mathfrak{h} \subseteq \mathfrak{g}$ is a subspace closed under the Lie bracket, we say that $\mathfrak{h}$ is a **Lie subalgebra** of $\mathfrak{g}$. Given Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ over the same field, a **morphism of Lie algebras** is a linear map $\phi : \mathfrak{g} \rightarrow \mathfrak{h}$ that preserves the Lie bracket in the sense of Equation (2.1). We say that a **representation of a Lie algebra** $\mathfrak{g}$ on a vector space V is a morphism of Lie algebras

$$\mathfrak{g} \rightarrow \mathfrak{gl}(V).$$

Note that V is over the same field as $\mathfrak{g}$. Equivalently, a representation of $\mathfrak{g}$ on V is a linear action of $\mathfrak{g}$ on V such that

$$[X, Y](v) = X(Y(v)) - Y(X(v)).$$

If $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ is a representation of $\mathfrak{g}$, we denote the action of $\mathfrak{g}$ on $v \in V$ by $\mathfrak{g} \cdot v$ or

$$X(v) := \rho(X)(v), \quad X \in \mathfrak{g}.$$

When the morphism ρ is clear from context, we often refer to V itself or $(V, \mathfrak{g})$ as the representation. A representation of a Lie algebra is **faithful** if it is injective or, equivalently, has zero kernel. Given a representation V of a Lie algebra $\mathfrak{g}$, we say that a subspace $W \subset V$ is **invariant** under $\mathfrak{g}$ if $X(w) \in W$ for all $w \in W$ and $X \in \mathfrak{g}$. If $W \subset V$ is invariant under $\mathfrak{g}$, then W gives rise to a representation $\mathfrak{g} \rightarrow \mathfrak{gl}(W)$ of $\mathfrak{g}$, and we say that W is a

subrepresentation of V . A nonzero representation V is said to be **irreducible** if V and 0 are the only invariant subspaces. If V and W are representations of $\mathfrak{g}$, then a linear map $f : V \rightarrow W$ is a **homomorphism** of $\mathfrak{g}$ -modules if it satisfies

$$f(X(v)) = X(f(v))$$

for any $X \in \mathfrak{g}$ and $v \in V$. If f is a bijection, then V and W are said to be **isomorphic** as representations of $\mathfrak{g}$.

Remark 2.2.12. Propositions 2.2.10 and 2.2.11 imply that any Lie group G gives rise to a Lie algebra, which is $\mathfrak{g} = T_e G$ with Lie bracket $[X, Y] := \text{ad}(X)(Y)$. Since $\text{ad} : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$ is linear and preserves the Lie bracket, it is a representation of $\mathfrak{g} = T_e G$. We refer to ad as the **adjoint representation** of $\mathfrak{g}$.

Now let G and H be Lie groups, with G simply connected. Denote the set of Lie group morphisms from G to H by $\text{Hom}_{\text{Lie-grp}}(G, H)$, and the set of Lie algebra morphisms from $\mathfrak{g} = T_e G$ to $\mathfrak{h} = T_e H$ by $\text{Hom}_{\text{Lie-alg}}(\mathfrak{g}, \mathfrak{h})$. The map

$$\begin{aligned} \text{Hom}_{\text{Lie-grp}}(G, H) &\rightarrow \text{Hom}_{\text{Lie-alg}}(\mathfrak{g}, \mathfrak{h}) \\ (\rho : G \rightarrow H) &\mapsto ((d\rho)_e : \mathfrak{g} \rightarrow \mathfrak{h}) \end{aligned}$$

is bijective. Injectivity will follow from Proposition 2.2.4 and surjectivity will follow from Proposition 2.2.8. This bijection implies that *representations of a simply connected Lie group are in one-to-one correspondence with representations of its Lie algebra*.

Example 2.2.13. [9, Section 8.1] Let G be a Lie group with associated Lie algebra $\mathfrak{g} = T_e G$. Recall that if V and W are representations of G , then $V \otimes W$ is a representation whose action of G is defined by

$$g(v \otimes w) = g(v) \otimes g(w), \quad g \in G.$$

We would like to describe V, W , and $V \otimes W$ as representations of $\mathfrak{g}$. Given $X \in \mathfrak{g}$, we can find a smooth path $\gamma : (-\varepsilon, \varepsilon) \rightarrow G$ such that $\gamma(0) = e$ and $\gamma'(0) = X$. By definition, the action of X on V is given by

$$X(v) = \left. \frac{d}{dt} \right|_{t=0} \gamma(t)(v).$$

The action of X on W is given similarly. For convenience, set $\gamma_t := \gamma(t)$. Applying the chain rule to $(\gamma_t(v) \otimes 1)(1 \otimes \gamma_t(w))$, the action of X on $V \otimes W$ is

$$\begin{aligned} X(v \otimes w) &= \left. \frac{d}{dt} \right|_{t=0} \gamma_t(v) \otimes \gamma_t(w) \\ &= \left. \frac{d}{dt} \right|_{t=0} \gamma_t(v) \otimes \gamma_0(w) + \gamma_0(v) \otimes \left. \frac{d}{dt} \right|_{t=0} \gamma_t(w), \end{aligned}$$

and so

$$X(v \otimes w) = X(v) \otimes w + v \otimes X(w). \quad (2.2)$$

Let $\rho : G \rightarrow \mathrm{GL}(V)$ be a representation of G . Then the dual representation $\rho^* : G \rightarrow \mathrm{GL}(V^*)$ of G is the transpose of $\rho(g^{-1})$. That is,

$$\rho^*(g) = \rho(g^{-1})^T, \quad g \in G.$$

Differentiating ρ^* at e , and using the fact that $(\gamma^{-1}(0))' = -X$, the action of X on V^* is

$$(d\rho^*)_e(X)(f) = -(d\rho)_e(X)^T(f), \quad f \in V^*.$$

◇

Remark 2.2.14. We define the action of a Lie algebra $\mathfrak{g}$ on the tensor product of two representations of $\mathfrak{g}$ by Equation (2.2). If $\phi : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ is a representation of $\mathfrak{g}$, then we define the dual representation $\phi^* : \mathfrak{g} \rightarrow \mathfrak{gl}(V^*)$ of $\mathfrak{g}$ by

$$\phi^*(X) := -\phi(X)^T, \quad X \in \mathfrak{g}.$$

2.2.3 Examples of Lie algebras

For each group in Section 2.2.1, we compute its tangent space at the identity to get its associated Lie algebra. These Lie algebras are all Lie subalgebras of $\mathfrak{gl}_n(F) = \mathrm{M}_n(F)$, where $\mathfrak{gl}_n(F) := T_e \mathrm{GL}_n(F)$. We use the matrix exponential

$$\exp(M) := e^M, \quad M \in \mathrm{M}_n(F),$$

to compute each of these Lie algebras.

Example 2.2.15. We show that the Lie algebra of $\mathrm{SL}_n(F)$ is

$$\mathfrak{sl}_n(F) = \{X \in \mathrm{M}_n(F) : \mathrm{Tr}(X) = 0\}.$$

If $X \in \mathfrak{sl}_n(F)$, then we can find a smooth path $\gamma : (-\varepsilon, \varepsilon) \rightarrow \mathrm{SL}_n(F)$ such that $\gamma(0) = I_n$ and $\gamma'(0) = X$. Since $\det(\gamma(t)) = 1$ for all $t \in (-\varepsilon, \varepsilon)$, the differential is

$$(d\det)_{I_n}(X) = \left. \frac{d}{dt} \right|_{t=0} \det(\gamma(t)) = 0.$$

We also have that

$$\det(I_n + tX) = 1 + t\operatorname{Tr}(X) + t^2p(t)$$

for some polynomial $p(t)$. Then

$$(d\det)_{I_n}(X) = \lim_{t \rightarrow 0} \frac{\det(I_n + tX) - \det(I_n)}{t} = \lim_{t \rightarrow 0} \operatorname{Tr}(X) + tp(t) = \operatorname{Tr}(X).$$

It follows that

$$0 = (d\det)_{I_n}(X) = \operatorname{Tr}(X).$$

Conversely, let $X \in M_n(F)$ with $\operatorname{Tr}(X) = 0$. Then for any $t \in \mathbb{R}$, we have that

$$\det(\exp(tX)) = e^{\operatorname{Tr}(tX)} = e^0 = 1,$$

where the first equality can be found in [11, Theorem 2.12]. Thus, we may consider the smooth path $\gamma : (-\varepsilon, \varepsilon) \rightarrow \operatorname{SL}_n(F)$ defined by $\gamma(t) = \exp(tX)$. Since $\gamma(0) = I_n$ and $\gamma'(0) = X$, we get that $X \in \mathfrak{sl}_n(F)$. $\diamond$

Example 2.2.16. We show that the Lie algebra of $\operatorname{Sp}_{2n}(F)$ is

$$\mathfrak{sp}_{2n}F = \{X \in M_{2n}(F) : X^T\Omega + \Omega X = 0\},$$

where

$$\Omega := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

If $X \in \mathfrak{sp}_{2n}(F)$, then we can find a smooth path $\gamma : (-\varepsilon, \varepsilon) \rightarrow \operatorname{Sp}_{2n}(F)$ such that $\gamma(0) = I_{2n}$ and $\gamma'(0) = X$. Differentiating the condition $\gamma(t)^T\Omega\gamma(t) = \Omega$ at 0, we get that

$$0 = \frac{d}{dt} \Big|_{t=0} \gamma(t)^T\Omega\gamma(t) = \gamma'(0)^T\Omega\gamma(0) + \gamma(0)^T\Omega\gamma'(0) = X^T\Omega + \Omega X.$$

Conversely, let $X \in M_{2n}(F)$ such that $X^T\Omega + \Omega X = 0$. Using the definition of the matrix exponential, we compute that

$$\exp(tX)^T\Omega\exp(tX) = \Omega$$

for all $t \in \mathbb{R}$. Then we get a smooth path $\gamma : (-\varepsilon, \varepsilon) \rightarrow \operatorname{Sp}_{2n}(F)$ defined by $\gamma(t) = \exp(tX)$. Since $\gamma(0) = I_{2n}$ and $\gamma'(0) = X$, we get that $X \in \mathfrak{sp}_{2n}(F)$. $\diamond$

Example 2.2.17. We show that the Lie algebra of $O_n(F)$ is

$$\mathfrak{o}_n(F) = \{X \in M_n(F) : X^T + X = 0\}.$$

If $X \in \mathfrak{o}_n(F)$, then we can find a smooth path $\gamma : (-\varepsilon, \varepsilon) \rightarrow O_n(F)$ such that $\gamma(0) = I_n$ and $\gamma'(0) = X$. Differentiating the relation $\gamma(t)^T \gamma(t) = \gamma(t) \gamma^T(t) = I_n$ at 0, we get that

$$0 = \left. \frac{d}{dt} \right|_{t=0} \gamma^T(t) \gamma(t) = \gamma'(0)^T \gamma(0) + \gamma(0)^T \gamma'(0) = X^T + X.$$

Conversely, let $X \in M_n(F)$ such that $X^T + X = 0$. Using properties of the matrix exponential, we get that

$$(e^{tX})^T (e^{tX}) = e^{t(X^T + X)} = e^0 = I_n$$

for any $t \in \mathbb{R}$. Similarly $(e^{tX})(e^{tX})^T = I_n$. Thus, we may consider the smooth path $\gamma : (-\varepsilon, \varepsilon) \rightarrow O_n(F)$ defined by $\gamma(t) = \exp(tX)$. Since $\gamma(0) = I_n$ and $\gamma'(0) = X$, we get that $X \in \mathfrak{o}_n(F)$.

Lastly, we verify that the Lie algebra of $SO_n(F)$ is $\mathfrak{so}_n(F) = \mathfrak{o}_n(F)$. The inclusion $\mathfrak{so}_n(F) \subseteq \mathfrak{o}_n(F)$ follows since $SO_n(F) \subset O_n(F)$. Given $X \in \mathfrak{o}_n(F)$, we have that $X^T + X = 0$. Then

$$0 = \text{Tr}(X^T + X) = \text{Tr}(X^T) + \text{Tr}(X) = 2\text{Tr}(X)$$

and so $\text{Tr}(X) = 0$. It follows that

$$\det(\exp(tX)) = e^{t\text{Tr}(X)} = e^0 = 1$$

for all $t \in \mathbb{R}$. Thus, we may consider the smooth path $\gamma : (-\varepsilon, \varepsilon) \rightarrow SO_n(F)$ defined by $\gamma(t) = \exp(tX)$. Since $\gamma(0) = I_n$ and $\gamma'(0) = X$, we get that $X \in \mathfrak{so}_n(F)$. $\diamond$

2.2.4 The exponential map

As demonstrated in Section 2.2.3, the matrix exponential plays a key role when constructing paths in a Lie group G with a given tangent vector X in $\mathfrak{g} = T_e G$. We generalize this construction to all Lie groups in this section. In particular, we introduce the notion of an exponential map associated to a Lie group. The exponential map captures the local group structure from $\mathfrak{g} = T_e G$ and is essential for proving Propositions 2.2.4 and 2.2.8. We start by studying one-parameter subgroups of G , which we later use to define the exponential map.

Throughout this section, G is a Lie group with corresponding Lie algebra $\mathfrak{g} = T_e G$. For any $g \in G$, consider the map $m_g : G \rightarrow G$ defined by $m_g(h) = gh$. Given $X \in \mathfrak{g}$, we define a vector field v_X on G by

$$v_X(g) = (dm_g)_e(X).$$

Here, $v_X(g)$ translates X to a tangent vector in $T_g G$. To see this, let $\gamma : (-\varepsilon, \varepsilon) \rightarrow G$ be a smooth path with $\gamma(0) = e$ and $\gamma'(0) = X$. Then

$$v_X(g) = \left. \frac{d}{dt} \right|_{t=0} m_g(\gamma(t)) = \left. \frac{d}{dt} \right|_{t=0} g\gamma(t) = g\gamma'(0) = gX$$

with $gX \in T_g G$. The vector field v_X is left-invariant since for $g, h \in G$ we have that

$$(dm_g)_e v_X(h) = gv_X(h) = ghX = v_X(gh) = v_X(m_g(h)).$$

On the other hand, if v is a left-invariant vector field on G , then $v(g) = (dm_g)_e v(e)$ for all $g \in G$. Thus v is determined by its value at e . If we denote the set of left-invariant vector fields on G by $\mathfrak{X}^L(G)$, then

$$\begin{aligned} \mathfrak{g} &\rightarrow \mathfrak{X}^L(G) \\ X &\mapsto v_X \end{aligned}$$

is a bijection with inverse given by $v \mapsto v(e)$, $v \in \mathfrak{X}^L(G)$. Under this identification, the Lie bracket on $\mathfrak{g}$ corresponds to the Lie bracket of vector fields.

By the Picard-Lindelöf Theorem, for any $X \in \mathfrak{g}$ there exists a unique differentiable map $\varphi : (-\varepsilon, \varepsilon) \rightarrow G$ satisfying $\varphi(0) = e$ and $\varphi'(t) = v_X(\varphi(t))$ for all $t \in (-\varepsilon, \varepsilon)$. In fact, the map φ is uniquely characterized by these two properties. We show that φ is a homomorphism whenever s, t and $s+t$ are in $(-\varepsilon, \varepsilon)$. Fixing s and letting t vary, consider the smooth paths $\alpha(t) = \varphi(s)\varphi(t)$ and $\beta(t) = \varphi(s+t)$. Note that $\alpha(0) = \varphi(s) = \beta(0)$. Moreover, we compute that

$$\alpha'(t) = \varphi(s)\varphi'(t) = \varphi(s)v_X(\varphi(t)) = \varphi(s)\varphi(t)X = v_X(\alpha(t))$$

and

$$\beta'(t) = \varphi'(s+t) = v_X(\varphi(s+t)) = v_X(\beta(t))$$

for all $t \in (-\varepsilon, \varepsilon)$. By uniqueness, we must have that $\alpha(t) = \beta(t)$ and so $\varphi(s)\varphi(t) = \varphi(s+t)$. We may then extend φ uniquely to a homomorphism

$$\varphi_X : \mathbb{R} \rightarrow G$$

with $\varphi'_X(t) = v_X(\varphi(t))$ for all $t \in \mathbb{R}$. The map φ_X is uniquely determined by the fact that it is a homomorphism from $\mathbb{R}$ to G with tangent vector $\varphi'_X(0) = X$.

If $\phi : \mathbb{R} \rightarrow G$ is a morphism of Lie groups, then we say that ϕ is a **one-parameter subgroup** of G . The analysis above shows that for any $X \in \mathfrak{g}$, the map φ_X is the unique one-parameter subgroup of G that satisfies $\varphi'_X(0) = X$. Conversely, if $\phi : \mathbb{R} \rightarrow G$ is a one-parameter subgroup of G , then $\phi(0) = e$. Thus, we may define a left-invariant vector field v on G by

$$v(e) = \phi'(0).$$

Thus, we have bijections between $\mathfrak{g}$ and $\mathfrak{X}^L(G)$ and the set of one-parameter subgroups of G .

For any $X \in \mathfrak{g}$, let φ_X be the unique one-parameter subgroup of G that satisfies $\varphi'_X(0) = X$. We define the **exponential map** of G by

$$\begin{aligned} \exp : \mathfrak{g} &\rightarrow G \\ X &\mapsto \varphi_X(1). \end{aligned}$$

Note that the exponential map is smooth. For any $\lambda \in \mathbb{R}$, consider the map $\tilde{\varphi}_\lambda : \mathbb{R} \rightarrow G$ defined by $\tilde{\varphi}_\lambda(t) = \varphi_X(\lambda t)$. Since $\tilde{\varphi}_\lambda$ is a morphism of Lie groups with $\tilde{\varphi}'_\lambda(0) = \lambda X$, by uniqueness of $\varphi_{\lambda X}$, we must have that $\varphi_X(\lambda t) = \varphi_{\lambda X}(t)$ for all $t \in \mathbb{R}$. It follows that

$$\exp(\lambda X) = \varphi_{\lambda X}(1) = \varphi_X(\lambda)$$

for all $\lambda \in \mathbb{R}$. This equality is equivalent to saying that the exponential map restricted to lines through the origin in $\mathfrak{g}$ gives one-parameter subgroups of G .

Example 2.2.18. [9, Section 8.3] Let F be either $\mathbb{R}$ or $\mathbb{C}$, and consider $G = \mathrm{GL}_n(F)$. For $X \in \mathfrak{g} = M_n(F)$, consider the morphism of Lie groups $\tilde{\varphi}_X : \mathbb{R} \rightarrow G$ defined by $\tilde{\varphi}_X(t) = e^{tX}$. Then

$$\tilde{\varphi}'_X(t) = X e^{tX}$$

and so $\tilde{\varphi}'_X(0) = X$. By uniqueness of φ_X , we must have that $\varphi_X(t) = e^{tX}$ for all $t \in \mathbb{R}$. It follows that

$$\exp(X) = \varphi_X(1) = e^X.$$

This implies that for any Lie group $H \subseteq \mathrm{GL}_n(F)$, its exponential map $\exp : \mathfrak{h} \rightarrow H$ is given by $\exp(X) = e^X, X \in \mathfrak{h}$. $\diamond$

Proposition 2.2.19. [9, Proposition 8.33] *The exponential map is the unique map from $\mathfrak{g}$ to G taking 0 to e whose differential at the origin*

$$(d\exp)_0 : T_0\mathfrak{g} = \mathfrak{g} \rightarrow T_e\mathfrak{g} = \mathfrak{g}$$

is the identity, and whose restrictions to the lines through the origin in $\mathfrak{g}$ are one-parameter subgroups of G .

Proof. The last statement is proven in the paragraph preceding Example 2.2.18. To show uniqueness, let $\psi : \mathfrak{g} \rightarrow G$ be another such map. By assumption, if $X \in \mathfrak{g}$ and $\lambda \in \mathbb{R}$, then $\psi(\lambda X) = \tilde{\psi}_X(\lambda)$ for some one-parameter subgroup $\tilde{\psi}_X : \mathbb{R} \rightarrow G$. We use uniqueness of φ_X to show that $\tilde{\psi}_X = \varphi_X$. Since $\tilde{\psi}_X$ is a homomorphism, it remains to show that $\tilde{\psi}'_X(0) = X$. Consider the homomorphism $f_X : \mathbb{R} \rightarrow \mathfrak{g}$ given by $f_X(\lambda) = \lambda X$. Since $\tilde{\psi}_X = \psi \circ f_X$ and $(d\psi)_0$ is the identity, we get that

$$(d\tilde{\psi}_X)_0 = d(\psi \circ f_X)_0 = (d\psi)_0 \circ (df_X)_0 = (df_X)_0 = f_X.$$

Thus

$$\tilde{\psi}'_X(0) = (d\tilde{\psi}_X)_0(1) = f_X(1) = X.$$

By uniqueness of φ_X , we must have that $\varphi_X = \tilde{\psi}_X$. Then

$$\psi(X) = \tilde{\psi}_X(1) = \varphi_X(1) = \exp(X),$$

which proves uniqueness of the exponential map.

Next, we show that $\exp(0) = e$. Consider the homomorphism $\psi : \mathbb{R} \rightarrow G$ defined by $\psi(t) = e$. Since ψ is constant, we get that $\psi'(0) = 0$. By uniqueness of φ_0 , we must have that $\varphi_0 = \psi$. Then $e = \psi(1) = \varphi_0(1) = \exp(0)$, as required.

It remains to show that $(d\exp)_0$ is the identity. Since $\varphi_X(t) = \exp(tX)$, we compute that

$$(d\exp)_0(X) = \left. \frac{d}{dt} \right|_{t=0} \exp(tX) = \left. \frac{d}{dt} \right|_{t=0} \varphi_X(t) = \varphi'_X(0) = X. \quad \square$$

Corollary 2.2.20. [9, Section 8.3] *The exponential map $\exp : \mathfrak{g} \rightarrow G$ is natural, in the sense that for any morphism $\rho : G \rightarrow H$ of Lie groups the diagram*

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{(d\rho)_e} & \mathfrak{h} \\ \exp \downarrow & & \downarrow \exp \\ G & \xrightarrow{\rho} & H \end{array}$$

commutes.

Proof. For $X \in \mathfrak{g}$ and $t \in \mathbb{R}$, we show that

$$\exp((d\rho)_e(tX)) = \rho(\exp(tX)).$$

The result then follows by taking $t = 1$. Note that

$$\exp((d\rho)_e(tX)) = \exp(t(d\rho)_e(X)) = \varphi_{(d\rho)_e(X)}(t)$$

is the unique one-parameter subgroup of H that satisfies $\varphi'_{(d\rho)_e(X)}(0) = (d\rho)_e(X)$. By uniqueness of $\varphi_{(d\rho)_e(X)}$, it suffices to show that $\sigma : \mathbb{R} \rightarrow H$ defined by $\sigma(t) = \rho(\exp(tX))$ is a homomorphism with $\sigma'(0) = (d\rho)_e(X)$. The map σ is a homomorphism since ρ and $t \mapsto \exp(tX)$ are. Since $\varphi_X(t) = \exp(tX)$, we compute that

$$\begin{aligned} \sigma'(0) &= \left. \frac{d}{dt} \right|_{t=0} \rho(\exp(tX)) \\ &= (d\rho)_{\exp(0)} \left(\left. \frac{d}{dt} \right|_{t=0} \exp(tX) \right) \\ &= (d\rho)_e \left(\left. \frac{d}{dt} \right|_{t=0} \varphi_X(t) \right) \\ &= (d\rho)_e(\varphi'_X(0)) \\ &= (d\rho)_e(X). \end{aligned}$$

We conclude that $\sigma(t) = \varphi_{(d\rho)_e(X)}(t)$ for all $t \in \mathbb{R}$. □

Proof of Proposition 2.2.4. [9, Section 8.3] Let $\rho : G \rightarrow H$ be a morphism of Lie groups. By Proposition 2.2.19, the differential $(d\exp)_0$ of $\exp : \mathfrak{g} \rightarrow G$ is the identity and hence a linear isomorphism. So, by the inverse function theorem for manifolds, there exists an open neighborhood $U \subset \mathfrak{g}$ of 0 such that $(\exp|_U)$ is a diffeomorphism and $\exp(U)$ is an open neighborhood of the identity $e \in G$. Now, assume that G is connected. Then by Proposition 2.2.2, the set $\exp(U)$ generates G as a group. Thus, by Corollary 2.2.20, if we understand $(d\rho)_e(U)$, then we know ρ in an open neighborhood of the identity, and hence everywhere. □

As shown in the proof of Proposition 2.2.4, the map $\exp : \mathfrak{g} \rightarrow G$ is a diffeomorphism on a small open neighborhood of $0 \in \mathfrak{g}$. Thus, for a small open neighborhood $U \subset G$ of $e \in G$, we define the **logarithm** map

$$\log : U \rightarrow \mathfrak{g}$$

to be the inverse of the exponential map. Then for $X, Y \in \mathfrak{g}$ close to zero, we define the bilinear operation

$$X * Y := \log(\exp(X) \exp(Y)).$$

This operation is only well-defined if X, Y are small and the logarithm is defined. In the case that G imbeds into $\mathrm{GL}(V)$, the logarithm map is

$$\log(g) = (g - I) - \frac{(g - I)^2}{2} + \frac{(g - I)^3}{3} - \dots$$

for $g \in G$ sufficiently close to $e \in G$.

Theorem 2.2.21 (Baker-Campbell-Hausdorff formula). *[9, Section 8.3] Let G be a Lie group that imbeds into $\mathrm{GL}(V)$. Given $X, Y \in \mathfrak{g}$, we have that*

$$X * Y = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}([X, [X, Y]] + \frac{1}{12}[Y, [X, Y]]) - \frac{1}{24}[Y, [X, [X, Y]]] \pm \dots$$

Proof. Using the formula

$$\exp(X) = 1 + X + \frac{X^2}{2} + \frac{X^3}{6} + \dots,$$

we can expand $X * Y$ into a power series and then group terms of the same degree. $\square$

Corollary 2.2.22. *[9, Proposition 8.41] Let G be a Lie group with Lie algebra $\mathfrak{g} = T_e G$, and $\mathfrak{h} \subset \mathfrak{g}$ a subspace. Then the subgroup of G generated by $\exp(\mathfrak{h})$ is a connected immersed Lie subgroup H of G with tangent space $T_e H = \mathfrak{h}$ if and only if $\mathfrak{h}$ is a Lie subalgebra.*

Proof. We provide a sketch of the proof, where more details can be found in [11, Theorem 5.20]. See [9, Theorem 19.26] for an alternative approach. One direction is immediate since if H is a Lie group, then $T_e H$ is a Lie algebra.

Conversely, suppose $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$. Since any finite-dimensional Lie algebra can be embedded in the Lie algebra $\mathfrak{gl}_n(\mathbb{R})$ [9, Appendix E], we may assume that G is a subgroup of $\mathrm{GL}_n(\mathbb{R})$. We begin by showing that H is path-connected. It suffices to show that for any $h \in H$, there exists a path joining h and the identity $e \in H$. If $h \in H$, then we may write

$$h = \exp(X_1) \cdots \exp(X_n)$$

for some $X_i \in \mathfrak{h}$. If $X \in \mathfrak{h}$, then $\exp(tX) \in \mathfrak{h}$ for all $t \in \mathbb{R}$. Thus, we may consider the path $\gamma : [0, 1] \rightarrow H$ defined by

$$\gamma(t) = \exp((1 - t)X_1) \cdots \exp((1 - t)X_n).$$

Since $\gamma(0) = h$ and $\gamma(1) = e$, we conclude that H is path-connected and hence connected.

Next, we show that $T_e H = \mathfrak{h}$. For $X \in \mathfrak{h}$, consider the smooth path $\gamma : (-\varepsilon, \varepsilon) \rightarrow H$ defined by $\gamma(t) = \exp(tX)$. Since $\gamma(0) = e$ and $\gamma'(0) = X$, we get that $X \in T_e H$. So, the inclusion $\mathfrak{h} \subset T_e H$ holds. It remains to show that $T_e H \subset \mathfrak{h}$. Since $\mathfrak{h}$ is a subspace of $\mathfrak{g}$, we may write $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp$, where $\mathfrak{h}^\perp$ is the orthogonal complement of $\mathfrak{h}$ with respect to the standard inner product on $\mathbb{R}^{n^2}$. Since $(d\exp)_0$ is a linear isomorphism, by the inverse function theorem for manifolds, there exists an open neighborhood $U \subset \mathfrak{g}$ of 0 such that $\exp|_U : U \rightarrow \exp(U)$ is a diffeomorphism and $\exp(U)$ is an open neighborhood of the identity $e \in G$. Next, let $Z \in T_e H$. Since $T_e H$ is the Lie algebra of H , we have that $\exp(tZ) \in H$ for all $t \in \mathbb{R}$. Since $U \subset \mathfrak{h} \oplus \mathfrak{h}^\perp$, for sufficiently small $t \in \mathbb{R}$, there exists open neighborhoods $W_1 \subset \mathfrak{h}$ and $W_2 \subset \mathfrak{h}^\perp$ of 0 such that

$$\exp(tZ) = \exp(X(t) + Y(t)) = \exp(X(t)) \exp(Y(t)),$$

where $X(t) \in W_1$ and $Y(t) \in W_2$ are continuous functions of t . Here, the second equality follows from Theorem 2.2.21. Since $X(t) \in \mathfrak{h}$, we get that $\exp(X(t)) \in H$. Thus, for small $t \in \mathbb{R}$, we must have that $\exp(Y(t)) \in H$. As argued in [11, Theorem 5.20], the function $Y(t)$ is constant. Since $Y(0) = 0$, we find that $Y(t) = 0$ for all t . So, for small t , we have that $\exp(tZ) = \exp(X(t))$. Since $\exp|_U$ is a diffeomorphism, we get that $tZ = X(t) \in \mathfrak{h}$ and so $Z \in \mathfrak{h}$. We conclude that $T_e H \subset \mathfrak{h}$. $\square$

Remark 2.2.23. The above corollary gives us a one-to-one correspondence between connected immersed Lie subgroups $H \subset G$ and Lie subalgebras $\mathfrak{h} \subset \mathfrak{g}$.

Lemma 2.2.24. *Let $\phi : G \rightarrow H$ be a Lie group morphism with G and H connected. If $(d\phi)_e : \mathfrak{g} \rightarrow \mathfrak{h}$ is a linear isomorphism then ϕ is a covering map.*

Proof. Note that ϕ is continuous since it is a Lie group morphism. We first show that ϕ is surjective. Since $(d\phi)_e$ is a linear isomorphism, by the inverse function theorem for manifolds, there exists an open neighborhood $U \subset G$ of $e \in G$ such that

$$\phi|_U : U \rightarrow \phi(U)$$

is a diffeomorphism and $\phi(U)$ is an open neighborhood of the identity $e \in H$. By Proposition 2.2.2, the set $\phi(U)$ generates H as a group. So, every element of H can be written as a finite product of elements in $\phi(U)$ and their inverses. Using the fact that $\phi|_U$ is surjective and ϕ is a group homomorphism, we get that ϕ is surjective.

Next, we show for each $h \in H$ that there exists an open neighborhood V of h such that $\phi^{-1}(V)$ is the union of disjoint open sets in G . By group invariance, it suffices to show this

property at the identity $e \in H$. Since $(d\phi)_e$ is a linear isomorphism, there exists an open neighborhood U of $e \in G$ such that ϕ maps U bijectively to an open neighborhood V of $e \in H$. For any $g \in G$, recall the diffeomorphism m_g from Lemma 2.2.1. Then for any $x \in \phi^{-1}(e)$, we have that

$$(\phi \circ m_x)(g) = \phi(xg) = \phi(x)\phi(g) = \phi(g).$$

So $\phi \circ m_x = \phi$ for all $g \in G$. It follows that

$$\phi^{-1}(V) = \bigcup_{x \in \phi^{-1}(e)} m_x(U),$$

where $m_x(U) := \{xu : u \in U\}$. Since m_x is a diffeomorphism, it is open. So $m_x(U)$ is open in G for all $x \in \phi^{-1}(e)$. It remains to show for $x, y \in \phi^{-1}(e)$ that if $x \neq y$, then $m_x(U) \cap m_y(U) = \emptyset$. We prove the contrapositive. If $m_x(U) \cap m_y(U) \neq \emptyset$, then there exists $u, u' \in U$ such that $xu = yu'$. It follows that $\phi(u) = \phi(x^{-1}yu') = \phi(u')$. Since $\phi|_U$ is injective, we get that $u = u'$ and so $x = y$. $\square$

Proof of Proposition 2.2.8. [9, Section 8.3] Let G and H be Lie groups, with G simply connected. By Proposition 2.2.6, if $\phi : \mathfrak{g} \rightarrow \mathfrak{h}$ is a linear map equal to the differential of some Lie group morphism, then ϕ is a Lie algebra morphism. Conversely, suppose that $\phi : \mathfrak{g} \rightarrow \mathfrak{h}$ is a Lie algebra morphism. Consider the Lie group $G \times H$ with Lie algebra $\mathfrak{g} \oplus \mathfrak{h}$. The Lie bracket for $\mathfrak{g} \oplus \mathfrak{h}$ is defined by

$$[(X, Y), (X', Y')] := ([X, X'], [Y, Y']).$$

Let

$$\Gamma_\phi := \{(X, Y) \in \mathfrak{g} \oplus \mathfrak{h} : \phi(X) = Y\}$$

be the graph of ϕ . If $(X, Y), (X', Y') \in \Gamma_\phi$ then

$$\phi([X, X']) = [\phi(X), \phi(X')] = [Y, Y']$$

and so $([X, X'], [Y, Y']) \in \Gamma_\phi$. Since Γ_ϕ is a subspace of $\mathfrak{g} \oplus \mathfrak{h}$ and is closed under the Lie bracket, it is a Lie subalgebra. Then, by Corollary 2.2.22, there exists a connected Lie subgroup $J \subset G \times H$ with tangent space $T_e J = \Gamma_\phi$. Next, define $p_1 : J \rightarrow G$ to be projection on the first factor. Since p_1 is a composition of Lie group morphisms, it is a Lie group morphism. We show that $(dp_1)_e : \Gamma_\phi \rightarrow \mathfrak{g}$ is a linear isomorphism. Given $(X, Y) \in \Gamma_\phi$, we can find smooth paths $\gamma_1 : (-\varepsilon, \varepsilon) \rightarrow G$ and $\gamma_2 : (-\varepsilon, \varepsilon) \rightarrow H$ initialized at X and Y , respectively. Then

$$(dp_1)_e(X, Y) = \left. \frac{d}{dt} \right|_{t=0} p_1(\gamma_1(t), \gamma_2(t)) = \left. \frac{d}{dt} \right|_{t=0} \gamma_1(t) = \gamma_1'(0) = X.$$

Thus $(dp_1)_e$ is projection on the first factor and hence injective. The map $(dp_1)_e$ is surjective since if $X \in \mathfrak{g}$, then $(X, \phi(X)) \in \Gamma_\phi$ with $(dp_1)_e(X, \phi(X)) = X$. We conclude that $(dp_1)_e$ is bijective and hence a linear isomorphism. Since Γ_ϕ and G are connected, we may apply Lemma 2.2.24 to get that $p_1 : J \rightarrow G$ is a covering map. Then since G is simply connected, the map p_1 is a Lie group isomorphism. It follows that $\varphi : G \rightarrow H$ defined by $\varphi = p_2 \circ p_1^{-1}$ is a Lie group morphism, where $p_2 : J \rightarrow H$ is projection on the second factor. Then for $X \in \mathfrak{g}$, we compute that

$$(d\varphi)_e(X) = ((dp_2)_e \circ (dp_1)_e^{-1})(X) = (dp_2)_e(X, \phi(X)) = \phi(X)$$

and so $(d\varphi)_e = \phi$. □

2.2.5 Semisimple Lie algebras

In this section we define semisimple Lie algebras. Semisimple Lie algebras are well behaved in the sense that many aspects of representation theory of finite groups still hold for them. For instance, every representation of a semisimple Lie algebra decomposes into a direct sum of irreducible representations. Moreover, every element of a semisimple Lie algebra $\mathfrak{g}$ has a decomposition, similar to that of a Jordan decomposition, which factors through any representation of $\mathfrak{g}$.

We start with some definitions that are analogous to those from group theory. If $H \subseteq \mathrm{GL}(V)$ is a Lie group, then its Lie bracket measures the extent to which matrix multiplication is commutative in $\mathfrak{h} = T_e H$. Precisely, if $X, Y \in \mathfrak{h}$ then X and Y commute with each other if and only if $[X, Y] = XY - YX = 0$. Given this, we say that a Lie algebra $\mathfrak{g}$ is **abelian** if $[X, Y] = 0$ for all $X, Y \in \mathfrak{g}$. Furthermore, we define the **center** of $\mathfrak{g}$ to be

$$Z(\mathfrak{g}) := \{X \in \mathfrak{g} : [X, Y] = 0 \text{ for all } Y \in \mathfrak{g}\}.$$

We say that a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is an **ideal** if $[X, Y] \in \mathfrak{h}$ for all $X \in \mathfrak{h}$ and $Y \in \mathfrak{g}$. Given two ideals $\mathfrak{h}$ and $\mathfrak{h}'$ of $\mathfrak{g}$, we define $[\mathfrak{h}, \mathfrak{h}']$ to be the subspace of $\mathfrak{g}$ generated by brackets of the form $[X, Y]$, where $X \in \mathfrak{h}$ and $Y \in \mathfrak{h}'$. The **derived series** $\{\mathcal{D}^k \mathfrak{g}\}$ of a Lie algebra $\mathfrak{g}$ is defined inductively by

$$\mathcal{D}^0 \mathfrak{g} = \mathfrak{g} \quad \text{and} \quad \mathcal{D}^k \mathfrak{g} = [\mathcal{D}^{k-1} \mathfrak{g}, \mathcal{D}^{k-1} \mathfrak{g}].$$

The derived series gives us a descending chain

$$\mathcal{D}^0 \mathfrak{g} \supset \mathcal{D}^1 \mathfrak{g} \supset \mathcal{D}^2 \mathfrak{g} \supset \dots$$

of ideals of $\mathfrak{g}$.

Proposition 2.2.25. [9, Section 9.1] If $\mathfrak{h}$ and $\mathfrak{h}'$ are ideals of a Lie algebra $\mathfrak{g}$, then $[\mathfrak{h}, \mathfrak{h}']$ is an ideal of $\mathfrak{g}$. It follows that each $\mathcal{D}^k \mathfrak{h}$ is an ideal of $\mathfrak{g}$.

Proof. We first show that $[\mathfrak{h}, \mathfrak{h}']$ is a Lie subalgebra of $\mathfrak{g}$. Since $\mathfrak{h}$ and $\mathfrak{h}'$ are ideals, we find that $[X, Y] \in \mathfrak{h} \cap \mathfrak{h}'$ for all $X \in \mathfrak{h}$ and $Y \in \mathfrak{h}'$. Then for $X, X' \in \mathfrak{h}$ and $Y, Y' \in \mathfrak{h}'$, we get that

$$[[X, Y], [X', Y']] \in [\mathfrak{h}, \mathfrak{h}'].$$

So $[\mathfrak{h}, \mathfrak{h}']$ is a Lie subalgebra of $\mathfrak{g}$. Next, let $X \in \mathfrak{g}, Y \in \mathfrak{h}$, and $Z \in \mathfrak{h}'$. By the Jacobi identity, we have that

$$[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]].$$

Since $\mathfrak{h}$ and $\mathfrak{h}'$ are ideals of $\mathfrak{g}$, we get that $[X, Y] \in \mathfrak{h}$ and $[X, Z] \in \mathfrak{h}'$. Then both terms on the right are in $[\mathfrak{h}, \mathfrak{h}']$. We conclude that $[X, [Y, Z]] \in [\mathfrak{h}, \mathfrak{h}']$ and so $[\mathfrak{h}, \mathfrak{h}']$ is an ideal of $\mathfrak{g}$. $\square$

We say that an ideal $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is **solvable** if $\mathcal{D}^k \mathfrak{h} = 0$ for some k . Finally, we say that a Lie algebra $\mathfrak{g}$ is **semisimple** if $\mathfrak{g}$ has no nonzero solvable ideals.

Example 2.2.26. [9, Section 9.3] For $n \geq 1$, the Lie algebras $\mathfrak{sl}_n(F)$ and $\mathfrak{sp}_n(F)$ are semisimple. If $n \neq 2$, then $\mathfrak{so}_n(F)$ is semisimple. $\diamond$

We show that a Lie algebra is semisimple if and only if it has no nonzero abelian ideals. To do this, we introduce the *radical* of a Lie algebra. First note that the sum of two solvable ideals is solvable. The **radical** of a Lie algebra $\mathfrak{g}$ is the unique maximal solvable ideal of $\mathfrak{g}$ given by

$$\text{rad}(\mathfrak{g}) := \sum_{\mathfrak{h} \text{ solvable}} \mathfrak{h}.$$

Note that $\text{rad}(\mathfrak{g}) \neq \emptyset$ since $0 \subset \mathfrak{g}$ is a solvable ideal.

Proposition 2.2.27. [9, Section 9.1] A Lie algebra $\mathfrak{g}$ is semisimple if and only if it has no nonzero abelian ideals.

Proof. By maximality of $\text{rad}(\mathfrak{g})$, the condition $\text{rad}(\mathfrak{g}) = 0$ is equivalent to the statement that $\mathfrak{g}$ has no nonzero solvable ideals. So, we show that $\text{rad}(\mathfrak{g}) = 0$ if and only if $\mathfrak{g}$ has no nonzero abelian ideals. Suppose $\text{rad}(\mathfrak{g}) = 0$ and let $\mathfrak{h}$ be an abelian ideal of $\mathfrak{g}$. Then $\mathfrak{h}$ is solvable since $\mathcal{D}^1 \mathfrak{h} = [\mathfrak{h}, \mathfrak{h}] = 0$. Since $\mathfrak{h}$ is solvable and $\text{rad}(\mathfrak{g}) = 0$, we must have that

$\mathfrak{h} = 0$. Conversely, suppose that $\mathfrak{g}$ has no nonzero abelian ideals. Since $\text{rad}(\mathfrak{g})$ is solvable, we have that $\mathcal{D}^k \text{rad}(\mathfrak{g}) = 0$ for some $k \in \mathbb{N}$. Then

$$0 = \mathcal{D}^k \text{rad}(\mathfrak{g}) = [\mathcal{D}^{k-1} \text{rad}(\mathfrak{g}), \mathcal{D}^{k-1} \text{rad}(\mathfrak{g})]$$

and so $\mathcal{D}^{k-1} \text{rad}(\mathfrak{g})$ is abelian. By our hypothesis, we must have that $\mathcal{D}^{k-1} \text{rad}(\mathfrak{g}) = 0$. Then, by the same argument, we get that $\mathcal{D}^{k-2} \text{rad}(\mathfrak{g}) = 0$. Continuing this process, we find that $\text{rad}(\mathfrak{g}) = \mathcal{D}^0 \text{rad}(\mathfrak{g}) = 0$. $\square$

Remark 2.2.28. Proposition 2.2.27 implies that a semisimple Lie algebra $\mathfrak{g}$ has trivial center. It follows that the adjoint representation $\text{ad} : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$ is faithful since $\ker(\text{ad}) = Z(\mathfrak{g}) = 0$.

Theorem 2.2.29. [9, Theorem 9.19] *Let V be a representation of a semisimple Lie algebra $\mathfrak{g}$. If $W \subset V$ is a subspace invariant under $\mathfrak{g}$, then there exists a subspace $W' \subset V$ complementary to W and invariant under $\mathfrak{g}$.*

Remark 2.2.30. Theorem 2.2.29 implies that $V \cong W \oplus W'$ as representations of $\mathfrak{g}$. We can continue applying Theorem 2.2.29 until V is isomorphic to a (finite) direct sum of irreducible representations of $\mathfrak{g}$.

Theorem 2.2.31. [9, Theorem 9.20] *Let $\mathfrak{g}$ be a semisimple Lie algebra. For any $X \in \mathfrak{g}$, there exists $X_s, X_n \in \mathfrak{g}$ such that $X = X_s + X_n$. Moreover, for any representation $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ we have that*

$$\rho(X)_s = \rho(X_s) \quad \text{and} \quad \rho(X)_n = \rho(X_n),$$

where $\rho(X)_s$ and $\rho(X)_n$ denote respectively the semisimple (i.e., diagonalizable) and nilpotent parts of the Jordan decomposition of $\rho(X)$.

Corollary 2.2.32. *Let $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ be a representation of the semisimple Lie algebra $\mathfrak{g}$. If $\psi : \mathfrak{g} \rightarrow \mathfrak{gl}(W)$ is a faithful representation and $X \in \mathfrak{g}$ such that $\psi(X)$ is diagonalizable, then $\rho(X)$ is diagonalizable.*

Proof. By Theorem 2.2.31, there exists $X_s, X_n \in \mathfrak{g}$ such that

$$\psi(X)_s = \psi(X_s) \quad \text{and} \quad \psi(X)_n = \psi(X_n)$$

with $X = X_s + X_n$. By linearity of ψ , we get that $\psi(X) = \psi(X)_s + \psi(X)_n$. Since $\psi(X)$ is diagonalizable, we must have that $\psi(X)_n = \psi(X)_n = 0$. By injectivity of ψ , we get that $X_n = 0$. It follows that $\rho(X) = \rho(X_s) = \rho(X)_s$ and so $\rho(X)$ is diagonalizable. $\square$

2.3 Representations of $\mathfrak{sl}_2$

2.3.1 Irreducible representations of $\mathfrak{sl}_2$

We start our analysis of finite-dimensional representations of semisimple Lie algebras by studying the irreducible representations of $\mathfrak{sl}_2$. In this section we completely classify the irreducible representations, and hence all representations, of $\mathfrak{sl}_2$. We show that $\mathfrak{sl}_2$ has a unique irreducible representation in every positive dimension. The study of $\mathfrak{sl}_2$ sets a framework for the classification of representations of $\mathfrak{sl}_n$ for higher $n \in \mathbb{N}$.

It will be helpful to work with the basis

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

for $\mathfrak{sl}_2$, that satisfies

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H.$$

Now let V be *any* representation of $\mathfrak{sl}_2$. The standard representation $\rho : \mathfrak{sl}_2 \rightarrow \mathfrak{gl}_2(\mathbb{C})$ of $\mathfrak{sl}_2$ is defined by $\rho(X) = X$. Since H is diagonal, it acts diagonally on the faithful standard representation. Thus, by Corollary 2.2.32, the action of H on V is diagonalizable. So, we get a decomposition

$$V = \bigoplus_{\beta \in \mathbb{C}} V_\beta,$$

where

$$V_\beta = \{v \in V : H(v) = \beta v\}.$$

In other words, V decomposes into eigenspaces for the action of H . Note that β is an eigenvalue for H if and only if $V_\beta \neq 0$. Given $v \in V_\beta$, we would like to know where $X(v)$ and $Y(v)$ land in the decomposition of V . Since the decomposition of V is obtained by the action of H , we just need to figure out where H sends $X(v)$ and $Y(v)$. Using the equality

$$[H, X](v) = H(X(v)) - X(H(v)),$$

we get that

$$\begin{aligned} H(X(v)) &= X(H(v)) + [H, X](v) \\ &= X(\beta v) + 2X(v) \\ &= (\beta + 2)X(v). \end{aligned}$$

This computation tells us that $X(v)$ is an eigenvector for H with eigenvalue $\beta+2$. Similarly,

$$\begin{aligned} H(Y(v)) &= Y(H(v)) + [H, Y](v) \\ &= Y(\beta v) - 2Y(v) \\ &= (\beta - 2)Y(v) \end{aligned}$$

tells us that $Y(v)$ is an eigenvector for H with eigenvalue $\beta - 2$. We summarize these facts with the following proposition.

Proposition 2.3.1. *[9, Section 11.1] Let V be a representation of $\mathfrak{sl}_2$. The action of X on V sends V_β to $V_{\beta+2}$. The action of Y on V sends V_β to $V_{\beta-2}$.*

For the remainder of this section, unless stated otherwise, V is a fixed finite-dimensional irreducible representation of $\mathfrak{sl}_2$.

Corollary 2.3.2. *[9, Section 11.1] The eigenvalues for H of an irreducible representation V of $\mathfrak{sl}_2$ form an unbroken string of complex numbers congruent to one another modulo 2.*

Proof. If β is an eigenvalue for H , then we can find an unbroken string of (nonzero) eigenspaces $V_{\beta-2k}$, where $\ell_1 \leq k \leq \ell_2$ for some $\ell_1, \ell_2 \in \mathbb{Z}$ satisfying $\ell_1 \leq 0 \leq \ell_2$. Proposition 2.3.1, together with the fact that each eigenspace $V_{\beta-2k}$ is invariant under H , shows that

$$\bigoplus_{k=\ell_1}^{\ell_2} V_{\beta-2k} \subseteq V$$

is invariant under $\mathfrak{sl}_2$. Since V is irreducible, the only subspaces of V that are invariant under the action of $\mathfrak{sl}_2$ are 0 and V itself. So, by irreducibility of V , we get that

$$V = \bigoplus_{k=\ell_1}^{\ell_2} V_{\beta-2k}.$$

We conclude that the eigenvalues for H are exactly $\beta - 2k$ for $\ell_1 \leq k \leq \ell_2$. □

Throughout this section, let n denote the eigenvalue for the action of H on V whose real part is maximized. Then $V_{n-2k} = 0$ for all integers $k < 0$. Thus, by Corollary 2.3.2, we may write

$$V = \bigoplus_{k \in \mathbb{N}} V_{n-2k}. \tag{2.3}$$

Note that $X(V_n) = 0$ since $V_{n+2} = 0$.

