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*Highlights (for review)

1-The multi-scale entropy analysis can unveil the self-similarity in time series.
2-The result of multi-scale entropy for fractional Gaussian noise is well modeled by a decreasing q-exponential function.
3-The Hurst exponent of a time series can be determined by the multi-scale entropy analysis.

Multi-scale entropy analysis and Hurst exponent

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Abstract

Several methods exist for measuring the complexity in a system through analysis of its associated time series. Multi-scale entropy appears as a successful method on this matter. It has been applied in many disciplines with great achievements. For example in analysis of the bio-signals, we are able to diagnose various diseases. However, in most versions for the multi-scale entropy the examined time series is analyzed qualitatively. In this study, we try to present a quantitative picture for the multi-scale entropy analysis. Particularly, we focus on finding relation between the result of the multi-scale analysis and the Hurst exponent which quantifies the persistence in time series. For this purpose, the fractional Gaussian noise time series with different Hurst exponents are analyzed by the multi-scale entropy method and the results are fitted to a decreasing q-exponential function. We observe remarkable relation between the function parameters and Hurst exponent. This function can simulate the result of analysis for the white noise to the $1/f$ noise.

Keywords: Multi-scale Entropy, Fractional Gaussian Noise, Hurst Exponent

1. Introduction

Time series give many information about the examined system. Such information are mostly coded as self-similar patterns in time series. The ex-

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istence of these patterns is usually due to the long range interaction between system components and/or long term memory in their dynamics. Every pattern exhibits an order or regularity in the system and allows us to predict the system's behavior statistically. This is what most of the scientists called complexity [1]. Albeit, some other definitions of the complexity exist which differ with the above explanation. Most of them deal with the amount of information that we need for understanding of the system behavior. Regarding to this perspective, the concept of entropy appears as a simple and powerful measure for quantifying the complexity. It can be found by simple googling many literature which discuss alternative methods for entropy estimation of the complexity in real world data. Here we try to reconcile these two treatments of the complexity.

Several methods have been proposed for identification of the self-similar pattern in time series [2]. The self-similarity may exist in time series graph which be revealed by the fractal and multi-fractal analysis [3, 4]. The scale-free distribution of values is another indication for self-similarity in time series [5]. The power law relationship for the auto-correlation function is the most well known feature which represents the self-similar pattern in time series. It deserves to note that the above aspects of self-similarity may be related to one another.

For the first time, It was H. E. Hurst a British hydrologist who pointed out to the