Proposition 2.3.3. [9, Claim 11.4] *Let V be an irreducible representation of $\mathfrak{sl}_2$, and $v \in V_n$ be any nonzero vector. Then $\{v, Y(v), Y^2(v), \dots\}$ span V . In particular, the set of vectors $Y^k(v)$ which are nonzero are linearly independent, and hence form a basis for V .*

Proof. For each $k \in \mathbb{N}$, we have that $Y^k(v) \in V_{n-2k}$. So, if $Y^k(v)$ is nonzero, then it is an eigenvector with eigenvalue $n - 2k$. It follows that the vectors $Y^k(v)$ which are nonzero are eigenvectors for different eigenvalues, and hence are linearly independent. Let

$$W := \text{span}_{\mathbb{C}}\{v, Y(v), Y^2(v), \dots\} \subseteq V.$$

Since $W \neq 0$ and V is irreducible, it suffices to show that W is invariant under $\mathfrak{sl}_2$. Invariance under Y and H follows since

$$Y(Y^k(v)) = Y^{k+1}(v)$$

and

$$H(Y^k(v)) = (n - 2k)Y^k(v).$$

We do induction on k to show that

$$X(Y^k(v)) = k(n - k + 1)Y^{k-1}(v). \quad (2.4)$$

Since $X(v) = 0$, Equation (2.4) holds for $k = 0$. Suppose that the formula holds for $k - 1$. Then

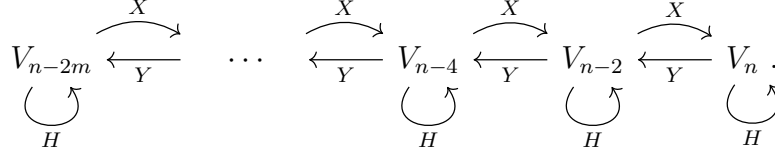
$$\begin{aligned} X(Y^k(v)) &= Y(X(Y^{k-1}(v))) + [X, Y]Y^{k-1}(v) \\ &= Y((k - 1)(n - (k - 1) + 1)Y^{k-2}(v)) + H(Y^{k-1}(v)) \\ &= (k - 1)(n - k + 2)Y^{k-1}(v) + (n - 2(k - 1))Y^{k-1}(v) \\ &= k(n - k + 1)Y^{k-1}(v). \end{aligned}$$

This shows that Equation (2.4) holds for all k , and hence that X is invariant under $\mathfrak{sl}_2$. $\square$

Remark 2.3.4. Since V is finite-dimensional, we must have that $Y^k(v) = 0$ for sufficiently large k . We can choose $m \in \mathbb{N}$ so that $m + 1$ is the minimal power of Y that annihilates v . Then

$$V = \text{span}_{\mathbb{C}}\{v, Y(v), Y^2(v), \dots, Y^m(v)\},$$

and the action of $\mathfrak{sl}_2$ on V can be visualized with the following diagram:



Corollary 2.3.5. [9, Section 11.1] *Let V be an irreducible representation of $\mathfrak{sl}_2$, and $v \in V_n$ be any nonzero vector. The eigenspaces V_{n-2k} are one dimensional with $V_{n-2k} = \text{span}_{\mathbb{C}}\{Y^k(v)\}$. It follows that the eigenvalues for the action of H on V each have multiplicity 1. Moreover, if $m+1$ is the minimal power of Y that annihilates v , then the set of eigenvalues for the action of H on V is $\{n-2k : 0 \leq k \leq m\}$.*

Proof. The first statement follows immediately from Remark 2.3.4 and the fact that the vectors $Y^k(v)$ live in distinct eigenspaces. Since $m+1$ is the minimal power of Y that annihilates v , we get that $Y^k(v) = 0$ for all $k \geq m+1$. By the first statement, we get that

$$V_{n-2k} = \text{span}_{\mathbb{C}}\{Y^k(v)\} = 0$$

for all $k \geq m+1$. Then Equation (2.3) becomes

$$V = \bigoplus_{k=0}^m V_{n-2k}.$$

Thus, the eigenvalues for H are $\{n-2k : 0 \leq k \leq m\}$. □

Corollary 2.3.6. [9, Section 11.1] *Let V be an irreducible representation of $\mathfrak{sl}_2$. Then the eigenvalues for the action of H on V are integral and symmetric about 0. In particular, the maximal eigenvalue n is a natural number.*

Proof. Let $m+1$ be the minimal power of Y that annihilates v . Then $Y^m(v) \neq 0$. By Equation (2.4), we may write

$$0 = X(Y^{m+1}(v)) = (m+1)(n-m)Y^m(v).$$

Since $m+1 \neq 0$ and $Y^m(v) \neq 0$, we must have that $n-m=0$. Then $n=m$ and so $n \in \mathbb{N}$. By Corollary 2.3.5, the eigenvalues for H are $n-2k$, where $0 \leq k \leq n$. □

Remark 2.3.7. By Corollary 2.3.6, we may write

$$V = V_n \oplus V_{n-2} \oplus \cdots \oplus V_{-n+2} \oplus V_{-n},$$

where $n \in \mathbb{N}$ is the maximum eigenvalue for the action of H on V . Since each eigenspace is one-dimensional, we have that $\dim V = n+1$. Theorem 2.2.29 and Corollary 2.3.6 together imply that the eigenvalues for the action of H on *any* representation of $\mathfrak{sl}_2$ are integral and symmetric about 0.

Corollary 2.3.8. [9, Section 11.1] Any representation W of $\mathfrak{sl}_2$ whose eigenvalues for H all have the same parity and occur with multiplicity one is irreducible.

Proof. By Theorem 2.2.29, we may write

$$W = \bigoplus W^k,$$

where each W^k is an irreducible representation of $\mathfrak{sl}_2$. Assume for contradiction that W is not irreducible. Then there are at least two nonzero irreducible representations, say W^k and W^ℓ , in the direct sum decomposition of W . By Corollary 2.3.6, the H eigenvalues of W^k and W^ℓ are respectively of the form

$$n_k, n_k - 2, \dots, -n_k + 2, -n_k$$

and

$$n_\ell, n_\ell - 2, \dots, -n_\ell + 2, -n_\ell$$

for some $n_k, n_\ell \in \mathbb{N}$. By assumption, these two strings of eigenvalues are of the same parity, and hence have non-empty intersection. But, this contradicts the fact that each eigenvalue occurs with multiplicity one. So, W must be irreducible. $\square$

We are almost ready to prove the classification theorem for $\mathfrak{sl}_2$ which states that for each $n \in \mathbb{N}$, there is a unique irreducible representation of $\mathfrak{sl}_2$. This representation is $(n+1)$ -dimensional and the eigenvalues for H are $n - 2k$, where $0 \leq k \leq n$. Before we prove this theorem, let's look at some examples.

Example 2.3.9. [9, Section 11.1] Consider the trivial one-dimensional representation

$$\begin{aligned} \mathfrak{sl}_2 &\rightarrow \mathfrak{gl}(\mathbb{C}) \\ g &\mapsto 0. \end{aligned}$$

Then $H(v) = 0$ for all $v \in \mathbb{C}$. So, the only eigenvalue for H is 0. Note that $\mathbb{C} \cong \text{Sym}^0 V$ as representations, where $V \cong \mathbb{C}^2$ is the standard representation of $\mathfrak{sl}_2$. $\diamond$

Example 2.3.10. [9, Section 11.1] Let $V \cong \mathbb{C}^2$ be the standard representation of $\mathfrak{sl}_2$ with standard basis

$$x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then $H(x) = x$ and $H(y) = -y$. So, the eigenvalues for H are 1 and -1 . Then $V_1 \cong \mathbb{C}x$ and $V_{-1} \cong \mathbb{C}y$. It follows that

$$V \cong \mathbb{C}x \oplus \mathbb{C}y \cong V_1 \oplus V_{-1}.$$

Note that $V = \text{Sym}^1 V$ as representations. $\diamond$

Example 2.3.11. [9, Section 11.1] Consider the adjoint representation $\text{ad} : \mathfrak{sl}_2 \rightarrow \mathfrak{gl}(\mathfrak{sl}_2)$. Then

$$\begin{aligned}\text{ad}(H)(X) &= [H, X] = 2X \\ \text{ad}(H)(Y) &= [H, Y] = -2Y \\ \text{ad}(H)(H) &= [H, H] = 0.\end{aligned}$$

So, the eigenvalues for H are $2, 0, -2$. Now consider $\text{Sym}^2 V$, where V is the standard representation of $\mathfrak{sl}_2$. If $\{x, y\}$ is the standard basis for V , then $\{x^2, xy, y^2\}$ is a basis for $\text{Sym}^2 V$. Then

$$H(x^2) = 2x^2, \quad H(xy) = 0, \quad H(y^2) = -2y^2.$$

So, the eigenvalues of H are $2, 0, -2$. Thus, the eigenvalues for H are the same for both of these representations. In fact, $\mathfrak{sl}_2 \cong \text{Sym}^2 V$ as representations. $\diamond$

The main observation to take away from the above examples is the fact that $\text{Sym}^0 V, \text{Sym}^1 V$, and $\text{Sym}^2 V$ are all irreducible representations of $\mathfrak{sl}_2$ whose dimensions and eigenvalues align with that of the classification theorem. This observation leads us to the following proposition.

Proposition 2.3.12. [9, Section 11.1] *Let V be the standard representation of $\mathfrak{sl}_2$. For each $n \in \mathbb{N}$, the n th symmetric power $\text{Sym}^n V$ of V is an irreducible representation of $\mathfrak{sl}_2$. The representation $\text{Sym}^n V$ is $(n + 1)$ -dimensional and the eigenvalues for H are $n - 2k$, where $0 \leq k \leq n$ (each of multiplicity one).*

Proof. Let $\{x, y\}$ be the standard basis for V . Then $\text{Sym}^n V$ has basis

$$\{x^{n-k}y^k : 0 \leq k \leq n\}.$$

So $\text{Sym}^n V$ is an $(n + 1)$ -dimensional representation of $\mathfrak{sl}_2$. We compute that

$$\begin{aligned}H(x^{n-k}y^k) &= H(x^{n-k})y^k + x^{n-k}H(y^k) \\ &= (n - k)x^{n-k}y^k - kx^{n-k}y^k \\ &= (n - 2k)x^{n-k}y^k.\end{aligned}$$

So, the eigenvalues for H are exactly $n - 2k$ for $0 \leq k \leq n$. Since $\text{Sym}^n V$ has a basis consisting of eigenvectors with distinct eigenvalues, the eigenspaces must be one-dimensional. We may then apply Corollary 2.3.8 to deduce that $\text{Sym}^n V$ is irreducible. $\square$

Theorem 2.3.13. [9, Section 11.1] *For each $n \in \mathbb{N}$, there is a unique irreducible representation of $\mathfrak{sl}_2$; the representation is $(n+1)$ -dimensional and the eigenvalues for H are $n-2k$, where $0 \leq k \leq n$ (each of multiplicity one).*

Proof. Existence follows immediately from Proposition 2.3.12. It remains to show uniqueness. Let V and V' be two irreducible representations of dimension $n+1$. We must show that $V \cong V'$ as representations. The representations V and V' have eigenspace decompositions

$$V = \bigoplus_{k=0}^n V_{n-2k}, \quad V' = \bigoplus_{k=0}^n V'_{n-2k}.$$

We can pick nonzero vectors $v \in V_n$ and $w \in V'_n$. Then $\{v, Y(v), Y^2(v), \dots, Y^n(v)\}$ is a basis for V , and $\{w, Y(w), Y^2(w), \dots, Y^n(w)\}$ is a basis for V' . Define

$$\begin{aligned} f : V &\rightarrow V' \\ Y^m(v) &\mapsto Y^m(w), \end{aligned}$$

and extend by linearity. The map f gives the desired isomorphism. $\square$

Remark 2.3.14. Our analysis above, together with Theorem 2.3.13, gives a bijection between irreducible representations of $\mathfrak{sl}_2$ and the natural numbers. In particular, we showed that any irreducible representation of $\mathfrak{sl}_2$ is a symmetric power of the standard representation. This bijection can also be rephrased in terms of the maximum eigenvalue for H . That is, for each $n \in \mathbb{N}$, there is a unique irreducible representation whose maximum eigenvalue for H is n .

2.3.2 Examples of representations of $\mathfrak{sl}_2$

We just learnt that the building blocks of representations of $\mathfrak{sl}_2$ are symmetric powers of the standard representation of $\mathfrak{sl}_2$. So, by Theorem 2.2.29, every representation of $\mathfrak{sl}_2$ is of the form

$$\mathrm{Sym}^{a_1} V \oplus \mathrm{Sym}^{a_2} V \oplus \dots \oplus \mathrm{Sym}^{a_k} V$$

for some $a_i \in \mathbb{N}$. For example, consider representations of the form

$$\mathrm{Sym}^a V \otimes \mathrm{Sym}^b V$$

for some $a, b \in \mathbb{N}$. We can ask whether $\mathrm{Sym}^a V \otimes \mathrm{Sym}^b V$ is irreducible, and, if not, how it decomposes into irreducible representations. We will answer this question shortly, after doing some explicit examples.

Example 2.3.15. [9, Section 11.2] Let W and W' be representations of $\mathfrak{sl}_2$. Suppose

$$W = \bigoplus W_\alpha \quad \text{and} \quad W' = \bigoplus W'_\beta$$

are the H eigenspace decompositions. If $w \in W_\alpha$ and $w' \in W'_\beta$, then

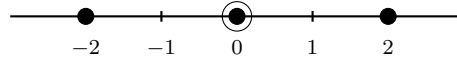
$$\begin{aligned} H(w \otimes w') &= H(w) \otimes w' + w \otimes H(w') \\ &= \alpha w \otimes w' + w \otimes \beta w' \\ &= (\alpha + \beta)(w \otimes w'). \end{aligned}$$

It follows that the H eigenspace decomposition of $W \otimes W'$ is

$$W \otimes W' = \bigoplus (W_\alpha \otimes W'_\beta),$$

where $W_\alpha \otimes W'_\beta$ is the eigenspace corresponding to eigenvalue $\alpha + \beta$. $\diamond$

Example 2.3.16. [9, Section 11.2] Let V be the standard representation of $\mathfrak{sl}_2$, and consider $V \otimes V$. The eigenvalues of V are $\{1, -1\}$. So, the eigenvalues of $V \otimes V$ are $\{2, 0, -2\}$, with 0 having multiplicity two. We represent the eigenvalues for the action of H on $V \otimes V$ with the following diagram:



Since the eigenvalues for H of an irreducible representation of $\mathfrak{sl}_2$ have multiplicity one, we know that $V \otimes V$ must be reducible. In this case, the only way to decompose $V \otimes V$ into irreducibles is

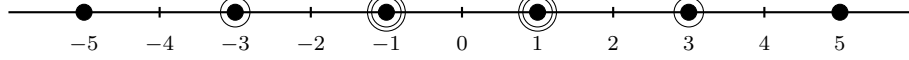
$$V \otimes V \cong \text{Sym}^2 V \oplus \text{Sym}^0 V.$$

Note that $\text{Sym}^0 V \cong \mathbb{C}$ is just the trivial representation. $\diamond$

Example 2.3.17. [9, Section 11.2] Consider the representation

$$\text{Sym}^2 V \otimes \text{Sym}^3 V,$$

where V is the standard representation of $\mathfrak{sl}_2$. The eigenvalues of $\text{Sym}^2 V$ are $\{2, 0, -2\}$, and the eigenvalues of $\text{Sym}^3 V$ are $\{3, 1, -1, -3\}$. So, the eigenvalues of $\text{Sym}^2 V \otimes \text{Sym}^3 V$ are $\{5, 3, 1, -1, -3, -5\}$. Here, 3 and -3 each have multiplicity two, while 1 and -1 each have multiplicity three. The diagram in this case is:



Thus, we get the decomposition

$$\mathrm{Sym}^2 V \otimes \mathrm{Sym}^3 V \cong \mathrm{Sym}^5 V \oplus \mathrm{Sym}^3 V \oplus \mathrm{Sym}^1 V.$$

◇

Remark 2.3.18. In general, for $a \geq b$ we have that

$$\mathrm{Sym}^a V \otimes \mathrm{Sym}^b V \cong \mathrm{Sym}^{a+b} V \oplus \mathrm{Sym}^{a+b-2} V \oplus \cdots \oplus \mathrm{Sym}^{a-b} V.$$

From this formula, we deduce that $\mathrm{Sym}^a V \otimes \mathrm{Sym}^b V$ is irreducible if and only if $a = 0$ or $b = 0$.

2.4 Representations of $\mathfrak{sl}_3$

2.4.1 The structure of $\mathfrak{sl}_3$

As to be illustrated through the study of $\mathfrak{sl}_3$, the general procedure for classifying irreducible representations of $\mathfrak{sl}_2$ can be naturally extended to $\mathfrak{sl}_n$ for higher $n \in \mathbb{N}$. In addition to using the $\mathfrak{sl}_2$ case as a framework, we rely heavily on explicit $\mathfrak{sl}_2$ results to classify the representations of $\mathfrak{sl}_n$. More precisely, there exists subalgebras of $\mathfrak{sl}_n$ that are isomorphic to $\mathfrak{sl}_2$. One of the techniques we use in the classification of representations of $\mathfrak{sl}_n$ is to restrict to these subalgebras, and then apply results from Section 2.3.1 to them. We now focus our attention to the study of $\mathfrak{sl}_3$, as it is the smallest example which shows exactly how to generalize to $\mathfrak{sl}_n$. At the end of this chapter, we arrive at a complete classification of the irreducible representations of $\mathfrak{sl}_3$.

We begin our analysis by choosing an appropriate basis for $\mathfrak{sl}_3$. To start, we find the analogue of the matrix

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

for $\mathfrak{sl}_3$. One key observation to make about H is that it is diagonalizable in any representation of $\mathfrak{sl}_2$. Also, H is a basis for the subalgebra of $\mathfrak{sl}_2$ consisting of (traceless) diagonal matrices. Given these observations, a natural replacement for H is the subalgebra

$$\mathfrak{h} := \left\{ \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix} : a_i \in \mathbb{C}, a_1 + a_2 + a_3 = 0 \right\} \subset \mathfrak{sl}_3.$$

Note that $\dim \mathfrak{h} = 2$ and a basis for $\mathfrak{h}$ is

$$\begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & & \\ & 1 & \\ & & -1 \end{pmatrix}.$$

The reason $\mathfrak{h}$ is the generalization we want, boils down to the fact that commuting diagonalizable matrices are simultaneously diagonalizable. Applying this fact to $\mathfrak{h}$, we find that the action of $\mathfrak{h}$ on the standard representation of $\mathfrak{sl}_3$ is diagonalizable. Then, by Corollary 2.2.32, we get that any representation of $\mathfrak{sl}_3$ admits an *eigenspace* decomposition from the action of $\mathfrak{h}$. Let's introduce some terminology before making this decomposition more explicit.

Let V be *any* representation of $\mathfrak{sl}_3$. We say that $v \in V$ is an **eigenvector** for $\mathfrak{h}$ if there exists $\beta \in \mathfrak{h}^*$ such that

$$H(v) = \beta(H)v \quad (2.5)$$

for all $H \in \mathfrak{h}$. So, an eigenvector for $\mathfrak{h}$ is an eigenvector (in the classical sense) for every $H \in \mathfrak{h}$. We say that $\beta \in \mathfrak{h}^*$ is an **eigenvalue** for $\mathfrak{h}$ if there exists a nonzero vector $v \in V$ satisfying Equation (2.5). Lastly, given an eigenvalue $\beta \in \mathfrak{h}^*$, we say that

$$V_\beta := \{v \in V : H(v) = \beta(H)v \text{ for all } H \in \mathfrak{h}\}$$

is the **eigenspace** for $\mathfrak{h}$ with eigenvalue β . Using the above terminology, since the action of $\mathfrak{h}$ on V is diagonalizable, we get an eigenspace decomposition

$$V = \bigoplus_{\beta \in \mathfrak{h}^*} V_\beta, \quad (2.6)$$

where β ranges over a finite subset of $\mathfrak{h}^*$. The next step is to find analogues of the matrices

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

for $\mathfrak{sl}_3$. Recall that X, Y , and H satisfy the relations

$$[H, X] = 2X \quad \text{and} \quad [H, Y] = -2Y.$$

Exploiting our new terminology, we may interpret these commutation relations as saying that X and Y are eigenvectors for the adjoint action of H on $\mathfrak{sl}_2$. Indeed, as in Example 2.3.11, the adjoint action of H on $\mathfrak{sl}_2$ gives the eigenspace decomposition

$$\mathfrak{sl}_2 \cong \mathbb{C}H \oplus \mathbb{C}X \oplus \mathbb{C}Y,$$

where $\mathbb{C}H$, $\mathbb{C}X$, and $\mathbb{C}Y$ are the eigenspaces corresponding to the eigenvalues 0, 2, and -2 , respectively. Given this observation, we want the analogues of X and Y for $\mathfrak{sl}_3$ to be eigenvectors (in the new sense) for the adjoint action of $\mathfrak{h}$ on $\mathfrak{sl}_3$. Applying Equation (2.6) to the adjoint representation of $\mathfrak{sl}_3$, we obtain an eigenspace decomposition

$$\mathfrak{sl}_3 \cong \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha$ is the eigenspace for $\mathfrak{h}$ with corresponding eigenvalue α . We simplify this decomposition with the following proposition.

Proposition 2.4.1. *The zero linear functional in $\mathfrak{h}^*$ is an eigenvalue for the adjoint action of $\mathfrak{h}$ on $\mathfrak{sl}_3$ with corresponding eigenspace equal to $\mathfrak{h}$.*

Proof. Let $0 : \mathfrak{h} \rightarrow \mathbb{C}$ be the zero linear functional. Since $\mathfrak{h}$ is abelian, we find that $\mathfrak{h} \subseteq \mathfrak{g}_0$. Because $\mathfrak{h} \neq 0$, the inclusion $\mathfrak{h} \subseteq \mathfrak{g}_0$ shows that 0 is an eigenvalue for $\mathfrak{h}$. For the other inclusion, let $X \in \mathfrak{g}_0$. Then $[H, X] = 0$ for all $H \in \mathfrak{h}$. Taking $H = \text{diag}(1, -1, 0)$ and setting $[H, X] = 0$, we find that X is necessarily diagonal. Hence, $\mathfrak{g}_0 \subseteq \mathfrak{h}$. $\square$

By Proposition 2.4.1, the eigenspace decomposition of $\mathfrak{sl}_3$ becomes

$$\mathfrak{sl}_3 \cong \mathfrak{h} \oplus \left(\bigoplus_{\substack{\alpha \in \mathfrak{h}^* \\ \alpha \neq 0}} \mathfrak{g}_\alpha \right).$$

We find an explicit description for the eigenspaces $\mathfrak{g}_\alpha$. Let $D = \text{diag}(a_1, a_2, a_3)$ and denote the (i, j) entry of $M \in M_3(\mathbb{C})$ by m_{ij} . Then $[D, M]$ is the matrix whose (i, j) entry is $(a_i - a_j)m_{ij}$. So $[D, M]$ is a multiple of M for all diagonal D if and only if all but one entry of M is 0. Note that the diagonal entries of $[D, M]$ are 0. Thus, if $[D, M]$ is a multiple of M , then the diagonal entries of M are also 0. It follows in this case, that the nonzero entry of M must be off of the diagonal. Set $E_{ij} \in M_3(\mathbb{C})$ to be the matrix whose entries are all 0 except in the (i, j) entry, where it is 1. Then the eigenvectors for $\mathfrak{h}$ are scalar multiples of the matrices E_{ij} for $i \neq j$. In other words, the E_{ij} with $i \neq j$ generate the eigenspaces $\mathfrak{g}_\alpha$ for the adjoint action of $\mathfrak{h}$ on $\mathfrak{sl}_3$. Let's find the eigenvalue corresponding to E_{ij} . For $i = 1, 2, 3$, define $L_i \in \mathfrak{h}^*$ by

$$L_i \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} = a_i.$$

Then

$$[H, E_{ij}] = (L_i - L_j)(H)E_{ij}$$

for all $H \in \mathfrak{h}$. So, the eigenvalue corresponding to E_{ij} is $L_i - L_j$. We conclude that the nonzero eigenvalues appearing in the eigenspace decomposition of $\mathfrak{sl}_3$ are the six linear functionals $L_i - L_j$, with corresponding eigenspaces $\mathfrak{g}_{L_i - L_j} \cong \mathbb{C}E_{ij}$. The eigenspace decomposition of $\mathfrak{sl}_3$ becomes

$$\mathfrak{sl}_3 \cong \mathfrak{h} \oplus \left(\bigoplus_{i \neq j} \mathfrak{g}_{L_i - L_j} \right) \cong \mathfrak{h} \oplus \left(\bigoplus_{i \neq j} \mathbb{C}E_{ij} \right).$$

When we do not have to use the explicit description of the eigenvalues and eigenspaces for the adjoint action of $\mathfrak{h}$ on $\mathfrak{sl}_3$, we use the notation $\mathfrak{g}_\alpha$. The analogues of X and Y for $\mathfrak{sl}_3$ are the six eigenvectors E_{ij} with $i \neq j$. In particular, the E_{ij} with $i < j$ play the role of X , and the E_{ij} with $j < i$ play the role of Y . This partition of the E_{ij} will be clear later on.

It will be helpful to write down generators for $\mathfrak{h}^*$. We naturally imbed $\mathfrak{h} \subset \mathbb{C}^3$ by

$$\mathfrak{h} = \{(a_1, a_2, a_3) \in \mathbb{C}^3 : a_1 + a_2 + a_3 = 0\}.$$

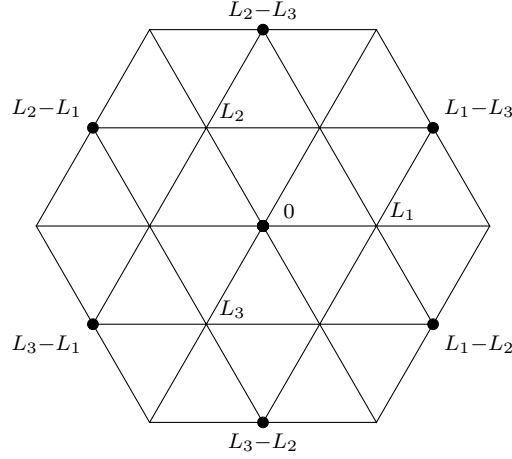
Then

$$\mathfrak{h}^* \cong \text{span}_{\mathbb{C}}\{L_1, L_2, L_3\} / \mathbb{C}(L_1 + L_2 + L_3),$$

where

$$\mathbb{C}(L_1 + L_2 + L_3) := \text{span}_{\mathbb{C}}\{L_1 + L_2 + L_3\}.$$

One advantage of $\mathfrak{sl}_3$ is that we can illustrate the eigenvalues for the action of $\mathfrak{h}$ with a picture. In the case of $\mathfrak{sl}_2$, the eigenvalues for $H = \text{diag}(1, -1)$ are integers, and so we draw them as points on a real line. Since $\dim \mathfrak{h} = 2$, the natural thing to do for $\mathfrak{sl}_3$ is to draw the eigenvalues in a plane. To retain symmetry and the relation $L_1 + L_2 + L_3 = 0$, we place the eigenvalues on a hexagonal lattice with the L_i on the third roots of unity in the plane. We illustrate the eigenvalues for the adjoint action of $\mathfrak{h}$ on $\mathfrak{sl}_3$ with the following picture:



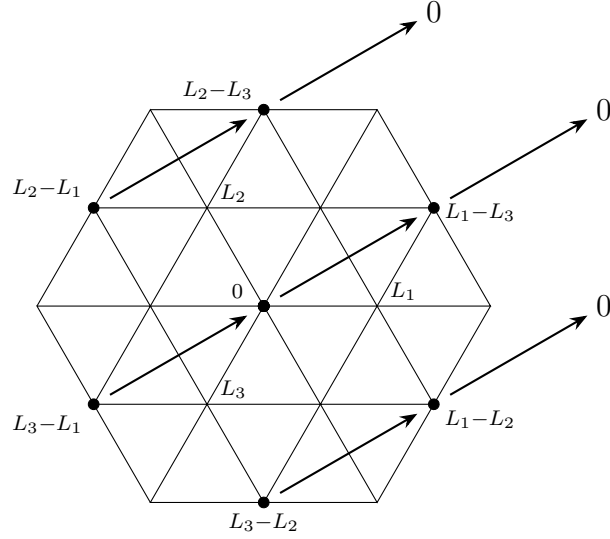
In this picture, the lattice represents a subspace of $\mathfrak{h}^*$ (integral combinations of the L_i), and the dots represent eigenvalues for the adjoint action of $\mathfrak{h}$. Most of the structure of $\mathfrak{sl}_3$ can be easily read off this picture. To see this, one should mentally associate to each eigenvalue its corresponding eigenspace. Under this association, the dots in the diagram represent the eigenspaces $\mathfrak{g}_\alpha$, with 0 representing $\mathfrak{h} = \mathfrak{g}_0$. Let's first consider the action of $\mathfrak{h}$ on $\mathfrak{sl}_3$. Since $\text{ad}(\mathfrak{h})(X) = \alpha X$ for all $X \in \mathfrak{g}_\alpha$, we have that

$$\text{ad}(\mathfrak{h})(\mathfrak{g}_\alpha) \subset \mathfrak{g}_\alpha.$$

So, $\mathfrak{h}$ carries each eigenspace $\mathfrak{g}_\alpha$ corresponding to a dot in the diagram into itself. Since $\mathfrak{h} = \mathfrak{g}_0$, we can interpret the action of $\mathfrak{h}$ in the diagram as translating the eigenvalues by 0. To see how the rest of $\mathfrak{sl}_3$ acts, let $X \in \mathfrak{g}_\alpha$ and $Y \in \mathfrak{g}_\beta$. We would like to see where $\text{ad}(X)$ sends Y . Since the decomposition of $\mathfrak{sl}_3$ is obtained by the adjoint action of $\mathfrak{h}$, we just need to figure out how $\mathfrak{h}$ acts on $\text{ad}(X)(Y)$. Using the Jacobi identity, we find that

$$\begin{aligned} [H, [X, Y]] &= [X, [H, Y]] + [[H, X], Y] \\ &= [X, \beta(H)Y] + [\alpha(H)X, Y] \\ &= (\alpha(H) + \beta(H))[X, Y] \end{aligned}$$

for all $H \in \mathfrak{h}$. This shows that $\text{ad}(\mathfrak{g}_\alpha)(\mathfrak{g}_\beta) = [\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$. That is, the action of $\mathfrak{g}_\alpha$ carries the eigenspace $\mathfrak{g}_\beta$ into $\mathfrak{g}_{\alpha+\beta}$. So, if $[X, Y] \neq 0$, then it is an eigenvector for $\mathfrak{h}$ with eigenvalue $\alpha + \beta$. We interpret the action of the eigenspace $\mathfrak{g}_\alpha$ in the diagram as translating the eigenvalues by α . For an explicit example, the action of $\mathfrak{g}_{L_1-L_3}$ can be illustrated with the following picture:



Remark 2.4.2. The action of $\mathfrak{g}_\alpha$ may annihilate some eigenspaces. In particular, if $\alpha + \beta$ is not an eigenvalue, then $\mathfrak{g}_{\alpha+\beta} = 0$ and hence $\text{ad}(\mathfrak{g}_\alpha)(\mathfrak{g}_\beta) = 0$. In the example above, the action of $\mathfrak{g}_{L_1-L_3}$ annihilates $\mathfrak{g}_{L_1-L_2}$, $\mathfrak{g}_{L_2-L_3}$, and itself.

2.4.2 Roots and weights

Now that we have an explicit basis for $\mathfrak{sl}_3$, let's continue our analysis of representations V of $\mathfrak{sl}_3$. Recall from Equation (2.6) that the action of $\mathfrak{h}$ gives us the decomposition

$$V = \bigoplus_{\beta \in \mathfrak{h}^*} V_\beta$$

of V into eigenspaces of $\mathfrak{h}$. The following proposition tells us how $\mathfrak{sl}_3$ acts on V . In particular, the action of $\mathfrak{g}_\alpha$ carries V_β into $V_{\alpha+\beta}$.

Proposition 2.4.3. [9, Lecture 12] *Let V be any representation of $\mathfrak{sl}_3$. If $X \in \mathfrak{g}_\alpha$ and $v \in V_\beta$, then $X(v) \in V_{\alpha+\beta}$.*

Proof. For any $H \in \mathfrak{h}$, we compute that

$$\begin{aligned} H(X(v)) &= X(H(v)) + [H, X](v) \\ &= X(\beta(H)v) + \alpha(H)X(v) \\ &= (\alpha(H) + \beta(H))X(v). \end{aligned}$$

This shows that $X(v) \in V_{\alpha+\beta}$. □

Remark 2.4.4. If $X(v) \neq 0$, then it is an eigenvector for $\mathfrak{h}$ with eigenvalue $\alpha + \beta$. The action of $\mathfrak{g}_\alpha$ may annihilate some eigenspaces V_β . In particular, if $\alpha + \beta$ is not an eigenvalue, then $V_{\alpha+\beta} = 0$ and hence $\mathfrak{g}_\alpha \cdot V_\beta = 0$.

We now introduce some common terminology used in representation theory. For an arbitrary representation V of $\mathfrak{sl}_3$, the eigenvalues $\beta \in \mathfrak{h}^*$ for the action of $\mathfrak{h}$ on V are called the **weights** of V . The corresponding eigenspaces V_β are called the **weight spaces**, and the eigenvectors in V_β are called **weight vectors**. It will be helpful to have separate terminology when talking about the adjoint representation of $\mathfrak{sl}_3$. The nonzero eigenvalues α for the action of $\mathfrak{h}$ on the adjoint representation are called **roots**. By convention, the zero linear functional is not a root; however, in some cases it may be helpful to adopt the notation $\mathfrak{g}_0 = \mathfrak{h}$. We denote the set of roots by R . The corresponding eigenspaces $\mathfrak{g}_\alpha$ are called **root spaces**, and the eigenvectors in $\mathfrak{g}_\alpha$ are called **root vectors**. From now on, we use this terminology when talking about $\mathfrak{sl}_3$. We can adopt this terminology for $\mathfrak{sl}_2$; however, in this chapter we refrain from doing so.

Recall from Corollary 2.3.2 that the eigenvalues for $H = \text{diag}(1, -1)$ of an irreducible representation of $\mathfrak{sl}_2$ are congruent to one another modulo 2. In the case of $\mathfrak{sl}_3$, we have:

Proposition 2.4.5. [9, Observation 12.6] *Let V be an irreducible representation of $\mathfrak{sl}_3$. Then the weights of V are congruent to one another by integral linear combinations of the roots $L_i - L_j \in R$.*

Proof. Let β be a weight of V , and consider

$$V' := \bigoplus_{\gamma \in \Lambda_R} V_{\beta+\gamma},$$

where

$$\Lambda_R := \left\{ \sum_{i \neq j} a_{ij}(L_i - L_j) : a_{ij} \in \mathbb{Z} \right\}.$$

Since $V_\beta \subset V'$, we get that $V' \neq 0$. The space V' is invariant under $\mathfrak{sl}_3$ since

$$\mathfrak{g}_\alpha \cdot V_{\beta+\gamma} \subset V_{\beta+\gamma+\alpha}$$

and $\gamma + \alpha \in \Lambda_R$ for any root $\alpha \in R$. By irreducibility of V , we must have that $V = V'$. It follows that the weights of V are of the form $\beta + \gamma$ for some $\gamma \in \Lambda_R$. $\square$

The lattice $\Lambda_R \subset \mathfrak{h}^*$ generated by the roots in R is called the **root lattice**. Thus, we may rephrase Proposition 2.4.5 as saying that the weights of an irreducible representation of $\mathfrak{sl}_3$ live in some translate of the root lattice or, equivalently, that they are congruent to one another modulo the root lattice. We prove later in this chapter that the weights of an arbitrary representation of $\mathfrak{sl}_3$ lie in the lattice generated by L_1, L_2 , and L_3 . Given this fact, the weights of a representation of $\mathfrak{sl}_3$ can be represented as dots on the vertices of the lattice in Figure (2.1). We refer to these pictures as **weight diagrams**. A priori, we draw the weights as dots on the vertices, rather than arbitrarily in the plane. The following results are not compromised by considering such pictures.

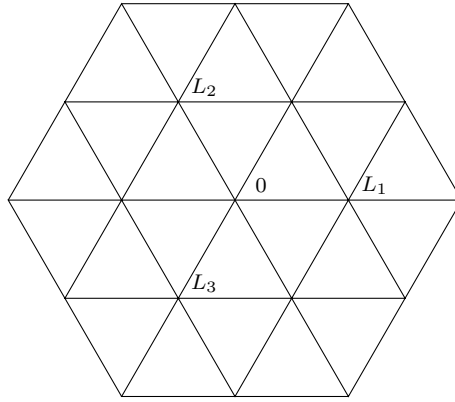


Figure 2.1: Skeleton of a weight diagram.

Proposition 2.4.6. [9, Lecture 12] In $\mathfrak{sl}_3$, the root lattice is equal to

$$\Lambda_R = \{a_1 L_1 + a_2 L_2 + a_3 L_3 \in \mathfrak{h}^* : a_i \in \mathbb{Z}, a_1 + a_2 + a_3 = 0\}.$$

Proof. Since $L_i - L_j$ and $L_j - L_i$ are linearly dependent, the roots $L_i - L_j$ with $i < j$ form a generating set for Λ_R . So, any element in Λ_R can be written as

$$a(L_1 - L_2) + b(L_2 - L_3) + c(L_1 - L_3)$$

for some $a, b, c \in \mathbb{Z}$. Simplifying, this element becomes

$$(a + c)L_1 + (b - a)L_2 + (-b - c)L_3.$$

Since $(a + c) + (b - a) + (-b - c) = 0$, we get the inclusion

$$\Lambda_R \subseteq \{a_1 L_1 + a_2 L_2 + a_3 L_3 : a_i \in \mathbb{Z}, a_1 + a_2 + a_3 = 0\}.$$

Now consider $w = a_1L_1 + a_2L_2 + a_3L_3$ for some $a_i \in \mathbb{Z}$ satisfying $a_1 + a_2 + a_3 = 0$. Making the substitution $a_1 = -a_2 - a_3$, we find that

$$w = (-a_2 - a_3)L_1 + a_2L_2 + a_3L_3 = -a_2(L_1 - L_2) - a_3(L_1 - L_3)$$

and so $w \in \Lambda_R$. □

In the analysis of $\mathfrak{sl}_2$, we chose a nonzero eigenvector v in the eigenspace corresponding to the maximum eigenvalue of a given irreducible representation. We saw that v was in the kernel of $X \in \mathfrak{sl}_2$, and that we could get the entire representation by repeatedly applying $Y \in \mathfrak{sl}_2$ to v . In other words, we were able to completely describe the representation in terms of the images of v under X and Y . We would like to do something analogous for $\mathfrak{sl}_3$. However, since the weights of an irreducible representation of $\mathfrak{sl}_3$ are congruent to one another modulo the root lattice, we no longer have the concept of a maximal weight. We must decide on what we mean by *maximal* in this case. The general idea for $\mathfrak{sl}_3$ is to pick a seemingly random direction in the weight diagram, and then say that the weight furthest in this direction is *maximal* (with respect to this choice). More precisely, we choose a linear functional

$$\ell : \Lambda_R \rightarrow \mathbb{R},$$

and extend it by linearity to a linear functional $\ell : \mathfrak{h}^* \rightarrow \mathbb{C}$. Then, for *any* representation of $\mathfrak{sl}_3$, we choose the weight ω for which the real part of $\ell(\omega)$ is maximal. We refer to this weight as the **extremal weight**. To ensure that the extremal weight is unique, we choose ℓ to be irrational with respect to the lattice Λ_R . Given how the roots $L_i - L_j$ are situated on the weight diagram, choosing ℓ to be irrational is equivalent to saying that the hyperplane $\ell = 0$ does not contain any of the points in Λ_R . We prove below that any vector in the weight space corresponding to ω is in the kernel of the action of $\mathfrak{g}_\alpha$ for every $\alpha \in R$ satisfying $\ell(\alpha) > 0$. In fact, any vector in this weight space is killed by half of the root spaces. This aligns with our story of $\mathfrak{sl}_2$.

Explicitly, define $\ell : \mathfrak{h}^* \rightarrow \mathbb{C}$ by

$$\ell(a_1L_1 + a_2L_2 + a_3L_3) = a_1c_1 + a_2c_2 + a_3c_3,$$

where $c_i \in \mathbb{R}$ satisfy

$$c_1 > c_2 > c_3 \quad \text{and} \quad c_1 + c_2 + c_3 = 0,$$

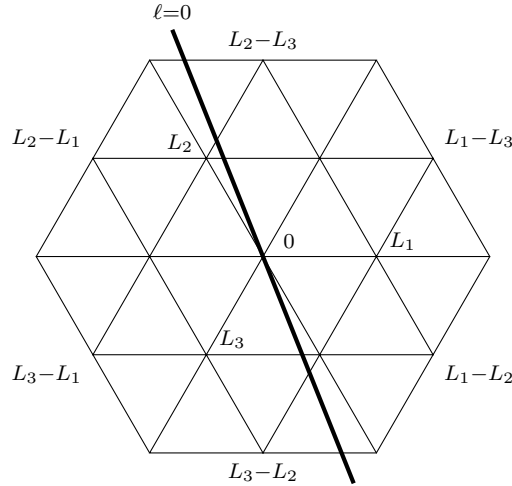
and are chosen so that ℓ is irrational with respect to Λ_R . Given this choice of ℓ , we can partition our set of roots R into **positive roots**

$$R^+ := \{\alpha \in R : \ell(\alpha) > 0\} = \{L_i - L_j : i < j\},$$

and **negative roots**

$$R^- := \{\alpha \in R : \ell(\alpha) < 0\} = \{L_i - L_j : j < i\}.$$

The weight spaces $\mathfrak{g}_\alpha$ for which $\alpha \in R^+$ are called **positive root spaces**. Likewise, the weight spaces $\mathfrak{g}_\alpha$ for which $\alpha \in R^-$ are called **negative root spaces**. We refer to the vectors in the positive root spaces as **positive root vectors**, and the vectors in the negative root spaces as **negative root vectors**. We may draw the hyperplane $\ell = 0$ on the weight diagram as follows:



Proposition 2.4.7. [9, Lecture 12] Let V be any representation of $\mathfrak{sl}_3$. Denote by ω the extremal weight of V . If $X \in \mathfrak{g}_\alpha$ such that $\alpha \in R^+$, then $X(V_\omega) = 0$.

Proof. First note, by Proposition 2.4.3, that $X(V_\omega) \subseteq V_{\omega+\alpha}$. Given $z \in \mathbb{C}$, let $\text{Re}(z)$ denote the real part of z . Since $\text{Re}(\ell(\alpha)) = \ell(\alpha) > 0$, we find that

$$\text{Re}(\ell(\omega + \alpha)) = \text{Re}(\ell(\omega)) + \text{Re}(\ell(\alpha)) > \text{Re}(\ell(\omega)).$$

But, we chose ω so that $\text{Re}(\ell(\omega))$ is maximal. Thus, the functional $\omega + \alpha$ is not a weight of V . It follows that $V_{\omega+\alpha} = 0$ and so $X(V_\omega) = 0$. $\square$

Corollary 2.4.8. [9, Lemma 12.9] Let V be any representation of $\mathfrak{sl}_3$. Then there exists a nonzero vector $v \in V$ with the properties that

- (i) v is a weight vector of V , i.e., $v \in V_\beta$ for some weight β ; and

(ii) v is killed by E_{12}, E_{13} , and E_{23} , i.e., v is killed by the positive root spaces.

Proof. Let ω be the extremal weight of V . Since V_ω is a weight space, we can find $v \in V_\omega$ such that $v \neq 0$. Then v is a weight vector and, by Proposition 2.4.7, is annihilated by the positive root spaces. $\square$

A vector with the properties in Corollary 2.4.8 is called a **highest weight vector**. The weight associated to a highest weight vector is called a **highest weight**. We saw that the extremal weight ω of any representation is a highest weight, and that any nonzero vector $v \in V_\omega$ is a highest weight vector. In particular, any $v \in V_\omega$ is killed by the E_{ij} with $i < j$. So, as stated earlier, the role of $X \in \mathfrak{sl}_2$ for $\mathfrak{sl}_3$ is played precisely by the vectors $E_{i,j}$ with $i < j$. Analogous to Proposition 2.3.3, we have the following result.

Theorem 2.4.9. [9, Claim 12.10, Proposition 12.11, Lecture 12] *Let V be any representation of $\mathfrak{sl}_3$, and $v \in V$ a highest weight vector. Define W to be the subspace of V generated by the images of v by successive applications of the three negative root vectors E_{21}, E_{31} , and E_{32} . Then we have the following:*

(i) W is an irreducible subrepresentation of V . In particular, if V is irreducible, then $V = W$.

(ii) If V is irreducible and $v \in V_\beta$, then $\dim V_\beta = 1$. More generally, we have that

$$\dim V_{\beta+k(L_2-L_1)} = 1 \quad \text{and} \quad \dim V_{\beta+k(L_3-L_2)} = 1$$

for all $k \in \mathbb{N}$.

(iii) If V is irreducible, then V has a unique highest weight that is equal to its extremal weight. It follows that V has a unique highest weight vector, up to scalars.

Proof. For part (i), it suffices to show that W is invariant under E_{12}, E_{23} , and E_{13} . Let w_n denote any word of length n or less in the letters E_{21} and E_{32} . Also, let W_n denote the vector space spanned by the vectors $w_n(v)$ for all possible words w_n . Since $[E_{32}, E_{21}] = E_{31}$, we have that

$$W = \bigcup_n W_n.$$

By induction on n , we show E_{12} and E_{23} carry W_n to W_{n-1} . For $n = 0$, we have that $W_0 = \mathbb{C}v$ and $W_{-1} = 0$. So, if $w \in W_0$ then $w = \lambda v$ for some $\lambda \in \mathbb{C}$. Since E_{12} and E_{23} kill v , we get that $E_{12}(w) = E_{23}(w) = 0$. Now suppose

$$E_{12}(W_{n-1}) \subset W_{n-2} \quad \text{and} \quad E_{23}(W_{n-1}) \subset W_{n-2}.$$

We can write any word w_n as $w_n = E_{21}(w_{n-1})$ or $w_n = E_{32}(w_{n-1})$ for some w_{n-1} . First suppose $w_n = E_{21}(w_{n-1})$. We may assume that $w_{n-1}(v) \neq 0$. Then $w_{n-1}(v)$ is a weight vector for some weight $\beta \in \mathfrak{h}^*$. Then

$$\begin{aligned} E_{12}(w_n(v)) &= E_{12}(E_{21}(w_{n-1}(v))) \\ &= E_{21}(E_{12}(w_{n-1}(v))) + [E_{12}, E_{21}](w_{n-1}(v)) \\ &\in E_{21}(W_{n-2}) + \beta([E_{12}, E_{21}])w_{n-1}(v) \\ &\subset W_{n-1} \end{aligned}$$

since $[E_{12}, E_{21}] \in \mathfrak{h}$. Similarly, we find that

$$\begin{aligned} E_{23}(w_n(v)) &= E_{23}(E_{21}(w_{n-1}(v))) \\ &= E_{21}(E_{23}(w_{n-1}(v))) + [E_{23}, E_{21}]w_{n-1}(v) \\ &\in E_{21}(W_{n-2}) \\ &\subset W_{n-1} \end{aligned}$$

since $[E_{23}, E_{21}] = 0$. The calculations for the case $w_n = E_{32}(w_{n-1})$ are of the same nature. Since W is invariant under E_{12} and E_{23} , it is invariant under $[E_{12}, E_{23}] = E_{13}$.

To show that W is irreducible, suppose

$$W = W_1 \oplus W_2$$

for some representations $W_i \subseteq W$ of $\mathfrak{sl}_3$. Since projection onto W_i commutes with the action of $\mathfrak{h}$, we get that

$$W_\beta = (W_1)_\beta \oplus (W_2)_\beta,$$

where $v \in V_\beta$. Since $\dim W_\beta = 1$ (see part (ii)), we may assume that $(W_1)_\beta = 0$. Then $v \in W_\beta = (W_2)_\beta$ and so $v \in W_2$. Since W_2 is a representation of $\mathfrak{sl}_3$, it is invariant under the negative root spaces. Hence, the images of $v \in W_2$ under successive applications of E_{21} , E_{31} , and E_{32} are in W_2 . Since W_2 is a vector space, it also contains the linear combinations of these images. So, we get the inclusion $W \subseteq W_2$. Thus $W = W_2$ is irreducible.

For part (ii), since V is irreducible we have that $W = V$. Once we apply the negative root vectors to v we leave the weight space V_β . Thus, since $V = W$, we must have that $V_\beta \cong \mathbb{C}v$. Every vector in $V_{\beta+k(L_2-L_1)}$ can be written as $E_{21}^k(v')$ for some $v' \in V_\alpha$. Since $V_\beta \cong \mathbb{C}v$, the space $V_{\beta+k(L_2-L_1)}$ is spanned by $\mathbb{C}E_{21}^k(v)$. Similarly, the space $V_{\beta+k(L_3-L_2)}$ is spanned by $\mathbb{C}E_{32}^k(v)$.

To prove part (iii), let ω be the extremal weight of V and w a highest weight vector of V with corresponding weight β . By part (i), the vector w produces a subrepresentation

W of V . Since V is irreducible and $W \neq 0$, we get that $V = W$. Then ω is a weight of W . By definition of W , we may write

$$\omega = \beta + \sum_i k_i \alpha_i$$

for some $k_i \in \mathbb{N}$ and $\alpha_i \in R^-$. Since $\alpha_i \in R^-$, we have that $\operatorname{Re}(\ell(-\alpha_i)) = -\ell(\alpha_i) > 0$. Then

$$\operatorname{Re}(\ell(\beta)) = \operatorname{Re}(\ell(\omega)) + \sum_i k_i \operatorname{Re}(\ell(-\alpha_i)) \geq \operatorname{Re}(\ell(\omega)).$$

But, we chose ω so that $\operatorname{Re}(\ell(\omega))$ is maximal. Thus $\beta = \omega$ and so the highest weight is unique. By part (ii), we have that $V_\omega \cong \mathbb{C}v$ for some nonzero $v \in V_\omega$. It follows that $w \in \mathbb{C}v$ and so V has a unique highest weight vector, up to scalars. $\square$

Remark 2.4.10. The above theorem makes clear that the role of $Y \in \mathfrak{sl}_2$ for $\mathfrak{sl}_3$ is played precisely by the vectors $E_{i,j}$ with $j < i$.

An important consequence of Theorem 2.4.9 is that the weights of an irreducible representation V of $\mathfrak{sl}_3$ lie in a third of the plane. This is depicted in Figure (2.2). Given part (iii) of Theorem 2.4.9, we consider the highest weight vector $v \in V_\omega$, where ω is the extremal weight of V . Then the weights of V lie in the section of the plane cut out by the lines through ω in the directions $L_2 - L_1$ and $L_3 - L_2$. So, the weight diagram will have a corner at ω . Viewing the weight diagram in terms of weight spaces, the line through ω in the direction $L_2 - L_1$ is obtained by starting at v and applying E_{21} successively. Similarly, the line through ω in the direction $L_3 - L_2$ is obtained by starting at v and applying E_{32} successively. Observe that there are no weights in the other two-thirds of the plane. This can be seen by acting the positive root spaces on V_ω and then noticing that the vectors in the resulting space can not be obtained by the images of v under the negative root vectors.

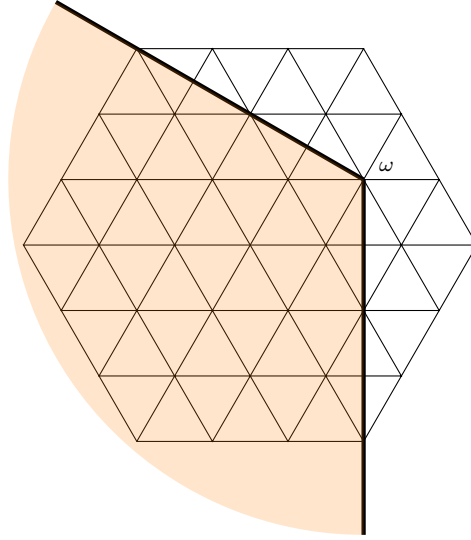


Figure 2.2: The shaded region represents where the weights of an irreducible representation lie.

2.4.3 Subalgebras isomorphic to $\mathfrak{sl}_2$

We now introduce subalgebras of $\mathfrak{sl}_3$ which are isomorphic to $\mathfrak{sl}_2$. Recall that the roots of $\mathfrak{sl}_3$ are

$$R = \{L_i - L_j : i \neq j\}.$$

So, for every $\alpha \in R$ we have that $-\alpha \in R$. If $\alpha = L_i - L_j$, then $\mathfrak{g}_\alpha \cong \mathbb{C}E_{ij}$ and $\mathfrak{g}_{-\alpha} \cong \mathbb{C}E_{ji}$. It follows that $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \cong \mathbb{C}H_{L_i-L_j}$, where

$$H_{L_i-L_j} := [E_{ij}, E_{ji}] = E_{ii} - E_{jj}.$$

Observe that since $H_{L_i-L_j}$ is a diagonal matrix, we get that $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subseteq \mathfrak{h}$. Moreover, the subalgebra

$$s_\alpha := \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \cong \mathbb{C}E_{ij} \oplus \mathbb{C}E_{ji} \oplus \mathbb{C}H_{L_i-L_j}$$

is isomorphic to $\mathfrak{sl}_2$ under the map

$$\begin{aligned} s_{L_i-L_j} &\rightarrow \mathfrak{sl}_2 \\ (E_{ij}, E_{ji}, H_{L_i-L_j}) &\mapsto (X, Y, H). \end{aligned}$$

Note that the basis $\{E_{ij}, E_{ji}, H_{L_i-L_j}\}$ for $s_{L_i-L_j}$ satisfies the commutation relations

$$[H_{ij}, E_{ij}] = 2E_{ij}, \quad [H_{ij}, E_{ji}] = -2E_{ji}, \quad [E_{ij}, E_{ji}] = H_{L_i-L_j}.$$

So, for each pair $(\alpha, -\alpha)$ of roots, we get a subalgebra of $\mathfrak{sl}_3$ which is isomorphic to $\mathfrak{sl}_2$. Restricting to these subalgebras, we use results from $\mathfrak{sl}_2$ to understand the weights of an irreducible representation of $\mathfrak{sl}_3$. We start by analysing the weights on the lines through the extremal weight ω in the directions $L_2 - L_1$ and $L_3 - L_2$.

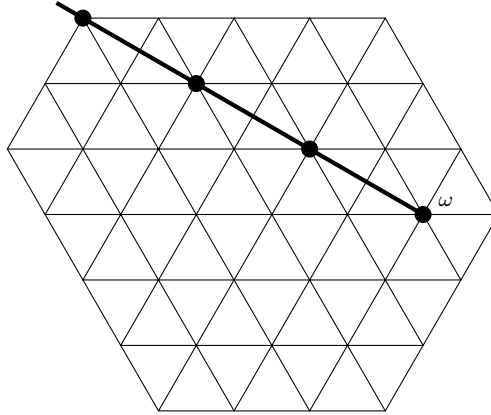
2.4.4 The Weyl group and weight lattice

Throughout this section, unless stated otherwise, V is a fixed irreducible representation of $\mathfrak{sl}_3$. In particular, the weight diagrams in this section are in reference to irreducible representations of $\mathfrak{sl}_3$.

Let V be an irreducible representation of $\mathfrak{sl}_3$ with extremal weight ω . For $k \in \mathbb{N}$, consider the vectors $E_{21}^k(v)$, where $v \in V_\omega$ is a highest weight vector. If $E_{21}^k(v) \neq 0$ then it spans the weight space $V_{\omega+k(L_2-L_1)}$. In this case, $\omega + k(L_2 - L_1)$ is a weight of V . Since V is finite-dimensional, we must have that $E_{21}^k(v) = 0$ for sufficiently large k . We can choose $m \in \mathbb{N}$ so that $m+1$ is the minimal power of E_{21} that annihilates v . Then $V_{\omega+k(L_2-L_1)} = 0$ for all $k > m$, and we get an unbroken string of (nonzero) weight spaces

$$V_{\omega+k(L_2-L_1)} \cong \mathbb{C}E_{21}^k(v), \quad 0 \leq k \leq m.$$

We represent the weights $\omega + k(L_2 - L_1)$ as dots on the weight diagram for V as follows:



We use our knowledge of representations of $\mathfrak{sl}_2$ to determine exactly where this string of weights terminates. To do this, consider the subspace

$$V' := \bigoplus_{k=0}^m V_{\omega+k(L_2-L_1)}.$$

Since V' is invariant under the action of $s_{L_2-L_1}$, it is a representation of $s_{L_2-L_1} \cong \mathfrak{sl}_2$. It follows that the eigenvalues

$$(\omega + k(L_2 - L_1))H_{L_2-L_1} = \omega(H_{L_2-L_1}) + 2k, \quad 0 \leq k \leq m,$$

with respect to $H_{L_2-L_1}$ are integral and symmetric about 0. So, the weight $\omega + m(L_2 - L_1)$ is obtained by reflecting ω through the hyperplane perpendicular to $H_{L_2-L_1}$. In other words, the weights

$$\omega + k(L_2 - L_1), \quad 0 \leq k \leq m,$$

are symmetric with respect to the line

$$\Omega_{L_2-L_1} := \{\beta \in \mathfrak{h}^* : \beta(H_{L_2-L_1}) = 0\}.$$

We can do the exact same analysis for the line through ω in the direction $L_3 - L_2$. Thus, our weight diagram becomes:

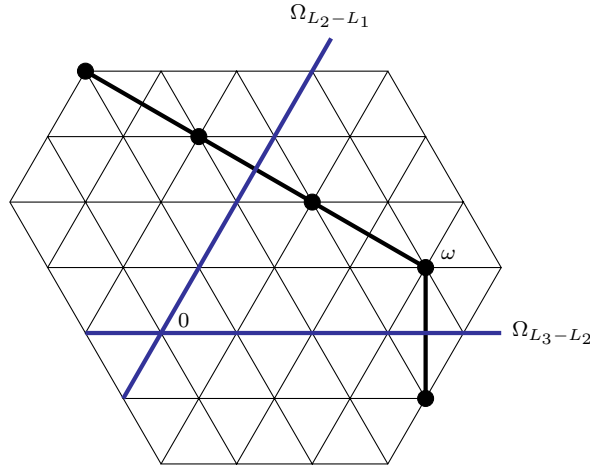


Figure 2.3: Utilizing the representation theory of $\mathfrak{sl}_2$ to understand the weights of V .

In general, for any root $L_i - L_j \in R$, we define the line

$$\Omega_{L_i-L_j} := \{\beta \in \mathfrak{h}^* : \beta(H_{L_i-L_j}) = 0\}.$$

Then the involution $W_{L_i-L_j} : \mathfrak{h}^* \rightarrow \mathfrak{h}^*$ defined by

$$W_{L_i-L_j}(\beta) = \beta - \frac{2\beta(H_{L_i-L_j})}{(L_i - L_j)(H_{L_i-L_j})}(L_i - L_j) = \beta - \beta(H_{L_i-L_j})(L_i - L_j)$$

is reflection in the line $\Omega_{L_i-L_j}$. Indeed, if we consider the example above, we get that

$$W_{L_2-L_1}(\omega) = \omega - \omega(H_{L_2-L_1})(L_2 - L_1) = \omega + m(L_2 - L_1)$$

since $m = -\omega(H_{L_2-L_1})$. The following proposition uses the integrality of the eigenvalues of representations of $\mathfrak{sl}_2$.

Proposition 2.4.11. *[9, Lecture 12] Let V be a representation of $\mathfrak{sl}_3$. If β is a weight of V , then $\beta(H_{L_i-L_j}) \in \mathbb{Z}$ for all i, j .*

Proof. If β is any weight of V , then

$$\bigoplus_{k \in \mathbb{Z}} V_{\beta+k(L_i-L_j)}$$

is invariant under $s_{L_i-L_j}$ and hence a representation of $s_{L_i-L_j} \cong \mathfrak{sl}_2$. Thus

$$(\beta + k(L_i - L_j))(H_{L_i-L_j}) \in \mathbb{Z}$$

whenever $V_{\beta+k(L_i-L_j)} \neq 0$. Taking $k = 0$, we get that $\beta(H_{L_i-L_j}) \in \mathbb{Z}$. $\square$

Consider the weight diagram in Figure (2.3). Let $E_{21}^m(v)$ be the last weight space on the line through ω in the direction $L_2 - L_1$. We can choose $v' \in E_{21}^m(v)$ so that $v' \neq 0$. Then $E_{21}(v') = 0$. Since there are no weights above the line through ω in the direction $L_2 - L_1$, we find that $E_{23}(v') = E_{13}(v') = 0$. So v' satisfies the properties in Corollary 2.4.8, aside from the exchange of E_{12} with E_{21} . If instead we choose $\ell : \mathfrak{h}^* \rightarrow \mathbb{C}$ so that $c_2 > c_1 > c_3$, then we can take v' to be the vector in Corollary 2.4.8. Indeed, carrying out the above analysis with respect to the vector v' , we find that the weights of V lie in the section of the plane cut out by the lines through $\omega + m(L_2 - L_1)$ in the directions $L_3 - L_1$ and $L_1 - L_2$. As before, there are no weights above these two lines. Moreover, the lines through $\omega + m(L_2 - L_1)$ in the directions $L_3 - L_1$ and $L_1 - L_2$ are symmetric about $\Omega_{L_3-L_1}$ and $\Omega_{L_1-L_2}$, respectively. As depicted in Figure (2.4), we continue this process until we form a closed polygon in the plane.

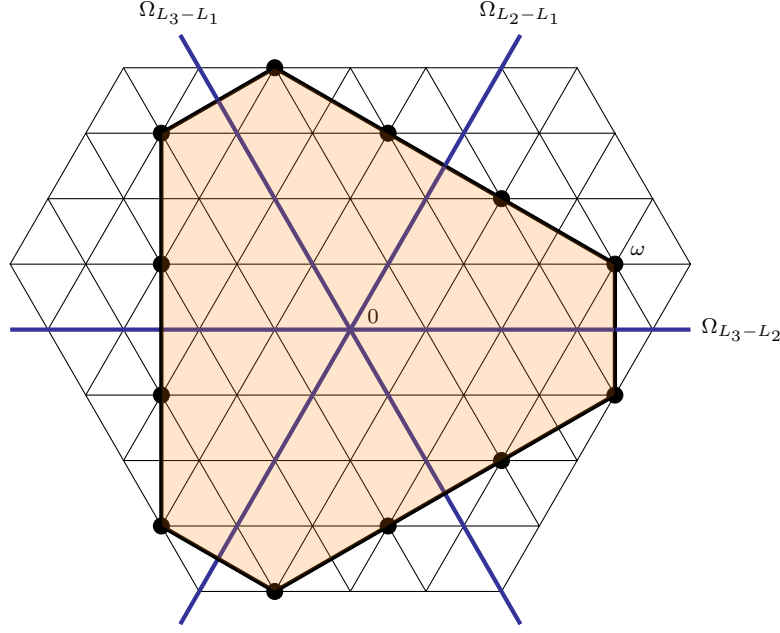


Figure 2.4: The weights of V are bounded by a polygon that is symmetric with respect to the hyperplanes $\Omega_{L_i-L_j}$ that have normal vector $L_i - L_j$.

The above analysis shows that the weights of V are bounded by a polygon symmetric with respect to the lines $\Omega_{L_i-L_j}$, and with one vertex at the extremal weight ω . Since V is irreducible, by Theorem 2.4.9, the weight spaces which lie on the border of the polygon are all of dimension 1. We also observed that the weights of V which lie on the border of the polygon are invariant under the involutions $W_{L_i-L_j}$. In fact, we prove shortly that the set of weights of a representation are invariant under each $W_{L_i-L_j}$. This leads us to the following definition. We define the **Weyl group**, denoted by $\mathfrak{W}$, to be the group generated by the involutions $W_{L_i-L_j}$. Observe that the action of $W_{L_i-L_j}$ on $\mathfrak{h}^*$ permutes L_i and L_j , while fixing L_k for $k \neq i, j$. So $W_{L_i-L_j}$ is the transposition that exchanges L_i and L_j . Since $\mathfrak{W}$ is generated by the transpositions $W_{L_i-L_j}$, it is exactly the symmetric group S_3 on the three letters L_1, L_2 , and L_3 .

Proposition 2.4.12. [9, Proposition 12.15] *Let V be a representation of $\mathfrak{sl}_3$. The weights of V are integral linear combinations of L_1, L_2 , and L_3 . That is, the weights of V lie in the lattice*

$$\Lambda_W = \text{span}_{\mathbb{Z}}\{L_1, L_2, L_3\} / \mathbb{Z}(L_1 + L_2 + L_3).$$

Proof. By Proposition 2.4.11, it suffices to show that

$$\{\beta \in \mathfrak{h}^* : \beta(H_{L_i-L_j}) \in \mathbb{Z} \text{ for all } i, j\} = \text{span}_{\mathbb{Z}}\{L_1, L_2, L_3\} / \mathbb{Z}(L_1 + L_2 + L_3).$$

The inclusion

$$\{\beta \in \mathfrak{h}^* : \beta(H_{L_i-L_j}) \in \mathbb{Z} \text{ for all } i, j\} \supseteq \text{span}_{\mathbb{Z}}\{L_1, L_2, L_3\} / \mathbb{Z}(L_1 + L_2 + L_3)$$

is immediate. For the other inclusion, let $\beta = \sum a_i L_i \in \mathfrak{h}^*$ so that $\beta(H_{L_i-L_j}) \in \mathbb{Z}$ for all i, j . Then $b_1 := a_1 - a_2$ and $b_2 := a_2 - a_3$ are integers. Using the identity $L_1 + L_2 + L_3 = 0$, we find that

$$\beta = b_1 L_1 + b_2 (L_1 + L_2). \quad \square$$

We refer to the lattice Λ_W as the **weight lattice** of $\mathfrak{sl}_3$.

Theorem 2.4.13. [9, Proposition 12.18] *Let V be an irreducible representation of $\mathfrak{sl}_3$ with extremal weight ω . The weights of V are exactly the linear functionals $\beta \in \Lambda_W$ which are congruent to ω modulo Λ_R , and lie in the convex hull of the images of ω under the Weyl group $\mathfrak{W}$.*

Proof. We already know that the weights of V satisfy these properties. In particular, by Proposition 2.4.12, the weights of V lie in Λ_W . By Proposition 2.4.5, the weights of V are congruent to one another modulo Λ_R . Thus, each weight of V is congruent to ω modulo Λ_R . Lastly, by the discussion directly above Proposition 2.4.12, we know that the weights of V are bounded by a polygon with vertices obtained by reflecting ω in the lines $\Omega_{L_i-L_j}$. However, this just means that the weights of V lie in the convex hull of the images of ω under $\mathfrak{W}$.

For the other direction, suppose $\gamma \in \Lambda_W$ lies in the convex hull and is congruent to ω modulo Λ_R . Given our assumptions on γ , we can find a weight β on the border of the convex hull such that $\gamma = \beta + s(L_3 - L_1)$ for some $s \in \mathbb{Z}$. If γ is on the border, then it is a weight and we are done. So, assume that γ is in the interior of the convex hull. Set $\alpha := L_3 - L_1$ and consider the subspace

$$V_{[\beta]} := \bigoplus_{k \in \mathbb{Z}} V_{\beta+k\alpha}.$$

Since β is on the border and γ is in the interior, either $V_{\beta+k\alpha} = 0$ for all $k > 0$ or $V_{\beta+k\alpha} = 0$ for all $k < 0$. Without loss of generality, assume that $V_{\beta+k\alpha} = 0$ for all $k < 0$. Since $V_{[\beta]}$ is invariant under s_α , it is a representation of $s_\alpha \cong \mathfrak{sl}_2$. It follows that the eigenvalues

$$S := \{(\beta(H_\alpha) + 2k : k \in \mathbb{N}, V_{\beta+k\alpha} \neq 0)\}$$

with respect to H_α are integral and symmetric about 0. By $\mathfrak{sl}_2$, since the integers in S are symmetric about 0 and have the same parity, they form an unbroken string of integers congruent to one another modulo 2. Thus, we have that

$$S = \{\beta(H_\alpha) + 2k : 0 \leq k \leq n\}$$

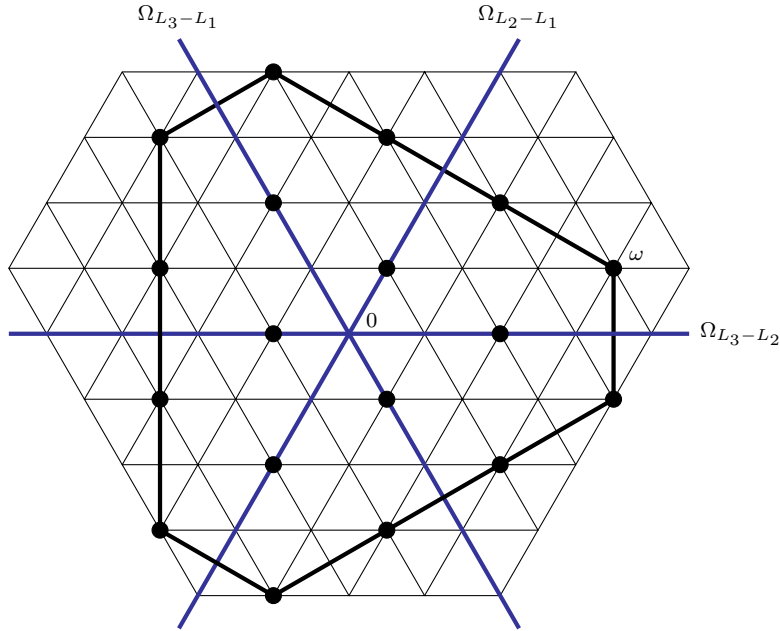
for some $n \in \mathbb{N}$. Then the weights corresponding to $V_{[\beta]}$ are $\beta + k\alpha$ for $0 \leq k \leq n$. Since S is symmetric about 0, we get that $n = -\beta(H_\alpha)$ and hence

$$W_\alpha(\beta) = \beta - \beta(H_\alpha)\alpha = \beta + n\alpha.$$

That is, the last weight $\beta + n\alpha$ in the unbroken string of weights is the reflection of β about the line Ω_α . Since the convex hull is symmetric with respect to Ω_α , and β lives on the border of the convex hull, the weight $W_\alpha(\beta) = \beta + n\alpha$ must live on the border across from β in the direction α . Since $\gamma = \beta + s\alpha$ lives in the convex hull, we necessarily have that $0 < s < n$. Thus $V_\gamma \neq 0$ and so γ is a weight of V . $\square$

Remark 2.4.14. A proof similar to that for Theorem 2.4.13 shows that the weights of any irreducible representation of V are invariant under the action of $\mathfrak{W}$.

By Theorem 2.4.13, the complete weight diagram for V looks like:



We would like to determine where in the convex hull the highest weight ω of V lives. Set $\alpha = L_i - L_j \in R^-$ with $j < i$. Since ω is the highest weight, $\omega + k\alpha$ is not a weight if $k < 0$. Thus, we can find an unbroken string of weights

$$\omega + k\alpha, \quad 0 \leq k \leq n,$$

for some $n \in \mathbb{N}$. Since

$$\bigoplus_{k=0}^n V_{\omega+k\alpha}$$

is a representation of $s_{L_i-L_j} \cong \mathfrak{sl}_2$, we must have that $n = -\omega(H_{L_i-L_j})$. It follows that $\omega(H_{L_i-L_j}) \in \mathbb{N}$ for all $i < j$. So, the highest weight ω lives in the **Weyl chamber**

$$\mathcal{W} := \{\beta \in \mathfrak{h}^* : \beta(H_{L_i-L_j}) \geq 0 \text{ for all } i < j\}.$$

So $\omega \in \mathcal{W} \cap \Lambda_W$. In the case of $\mathfrak{sl}_3$, the description of $\mathcal{W}$ is simple.

Proposition 2.4.15. [9, Section 13.1] *We have that*

$$\mathcal{W} = \{a_1 L_1 + a_2 L_2 + a_3 L_3 \in \mathfrak{h}^* : a_1 \geq a_2 \geq a_3\}.$$

Proof. Let $\beta = a_1 L_1 + a_2 L_2 + a_3 L_3 \in \mathcal{W}$. Since $\beta(H_{L_i-L_j}) = a_i - a_j$, we get that $a_i \geq a_j$ for all $i < j$. The other inclusion is immediate. $\square$

Remark 2.4.16. It follows that

$$\mathcal{W} \cap \Lambda_W = \{a_1 L_1 + a_2 L_2 + a_3 L_3 \in \mathfrak{h}^* : a_i \in \mathbb{Z}, a_1 \geq a_2 \geq a_3\}.$$

Proposition 2.4.17. [9, Section 13.1] *We have that*

$$\mathcal{W} \cap \Lambda_W = \text{span}_{\mathbb{N}}\{L_1, L_1 + L_2\},$$

where $L_1, L_1 + L_2 \in \mathfrak{h}^*$. In particular, every element in $\mathcal{W} \cap \Lambda_W$ can be written as $aL_1 - bL_3$ for some $a, b \in \mathbb{N}$.

Proof. Let $a, b \in \mathbb{N}$. Then

$$aL_1 + b(L_1 + L_2) = (a+b)L_1 + bL_2,$$

where $a+b, b \in \mathbb{Z}$ and $a+b > b$. So, the inclusion

$$\mathcal{W} \cap \Lambda_W \supseteq \text{span}_{\mathbb{N}}\{L_1, L_1 + L_2\}$$

holds. Conversely, let $a_1L_1 + a_2L_2 + a_3L_3 \in \mathcal{W} \cap \Lambda_W$. Then $b_1 := a_1 - a_2, b_2 := a_2 - a_3 \in \mathbb{N}$ and

$$a_1L_1 + a_2L_2 + a_3L_3 = b_1L_1 + b_2(L_1 + L_2),$$

give us the other inclusion. Since $L_1 + L_2 = -L_3$, any element in $\mathcal{W} \cap \Lambda_W$ can be written as

$$aL_1 + b(L_1 + L_2) = aL_1 - bL_3$$

for some $a, b \in \mathbb{N}$. □

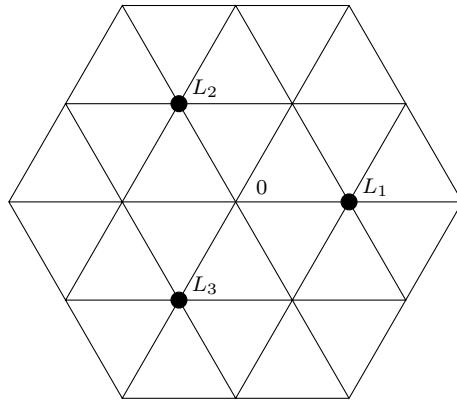
2.4.5 Irreducible representations of $\mathfrak{sl}_3$

We are almost ready to prove the classification theorem for $\mathfrak{sl}_3$ which states that the irreducible representations of $\mathfrak{sl}_3$ are in bijection with the points of $\mathcal{W} \cap \Lambda_W$. More precisely, for any $a, b \in \mathbb{N}$ there exists an irreducible representation of $\mathfrak{sl}_3$ with highest weight equal to $aL_1 - bL_3$. Before we prove this, let's look at some examples.

Example 2.4.18. [9, Section 13.1] Let $V \cong \mathbb{C}^3$ be the standard representation of $\mathfrak{sl}_3$ with standard basis $\{e_1, e_2, e_3\}$. Then $H(e_i) = L_i(H)e_i$ for all $H \in \mathfrak{h}$. So, the weights of V are L_1, L_2 , and L_3 . The weight space decomposition of V is

$$V \cong V_{L_1} \oplus V_{L_2} \oplus V_{L_3}.$$

Since $\ell(L_i) = c_i$ and $c_1 > c_2 > c_3$, the highest weight of V is L_1 . The weight diagram for V is:

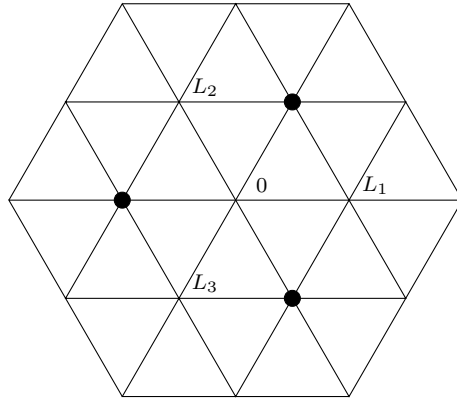


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Example 2.4.19. [9, Section 13.1] Let V be the standard representation of $\mathfrak{sl}_3$. Then the weights of the dual representation V^* are just the negatives of the weights of V . Thus V has weight space decomposition

$$V^* \cong V_{-L_1} \oplus V_{-L_2} \oplus V_{-L_3}.$$

Since $\ell(-L_i) = -c_i$ and $c_1 > c_2 > c_3$, the highest weight of V^* is $-L_3$. The weight diagram for V is:

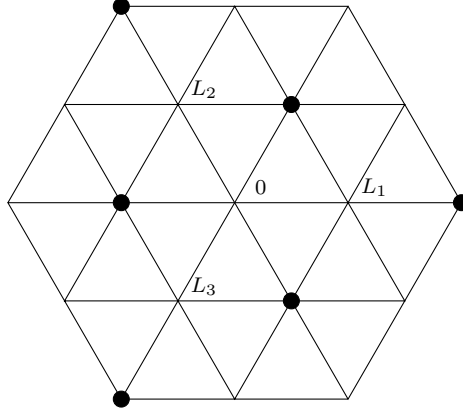


Consider the representation $\bigwedge^2 V$. The weights of $\bigwedge^2 V$ are $L_1 + L_2$, $L_1 + L_3$, and $L_2 + L_3$. Using the relation $L_1 + L_2 + L_3 = 0$, we see that the weights of V^* and $\bigwedge^2 V$ are the same. Since V^* and $\bigwedge^2 V$ are irreducible, and share the same highest weight, they are isomorphic as representations of $\mathfrak{sl}_3$. $\diamond$

Example 2.4.20. [9, Section 13.1] Let V be the standard representation of $\mathfrak{sl}_3$. The weights of $\text{Sym}^2 V$ are

$$\{L_1 + L_2, L_1 + L_3, L_2 + L_3, 2L_1, 2L_2, 2L_3\}.$$

Thus, the highest weight of $\text{Sym}^2 V$ is $2L_1$. The weight diagram in this case is:



Similarly, we can consider the representation $\text{Sym}^2 V^*$ whose weights are

$$\{-L_1 - L_2, -L_1 - L_3, -L_2 - L_3, -2L_1, -2L_2, -2L_3\}.$$

The highest weight of $\text{Sym}^2 V^*$ is $-2L_3$. Note that $\text{Sym}^2 V$ and $\text{Sym}^2 V^*$ are irreducible. In fact, $\text{Sym}^n V$ and $\text{Sym}^n V^*$ are irreducible with highest weights equal to nL_1 and $-nL_3$, respectively. $\diamond$

Proposition 2.4.21. [9, Observation 13.2] *Let V and W be representations of $\mathfrak{sl}_3$ with highest weight vectors $v \in V_\alpha$ and $w \in W_\beta$, respectively. Then $v \otimes w$ is a highest weight vector of $V \otimes W$ with corresponding weight $\alpha + \beta$.*

Proof. Since v and w are highest weight vectors, they are killed by the positive root spaces. So $E_{ij}(v) = E_{ij}(w) = 0$ for all $i < j$. It follows that

$$E_{ij}(v \otimes w) = E_{ij}(v) \otimes w + v \otimes E_{ij}(w) = 0$$

for all $i < j$. Moreover, we compute that

$$\begin{aligned} H(v \otimes w) &= H(v) \otimes w + v \otimes H(w) \\ &= \alpha(H)v \otimes w + v \otimes \beta(H)w \\ &= (\alpha(H) + \beta(H))(v \otimes w) \end{aligned}$$

for all $H \in \mathfrak{h}$. $\square$

Theorem 2.4.22. [9, Theorem 13.1] *For any pair of natural numbers a, b there exists a unique irreducible representation $\Gamma_{a,b}$ of $\mathfrak{sl}_3$ with highest weight $aL_1 - bL_3$. The weights of $\Gamma_{a,b}$ consist of those elements of Λ_W congruent to $aL_1 - bL_3$ modulo Λ_R and lying in the convex hull of the the images of $aL_1 - bL_3$ under the Weyl group $\mathfrak{W}$. This gives a bijection between $\mathcal{W} \cap \Lambda_W$ and the set of irreducible representations of $\mathfrak{sl}_3$.*

Proof. We first show existence. Let V be the standard representation of $\mathfrak{sl}_3$. Then aL_1 is the highest weight of $\text{Sym}^a V$, and $-bL_3$ is the highest weight of $\text{Sym}^b V^*$. By Proposition 2.4.21, the functional $aL_1 - bL_3$ is a highest weight of $\text{Sym}^a V \otimes \text{Sym}^b V^*$. Then, by Theorem 2.4.9, we can find an irreducible subrepresentation

$$\Gamma_{a,b} \subset \text{Sym}^a V \otimes \text{Sym}^b V^*$$

with highest weight $aL_1 - bL_3$. By Theorem 2.4.13, the weights of $\Gamma_{a,b}$ are as desired. The bijection follows by the discussion immediately above Proposition 2.4.15 and the fact that $aL_1 - bL_3 \in \mathcal{W} \cap \Lambda_W$.

We now show uniqueness. Let V and U be irreducible representations of $\mathfrak{sl}_3$ with highest weight vectors v and u , respectively. Assume that v and u have the same corresponding weight $\beta \in \mathfrak{h}^*$. Then (v, u) is a highest weight vector of $V \oplus U$ with corresponding weight β . Let W be the subspace of $V \oplus U$ generated by the images of (v, u) by successive application of the negative root vectors. By Theorem 2.4.9, W is an irreducible subrepresentation. Consider the projection maps

$$\pi_V : W \rightarrow V \quad \text{and} \quad \pi_U : W \rightarrow U,$$

which are homomorphisms of $\mathfrak{sl}_3$ -modules. These maps are nonzero since $\pi_V(v, u) = v$ and $\pi_U(v, u) = u$. Since V is irreducible and $\text{Im}(\pi_V)$ is a subrepresentation of V , we must have that $\text{Im}(\pi_V) = V$. Similarly $\text{Im}(\pi_U) = U$. Next, we have that $\ker(\pi_V)$ and $\ker(\pi_U)$ are subrepresentations of W . Since W is irreducible and $\pi_V \neq 0$, we must have that $\ker(\pi_V) = 0$. Similarly $\ker(\pi_U) = 0$. It follows that $V \cong W \cong U$ as representations. $\square$

2.5 Representations of semisimple Lie algebras

2.5.1 Irreducible representations of semisimple Lie algebras

The procedure for classifying irreducible representations of $\mathfrak{sl}_n$ is basically identical to that of $\mathfrak{sl}_3$. In this section, we see how the process for classifying representations of $\mathfrak{sl}_3$ generalizes to an algorithmic procedure for classifying representations of any complex semisimple Lie algebra. We illustrate this general paradigm by applying it to $\mathfrak{sl}_n$. Some of the claims made in this section are only proven for $\mathfrak{sl}_n$.

Throughout this section, let $\mathfrak{g}$ be a complex semisimple Lie algebra. We start by finding a maximal abelian subalgebra $\mathfrak{h} \subset \mathfrak{g}$ such that $\text{ad}(H)$ is diagonalizable for all $H \in \mathfrak{h}$. We

refer to the subalgebra $\mathfrak{h}$ as a **Cartan subalgebra** of $\mathfrak{g}$. Note that $\mathfrak{h}$ is unique up to an automorphism [9, Appendix D.3]. So, without loss of generality, we may speak of *the* Cartan subalgebra of $\mathfrak{g}$. There are two immediate properties that Cartan subalgebras satisfy. The first property is that elements in the image $\text{ad}(\mathfrak{h})$ are simultaneously diagonalizable. To see this, note that since $\mathfrak{h}$ is abelian, we get that

$$[\text{ad}(H), \text{ad}(H')] = \text{ad}([H, H']) = 0$$

for all $H, H' \in \mathfrak{h}$. Since the elements in $\text{ad}(\mathfrak{h})$ are diagonalizable and commute with one another, they are simultaneously diagonalizable. The second property is that $\mathfrak{h}$ acts diagonally on all representations of $\mathfrak{g}$. This follows from Remark 2.2.28, Corollary 2.2.32, and the fact that $\mathfrak{h}$ acts diagonally on the adjoint representation of $\mathfrak{g}$. We already saw two examples of Cartan subalgebras. The Cartan subalgebra of $\mathfrak{sl}_2$ is $\mathbb{C}H$, where $H = \text{diag}(1, -1)$. And, the Cartan subalgebra of $\mathfrak{sl}_3$ is the subalgebra of diagonal matrices.

Proposition 2.5.1. [9, Section 15.1] *The Cartan subalgebra of $\mathfrak{sl}_n$ is*

$$\mathfrak{h} := \{\text{diag}(a_1, \dots, a_n) : a_i \in \mathbb{C}, a_1 + \dots + a_n = 0\}.$$

Proof. Since diagonal matrices commute, we have that $[\mathfrak{h}, \mathfrak{h}] = 0$ and so $\mathfrak{h}$ is abelian. We show $\mathfrak{h}$ is maximal among abelian subalgebras of $\mathfrak{sl}_n$. Set $\mathfrak{g} := \mathfrak{sl}_n$ and note that $\mathfrak{h} = C_{\mathfrak{g}}(\mathfrak{h})$, where

$$C_{\mathfrak{g}}(\mathfrak{h}) := \{X \in \mathfrak{g} : [\mathfrak{h}, X] = 0\}$$

is the **centralizer** of $\mathfrak{h}$ in $\mathfrak{g}$. Suppose $\mathfrak{h}'$ is an abelian subalgebra of $\mathfrak{g}$ such that $\mathfrak{h} \subset \mathfrak{h}'$. Since $\mathfrak{h}'$ is abelian and $\mathfrak{h} \subset \mathfrak{h}'$, we have that $[\mathfrak{h}, \mathfrak{h}'] = 0$. Then $\mathfrak{h}' \subset C_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$ and so $\mathfrak{h}' = \mathfrak{h}$. Lastly, by Theorem 2.2.31, since the elements in $\mathfrak{h}$ are diagonal, we get that $\text{ad}(H)$ is diagonalizable for all $H \in \mathfrak{h}$. Alternatively, since $\mathfrak{h}$ consists of diagonal matrices, it acts diagonally on the faithful standard representation of $\mathfrak{g}$. Thus, by Corollary 2.2.32, $\mathfrak{h}$ acts diagonally on all representations of $\mathfrak{g}$. $\square$

Remark 2.5.2. Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{sl}_n$ consisting of diagonal matrices. For each $0 \leq i \leq n$, define $L_i \in \mathfrak{h}^*$ by

$$L_i(\text{diag}(a_1, \dots, a_n)) = a_i.$$

We may embed $\mathfrak{h} \subset \mathbb{C}^n$ by

$$\mathfrak{h} = \{(a_1, \dots, a_n) \in \mathbb{C}^n : a_1 + \dots + a_n = 0\}.$$

Then

$$\mathfrak{h}^* \cong \text{span}_{\mathbb{C}}\{L_1, \dots, L_n\} / \mathbb{C}(L_1 + \dots + L_n),$$

where

$$\mathbb{C}(L_1 + \cdots + L_n) := \text{span}_{\mathbb{C}}\{L_1 + \cdots + L_n\}.$$

The structure of $\mathfrak{g}$ can be described by the adjoint action of its Cartan subalgebra. More precisely, since $\mathfrak{h}$ acts diagonally on the adjoint representation of $\mathfrak{g}$, we get a decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\substack{\alpha \in \mathfrak{h}^* \\ \alpha \neq 0}} \mathfrak{g}_\alpha \right),$$

where

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g} : \text{ad}(H)(X) = \alpha(H)X \text{ for all } H \in \mathfrak{h}\}.$$

We refer to this decomposition as the **Cartan decomposition** of $\mathfrak{g}$. Note that the direct sum in this decomposition is over a finite set and

$$\mathfrak{h} = \{X \in \mathfrak{g} : \text{ad}(H)(X) = 0 \text{ for all } H \in \mathfrak{h}\}.$$

We refer to the $\alpha \in \mathfrak{h}^*$ in the Cartan decomposition as the **roots** of $\mathfrak{g}$, and the corresponding spaces $\mathfrak{g}_\alpha$ as the **root spaces**. The elements in the root spaces are called **root vectors**. By convention, the zero functional $0 \in \mathfrak{h}^*$ is not a root; however, in some cases it may be helpful to adopt the notation $\mathfrak{g}_0 = \mathfrak{h}$. We denote the set of roots by R .

Proposition 2.5.3. [9, Section 15.1] *Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{sl}_n$ consisting of diagonal matrices. Define $E_{ij} \in M_n(\mathbb{C})$ to be the matrix with 1 in the (i, j) -entry, and 0 elsewhere. Then the corresponding Cartan decomposition for $\mathfrak{sl}_n$ is*

$$\mathfrak{sl}_n = \mathfrak{h} \oplus \bigoplus_{i \neq j} \mathbb{C}E_{ij}, \quad 0 \leq i, j \leq n,$$

where $\mathbb{C}E_{ij}$ is the root space with root $L_i - L_j$. It follows that the set of roots is

$$R = \{L_i - L_j : i \neq j\}.$$

Proof. By the discussion immediately after Proposition 2.4.1, the matrices E_{ij} generate the root spaces for the adjoint action of $\mathfrak{h}$ on $\mathfrak{sl}_n$. Since

$$[H, E_{ij}] = (L_i - L_j)(H)E_{ij}$$

for all $H \in \mathfrak{h}$, the root corresponding to $\mathbb{C}E_{ij}$ is $L_i - L_j$. Setting $\mathfrak{g} := \mathfrak{sl}_n$, we may write

$$\mathfrak{sl}_n = \mathfrak{h} \oplus \left(\bigoplus_{i \neq j} \mathbb{C}E_{ij} \right) \cong \mathfrak{h} \oplus \left(\bigoplus_{i \neq j} \mathfrak{g}_{L_i - L_j} \right). \quad \square$$

Proposition 2.5.4. [9, Facts 14.2, Section 14.1] Let $\mathfrak{g}$ be a semisimple Lie algebra with Cartan subalgebra $\mathfrak{h}$. Then

- (i) each root space $\mathfrak{g}_\alpha$ is one-dimensional;
- (ii) R generates a lattice $\Lambda_R \subset \mathfrak{h}^*$ of rank equal to $\dim \mathfrak{h}$;
- (iii) R is symmetric about the origin, i.e., if $\alpha \in R$ is a root, then $-\alpha \in R$ is a root as well;
- (iv) the adjoint action of $\mathfrak{g}_\alpha$ carries the root space $\mathfrak{g}_\beta$ into $\mathfrak{g}_{\alpha+\beta}$.

Proof. We prove the proposition for $\mathfrak{g} = \mathfrak{sl}_n$. Since $\mathfrak{g}_{L_i - L_j} \cong \mathbb{C}E_{ij}$, we have that $\dim \mathfrak{g}_{L_i - L_j} = 1$. Since $L_i - L_j$ and $L_j - L_i$ are linearly dependent, the roots $L_i - L_j$ with $i < j$ form a generating set for Λ_R . Since there are $n - 1$ roots of the form $L_i - L_j$ with $i < j$, the rank of Λ_R is equal to $n - 1 = \dim \mathfrak{h}$. If $L_i - L_j \in R$ then $-(L_i - L_j) = L_j - L_i \in R$. Lastly, if $X \in \mathfrak{g}_\alpha$ and $Y \in \mathfrak{g}_\beta$, then

$$[H, [X, Y]] = (\alpha(H) + \beta(H))[X, Y]$$

for all $H \in \mathfrak{h}$. □

We refer to Λ_R from Proposition 2.5.4 as the **root lattice** of $\mathfrak{g}$. Now let V be *any* representation of $\mathfrak{g}$. Since $\mathfrak{h}$ acts diagonally on V , we get a decomposition

$$V = \bigoplus_{\beta \in \mathfrak{h}^*} V_\beta,$$

where

$$V_\beta := \{v \in V : H(v) = \beta(H)v \text{ for all } H \in \mathfrak{h}\}.$$

We refer to the $\beta \in \mathfrak{h}^*$ in the decomposition as the **weights** of V , and the corresponding spaces V_β as the **weight spaces**. The elements in the weight spaces are called **weight vectors**. This decomposition encodes how $\mathfrak{h}$ acts on V . To see how the rest of $\mathfrak{g}$ acts on V , let $X \in \mathfrak{g}_\alpha$ and $v \in V_\beta$. Then

$$H(X(v)) = (\alpha(H) + \beta(H))X(v)$$

for all $H \in \mathfrak{h}$. So, the action of $\mathfrak{g}_\alpha$ carries V_β into $V_{\alpha+\beta}$. Using this, we show that the weights of an irreducible representation of $\mathfrak{g}$ lie in a translate of Λ_R .

Proposition 2.5.5. [9, Section 14.1] Let $\mathfrak{g}$ be a semisimple Lie algebra and V be an irreducible representation of $\mathfrak{g}$. Then the weights of V are congruent to one another modulo the root lattice Λ_R .

Proof. The proof is identical to that for Proposition 2.4.5. $\square$

The next step is to find a subalgebra s_α of $\mathfrak{g}$ that is isomorphic to $\mathfrak{sl}_2$ for each $\alpha \in R$. Let $\mathfrak{g}_\alpha \subset \mathfrak{g}$ be any root space. By Proposition 2.5.4, we get that $\mathfrak{g}_{-\alpha}$ is a root space with $\dim \mathfrak{g}_\alpha = \dim \mathfrak{g}_{-\alpha} = 1$. Then $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is a subspace of $\mathfrak{g}_0 = \mathfrak{h}$ of dimension at most one. Since the adjoint action of $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ carries each of $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ into itself, the direct sum

$$s_\alpha := \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$$

is a subalgebra of $\mathfrak{g}$.

Proposition 2.5.6. [9, Facts 14.6] We have that $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \neq 0$ and $[[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}], \mathfrak{g}_\alpha] \neq 0$.

Proof. We prove this proposition for $\mathfrak{g} = \mathfrak{sl}_n$. Set $\alpha = L_i - L_j$. Then $E_{ij} \in \mathfrak{g}_\alpha$ and $E_{ji} \in \mathfrak{g}_{-\alpha}$ with $[E_{ij}, E_{ji}] = E_{ii} - E_{jj}$. Since $H_\alpha := E_{ii} - E_{jj} \neq 0$, we get that $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \neq 0$. Lastly, since $[H_\alpha, E_{ij}] = 2E_{ij} \neq 0$, we get that $[[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}], \mathfrak{g}_\alpha] \neq 0$. $\square$

As shown in [14, Theorem 8.31], Proposition 2.5.6 implies that there is a basis $\{X_\alpha, Y_\alpha, H_\alpha\}$ for s_α , with $X_\alpha \in \mathfrak{g}_\alpha$, $Y_\alpha \in \mathfrak{g}_{-\alpha}$, and $H_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$, that satisfies

$$[H_\alpha, X_\alpha] = 2X_\alpha, \quad [H_\alpha, Y_\alpha] = -2Y_\alpha, \quad [X_\alpha, Y_\alpha] = H_\alpha.$$

Under this basis, it follows that $s_\alpha \cong \mathfrak{sl}_2$. In the case of $\mathfrak{sl}_n$, if $\alpha = L_i - L_j$ then we take the basis $X_\alpha = E_{ij}$, $Y_\alpha = E_{ji}$, and $H_\alpha = H_{L_i - L_j}$. Then

$$s_{L_i - L_j} = \mathbb{C}E_{ij} \oplus \mathbb{C}E_{ji} \oplus \mathbb{C}H_{L_i - L_j} \cong \mathfrak{sl}_2.$$

Next, we use the integrality of the eigenvalues of representations of $\mathfrak{sl}_2$. For any representation of s_α , and hence any representation of $\mathfrak{g}$, the eigenvalues for the action of H_α must be integers. In other words, for any weight $\beta \in \mathfrak{h}^*$ of any representation of $\mathfrak{g}$, we have that $\beta(H_\alpha) \in \mathbb{Z}$ for all $\alpha \in R$. Just as in Proposition 2.4.11, the integrality of the $\beta(H_\alpha)$ can be seen by considering the representation

$$\bigoplus_{k \in \mathbb{Z}} V_{\beta + k\alpha}$$

of $s_\alpha \cong \mathfrak{sl}_2$, where V is a representation of $\mathfrak{g}$. It follows that all weights of all representations of $\mathfrak{g}$ lie in

$$\Lambda_W := \{\beta \in \mathfrak{h}^* : \beta(H_\alpha) \in \mathbb{Z} \text{ for all } \alpha \in R\}.$$

In particular, since R is the set of weights for the adjoint representation of $\mathfrak{g}$, we get that $R \subset \Lambda_W$ and hence $\Lambda_R \subset \Lambda_W$. We refer to Λ_W as the **weight lattice** of $\mathfrak{g}$.

Proposition 2.5.7. [9, Section 15.1] *If $\mathfrak{g} = \mathfrak{sl}_n$ then*

$$\Lambda_R = \{a_1 L_1 + \cdots + a_n L_n \in \mathfrak{h}^* : a_i \in \mathbb{Z}, a_1 + \cdots + a_n = 0\}$$

and

$$\Lambda_W = \text{span}_{\mathbb{Z}}\{L_1, \dots, L_n\} / \mathbb{Z}(L_1 + \cdots + L_n).$$

Proof. This follows by direct generalizations of the proofs for Propositions 2.4.6 and 2.4.12. $\square$

In addition to integrality, we use the fact that the eigenvalues of representations of $\mathfrak{sl}_2$ are symmetric about 0. For any $\alpha \in R$, we define the involution $W_\alpha : \mathfrak{h}^* \rightarrow \mathfrak{h}^*$ by

$$W_\alpha(\beta) = \beta - \frac{2\beta(H_\alpha)}{\alpha(H_\alpha)}\alpha = \beta - \beta(H_\alpha)\alpha.$$

Then W_α is reflection in the hyperplane

$$\Omega_\alpha := \{\beta \in \mathfrak{h}^* : \beta(H_\alpha) = 0\}.$$

Furthermore, we define the **Weyl group** $\mathfrak{W}$ of $\mathfrak{g}$ to be the group generated by the involutions W_α . There is an inner product on $\mathfrak{g}$ called the *Killing form*. By restriction, we get an inner product on $\mathfrak{h}$, and hence on $\mathfrak{h}^*$, which are both called the *Killing form*. Up to scalars, the Killing form on $\mathfrak{h}^*$ is the unique inner product such that the line spanned by each root $\alpha \in R$ is perpendicular to the hyperplane Ω_α [9, Section 14.1]. Thus, we can say that the Weyl group is the group generated by the reflections in the hyperplanes perpendicular to the roots in R . More details about the Killing form of an arbitrary complex semisimple Lie algebra can be found in [9]. In the case of $\mathfrak{sl}_n$, the **Killing form** on $\mathfrak{h}^*$ is defined by

$$B\left(\sum_i a_i L_i, \sum_i b_i L_i\right) = \frac{1}{2n} \left(\sum_i a_i b_i - \frac{1}{n} \sum_{ij} a_i b_j \right).$$

With respect to the Killing form, for any $L_i - L_j \in R$, the hyperplane $\Omega_{L_i - L_j}$ is perpendicular to the line $\mathbb{C}(L_i - L_j)$ spanned by $L_i - L_j$.

Proposition 2.5.8. [9, Section 15.1] If $\mathfrak{g} = \mathfrak{sl}_n$ then $\mathfrak{W} \cong S_n$.

Proof. Let $\alpha = L_i - L_j$ be any root. Then W_α is reflection in the hyperplane perpendicular to $\mathbb{C}(L_i - L_j)$. So, the action of W_α on $\mathfrak{h}^*$ permutes L_i and L_j , while fixing L_k for $k \neq i, j$. This is equivalent to saying that W_α is the transposition that exchanges L_i and L_j . Since $\mathfrak{W}$ is generated by the transpositions W_α , it is isomorphic to the symmetric group S_n on n letters. Here, it makes sense to label the n letters as $L_1, \dots, L_n$. $\square$

Proposition 2.5.9. [9, Section 14.1] The set of weights of any representation of $\mathfrak{g}$ is invariant under the Weyl group $\mathfrak{W}$.

Proof. It suffices to show that the weights of any representation V of $\mathfrak{g}$ are invariant under W_α for arbitrary $\alpha \in R$. Let β be a weight of V . Then for any $\alpha \in R$, the subspace

$$V_{[\beta]} := \bigoplus_{k \in \mathbb{Z}} V_{\beta + k\alpha}$$

is invariant under s_α and hence a representation of $s_\alpha \cong \mathfrak{sl}_2$. It follows that the eigenvalues

$$S := \{(\beta + k\alpha)(H_\alpha) = \beta(H_\alpha) + 2k : k \in \mathbb{Z}, V_{\beta + k\alpha} \neq 0\}$$

with respect to H_α are integral and symmetric about 0. Choose $k \in \mathbb{Z}$ so that $\beta(H_\alpha) + 2k$ is the largest integer in S . Since the elements in S are symmetric about 0, the smallest integer in S is $-\beta(H_\alpha) - 2k$. By $\mathfrak{sl}_2$, every integer that is between $-\beta(H_\alpha) - 2k$ and $\beta(H_\alpha) + 2k$, and of the same parity as $\beta(H_\alpha) + 2k$, also lies in S . Since the values in S lie between $-\beta(H_\alpha) - 2k$ and $\beta(H_\alpha) + 2k$, and are all of the same parity, they must form an unbroken string of integers congruent to one another modulo 2. The point is, after possibly replacing β with a translate by a multiple of α , we may assume that

$$S = \{\beta(H_\alpha) + 2k : 0 \leq k \leq m\}$$

for some $m \in \mathbb{N}$. Then the weights corresponding to weight spaces in $V_{[\beta]}$ are $\beta + k\alpha$ for $0 \leq k \leq m$. Since S is symmetric about 0, we find that $m = -\beta(H_\alpha)$. Then

$$\beta + m\alpha = \beta - \beta(H_\alpha)\alpha = W_\alpha(\beta).$$

So, the weight $\beta + m\alpha$ is the reflection of β in Ω_α . Since $W_\alpha(\alpha) = -\alpha$, we get that

$$W_\alpha(\beta + k\alpha) = W_\alpha(\beta) - k\alpha$$

for all $0 \leq k \leq m$. This shows that the weights corresponding to weight spaces in $V_{[\beta]}$ are invariant under W_α . The result follows since if $V = \bigoplus V_\beta$ is the weight space decomposition of V , then

$$V = \bigoplus_{\beta} V_\beta = \sum_{\beta} V_{[\beta]}. \quad \square$$

Next in the analysis, we choose a linear functional $\ell : \Lambda_R \rightarrow \mathbb{R}$ and extend it by linearity to a linear functional $\ell : \mathfrak{h}^* \rightarrow \mathbb{C}$. We choose ℓ in such a way so that the hyperplane $\ell = 0$ does not contain any of the points in Λ_R . We then partition the set of roots into

$$R = R^+ \cup R^-,$$

where $R^+ := \{\alpha \in R : \ell(\alpha) > 0\}$ and $R^- := \{\alpha \in R : \ell(\alpha) < 0\}$. We refer to the roots in R^+ as the **positive roots**, and the roots in R^- as the **negative roots**. In the case of $\mathfrak{sl}_n$, we define $\ell : \mathfrak{h}^* \rightarrow \mathbb{C}$ by

$$\ell(a_1 L_1 + \cdots + a_n L_n) = a_1 c_1 + \cdots + a_n c_n,$$

where $c_i \in \mathbb{R}$ satisfy

$$c_1 > \cdots > c_n \quad \text{and} \quad c_1 + \cdots + c_n = 0,$$

and are chosen so that ℓ is irrational with respect to Λ_R . Given this choice of ℓ , we partition $R = \{L_i - L_j : i \neq j\}$ into

$$R^+ = \{L_i - L_j \in R : i < j\} \quad \text{and} \quad R^- = \{L_i - L_j : j < i\}.$$

Now let V be any representation of $\mathfrak{g}$. If $v \in V$ is a nonzero weight vector and in the kernel of $\mathfrak{g}_\alpha$ for all $\alpha \in R^+$, we say that v is a **highest weight vector** of V . We refer to the weight ω of V for which $\text{Re}(\ell(\omega))$ is maximal as the **extremal weight**. By irrationality of ℓ , the extremal weight is unique.

Theorem 2.5.10. [9, Proposition 14.13] *For any semisimple Lie algebra $\mathfrak{g}$, we have the following:*

- (i) *Every representation V of $\mathfrak{g}$ possesses a highest weight vector.*
- (ii) *The subspace W of V generated by the images of a highest weight vector v under successive applications of root spaces $\mathfrak{g}_\alpha$ for $\alpha \in R^-$ is an irreducible subrepresentation. In particular, if V is irreducible, then $V = W$.*

(iii) *An irreducible representation V possesses a unique highest weight, that is equal to its extremal weight. It follows that V has a unique highest weight vector, up to scalars.*

Proof. The proof of part (iii) is identical to that for Theorem 2.4.9. For part (i), choose any nonzero vector $v \in V_\omega$, where ω is the extremal weight for V . Then for any $\alpha \in R^+$, we have that $\mathfrak{g}_\alpha \cdot V_\omega \subset V_{\omega+\alpha}$. Since ω was chosen so that $\text{Re}(\ell(\omega))$ is maximal, we must have that $V_{\omega+\alpha} = 0$. So $\mathfrak{g}_\alpha \cdot V_\omega = 0$ and v is necessarily in the kernel of $\mathfrak{g}_\alpha$ for all $\alpha \in R^+$. We conclude that v is a highest weight vector of V .

To prove part (ii), we follow a similar strategy as in part (i) of Theorem 2.4.9. It suffices to show that W is invariant under the positive root spaces. Let w_n denote any word of length n or less in the elements of $\mathfrak{g}_\alpha$ for $\alpha \in R^-$. Also, let W_n denote the subspace spanned by the vectors $w_n(v)$ for all possible w_n . Then

$$W = \bigcup_n W_n.$$

We do induction on n to show that $X(W_n) \subset W_n$ for all $\alpha \in R^+$ and $X \in \mathfrak{g}_\alpha$. If $n = 0$ then $W_0 = \mathbb{C}v$. By part (i), the highest weight vector v is killed by the positive root spaces. Thus $X(\lambda v) = \lambda X(v) = 0$ for all $\lambda v \in W_0$. So $X(W_0) = 0$ for all $X \in \mathfrak{g}_\alpha$. Now suppose $X(W_{n-1}) \subset W_{n-1}$ for all $\alpha \in R^+$ and $X \in \mathfrak{g}_\alpha$. We can write any generator $w_n(v)$ in the form $Y(w)$, where $w \in W_{n-1}$ and $Y \in \mathfrak{g}_{\alpha'}$ for some $\alpha' \in R^-$. Then

$$X(w_n(v)) = X(Y(w)) = Y(X(w)) + [X, Y](w),$$

where $[X, Y] \in \mathfrak{g}_{\alpha+\alpha'}$. We have three cases to consider for $\alpha + \alpha'$. By definition, if $\alpha + \alpha' \in R^-$ then $[X, Y](w) \in W_n$. If $\alpha + \alpha' \in R^+$, by the inductive hypothesis, we get that $[X, Y](W_{n-1}) \subset W_{n-1}$. It follows that $[X, Y](w) \in W_{n-1} \subset W_n$. Lastly, if $\alpha + \alpha' = 0$ then $[X, Y] \in \mathfrak{h}$ and so $[X, Y](w) \in W_{n-1} \subset W_n$. By the inductive hypothesis, $X(w) \in W_{n-1}$ and hence $Y(X(w)) \in W_n$. Since both $Y(X(w))$ and $[X, Y](w)$ are in W_n , we get that $X(w_n(v)) \in W_n$. It follows that W is invariant under $\mathfrak{g}$ since each W_n is.

To show that W is irreducible, assume $W = W_1 \oplus W_2$ for some representations $W_i \subseteq W$ of $\mathfrak{g}$. Then

$$W_\omega = (W_1)_\omega \oplus (W_2)_\omega.$$

By above, we must have that $W_\omega \cong \mathbb{C}v$ and hence $\dim W_\omega = 1$. Thus, we may assume that $(W_1)_\omega = 0$. Then $v \in W_\omega = (W_2)_\omega$ and so $v \in W_2$. Since W_2 is a representation of $\mathfrak{g}$, it is invariant under the negative root spaces. Hence, the images of $v \in W_2$ under successive applications of the root spaces $\mathfrak{g}_\alpha$ for $\alpha \in R^-$ are in W_2 . Thus $W \subseteq W_2$ and so $W = W_2$. We conclude that W is irreducible. $\square$

Remark 2.5.11. In the case of $\mathfrak{sl}_n$, the subspace W above is generated by images of v under successive applications of the root spaces $\mathfrak{g}_{L_i-L_j} \cong \mathbb{C}E_{ij}$ for $j < i$.

Lemma 2.5.12. [9, Section 14.1] *Let V be an irreducible representation of $\mathfrak{g}$ with extremal weight ω . Then every vertex of the convex hull of the weights of V must be conjugate to ω under the Weyl group $\mathfrak{W}$. Moreover, for every $\alpha \in R^+$, we have that $\omega(H_\alpha) \in \mathbb{N}$.*

Proof. Let $v \in V_\omega$ be nonzero. Then by Theorem 2.5.10, the set of weights of V is contained in the cone $\omega + C^-$, where C^- is the positive real cone spanned by the roots $\alpha \in R^-$ such that $\mathfrak{g}_\alpha(v) \neq 0$. Conversely, since ω is the highest weight of V , we have that $V_{\omega+k\alpha} = 0$ for all $\alpha \in R^-$ and $k < 0$. Thus, for any $\alpha \in R^-$, the set of weights of V contains the unbroken string

$$\omega + k\alpha, \quad 0 \leq k \leq m,$$

for some $m \in \mathbb{N}$. We may choose m so that $V_{\omega+(m+1)\alpha} = 0$. Then

$$\bigoplus_{k=0}^m V_{\omega+k\alpha}$$

is a representation of $s_\alpha \cong \mathfrak{sl}_2$. It follows that the eigenvalues

$$\omega(H_\alpha) + 2k, \quad 0 \leq k \leq m,$$

with respect to H_α are integral and symmetric about 0. Thus $\omega + n\alpha = \omega - \omega(H_\alpha)\alpha$. So, any vertex of the convex hull of the weights of V that is adjacent to ω is of the form

$$\omega - \omega(H_\alpha)\alpha = W_\alpha(\omega)$$

for some $\alpha \in R$. Applying this analysis to each successive vertex gives the first statement. From above, we have that $-\omega(H_\alpha) \in \mathbb{N}$ for all $\alpha \in R^-$. Thus $\omega(H_\alpha) \in \mathbb{N}$ for all $\alpha \in R^+$. $\square$

Theorem 2.5.13. [9, Section 14.1] *Let V be an irreducible representation of $\mathfrak{g}$ with extremal weight ω . The weights of V are exactly the linear functionals $\beta \in \Lambda_W$ which are congruent to ω modulo Λ_R and lie in the convex hull of the images of ω under the Weyl group $\mathfrak{W}$.*

Proof. We provide a sketch of the proof. The argument directly above Proposition 2.5.7 shows that the weights of V lie in Λ_W . By Proposition 2.5.5, the weights of V are congruent to one another modulo Λ_R . In particular, the weights of V are congruent to ω modulo Λ_R . By Lemma 2.5.12, the vertices of the convex hull of the weights of V are obtained by

the reflection of ω in the hyperplanes Ω_α . Thus, the set of weights of V must lie in the convex hull of the images of ω under the Weyl group $\mathfrak{W}$. Conversely, let β be a weight and $\alpha \in R$. As demonstrated in the proof of Proposition 2.5.9, the intersection of the set of weights of V with $\{\beta + k\alpha : k \in \mathbb{Z}\}$ is an unbroken string. So, the set of weights of V is “convex” in the sense that for any two weights, every functional on the line connecting them in the direction of some $\alpha \in R$ is also a weight. Since any element in the intersection of the convex hull with $\omega + \Lambda_R$ is of the form $\beta + k\alpha$, it is necessarily a weight. $\square$

Let V be an irreducible representation of $\mathfrak{g}$. By Lemma 2.5.12, we have that $\omega(H_\alpha) \in \mathbb{N}$ for all $\alpha \in R^+$. It follows that the highest weight ω lives in

$$\mathcal{W} := \{\beta \in \mathfrak{h}^* : \beta(H_\alpha) \geq 0, \text{ for all } \alpha \in R^+\}.$$

So $\omega \in \mathcal{W} \cap \Lambda_W$. We refer to $\mathcal{W}$ as the **Weyl chamber** of $\mathfrak{g}$.

Proposition 2.5.14. [9, Section 15.1] *If $\mathfrak{g} = \mathfrak{sl}_n$ then*

$$\mathcal{W} = \{a_1 L_1 + \cdots + a_n L_n \in \mathfrak{h}^* : a_1 \geq \cdots \geq a_n\}$$

and

$$\mathcal{W} \cap \Lambda_W = \text{span}_{\mathbb{N}} \left\{ \sum_{1 \leq i \leq k} L_i \in \mathfrak{h}^* : k < n \right\}.$$

Proof. The first equality follows by a direct generalization of the proof for Proposition 2.4.15. By the first equality and Proposition 2.5.7, we get that

$$\mathcal{W} \cap \Lambda_W = \{a_1 L_1 + \cdots + a_n L_n \in \mathfrak{h}^* : a_i \in \mathbb{Z}, a_1 \geq \cdots \geq a_n\}.$$

Using this, and a direct generalization of the proof for Proposition 2.4.17, the second equality holds. $\square$

We are now ready to state the classification theorem for irreducible representations of an arbitrary complex semisimple Lie algebra.

Theorem 2.5.15. [9, Theorem 14.18] *For any $\beta \in \mathcal{W} \cap \Lambda_W$ there exists a unique irreducible representation Γ_β of $\mathfrak{g}$ with highest weight β . The weights of Γ_β consist of those elements of Λ_W congruent to β modulo Λ_R and lying in the convex hull of the images of β under the Weyl group $\mathfrak{W}$. This gives a bijection between $\mathcal{W} \cap \Lambda_W$ and the set of irreducible representations of $\mathfrak{g}$.*

Proof. Uniqueness follows by replacing Theorem 2.4.9 with Theorem 2.5.10 in the proof of uniqueness for Theorem 2.4.22. We prove existence in Section 2.5.2 for $\mathfrak{g} = \mathfrak{sl}_n$. $\square$

2.5.2 Irreducible representations of $\mathfrak{sl}_n$

Before we prove Theorem 2.5.15 for $\mathfrak{g} = \mathfrak{sl}_n$, let's do some examples.

Example 2.5.16. [9, Section 15.2] Let $V \cong \mathbb{C}^n$ be the standard representation of $\mathfrak{sl}_n$ with standard basis $\{e_1, \dots, e_n\}$. Then

$$H(e_i) = L_i(H)e_i$$

for all $H \in \mathfrak{h}$. So, the set of weights of V is $\{L_1, \dots, L_n\}$. Since $\ell(L_i) = c_i$ and $c_1 \geq \dots \geq c_n$, the highest weight of V is L_1 . $\diamond$

Example 2.5.17. [9, Section 15.2] Consider $\bigwedge^k V$ where V is the standard representation of $\mathfrak{sl}_n$. The weights of $\bigwedge^k V$ are sums of distinct weights of V of length k . So, the highest weight of $\bigwedge^k V$ is $L_1 + \dots + L_k$. It follows that the highest weight of $\text{Sym}^a(\bigwedge^k V)$ is $a(L_1 + \dots + L_k)$. $\diamond$

Proposition 2.5.18. [9, Observation 13.2] Let V and W be representations of $\mathfrak{g}$ with highest weight vectors $v \in V_\alpha$ and $w \in W_\beta$, respectively. Then $v \otimes w$ is a highest weight vector of $V \otimes W$ with corresponding weight $\alpha + \beta$.

Proof. The proof is similar to that for Proposition 2.4.21. Since v and w are highest weight vectors, they are killed by the positive roots spaces. So, for any $\alpha \in R^+$, we have that $X(v) = X(w) = 0$ for all $X \in \mathfrak{g}_\alpha$. It follows that

$$X(v \otimes w) = X(v) \otimes w + v \otimes X(w) = 0$$

for all $X \in \mathfrak{g}_\alpha$. Moreover, we compute that

$$H(v \otimes w) = (\alpha(H) + \beta(H))(v \otimes w)$$

for all $H \in \mathfrak{h}$. $\square$

Theorem 2.5.19. [9, Section 15.1, Section 15.2] For any $(a_1, \dots, a_{n-1}) \in \mathbb{N}^{n-1}$ there exists a unique irreducible representation $\Gamma_{a_1, \dots, a_{n-1}}$ of $\mathfrak{sl}_n$ with highest weight

$$\beta := a_1 L_1 + a_2(L_1 + L_2) + \dots + a_{n-1}(L_1 + \dots + L_{n-1}).$$

The weights of $\Gamma_{a_1, \dots, a_{n-1}}$ consist of those elements of Λ_W congruent to β modulo Λ_R and lying in the convex hull of the images of β under the Weyl group $\mathfrak{W}$. This gives a bijection between $\mathcal{W} \cap \Lambda_W$ and the set of irreducible representations of $\mathfrak{sl}_n$.

Proof. Uniqueness follows from Theorem 2.5.15. To show existence, let $V \cong \mathbb{C}^n$ be the standard representation of $\mathfrak{sl}_n$. For each $1 \leq k \leq n-1$, the representation $\text{Sym}^{a_k}(\bigwedge^k V)$ of $\mathfrak{sl}_n$ has highest weight equal to

$$a_k \sum_{i=1}^k L_i = a_k(L_1 + \cdots + L_k).$$

Then, by Proposition 2.5.18, the functional β is a highest weight of

$$\text{Sym}^{a_1} V \otimes \text{Sym}^{a_2}(\bigwedge^2 V) \otimes \cdots \otimes \text{Sym}^{a_{n-1}}(\bigwedge^{n-1} V).$$

By Theorem 2.5.10, we can find an irreducible subrepresentation

$$\Gamma_{a_1, \dots, a_{n-1}} \subset \text{Sym}^{a_1} V \otimes \text{Sym}^{a_2}(\bigwedge^2 V) \otimes \cdots \otimes \text{Sym}^{a_{n-1}}(\bigwedge^{n-1} V)$$

of $\mathfrak{sl}_n$ with highest weight β . By Theorem 2.5.13, the weights of $\Gamma_{a_1, \dots, a_{n-1}}$ are as desired. Note, by Proposition 2.5.14, that $\beta \in \mathcal{W} \cap \Lambda_W$. Thus, each point in $\mathcal{W} \cap \Lambda_W$ is the highest weight of a unique irreducible representation of $\mathfrak{sl}_n$. Conversely, by the discussion immediately after Theorem 2.5.13, every irreducible representation of $\mathfrak{sl}_n$ has unique highest weight in $\mathcal{W} \cap \Lambda_W$. This proves the bijection. $\square$

Chapter 3

Geometric invariant theory

3.1 Introduction

The building blocks of geometric invariant theory (GIT) are comprised of two objects: groups and varieties. Geometric invariant theory is a branch of abstract algebra that brings these two objects together by studying the actions of groups on varieties from the point of view of their effect on functions. Specifically, if G is a group that acts on a variety V over a field k , we may consider the algebra $k[V]^G$ of G -invariant polynomials. Representation theory is directly tied to invariant theory since representations of groups yield examples of group actions on varieties. One reason we study the invariants $k[V]^G$ is to parametrize orbits or, more precisely, to construct a quotient variety. Specifically, the commutative algebra $k[V]^G$ gives rise to the quotient space $V//G := \text{Spec}(k[V]^G)$ and quotient map $\pi : V \rightarrow V//G$. In general, $V//G$ does not behave like a true orbit space and so we often eliminate certain orbits from consideration in order to construct a quotient with good structure. Explicitly, we restrict π to the set of G -stable points V^s , which are the points that have well-behaved orbits. At its heart, geometric invariant theory is the study of $k[V]^G$ and its interactions with $k[V]$. One property of (V, G) that we care about, called *cofree*, is when $k[V]$ is free as a $k[V]^G$ -module. The geometric description of cofree is that $V//G$ is smooth and π behaves as nicely as possible. In this thesis, we focus on intrinsically characterizing irreducible representations of $\text{SL}_n(\mathbb{C}) \times \text{SL}_m(\mathbb{C})$ that are cofree via a geometric property called *pure*. The problem of characterizing cofree representations is a motivating problem for a field of research that has been active for over 70 years. Since the Lie groups that we consider in this thesis are over $\mathbb{C}$, we write SL_n to mean $\text{SL}_n(\mathbb{C})$.

3.2 Preliminaries

Throughout, let k be an algebraically closed field of characteristic 0. We consider finite-dimensional *representations* V over k of a *reductive group* G . We say that a group G over k is an **affine algebraic group** (or linear algebraic group) if it is isomorphic to a subgroup of $\mathrm{GL}_n(k)$ and has the structure of an affine variety. Further, we say that the group G over k is **reductive** if it is an affine algebraic group and satisfies the property that every finite-dimensional representation of G decomposes into a direct sum of irreducible representations of G . A **representation of a reductive group** G is a group homomorphism $G \rightarrow \mathrm{GL}(V)$ such that the action map $G \times V \rightarrow V$ is a regular map of affine varieties (i.e., polynomial). We may also say that the action of G on V is **regular** when $G \times V \rightarrow V$ is regular. When we write (V, G) we are referring to the representation $G \rightarrow \mathrm{GL}(V)$, however, we often just write V for brevity when the group action is clear.

Remark 3.2.1. The majority of the content in Chapter 3 is also applicable to regular actions of an algebraic group on an algebraic variety. However, we restrict to the case of representations of reductive groups as it is most relevant to this thesis. Some statements in this chapter are stated more generally for a **quasi-affine variety**, which is an open subset of an affine variety.

Example 3.2.2. All simple and semisimple Lie groups over k are reductive. Relevant examples of this include SL_n and $\mathrm{SL}_n \times \mathrm{SL}_m$. $\diamond$

Viewing V as an affine variety, recall that $k[V]$ is the **coordinate ring** of V . For an explicit description, $k[V]$ is polynomials in the dual basis of a fixed basis for V . That is, if $\{x_1, \dots, x_n\}$ is a basis for V and $\{x_1^*, \dots, x_n^*\}$ is the associated dual basis for V^* , then $k[V] = k[x_1^*, \dots, x_n^*]$. Then V can be viewed as the affine variety $\mathrm{Spec}(k[x_1^*, \dots, x_n^*]) \cong \mathbb{A}_k^n$. The symmetric algebra $\mathrm{Sym}(V^*)$ of V^* gives a coordinate free description of V as an affine variety because $k[x_1^*, \dots, x_n^*] = \mathrm{Sym}(V^*)$. Now, the linear group action of G on V induces an action on $k[V]$ given by

$$(g \cdot f)(v) := f(g^{-1}v), \quad g \in G, f \in k[V], v \in V.$$

We define the space of **invariant polynomials** for (V, G) , denoted $k[V]^G$, to be the set of polynomials invariant under this group action. That is,

$$k[V]^G := \{f \in k[V] : g \cdot f = f \text{ for all } g \in G\}.$$

One may interpret the invariant polynomials as polynomials which are fixed under the “change of variables” due to the action of G . Indeed, given $f \in k[V]^G$, observe that

$f(v) = f(g(v))$ for all $g \in G$ since $f(v) = (g^{-1} \cdot f)(v) = f(g(v))$. Note that constants are always invariant, and so $k \subseteq k[V]^G$.

Example 3.2.3. Let $V = \mathbb{A}_{\mathbb{C}}^2$ and $G = \mathbb{C}^*$. Consider the action of G on V given by

$$\lambda \cdot (a, b) = (\lambda a, \lambda^{-1} b).$$

Writing $k[V] = k[x, y]$, observe that $xy \in k[V]^G$ since

$$(\lambda \cdot xy)(a, b) = (xy)(\lambda^{-1} \cdot (a, b)) = (xy)(\lambda^{-1} a, \lambda b) = \lambda^{-1} a \lambda b = ab = (xy)(a, b)$$

for all $\lambda \in G$ and $(a, b) \in V$. That is, $\lambda \cdot xy = xy$ for all $\lambda \in G$. So, $k[xy] \subseteq k[V]^G$. Now, let $f \in k[V]^G$ and write

$$f(x, y) = \sum_{i=0}^n \sum_{j=0}^{n'} a_{ij} x^i y^j, \quad a_{ij} \in k.$$

Note that $f \in k[V]^G$ if and only if $f(x, y) = f(\lambda x, \lambda^{-1} y)$ for all $\lambda \in G$. This is equivalent to having

$$\sum_{i=0}^n \sum_{j=0}^{n'} a_{ij} x^i y^j = \sum_{i=0}^n \sum_{j=0}^{n'} a_{i,j} \lambda^{i-j} x^i y^j$$

for all $\lambda \in G$. This equality forces $i = j$ and $n = n'$. So, $f \in k[V]^G$ if it takes the form

$$f(x, y) = \sum_{i=0}^n b_i (xy)^i \in k[x, y].$$

We conclude that $k[V]^G = k[xy]$, which is a polynomial ring in one variable and hence finitely generated. $\diamond$

Example 3.2.4. Let V be a representation of G such that G acts transitively on $V \setminus \{0\}$. We show that $k[V]^G = k$. Let $f \in k[V]$ be non-constant so that $f(v) \neq f(v')$ for some $v, v' \in V$ with $v \neq v'$. Since f is continuous, we may choose v and v' to be nonzero. Then, by transitivity, there exists $g \in G$ such that $g^{-1}v = v'$. It follows that

$$(g \cdot f)(v) = f(g^{-1}v) = f(v') \neq f(v)$$

and so $f \notin k[V]^G$. So, the only invariant polynomials are the constants. That is, $k[V]^G = k$. Note that the standard representation of SL_n is an example of this particular situation. $\diamond$

Given that $k[V]$ is a k -algebra, a natural question to ask is: when is $k[V]^G$ finitely generated? Indeed, this has been an important problem in GIT and is known as Hilbert's 14th problem [19, pg. 171]. Nagata [17, pg. 372] showed that $k[V]^G$ is a finitely generated k -algebra when G is reductive (the case we are considering). It follows that the inclusion $\pi^* : k[V]^G \hookrightarrow k[V]$ of finitely generated k -algebras induces a morphism

$$\pi : V \rightarrow \operatorname{Spec}(k[V]^G) =: V//G$$

of affine varieties. We refer to π as the affine **GIT quotient** associated to π^* . Since $k[V]^G$ is finitely generated, we may write $k[V]^G = k[f_1, \dots, f_r]$ for some $f_i \in k[V]^G$. Because there may be relations between the f_i , we have that $V//G \subseteq k^r$ with equality not always occurring. An explicit description of π is then

$$\pi : V \rightarrow V//G, \quad v \mapsto (f_1(v), \dots, f_r(v)).$$

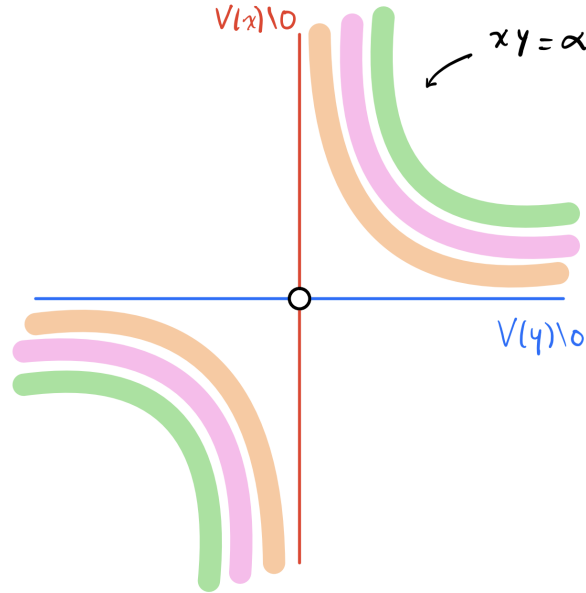
Example 3.2.5. Consider Example 3.2.3, where $V = \mathbb{A}_{\mathbb{C}}^2$ and $G = \mathbb{C}^*$. Writing $k[V] = k[x, y]$, we found that $k[V]^G = k[xy]$. So, $V//G = \operatorname{Spec}(k[xy]) = \mathbb{A}_{\mathbb{C}}$ and

$$\pi : \mathbb{A}_{\mathbb{C}}^2 \rightarrow \mathbb{A}_{\mathbb{C}}^2 // \mathbb{C}^* = \mathbb{A}_{\mathbb{C}}, \quad (a, b) \mapsto (xy)(a, b) = ab.$$

Observe that π is a surjective map. As a means to foreshadow the next discussion, let's compute the orbits of the action of G on V . Recalling that the action is $\lambda \cdot (a, b) = (\lambda a, \lambda^{-1}b)$, the orbits are:

- $G \cdot (0, 0) = \{(0, 0)\}$ is the origin.
- For $a \neq 0$, $G \cdot (a, 0) = \{(x, 0) : x \neq 0\} = V(y) \setminus \{(0, 0)\}$ is the punctured x -axis.
- For $b \neq 0$, $G \cdot (0, b) = \{(0, y) : y \neq 0\} = V(x) \setminus \{(0, 0)\}$ is the punctured y -axis.
- For $ab \neq 0$, $G \cdot (a, b) = \{(x, y) : xy = ab\} = V(xy - ab)$ is the hyperbola defined by the equation $xy = ab$. So, for any $\alpha \in \mathbb{C}^*$, the hyperbola $xy = \alpha$ is an orbit.

The above orbits can be depicted in an illustration as follows:



Observe that the first three orbits all get mapped to $\{(0, 0)\}$ under π . Whereas, the orbit $V(xy - \alpha)$ for $\alpha \in \mathbb{C}^*$ gets mapped to $\{\alpha\}$ under π . Given this, we see that π identifies orbits whose closures meet. For instance, $V(x) \setminus \{(0, 0)\}$ and $V(y) \setminus \{(0, 0)\}$ have the origin in their closure and so they get identified as the origin under the quotient map. Because the orbit $V(xy - \alpha)$ depends on the parameter α , its image $\{\alpha\}$ under π can be viewed as the orbit $V(xy - \alpha)$ itself. As such, $V//G$ is *sort* of like an orbit space; it collapses orbits into points. But $V//G$ is *not* an orbit space because 3 orbits get identified to the same point. Namely, the pre-image $\pi^{-1}(0, 0)$ is not a single orbit, but rather a union of 3 orbits. $\diamond$

Next, we consider the usual quotient map

$$q : V \rightarrow V/G, \quad v \rightarrow G \cdot v,$$

where V/G is the **orbit space** consisting of all orbits of V under the action of G . Some facts to note about q are:

1. $q : V \rightarrow V/G$ is a **topological quotient**; it is surjective and satisfies the property that: $U \subseteq V/G$ is open (resp. closed) if and only if the preimage $q^{-1}(U)$ is open (resp. closed).

2. $q : V \rightarrow V/G$ is a **categorical quotient**; it is G -invariant and satisfies the universal property that any other G -invariant morphism factors uniquely through it.
3. The fibre of a point in V/G under q is a single orbit.

Properties 1 and 2 listed above also hold for the GIT quotient. Both of these properties follow from the fact that a GIT quotient is a *good quotient*. Although the GIT quotient behaves in many of the same ways as the usual quotient map, in general, they are not the same. This is because property 3 listed above does not hold in general for GIT quotients. That is, in general, $V//G$ does not parametrize distinct orbits. For example, $V//G$ in Example 3.2.5 identifies multiple distinct orbits. However, if $V//G$ behave like a true orbit space, then we say that π is a *geometric quotient*.

Definition 3.2.6. [2, Section 1.2.3] Let G be a reductive group acting on a quasi-affine variety X and let Y be a quasi-affine variety. A regular map $\varphi : X \rightarrow Y$ is a **good quotient** for the action of G on X if

1. φ is G -invariant, or equivalently, φ is constant on orbits.
2. φ is surjective.
3. φ is affine; if U is open and isomorphic to an affine variety, then $\varphi^{-1}(U)$ is open and isomorphic to an affine variety.
4. If $U \subseteq Y$ is open, then $k[U] \cong k[\varphi^{-1}(U)]^G$.
5. If C is a G -invariant closed subset of X , then $\varphi(C)$ is closed in Y .
6. If C_1 and C_2 are disjoint G -invariant closed subsets of X , then $\varphi(C_1) \cap \varphi(C_2) = \emptyset$.

Theorem 3.2.7. [19, Section 4.4] The GIT quotient $\pi : V \rightarrow V//G$ is a good quotient, categorical quotient, and topological quotient.

Sketch. It is shown in [19, Section 4.4] that π is a good quotient. Since any good quotient is a categorical quotient ([2, Corollary 1.2.3.8]), π is also a categorical quotient. By [2, Corollary 1.2.3.8], since π is a good quotient, $V//G$ carries the quotient topology with respect to the map π . This, together with the fact that π is surjective, means that π is a topological quotient. $\square$

Proposition 3.2.8. [2, Proposition 1.2.3.9] For any open $U \subseteq V//G$, the restriction $\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \rightarrow U$ is a good quotient for the restricted G -action.

Lemma 3.2.9. *If $\varphi : X \rightarrow Y$ is a good quotient for the action of G on X , then φ is constant on orbit closures. That is, $\varphi(\overline{G \cdot x}) = \varphi(x)$ for all $x \in X$.*

Proof. Since φ is continuous, $\varphi^{-1}(\varphi(x))$ is closed. Then $G \cdot x \subseteq \varphi^{-1}(\varphi(x))$ implies that $\overline{G \cdot x} \subseteq \varphi^{-1}(\varphi(x))$. So, $\varphi(\overline{G \cdot x}) = \varphi(x)$. $\square$

Corollary 3.2.10. *If $\varphi : X \rightarrow Y$ is a good quotient for the action of G on X , then $\varphi(x_1) = \varphi(x_2)$ if and only if $\overline{G \cdot x_1} \cap \overline{G \cdot x_2} \neq \emptyset$. In other words, good quotients identify points whose closures meet.*

Proof. By Lemma 3.2.9, if $\varphi(x_1) = \varphi(x_2)$ for some $x_1, x_2 \in X$, then $\varphi(\overline{G \cdot x_1}) = \varphi(\overline{G \cdot x_2})$. Since the action of G on X is continuous, the closures of orbits are G -invariant. So, by Property 6 of a good quotient, $\overline{G \cdot x_1} \cap \overline{G \cdot x_2} \neq \emptyset$. Conversely, suppose $\overline{G \cdot x_1} \cap \overline{G \cdot x_2} \neq \emptyset$. Then there exists $x \in X$ with $x \in \overline{G \cdot x_1}$ and $x \in \overline{G \cdot x_2}$. By Lemma 3.2.9, $\varphi(x) = \varphi(x_1)$ and $\varphi(x) = \varphi(x_2)$. So, $\varphi(x_1) = \varphi(x_2)$. $\square$

Lemma 3.2.11. [2, Corollary 1.2.3.7] *Let $\varphi : X \rightarrow Y$ be a good quotient for the action of G on X . For every $x \in X$, the boundary $\overline{G \cdot x} \setminus G \cdot x$ of $G \cdot x$ is a union of orbits of strictly smaller dimension than $\dim G \cdot x$ and $\overline{G \cdot x}$ contains a unique closed orbit. It follows that for every $y \in Y$, the fibre $\varphi^{-1}(y)$ contains a unique closed orbit.*

Proof. Since $\overline{G \cdot x}$ is G -invariant, it is a union of orbits, one of which is $G \cdot x$. So, $\overline{G \cdot x} \setminus G \cdot x$ is a union of orbits. Since $G \cdot x$ is open in $\overline{G \cdot x}$, the boundary $\overline{G \cdot x} \setminus G \cdot x$ is closed in X and has strictly lower dimension than $\dim G \cdot x$. So, the orbits in $\overline{G \cdot x} \setminus G \cdot x$ have dimension strictly less than $\dim G \cdot x$. Now, let $G \cdot v$ be an orbit in $\overline{G \cdot x} \setminus G \cdot x$ of minimal dimension. Similar to above, if $G \cdot v$ is not closed, then $\overline{G \cdot v} \setminus G \cdot v$ is a non-empty union of orbits of strictly smaller dimension than $\dim G \cdot v$. However, since $\overline{G \cdot x} \setminus G \cdot x$ is closed in X , these orbits are also contained in $\overline{G \cdot x} \setminus G \cdot x$, which contradicts minimality of $\dim G \cdot v$. So, $G \cdot v$ is closed in X . We conclude that $\overline{G \cdot x}$ contains a closed orbit with dimension strictly less than $\dim G \cdot x$. For $y \in Y$, since φ is surjective, there exists $x \in \varphi^{-1}(y)$. Then $\overline{G \cdot x} \subseteq \varphi^{-1}(y)$. By what we have just shown, $\varphi^{-1}(y)$ contains a closed orbit. For uniqueness, suppose C_1 and C_2 are two closed orbits contained in $\overline{G \cdot x}$, and therefore $\varphi^{-1}(y)$. Then $\varphi(C_1) = y = \varphi(C_2)$. Since $\varphi(C_1) \cap \varphi(C_2) \neq \emptyset$, it follows that C_1 and C_2 are not disjoint. And so $C_1 = C_2$. $\square$

Definition 3.2.12. [2, Definition 1.2.3.1] *If $\varphi : X \rightarrow Y$ is a good quotient for the action of G on X and the fibre of each point in Y under φ is a single orbit, then we say that φ is a **geometric quotient**.*

Proposition 3.2.13. [19, Theorem 4.10] *If $\varphi : X \rightarrow Y$ is a good quotient for the action of G on X , then φ is a geometric quotient if and only if all of the orbits of G in X are closed. In particular, the GIT quotient $\pi : V \rightarrow V//G$ is a geometric quotient if and only if all of the orbits of G in V are closed.*

Proof. Suppose φ is geometric and let $G \cdot x$ be an orbit. Since φ is geometric, $\varphi^{-1}(y) = G \cdot x$, where $y := \varphi(x)$. Since $\varphi^{-1}(y)$ is closed, $G \cdot x$ is closed. Next, suppose all of the orbits are closed and let $y \in Y$. By Lemma 3.2.11, there exists a unique closed orbit C with $C \subseteq \varphi^{-1}(y)$. Since φ is G -invariant, we may write

$$\varphi^{-1}(y) = \bigcup_{x \in \varphi^{-1}(y)} G \cdot x.$$

Since all of the orbits are closed and $\varphi^{-1}(y)$ contains a unique closed orbit, we get that $C = G \cdot x$ for all $x \in \varphi^{-1}(y)$. And so $\varphi^{-1}(y) = C$. This shows that every fibre of φ is a single orbit. This result holds for the GIT quotient because it is a good quotient, as shown in Theorem 3.2.7. $\square$

Remark 3.2.14. To distinguish when the GIT quotient is good or geometric, we write $\pi : V \rightarrow V//G$ for good quotients and $\pi : V \rightarrow V/G$ for geometric quotients. In other words, when the action of G on V is closed, we write V/G instead of $V//G$. This notation encompass the fact that $\text{Spec}(k[V]^G)$ behaves like a true orbit space when all of the orbits are closed. Indeed, when π is geometric, each point in $V//G$ can be viewed as an orbit via the set bijection

$$\{G \cdot v : v \in V\} \rightarrow V//G, \quad G \cdot v \mapsto (f_1(v), \dots, f_r(v)),$$

where $k[V]^G = k[f_1, \dots, f_r]$. And so, $V//G$ parametrizes unique orbits.

3.3 Stable points

In general, GIT quotients are not geometric quotients. An example of this was seen in Example 3.2.5. However, we can get rid of the “bad” orbits, or “unstable points”, and restrict to nice points which we call *stable*. The restriction of $\pi : V \rightarrow V//G$ to the open subset $V^s(G) \subseteq V$ of *stable points* is a geometric quotient. That is, the restriction of π to stable points ensures that every fibre is a single orbit, while maintaining quotient properties. An orbit is “bad” when it gets identified with another orbit under π , which, by Corollary 3.2.10, means that their orbit closures meet. An instance of this occurrence

is when an orbit is not closed; an orbit that is not closed gets identified with the unique closed orbit in its closure. As such, we consider the following definition:

Definition 3.3.1. [8, Definition 1.1] Let V be a representation of a reductive group G . A point $v \in V$ is G -**stable** if $G \cdot v$ is closed and v is not contained in the closure of any other orbit. Denote the set of G -stable points by $V^s(G)$. We say that the representation V is **stable** if $V^s(G) \neq \emptyset$. When the group action is clear, we may write V^s to mean $V^s(G)$.

Example 3.3.2. Recall Example 3.2.5. Since the origin and two punctured axes get identified, we expect the points in these orbits to be not stable. Indeed, the origin is not stable because it is contained in the closure of the punctured axes. Since the punctured axes are not closed, the points in them are not stable. The remaining points, which are in the hyperbola orbits, are stable. That is,

$$V^s(G) = \bigcup_{\alpha \in \mathbb{C}^*} V(xy - \alpha) = \{(x, y) \in \mathbb{A}_{\mathbb{C}}^2 : xy \neq 0\}.$$

◇

Example 3.3.3. Consider the action of $G = \mathbb{C}^*$ on $V = \mathbb{A}_{\mathbb{C}}^n$ given by

$$\lambda \cdot (a_1, \dots, a_n) = (\lambda a_1, \dots, \lambda a_n).$$

Then $\mathbb{C}[V]^G = \mathbb{C}$ so that $V//G$ is a point. This means that π is a constant map with unique fibre being all of V . So, π is not geometric with all points being not stable. Indeed, for non-zero $v \in V$, $G \cdot v$ is the line through 0 and v that is punctured at 0. So, $G \cdot v$ is not closed. And 0 is contained in the closure of $G \cdot v$ for every $v \neq 0$. We conclude that $V^s = \emptyset$.

◇

Remark 3.3.4. Recall that the **stabilizer** of $v \in V$ with respect to the action of G is

$$G_v := \{g \in G : g \cdot v = v\}.$$

Note that the generic stabilizer has minimal dimension [19, pg. 151]. So, the dimension of the generic stabilizer is 0 if and only if there exists a point whose stabilizer has dimension 0 if and only if there exists a finite stabilizer.

Proposition 3.3.5. [7, Proposition 2.6] If $v \in V^s$, then $\pi^{-1}(\pi(v)) = G \cdot v$ and G_v has minimal dimension. Moreover, the set V^s of G -stable points is open and G -invariant.

Proof. To show that V^s is G -invariant, let $v \in V^s$ and $g \in G$. Then $G \cdot (g \cdot v) = G \cdot v$ is closed. Since orbit closures are G -invariant, if $g \cdot v$ was in the closure of some other orbit, then so would v be. So, V^s is G -invariant. Next, we show that $\pi^{-1}(\pi(v)) = G \cdot v$ for any $v \in V^s$. Since π is G -invariant, $\pi(G \cdot v) = \pi(v)$ and so $G \cdot v \subseteq \pi^{-1}(\pi(v))$. For the other inclusion, assume for contradiction that there exists $y \in \pi^{-1}(\pi(v)) \setminus G \cdot v$. By Corollary 3.2.10, $\pi(y) = \pi(v)$ is equivalent to $\overline{G \cdot y} \cap G \cdot v \neq \emptyset$. So, there exists $z \in \overline{G \cdot y}$ such that $z = g \cdot v$ for some $g \in G$. Since orbit closures are G -invariant, $v \in \overline{G \cdot y}$. However, since $y \notin G \cdot v$, we have that $v \notin G \cdot y$. So, $v \in \overline{G \cdot y} \setminus G \cdot y$. In other words, v is contained in the closure of some other orbit, which is a contradiction. We conclude that $G \cdot v = \pi^{-1}(\pi(v))$.

Let d be the minimal stabilizer dimension. We show that $\dim G_v = d$ for $v \in V^s$. As noted in [23, pg. 840], there exists a dense open $A \subseteq V//G$ such that $\dim \pi^{-1}(a) = \dim V - \dim V//G$ for all $a \in A$. Since π is a topological quotient, we know that $\pi^{-1}(A)$ is open in V . Moreover, since the dimension of the stabilizer subgroup viewed as the function $v \mapsto \dim G_v$ is upper semi-continuous, the set

$$V_{>d} := \{v \in V : \dim G_v > d\}$$

is closed in V . And so its complement

$$X \setminus V_{>d} = \{v \in V : \dim G_v \leq d\} = \{v \in V : \dim G_v = d\}$$

is open in V . Since V is irreducible, all of its non-empty open subsets are dense and so $\pi^{-1}(A) \cap (X \setminus V_{>d}) \neq \emptyset$. Let z be in this intersection. Then $\dim G_v \geq \dim G_z$. Also, $\pi(z) \in A$ and so $\dim \pi^{-1}(\pi(z)) = \dim V - \dim V//G$. And in general, any fibre has dimension at least $\dim V - \dim V//G$ [23, pg. 840]. Hence

$$\dim \pi^{-1}(\pi(v)) \geq \dim V - \dim V//G = \dim \pi^{-1}(\pi(z)).$$

Note that $\dim G = \dim G_x + \dim G \cdot x$ for all $x \in V$ [19, pg. 151]. It follows that

$$\begin{aligned} \dim \pi^{-1}(\pi(z)) &\geq \dim G \cdot z \\ &= \dim G - \dim G_z \\ &\geq \dim G - \dim G_v \\ &= \dim G \cdot v \\ &= \dim \pi^{-1}(\pi(v)) \\ &\geq \dim \pi^{-1}(\pi(z)). \end{aligned}$$

We conclude that $\dim G_v = \dim G_z = d$.

Lastly, we show that V^s is open. We do this by showing that

$$V^s = V \setminus \pi^{-1}(\pi(V_{>d}))$$

and that $\pi^{-1}(\pi(V_{>d}))$ is closed in V . Since $V_{>d}$ is a G -invariant closed subset and π is a good quotient, $\pi(V_{>d})$ is closed in $V//G$. Since π is continuous, $\pi^{-1}(\pi(V_{>d}))$ is closed in V . Now, let $v \in V^s$. To show $v \notin \pi^{-1}(\pi(V_{>d}))$, it is equivalent to show that $\overline{G \cdot x} \cap G \cdot v = \emptyset$ for all $x \in V_{>d}$. So, let $x \in V_{>d}$ and assume for contradiction that $\overline{G \cdot x} \cap G \cdot v \neq \emptyset$. Then there exists $g \cdot v \in \overline{G \cdot x}$ for some $g \in G$. Then $v \in \overline{G \cdot x}$. Since $\dim G_v = d$ is minimal, $\dim G \cdot v$ is maximal. And since $\dim G_x$ is not minimal, $\dim G \cdot x$ is not maximal. So, $G \cdot v$ and $G \cdot x$ are distinct orbits. So, $v \in \overline{G \cdot x} \setminus G \cdot x$, which contradicts that $v \in V^s$. We conclude that $V^s \subseteq (\pi^{-1}(\pi(V_{>d}))^c$. For the other inclusion, we show $(V^s)^c \subseteq \pi^{-1}(\pi(V_{>d}))$. Let $v \in (V^s)^c$ so that either $G \cdot v$ is not closed or v is in the closure of another orbit. If $G \cdot v$ is not closed, then by Lemma 3.2.11, $\overline{G \cdot v}$ contains a closed orbit $G \cdot y$ with $\dim G \cdot y < \dim G \cdot v$. Then $\dim G_y > \dim G_v$ and hence $y \in V_{>d}$. Since π is constant on orbit closures, $\pi(y) = \pi(v)$ and so $v \in \pi^{-1}(\pi(V_{>d}))$. On the other hand, if v is in the closure of another orbit, say $v \in \overline{G \cdot x} \setminus G \cdot x$, then $G \cdot x$ is not closed and hence $\overline{G \cdot x}$ contains an orbit $G \cdot z$ with $\dim G \cdot z < \dim G \cdot x$. Similar to above, $x \in V_{>d}$ and $\pi(v) = \pi(x)$, and so $v \in \pi^{-1}(\pi(V_{>d}))$. $\square$

Lemma 3.3.6. *Let V be a representation of a reductive group G . Then $\pi(V^s)$ is open in $V//G$.*

Proof. Since π is a topological quotient it is enough to show that the preimage $\pi^{-1}(\pi(V^s))$ is open in V . We claim that $\pi^{-1}(\pi(V^s)) = V^s$, which is open by Proposition 3.3.5. By Proposition 3.3.5, if $v \in V^s$, then $\pi^{-1}(\pi(v)) = G \cdot v$ and so $v \in G \cdot v \subseteq \pi^{-1}(\pi(V^s))$. For the other inclusion, let $v \in \pi^{-1}(\pi(V^s))$. This means that $\pi(v) = \pi(x)$ for some $x \in V^s$. And so $v \in \pi^{-1}(\pi(x)) = G \cdot x$. Since $x \in V^s$ and V^s is G -invariant, $G \cdot x \subseteq V^s$ and so $v \in V^s$. $\square$

Proposition 3.3.7. *Let V be a representation of a reductive group G . Set d to be the minimal stabilizer dimension. If V is stable, then*

$$\dim V//G = \dim V - \dim G + d.$$

Proof. Since V is stable, $V^s \neq \emptyset$ and so $\pi(V^s) \neq \emptyset$. By Lemma 3.3.6, $\pi(V^s)$ is open in $V//G$. As noted in [23, pg. 840], there exists a dense open $A \subseteq V//G$ such that $\dim \pi^{-1}(a) = \dim V - \dim V//G$ for all $a \in A$. Since A is dense and $\pi(V^s) \neq \emptyset$ is open, there exists $y \in A \cap \pi(V^s)$. Then $\dim \pi^{-1}(y) = \dim V - \dim V//G$. Given this, note that the equation

$\dim V//G = \dim V - \dim G + d$ is equivalent to $\dim \pi^{-1}(y) = \dim G - d$, which is equivalent to $\dim \pi^{-1}(y)$ being equal to the maximal orbit dimension. The last equivalence follows from that fact that $\dim G \cdot v = \dim G - \dim G_v$ for all $v \in V$ [19, pg. 151] and so $\dim G_v$ is minimal if and only if $G \cdot v$ is maximal. So, it is enough to show that $\dim \pi^{-1}(y)$ is equal to the maximal orbit dimension. Since $y \in \pi(V^s)$, there exists $x \in V^s$ such that $\pi(x) = y$. By Proposition 3.3.5, we know that $\dim G \cdot x$ is the maximal orbit dimension and $G \cdot x = \pi^{-1}(\pi(x)) = \pi^{-1}(y)$. $\square$

Remark 3.3.8. The definition of a stable point, and therefore stable representation or action, is not consistent throughout the literature. For instance, it is common to define a stable point as a point whose orbit is closed and stabilizer is finite; see [15, pg.41]. As noted in Proposition 3.3.10 below, this definition of stable is equivalent to Definition 3.3.1 when the dimension of the generic stabilizer is 0. More generally, stable points are usually defined to be points that have a closed orbit and satisfy some open condition. We need V^s to be open so that $\pi|_{V^s}$ is a good quotient; see Proposition 3.2.8. And the points in V^s having closed orbits is needed so that $\pi(V^s) \subseteq V//G$ acts like a true orbit space, or equivalently, so that $\pi|_{V^s}$ is geometric; see Proposition 3.2.13. These two defining features of a stable point are highlighted in the definition of a stable action in [19, pg. 156]: the action of G on V is called *stable* if there exists in V a non-empty open subset such that the orbits of its points are closed. And they are also highlighted in [24, pg. 463]: a representation (V, G) is stable if and only if V contains a dense open subset consisting entirely of closed orbits.

Lemma 3.3.9. *Let V be a representation of a reductive group G . If $v \in V$ such that G_v has minimal dimension, then v is not contained in the closure of any other orbit. In particular, if G_v is finite, then v is not contained in the closure of any other orbit.*

Proof. Note that $\dim G = \dim G_v + \dim G \cdot v$ [19, pg. 151]. So, G_v having minimal dimension means that $G \cdot v$ has maximal dimension. Now, assume for contradiction that v is in the closure of some other orbit, say $G \cdot y$. This means that $v \in \overline{G \cdot y} \setminus G \cdot y$. However, by Lemma 3.2.11, we know that the boundary $\overline{G \cdot y} \setminus G \cdot y$ is a union of orbits with dimension strictly less than $\dim G \cdot y$. Since $G \cdot v$ is contained in this union, $G \cdot v$ is equal to one of these orbits and so $\dim G \cdot v < \dim G \cdot y$. This contradicts the fact that $\dim G \cdot v$ has maximal dimension. So, v cannot be in the closure of another orbit. $\square$

Proposition 3.3.10. *Let V be a representation of a reductive group G such that the generic stabilizer has dimension 0. Given $v \in V$, $G \cdot v$ is closed and G_v is finite if and only if $G \cdot v$ is closed and v is not contained in the closure of any other orbit. In other words, $v \in V$ is stable if and only if $G \cdot v$ is closed and G_v is finite.*

Proof. The stabilizer G_v being finite is equivalent to $\dim G_v = 0$. So, G_v has minimal dimension. Then, by Lemma 3.3.9, v is not contained in the closure of any other orbit. Conversely, suppose that v is stable. From Proposition 3.3.5, G_v has minimal dimension and so $\dim G_v = 0$. This means G_v is finite. $\square$

Corollary 3.3.11. *Let V be a representation of a reductive group G . If the generic stabilizer has dimension 0, then V is stable*

Proof. By assumption, there exists $v \in V$ with $\dim G_v = 0$. From Lemma 3.2.11, let $G \cdot x$ be the unique closed orbit in $\pi^{-1}(\pi(v))$. Then, by Corollary 3.2.10, $\overline{G \cdot v} \cap G \cdot x \neq \emptyset$. So, there exists $z \in \overline{G \cdot v}$ with $G \cdot z = G \cdot x$ closed. Note that $\{v \in V : \dim G_v = 0\}$ is closed and G -invariant. And so $z \in \overline{G \cdot v} \subseteq \{v \in V : \dim G_v = 0\}$. This shows that G_z is finite. By Proposition 3.3.10, since G_z is finite and $G \cdot z$ is closed, z is stable and hence V is stable. $\square$

Lemma 3.3.12. *Let $V \neq 0$ be representation of G . If $\dim V//G = 0$, then V is not stable.*

Proof. Since $\dim V//G = 0$, $\text{Spec}(k[V]^G) = \text{Spec}(k)$ is a point and so the map $\pi : V \rightarrow V//G$ is constant, say $\pi(v) = x$ for all $v \in V$. Then $\pi^{-1}(x) = V$ and so, by Lemma 3.2.11, V contains a unique closed orbit. This means that 0 is the only closed orbit. So, every non-zero point in V is not stable as its orbit is not closed. Moreover, since $V \neq 0$, $\pi(0) = \pi(v)$ for some $v \in V \setminus 0$ and so $0 \in \overline{G \cdot v} \setminus G \cdot v$, which means 0 is not stable. We conclude that V has no stable points. $\square$

Proposition 3.3.13. [19, pg. 186] *Let V be a representation of a reductive group G . Then*

$$\pi|_{V^s} : V^s \rightarrow \pi(V^s)$$

is a geometric quotient.

Proof. We know from Lemma 3.3.6 that $\pi(V^s)$ is open. In the proof of Lemma 3.3.6, we showed that $\pi^{-1}(\pi(V^s)) = V^s$. By Proposition 3.2.8, the restriction of π to $\pi^{-1}(\pi(V^s)) = V^s$ is a good quotient. That is, $\pi|_{V^s} : V^s \rightarrow \pi(V^s)$ is a good quotient. Also, by definition of V^s , all of the orbits of G in V^s are closed and so by Proposition 3.2.13, $\pi|_{V^s}$ is geometric. $\square$

In addition to stable points providing us with a true GIT orbit space, stable points are relevant to the main theorem of this thesis, which works towards a characterization of stable *cofree* representations of the reductive group $\text{SL}_n \times \text{SL}_m$.

3.4 History of geometric invariant theory

Classically, invariant theory dealt with finding explicit descriptions of algebraic expressions, such as polynomial functions, that do not change, or are invariant, under the transformations of a given group. Such descriptions are extremely useful in mathematics as they often reflect intrinsic properties of the object under study. Prior to the 1960's, invariant theory only addressed the situation of representations of linear groups, and then later broadened its interpretation for algebraic groups acting on algebraic varieties [19, pg. 131, pg. 166]. For us, we consider the classical case of a finite-dimensional representation V of an algebraic group G . From this we can then consider the subspace $k[V]^G$ of all polynomial functions which are invariant under this group action. Some of the early problems in invariant theory were concerned with presenting a finite generating set and a finite system of defining relations of $k[V]^G$ [19, pg.238].

Example 3.4.1. Consider the action of $G = \langle -I \rangle \subset \mathrm{GL}_2(\mathbb{C})$ on $V = \mathbb{A}_{\mathbb{C}}^2$ via the standard action of left multiplication. If $\mathbb{C}[V] = \mathbb{C}[x, y]$, we compute that $\mathbb{C}[V]^G = \mathbb{C}[x^2, xy, y^2]$. Noting the relation $x^2y^2 = (xy)^2$, we may write $\mathbb{C}[V]^G \cong \mathbb{C}[a, b, c]/\langle ac - b^2 \rangle$. $\diamond$

Generally speaking, the algebra of invariants $k[V]^G$ can have a structure that is complicated or what seems to be impossible to describe explicitly. Even when we can describe $k[V]^G$ explicitly, it can be difficult to extract any useful information from its generators and relations. Often, problems in invariant theory seem to be only well understood for a few linear groups, or certain classes of groups [19, pg. 238]. That being said, invariant theory aims to characterize groups by various properties that are seen as “good” through an invariant theory lens. We list some of these properties in the following definition.

Definition 3.4.2. [19, pg. 239] Let V be finite-dimensional representation of a reductive group G . We say that (V, G) or, for shorthand, V is

- (FA) **coregular** if $k[V]^G$ is a (free) polynomial ring.
- (ED) **equidimensional** if all fibres of $\pi : V \rightarrow V//G$ have the same dimension.
- (FM) **cofree** if $k[V]$ is a free $k[V]^G$ -module.

Example 3.4.3. Consider Example 3.2.5, where $G = \mathbb{C}^*$ and $V = \mathbb{A}_{\mathbb{C}}^2$. Writing $\mathbb{C}[V] = \mathbb{C}[x, y]$, we have that $\mathbb{C}[V]^G = \mathbb{C}[xy]$ and so V is coregular. We can write the fibre $\pi^{-1}(0)$ as the union of the punctured x -axis, punctured y -axis, and origin, and so it has dimension 1. The fibre $\pi^{-1}(c) = V(xy - c)$ with $c \neq 0$ also has dimension 1. So, V is equidimensional. Lastly, $\mathbb{C}[x, y]$ is free as a $\mathbb{C}[xy]$ -module with basis $\{1, x, y, x^2, y^2, x^3, y^3, \dots\}$. So, V is cofree. $\diamond$

Example 3.4.4. Consider Example 3.4.1, where $G = \langle -I \rangle \subset \mathrm{GL}_2(\mathbb{C})$ and $V = \mathbb{A}_{\mathbb{C}}^2$. Since $\mathbb{C}[V]^G = \mathbb{C}[x^2, xy, y^2]$ has the relation $x^2y^2 = (xy)^2$, it is not a polynomial ring. So, V is not coregular. Any potential $\mathbb{C}[x^2, xy, y^2]$ -basis of $\mathbb{C}[x, y]$ contains a $\mathbb{C}$ -basis of $\mathrm{span}_{\mathbb{C}}\{1, x, y\}$, and such a basis is not $\mathbb{C}[x^2, xy, y^2]$ linearly independent, and so no such basis exists. So, V is not cofree. Every fibre is finite because it is a union of orbits, all of which are finite. Since each fibre has the same dimension of 0, V is equidimensional. $\diamond$

Remark 3.4.5. Note that $\pi(0)$ is a nonsingular point if and only if $V//G$ is smooth [19, Proposition 4.11]. This equivalence follows because if $\pi(0)$ is nonsingular and $\mathcal{M}$ is the minimum number of generators of $k[V]^G$, then for all $y \in V//G$ we have that

$$\dim V//G \leq \dim T_y(V//G) \leq \mathcal{M} = \dim T_{\pi(0)}(V//G) = \dim V//G.$$

And so $\dim T_y(V//G) = \dim V//G$ for all $y \in V//G$, which means that $V//G$ is smooth. Also, if $V//G$ is smooth, then $\dim V//G = \dim T_{\pi(0)}(V//G) = \mathcal{M}$. However, we may also write $k[V]^G = k[f_1, \dots, f_{\mathcal{M}}]$ so that

$$V//G = \mathrm{Spec}(k[f_1, \dots, f_{\mathcal{M}}]) \subseteq \mathbb{A}_k^{\mathcal{M}}.$$

Then $\dim V//G = \mathcal{M}$ implies that $V//G = \mathbb{A}_k^{\mathcal{M}}$. Thus, $V//G$ is affine space if and only if $V//G$ is smooth. Recalling that we may write $k[V]^G$ as a polynomial ring possibly quotiented by some relations, we conclude that $k[V]^G$ is coregular if and only if there are no relations among the generators of $k[V]^G$ if and only if $V//G$ is affine space if and only if $V//G$ is smooth. So, $V//G$ is smooth (i.e., affine space) if and only if $k[V]^G$ is coregular.

Remark 3.4.6. It was shown by Schwarz that the satisfaction of both (FA) and (ED) is equivalent to (FM). That is, V is cofree if and only if $V//G$ is smooth (equivalently, V is coregular) and V is equidimensional [21, Proposition 17.29]. Note that this result uses the fact that V being equidimensional is equivalent to $\dim \pi^{-1}(\pi(0)) = \dim V - \dim V//G$ [23, pg. 840]. Moreover, by the Magic Flatness Theorem, V is cofree if and only if $V//G$ is smooth and π is flat.

The properties (FA) through (FM) reflect “nice” characteristics that (V, G) can possess, and thus, it is natural to ask when representations of a given group satisfy them. Shephard and Todd [22] and Chevalley [3] proved the following intrinsic characterization of cofree representations for finite groups: V is cofree if and only if G is generated by pseudoreflections if and only if $V//G$ is smooth. This characterization of cofree representations of finite groups is known as the *Chevalley-Shephard-Todd Theorem*. Recall from Remark 3.4.6 that V is cofree if and only if $V//G$ is smooth and V is equidimensional. In the finite group case, V

is automatically equidimensional, and V being cofree is equivalent to $V//G$ being smooth. Popov [23, pg. 839] conjectured that for connected reductive groups, V being cofree is equivalent to V being equidimensional. In other words, Popov conjectured that the condition of $V//G$ being smooth is superfluous. Wehlau [23] proved that Popov’s conjecture holds true for representations of a torus. Also, Littelmann proved Popov’s conjecture for irreducible representations of connected semisimple algebraic groups [13, pg. 202]. In contrast to the Chevalley-Shepard-Todd Theorem, if G is a connected reductive group, then there does not exist a precise group-theoretic characterization of when V is cofree. However, several classifications of cofree representations have been found for certain connected reductive groups. For example, the cofree representations of simple Lie groups were classified independently by Schwarz [20] and Adamovich and Golovina [1], while Littelmann [13] classified irreducible cofree representations of semisimple groups. Unlike the Chevalley-Shepard-Todd Theorem, the above results for connected reductive groups consist only of a list of cofree representations, rather than a general group-theoretic characterization.

3.5 Pure representations

In 2019, Edidin, Satriano, and Whitehead stated a conjecture in [8] that intrinsically characterizes irreducible cofree representations of reductive groups. Specifically, the conjecture states the following:

Conjecture 3.5.1. [8, Question 1.9] *If V is a stable irreducible representation of a connected reductive group, then V is cofree if and only if V is pure.*

In contrast to a list of all cofree representations for a given group, this conjecture is an intrinsic characterization of cofree representations and hence may be viewed as an analogue of the Chevalley–Shepard–Todd Theorem for connected reductive groups. In [8, Theorem 1.3], the conjecture was verified for irreducible representations of simple Lie groups. We now discuss the notion of *pure*. In algebraic geometry, there is a notion of purity, which essentially means that when something happens, it happens in a particular codimension. The conjecture above and definition of *pure* from [8] is inspired by purity, and in particular, the *Purity of the Branch Locus Theorem* [25, 16]. Refining to a specific relevant case, this theorem states the following:

Theorem 3.5.2 (Purity of the Branch Locus Theorem). ([25] and [16]) *If X and Y are smooth varieties and $f : X \rightarrow Y$ is finite, then the branch locus (image of the ramification locus) is pure of codimension-one, meaning that every irreducible component of the branch locus has codimension-one.*

Recall from Remark 3.4.6 that if V is cofree, then $V//G$ is smooth. In that case, $\pi : V \rightarrow V//G$ is a map between smooth varieties. However, in general, π is not finite as it does not have finite fibres. Thus, we cannot apply the Purity of the Branch Locus Theorem to π . Given this, in order to seek out a necessary condition for $V//G$ to be smooth, the authors in [8] made sense of a geometric invariant theoretic analogue of the Purity of the Branch Locus Theorem. A natural invariant theoretic analogue of the branch locus is the image of the strictly semi-stable locus $V^{sss} := V \setminus V^s$ under π . This is because, similar to the branch locus, the fibres of points in $\pi(V^{sss})$ behave in an undesirable way, in the sense that $V//G$ does not always behave like a true orbit space.

Definition 3.5.3. [8, Definition 1.1] Let V be a representation of a reductive group G . We denote by $V^{sss}(G)$ the closed subset $V \setminus V^s$. Points $v \in V^{sss}(G)$ are called **G -strictly semi-stable**. When the group action is clear, we may write V^{sss} to mean $V^{sss}(G)$.

Since π is not finite, we instead consider the *quotient stack* $[V//G]$ and the following factorization:

$$\begin{array}{ccc} V & & \\ \pi \downarrow & \searrow & \\ & [V//G] & \\ & \swarrow \eta & \\ V//G & & \end{array}$$

For our purposes, all we really need to know about $[V//G]$ is that it can be thought of as the space $V//G$ with more information retained. In comparison to $V//G$, the stack $[V//G]$ only has one point for every orbit, and so it also keeps track of the stabilizers of orbits. The map $V \rightarrow [V//G]$ is a smooth covering map and η is the canonical map that sends each orbit to its corresponding point. The point is that if $V//G$ is smooth (e.g., V cofree), then $\eta : [V//G] \rightarrow V//G$ is a map between smooth spaces and is almost finite (it satisfies almost all of the properties that a finite stack map does). So, via the above factorization, an invariant theoretic analogue of the Purity of the Branch Locus Theorem is: if $\pi : V \rightarrow V//G$ is a map between smooth varieties with V cofree, then $\pi(V^{sss})$ is pure of codimension-one. The conjecture in [8] suggests that this is true for stable irreducible representations of connected reductive groups, and further suggests that the converse is true. As a result, we have the following definition:

Definition 3.5.4. [8, Definition 1.2] Let V be a stable representation of a connected reductive group G . We say that V is **pure** if $\pi(V^{sss})$ is pure of codimension-one in $V//G$, meaning that every irreducible component of $\pi(V^{sss})$ has codimension-one.

Example 3.5.5. Recall Example 3.3.2, where $V^s(G)$ consists of hyperbolas. Since $V^s(G) \neq \emptyset$, V is stable. Moreover, V^{sss} consists of the origin and two punctured axes, all of which collapse to 0 under π . Then $\pi(V^{sss}) = 0$ has codimension-one in $V//G = \mathbb{A}_{\mathbb{C}}$ and so V is pure. $\diamond$

Definition 3.5.6. [8, Part II. Section 2] Let V be a representation of a reductive group G . We say that the representation V is **polar** if there exists a subspace $\mathfrak{c} \subset V$ such that the map $\mathfrak{c} \rightarrow \text{Spec}(k[V]^G)$ is finite and surjective. In such a case, we call the subspace $\mathfrak{c}$ a **Cartan subspace** of V . Dadok and Kac proved that if V is polar with Cartan subspace $\mathfrak{c}$ then $W = N_G(\mathfrak{c})/Z_G(\mathfrak{c})$ is finite and $k[V]^G = k[\mathfrak{c}]^W$ [6, Theorem 2.9].

Polar representations are particularly nice because they allows us to utilize the Chevalley-Shepard-Todd Theorem. Specifically, if V is a polar representations of a connected group G with $k[V]^G \cong k[\mathfrak{c}]^W$, then W is generated by pseudoreflections [6, Theorem 2.10] and so, by the Chevalley-Shepard-Todd Theorem, we get that $k[V]^G \cong k[\mathfrak{c}]^W$ is cofree. Edidin, Satriano, and Whitehead also proved the following relationship between polar and pure representations.

Proposition 3.5.7. [8, Proposition 2.1] *Let V be a representation of a reductive group G . If V is a stable polar representation, then V is pure.*

The aim of this thesis is to work towards the verification of Conjecture 3.5.1 for irreducible representations of connected semisimple Lie groups. The main result of this thesis is the verification of Conjecture 3.5.1 for a large class of irreducible representations of the connected semisimple Lie group $\text{SL}_n \times \text{SL}_m$. Specifically, Conjecture 3.5.1 holds for irreducible representations $V \otimes W$ of $\text{SL}_n \times \text{SL}_m$ satisfying $\dim V \geq n^2$ and $\dim W \geq m^2$; see Theorem 4.8.1. So, one could say, that Conjecture 3.5.1 holds generically for representations of $\text{SL}_n \times \text{SL}_m$. It is expected that a proof that works for all irreducible representations of $\text{SL}_n \times \text{SL}_m$ will naturally generalize to any connected semisimple Lie group. Note that all irreducible cofree representations of $\text{SL}_n \times \text{SL}_m$ are reflected in a table from [13], and so the main step in verifying Conjecture 3.5.1 is to find the complete list of pure representations of $\text{SL}_n \times \text{SL}_m$. The conjecture will then be proven if the two lists agree when restricting to stable representations. In Section 3.6, we write down the complete list of irreducible cofree representations of $\text{SL}_n \times \text{SL}_m$ (see Table 3.1) and discuss the direction of Conjecture 3.5.1 which says that stable cofree representations of $\text{SL}_n \times \text{SL}_m$ are pure.

3.6 Cofree representations of $\mathrm{SL}_n \times \mathrm{SL}_m$

Recall from Section 3.4 that Littelmann classified all irreducible cofree representations of semisimple Lie groups and that this classification is reflected via a table of such representations [13]. In particular, Littelmann classified all *reflection-reduced* irreducible cofree representations in a table [13, pg. 198-201]. Going forward, we refer to this table as **Littelmann's table**. From Littelmann's table, one can then deduce all irreducible cofree representations of a given semisimple Lie group. In this section, we go into detail of what this exactly means in order to write down the complete list of all irreducible cofree representations of $\mathrm{SL}_n \times \mathrm{SL}_m$; see Table 3.1. We then discuss the direction of Conjecture 3.5.1 which says that stable irreducible cofree representations of $\mathrm{SL}_n \times \mathrm{SL}_m$ are pure.

Definition 3.6.1. [13, Lemma 2, pg. 196] Let W be a representation of a semisimple Lie group H and consider $(V, G) := (k^m \otimes W, \mathrm{SL}_m \times H)$, with $m \geq 1$. We say that (V', G') can be **castled** to (V, G) if (V', G') takes the form

$$(V', G') = (k^{m'} \otimes W^*, \mathrm{SL}_{m'} \times H)$$

for some m' satisfying $1 \leq m \leq m'$ and $m + m' = \dim W$. We call the transition from (V, G) to (V', G') a **castling transformation**. Two representations (V, G) and (V', G') are said to be **equivalent**, written $(V, G) \approx (V', G')$, if they differ from G by an outer automorphism. Further, we say that (V, G) and (V', G') belong to the same **reflection class** if they can be transformed into equivalent representations by a sequence of castling transformations. Alternatively, the reflection class of (V, G) is defined to be the equivalence class of (V, G) under the equivalence relation generated by castling and outer automorphisms of the group G .

Example 3.6.2. [13, Section 1.2] We show that $\rho : G \rightarrow \mathrm{GL}(V)$ and $\rho^* : G \rightarrow \mathrm{GL}(V^*)$ are equivalent representations, where G is a semisimple Lie group and $\dim V = \dim V^* = n$. Consider the outer automorphism (that is not inner)

$$\phi : \mathrm{GL}_n(\mathbb{C}) \rightarrow \mathrm{GL}_n(\mathbb{C}), \quad A \mapsto (A^{-1})^T.$$

The diagram

$$\begin{array}{ccccc} G & \xrightarrow{\rho} & \mathrm{GL}_n(\mathbb{C}) & \xrightarrow{\cong} & \mathrm{GL}(V) \\ & \searrow \rho^* & \downarrow \phi & & \\ & & \mathrm{GL}_n(\mathbb{C}) & \xrightarrow{\cong} & \mathrm{GL}(V^*) \end{array}$$

commutes as $(\phi \circ \rho)(g) = \rho(g^{-1})^T = \rho^*(g)$. Then $\rho^* = \phi \circ \rho$ means that $(V, G) \approx (V^*, G)$. Extending this, the following representations are equivalent:

$$\left(\bigotimes V_i, G_1 \times \cdots \times G_s \right) \approx \left(\bigotimes \overline{V}_i, G_1 \times \cdots \times G_s \right)$$

where $\overline{V}_i$ means V or V^* for all i . ◇

Example 3.6.3. Let V be a non-trivial irreducible representation of a simple Lie group G and set $n := \dim V$. Using $W = V, H = G, m = 1$, and $m' = n - 1$ in Definition 3.6.1, we get that

$$(V', G') := (k^{n-1} \otimes V^*, \mathrm{SL}_{n-1} \times G)$$

can be castled to $(V, G) \cong (k \otimes V, \mathrm{SL}_1 \times G)$. Similarly,

$$(V'', G'') := (k^{m^2-m-1} \otimes k^{m-1} \otimes V, \mathrm{SL}_{m^2-m-1} \times \mathrm{SL}_{m-1} \times G)$$

can be castled to (V', G') . In this example, (V', G') and (V'', G'') , and their duals, are all in the reflection class of (V, G) . ◇

The following proposition helps us understand Littelmann's table tells us how to extra all irreducible cofree representation of $\mathrm{SL}_n \times \mathrm{SL}_m$ from the table.

Proposition 3.6.4. [13] *Let V be an irreducible representation of a semisimple Lie group G . Set $\dim V = n$. Then the following hold:*

- (a) *If (V', G') is in the reflection class of (V, G) , then $V//G \cong V'//G'$ [13, Lemma 2].*
- (b) *Up to duals, there is exactly one representation in the reflection class of (V, G) with minimum dimension, which we call **reflection-reduced** [13, pg. 196].*
- (c) *If $\dim V//G \leq 1$, then all representations in the reflection class of (V, G) are polar [13, Section 1.3].*
- (d) *If $\dim V//G \geq 2$, then at most the reflection-reduced representation in the reflection class of (V, G) can be polar [13, Lemma 3].*
- (e) *If $\dim V//G \leq 2$, then all representations in the reflection class of (V, G) are cofree [13, Remark 1.2].*
- (f) *If $\dim V//G \geq 3$, then the reflection-reduced representation in the reflection class of (V, G) is cofree [13, Lemma 2(b)].*

(g) If $\dim V//G \geq 3$ and V is cofree, then $(k^{n-1} \otimes V, \mathrm{SL}_{n-1} \times G)$ is not equidimensional and hence not cofree [13, Lemma 2(b)].

Proposition 3.6.5. ([8] and [13]) *The properties cofree and polar are invariant under duals. That is, (V, G) is cofree (resp. polar) if and only if (V^*, G) is cofree (resp. polar).*

Proof. Since (V, G) and (V^*, G) are in the same reflection class, we have that $V//G \cong V^*//G$. So, smoothness (aka coregularity) is preserved by duals. Alternatively, a fixed isomorphism $V \rightarrow V^*$ descends to an isomorphism of quotients $V//G \rightarrow V^*//G$. So, equidimensionality and hence cofreeness are preserved. And polar is preserved because we can map the Cartan subalgebra $\mathfrak{c} \subset V$ to an isomorphic Cartan subalgebra $\mathfrak{c}^* \subset V^*$ with the same properties. $\square$

Recall that a representation (V, G) is called **locally faithful** if the kernel of the representations homomorphism $\rho : G \rightarrow \mathrm{GL}(V)$ is finite. Let (V, G) be a reflection-reduced, non-trivial, irreducible, locally faithful representation of a simply connected semisimple algebraic group. Then, by [13, Theorem 1.4], the following statements are equivalent:

- (a) (V, G) is coregular.
- (b) (V, G) is equidimensional.
- (c) (V, G) is cofree.
- (d) (V, G) is equivalent to one of the representation in Littelmann's table.

For the reminder of this section, we consider the case when $G = \mathrm{SL}_n \times \mathrm{SL}_m$. Note that the kernel of every irreducible representation of $\mathrm{SL}_n \times \mathrm{SL}_m$ is contained in the centre $Z(\mathrm{SL}_n) \times Z(\mathrm{SL}_m)$, which is a finite subgroup. Hence, all irreducible representations of $\mathrm{SL}_n \times \mathrm{SL}_m$ are locally faithful. The goal is to extract and write down all irreducible cofree representations $V \otimes W$ of $\mathrm{SL}_n \times \mathrm{SL}_m$ from Littelmann's table. The representations with group $G = A_n + A_m = \mathrm{SL}_n \times \mathrm{SL}_m$ in Littelmann's table are cofree, and so we add these and their duals to our list. We must also consider representations of $\mathrm{SL}_n \times \mathrm{SL}_m$ that arise from the reflection-reduced representations of $A_n = \mathrm{SL}_n$ in Littelmann's table. In other words, if (V, SL_n) is a representation in the table, we must consider

$$(k^{d-1} \otimes V, \mathrm{SL}_{d-1} \otimes \mathrm{SL}_n), \quad \dim V = d$$

as a potential cofree representation. By parts (e) and (g) of Proposition 3.6.4, we only need to consider such representation when $\dim V//G \leq 2$. So, if (V, SL_n) satisfies $\dim V//\mathrm{SL}_n \leq 2$,

we add $(k^{d-1} \otimes V, \mathrm{SL}_{d-1} \otimes \mathrm{SL}_n)$ and its dual to our list. This concludes the full list of irreducible cofree representations of $\mathrm{SL}_n \times \mathrm{SL}_m$, which are recorded in Table 3.1.

Note that the last column of Table 3.1 records whether or not the representation in that row is polar or not. The polarity of cofree representations of $\mathrm{SL}_n \times \mathrm{SL}_m$ is listed in Littelmann's table. For representations in Table 3.1 arising from reflection-reduced representations of SL_n , we determine their polarity by using parts (c) and (d) of Proposition 3.6.4. In particular, if (V, G) is in Littelmann's table, it is reflection-reduced, and so the representation $(k^{d-1} \otimes V, \mathrm{SL}_{d-1} \times \mathrm{SL}_n)$ is polar if and only if $\dim V//G \leq 1$. Note from Proposition 3.6.5 that the property polar is invariant under duals.

Conjecture 3.6.6. *Let $V \otimes W$ be an irreducible representation of $\mathrm{SL}_n \times \mathrm{SL}_m$. If $V \otimes W$ is stable and cofree, then $V \otimes W$ is pure.*

We sketch the general idea to prove this conjecture. All cofree irreducible representations $V \otimes W$ of $\mathrm{SL}_n \times \mathrm{SL}_m$ are in Table 3.1. Note that all but 4 of the representations, and their duals, in Table 3.1 are polar. And so, by Proposition 3.5.7, if these representations are stable, then they are pure. Given this, to prove Conjecture 3.6.6, it is enough to verify that the 4 exceptional representations, and their duals, are either not stable or pure. In Lemma 3.6.7, we show that if the exceptional representation $\mathrm{SL}_2 \otimes \mathrm{AdSL}_2$ is stable, then it is pure. The other 3 exceptional representations are stable, however, the current techniques we have to show that a given representation is pure, which essentially comes down to a direct computation of $\dim \pi(V^{sss})$, is not always feasible to do by hand. Interesting future work aims to find new techniques to show that a given representation is pure with ease.

Lemma 3.6.7 (K). *If $\mathrm{SL}_2 \otimes \mathrm{AdSL}_2 \cong \mathrm{SL}_2 \otimes \mathrm{Sym}^2 \mathrm{SL}_2$ is stable, then it is pure.*

Proof. Set $V = \mathrm{SL}_2 \otimes \mathrm{AdSL}_2$ and $G = \mathrm{SL}_2 \times \mathrm{SL}_2$ and assume (V, G) is stable. From Littelmann's table on page 198, we know that $\dim V//G = 1$. Moreover, note that $\pi(V^{sss})$ is a proper subset of $V//G$ because otherwise we would have that $\pi(V^{sss}) = \pi(V)$ by surjectivity of π , which contradicts the fact that V is stable. Since V^{sss} is closed and G -invariant, $\pi(V^{sss}) \subsetneq V//G$ is closed. It follows that $\dim \pi(V^{sss}) < \dim V//G = 1$ and so $\dim \pi(V^{sss}) = 0$. We conclude that every irreducible component of $\pi(V^{sss})$ has dimension 0, or equivalently codimension 1. So, V is pure. $\square$

Corollary 3.6.8 (K). *Let $V \otimes W$ be an irreducible representation of $\mathrm{SL}_n \times \mathrm{SL}_m$. If $V \otimes W$ is stable and cofree, then $V \otimes W$ is pure, as long as $V \otimes W$ is not one of the 3 exceptional representations, or their duals:*

$$\mathrm{SL}_9 \otimes \mathrm{Sym}^3 \mathrm{SL}_3 \quad \mathrm{SL}_4 \otimes \mathrm{Sym}^4 \mathrm{SL}_2 \quad \mathrm{Sym}^2 \mathrm{SL}_2 \otimes \bigwedge^2 \mathrm{SL}_5.$$

Highest weight	Group	Constraints on n	Polar
(ω_1, ω_1)	$\mathrm{SL}_{n-1} \times \mathrm{SL}_n$		Yes
(ω_1, ω_2)	$\mathrm{SL}_{(1/2)(n-1)n-1} \times \mathrm{SL}_n$		Yes
$(\omega_1, 2\omega_1)$	$\mathrm{SL}_{(1/2)n(n+1)-1} \times \mathrm{SL}_n$		Yes
$(\omega_1, \omega_1 + \omega_{n-1})$	$\mathrm{SL}_{n^2-2} \times \mathrm{SL}_n$	$n = 2$	No
(ω_1, ω_2)	$\mathrm{SL}_2 \times \mathrm{SL}_n$	$n \geq 5, n \text{ odd}$	Yes
(ω_1, ω_2)	$\mathrm{SL}_2 \times \mathrm{SL}_n$	$n = 4, 6, 8$	Yes
(ω_1, ω_3)	$\mathrm{SL}_{55} \times \mathrm{SL}_8$		Yes
(ω_1, ω_3)	$\mathrm{SL}_{34} \times \mathrm{SL}_7$		Yes
(ω_1, ω_3)	$\mathrm{SL}_{19} \times \mathrm{SL}_6$		Yes
$(\omega_1, 3\omega_1)$	$\mathrm{SL}_9 \times \mathrm{SL}_3$		No
$(\omega_1, 4\omega_1)$	$\mathrm{SL}_4 \times \mathrm{SL}_2$		No
$(\omega_1, 3\omega_1)$	$\mathrm{SL}_3 \times \mathrm{SL}_2$		Yes
(ω_1, ω_3)	$\mathrm{SL}_2 \times \mathrm{SL}_6$		Yes
(ω_1, ω_2)	$\mathrm{SL}_3 \times \mathrm{SL}_6$		Yes
(ω_1, ω_2)	$\mathrm{SL}_n \times \mathrm{SL}_5$	$n = 3, 4, 5$	Yes
$(2\omega_1, \omega_2)$	$\mathrm{SL}_2 \times \mathrm{SL}_5$		No
(ω_2, ω_2)	$\mathrm{SL}_4 \times \mathrm{SL}_4$		Yes
(ω_1, ω_2)	$\mathrm{SL}_3 \times \mathrm{SL}_4$		Yes
$(2\omega_1, \omega_2)$	$\mathrm{SL}_2 \times \mathrm{SL}_4$		Yes
$(\omega_1, 2\omega_1)$	$\mathrm{SL}_2 \times \mathrm{SL}_4$		Yes
$(\omega_1, 2\omega_1)$	$\mathrm{SL}_3 \times \mathrm{SL}_3$		Yes
$(\omega_1, 2\omega_1)$	$\mathrm{SL}_2 \times \mathrm{SL}_3$		Yes
$(\omega_1, 3\omega_1)$	$\mathrm{SL}_2 \times \mathrm{SL}_2$		Yes
$(2\omega_1, 2\omega_1)$	$\mathrm{SL}_2 \times \mathrm{SL}_2$		Yes
$(\omega_1, 2\omega_1)$	$\mathrm{SL}_2 \times \mathrm{SL}_2$		Yes

Table 3.1: All cofree irreducible representations of $\mathrm{SL}_n \times \mathrm{SL}_m$, up to duals.

Chapter 4

Characterizing pure representations of $\mathrm{SL}_n \times \mathrm{SL}_m$

4.1 Introduction

Throughout, let k be an algebraically closed field of characteristic 0 and let G be a reductive group. Recall, that a representation V of G is said to be cofree if $k[V]$ is free as a $k[V]^G$ -module. Equivalently, V is cofree if $V//G$ is smooth and the fibres of $\pi : V \rightarrow V//G$ all have the same dimension. The classification of cofree representations for various groups has been a motivating problem for a field of research that has been active for over 70 years. In the case of finite groups, the Chevalley-Shepard-Todd Theorem gives a beautiful intrinsic characterization of cofree representations: V is cofree if and only if G is generated by pseudoreflections. This theorem was first proved by Shephard and Todd [22] with a case-by-case proof, and then soon afterwards by Chevalley [3] with a uniform proof. In contrast to the Chevalley-Shepard-Todd Theorem, if G is a connected reductive group, then there does not exist a precise group-theoretic characterization of when V is cofree. However, several classifications of cofree representations have been found for certain connected reductive groups. For instance, the cofree representations of simple Lie groups were classified independently by Schwarz [20] and later by Adamovich and Golovina [1]. Moreover, Littelmann [13] classified irreducible cofree representations of semisimple groups. Unlike the Chevalley-Shepard-Todd Theorem, the above results for connected reductive groups consist only of an explicit list of cofree representations, rather than a general group-theoretic characterization. Indeed, Popov explicitly pointed out that a general group-theoretical characterization of cofree representations is unknown [18, pg. 403].

In 2019, Edidin, Satriano, and Whitehead formulated a conjecture for characterizing cofree representations of connected reductive groups [8]. Specifically, the conjecture states that for stable irreducible representations of connected reductive groups, V is cofree if and only if V is pure; see Conjecture 3.5.1. Further, Edidin, Satriano, and Whitehead, verified their conjecture for irreducible representations of simple Lie groups [8, Theorem 1.3]. The aim of this thesis is to work towards a verification of this conjecture by first proving it for the group $\mathrm{SL}_n \times \mathrm{SL}_m$, whose proof is expected to generalize to all connected semisimple Lie groups. The main results of this thesis come together in Chapter 4, utilizing the previous chapters as background theory.

We now briefly describe the contents of each section in this chapter, which ultimately aim to verify Conjecture 3.5.1 for irreducible representations of $\mathrm{SL}_n \times \mathrm{SL}_m$. The main results of this thesis are highlighted in Section 4.3; see Theorem 4.5.4 and Theorem 4.8.1. In particular, Theorem 4.8.1 verifies Conjecture 3.5.1 for irreducible representations $V \otimes W$ of $\mathrm{SL}_n \times \mathrm{SL}_m$ satisfying $\dim V \geq n^2$ and $\dim W \geq m^2$. The proof of Theorem 4.8.1 boils down to finding a finite list of representations that contains all possible pure representations. We find this list by utilizing Theorem 4.5.4, which gives an upper bound on the dimension of a pure representation of $G_1 \times G_2$, with G_i connected reductive Lie groups. This bound then allows us to turn the problem of classifying pure representations into a finite one, up to infinite families of representations. Section 4.4 describes a necessary condition for pure representations from [8, Proposition 3.1], which essentially says that there exists a hyperplane with “enough” weights on it. We formulate this necessary condition in 3 ways: the first as one relating to the dimension of the 0-weight space of a representation, the second relates hyperplanes and a representation of G , and the third relates hyperplanes and a representation of $\mathrm{Lie}(G)$ in the case that G is a connected reductive Lie group (e.g., SL_n and $\mathrm{SL}_n \times \mathrm{SL}_m$). The beginning of Section 4.4 sets up the background and notation in order to understand this necessary condition. Section 4.5 is devoted to the proof of Theorem 4.5.4. The two following sections are purely computation, and their results are utilized in the proof of Theorem 4.8.1. Specifically, Section 4.6 classifies all representations of SL_n that are bounded above by certain functions. And Section 4.7 finds upper bounds on the number of weights in a hyperplane for specific irreducible representations of SL_n . Section 4.8 is devoted to the proof of Theorem 4.8.1, which ties in the previous sections. As mentioned earlier, the proof of Theorem 4.8.1 boils down to first finding a finite list of representations that contains all possible pure representations, and then determining which of them are pure or not. These representations are shown to not be pure by utilizing the methods explained in Section 4.9. In Section 3.6, we write down the list of all irreducible cofree representations of $\mathrm{SL}_n \times \mathrm{SL}_m$, those of which are classified in [13]. Lastly, Section 4.10 discusses the remaining cases, namely when $V \otimes W$ satisfies $\dim V < n^2$ or $\dim W < m^2$.

4.2 Notation

We present notation and conventions that are used throughout Chapter 4. All representations are finite-dimensional. Unless specified otherwise, V is a representation of a reductive group G over k , where k is an algebraical closed field of characteristic 0. In the case that V is a representation of a Lie group G , we take the ground field to be $\mathbb{C}$. In particular, SL_n and $\mathrm{SL}_n \times \mathrm{SL}_m$ are over $\mathbb{C}$, and their representations are over $\mathbb{C}$. We define a **torus** over k to be a group isomorphic to $(\mathbb{G}_m)^n$ for some n . For a reductive group G , a **maximal torus** of G is a torus that is a subgroup of G and is maximal among other such torus, with respect to set inclusion. For a connected Lie group G and maximal torus $T \subseteq G$, the **Weyl group** of G is $W(T, G) := N_G(T)/T$, where $N_G(T)$ is the normalizer of the torus. In the case of semisimple Lie algebras, the **Weyl group** $\mathfrak{W}$ of $\mathfrak{g} := \mathrm{Lie}(G)$ is as defined in Section 2.5.1.

In this chapter, we are mostly concerned with representations of SL_n and often work with them from the point of view of them being representations of $\mathfrak{sl}_n$. As such, we adopt the notation used in Section 2.5. When it may not be clear from context, we add subscripts to our typical notation. For example, we may use $\mathfrak{h}_n$ instead of $\mathfrak{h}$ when referring to the Cartan subalgebra of $\mathfrak{sl}_n$. Recall from Theorem 2.5.19 that the highest weight of an irreducible representation of $\mathfrak{sl}_n$ takes the form

$$a_1 L_1 + a_2 (L_1 + L_2) + \cdots + a_{n-1} (L_1 + \cdots + L_{n-1}),$$

for some $a_i \in \mathbb{N}$. For $1 \leq i \leq n-1$, we define the **fundamental weight** ω_i to be

$$\omega_i := L_1 + \cdots + L_i.$$

Recall that $L_1 + \cdots + L_n = 0$. We may refer to representations of $\mathfrak{sl}_n$ via their weights in terms of fundamental weights. For example, ω_1 is the highest weight of the standard representation $\mathbb{C}^n$, $3\omega_1$ is the highest weight of the representation $\mathrm{Sym}^3 \mathbb{C}^n$, and $\omega_1 + \omega_{n-1}$ is the highest weight of the adjoint representation.

When referring to specific representations of SL_n , through abuse of notation, we often use SL_n to denote the standard representation $\mathbb{C}^n$ of SL_n . For example, we may denote the representation $\mathrm{Sym}^a \mathbb{C}^n$ of SL_n as $\mathrm{Sym}^a \mathrm{SL}_n$. Similarly, we may denote the representation $\bigwedge^a \mathbb{C}^n$ of SL_n as $\bigwedge^a \mathrm{SL}_n$. This notation becomes particularly useful when having to refer to representations of different groups. Since we only work with representations of SL_n and $\mathrm{SL}_n \times \mathrm{SL}_m$ in this chapter, this notation may not be particularly useful. However, we will adopt the notation as it will be helpful for extending this work to other groups.

We go through the structure, and therefore notation, for irreducible representations of $\mathrm{SL}_n \times \mathrm{SL}_m$. Every irreducible representation of $\mathrm{SL}_n \times \mathrm{SL}_m$ is of the form $V \otimes W$, where V and W are irreducible representations of SL_n and SL_m , respectively. We often work with representations of $\mathrm{SL}_n \times \mathrm{SL}_m$ from the point of view of them being representations of $\mathrm{Lie}(\mathrm{SL}_n \times \mathrm{SL}_m) = \mathfrak{sl}_n \times \mathfrak{sl}_m$. The highest weight of an irreducible representation $V \otimes W$ of $\mathfrak{sl}_n \times \mathfrak{sl}_m$ takes the form (ω, ω') , where ω is the highest weight of V and ω' is the highest weight of W . The Cartan subalgebra of $\mathfrak{sl}_n \times \mathfrak{sl}_m$ is $\mathfrak{h}_n \times \mathfrak{h}_m$, but sometimes we just use $\mathfrak{h}$ when clear. The Weyl group of $\mathfrak{sl}_n \times \mathfrak{sl}_m$ is the product of symmetric groups $S_n \times S_m$, which we denote by $\mathfrak{W}$ when clear.

4.3 Main results

In this section, we present an overview and a simplified version of the two main results of this thesis. The first main result provides an upper bound on $\dim V$ when V is pure:

Dimension bound on pure representations (Theorem 4.5.4) *Let $V = V_1 \otimes V_2$ be a representation of $G_1 \times G_2$ with G_i connected reductive Lie groups, with non-trivial tensor factors. If V is pure, then $\dim V$ is roughly bounded above by a cubic in $\mathrm{rk}(G)$.*

The main idea behind the proof of Theorem 4.5.4 is to use Proposition 4.4.20, which says that there exists a hyperplane $H \subseteq \mathfrak{h}^*$ satisfying

$$|H| \geq \dim V - \dim G - 1.$$

Here, $|H|$ denotes the number of weights of (V, G) in H counting multiplicity. We move the hyperplane H around by considering the Weyl group conjugates wH for each $w \in \mathfrak{W}$. There are at least $\mathrm{rk}(G)$ distinct conjugates and they all satisfy $|wH| = |H|$. An inclusion-exclusion argument on the intersection of these $\mathrm{rk}(G)$ conjugates, which is 0, gives an upper bound on the dimension of the 0-weight space of V . In turn, this allows us to obtain an upper bound on $\dim V$. The second main result tells us that irreducible representations of $\mathrm{SL}_n \times \mathrm{SL}_m$ of large dimension relative to $\dim(\mathrm{SL}_n \times \mathrm{SL}_m)$ are not pure:

Large representations are not pure (Theorem 4.8.1) *Let $V \otimes W$ be an irreducible representation of $\mathrm{SL}_n \times \mathrm{SL}_m$ satisfying $\dim V \geq n^2$ and $\dim W \geq m^2$. Then $V \otimes W$ is not pure and Conjecture 3.5.1 holds.*

To prove Theorem 4.8.1, we first utilize Theorem 4.5.4 to obtain a finite list containing all possible pure representations satisfying the constraints $\dim V \geq n^2$ and $\dim W \geq m^2$. Representations whose dimension is outside of these dimension constraints are not pure and

so we only need to check the purity of the representations in our list. To show that the list does not contain any pure representations, we show that each representation in it is not pure by utilizing the methods from Section 4.9. Littelmann classified all cofree representations of $\mathrm{SL}_n \times \mathrm{SL}_m$, all of which can be found in Table 3.1. Because no representations in Table 3.1 satisfy the dimension constraints $\dim V \geq n^2$ and $\dim W \geq m^2$, Conjecture 3.5.1 holds vacuously.

4.4 Necessary condition for purity

In this section, we set up the notation and background needed to state a necessary condition for a representation to be pure. We then describe how this condition can be interpreted as one regarding weights and hyperplanes. This condition is from [8, Proposition 3.1] and is restated in this thesis in two equivalent ways: The first being Proposition 4.4.6 in the language of 1-parameter subgroups and the second being Proposition 4.4.17 in the language of hyperplanes and representations of G . In the case that G is a connected reductive Lie group (e.g., SL_n and $\mathrm{SL}_n \times \mathrm{SL}_m$), Proposition 4.4.17 can be restated as one relating to hyperplanes and representations of $\mathrm{Lie}(G)$; see Proposition 4.4.20. This necessary condition is fundamental to the proof of Theorem 4.8.1. In particular, we use [8, Proposition 3.1] to find an upper bound on $\dim V$ when V is pure (Theorem 4.5.4), as well as to directly show that a given representation is not pure (see Section 4.9.2).

4.4.1 1-parameter subgroups and characters

Let V be a representation of a reductive group G and fix a maximal torus T of G . For Chapter 4, we define a **1-parameter subgroup** of T to be a morphism of Lie groups $\lambda : \mathbb{G}_m \rightarrow T$, where $\mathbb{G}_m$ is the **multiplicative group** of k with underlying set $k^* := k \setminus \{0\}$. We let $N(T)$ denote the group of 1-parameter subgroups of T .

Example 4.4.1. Consider when $G = \mathrm{GL}_n$ and T is the maximal torus of G consisting of diagonal matrices. Here, $\mathbb{G}_m = \mathbb{C}^*$. As noted in [4, Section 1.1], every 1-parameter subgroup λ of $T \cong (\mathbb{C}^*)^n$ takes the form

$$\lambda(t) = \mathrm{diag}(t^{a_1}, \dots, t^{a_n}), \quad t \in \mathbb{C}^*, \quad a_i \in \mathbb{Z}.$$

Given this, we see that $N(T) \cong \mathbb{Z}^n$. Here, we say that λ is the 1-parameter subgroup defined by $(a_1, \dots, a_n)$. For every integer $n \geq 2$, define T_n to be the maximal torus of SL_n consisting of diagonal matrices. The 1-parameter subgroups of T_n take the same form as above, but with the further requirement that $a_1 + \dots + a_n = 0$. $\diamond$

Example 4.4.2. Consider when $G = \mathrm{SL}_n \times \mathrm{SL}_m$. In this thesis, we will always fix the maximal torus of G to be $T_n \times T_m$. By the above example, every 1-parameter subgroup λ of $T_n \times T_m$ takes the form

$$\lambda(t) = (\mathrm{diag}(t^{a_1}, \dots, t^{a_n}), \mathrm{diag}(t^{b_1}, \dots, t^{b_m})), \quad t \in \mathbb{C}^*, \quad a_i, b_i \in \mathbb{Z},$$

where $a_1 + \dots + a_n = b_1 + \dots + b_m = 0$. ◇

Any representation V of G has an induced $\mathbb{G}_m$ action via

$$t \cdot v := \lambda(t) \cdot v, \quad t \in \mathbb{G}_m, \quad v \in V,$$

where $\lambda(t) \cdot v$ is the action of G on V . We call this the **action** of λ on V . Using the fact that λ is a group homomorphism, it can be verified that this action is indeed a torus representation of $\mathbb{G}_m$. Since λ is a torus representation, it is well known [4, Section 1.1] that the action of λ on V gives V an eigenspace decomposition. More precisely, we may write

$$V = \bigoplus_{\chi \in X(T)} V_\lambda^\chi, \quad V_\lambda^\chi := \{v \in V : \lambda(t) \cdot v = \chi(\lambda(t))v \text{ for all } t \in \mathbb{G}_m\},$$

where $\chi : T \rightarrow \mathbb{G}_m$ is a character of T ; recall that a **character of a torus** T is a group homomorphism $T \rightarrow \mathbb{G}_m$. We refer to such characters appearing in the decomposition of V as **weights** with respect to λ . The collection of all characters of T , denoted by $X(T)$, is an abelian group called the **character lattice** of T .

Example 4.4.3. Let $G = \mathrm{GL}_n$ and fix T to be the maximal torus of G consisting of diagonal matrices. As noted in [4, Section 1.1], every character χ of T takes the form

$$\chi(\mathrm{diag}(t_1, \dots, t_n)) = t_1^{b_1} \cdots t_n^{b_n}, \quad b_i \in \mathbb{Z}.$$

So, $X(T) \cong \mathbb{Z}^n$. Here, we say that χ is the character defined by $(b_1, \dots, b_n)$. For $G = \mathrm{SL}_n$, the characters of T_n are the restrictions of the characters of GL_n above and

$$X(T_n) \cong \mathbb{Z}^n / \langle (1, \dots, 1) \rangle \cong \mathbb{Z}^{n-1}.$$

Moreover, every character of T_n can be written uniquely in this form if we insist that, say $b_1 = 0$. ◇

Example 4.4.4. Consider when $G = \mathrm{SL}_n \times \mathrm{SL}_m$. Every character χ of $T_n \times T_m$ takes the form

$$\chi(\mathrm{diag}(t_1, \dots, t_n), \mathrm{diag}(\ell_1, \dots, \ell_m)) = t_1^{b_1} \cdots t_n^{b_n} \ell_1^{c_1} \cdots \ell_m^{c_m}, \quad b_i, c_i \in \mathbb{Z}.$$

◇

Now, consider the group homomorphism $\chi \circ \lambda : \mathbb{G}_m \rightarrow \mathbb{G}_m$. Since any torus over k is isomorphic to $(\mathbb{G}_m)^n$ for some n , one can show that $\chi \circ \lambda$ is of the form $t \rightarrow t^r$ for some $r \in \mathbb{Z}$. We define the inner product $\langle \lambda, \chi \rangle := r$, thus giving us a notion of non-negativity. Namely, we say a weight χ is **non-negative** with respect to λ , and write $\chi \geq 0$, if $\langle \lambda, \chi \rangle \geq 0$. We define $V_\lambda^{\geq 0}$ to be the subspace of V spanned by the basis vectors of V_λ^χ for each $\chi \geq 0$. Likewise, we define V_λ^0 to be the fixed points of V under the action of λ and call it the **0-weight space** of λ . That is,

$$V_\lambda^0 := \{v \in V : \lambda(t) \cdot v = v \text{ for all } t \in \mathbb{G}_m\}.$$

Example 4.4.5. Recall from Section 2.2.3, that $\mathfrak{sl}_3$ consists of $X \in M_3(\mathbb{C})$ satisfying $\text{Tr}(X) = 0$. Let $V = \mathfrak{sl}_3$ be the adjoint representation of $G = \text{SL}_3$ from Section 2.2.2 and consider the maximal torus T_3 . Consider the 1-parameter subgroup λ defined by $(1, 0, -1)$. Given $X = (x_{ij}) \in \mathfrak{sl}_3$, we compute that

$$\lambda(t) \cdot X = \lambda(t)X\lambda(t)^{-1} = \begin{pmatrix} x_{11} & tx_{12} & t^2x_{13} \\ t^{-1}x_{21} & x_{22} & tx_{23} \\ t^{-2}x_{31} & t^{-1}x_{32} & x_{33} \end{pmatrix}, \quad t \in \mathbb{C}^*.$$

Observe that $X \in V_\lambda^0$ exactly when X is diagonal. Since V_λ^0 consists of diagonal matrices with zero trace, $\dim V_\lambda^0 = 2$. $\diamond$

We now present the necessary condition for a representation to be pure from [8, Proposition 3.1]. As mentioned earlier, we use this condition to find an upper bound on $\dim V$ when V is pure and, additionally, to show that a given explicit representation is not pure.

Proposition 4.4.6. [8, Proposition 3.1] *Let V be a representation of a reductive group G and fix a maximal torus T of G . If V is pure, then there exists a non-trivial 1-parameter subgroup λ of T such that*

$$\dim V_\lambda^0 \geq \dim V - \dim G - 1.$$

Proof. We provide the sketch of the proof found in [8, Proposition 3.1]. Note that $V^{ss}(G) = GV^{ss}(T)$. Recall from Lemma 3.3.9 that the set of points with closed orbit and finite stabilizer is contained in $V^s(T)$. Thus, every point in $V^{ss}(T)$ has an orbit that is not closed or infinite stabilizer. And so, by the Hilbert-Mumford criterion, for every $v \in V^{ss}(T)$, there exists a non-trivial $\lambda \in N(T)$ for which v admits only non-negative weights. And so, $v \in V_\lambda^{\geq 0}$. Thus, we may write

$$V^{ss}(G) = GV^{ss}(T) \subseteq \bigcup_{\lambda \in N(T)} GV_\lambda^{\geq 0},$$

where the union is over non-trivial $\lambda \in N(T)$. Since V is finite-dimensional, there are only finitely many distinct subspaces $V_\lambda^{\geq 0}$ as λ runs through the elements of $N(T)$. Given an irreducible component C of $V^{ss}(G)$, we can find non-trivial λ such that $C \subseteq GV_\lambda^{\geq 0}$. Since V is pure, we know that $\dim \pi(C) = \dim V//G - 1$. Moreover, $\pi(C) \subseteq \pi(GV_\lambda^{\geq 0}) = \pi(V_\lambda^0)$ and so

$$\dim V_\lambda^0 \geq \dim \pi(V_\lambda^0) \geq \dim V//G - 1 = \dim V - \dim G + d - 1 \geq \dim V - \dim G - 1,$$

where d is the dimension of the generic stabilizer of V . $\square$

Example 4.4.7. Consider Example 4.4.5, where $V = \mathfrak{sl}_3$ and $G = \mathrm{SL}_3$. We know from [8, Table 1] that V is pure. It was shown in Example 4.4.5 that if λ is the 1-parameter subgroup defined by $(1, 0, -1)$, then $V_\lambda^0 = 2$. And, indeed

$$\dim V - \dim G - 1 = 8 - (3^2 - 1) - 1 = 2 = \dim V_\lambda^0.$$

$\diamond$

Section 4.9.2 contains explicit examples for how to show that a given representations is not pure by Proposition 4.4.6.

4.4.2 Weights of a group representation

In this section, we provide a different interpretation of V_λ^0 using the notion of weights of (V, G) . Then, in the case that G is a connected reductive Lie group, we describe the one-to-one correspondence between weights of (V, G) and weights of $(V, \mathfrak{g})$, where $\mathfrak{g} := \mathrm{Lie}(G)$. Ultimately, for our purposes, this correspondence allows us to utilize our knowledge from Chapter 2 when using Proposition 4.4.6. See Section 2.2.2 and Section 2.5 for the background and notation of Lie algebras and their representations. Fixing a maximal torus T of G , we have the following eigenspace decomposition:

$$V = \bigoplus_{\chi \in X(T)} V^\chi, \quad V^\chi := \{v \in V : t \cdot v = \chi(t)v \text{ for all } t \in T\}.$$

The action $t \cdot v$ appearing in V^χ is the action of G on V . We refer to the characters χ appearing in this decomposition of V as **weights** of (V, G) . Given a 1-parameter subgroup $\lambda : \mathbb{G}_m \rightarrow T$, we can rewrite V_λ^0 as follows:

$$V_\lambda^0 = \bigoplus_{\substack{\chi \in X(T) \\ \langle \lambda, \chi \rangle = 0}} V^\chi. \tag{4.1}$$

So, if we already know the weights of V , to determine V_λ^0 we just need to see which χ satisfy $\langle \lambda, \chi \rangle = 0$.

Example 4.4.8. Consider $V = \mathfrak{sl}_3$ and $G = \mathrm{SL}_3$ as in Example 4.4.5. If $v = (x_{ij}) \in V$ and $t = \mathrm{diag}(t_1, t_2, t_3)$, then

$$t \cdot v = tv t^{-1} = \begin{pmatrix} x_{11} & t_1 t_2^{-1} x_{12} & t_1 t_3^{-1} x_{13} \\ t_2 t_1^{-1} x_{21} & x_{22} & t_2 t_3^{-1} x_{23} \\ t_3 t_1^{-1} x_{31} & t_3 t_2^{-1} x_{32} & x_{33} \end{pmatrix}.$$

To compute the decomposition of V into V^χ , we want to see for which $(b_1, b_2, b_3) \in \mathbb{Z}^3$ there exists a non-zero $v \in V$ with $t \cdot v = t_1^{b_1} t_2^{b_2} t_3^{b_3} v$. Observing the powers of the t_i in $t \cdot v$, the weights of V can be represented as the following set:

$$\{(0, 0, 0), (1, -1, 0), (-1, 1, 0), (0, 1, -1), (0, -1, 1), (1, 0, -1), (-1, 0, 1)\},$$

where each tuple defines a character. If λ is defined by $(1, 0, -1)$, we see that $\langle \lambda, \chi \rangle = 0$ only when χ is defined by $(0, 0, 0)$, in which case $V_\lambda^0 = V^\chi$ consists of diagonal matrices. Note that this computation of V_λ^0 agrees with the result from Example 4.4.5. $\diamond$

Remark 4.4.9. If G is a connected affine algebraic group over k , we define a **Cartan subgroup** of G to be a closed connected subgroup whose Lie algebra is a Cartan subalgebra of $\mathfrak{g}$. If we further insist that G is reductive (e.g., connected semisimple Lie group), then the Cartan subgroups of G are exactly the maximal tori of G . Since k is an algebraically closed field, any two maximal tori of an affine algebraic group G are G -conjugate, just as any two Cartan subalgebras are conjugate.

Proposition 4.4.10. *Let G be a connected reductive Lie group and $\rho : G \rightarrow \mathrm{GL}(V)$ a representation of G . Consider the corresponding representation $(d\rho)_e : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ of $\mathfrak{g}$. There is a bijection between the weights of (V, G) and the weights of $(V, \mathfrak{g})$. Moreover, if χ is a weight of (V, G) corresponding to the weight β of $(V, \mathfrak{g})$, then $V^\chi = V_\beta$.*

Proof. Fix a maximal torus T of G . Let $\chi : T \rightarrow \mathbb{C}^*$ be a weight of (V, G) and consider $(d\chi)_e : \mathfrak{h} \rightarrow \mathbb{C}$, where $\mathfrak{h} := \mathrm{Lie}(T)$. By Remark 4.4.9, since G is connected and reductive, $\mathfrak{h}$ is indeed a Cartan subalgebra of $\mathfrak{g}$. Recall that

$$V_{(d\chi)_e} = \{v \in V : H(v) = (d\chi)_e(H)v \text{ for all } H \in \mathfrak{h}\}$$

where $H(v) = (d\rho)_e(H)v$. To show $V^\chi \subseteq V_{(d\chi)_e}$, let $v \in V^\chi$ so that $\rho(t)v = \chi(t)v$ for all $t \in T$. Given $H \in \mathrm{Lie}(T)$, we can find a smooth path $\gamma : (-\varepsilon, \varepsilon) \rightarrow T$ such that $\gamma(0) = e$ and $\gamma'(0) = H$. Then

$$(d\chi)_e(H)v = \left. \frac{d}{dt} \right|_0 \chi(\gamma(t))v = \left. \frac{d}{dt} \right|_0 \rho(\gamma(t))v = (d\rho)_e(H)v = H(v)$$

and so $v \in V_{(d\chi)_e}$. Note that since $0 \neq V^\chi \subseteq V_{(d\chi)_e}$ and $(d\chi)_e \in \mathfrak{h}^*$, $(d\chi)_e$ is a weight of $(V, \mathfrak{g})$. Using the fact that

$$\bigoplus_{\chi \in X(T)} V^\chi = \bigoplus_{\beta \in \Lambda_W} V_\beta$$

and $V^\chi \subseteq V_{(d\chi)_e}$ for all weights χ , we have that $V^\chi = V_{(d\chi)_e}$. This, together with the direct sum equality above, tells us that for every weight β of $(V, \mathfrak{g})$ we can find a weight χ of (V, G) such that $\beta = (d\chi)_e$. We conclude that there is a bijection between the weights of (V, G) and the weights of $(V, \mathfrak{g})$, with the natural bijection being $\chi \mapsto (d\chi)_e$. $\square$

Example 4.4.11. Let V be a representation of $G = \mathrm{SL}_n$. Let $\chi \in X(T_n)$ be a weight of (V, SL_n) defined by $(b_1, \dots, b_n) \in \mathbb{Z}^n$. Given $H \in \mathrm{Lie}(T) \subset \mathfrak{sl}_n$, we can find a smooth path $\gamma : (-\varepsilon, \varepsilon) \rightarrow T_n$, such that $\gamma(0) = e$ and $\gamma'(0) = H$. Write $H = \mathrm{diag}(a_1, \dots, a_n)$ and

$$\gamma(t) = \mathrm{diag}(\gamma_1(t), \dots, \gamma_n(t)),$$

where $\gamma_i : (-\varepsilon, \varepsilon) \rightarrow \mathbb{C}$ satisfies $\gamma_i(0) = 1$ and $\gamma'_i(0) = a_i$ for all i . We compute that

$$\begin{aligned} (d\chi)_e(H) &= \left. \frac{d}{dt} \right|_0 (\chi(\gamma(t))) \\ &= \left. \frac{d}{dt} \right|_0 \gamma_1(t)^{b_1} \cdots \gamma_n(t)^{b_n} \\ &= \sum_{i=1}^n \left. \frac{d}{dt} \right|_0 \gamma_i(t)^{b_i} \\ &= \sum_{i=1}^n b_i \gamma_i(0)^{b_i-1} \gamma'_i(0) \\ &= \sum_{i=1}^n b_i a_i \\ &= (b_1 L_1(H) + \cdots + b_n L_n(H)) \end{aligned}$$

and so $(d\chi)_e = b_1 L_1 + \cdots + b_n L_n$. This, together with Proposition 4.4.10, tells us that

$$(b_1, \dots, b_n) \mapsto b_1 L_1 + \cdots + b_n L_n$$

is a bijection between the weights of (V, SL_n) and the weights of $(V, \mathfrak{sl}_n)$. Note that if $(b, \dots, b) \in \langle (1, \dots, 1) \rangle$ is a weight of (V, SL_n) , then it corresponds to the 0 weight of $(V, \mathfrak{sl}_n)$ since

$$bL_1 + \cdots + bL_n = b(L_1 + \cdots + L_n) = b(0) = 0.$$

$\diamond$

Example 4.4.12. Consider $V = \mathfrak{sl}_3$ and $G = \mathrm{SL}_3$ as in Example 4.4.8. We see that the weights of (V, G) correspond to the weights

$$0, L_1 - L_2, L_2 - L_1, L_2 - L_3, L_3 - L_2, L_1 - L_3, L_3 - L_1$$

of $(V, \mathfrak{g})$, which we are familiar with from Section 2.4. $\diamond$

Proposition 4.4.13. *Fix a maximal torus T of a connected reductive Lie group G . Given $\lambda \in N(T)$ and $\chi \in X(T)$, we have that $\langle \lambda, \chi \rangle = 0$ if and only if $(d(\chi \circ \lambda))_e = 0$.*

Proof. Recall that $\chi \circ \lambda : \mathbb{C}^* \rightarrow \mathbb{C}^*$ is given by $(\chi \circ \lambda)(t) = t^r$ for some $r \in \mathbb{Z}$. Given $a \in \mathbb{C}$, we can find a smooth path $\gamma : (-\varepsilon, \varepsilon) \rightarrow \mathbb{C}^*$, such that $\gamma(0) = 1$ and $\gamma'(0) = a$. We compute that

$$(d(\chi \circ \lambda))_e(a) = \left. \frac{d}{dt} \right|_0 (\chi \circ \lambda)(\gamma(t)) = \left. \frac{d}{dt} \right|_0 \gamma(t)^r = r\gamma(0)^{r-1}\gamma'(0) = ra.$$

So, $(d(\chi \circ \lambda))_e = r$. It follows that $(d(\chi \circ \lambda))_e = 0$ if and only if $r = 0$ if and only if $\langle \lambda, \chi \rangle = 0$. $\square$

Corollary 4.4.14. *Let V be a representation of a connected reductive Lie group G . Fix a maximal torus T of G . Given $\lambda \in N(T)$, we have that*

$$V_\lambda^0 = \bigoplus_{\substack{\chi \in X(T) \\ (d(\chi \circ \lambda))_e = 0}} V_{(d\chi)_e}.$$

Proof. This follows immediately from Proposition 4.4.10 and Proposition 4.4.13. $\square$

Remark 4.4.15. Taking into account Chapter 2, if we are given an explicit representation (V, G) , we often know what the weights β of $(V, \mathfrak{g})$ are. And so, by Corollary 4.4.14, if we already know the weights β of $(V, \mathfrak{g})$, then to compute V_λ^0 we just need to figure out which ones satisfy $(d(\chi \circ \lambda))_e = 0$ under the bijection $\chi \mapsto \beta = (d\chi)_e$.

Example 4.4.16. Consider the representation $V = \mathrm{Sym}^2 \mathrm{SL}_3$. We know that the weights of $(V, \mathfrak{sl}_3)$ are

$$2L_1, 2L_2, 2L_3, L_1 + L_2, L_1 + L_3, L_2 + L_3.$$

Note that if χ is a weight of (V, G) , then

$$(d(\chi \circ \lambda))_e = (d\chi)_{\lambda(e)} \circ (d\lambda)_e = (d\chi)_e \circ (d\lambda)_e.$$

Consider $\lambda : \mathbb{C}^* \rightarrow T_3$ given by $\lambda(t) = \text{diag}(t, 1, t^{-1})$. We compute V_λ^0 using Corollary 4.4.14. For $a \in \mathbb{C}$, we can find a smooth path $\gamma : (-\varepsilon, \varepsilon) \rightarrow \mathbb{C}^*$ such that $\gamma(0) = 1$ and $\gamma'(0) = a$. We compute that

$$(d\lambda)_e(a) = \left. \frac{d}{dt} \right|_0 \lambda(\gamma(t)) = \left. \frac{d}{dt} \right|_0 \text{diag}(\gamma(t), 1, \gamma(t)^{-1}) = \text{diag}(\gamma'(0), 0, -\frac{\gamma'(0)}{\gamma(0)^2}) = \text{diag}(a, 0, -a).$$

Now, we just need to check which weights of $(V, \mathfrak{sl}_3)$ map $\text{diag}(a, 0, -a)$ to 0 for all $a \in k$. Observe that $2L_2$ and $L_1 + L_3$ are the only such weights, and indeed

$$2L_2(\text{diag}(a, 0, -a)) = 2(0) = 0, \quad (L_1 + L_3)(\text{diag}(a, 0, -a)) = a + (-a) = 0.$$

It follows that $V_\lambda^0 = V_{2L_2} \oplus V_{L_1+L_3} = V^{(0,2,0)} \oplus V^{(1,0,1)}$. $\diamond$

4.4.3 Hyperplanes arising from 1-parameter subgroups

The purpose of this section is to translate Proposition 4.4.6 into the language of hyperplanes and weights. In particular, we will see that $\dim V_\lambda^0$ is equal to the number of (V, G) weights in some hyperplane in $X(T) \otimes \mathbb{R}$; see Proposition 4.4.17. In the case that G is a connected reductive Lie group, we also have that V_λ^0 is equal to the number of $(V, \mathfrak{g})$ weights in some hyperplane in $\mathfrak{h}^* = \text{Lie}(T)^*$; see Proposition 4.4.20. Let V be a representation of a reductive group G and fix a maximal torus T of G . To each non-trivial 1-parameter subgroup $\lambda : \mathbb{G}_m \rightarrow T$, we can associate to it a hyperplane

$$H_\lambda^T := \{\chi \otimes 1 : \langle \lambda, \chi \rangle = 0\} \subset X(T) \otimes \mathbb{R}.$$

Alternatively, by identifying $\text{span}_{\mathbb{R}} X(T) \cong X(T) \otimes \mathbb{R}$ and extending $\langle \lambda, \cdot \rangle$ linearly to $\text{span}_{\mathbb{R}} X(T)$, we can think of H_λ^T as living inside of $\text{span}_{\mathbb{R}} X(T)$. Note that H_λ^T is a hyperplane through 0. We will often suppress notation and write

$$H_\lambda^T = \{\chi \in X(T) : \langle \lambda, \chi \rangle = 0\}.$$

Given a hyperplane H in $X(T) \otimes \mathbb{R}$, we write $|H|_V$ to mean the number of weights of (V, G) in H counting multiplicity, where the multiplicity of χ is $\dim V^\chi$. Then, by Equation 4.1, it follows that

$$\dim V_\lambda^0 = \sum_{\substack{\chi \in X(T) \\ \langle \lambda, \chi \rangle = 0}} \dim V^\chi = |H_\lambda^T|_V.$$

Note that since $\dim V^\chi = 0$ if χ is not a weight, this sum ranges over weights $\chi \in X(T)$ satisfying $\langle \lambda, \chi \rangle = 0$. Proposition 4.4.6 can then be interpreted as follows:

Proposition 4.4.17 (Restatement of Proposition 4.4.6). *Let V be a representation of a reductive group G and fix a maximal torus T of G . If V is pure, then there exists a non-trivial 1-parameter subgroup λ of T such that*

$$|H_\lambda^T|_V \geq \dim V - \dim G - 1.$$

That is, if V is pure, then we can find a hyperplane with “enough” weights in it, with “enough” being $\dim V - \dim G - 1$.

Next, we translate this condition to one for a hyperplane in $\mathrm{Lie}(T)^*$. For the remainder of this section, let G be a connected reductive Lie group. Recall from Section 2.5.1 or [9, Section 14.2] the Killing form B on $\mathfrak{h}^*$, where $\mathfrak{h} = \mathrm{Lie}(T)$. The restriction of B to $\mathrm{span}_{\mathbb{R}}\Lambda_R$ is positive definite [9, Corollary 14.30] and so $(\mathrm{span}_{\mathbb{R}}\Lambda_R, B)$ is a real inner product space. Identifying the dual of $\mathbb{C}$ with $\mathbb{C}$ via $\mapsto f(1)$, the dual of $(d\lambda)_e : \mathbb{C} \rightarrow \mathfrak{h}$ is the linear map

$$(d\lambda)_e^* : \mathfrak{h}^* \rightarrow \mathbb{C}, \quad (d\lambda)_e^*(\beta) = (\beta \circ (d\lambda)_e)(1).$$

By the Riesz Representation Theorem, there exists a unique vector $\alpha_\lambda \in \mathrm{span}_{\mathbb{R}}\Lambda_R$, such that

$$(d\lambda)_e^*(\beta) = B(\beta, \alpha_\lambda)$$

for all $\beta \in \mathrm{span}_{\mathbb{R}}\Lambda_R$. Since $(d\lambda)_e^*$ and $B(\cdot, \alpha_\lambda)$ are both $\mathbb{C}$ -linear functions equal on $\mathrm{span}_{\mathbb{R}}\Lambda_R$, they are equal on $\mathrm{span}_{\mathbb{C}}\Lambda_R = \mathfrak{h}^*$. Note that equality $\mathrm{span}_{\mathbb{C}}\Lambda_R = \mathfrak{h}^*$ follows by Proposition 2.5.4.

Proposition 4.4.18. *Fix a maximal torus T of a connected reductive Lie group G . Given $\lambda \in N(T)$ and $\chi \in X(T)$, we have that $\langle \lambda, \chi \rangle = 0$ if and only if $B((d\chi)_e, \alpha_\lambda) = 0$.*

Proof. By Proposition 4.4.13, we have that $\langle \lambda, \chi \rangle = 0$ if and only if $(d\chi)_e \circ (d\lambda)_e = 0$. Since a linear function $\mathbb{C} \rightarrow \mathbb{C}$ is determined by its value at 1, $(d\chi)_e \circ (d\lambda)_e = 0$ is equivalent to $(d\lambda)_e^*((d\chi)_e) = ((d\chi)_e \circ (d\lambda)_e)(1) = 0$. By the above paragraph, $(d\lambda)_e^*((d\chi)_e) = 0$ is equivalent to $B((d\chi)_e, \alpha_\lambda) = 0$. $\square$

By Proposition 4.4.18, we may rewrite H_λ^T as

$$H_\lambda^T = \{\chi \in X(T) : B((d\chi)_e, \alpha_\lambda) = 0\}.$$

Given a hyperplane H in $\mathfrak{h}^*$, we write $|H|_V$ to mean the number of weights of $(V, \mathfrak{g})$ in H counting multiplicity, where the multiplicity of β is $\dim V_\beta$.

Corollary 4.4.19. *Let V be a representation of a connected reductive Lie group G . Fix a maximal torus T of G and set $\mathfrak{h} = \text{Lie}(T)$. Given a non-trivial $\lambda \in N(T)$, define the hyperplane*

$$H_\lambda^{\mathfrak{h}^*} := \{\beta \in \mathfrak{h}^* : B(\beta, \alpha_\lambda) = 0\}.$$

Then $|H_\lambda^T|_V = |H_\lambda^{\mathfrak{h}^*}|_V$.

Proof. By Proposition 4.4.10 and Proposition 4.4.18, we have that

$$|H_\lambda^T|_V = \sum_{\substack{\chi \in X(T) \\ \langle \lambda, \chi \rangle = 0}} \dim V^\chi = \sum_{\substack{\chi \in X(T) \\ \langle \lambda, \chi \rangle = 0}} V_{(d\chi)_e} = \sum_{\substack{\chi \in X(T) \\ B((d\chi)_e, \alpha_\lambda) = 0}} V_{(d\chi)_e} = \sum_{\substack{\beta \in \mathfrak{h}^* \\ B(\beta, \alpha_\lambda) = 0}} V_\beta = |H_\lambda^{\mathfrak{h}^*}|_V.$$

□

Note that $H_\lambda^{\mathfrak{h}^*}$ is a hyperplane through 0. We make a final restatement of [8, Proposition 3.1].

Proposition 4.4.20 (Connected reductive Lie group version of Proposition 4.4.17). *Let V be a representation of a connected reductive Lie group G . Fix a maximal torus T of G and set $\mathfrak{h} = \text{Lie}(T)$. If V is pure, then there exists a non-trivial 1-parameter subgroup λ of T such that*

$$|H_\lambda^{\mathfrak{h}^*}|_V \geq \dim V - \dim G - 1.$$

Remark 4.4.21. Since $|H_\lambda^T|_V = |H_\lambda^{\mathfrak{h}^*}|_V$ when G is a connected reductive Lie group, through abuse of notation, we often simplify notation and denote both hyperplanes simply by H_λ . When the representation V of G is clear from context, we often write $|H_\lambda|$ to mean $|H_\lambda|_V$.

Example 4.4.22. Consider $G = \text{SL}_n$ and the maximal torus T_n . Let $\lambda \in N(T_n)$ be defined by $(a_1, \dots, a_n) \in \mathbb{Z}^n$, where $\sum a_i = 0$ and $a_1 \cdots a_n \neq 0$. Given $x \in \mathbb{C}$ we can find a smooth path $\gamma : (-\varepsilon, \varepsilon) \rightarrow \mathbb{C}^*$ such that $\gamma(0) = 1$ and $\gamma'(0) = x$. Then

$$\begin{aligned} (d\lambda)_e(x) &= \left. \frac{d}{dt} \right|_0 \lambda(\gamma(t)) \\ &= \left. \frac{d}{dt} \right|_0 \text{diag}(\gamma(t)^{a_1}, \dots, \gamma(t)^{a_n}) \\ &= \text{diag}(a_1 \gamma(0)^{a_1-1} \gamma'(0), \dots, a_n \gamma(0)^{a_n-1} \gamma'(0)) \\ &= x \text{diag}(a_1, \dots, a_n). \end{aligned}$$

Recall from Proposition 2.5.7 that

$$\Lambda_R = \{a_1 L_1 + \cdots + a_n L_n \in \mathfrak{h}^* : a_i \in \mathbb{Z}, a_1 + \cdots + a_n = 0\}.$$

Setting $\alpha_\lambda := 2n \sum a_i L_i$, we have that $\alpha_\lambda \in \Lambda_R$. Given $\beta = b_1 L_1 + \cdots + b_n L_n \in \mathfrak{h}^*$, we compute that

$$(d\lambda)_e^*(\beta) = (\beta \circ (d\lambda)_e)(1) = (b_1 L_1 + \cdots + b_n L_n)(\text{diag}(a_1, \dots, a_n)) = a_1 b_1 + \cdots + a_n b_n.$$

Recall the Killing form B on $\mathfrak{sl}_n$ from Section 2.5.1. Using that $\sum a_i = 0$, we have that

$$B(\beta, \alpha_\lambda) = \frac{1}{2n} \left(\sum_i (2n) a_i b_i - \frac{1}{n} \sum_j b_j \sum_i (2n) a_i \right) = \sum_i a_i b_i.$$

So α_λ is the unique vector satisfying $(d\lambda)_e^*(\beta) = B(\beta, \alpha_\lambda)$ for all $\beta \in \mathfrak{h}^*$. Since scalar multiplying the normal vector of a hyperplane through 0 does not change it, we can rescale and take $\alpha_\lambda = \sum a_i L_i$ if we want. We conclude that

$$H_\lambda^{\mathfrak{h}^*} = \{\beta \in \mathfrak{h}^* : B(\beta, \sum a_i L_i) = 0\} = \{b_1 L_1 + \cdots + b_n L_n \in \mathfrak{h}^* : \sum_i a_i b_i = 0\}.$$

Since the normal vector of both H_λ^T and $H_\lambda^{\mathfrak{h}^*}$ is determined by the a_i (i.e., λ), through abuse of notation, we often write $H_\lambda^T = (a_1, \dots, a_n)^\perp$ and $H_\lambda^{\mathfrak{h}^*} = (a_1, \dots, a_n)^\perp$. $\diamond$

4.5 Bounding the dimension of a pure representation

In this section, we prove the first main result of this thesis, which is an upper bound on $\dim(V \otimes W)$ when $V \otimes W$ is a pure representation of a connected reductive Lie group; see Theorem 4.5.4. This result is analogous to Popov's [19, Theorem 8.9], which says that if V is a cofree representation of a semisimple Lie group G , then $\dim V \leq 2 \dim G$. For irreducible representations $V \otimes W$ of $\text{SL}_n \times \text{SL}_m$, if $\dim V \geq n^2$ and $\dim W \geq m^2$, then Theorem 4.5.4 allows us to obtain a finite list that contains all pure representations of $\text{SL}_n \times \text{SL}_m$. And so, this bound turns the problem of classifying pure representations of $\text{SL}_n \times \text{SL}_m$ into a finite one; see Section 4.8. To prove Theorem 4.5.4, we first prove the following two lemmas regarding representations of finite groups.

Lemma 4.5.1. *Let G be non-trivial finite group and V an irreducible representation of G over k . Consider the maps*

$$\phi : k[G] \rightarrow \text{End}(V), \quad \phi\left(\sum \lambda_i g_i\right)(v) := \sum \lambda_i (g_i \cdot v)$$

and

$$\omega : k[G] \rightarrow k, \quad \omega\left(\sum \lambda_i g_i\right) := \sum \lambda_i,$$

where $\lambda_i \in k, g_i \in G$, and $v \in V$. If V is non-trivial, then for all $v \in V \setminus \{0\}$, we have that $\text{ann}(v) \not\subseteq \ker(\omega)$, where

$$\text{ann}(v) := \{g \in \mathbb{C}[G] : g \cdot v = 0\}.$$

Proof. We prove the contrapositive of the statement. Suppose there exists $v \in V \setminus \{0\}$ such that $\text{ann}(v) \subseteq \ker(\omega)$. We claim that V is the trivial representation. To show this, we first show that $\text{ann}(v)$ is a maximal left ideal in $k[G]$. Since

$$k[G] \rightarrow k[G] \cdot v, \quad x \mapsto x \cdot v$$

is a surjective left $k[G]$ -module map, by the first isomorphism theorem, we get that

$$k[G] \cdot v \cong k[G]/\text{ann}(v).$$

Since $k[G] \cdot v$ is a non-zero G -invariant subspace of V , by irreducibility, we get that $V = k[G] \cdot v$. So, $V \cong k[G]/\text{ann}(v)$. Moreover, since V is an irreducible representation of G , V is a simple $k[G]$ -module. Thus, $k[G]/\text{ann}(v)$ is a simple $k[G]$ -module, and so $\text{ann}(v)$ is a maximal left ideal.

Since $\text{ann}(v)$ is maximal, we get that $\text{ann}(v) = \ker(\omega)$. Now, since ω is a surjective module morphism, by the first isomorphism theorem, we get that

$$k \cong k[G]/\ker(\omega) = k[G]/\text{ann}(v).$$

By the universal property of quotients, there exists a $k[G]$ -module map θ such that

$$\begin{array}{ccc} k[G] & \xrightarrow{\omega} & k \cong k[G]/\text{ann}(v) \\ \phi \downarrow & & \swarrow \theta \\ \text{End}(V) & & \\ \text{ev}_v \downarrow & \nwarrow & \\ V & & \end{array}$$

commutes. Here, ev_v evaluates endomorphisms in $\text{End}(V)$ at v . Since $\text{ev}_v \circ \phi \neq 0$, we get that $\theta \neq 0$. Since V and k are irreducible representations of G and θ is a non-zero $k[G]$ -module map, by Schur's lemma, we get that $k \cong V$ as representations. So, V is trivial. $\square$

Lemma 4.5.2. *Let G_1 and G_2 be non-trivial finite groups with non-trivial irreducible representations V_1 and V_2 over k , respectively. Set $G := G_1 \times G_2$ and $V := V_1 \oplus V_2$. Then for all $v = (v_1, v_2) \in V$, we have that*

$$\text{span}(Gv) = \text{span}(G_1v_1) \oplus \text{span}(G_2v_2).$$

Proof. It is immediate by linear algebra that $\text{span}(Gv)$ is contained in the right hand side. Let $v = (v_1, v_2) \in V$ and first consider the case when $v_1 = 0$ or $v_2 = 0$. We may assume $v_2 = 0$. Then $\text{span}(G_2v_2) = 0$ and

$$Gv = \{(g_1, g_2)(v_1, 0) : g_1 \in G_1, g_2 \in G_2\} = \{(g_1v_1, 0) : g_1 \in G_1\}.$$

Let $x \in \text{span}(G_1v_1) \oplus \text{span}(G_2v_2)$. Since $\text{span}(G_2v_2) = 0$, we may write

$$x = \left(\sum \lambda_i h_i v_1, 0\right) = \sum \lambda_i (h_i v_1, 0)$$

for some $\lambda_i \in k$ and $h_i \in G_1$. By the expression of Gv above, we see that x is in $\text{span}(Gv)$.

Next, consider the case when $v_1 \neq 0$ and $v_2 \neq 0$. It suffices to show that $\text{span}(G_1v_1) \oplus 0$ and $0 \oplus \text{span}(G_2v_2)$ are contained in $\text{span}(Gv)$. Further, by symmetry, it is enough to show that $\text{span}(G_1v_1) \oplus 0 \subseteq \text{span}(Gv)$. Since $\text{span}(Gv)$ is a subspace of V , we just need to show that $(w, 0) \in \text{span}(Gv)$ for any $w \in G_1v_1$. So, let $w \in G_1v_1$ and say $w = gv_1$ for some $g \in G_1$. Since $v_2 \neq 0$, by Lemma 4.5.1, we have that $\text{ann}(v_2) \not\subseteq \ker(\omega)$. This means that there exists $x \in k[G_2]$ such that $x \cdot v_2 = 0$ and $x \notin \ker(\omega)$. Let $x = \sum \lambda_i g_i$ where $g_i \in G_2, \lambda_i \in \mathbb{C}$. Since $x \notin \ker(\omega), \lambda := \sum \lambda_i \neq 0$ and $x \neq 0$. Then

$$\text{span}(Gv) \ni \sum_i \lambda_i (e, g_i)(v_1, v_2) = \sum_i \lambda_i (v_1, g_i v_2) = (\lambda v_1, x \cdot v_2) = (\lambda v_1, 0)$$

and hence $(v_1, 0) = (1/\lambda)(\lambda v_1, 0) \in \text{span}(Gv)$. So, $(w, 0) = (g, e)(v_1, 0) \in \text{span}(Gv)$, as $\text{span}(Gv)$ is G -invariant. $\square$

Notation 4.5.3 (For Theorem 4.5.4 below). If $V = V_1 \otimes V_2$ is a pure representation of a connected reductive Lie group $G = G_1 \times G_2$, then by Proposition 4.4.20 there exists a hyperplane $H \subseteq \mathfrak{h}^*$ through 0 such that

$$|H| \geq \dim V - (\dim G + 1).$$

The hyperplane H is defined by some normal vector $v = (v_1, v_2) \in \mathfrak{h}^*$, and we write $H = \{v\}^\perp$ to represent this.

Consider the hyperplane H from Notation 4.5.3. Moving H around by the Weyl group $\mathfrak{W}$ results in new hyperplanes of the form $wH, w \in \mathfrak{W}$. We call such hyperplanes **conjugates** of H under $\mathfrak{W}$. We also say that wH is a $\mathfrak{W}$ -conjugate of H . Since weights and their multiplicities are invariant under $\mathfrak{W}$, we have that $|H| = |wH|$ for all $w \in \mathfrak{W}$. The idea behind Theorem 4.5.4 is to use these conjugates to get a relationship between the multiplicity of 0 in V and $\dim V$, which ultimately results in an upper bound for $\dim V$.

Theorem 4.5.4 (K). *Let $V = V_1 \otimes V_2$ be a representation of $G_1 \times G_2$, with G_i connected reductive Lie groups. Assume V_1 and V_2 are non-trivial. If V is pure, then*

$$\dim V \leq \begin{cases} \operatorname{rk}(G)(\dim G + 1) + \dim(V_1)_0 \dim(V_2)_0 & v_1, v_2 \neq 0 \\ \operatorname{rk}(G_1)(\dim G + 1) + \dim(V_1)_0 \dim V_2 & v = (v_1, 0) \\ \operatorname{rk}(G_2)(\dim G + 1) + \dim V_1 \dim(V_2)_0 & v = (0, v_2) \end{cases}.$$

Here, $v = (v_1, v_2)$ is the normal vector of H , where H is the hyperplane from Notation 4.5.3, and $(V_i)_0$ denotes the 0-weight space of V_i .

Proof. Let $\mathfrak{h}_{G_i}$ be the Cartan subalgebra of $\operatorname{Lie}(G_i)$. First consider the case when $v_1, v_2 \neq 0$. We claim that H has at least $\operatorname{rk}(G)$ linearly independent conjugates under $\mathfrak{W}$. By [9, Lemma 14.31], we know that $\mathfrak{h}_{G_i}^*$ is an irreducible representation of $\mathfrak{W}_i$. Thus, $\operatorname{span}(\mathfrak{W}_i v_i) = \mathfrak{h}_{G_i}^*$ and so, by Lemma 4.5.2, we get that

$$\operatorname{span}(\mathfrak{W}v) = \operatorname{span}(\mathfrak{W}_1 v_1) \oplus \operatorname{span}(\mathfrak{W}_2 v_2) = \mathfrak{h}_{G_1}^* \oplus \mathfrak{h}_{G_2}^* = \mathfrak{h}^*.$$

This implies that there are $\dim \mathfrak{h}^* = \operatorname{rk}(G)$ linearly independent vectors in $\operatorname{span}(\mathfrak{W}v)$. If $x \in \mathfrak{h}^*$ is one of these $\operatorname{rk}(G)$ linearly independent vectors, then one can show that $\{x\}^\perp$ is a $\mathfrak{W}$ -conjugate of H . So, there are at least $\operatorname{rk}(G)$ linearly independent conjugates under $\mathfrak{W}$, as desired. From here, we can apply the rest of the proof of [8, Proposition 3.5] to get that there are at most $\operatorname{rk}(G)(\dim G + 1)$ non-zero weights of V counted with multiplicity. Specifically, let $H_1, \dots, H_{\operatorname{rk}(G)}$ be the $\operatorname{rk}(G)$ such conjugates of H . By inclusion-exclusion,

$$\begin{aligned} |H_1 \cap H_2| &= |H_1| + |H_2| - |H_1 \cup H_2| \\ &= 2|H| - |H_1 \cup H_2| \\ &\geq 2(\dim V - (\dim G + 1)) - \dim V \\ &= \dim V - 2(\dim G + 1). \end{aligned}$$

Here, we are using the fact that $|H| = |H_i|$. By induction on n , we get that

$$|H_1 \cap \dots \cap H_n| \geq \dim V - n(\dim G + 1).$$

So, $H_1 \cap \cdots \cap H_{\text{rk}(G)} = \{0\}$ contains at least $\dim V - \text{rk}(G)(\dim G + 1)$ weights counted with multiplicity. This means that

$$\dim(V_1)_0 \dim(V_2)_0 \geq \dim V - \text{rk}(G)(\dim G + 1),$$

where $\dim(V_1)_0 \dim(V_2)_0$ is the multiplicity of the 0 weight of V , as desired.

Next, consider the case when $v = (v_1, 0)$. The case $v = (0, v_2)$ follows by symmetry. In this case, we have that $H = H' \oplus \mathfrak{h}_{G_2}^*$, where $H' := \{v_1\}^\perp$. Similar to above, we may write

$$\text{span}(\mathfrak{W}v) = \text{span}(\mathfrak{W}_1 v_1) \oplus \text{span}(\mathfrak{W}_2 0) = \mathfrak{h}_{G_1}^* \oplus 0 \cong \mathfrak{h}_{G_1}^*,$$

which implies that H has at least $\dim \mathfrak{h}_{G_1}^* = \text{rk}(G_1)$ linearly independent conjugates under $\mathfrak{W}$, say $H_1, \dots, H_{\text{rk}(G_1)}$. Write $H_i = w_i H$ for some $w_i = (w_{i1}, w_{i2}) \in \mathfrak{W}$ so that the normal vector of H_i is $w_i v$. Then

$$H_i = w_i H = (w_{i1}, w_{i2})(H' \oplus \mathfrak{h}_{G_2}^*) = w_{i1} H' \oplus w_{i2} \mathfrak{h}_{G_2}^* = w_{i1} H' \oplus \mathfrak{h}_{G_2}^*$$

and so $H_1 \cap \cdots \cap H_{\text{rk}(G_1)} = \bigcap_i w_{i1} H' \oplus \mathfrak{h}_{G_2}^*$. Since the $w_i v = (w_{i1} v_1, 0)$ are linearly independent in $\mathfrak{h}^*$, the $w_{i1} v_1$ are linearly independent in $\mathfrak{h}_{G_1}^*$. It follows that

$$\left(\bigcap_i w_{i1} H' \right)^\perp = \sum_i (w_{i1} H')^\perp = \sum_i w_{i1} (H')^\perp = \sum_i w_{i1} \text{span}\{v_1\}$$

contains at least $\text{rk}(G_1)$ linearly independent vectors and so $(\bigcap_i w_{i1} H')^\perp = \mathfrak{h}_{G_1}^*$. Thus $\bigcap_i w_{i1} H' = \{0\}$ so that $H_1 \cap \cdots \cap H_{\text{rk}(G_1)} = 0 \oplus \mathfrak{h}_{G_2}^*$. Applying an inclusion-exclusion argument similar to above, we get that $0 \oplus \mathfrak{h}_{G_2}^*$ contains at least $\dim V - \text{rk}(G_1)(\dim G + 1)$ weights counted with multiplicity. This means that

$$\dim(V_1)_0 \dim V_2 \geq \dim V - \text{rk}(G_1)(\dim G + 1),$$

as desired. □

4.6 Dimension results for SL_n

We consider two explicit functions in the variable n , and for each function we find all irreducible representations of SL_n whose dimension is bounded above by the function. The gathering of these lists of representations is for the proof of Theorem 4.8.1. Throughout,

let V be a non-trivial irreducible representation of SL_n . Recall from Section 2.5.2 that the highest weight vector of $(V, \mathfrak{sl}_n)$ takes the form

$$c_1 L_1 + c_2(L_1 + L_2) + \cdots c_{n-1}(L_1 + \cdots + L_{n-1})$$

for some $c_i \in \mathbb{N}$. Since V is uniquely determined by its highest weight, we write $V = \Gamma_{(c_1, \dots, c_{n-1})}$. We refer to the fundamental weights $\omega_i = L_1 + \cdots + L_i$ for $1 \leq i \leq n-1$ as rays since they are the rays of the Weyl chamber of $\mathfrak{sl}_n$. Setting $\mathbf{c} = (c_1, \dots, c_{n-1})$, as noted in [9, Section 15.3], the dimension of $V = \Gamma_{\mathbf{c}}$ is

$$d_{\mathbf{c}} := \dim \Gamma_{\mathbf{c}} = \prod_{1 \leq i < j \leq n} \frac{c_i + \cdots + c_{j-1} + j - i}{j - i}.$$

For $1 \leq i < j \leq n$, define $\mathbf{c}_{[i,j]} := (c_i + \cdots + c_{j-1} + j - i)/(j - i)$ so that

$$d_{\mathbf{c}} = \prod_{1 \leq i < j \leq n} \mathbf{c}_{[i,j]}.$$

This section contains results for when $\dim V$ is bounded above by one of the following functions:

$$f(n) := \frac{1}{4}(n^3 + 3n), \quad h(n) := \frac{n^2(n^2 + 3)}{4n - 2}.$$

Where these functions come from, and why we are interested in bounding $\dim V$ by them, will become evident in Section 4.8. This section is purely computational and is a means to make the following sections flow with ease.

Note that $f(n) < h(n) < n^3$ for all $n \geq 2$. By [8, Lemma 4.4, Proposition 4.5], if at least 3 of the c_i are non-zero, then $\dim V \geq n^3$. Given this, if $\dim V \leq f(n)$ or $\dim V \leq h(n)$, then at most 2 of the c_i are non-zero, which means that the highest weight of V consists of at most 2 rays. In this section, we produce a short list that contains all possible representations of SL_n whose dimension is bounded above by $f(n)$ or $h(n)$. In the case that the highest weight has 2 rays, the list obtained is in fact a complete classification of such representations.

Proposition 4.6.1 (K). *Let V be a non-trivial irreducible representation of SL_n whose highest weight vector is*

$$m_1 \sum_{i=1}^a L_i + m_2 \sum_{i=1}^b L_i$$

with $1 \leq a < b \leq n-1$ and $m_i \geq 1$. Then for all $n \geq 3$, $\dim V \leq h(n)$ if and only if V is the adjoint representation $\Gamma_{(1,0,\dots,0,1)}$ or $V = \Gamma_{(1,1,0)}$ or $V = \Gamma_{(0,1,1)}$.

Proof. Given the highest weight vector of V , if $V = \Gamma_{\mathbf{c}}$ with $\mathbf{c} = (c_1, \dots, c_{n-1})$ then $c_a = m_1, c_b = m_2$, and $c_i = 0$ for all $i \neq a, b$. The backwards direction follows by direct computation and the fact that $\dim \Gamma_{(1,0,\dots,0,1)} = n^2 - 1 \leq h(n)$ for all $n \geq 2$.

Let's first prove the forward direction for $n \in \{3, 4\}$. In these cases, we just need to show that $m_1 = m_2 = 1$. Note that $d_{(1,2)} > h(3)$ and $d_{(1,2,0)}, d_{(1,0,2)} > h(4)$. Since increasing any entry in $\mathbf{c}$ strictly increases $d_{\mathbf{c}}$ and $d_{\mathbf{c}}$ is invariant under reflection, it follows that $m_1 = m_2 = 1$.

Now assume $n \geq 5$. We first show that if $c_1 = 0$ or $c_{n-1} = 0$ then $d_{\mathbf{c}} > h(n)$. Since any entry in $\mathbf{c}$ strictly increases $d_{\mathbf{c}}$, it suffices to prove this when $m_1 = m_2 = 1$. We do induction on $n \geq 5$. The base case $n = 5$ follows by direct computations. For $n \geq 5$, define $\mathcal{T}_n$ to be the set of sequences in $\mathbb{N}^{n-1}$ with exactly 2 ones and all other entries 0. The inductive hypothesis tells us that if $\mathbf{b} = (b_1, \dots, b_{n-1}) \in \mathcal{T}_n$ with $b_1 = 0$ or $b_{n-1} = 0$ then $d_{\mathbf{b}} > h(n)$. Let $\mathbf{c} = (c_1, \dots, c_n) \in \mathcal{T}_{n+1}$ such that $c_1 = 0$ or $c_n = 0$. We have 2 cases to consider, namely, when $c_1 = c_n = 0$ or $c_1 = 0, c_n = 1$. In each case, we show that there is a sequence $\mathbf{c}' \in \mathcal{T}_n$ such that $d_{\mathbf{c}}/d_{\mathbf{c}'} > h(n+1)/h(n)$ and $d_{\mathbf{c}'} > h(n)$. This implies that $d_{\mathbf{c}} > h(n+1)$, which completes the induction.

Case I. $c_1 = c_n = 0$. In this case, consider $\mathbf{c}' := (c_1, \dots, c_{n-1}) \in \mathcal{T}_n$. Since $c_1 = 0$, by the inductive hypothesis, we have that $d_{\mathbf{c}'} > h(n)$. Note that $\mathbf{c}'_{[i,j]} = \mathbf{c}_{[i,j]}$ for $1 \leq i < j \leq n$, and so $d_{\mathbf{c}}/d_{\mathbf{c}'}$ is the product of $\mathbf{c}_{[i,n+1]}$ over $1 \leq i \leq n$. Then

$$\frac{d_{\mathbf{c}}}{d_{\mathbf{c}'}} \geq \mathbf{c}_{[1,n+1]} \mathbf{c}_{[2,n+1]} = \frac{(n+2)(n+1)}{n(n-1)} > \frac{h(n+1)}{h(n)}.$$

Case II. $c_1 = 0$ and $c_n = 1$. First consider when $c_{n-1} = 1$. Then $\mathbf{c}' := (c_2, \dots, c_n) = (0, \dots, 0, 1, 1) \in \mathcal{T}_n$ with $d_{\mathbf{c}'} > h(n)$. Note that $\mathbf{c}'_{[i,j]} = \mathbf{c}_{[i,j]}$ for $2 \leq i < j \leq n+1$. Thus $d_{\mathbf{c}}/d_{\mathbf{c}'}$ is the product of $\mathbf{c}_{[1,j]}$ over $2 \leq j \leq n+1$ and

$$\frac{d_{\mathbf{c}}}{d_{\mathbf{c}'}} \geq \mathbf{c}_{[1,n]} \mathbf{c}_{[1,n+1]} = \frac{n+2}{n-1} > \frac{h(n+1)}{h(n)}.$$

Next, consider the case when $c_{n-1} = 0$ so that $\mathbf{c} = (0, c_2, \dots, c_{n-2}, 0, 1)$. Let k be such that $2 \leq k \leq n-2$ and $c_k = 1$. Then $\mathbf{c}' := (c_1, \dots, c_{n-2}, c_n) = (0, c_2, \dots, c_{n-2}, 1) \in \mathcal{T}_n$ with $d_{\mathbf{c}'} > h(n)$. Note that $\mathbf{c}'_{[i,j]} = \mathbf{c}_{[i,j]}$ for $1 \leq i < j \leq n-1$. Using this, we compute that

$$\frac{d_{\mathbf{c}}}{d_{\mathbf{c}'}} = \frac{\prod_{1 \leq i \leq n-1} \mathbf{c}_{[i,n]} \prod_{1 \leq i \leq n} \mathbf{c}_{[i,n+1]}}{\prod_{1 \leq i \leq n-1} \mathbf{c}'_{[i,n]}} = \frac{(n+2)(n+1-k)}{(n-k)(n+2-k)} > \frac{h(n+1)}{h(n)}.$$

This completes the case work. So far, for $n \geq 5$, we have shown that if $d_{\mathbf{c}} \leq h(n)$ then $c_1 \neq 0$ and $c_{n-1} \neq 0$. This means that $a = 1$ and $b = n - 1$, and so $\mathbf{c} = (m_1, 0, \dots, 0, m_2)$. It remains to show that $m_1 = m_2 = 1$. It suffices to show that $d_{\mathbf{c}} > h(n)$ for $\mathbf{c} = (1, 0, \dots, 0, 2)$. In this case, we compute that

$$d_{\mathbf{c}} = \frac{n+2}{n-1} \prod_{2 \leq j \leq n-1} \frac{j}{j-1} \prod_{2 \leq i \leq n-1} \frac{2+n-i}{n-i} = \frac{1}{2} n(n-1)(n+2).$$

And we see that $d_{\mathbf{c}} > h(n)$ for $n \geq 5$, as desired. □

Lemma 4.6.2 (K). *Let $n \geq 4$ and fix an integer $b \geq 1$. For each $1 \leq k \leq n-1$, define $\mathbf{c}^k := (c_1, \dots, c_{n-1})$ where $c_k = b$ and $c_i = 0$ for $i \neq k$. Note that each $\mathbf{c}^k$ defines a representation of SL_n whose highest weight is the ray $b(L_1 + \dots + L_k)$. If $2 \leq k+1 \leq \lceil (n-1)/2 \rceil$, then $d_{\mathbf{c}^k} < d_{\mathbf{c}^{k+1}}$.*

Proof. Using the dimension formula for representations of SL_n , we have that

$$d_{\mathbf{c}^k} = \prod_{j=k+1}^n \prod_{i=1}^k \mathbf{c}_{[i,j]}^k = \left(\prod_{i=1}^k \mathbf{c}_{[i,k+1]}^k \right) \left(\prod_{j=k+2}^n \prod_{i=1}^k \mathbf{c}_{[i,j]}^k \right)$$

and

$$d_{\mathbf{c}^{k+1}} = \prod_{j=k+2}^n \prod_{i=1}^{k+1} \mathbf{c}_{[i,j]}^{k+1} = \left(\prod_{j=k+2}^n \mathbf{c}_{[k+1,j]}^{k+1} \right) \left(\prod_{j=k+2}^n \prod_{i=1}^k \mathbf{c}_{[i,j]}^{k+1} \right).$$

Note that $k+2 \leq n$ follows by our assumptions on n and k . Also note that $\mathbf{c}_{[i,j]}^k = \mathbf{c}_{[i,j]}^{k+1}$ whenever $1 \leq i \leq k$ and $k+2 \leq j \leq n$. So

$$\frac{d_{\mathbf{c}^{k+1}}}{d_{\mathbf{c}^k}} = \frac{\prod_{j=k+2}^n \mathbf{c}_{[k+1,j]}^{k+1}}{\prod_{i=1}^k \mathbf{c}_{[i,k+1]}^k}.$$

Since $2 \leq k+1 \leq \lceil (n-1)/2 \rceil$, we have that $2k+2 \leq n$ and so we may write

$$\prod_{j=k+2}^n \mathbf{c}_{[k+1,j]}^{k+1} = \prod_{j=2k+2}^n \mathbf{c}_{[k+1,j]}^{k+1} \prod_{j=k+2}^{2k+1} \mathbf{c}_{[k+1,j]}^{k+1}.$$

Doing a change of variables with $j = 2(k+1) - i$, we get that

$$\prod_{j=k+2}^{2k+1} \mathbf{c}_{[k+1,j]}^{k+1} = \prod_{i=1}^k \mathbf{c}_{[i,k+1]}^k$$

and so

$$\frac{d_{\mathbf{c}^{k+1}}}{d_{\mathbf{c}^k}} = \prod_{j=2k+2}^n \mathbf{c}_{[k+1,j]}^{k+1} > 1.$$

This product is strictly greater than one because $b > 0$ implies that $\mathbf{c}_{[k+1,j]}^{k+1} > 1$ for any j . $\square$

Proposition 4.6.3 (K). *Let V be a non-trivial irreducible representation of SL_n whose highest weight vector is*

$$m_1 \sum_{i=1}^a L_i + m_2 \sum_{i=1}^b L_i$$

with $1 \leq a < b \leq n-1$ and $m_i \geq 1$. Then for all $n \geq 3$, $\dim V \leq f(n)$ if and only if V is the adjoint representation.

Proof. Since $f(n) < h(n)$ for all $n \geq 3$, we may apply Proposition 4.6.1 to get that V is the adjoint representation or $\Gamma_{(1,1,0)}$ or $\Gamma_{(0,1,1)}$. However, $\dim \Gamma_{(1,1,0)} = \dim \Gamma_{(0,1,1)} = 20 > 19 = f(4)$ and so V is the adjoint representation. $\square$

Proposition 4.6.4 (K). *Let V be a non-trivial irreducible representation of SL_n whose highest weight vector is*

$$b \sum_{i=1}^a L_i$$

with $1 \leq a \leq n-1$ and $b \geq 1$. If $\dim V \leq h(n)$ then, after possible replacing V with V^* , we have that $a \leq 6$ and $b \in \{1, 2, 3\}$. For $n \geq 5$, if $\dim V \leq h(n)$ and $a \notin \{1, n-1\}$, then $b = 1$.

Proof. It was shown in [8, Lemma 4.2] that if the highest weight of V has one ray and $\dim V \leq n^3$, then $a \leq 6$. So, in our case, we have that $a \leq 6$. Next, we show that $b \leq 3$. Let $\mathbf{c} = (c_1, \dots, c_{n-1})$ be the highest weight of V where $c_k = b$ for some $1 \leq k \leq n-1$ and $c_i = 0$ for $i \neq k$. Since increasing any entry of $\mathbf{c}$ increases $d_{\mathbf{c}}$, it suffices to show that $d_{\mathbf{c}} > h(n)$ for all $n \geq 2$ when $b = 4$. We compute that

$$d_{(4,0,\dots,0)} = \frac{1}{24}n(n+1)(n+2)(n+3) > h(n)$$

for all $n \geq 2$. When $n \in \{2, 3\}$, we have that $\mathbf{c} \in \{(4), (4, 0), (0, 4)\}$ and so $d_{(4,0,\dots,0)} = d_{\mathbf{c}}$. For $n \geq 4$, $d_{\mathbf{c}} \geq d_{(4,0,\dots,0)}$ follows directly from Lemma 4.6.2 and the fact that $d_{\mathbf{c}}$ is invariant under reflection.

For the second statement, assume $n \geq 5$ and $a \notin \{1, n-1\}$. Then the highest weight of V takes the form $\mathbf{c} = (0, c_2, \dots, c_{n-2}, 0)$ where $c_k = b$ for some $2 \leq k \leq n-2$ and $c_i = 0$ for $i \neq k$. Similar to above, it is enough to show that $d_{\mathbf{c}} > h(n)$ when $\mathbf{c} = (0, 2, 0, \dots, 0)$. And indeed, we see that

$$d_{\mathbf{c}} = \prod_{j=3}^n \mathbf{c}_{[1,j]} \mathbf{c}_{[2,j]} = \prod_{j=3}^n \frac{j(j+1)}{(j-1)(j-2)} = \frac{1}{12}(n-1)n^2(n+1) > h(n)$$

for $n \geq 5$. □

4.7 Hyperplane results for SL_n

Throughout this section, we use the same notation as in Remark 4.4.21. In particular, the hyperplanes in $\mathfrak{h}^*$ that arise from non-trivial $\lambda \in N(T_n)$ are denoted by H_λ and $|H_\lambda|$ denotes the number of weights in H_λ , counting multiplicity. Since SL_n is a simply connected reductive Lie group, we use our knowledge of representations of $\mathfrak{sl}_n$ when considering hyperplanes $H_\lambda \subseteq \mathfrak{h}^*$ as in Example 4.4.22. In particular, for specific representations of SL_n , we compute an upper bound on $|H_\lambda|$, where from Example 4.4.22 we know that

$$H_\lambda = \{b_1 L_1 + \dots + b_n L_n \in \mathfrak{h}^* : \sum a_i b_i = 0\}$$

for $H_\lambda = (a_1, \dots, a_n)^\perp$. Since λ is non-trivial and defined by $(a_1, \dots, a_n)$, the $a_i \in \mathbb{Z}$ are not all zero and $\sum a_i = 0$. Again, we note that this section is purely computational and is a means to make the following sections flow with ease. The computations in this section are used in the proof of Theorem 4.8.1, specifically in Section 4.9.2 where Proposition 4.4.20 is utilized.

Remark 4.7.1. Say H_λ has normal vector $v = \sum a_i L_i$ so that $H_\lambda = (a_1, \dots, a_n)^\perp$. A permutation of the a_i in $\sum a_i L_i$, which results in a new hyperplane $H_{\lambda'}$ for some $\lambda' \in N(T)$, is equivalent to the action of some Weyl group $\mathfrak{W} = S_n$ element on $\sum a_i L_i$. The resulting hyperplane $wH_\lambda = H_{\lambda'}$ for $w \in \mathfrak{W}$ has normal vector wv and is a $\mathfrak{W}$ -conjugate of H_λ that satisfies $|H_\lambda| = |wH_\lambda|$. Thus, when utilizing Proposition 4.4.20, it is enough to consider the hyperplanes $(a_1, \dots, a_n)^\perp$, up to permutation.

Lemma 4.7.2 (K). *If $n \geq 30$ and λ is any non-trivial 1-parameter subgroup of T_n , then the number of $\bigwedge^3 \mathrm{SL}_n$ weights in H_λ satisfies*

$$|H_\lambda| < \dim \bigwedge^3 \mathrm{SL}_n - n^2 = \binom{n}{3} - n^2$$

as long as $H_\lambda \neq (1, 0, \dots, 0, -1)^\perp$, up to permutation.

Proof. First consider the case when $H_\lambda = (1, 0, \dots, 0, -1)^\perp$. Recall that the weights of $\bigwedge^3 \mathrm{SL}_n$ all have multiplicity 1 and are of the form $L_i + L_j + L_k$ with $1 \leq i, j, k \leq n$ and i, j, k all distinct. Then $L_i + L_j + L_k \in H_\lambda$ means that either $2 \leq i, j, k \leq n-1$ or $i = 1, j = n$. It follows that

$$|H_\lambda| = \binom{n-2}{3} + (n-2) > \binom{n}{3} - n^2$$

for all $n \geq 2$, as desired.

Now, let λ be any 1-parameter subgroup of T_n such that, up to permutation, $H_\lambda \neq (1, 0, \dots, 0, -1)^\perp$. We may write $H_\lambda = \{v\}^\perp$, for some non-zero $v = \sum a_i L_i$ with $a_i \in \mathbb{Z}$ and $\sum a_i = 0$. Let z be the number of a_i that are zero and p be the number of a_i that are positive, so that $n - z - p$ is the number of a_i that are negative. Since $H_\lambda \neq (1, 0, \dots, 0, -1)^\perp$, we have that $0 \leq z \leq n-3$. Also note that $1 \leq p \leq n-1$. If $L_i + L_j + L_k \in H_\lambda$, then (a_i, a_j, a_k) falls into one of the four cases below. Note here that we can assume that $a_i \leq a_j \leq a_k$.

- (1) $a_i = a_j = a_k = 0$
- (2) $a_i < 0, a_j = 0, a_k > 0$
- (3) $a_i < 0, a_j < 0, a_k > 0$
- (4) $a_i < 0, a_j > 0, a_k > 0$

Counting the number of possible weights $L_i + L_j + L_k$ in each case, gives

$$|H_\lambda| \leq \binom{z}{3} + pz(n - z - p) + p \binom{n - z - p}{2} + (n - z - p) \binom{p}{2}.$$

Set the expression on the right to be $f_n(p, z)$. For fixed $n \geq 63$, we compute the absolute maximum of $f_n(p, z)$ over $\mathbb{R}^2$ on the rectangle $1 \leq p \leq n-1, 0 \leq z \leq n-3$ to be

$$f_n(3/2, n-3) = \frac{1}{24}(4n^3 - 48n^2 + 242n - 375).$$

So, for $n \geq 63$, we have

$$|H_\lambda| \leq f_n(3/2, n-2) < \binom{n}{3} - n^2.$$

For each $30 \leq n \leq 62$, one can verify that

$$|H_\lambda| \leq \max_{(p,z) \in \mathbb{Z}^2} f_n(p, z) < \binom{n}{3} - n^2.$$

□

Lemma 4.7.3 (K). *If $n \geq 18$ and λ is any non-trivial 1-parameter subgroup of T_n , then the number of $\text{Sym}^3 \text{SL}_n$ weights in H_λ satisfies*

$$|H_\lambda| < \dim \text{Sym}^3 \text{SL}_n - n^2 = \binom{n+2}{3} - n^2$$

as long as $H_\lambda \neq (1, 0, \dots, 0, -1)^\perp$, up to permutation.

Proof. First consider the case when $H_\lambda = (1, 0, \dots, 0, -1)^\perp$. Recall that the weights of $\text{Sym}^3 \text{SL}_n$ all have multiplicity 1 and are of the form $L_i + L_j + L_k$ with $1 \leq i, j, k \leq n$ not necessarily distinct. The number of $3L_i \in H_\lambda$ is $n - 2$, the number of $2L_i + L_j \in H_\lambda$ with $i \neq j$ is $(n - 2)(n - 3)$, and the number of $L_i + L_j + L_k \in H_\lambda$ with i, j, k distinct is $\binom{n-2}{3} + (n - 2)$. It follows that

$$|H_\lambda| = 2(n - 2) + (n - 2)(n - 3) + \binom{n - 2}{3} > \binom{n + 2}{3} - n^2$$

for all $n \geq 3$, as desired.

Now, let λ be any 1-parameter subgroup of T_n , such that, up to permutation, $H_\lambda \neq (1, 0, \dots, 0, -1)^\perp$. Then $H_\lambda = \{v\}^\perp$ for some non-zero $v = \sum a_i L_i$ with $a_i \in \mathbb{Z}$ and $\sum a_i = 0$. Let z be the number of a_i that are zero and p be the number of a_i that are positive, so that $n - p - z$ is the number of a_i that are negative. Note that $0 \leq z \leq n - 3$ and $1 \leq p \leq n - 1$. If $L_i + L_j + L_k \in H_\lambda$, then (a_i, a_j, a_k) falls into one of the four cases below. Note here that we can assume that $a_i \leq a_j \leq a_k$.

- (1) $a_i = a_j = a_k = 0$
- (2) $a_i < 0, a_j = 0, a_k > 0$
- (3) $a_i < 0, a_j < 0, a_k > 0$
- (4) $a_i < 0, a_j > 0, a_k > 0$

Counting the number of possible weights $L_i + L_j + L_k$ in each case, gives

$$|H_\lambda| \leq \binom{z+2}{3} + zp(n-z-p) + p \binom{n-z-p+1}{2} + (n-p-z) \binom{p+1}{2}.$$

Set the expression on the right to be $f_n(z, p)$. Note that all of the cases, with the exception of case (2), involve counting with repetition.

For fixed $n \geq 27$, we compute the absolute maximum of $f_n(z, p)$ over $\mathbb{R}^2$ on the rectangle $1 \leq p \leq n-1, 0 \leq z \leq n-3$ to be

$$f_n(n-3, (3/2-3n)/(1-2n)) = \frac{1}{24}(4n^3 - 24n^2 + 98n - 51).$$

So, for $n \geq 27$, we have

$$|H_\lambda| \leq f_n(n-3, (3/2-3n)/(1-2n)) < \binom{n+2}{3} - n^2.$$

For each $18 \leq n \leq 26$, one can verify that

$$|H_\lambda| \leq \max_{(z,p) \in \mathbb{Z}^2} f_n(p, z) < \binom{n+2}{3} - n^2.$$

□

4.8 Main case

In this section, we study irreducible representations $V \otimes W$ of $\mathrm{SL}_n \times \mathrm{SL}_m$ with $\dim V \geq n^2$ and $\dim W \geq m^2$. The goal of this section is to prove Theorem 4.8.1, which says that such representations are not pure. Throughout, we assume that $n \leq m$. Using the same notation as in Section 2.5.1, we recall that the 0-weight space of V and W is denoted by V_0 and W_0 , respectively.

Theorem 4.8.1 (K). *Let $V \otimes W$ be an irreducible representation of $\mathrm{SL}_n \times \mathrm{SL}_m$. If $\dim V \geq n^2$ and $\dim W \geq m^2$, then $V \otimes W$ is not pure. Moreover, Conjecture 3.5.1 holds when $\dim V \geq n^2$ and $\dim W \geq m^2$.*

Proof. The remainder of Section 4.8 is devoted to the proof of the first statement, which boils down to showing that all of the representations in Table 4.5 are not pure. For the

second statement, note that, up to infinite families, there are finitely many cofree irreducible representations of $\mathrm{SL}_n \times \mathrm{SL}_m$, all of which are listed in Table 3.1. Section 3.6 goes into detail of how this list was obtained from [13]. Looking at Table 3.1, we see that none of the representations satisfy $\dim V \geq n^2$ and $\dim W \geq m^2$. So, Conjecture 3.5.1 holds vacuously. $\square$

Sections 4.8.1-4.8.4 are devoted to the proof of the first statement of Theorem 4.8.1. The general strategy to prove Theorem 4.8.1 goes as follows. We first utilize Proposition 4.5.4 to obtain a finite list that contains all possible pure irreducible representations of $\mathrm{SL}_n \times \mathrm{SL}_m$ satisfying $\dim V \geq n^2$ and $\dim W \geq m^2$; meaning, if V is pure, then it will be in this list. Then, using techniques from Section 4.9, we show that every representation in this list is not pure. This list is represented by Table 4.5.

As we are classifying pure representations, for the computations in this section we assume that $V \otimes W$ is pure. Then, by Proposition 4.4.20, we can find a hyperplane $H \subseteq \mathfrak{h}^*$ through 0 such that

$$|H| \geq \dim(V \otimes W) - (\dim(\mathrm{SL}_n \times \mathrm{SL}_m) + 1) = \dim V \dim W - (n^2 + m^2 - 1). \quad (4.2)$$

We may write $H = \{x\}^\perp$ for some $x = (v, w) \in \mathfrak{h}^*$, with both v and w not zero. In order to obtain the finite list mentioned above, we consider the three following cases for v : (i) $v, w \neq 0$ (ii) $w = 0$ (iii) $v = 0$. We obtain a finite list in each case, and then combine the lists into one final list that appears in Section 4.8.4; see Table 4.5.

4.8.1 Hyperplane case: $x = (v, w)$ with $v, w \neq 0$

By Theorem 4.5.4, we have that

$$\dim V \dim W \leq (n + m - 2)(n^2 + m^2 - 1) + \dim V_0 \dim W_0.$$

We consider two sub cases, namely, when $\dim V_0 \dim W_0$ is zero or not.

Subcase I. Suppose $\dim V_0 \dim W_0 = 0$. Then, using that $\dim V \geq n^2$,

$$\dim W \leq \frac{1}{\dim V} (n + m - 2)(n^2 + m^2 - 1) \leq \frac{1}{n^2} (n + m - 2)(n^2 + m^2 - 1).$$

This upper most bound is a decreasing function in n when $m \geq 3$ and $2 \leq n \leq m$, in which case we have that

$$\dim W \leq \frac{1}{4}(m^3 + 3m) = f(m).$$

We take care of the case $m = 2$ right now. Since $\dim W \geq m^2$, we have that $m^2 \leq f(m)$. But this inequality is not satisfied when $m = 2$. This contradiction shows that these $\mathrm{SL}_2 \times \mathrm{SL}_2$ representations are not pure.

Going forward in this case, assume $m \geq 3$. As explained in the beginning of Section 4.6, since $\dim W \leq f(m)$, the highest weight of W has either one or two rays. Since $\dim W \geq m^2$, we know that W cannot be the adjoint representation and so Proposition 4.6.3 tells us that the highest weight of W has exactly one ray. Recalling that $f(m) < h(n)$, we may then apply Proposition 4.6.4 to get constraints on the highest weight of W . Specifically, after possibly replacing W with W^* , the highest weight of W takes the form

$$b \sum_{i=1}^a L_i$$

with $1 \leq a \leq 6$ and $b \in \{1, 2, 3\}$. Furthermore, if $m \geq 5$ and $a \notin \{1, m-1\}$, then $b = 1$. Using these highest weight constraints, we obtain Table 4.1, which classifies irreducible representations W of SL_m satisfying $m^2 \leq \dim W \leq f(m)$, up to W^* .

Highest weight	Constraints on m
$3\omega_1$	$m \geq 5$
ω_3	$m \geq 9$
ω_4	$8 \leq m \leq 11$
ω_5	$m \in \{9, 10\}$
ω_6	$m \in \{9, 10\}$

Table 4.1: All irreducible representations W of SL_m satisfying $m^2 \leq \dim W \leq f(m)$, up to W^* .

Next, for each W in Table 4.1, we find all irreducible representation V of SL_n such that the paring $V \otimes W$ satisfies the bound

$$n^2 m^2 \leq \dim V \dim W \leq (n + m - 2)(n^2 + m^2 - 1) \quad (4.3)$$

and constraints $2 \leq n \leq m$ and $\dim V_0 \dim W_0 = 0$. Note that all of the representations in Table 4.1 satisfy $\dim W_0 = 0$, and so we do not need to worry about if $\dim V_0 = 0$ or not. In Example 4.8.3, we sketch a way to find V when W has highest weight $3\omega_1$. Finding V for the other W can be computed in a similar manner.

Remark 4.8.2. The factors V and W are obtained strictly via dimension means, and so we need to remember to consider V^* and W^* when forming our list of possible pure representations. That is, if $V \otimes W$ appears in the list, then $V^* \otimes W$, $V^* \otimes W^*$, and $V \otimes W^*$ should also appear in the list. Further, if one of the factors V is a representation of SL_2 , then V is self-dual, in which case we do not need to add the representations with factors V^* to the list; that is, we only need $V \otimes W$ and $V \otimes W^*$.

Example 4.8.3. Take $W = \mathrm{Sym}^3 \mathrm{SL}_m$ with $m \geq 5$. Using our bound from Theorem 4.5.4, we have that

$$n^2 \leq \dim V \leq \frac{6(n+m-2)(n^2+m^2-1)}{m(m+1)(m+2)}.$$

Using $n \leq m$, one can show that $n = 2$ if and only if

$$n^2 \leq \frac{6(n+m-2)(n^2+m^2-1)}{m(m+1)(m+2)}.$$

So we know that $n = 2$, and hence $V = \mathrm{Sym}^k \mathrm{SL}_2$ for some $k \in \mathbb{N}$. Substituting $n = 2$ and $\dim V = k + 1$ into the above bound gives

$$3 \leq k \leq \frac{6(m^2+3)}{(m+1)(m+2)} - 1.$$

Note that the right hand side of this inequality is bounded strictly above by 5, and so $k \in \{3, 4\}$. When $k = 3$ this inequality is satisfied if and only if $m \geq 5$. When $k = 4$ this inequality is satisfied if and only if $m \geq 15$. So we add $\mathrm{Sym}^3 \mathrm{SL}_2 \otimes \mathrm{Sym}^3 \mathrm{SL}_m$ with $m \geq 5$ and $\mathrm{Sym}^4 \mathrm{SL}_2 \otimes \mathrm{Sym}^3 \mathrm{SL}_m$ with $m \geq 15$ to our list. Regarding Remark 4.8.2, since the V here are representations of SL_2 they are self dual, and so we just need to represent $V \otimes W$ and $V \otimes W^*$ in the final list for each V . $\diamond$

Carrying out similar calculations as in Example 4.8.3 for the remaining W in Table 4.1, we get Table 4.2, which classifies irreducible representations satisfying Equation 4.3 and the constraints $2 \leq n \leq m$ and $\dim V_0 \dim W_0 = 0$. For simplicity, Table 4.2 is up to W^* .

Subcase II. Suppose $\dim V_0 \dim W_0 \neq 0$. This means that zero is a weight of both V and W . In this case,

$$\dim W \leq \frac{1}{\dim V}(n+m-2)(n^2+m^2-1) + \frac{\dim V_0}{\dim V} \dim W_0.$$

By [8, Lemma 3.7], we know that $\dim V_0 \leq \dim V/n$ and $\dim W_0 \leq \dim W/m$. This, together with $\dim V \geq n^2$, gives

$$\dim W \leq \frac{1}{n^2}(n+m-2)(n^2+m^2-1) + \frac{\dim W}{nm}.$$

Highest weight	Constraints on n	Constraints on m
$(3\omega_1, 3\omega_1)$	$n = 2$	$m \geq 5$
$(4\omega_1, 3\omega_1)$	$n = 2$	$m \geq 15$
$(3\omega_1, \omega_3)$	$n = 2$	$m \geq 9$
$(4\omega_1, \omega_3)$	$n = 2$	$m \geq 9$
$(5\omega_1, \omega_3)$	$n = 2$	$m \geq 9$
$(6\omega_1, \omega_3)$	$n = 2$	$9 \leq m \leq 21$
$(7\omega_1, \omega_3)$	$n = 2$	$9 \leq m \leq 12$
$(8\omega_1, \omega_3)$	$n = 2$	$m = 9$
$(3\omega_1, \omega_3)$	$n = 3$	$m = 9$
$(3\omega_2, \omega_3)$	$n = 3$	$m = 9$
$(3\omega_1, \omega_4)$	$n = 2$	$8 \leq m \leq 11$
$(4\omega_1, \omega_4)$	$n = 2$	$m \in \{8, 9\}$
$(5\omega_1, \omega_4)$	$n = 2$	$m \in \{8, 9\}$
$(6\omega_1, \omega_4)$	$n = 2$	$m = 8$
$(3\omega_1, \omega_5)$	$n = 2$	$m = 10$

Table 4.2: All irreducible representations $V \otimes W$ of $\mathrm{SL}_n \times \mathrm{SL}_m$ satisfying Equation 4.3 and $2 \leq n \leq m$ and $\dim V_0 \dim W_0 = 0$, up to W^* .

Solving for $\dim W$, we get the bound

$$\dim W \leq \frac{m(n+m-2)(n^2+m^2-1)}{n^2m-n}.$$

Note that this upper bound is a decreasing function in n for all $2 \leq n \leq m$. And so

$$\dim W \leq \frac{m^2(m^2+3)}{4m-2} = h(m)$$

for $m \geq 2$. Again, as noted in Section 4.6, since $\dim W \leq h(m)$ the highest weight of W has at most 2 rays. Since $\dim W \geq m^2$, W is not the adjoint representation. If the highest weight of W has two rays, then Proposition 4.6.1 tells us that, up to W^* , $W = \Gamma_{(1,1,0)}$. But zero is not a weight of $\Gamma_{(1,1,0)}$, and so the highest weight of W has exactly one ray. Thus, we can apply Proposition 4.6.4 to get that the highest weight of W takes the form

$$b \sum_{i=1}^a L_i$$

with $1 \leq a \leq 6$ and $b \in \{1, 2, 3\}$. Further, if $m \geq 5$ and $a \notin \{1, m-1\}$, then $b = 1$. Using these highest weight constraints and the conditions on W , one finds that $W \in \{\Gamma_{(3,0)}, \Gamma_{(0,3)}, \Gamma_{(0,2,0)}\}$. Theorem 4.5.4 and $\dim V_0 \leq \dim V/n$ tell us that

$$n^2 \leq \dim V \leq \frac{n(n+m-2)(n^2+m^2-1)}{n \dim W - \dim W_0}.$$

For each representation $W \in \{\Gamma_{(3,0)}, \Gamma_{(0,3)}, \Gamma_{(0,2,0)}\}$, we see that there are no representations V of SL_n that satisfy this inequality, $\dim V_0 \neq 0$, and $n \leq m$. For example, if we take $W = \Gamma_{(3,0)}$, then $\dim W = 10$ and $\dim W_0 = 1$. Since $n \leq 3$, we consider when $n = 2$ and $n = 3$. If $n = 2$, then

$$\frac{n(n+m-2)(n^2+m^2-1)}{n \dim W - \dim W_0} = \frac{72}{19} \approx 3.789 < 4$$

and so the inequality is not satisfied. If $n = 3$, then

$$\frac{n(n+m-2)(n^2+m^2-1)}{n \dim W - \dim W_0} = \frac{144}{29} \approx 4.965 < 9$$

and so the inequality is not satisfied. So, no such V exist when $W = \Gamma_{(3,0)}$. Doing similar computations for the other W , we conclude that there are no pure representations $V \otimes W$ in this case.

4.8.2 Hyperplane case: $x = (v, 0)$ with $v \neq 0$

We show that this case cannot happen. Meaning, if $V \otimes W$ is pure with $\dim V \geq n^2$ and $\dim W \geq m^2$, then the hyperplane H from Equation 4.2 is not of the form $H = \{(v, 0)\}^\perp$; see Proposition 4.8.5. Assume that $V \otimes W$ is pure and that the normal vector of H takes the form $x = (v, 0)$. Then $H = H' \oplus \mathfrak{h}_m^*$, where $H' := \{v\}^\perp \subseteq \mathfrak{h}_n^*$ is a hyperplane through 0. Using the bound in Equation 4.2 on $|H|$ and $\dim W \geq m^2$, we compute that

$$|H'| = \frac{|H|}{\dim W} \geq \frac{\dim V \dim W - (\dim G + 1)}{\dim W} \geq \dim V - \frac{n^2 + m^2 - 1}{m^2}.$$

Since $n \leq m$, we have that $(n^2 + m^2 - 1)/m^2 < 2$ and so

$$|H'| \geq \dim V - 1.$$

The constraint $|H'| \geq \dim V - 1$ says that the hyperplane H' contains all of the weights of V or all but one. The equality $|H'| = \dim V - 1$ is only possible when V is the trivial

representation. For V not the trivial representation, the inequality $|H'| \geq \dim V - 1$ is impossible due to the symmetry of the weight diagram; see Lemma 4.8.4. We conclude that the normal vector of H cannot take the form $x = (v, 0)$ if $V \otimes W$ is pure with no trivial factors, thus proving Proposition 4.8.5.

Lemma 4.8.4 (K). *Let V be an irreducible representation of SL_n . If there exists a hyperplane $H \subseteq \mathfrak{h}^*$ through 0 such that $|H| \geq \dim V - 1$, then V is the trivial representation.*

Proof. Assume $|H| \geq \dim V - 1$ to be true. Note that V cannot be the standard representation since otherwise H would contain $n - 1$ linearly independent weights, contradicting that $\dim H = n - 2$. Using $|H| \geq \dim V - 1$ in an argument similar to as in Theorem 4.5.4, we get

$$\dim V \leq \mathrm{rk}(\mathrm{SL}_n) + \dim V_0 = n - 1 + \dim V_0.$$

Applying [8, Lemma 3.7] to V , we obtain the bound $\dim V_0 \leq \dim V/n$ and so

$$\dim V \leq n - 1 + \frac{\dim V}{n}$$

from which it follows that $\dim V \leq n$. Since V is not the standard representations, we get that $\dim V < n$. The only representation that satisfies $\dim V < n$ is the trivial representation. This shows that $|H| \geq \dim V - 1$ is only possible if V is trivial, in which case we actually have that $|H| = \dim V$. $\square$

Proposition 4.8.5 (K). *Let $V \otimes W$ be an irreducible representation of $\mathrm{SL}_n \times \mathrm{SL}_m$ with $\dim V \geq n^2$ and $\dim W \geq m^2$. There does not exist a hyperplane $H \subseteq \mathfrak{h}^*$ through 0 of the form $H = H' \oplus \mathfrak{h}_m^*$, where $H' \subseteq \mathfrak{h}_n^*$ is a hyperplane through 0, such that*

$$|H| \geq \dim(V \otimes W) - (\dim(\mathrm{SL}_n \times \mathrm{SL}_m) + 1).$$

4.8.3 Hyperplane case: $x = (0, w)$ with $w \neq 0$

By Theorem 4.5.4, we have that

$$\dim V \dim W \leq (m - 1)(n^2 + m^2 - 1) + \dim V \dim W_0.$$

Using $\dim V \geq n^2$ and $\dim W_0 \leq \dim W/m$ gives

$$\dim W \leq \frac{m(n^2 + m^2 - 1)}{n^2}.$$

This upper bound is a decreasing function in n for all $2 \leq n \leq m$. And so,

$$\dim W \leq \frac{1}{4}(m^3 + 3m) = f(m)$$

for all $m \geq 2$. Representations satisfying $m^2 \leq \dim W \leq f(m)$ are already classified in Table 4.1. Since the representations in this table all satisfy $\dim W_0 = 0$, we can slightly refine this list by using

$$\dim V \dim W \leq (m-1)(n^2 + m^2 - 1).$$

In particular, using that $\dim V \geq n^2$, we get

$$\dim W \leq \frac{(m-1)(n^2 + m^2 - 1)}{n^2}.$$

This upper bound is a decreasing function in n for all $2 \leq n \leq m$, and so

$$\dim W \leq \frac{1}{4}(m-1)(m^2 + 3) =: g(m)$$

for all $m \geq 2$. The representations from Table 4.1 that satisfy this stricter bound $g(m)$ are listed in Table 4.3.

Highest weight	Constraints on m
$3\omega_1$	$m \geq 9$
ω_3	$m \geq 9$
ω_4	$8 \leq m \leq 10$
ω_5	$m = 9$
ω_6	$m \in \{9, 10\}$

Table 4.3: All irreducible representations W of SL_m satisfying $m^2 \leq \dim W \leq g(m)$, up to W^* .

Similar to as in Section 4.8.1, for each W in Table 4.3, we find all representations V of SL_n so that $V \otimes W$ satisfies

$$n^2 m^2 \leq \dim V \dim W \leq (m-1)(n^2 + m^2 - 1) \tag{4.4}$$

and $2 \leq n \leq m$. These representations are classified in Table 4.4, up to W^* .

Highest weight	Constraints on n	Constraints on m
$(3\omega_1, 3\omega_1)$	$n = 2$	$m \geq 9$
$(4\omega_1, 3\omega_1)$	$n = 2$	$m \geq 21$
$(3\omega_1, \omega_3)$	$n = 2$	$m \geq 9$
$(4\omega_1, \omega_3)$	$n = 2$	$m \geq 9$
$(5\omega_1, \omega_3)$	$n = 2$	$m \geq 9$
$(6\omega_1, \omega_3)$	$n = 2$	$9 \leq m \leq 15$
$(7\omega_1, \omega_3)$	$n = 2$	$m = 9$
$(3\omega_1, \omega_4)$	$n = 2$	$8 \leq m \leq 10$
$(4\omega_1, \omega_4)$	$n = 2$	$m \in \{8, 9\}$
$(5\omega_1, \omega_4)$	$n = 2$	$m = 8$

Table 4.4: All irreducible representations $V \otimes W$ of $\mathrm{SL}_n \times \mathrm{SL}_m$ satisfying Equation 4.4 and $2 \leq n \leq m$, up to W^* .

4.8.4 Non-purity of final list

Sections 4.8.1, 4.8.2, and 4.8.3 show that if $V \otimes W$ is a pure irreducible representation of $\mathrm{SL}_n \times \mathrm{SL}_m$ that satisfies $\dim V \geq n^2$ and $\dim W \geq m^2$, then $V \otimes W$ lies in Table 4.2 or Table 4.4. Note that Table 4.4 is completely contained in Table 4.2. Given this, to prove Theorem 4.8.1, it suffices to show that every representation in Table 4.2 is not pure. In other words, all possible pure representations are reflected in Table 4.2. Recall that the representations $V \otimes W$ listed in Table 4.2 are up to W^* , and so we need to also show that all $V \otimes W^*$ are not pure; please see Remark 4.8.6 for why the $V \otimes W^*$ are not pure.

Table 4.5 below is a copy of Table 4.2 with an additional column labelled “Method” that indicates which method was used to show that the representation in question is not pure. There are two main techniques used to show that a representation is not pure. If a specific lemma is listed under “Method”, then we used what we call the “counting weights on hyperplanes method”. Essentially, this method shows that for all hyperplanes in $\mathfrak{h}^*$ through 0, there are not “enough” weights in it; here “enough” is in reference to Equation 4.2. This method is explained in detail in Section 4.9.2. The second method is called the “toral method” and is explained in detail in Section 4.9.1. This method fixes a maximal torus and considers all elements in the torus of a specific order. If order r is used, it will be indicated under “Method” by writing “Toral order r ”. For example, “Toral order 3” means that the toral method was used with elements of order 3.

Highest weight	Constraints on n	Constraints on m	Method
$(3\omega_1, 3\omega_1)$	$n = 2$	$m \geq 5$	Lemma 4.9.8
$(4\omega_1, 3\omega_1)$	$n = 2$	$m \geq 15$	Lemma 4.9.9
$(3\omega_1, \omega_3)$	$n = 2$	$m \geq 9$	Lemma 4.9.7
$(4\omega_1, \omega_3)$	$n = 2$	$m \geq 9$	Lemma 4.9.7
$(5\omega_1, \omega_3)$	$n = 2$	$m \geq 9$	Lemma 4.9.7
$(6\omega_1, \omega_3)$	$n = 2$	$9 \leq m \leq 21$	Toral order 4
$(7\omega_1, \omega_3)$	$n = 2$	$9 \leq m \leq 12$	Toral order 4
$(8\omega_1, \omega_3)$	$n = 2$	$m = 9$	Toral order 4
$(3\omega_1, \omega_3)$	$n = 3$	$m = 9$	Toral order 2
$(3\omega_2, \omega_3)$	$n = 3$	$m = 9$	Toral order 2*
$(3\omega_1, \omega_4)$	$n = 2$	$8 \leq m \leq 11$	Toral order 3
$(4\omega_1, \omega_4)$	$n = 2$	$m \in \{8, 9\}$	Toral order 3
$(5\omega_1, \omega_4)$	$n = 2$	$m \in \{8, 9\}$	Toral order 3
$(6\omega_1, \omega_4)$	$n = 2$	$m = 8$	Toral order 3
$(3\omega_1, \omega_5)$	$n = 2$	$m = 10$	Toral order 3

Table 4.5: Irreducible representations $V \otimes W$ of $\mathrm{SL}_n \times \mathrm{SL}_m$ that satisfy $\dim V \geq n^2$ and $\dim W \geq m^2$, up to W^* , that are verified to not be pure to prove Theorem 4.8.1.

Remark 4.8.6. As noted in Remark 4.9.2 and Remark 4.9.6, if a representation V of G is shown to not be pure via the two methods in Section 4.9, then V^* is not pure. So, in either method, it is enough to show that either V or V^* is not pure. Sometimes it is easier to work with V^* to show that V is not pure. If we use V^* to show V is not pure, we add the symbol $*$ under “Method”. For example, for the representation $(3\omega_2, \omega_3)$ in Table 4.5, it is easier to show that its dual $(3\omega_1, \omega_6)$ is not pure; this is indicated in Table 4.5 by “Toral order 2*”.

Note that Table 4.5 only contains representations of $\mathrm{SL}_2 \times \mathrm{SL}_m$ and $\mathrm{SL}_3 \times \mathrm{SL}_m$. For the $\mathrm{SL}_2 \times \mathrm{SL}_m$ representations $V \otimes W$, we have that $(V \otimes W)^* \cong V \otimes W^*$; so, showing $V \otimes W$ is not pure via these methods shows that $V \otimes W^*$ is also not pure. For the $\mathrm{SL}_3 \times \mathrm{SL}_m$ representations $V \otimes W$ in Table 4.5, note that the dual $(V \otimes W^*)^*$ appears in Table 4.5, and so $V \otimes W^*$ is also not pure.

4.9 Methods to show a representation is not pure

We introduce two methods that be used to show that a given representation is not pure. As noted in Section 4.8.4, these two methods are called the “toral method” and the “counting weights on hyperplanes method”.

4.9.1 Toral method

Throughout, let V be a representation of a semisimple Lie group G and T a maximal torus of G . In this section, we describe what we call the toral method, which is a method that can be used to show that a representation is not pure. This method involves the computation of $\dim V^t$, where

$$V^t := \{v \in V : t \cdot v = v\},$$

for a specific collection of toral elements $t \in T$. Proposition 4.9.1 below, which outlines the toral method, is a restatement of [8, Lemma 3.4].

Proposition 4.9.1. [8, Lemma 3.4] *Let T be a maximal torus of a semisimple Lie group G and V a representation of G . Fix an integer $r \geq 2$. If*

$$\dim V^t \leq \dim V - \dim G - 2$$

for all $t \in T$ of order r , up to the action of the Weyl group, then V is not pure.

Proof. Every non-trivial 1-parameter subgroup $\lambda : \mathbb{C}^* \rightarrow T$ contains some toral element of order r . That is, for every 1-parameter subgroup λ of T , there exists $z \in \mathbb{C}^*$ such that $t_\lambda := \lambda(z)$ has order r . Recall from Section 4.4.1 that

$$V_\lambda^0 = \{v \in V : \lambda(z) \cdot v = v \text{ for all } z \in \mathbb{C}^*\} = \bigcap_{z \in \mathbb{C}^*} V^{\lambda(z)},$$

where $V^{\lambda(z)} := \{v \in V : \lambda(z) \cdot v = v\}$. Then $\dim V_\lambda^0 \leq \dim V^{t_\lambda}$ for all λ and so

$$\dim V_\lambda^0 \leq \max\{\dim V^t : t \in T, \text{ ord}(t) = r\}$$

for all λ . By assumption, let $\{t_1, \dots, t_k\}$ be the complete set of Weyl group representations of order r satisfying

$$\dim V^{t_i} \leq \dim V - \dim G - 2, \quad 1 \leq i \leq k.$$

Then every $t \in T$ with $\text{ord}(t) = r$ can be written as $gt_i g^{-1}$ for some representative t_i and $g \in N_G(T)$, as $\text{ord}(gt_i g^{-1}) = \text{ord}(t_i)$. In which case, we have that $V^t = gV^{t_i}$ and hence $\dim V^t = \dim V^{t_i}$. This, together with the assumption, tells us that

$$\dim V_\lambda^0 \leq \max\{\dim V^t : t \in T, \text{ord}(t) = r\} \leq \dim V - \dim G - 2$$

for all λ . Thus, by Proposition 4.4.6, V is not pure. $\square$

In general, Proposition 4.9.1 is usually the quickest way to show that a given representation is not pure. This is because, in many cases, Software packages like LiE are able to compute $\dim V^t$. Many of the representations in Table 4.5 were shown to not be pure by using code we implemented in Macaulay2; see Appendix A. Currently, the toral method has limitations when working with infinite families of representations. This is because it can be tough to find an order that can be used to show that all representations in a given family are not pure. In some cases, such an order may not even exist, and rather, one would need to use multiple different orders of toral elements when working with a given family.

Remark 4.9.2. If V is shown to not be pure via Proposition 4.9.1, then V^* is also not pure. This comes down to the fact that the proof of Proposition 4.9.1 utilizes Proposition 4.4.6. In particular, if the hypothesis of Proposition 4.9.1 is satisfied, then we know $\dim V_\lambda^0 \leq \dim V - \dim G - 2$ for all non-trivial 1-parameter subgroups λ of T . This is equivalent to the restatement Proposition 4.4.17 that says

$$|H_\lambda| \leq \dim V - \dim G - 2$$

for all non-trivial 1-parameter subgroups λ of T . Since the number of V weights that a hyperplane contains is equal to the number of V^* weight that it contains, this hyperplane inequality also holds for the representation V^* ; see also Remark 4.9.6. And so V^* is also not pure.

For the remainder of this section, we illustrate the toral method by using it to show that certain representations of SL_n and $\text{SL}_n \times \text{SL}_m$ are not pure. The representations in Table 4.5 that were shown to not be pure by the toral method utilized a computer. Unfortunately, the rank of G for the representations in Table 4.5 are too big to do clean computations by hand. So, we will choose some small dimension V and small rank G to illustrate the toral method by hand. Since we are looking at representations of $\text{SL}_n \times \text{SL}_m$, we first review what form toral elements of $T_n \times T_m$ of order r take.

Up to the action of the Weyl group, toral elements of T_n of order s take the form

$$t = \text{diag}(\underbrace{z_s^0, \dots, z_s^0}_{k_0}, \underbrace{z_s, \dots, z_s}_{k_1}, \dots, \underbrace{z_s^{s-1}, \dots, z_s^{s-1}}_{k_{s-1}}),$$

where $z_s := e^{\frac{2\pi i}{s}}$ is a primitive s th root of unity and k_i is the number of z_s^i in t . Note that $\sum k_i = n$. Since $\det(t) = 1$, we have the constraint $\sum_{i=1}^{s-1} ik_i \equiv 0 \pmod{s}$. Observe that $t^s = I$. But to ensure that $\text{ord}(t) = s$, we further need the constraint that the least common multiple of the orders of all diagonal entries in t is s . Now, a toral element of $T_n \times T_m$ of order r is $(t_1, t_2) \in \text{SL}_n \times \text{SL}_m$, such that

$$\text{lcm}(\text{ord}(t_1), \text{ord}(t_2)) = r.$$

Note that we can write t_i in the same way as t above, along with both of the constraints, by replacing s with $\text{ord}(t_i)$.

Example 4.9.3. We show that $\text{Sym}^5 \text{SL}_2$ is not pure using the toral method with order 2. Note that the only element in T_2 of order 2 is

$$t = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Given a pure tensor $v_1 \otimes v_2 \otimes v_3 \otimes v_4 \otimes v_5$ in $\text{Sym}^5 \text{SL}_2$, we compute that

$$t \cdot (v_1 \otimes v_2 \otimes v_3 \otimes v_4 \otimes v_5) = (-1)^5 (v_1 \otimes v_2 \otimes v_3 \otimes v_4 \otimes v_5) = -(v_1 \otimes v_2 \otimes v_3 \otimes v_4 \otimes v_5).$$

Extending the action of t on $\text{Sym}^5 \text{SL}_2$ linearly, we get that $t \cdot v = -v$ for all $v \in \text{Sym}^5 \text{SL}_2$. So, $(\text{Sym}^5 \text{SL}_2)^t = 0$. Further, since

$$0 \leq 1 = \dim \text{Sym}^5 \text{SL}_2 - \dim \text{SL}_2 - 2,$$

we may conclude from Proposition 4.9.1 that $\text{Sym}^5 \text{SL}_2$ is not pure. $\diamond$

Example 4.9.4. We show that $\text{Sym}^4 \text{SL}_3$ is not pure using the toral method with order 3. Let ω be the primitive 3rd root of unity. So, using the notation above, $\omega = z_3$. Up to the action of the Weyl group, the only toral elements in T_3 of order 3 are

$$t_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad t_2 = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{pmatrix}, \quad t_3 = \begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}.$$

We show that

$$(\mathrm{Sym}^4 \mathrm{SL}_3)^{t_i} \leq 5 = \dim \mathrm{Sym}^4 \mathrm{SL}_3 - \dim \mathrm{SL}_3 - 2$$

for all t_i . Then, by Proposition 4.9.1, we may conclude that $\mathrm{Sym}^4 \mathrm{SL}_3$ is not pure. Similar to the previous example, we compute that

$$t_2 \cdot v = \omega^4 v = \omega v, \quad t_3 \cdot v = (\omega^2)^4 v = \omega^2 v$$

for all $v \in \mathrm{Sym}^4 \mathrm{SL}_3$. So, $\dim(\mathrm{Sym}^4 \mathrm{SL}_3)^{t_i} = 0 \leq 5$ for $i = 2, 3$, as desired. To compute $(\mathrm{Sym}^4 \mathrm{SL}_3)^{t_1}$, we write

$$\mathrm{Sym}^4 \mathrm{SL}_3 = \mathrm{Sym}^4(\mathrm{SL}_1 \oplus \mathrm{SL}_1 \oplus \mathrm{SL}_1).$$

The action of t_1 on $\mathrm{SL}_1 \oplus \mathrm{SL}_1 \oplus \mathrm{SL}_1$ is given by

$$t_1 \cdot (x + y + z) = x + \omega y + \omega^2 z.$$

We then use the general formula

$$\mathrm{Sym}^a(V \oplus W) = \bigoplus_{i+j=a} \mathrm{Sym}^i V \otimes \mathrm{Sym}^j W$$

iteratively on $\mathrm{Sym}^4 \mathrm{SL}_3$. That is,

$$\begin{aligned} \mathrm{Sym}^4 \mathrm{SL}_3 &= \mathrm{Sym}^4(\mathrm{SL}_1 \oplus \mathrm{SL}_1 \oplus \mathrm{SL}_1) \\ &= \bigoplus_{i+j=4} \mathrm{Sym}^i \mathrm{SL}_1 \otimes \mathrm{Sym}^j(\mathrm{SL}_1 \oplus \mathrm{SL}_1) \\ &= \bigoplus_{i+j=4} \bigoplus_{k+\ell=j} \mathrm{Sym}^i \mathrm{SL}_1 \otimes \mathrm{Sym}^k \mathrm{SL}_1 \otimes \mathrm{Sym}^\ell \mathrm{SL}_1. \end{aligned}$$

The action of t_1 on an element in a single factor of this decomposition for fixed (i, j, k, ℓ) has the eigenvalue

$$1^i \omega^k \omega^{2\ell} = \omega^k \omega^{2\ell}.$$

So, the only way $t_1 \cdot v = v$ is if v has non-zero components only in factors where $k + 2\ell \equiv 0 \pmod{3}$. In other words,

$$(\mathrm{Sym}^4 \mathrm{SL}_3)^{t_1} = \bigoplus_{i+j=4} \bigoplus_{\substack{k+\ell=j \\ k+2\ell \equiv 0 \pmod{3}}} \mathrm{Sym}^i \mathrm{SL}_1 \otimes \mathrm{Sym}^k \mathrm{SL}_1 \otimes \mathrm{Sym}^\ell \mathrm{SL}_1.$$

Observe that $(\mathrm{Sym}^4 \mathrm{SL}_3)^{t_1}$ is comprised of 5 factors each of dimension 1. These factors correspond to $(i, j, k, \ell) \in \{(0, 4, 2, 2), (1, 3, 0, 3), (1, 3, 3, 0), (2, 2, 1, 1), (4, 0, 0, 0)\}$. So, $\dim(\mathrm{Sym}^4 \mathrm{SL}_3)^{t_1} = 5$, as desired. $\diamond$

Next, we present a general formula that helps us compute the fixed points of a toral element of $\mathrm{SL}_n \times \mathrm{SL}_m$. Let $V \otimes W$ be an irreducible representation of $\mathrm{SL}_n \times \mathrm{SL}_m$ and suppose $t = (t_1, t_2) \in T_n \times T_m$ has order r . Note that $\mathrm{ord}(t) = r$ is equivalent to $\mathrm{lcm}(\mathrm{ord}(t_1), \mathrm{ord}(t_2)) = r$. Set $d_i := \mathrm{ord}(t_i)$. As noted above, t_i is of the form

$$t_i = \mathrm{diag}(z_{d_i}^0, \dots, z_{d_i}^0, z_{d_i}, \dots, z_{d_i}, \dots, z_{d_i}^{d_i-1}, \dots, z_{d_i}^{d_i-1}),$$

where $z_{d_i} = e^{\frac{2\pi i}{d_i}}$ is a primitive d_i th root of unity. To simplify notation, set $\omega_i := z_{d_i}$. We define the eigenspaces

$$V^{\omega_1^i} := \{v \in V : t_1 \cdot v = \omega_1^i v\} \quad \text{and} \quad W^{\omega_2^j} := \{w \in W : t_2 \cdot w = \omega_2^j w\}.$$

Then we can decompose V and W into these eigenspaces as follows:

$$V = \bigoplus_{0 \leq i \leq d_1-1} V^{\omega_1^i} \quad \text{and} \quad W = \bigoplus_{0 \leq j \leq d_2-1} W^{\omega_2^j}.$$

And so

$$V \otimes W = \bigoplus_{\substack{0 \leq i \leq d_1-1 \\ 0 \leq j \leq d_2-1}} V^{\omega_1^i} \otimes W^{\omega_2^j}.$$

Consider a pure tensor $v \otimes w \in V^{\omega_1^i} \otimes W^{\omega_2^j}$. Then $v \otimes w \in (V \otimes W)^t$ if and only if $t \cdot (v \otimes w) = v \otimes w$, where we compute that

$$t \cdot (v \otimes w) = t_1 \cdot v \otimes t_2 \cdot w = \omega_1^i v \otimes \omega_2^j w = \omega_1^i \omega_2^j v \otimes w.$$

It follows that $v \otimes w \in (V \otimes W)^t$ if and only if $\omega_1^i \omega_2^j = 1$. Since $\mathrm{lcm}(d_1, d_2) = r$, there exists $x_1, x_2 \in \mathbb{Z}$ such that $r = d_1 x_1 = d_2 x_2$. Then, for a primitive r th root of unity z_r , we have that

$$\omega_1^i \omega_2^j = (z_r^{x_1})^i (z_r^{x_2})^j = z_r^{x_1 i + x_2 j}.$$

Note that $x_1 i + x_2 j \equiv 0 \pmod{r}$ is equivalent to $i/d_1 + j/d_2 \in \mathbb{Z}$, which is equivalent to $d_2 i + d_1 j \equiv 0 \pmod{d_1 d_2}$. So, $v \otimes w \in (V \otimes W)^t$ if and only if $x_1 i + x_2 j \equiv 0 \pmod{r}$ if and only if $d_2 i + d_1 j \equiv 0 \pmod{d_1 d_2}$. By the linearity of the action of t on $V \otimes W$, it follows that

$$(V \otimes W)^t = \bigoplus_{(i,j) \in S} V^{\omega_1^i} \otimes W^{\omega_2^j}, \tag{4.5}$$

where

$$S := \{0 \leq i \leq d_1 - 1, 0 \leq j \leq d_2 - 1 : d_2 i + d_1 j \equiv 0 \pmod{d_1 d_2}\}.$$

When computing $(V \otimes W)^t$, Equation 4.5 says that we only need to do eigenspace computations for representations of the special linear group. In the next example, we illustrate how to compute these eigenspaces using the same technique as in Example 4.9.3 and Example 4.9.4. The general algorithm for this technique appears in Appendix A for the representations $\text{Sym}^k \text{SL}_n$ and $\bigwedge^k \text{SL}_n$.

Example 4.9.5. We show that $\text{Sym}^3 \text{SL}_2 \otimes \text{Sym}^3 \text{SL}_3$ is not pure using the toral method with order 2. To simplify notation, set $V := \text{Sym}^3 \text{SL}_2$ and $W := \text{Sym}^3 \text{SL}_3$. For $t = (t_1, t_2) \in T_2 \times T_3$ to have order 2, it must lie in one of the following cases: $\text{ord}(t_1) = 1$ and $\text{ord}(t_2) = 2$, $\text{ord}(t_1) = 2$ and $\text{ord}(t_2) = 1$, or $\text{ord}(t_1) = \text{ord}(t_2) = 2$. In each of these cases, there is exactly one toral element that satisfies the order constraints, up to permutation of the diagonal. For each of these toral elements t , we show that

$$\dim(V \otimes W)^t \leq \dim(V \otimes W) - \dim(\text{SL}_2 \times \text{SL}_3) - 2 = 27.$$

Then, by Proposition 4.9.1, we may conclude that $V \otimes W$ is not pure. We show this inequality case-by-case for each t .

Case 1: When $\text{ord}(t_1) = 1$ and $\text{ord}(t_2) = 2$, we have that

$$t_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad t_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

By Equation 4.5, we just need to compute $\dim(V^{\omega_1^0} \otimes W^{\omega_2^0})$. Since $\omega_1^0 = 1$ and $t_1 = I$, $V^{\omega_1^0} = V$. Next, since $\omega_2^0 = 1$,

$$W^{\omega_2^0} = \{w \in W : t_2 \cdot w = w\} = W^{t_2}.$$

Similar as to in Example 4.9.4, we write

$$W = \text{Sym}^3(\text{SL}_1 \oplus \text{SL}_2) = \bigoplus_{k+\ell=3} \text{Sym}^k \text{SL}_1 \otimes \text{Sym}^\ell \text{SL}_2.$$

We see then that that action of t_2 on a point in $\text{Sym}^k \text{SL}_1 \otimes \text{Sym}^\ell \text{SL}_2$ has eigenvalue $(1)^k(-1)^\ell = (-1)^\ell$. So, the only way $t_2 \cdot w = w$ is if w has non-zero components only in factors where ℓ is even. That is,

$$W^{t_2} = (\text{Sym}^1 \text{SL}_1 \otimes \text{Sym}^2 \text{SL}_2) \oplus (\text{Sym}^3 \text{SL}_1 \otimes \text{Sym}^0 \text{SL}_2)$$

and so $\dim W^{t_2} = 3 + 1 = 4$. We conclude that

$$\dim(V \otimes W)^t = \dim(V^{\omega_1^0} \otimes W^{\omega_2^0}) = \dim(V \otimes W^{t_2}) = 4(4) = 16 < 27.$$

Case 2: When $\text{ord}(t_1) = 2$ and $\text{ord}(t_2) = 1$, we have that

$$t_1 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad t_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Similar to the above case, we just need to compute $\dim(V^{\omega_1^0} \otimes W^{\omega_2^0})$. Since $\omega_2^0 = 1$ and $t_2 = I$, $W^{\omega_2^0} = W$. Also,

$$V^{\omega_1^0} = \{v \in V : t_1 \cdot v = v\} = V^{t_1}.$$

The action of t_1 on a point in V has eigenvalue $(-1)^3 = -1$ and so $V^{t_1} = 0$. In this case,

$$\dim(V \otimes W)^t = \dim(V^{\omega_1^0} \otimes W^{\omega_2^0}) = \dim(0 \otimes W) = 0 < 27.$$

Case 3: When $\text{ord}(t_1) = \text{ord}(t_2) = 2$, we have that

$$t_1 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad t_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

By Equation 4.5, we just need to compute $\dim(V^{\omega_1^i} \otimes W^{\omega_2^j})$ for $i, j \in \{0, 1\}$ satisfying $2i + 2j \equiv 0 \pmod{4}$. The condition $2i + 2j \equiv 0 \pmod{4}$ is equivalent to $i + j$ being even, so $(i, j) = (0, 0)$ or $(i, j) = (1, 1)$. Similar to the other cases, when $(i, j) = (0, 0)$, we compute that

$$V^{\omega_1^0} = V^{t_1} = \{v \in V : (-1)^3 v = v\} = 0$$

and so $\dim(V^{\omega_1^0} \otimes W^{\omega_2^0}) = 0$. For $(i, j) = (1, 1)$, note that $\omega_1^1 = \omega_2^1 = e^{\pi i} = -1$. From the previous computation, we know that $t_1 \cdot v = -v$ for all $v \in V$ and so

$$V^{\omega_1^1} = \{v \in V : t_1 \cdot v = \omega_1^1 v\} = \{v \in V : -v = -v\} = V.$$

To compute $W^{\omega_2^1} = \{w \in W : t_2 \cdot w = -w\}$ we write

$$W = \text{Sym}^3(\text{SL}_1 \oplus \text{SL}_2) = \bigoplus_{k+\ell=3} \text{Sym}^k \text{SL}_1 \otimes \text{Sym}^\ell \text{SL}_2.$$

The action of t_2 on a vector in $\text{Sym}^k \text{SL}_1 \otimes \text{Sym}^\ell \text{SL}_2$ has eigenvalue $(1)^k (-1)^\ell = (-1)^\ell$. So, the only way $t_2 \cdot w = -w$ is if w has non-zero components only in factors where ℓ is odd. This means that

$$W^{\omega_2^1} = (\text{Sym}^0 \text{SL}_1 \otimes \text{Sym}^3 \text{SL}_2) \oplus (\text{Sym}^2 \text{SL}_1 \otimes \text{Sym}^1 \text{SL}_2)$$

and so $\dim W^{\omega_2^1} = 4 + 2 = 6$. We conclude that

$$\dim(V \otimes W)^t = \dim(V^{\omega_1^0} \otimes W^{\omega_2^0}) + \dim(V^{\omega_1^1} \otimes W^{\omega_2^1}) = 0 + 6 \dim V = 24 < 27.$$

◇

4.9.2 Counting weights on hyperplanes

For the notation in this section, please refer to Section 4.4 and Section 4.7. In this section, we describe a combinatorial method that can be used to show that a given representation is not pure. Consider the contrapositive of Proposition 4.4.17.

Contrapositive of Proposition 4.4.17 *Let V be a representation of a reductive group G and fix a maximal torus T of G . For every non-trivial 1-parameter subgroup λ of T , if*

$$|H_\lambda^T| < \dim V - \dim G - 1,$$

then V is not pure.

Proposition 4.4.17 roughly says that a representation cannot be pure if there does not exist a hyperplane that contains “enough” weights, with “enough” being $\dim V - \dim G - 1$. In practice, given an arbitrary H_λ , one can often compute the exact value of $|H_\lambda|$, or a decent upper bound on $|H_\lambda|$. For low rank G and low $\dim V$, this can usually be done by hand without too much struggle. In the case that G is a connected reductive Lie group, recall that Proposition 4.4.17 and Proposition 4.4.20 are essentially the same. In other words, $|H_\lambda^{\mathfrak{h}^*}| = |H_\lambda^T|$ for $\mathfrak{h} = \text{Lie}(T)$ and so it does not matter if we do hyperplane computations in $\mathfrak{h}^*$ or T . Also, recall that since $|wH_\lambda^{\mathfrak{h}^*}| = |H_\lambda^{\mathfrak{h}^*}|$ for all $w \in \mathfrak{W}$, it is enough to only consider hyperplanes of the form $H_\lambda^{\mathfrak{h}^*}$ for non-trivial $\lambda \in N(T)$, up to $\mathfrak{W}$ -conjugates, when applying Proposition 4.4.17.

Remark 4.9.6. If a representation V is shown to not be pure by the hyperplane method described above, then V^* will also not be pure. This is because for any hyperplane H , the number of V weights that H contains is equal to the number of V^* weights that H contains. The number of weights is the same because the weights of V^* are the negatives of the weights of V . Note that the toral method utilizes this method itself; see Remark 4.9.2.

The rest of this section is devoted to examples that demonstrate how to use Proposition 4.4.17 to show that a given representation is not pure. Note that all of our examples below are for the connected reductive Lie group $\text{SL}_n \times \text{SL}_m$ and so we are able to do our hyperplane computations in $\mathfrak{h}^*$.

Lemma 4.9.7 (K). *For $m \geq 9$ and $k \in \{3, 4, 5\}$, the representation $\text{Sym}^k \text{SL}_2 \otimes \bigwedge^3 \text{SL}_m$ is not pure.*

Proof. For $9 \leq m \leq 29$ and $k \in \{3, 4, 5\}$, the toral method with order 5 can be used to show that the representation is not pure. So, assume $m \geq 30$ from now on. We show that

for every $m \geq 30$ and every non-trivial 1-parameter subgroup λ of $T_2 \times T_m$,

$$|H_\lambda| < \dim(\mathrm{Sym}^k \mathrm{SL}_2 \otimes \bigwedge^3 \mathrm{SL}_m) - \dim(\mathrm{SL}_2 \times \mathrm{SL}_m) - 1.$$

Then, by Proposition 4.4.17, we may conclude that $\mathrm{Sym}^k \mathrm{SL}_2 \otimes \bigwedge^3 \mathrm{SL}_m$ is not pure for $m \geq 30$. To show this strict inequality, we consider the three possible forms that H_λ can take. Note that

$$\dim(\mathrm{Sym}^k \mathrm{SL}_2 \otimes \bigwedge^3 \mathrm{SL}_m) - \dim(\mathrm{SL}_2 \times \mathrm{SL}_m) - 1 = (k+1) \binom{m}{3} - m^2 - 3 =: g(k, m).$$

Case I. Assume $H_\lambda = \{(v, 0)\}^\perp$ for some non-zero $v \in \mathfrak{h}_2^*$. Then $H_\lambda = H_{\lambda_v} \oplus \mathfrak{h}_m^*$, where $H_{\lambda_v} := \{v\}^\perp$. Note that

$$|H_\lambda| = |H_{\lambda_v}| \dim \bigwedge^3 \mathrm{SL}_m = |H_{\lambda_v}| \binom{m}{3}.$$

The only hyperplane in $\mathfrak{h}_2^*$ containing the origin is 0, so $H_{\lambda_v} = 0$. For $k \in \{3, 5\}$, we have that $|H_{\lambda_v}| = 0$ since 0 is not a weight of $\mathrm{Sym}^k \mathrm{SL}_2$. In this case,

$$|H_\lambda| = 0 < g(k, m).$$

For $k = 4$, we have that $|H_{\lambda_v}| = 1$ since 0 is a weight of $\mathrm{Sym}^4 \mathrm{SL}_2$ with multiplicity one, and so

$$|H_\lambda| = \dim \bigwedge^3 \mathrm{SL}_m < g(4, m).$$

Case II. Assume $H_\lambda = \{(0, v)\}^\perp$ for some non-zero $v \in \mathfrak{h}_m^*$. Then $H_\lambda = \mathfrak{h}_2^* \oplus H_{\lambda_v}$ where $H_{\lambda_v} := \{v\}^\perp$. Note that

$$|H_\lambda| = \dim \mathrm{Sym}^k \mathrm{SL}_2 |H_{\lambda_v}| = (k+1) |H_{\lambda_v}|.$$

First consider the case when $H_{\lambda_v} = (1, 0, \dots, 0, -1)^\perp$. In the proof of Lemma 4.7.2, we found that

$$|H_{\lambda_v}| = \binom{m-2}{3} + (m-2),$$

and one can verify that

$$|H_\lambda| = (k+1) |H_{\lambda_v}| < g(k, m)$$

for $k \in \{3, 4, 5\}$. Now assume that $H_{\lambda_v} \neq (1, 0, \dots, 0, -1)^\perp$, up to permutation. By Lemma 4.7.2, we know that

$$|H_{\lambda_v}| < \dim \bigwedge^3 \mathrm{SL}_m - m^2$$

and we compute that

$$|H_\lambda| < (k+1)(\dim \bigwedge^3 \mathrm{SL}_m - m^2) < g(k, m)$$

for $k \in \{3, 4, 5\}$.

Case III. Assume $H_\lambda = \{(v, w)\}^\perp$ for some non-zero $v \in \mathfrak{h}_2^*$ and $w \in \mathfrak{h}_m^*$. Then $v = aL_1$ for some non-zero $a \in \mathbb{Z}$ and $w = \sum_{i=1}^m a_i L_i$ with $\sum_{i=1}^m a_i = 0$ and $a_i \in \mathbb{Z}$. Let $k \in \{3, 4, 5\}$ and recall that the weights of $\mathrm{Sym}^k \mathrm{SL}_2$ are all of multiplicity one, and of the form

$$\omega_{kr} := (k+1) - 2r, \quad 0 \leq r \leq k+1.$$

For any such ω_{kr} and weight ω of $\bigwedge^3 \mathrm{SL}_m$, we have that $(\omega_{kr}, \omega) \in H_\lambda$ if and only if $\omega_{kr}a = -\langle \omega, w \rangle$. Since $a \neq 0$, we have that $\omega_{kr}a \neq \omega_{kr'}a$ for distinct r, r' . This means that for any weight ω of $\bigwedge^3 \mathrm{SL}_m$, $(\omega_{kr}, \omega) \in H$ for at most one r . This fact implies that $|H_\lambda| \leq \dim \bigwedge^3 \mathrm{SL}_m$ and so

$$|H_\lambda| \leq \dim \bigwedge^3 \mathrm{SL}_m < g(k, m)$$

for all $m \geq 30$, as desired. □

Lemma 4.9.8 (K). *For $m \geq 5$, the representation $\mathrm{Sym}^3 \mathrm{SL}_2 \otimes \mathrm{Sym}^3 \mathrm{SL}_m$ is not pure.*

Proof. For $5 \leq m \leq 26$, the toral method can be used to show that the representation is not pure. Next, we show that for every $m \geq 27$ and every non-trivial 1-parameter subgroup λ of $T_2 \times T_m$,

$$|H_\lambda| < \dim(\mathrm{Sym}^3 \mathrm{SL}_2 \otimes \mathrm{Sym}^3 \mathrm{SL}_m) - \dim(\mathrm{SL}_2 \times \mathrm{SL}_m) - 1.$$

Then, by Proposition 4.4.17, we may conclude that $\mathrm{Sym}^3 \mathrm{SL}_2 \otimes \mathrm{Sym}^3 \mathrm{SL}_m$ is not pure for $m \geq 27$. Again, to show this strict inequality, we consider the three possible forms that H_λ can take. Note that

$$\dim(\mathrm{Sym}^3 \mathrm{SL}_2 \otimes \mathrm{Sym}^3 \mathrm{SL}_m) - \dim(\mathrm{SL}_2 \times \mathrm{SL}_m) - 1 = 4 \dim \mathrm{Sym}^3 \mathrm{SL}_m - m^2 - 3 =: g(m).$$

Case I. Assume $H_\lambda = \{(v, 0)\}^\perp$ for some non-zero $v \in \mathfrak{h}_2^*$. Then $H_\lambda = H_{\lambda_v} \oplus \mathfrak{h}_m^*$, where $H_{\lambda_v} := \{v\}^\perp$. The only hyperplane in $\mathfrak{h}_2^*$ containing the origin is 0, so $H_{\lambda_v} = 0$. It follows that $|H_{\lambda_v}| = 0$ since 0 is not a weight of $\text{Sym}^3 \text{SL}_2$. Thus

$$|H_\lambda| = |H_{\lambda_v}| \dim \text{Sym}^3 \text{SL}_m = 0 < g(m)$$

for all m , as desired.

Case II. Assume $H_\lambda = \{(0, v)\}^\perp$ for some non-zero $v \in \mathfrak{h}_m^*$. Then $H_\lambda = \mathfrak{h}_2^* \oplus H_{\lambda_v}$ where $H_{\lambda_v} := \{v\}^\perp$, and so

$$|H_\lambda| = \dim \text{Sym}^3 \text{SL}_2 |H_{\lambda_v}| = 4|H_{\lambda_v}|.$$

First consider the case when $H_{\lambda_v} = (1, 0, \dots, 0, -1)^\perp$. As computed in Lemma 4.7.3,

$$|H_{\lambda_v}| = 2(m-2) + (m-2)(m-3) + \binom{m-2}{3}$$

and we see that $|H_\lambda| = 4|H_{\lambda_v}| < g(m)$ for all m . Now assume that $H_{\lambda_v} \neq (1, 0, \dots, 0, -1)^\perp$, up to permutation. By Lemma 4.7.3, we have that

$$|H_{\lambda_v}| < \dim \text{Sym}^3 \text{SL}_m - m^2$$

for all $m \geq 27$ and we compute that

$$|H_\lambda| = 4|H_{\lambda_v}| < 4 \dim \text{Sym}^3 \text{SL}_m - 4m^2 < g(m).$$

Case III. Assume $H_\lambda = \{(v, w)\}^\perp$ for some non-zero $v \in \mathfrak{h}_2^*$ and $w \in \mathfrak{h}_m^*$. Then $v = aL_1$ for some non-zero $a \in \mathbb{Z}$ and $w = \sum_{i=1}^m a_i L_i$ with $\sum_{i=1}^m a_i = 0$ and $a_i \in \mathbb{Z}$. Recall that the weights of $\text{Sym}^3 \text{SL}_2$ are $\{-3L_1, -L_1, L_1, 3L_1\}$. For any $r \in \{-3, -1, 1, 3\}$ and weight ω_2 of $\text{Sym}^3 \text{SL}_m$, we have that $(rL_1, \omega_2) \in H$ if and only if $ra = -\langle \omega_2, w \rangle$. Since $a \neq 0$, we have that $ra \neq r'a$ for distinct $r, r' \in \{-3, -1, 1, 3\}$. This means that for any weight ω_2 of $\text{Sym}^3 \text{SL}_m$, $(\omega_1, \omega_2) \in H_\lambda$ for at most one weight ω_1 of $\text{Sym}^3 \text{SL}_2$. This fact implies that $|H_\lambda| \leq \dim \text{Sym}^3 \text{SL}_m$. And we see that $|H_\lambda| \leq \dim \text{Sym}^3 \text{SL}_m < g(m)$. $\square$

Lemma 4.9.9 (K). *For $m \geq 15$, the representation $\text{Sym}^4 \text{SL}_2 \otimes \text{Sym}^3 \text{SL}_m$ is not pure.*

Proof. For $15 \leq m \leq 26$, the toral method can be used to show that the representation is not pure. Next, for every $m \geq 27$ and every non-trivial 1-parameter subgroup λ of $T_2 \times T_m$, we show that

$$|H_\lambda| < \dim(\text{Sym}^4 \text{SL}_2 \otimes \text{Sym}^3 \text{SL}_m) - \dim(\text{SL}_2 \times \text{SL}_m) - 1.$$

Then, by Proposition 4.4.17, we may conclude that $\text{Sym}^4 \text{SL}_2 \otimes \text{Sym}^3 \text{SL}_m$ is not pure for $m \geq 27$. Again, we consider the three possible forms that H_λ can take. Note that

$$\dim(\text{Sym}^4 \text{SL}_2 \otimes \text{Sym}^3 \text{SL}_m) - \dim(\text{SL}_2 \times \text{SL}_m) - 1 = 5 \dim \text{Sym}^3 \text{SL}_m - m^2 - 3 =: g(m).$$

Case I. Assume $H_\lambda = \{(v, 0)\}^\perp$ for some non-zero $v \in \mathfrak{h}_2^*$. Then $H_\lambda = H_{\lambda_v} \oplus \mathfrak{h}_m^*$, where $H_{\lambda_v} := \{v\}^\perp$. The only hyperplane in $\mathfrak{h}_2^*$ containing the origin is 0, so $H_{\lambda_v} = 0$. It follows that $|H_{\lambda_v}| = 1$ since 0 is a weight of $\text{Sym}^3 \text{SL}_2$ with multiplicity 1. We find that

$$|H_\lambda| = |H_{\lambda_v}| \dim \text{Sym}^3 \text{SL}_m = \dim \text{Sym}^3 \text{SL}_m < g(m)$$

for all m , as desired.

Case II. Assume $H_\lambda = \{(0, v)\}^\perp$ for some non-zero $v \in \mathfrak{h}_m^*$. Then $H_\lambda = \mathfrak{h}_2^* \oplus H_{\lambda_v}$ where $H_{\lambda_v} := \{v\}^\perp$, and so

$$|H_\lambda| = \dim \text{Sym}^4 \text{SL}_2 |H_{\lambda_v}| = 5 |H_{\lambda_v}|.$$

First consider the case when $H_{\lambda_v} = (1, 0, \dots, 0, -1)^\perp$. As computed in Lemma 4.7.3,

$$|H_{\lambda_v}| = 2(m-2) + (m-2)(m-3) + \binom{m-2}{3}$$

and we see that $|H_\lambda| = 5 |H_{\lambda_v}| < g(m)$ for all m . Now assume that $H_{\lambda_v} \neq (1, 0, \dots, 0, -1)^\perp$, up to permutation. By Lemma 4.7.3, we have that

$$|H_{\lambda_v}| < \dim \text{Sym}^3 \text{SL}_m - m^2$$

for all $m \geq 27$ and we compute that

$$|H_\lambda| = 5 |H_{\lambda_v}| < 5 \dim \text{Sym}^3 \text{SL}_m - 5m^2 < g(m).$$

Case III. Assume $H_\lambda = \{(v, w)\}^\perp$ for some non-zero $v \in \mathfrak{h}_2^*$ and $w \in \mathfrak{h}_m^*$. Using the same argument as in Lemma 4.9.8, we have that $|H_\lambda| \leq \dim \text{Sym}^3 \text{SL}_m$. And we see that $|H_\lambda| \leq \dim \text{Sym}^3 \text{SL}_m < g(m)$. $\square$

4.10 Remaining cases

The general strategy to prove Conjecture 3.5.1 in its entirety for representations of $\text{SL}_n \times \text{SL}_m$ is to write down all irreducible pure representations of $\text{SL}_n \times \text{SL}_m$. This is because we

already know all of the irreducible cofree representations of $\mathrm{SL}_n \times \mathrm{SL}_m$; see Table 3.1. In Section 4.8, we showed that Conjecture 3.5.1 holds for irreducible representations $V \otimes W$ of $\mathrm{SL}_n \times \mathrm{SL}_m$ satisfying $\dim V \geq n^2$ and $\dim W \geq m^2$. We now discuss the remaining cases, namely when $V \otimes W$ satisfies $\dim V < n^2$ or $\dim W < m^2$. Specifically, we discuss two conjectures, which are just specific situations of Conjecture 3.5.1 for $\mathrm{SL}_n \times \mathrm{SL}_m$. If these two conjectures are eventually shown to be true, then together with Theorem 4.8.1 we have a complete proof of Conjecture 3.5.1 for representations of $\mathrm{SL}_n \times \mathrm{SL}_m$.

Conjecture 4.10.1. *Let $V \otimes W$ be an irreducible representation of $\mathrm{SL}_n \times \mathrm{SL}_m$. If $\dim V < n^2$ and $\dim W < m^2$, then Conjecture 3.5.1 holds.*

As noted in [19, pg. 234], up to duals, the only irreducible representations SL_n that have dimension strictly less than n^2 are

$$\mathrm{SL}_n \quad \mathrm{AdSL}_n \quad \mathrm{Sym}^2 \mathrm{SL}_n \quad \bigwedge^2 \mathrm{SL}_n \quad \bigwedge^3 \mathrm{SL}_6 \quad \bigwedge^3 \mathrm{SL}_7 \quad \bigwedge^3 \mathrm{SL}_8.$$

Pairing up all possible combinations of these representations and their duals, we arrive at a finite list of all representations $V \otimes W$ of $\mathrm{SL}_n \times \mathrm{SL}_m$ satisfying $\dim V < n^2$ and $\dim W < m^2$. Comparing this list to Table 3.1, we see that all irreducible cofree representations of $\mathrm{SL}_n \times \mathrm{SL}_m$ are contained in this list, with the exception of

$$\mathrm{SL}_2 \otimes \mathrm{Sym}^3 \mathrm{SL}_2 \quad \mathrm{SL}_3 \otimes \mathrm{Sym}^3 \mathrm{SL}_2 \quad \mathrm{SL}_9 \otimes \mathrm{Sym}^3 \mathrm{SL}_3 \quad \mathrm{SL}_4 \otimes \mathrm{Sym}^4 \mathrm{SL}_2$$

and their duals. Thus, to prove Conjecture 4.10.1, we need to verify that the stable cofree representations appearing in this list are pure and that all other representations in the list are not pure. Although this is a finite problem, we run into the same issue as in Conjecture 3.6.6 when it comes to having to show by hand that a given representation is pure. Moreover, there are many representations in this list that we expect to be not pure for which our current methods to show not pure are difficult to work with. For example, consider the representation $\mathrm{SL}_n \otimes \bigwedge^2 \mathrm{SL}_m$ that appears in our list. If $n \geq \dim \bigwedge^2 \mathrm{SL}_m$, then the representation is not stable and so we may assume that $n < \dim \bigwedge^2 \mathrm{SL}_m$. Looking at Table 3.1, we see that there are representations in this family that are cofree. In particular, there are two infinite sub-families that are cofree:

$$\mathrm{SL}_{\frac{1}{2}(n-1)n-1} \otimes \bigwedge^2 \mathrm{SL}_n, \quad n \geq 2 \qquad \mathrm{SL}_2 \otimes \bigwedge^2 \mathrm{SL}_m, \quad m \geq 5, \quad m \text{ odd}.$$

However, we run into issues when working with the representations in the family that are not cofree. Since n and m are both independent variables here, it is really tough to utilize our current methods to deduce non-purity.

Conjecture 4.10.2. *Let $V \otimes W$ be an irreducible representation of $\mathrm{SL}_n \times \mathrm{SL}_m$. If $\dim V < n^2$ and $\dim W \geq m^2$, then Conjecture 3.5.1 holds.*

We know from Theorem 4.8.1 that there are no irreducible cofree representations $V \otimes W$ of $\mathrm{SL}_n \times \mathrm{SL}_m$ satisfying $\dim V \geq n^2$ and $\dim W \geq m^2$. This, together with the analysis above, tells us that up to duals the only cofree representations satisfying the dimension constraints $\dim V < n^2$ and $\dim W \geq m^2$ are:

$$\mathrm{SL}_2 \otimes \mathrm{Sym}^3 \mathrm{SL}_2 \quad \mathrm{SL}_3 \otimes \mathrm{Sym}^3 \mathrm{SL}_2 \quad \mathrm{SL}_9 \otimes \mathrm{Sym}^3 \mathrm{SL}_3 \quad \mathrm{SL}_4 \otimes \mathrm{Sym}^4 \mathrm{SL}_2$$

Given this, we expect that almost all representations of $\mathrm{SL}_n \times \mathrm{SL}_m$ that satisfy $\dim V < n^2$ and $\dim W \geq m^2$ are not pure. Similar to the proof of Theorem 4.8.1, the strategy here is first find a finite list of representations that contains all pure representations of $\mathrm{SL}_n \times \mathrm{SL}_m$ satisfying $\dim V < n^2$ and $\dim W \geq m^2$. Since we already know all irreducible representations of SL_n with dimension strictly less than n^2 , we already know the finitely many options for V . The issue comes down to not being able to reduce the second factor W to finitely many options when V is an infinite family. If we assume that $V \otimes W$ is pure for a fixed V , then by Theorem 4.5.4

$$\dim W \leq \frac{q(n, m)}{\dim V}$$

for some rational function q . The issue here is that we are unable to further bound $q(n, m)/\dim V$ by something in just m when V is one of the infinite families. This, together with the fact that there is no relationship between n and m , means that this bound does not tell us anything useful about $\dim W$. Note that this phenomenon did not occur in the main case when $\dim V \geq n^2$ and $\dim W \geq m^2$, as we were able to assume $n \leq m$.

We finish this section by stating some hyperplane results that may be helpful, in combination with future work, to prove these conjectures.

Proposition 4.10.3 (K). *Let $V_1 \otimes V_2$ be an irreducible representation of $G_1 \times G_2$, with G_i connected simple Lie groups. Denote the Cartan subalgebra of $\mathrm{Lie}(G_i)$ by $\mathfrak{h}_i$. For any hyperplane $H = \{(v_1, v_2)\}^\perp \subseteq \mathfrak{h}_1^* \oplus \mathfrak{h}_2^*$ though 0,*

$$|H| \leq \begin{cases} (\max \# \text{ weights on a hyperplane in } \mathfrak{h}_1^*) \dim V_2 & v_1 \neq 0 \\ (\max \# \text{ weights on a hyperplane in } \mathfrak{h}_2^*) \dim V_1 & v_2 \neq 0 \end{cases}.$$

Proof. Let W_i be the set of weights for V_i . The following proof is symmetric with respect to v_1 and v_2 , so we may assume that $v_2 \neq 0$. We can think of the weight diagram of $V_1 \otimes V_2$

as copies of W_2 indexed by the weights of V_1 . That is, copies of $\{\omega\} \times W_2$, where ω is a weight of V_1 . For fixed ω , the number of weights in H of the form (ω, β) is

$$|H \cap (\{\omega\} \times W_2)| = |H \cap (\{\omega\} \times \mathfrak{h}_2^*)|.$$

By definition,

$$H \cap (\{\omega\} \times \mathfrak{h}_2^*) = \{(\omega, y) : y \in \mathfrak{h}_2^*, -\langle \omega, v_1 \rangle = \langle y, v_2 \rangle\}.$$

Since $-\langle \omega, v_1 \rangle$ is a constant and $v_2 \neq 0$, we get that

$$H \cap (\{\omega\} \times \mathfrak{h}_2^*) = \{\omega\} \times H',$$

where $H' \subseteq \mathfrak{h}_2^*$ is a hyperplane, not necessarily through 0. It follows that

$$|H| = \sum_{\omega \in W_1} |H \cap (\{\omega\} \times \mathfrak{h}_2^*)| = \sum_{\omega \in W_1} \text{mult}(\omega) |H'|.$$

Bounding $|H'|$ by the maximum number $\mathcal{M}_2$ of weights on a hyperplane in $\mathfrak{h}_2^*$, we get

$$|H| \leq \mathcal{M}_2 \sum_{\omega \in W_1} \text{mult}(\omega) = \mathcal{M}_2 \dim V_1,$$

as desired. □

Lemma 4.10.4 (K). *Consider the setting of SL_n . Write $v \in \mathfrak{h}^*$ as*

$$v = \sum a_i L_i, \quad a_i \in \mathbb{C}$$

and let $\sigma_1, \sigma_2 \in \mathfrak{W}$. Then $\sigma_1 v = \sigma_2 v$ if and only if $a_{\sigma_1^{-1}(i)} = a_{\sigma_2^{-1}(i)}$ for all i .

Proof. The backwards direction is immediate, so it remains to prove the forward direction. Since $\sigma_1 v = \sigma_2 v$, we have that

$$\sum a_{\sigma_1^{-1}(i)} L_i - \sum a_{\sigma_2^{-1}(i)} L_i = 0.$$

This, together with $\sum L_i = 0$, implies that

$$0 = \sum_{i=1}^{n-1} [(a_{\sigma_1^{-1}(i)} - a_{\sigma_2^{-1}(i)}) - (a_{\sigma_1^{-1}(n)} - a_{\sigma_2^{-1}(n)})] L_i.$$

Since $L_1, \dots, L_{n-1}$ are linearly independent, it follows that

$$\lambda := a_{\sigma_1^{-1}(n)} - a_{\sigma_2^{-1}(n)} = a_{\sigma_1^{-1}(i)} - a_{\sigma_2^{-1}(i)}, \quad 1 \leq i \leq n.$$

The action of $\sigma \in \mathfrak{W}$ on v just rearranges the ordering of $(a_1, \dots, a_n)$, and so $\sum a_i$ is invariant under $\mathfrak{W}$. Thus,

$$\sum a_{\sigma_2^{-1}(i)} = \sum a_{\sigma_1^{-1}(i)} = \sum (a_{\sigma_2^{-1}(i)} + \lambda) = \sum a_{\sigma_2^{-1}(i)} + n\lambda$$

and hence $n\lambda = 0$. This means $\lambda = 0$ and so $a_{\sigma_1^{-1}(i)} = a_{\sigma_2^{-1}(i)}$ for all i . $\square$

Lemma 4.10.5 (K). *Let V be an irreducible representation of $\mathfrak{sl}_n$. If ω is a weight of V , then*

$$\text{mult}(\omega) \leq \frac{\dim V}{n}.$$

Proof. The result is proven for the case $\omega = 0$ in [8, Lemma 3.7]. Assume $\omega \neq 0$. Since weights and their multiplicities are invariant under the Weyl group $\mathfrak{W}$, for every $\sigma \in \mathfrak{W}$, $\sigma \cdot \omega$ is a weight of V and $\text{mult}(\sigma \cdot \omega) = \text{mult}(\omega)$. So

$$|\mathfrak{W} \cdot \omega| \text{mult}(\omega) \leq \dim V.$$

The result then follows if $|\mathfrak{W} \cdot \omega| \geq n$. To show that $|\mathfrak{W} \cdot \omega| \geq n$, by the orbit-stabilizer Theorem, it suffices to bound the size of the stabilizer $\text{Stab}_{\mathfrak{W}}(\omega)$. Since $\omega \neq 0$, we may write

$$\omega = \sum a_i L_i, \quad a_i \in \mathbb{Z},$$

where not all a_i are equal. By reordering $L_1, \dots, L_n$, we may assume that $a_1 \geq a_2 \geq \dots \geq a_n$. This reordering ensures that equal a_i are next to each other in the tuple $\mathbf{a} := (a_1, \dots, a_n)$. Let k be the number of distinct a_i in $\mathbf{a}$ and label them as $b_1 > b_2 > \dots > b_k$. Then, we may write

$$\mathbf{a} = (\underbrace{b_1, \dots, b_1}_{n_1}, \underbrace{b_2, \dots, b_2}_{n_2}, \dots, \underbrace{b_k, \dots, b_k}_{n_k}),$$

where n_i is the number of b_i appearing in $\mathbf{a}$. Note that $k \geq 2$ since $\omega \neq 0$, and so $0 < n_k < n$. Also note that we can view that action of $\sigma \in \mathfrak{W}$ on ω as σ rearranging the elements of $\mathbf{a}$. By Lemma 4.10.4, $\sigma \in \text{Stab}_{\mathfrak{W}}(\omega)$ if and only if $a_i = a_{\sigma(i)}$ for all i . This is equivalent to saying that σ does not interchange any two distinct b_i . In other words, $\sigma \in \text{Stab}_{\mathfrak{W}}(\omega)$ if and only if σ permutes the first n_1 elements of $\mathbf{a}$ among each other, the next n_2 elements of $\mathbf{a}$ among each other, and so on. This means that

$$|\text{Stab}_{\mathfrak{W}}(\omega)| = n_1! n_2! \dots n_k!$$

and so

$$|\mathfrak{W} \cdot \omega| = \frac{|\mathfrak{W}|}{|\text{Stab}_{\mathfrak{W}}(\omega)|} = \frac{n!}{n_1!n_2! \cdots n_k!}.$$

Note that this is a multinomial coefficient, which has the known expansion

$$\frac{n!}{n_1!n_2! \cdots n_k!} = \binom{n_1}{n_1} \binom{n_1 + n_2}{n_2} \cdots \binom{n_1 + n_2 + \cdots + n_k}{n_k}.$$

Since $\sum n_i = n$ and $0 < n_k < n$, it follows that

$$|\mathfrak{W} \cdot \omega| \geq \binom{n}{n_k} \geq n.$$

□

The bound in Lemma 4.10.5 can not be improved since equality is achieved for the standard representation of SL_n . If we require that V is not the standard representation, we may be able to get a better bound.

Proposition 4.10.6 (K). *Let V be an irreducible representation of SL_n . If $H \subset \mathfrak{h}^*$ is a hyperplane, not necessarily through 0, then*

$$|H| \leq \frac{n-1}{n} \dim V.$$

Proof. Assume for contradiction that there exists a hyperplane $H \subset \mathfrak{h}^*$ such that

$$|H| > c_n \dim V, \quad c_n := \frac{n-1}{n}.$$

Let H be defined by the normal vector $v \in \mathfrak{h}^*$ and shift $a \in k^*$. That is,

$$H = \{x \in \mathfrak{h}^* : v \cdot x = a\}.$$

Given $\sigma \in \mathfrak{W}$, consider the Weyl group conjugate σH of H . Note that σH is the hyperplane defined by the normal vector σv and shift $a \in k^*$, and $|H| = |\sigma H|$. The representation defined by the action of $\mathfrak{W}$ on $\mathfrak{h}^*$ is irreducible [9, Lemma 14.31], and so the linear span of $\mathfrak{W}v$ is all of $\mathfrak{h}^*$ as it is a non-zero $\mathfrak{W}$ -invariant subspace. This implies that the number of linearly independent vectors of the form σv with $\sigma \in \mathfrak{W}$ is $\dim \mathfrak{h}^* = n-1$. Let $\sigma_1 v, \dots, \sigma_{n-1} v$ be such linearly independent vectors and define $H_i := \sigma_i H$. Note that

$x \in H_1 \cap \cdots \cap H_{n-1}$ if and only if $(\sigma_i v) \cdot x = a$ for all $1 \leq i \leq n-1$. The latter is a system of $n-1$ linearly independent equations, so its solution is a point in $\mathfrak{h}^*$. That is,

$$H_1 \cap \cdots \cap H_{n-1} = \{x\}$$

for some $x \in \mathfrak{h}^*$. We use the inclusion-exclusion principle to compute a bound on $\text{mult}(x)$. Since $\dim V \geq |H_1 \cup H_2|$, we get that

$$|H_1 \cap H_2| \geq |H_1| + |H_2| - \dim V > (2c_n - 1) \dim V.$$

One can show via induction that

$$|H_1 \cap \cdots \cap H_k| > (kc_n - (k-1)) \dim V$$

for all $1 \leq k \leq n-1$. Then

$$|\{x\}| = |H_1 \cap \cdots \cap H_{n-1}| > ((n-1)c_n - (n-2)) \dim V,$$

together with Lemma 4.10.5, tells us that

$$((n-1)c_n - (n-2)) \dim V < \text{mult}(x) \leq \frac{\dim V}{n}.$$

Since $(n-1)c_n - (n-2) < (1/n)$ has no solutions, this gives a contradiction. □

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APPENDICES

Appendix A

Macaulay2 code for the toral method

The following pages contain Macaulay2 code that may be used to determine if a given representations of the form

$$\mathrm{Sym}^a \mathrm{SL}_n \otimes \mathrm{Sym}^b \mathrm{SL}_m, \quad \mathrm{Sym}^a \mathrm{SL}_n \otimes \bigwedge^b \mathrm{SL}_m, \quad \bigwedge^a \mathrm{SL}_n \otimes \bigwedge^b \mathrm{SL}_m$$

is not pure via the toral method in Section [4.9.1](#). The end result outputted is either inconclusive or tells us that the given representation is not pure. This code is available by contacting the author directly.

```

1  -----
2  -- File: ToralMethod.m2
3  -- Author: Nicole Kitt
4  -----
5  -- Description: This file contains code that is used to determine whether a
6  -- given representation of  $SL_n \times SL_m$  is pure via the "toral method".
7  -- Currently, the code only works for representations that are of the form
8  --  $V \otimes W$ , where  $V$  and  $W$  are  $Sym^a$  or  $Wedge^b$  of the standard representation.
9  -----
10
11 -----
12 -- common dimension formulas
13 -----
14 -- Computes the dimension of the group  $SL_n \times SL_m$ 
15 dimG = (n,m) -> (n^2-1) + (m^2-1)
16
17 -- Computes the dimension of the representation  $Sym^a SL_n$ 
18 dimS = (a,n) -> binomial(n+a-1,a)
19
20 -- Computes the dimension of the representation  $Wedge^a SL_n$ 
21 dimW = (a,n) -> binomial(n,a)
22
23 -----
24 -- specific partition functions + generating toral elements
25 -----
26 -- Outputs list of all sequences  $(e_0, \dots, e_{r-1})$  where  $e_i \in \mathbb{N}$  and  $\sum_i e_i = n$ .
27 -- i.e., "Par" outputs a list of r-tuples which are partitions of n into r things.
28 -- This function is the basis for enumerating all toral elements of  $SL_n$ , up to
29 -- the action of Weyl group, of some fixed order r.
30 Par = (n,r) -> (
31   if r==1 then {sequence(n)}
32   else if r>=2 then (
33     L := {};
34     for e in 0..n do {
35       P := Par(n-e,r-1);
36       N := #P;
37       for i in 0..(N-1) do ( L = append(L, sequence(e)|Pi) );
38     }
39   )
40 )
41
42 -- Outputs ord( $w^k$ ) where w is a primitive rth root of unity and k is non-negative integer.
43 RootUnityOrd = (k,r) -> (
44   r/gcd(r,k)
45 )
46
47
48
49
50

```

```

51 -- Enumerates all toral elements of  $SL_n$ , up to action of Weyl group, of some fixed order  $r$ .
52 -- Output is a list of  $r$ -tuples  $(e_0, \dots, e_{r-1})$ , where  $e_i$  is the number of  $w^i$ 's in the
53 -- toral element  $(w^0, \dots, w^0, w^1, \dots, w^1, w^{r-1}, \dots, w^{r-1})$ .
54 Toral = (n,r) -> (
55   if r==1 then {sequence(n)} -- only toral element of order 1 is nxn identity
56   else if r>=2 then (
57     F := {}; -- this will be the first initial list of possible toral elements
58     P := Par(n,r);
59     N := #P;
60     for i in 0..(N-1) do {
61       if sum(1..(r-1), k -> k*(P_i)_k)%r == 0 then F = append(F, P_i); -- F contains tuples
whose corresponding matrix has det = 1
62       T := {}; -- this will be the final list that contains all toral elements or order r
63       M := #F;
64       for i in 0..(M-1) do { -- this loop picks out elements of F that have order r
65         SeqOrds := (); -- this will be the sequence whose entries are the orders of diagonal
entries of matrix corresponding to F_i, no repeats
66         for k in 0..(r-1) do {
67           if (F_i)_k != 0 then SeqOrds = append(SeqOrds, RootUnityOrd(k,r)); -- gathering
orders of diagonal elements
68           if lcm(SeqOrds) == r then T = append(T, F_i); -- need the least common multiple of the
orders of all diagonal entries to be r
69         };
70       }
71     }
72 )
73
74 -- Outputs a list of all possible (r-1)-tuples in which the 1st entry  $k_0$  is the first entry
from an element of Par(a,2), 2nd entry  $k_1$  is the first entry from an element of Par(a-
 $k_0$ ,2), 3rd entry  $k_2$  is the first entry from an element of Par(a- $k_0$ - $k_1$ ,2), ect.
75 -- Tuple is of form  $(k_0, \dots, k_{r-2})$  and function only works for  $r \geq 2$ .
76 MulPar = (a,r) -> (
77   J := Par(a,r);
78   N := #J;
79   if r==2 then (
80     L := {};
81     for i in 0..(N-1) do { L = append(L, sequence((J_i)_0) );
82     }
83   )
84   else if r>=3 then (
85     L' := {};
86     for q in 0..a do {
87       J' := MulPar(a-q,r-1);
88       N' := #(J');
89       for i in 0..(N'-1) do ( L' = append(L', sequence(q)|(J')_i)
90     );
91     }
92   )
93 )
94
95 -- Outputs all possible pairs (a,b) such that  $\text{lcm}(a,b) = r$  and  $1 \leq a, b \leq r$ .

```

```

96 -- Used to generate toral elements of SL_n x SL_m of order r.
97 ToralOrderPairs = (r) -> (
98   L := {};
99   for a in 1..r do
100     for b in 1..r do
101       {if lcm(a,b) == r then L = append(L, (a,b))};
102     L
103   )
104
105 -----
106 -- eigenspace dimensions
107 -----
108 -- computes the dimension of (Sym^a SL_n)^{w^i}, e=(e_0,...,e_{r-1}), 0<=i<=r-1, r>=1.
109 dimSE = method(Dispatch => {Thing, Thing, Thing, Thing});
110 dimSE(ZZ,ZZ,Sequence,ZZ) := (a,n,e,i) -> (
111   L := {};
112   r := #e;
113   if r==1 then (if i==0 then return dimS(a,n) else return "this eigenspace does not exist") -
114   -- here e=(e_0), so only one e.space
115   else if r>=2 then (
116     lastEntry := (e_{r-1})_0; -- last entry in the sequence e
117     MP := MulPar(a,r);
118     N := #MP;
119     for j in 0..(N-1) do (
120       Seq := (MP)_j;
121       num := #Seq; -- length of the sequence Seq, this is just r-1 though
122       SeqS := sum(0..(num-1), k -> Seq_k); -- summing all entries of the sequence Seq together
123       t := (a - SeqS);
124       if (sum(0..(r-2), k -> k*(Seq_k)) + (r-1)*t)%r == i then L = append(L, product(r-1, k ->
125         dimS(Seq_k,e_k))*dimS(t,lastEntry)) -- product goes 0 to r-2
126       );
127     );
128     sum L
129   )
130
131 -- computes the dimension of (Wedge^a SL_n)^{w^i}, e=(e_0,...,e_{r-1}), 0<=i<=r-1, r>=1.
132 dimWE = method(Dispatch => {Thing, Thing, Thing, Thing});
133 dimWE(ZZ,ZZ,Sequence,ZZ) := (a,n,e,i) -> (
134   L := {};
135   r := #e;
136   if r==1 then (if i==0 then return dimW(a,n) else return "this eigenspace does not exist")
137   else if r>=2 then (
138     lastEntry := (e_{r-1})_0;
139     MP := MulPar(a,r);
140     N := #MP;
141     for j in 0..(N-1) do (
142       Seq := (MP)_j;
143       num := #Seq;
144       SeqS := sum(0..(num-1), k -> Seq_k);
145       t := (a - SeqS);

```

```

144     if (sum(0..(r-2), k -> k*(Seq_k)) + (r-1)*t)%r == i then L = append(L, product(r-1, k ->
dimW(Seq_k,e_k))*dimW(t,lastEntry))
145   );
146   );
147   sum L
148 )
149
150 -----
151 -- toral method functions
152 -----
153 -- applying toral method of order r to representation Sym^a SL_n \tensor Sym^b SL_m.
154 ToralBoundSS = (a,n,b,m,r) -> (
155   L := {}; -- this will be the list of t satisfying inequality below; if empty can deduce not
pure.
156   P := ToralOrderPairs(r); -- possible pairs (x,y) such that ord(t_1) = x and ord(t_2)=y and
ord(t)=r
157   N := #P;
158   for i in 0..(N-1) do { -- going through each possible pairing (x,y)
159     ord1 := (P_i)_0; -- order of t_1, which is x
160     ord2 := (P_i)_1; -- order of t_2, which is y
161     Toral1 := Toral(n,ord1); -- list of possible toral elements of order x
162     Toral2 := Toral(m,ord2); -- list of possible toral elements of order y
163     num1 := #Toral1; -- number of toral elements of order x
164     num2 := #Toral2; -- number of toral elements of order y
165     for i in 0..(num1 -1) do -- checking inequality on each toral element t = (t_1,t_2)
166       for j in 0..(num2 -1) do
167         {Seq1 := Toral1_i; -- t_1
168         Seq2 := Toral2_j; -- t_2
169         J := {}; -- sum this list later to get dimension of fixed pts of t
170         for k in 0..(ord1 - 1) do
171           for l in 0..(ord2 - 1) do
172             {if (ord2*k + ord1*l)%(ord1*ord2) == 0 then J =
append(J,dimSE(a,n,Seq1,k)*dimSE(b,m,Seq2,l)); -- computing dim of e.spaces
173             J = sum J; -- dimension of fixed pts of t
174             if J >= dimS(a,n)*dimS(b,m) - dimG(n,m) - 1 then L = append(L,{Seq1,Seq2, ord1,ord2});
175             };
176           };
177       L -- If L = {}, then representation is not pure
178     }
179
180 -- applying toral method of order r to representation Sym^a SL_n \tensor Wedge^b SL_m.
181 ToralBoundSW = (a,n,b,m,r) -> (
182   L := {}; -- this will be the list of t satisfying inequality below; if empty can deduce not
pure.
183   P := ToralOrderPairs(r); -- possible pairs (x,y) such that ord(t_1) = x and ord(t_2)=y and
ord(t)=r
184   N := #P;
185   for i in 0..(N-1) do { -- going through each possible pairing (x,y)
186     ord1 := (P_i)_0; -- order of t_1, which is x
187     ord2 := (P_i)_1; -- order if t_2, which is y
188     Toral1 := Toral(n,ord1); -- list of possible toral elements of order x

```

```

189   Toral2 := Toral(m,ord2); -- list of possible toral elements of order y
190   num1 := #Toral1;
191   num2 := #Toral2;
192   for i in 0..(num1 -1) do -- checking inequality on each toral element
193   for j in 0..(num2 -1) do
194     {Seq1 := Toral1_i; -- t_1
195      Seq2 := Toral2_j; -- t_2
196      J :={};
197      for k in 0..(ord1 - 1) do
198      for l in 0..(ord2 - 1) do
199        {if (ord2*k + ord1*l)%(ord1*ord2) == 0 then J =
append(J,dimSE(a,n,Seq1,k)*dimWE(b,m,Seq2,l)); -- computing dim of e.spaces
200         J = sum J; -- dimension of fixed pts of irrep under t
201         if J >= dimS(a,n)*dimW(b,m) - dimG(n,m) - 1 then L = append(L,{Seq1,Seq2, ord1,ord2});
202         };
203       };
204     L -- If L = {}, then representation is not pure
205   }
206
207 -- applying toral method of order r to representation Wedge^a SL_n \tensor Wedge^b SL_m.
208 ToralBoundWW = (a,n,b,m,r) -> (
209   L := {}; -- this will be the list of t satisfying inequality below; if empty can deduce not
pure.
210   P := ToralOrderPairs(r); -- possible pairs (x,y) such that ord(t_1) = x and ord(t_2)=y and
ord(t)=r
211   N := #P;
212   for i in 0..(N-1) do { -- going through each possible pairing (x,y)
213     ord1 := (P_i)_0; -- order of t_1, which is x
214     ord2 := (P_i)_1; -- order if t_2, which is y
215     Toral1 := Toral(n,ord1); -- list of possible toral elements of order x
216     Toral2 := Toral(m,ord2); -- list of possible toral elements of order y
217     num1 := #Toral1;
218     num2 := #Toral2;
219     for i in 0..(num1 -1) do -- checking inequality on each toral element
220     for j in 0..(num2 -1) do
221       {Seq1 := Toral1_i; -- t_1
222        Seq2 := Toral2_j; -- t_2
223        J :={};
224        for k in 0..(ord1 - 1) do
225        for l in 0..(ord2 - 1) do
226          {if (ord2*k + ord1*l)%(ord1*ord2) == 0 then J =
append(J,dimWE(a,n,Seq1,k)*dimWE(b,m,Seq2,l)); -- computing dim of e.spaces
227           J = sum J; -- dimension of fixed pts of irrep under t
228           if J >= dimW(a,n)*dimW(b,m) - dimG(n,m) - 1 then L = append(L,{Seq1,Seq2, ord1,ord2});
229           };
230        };
231       L -- If L = {}, then representation is not pure
232     }

```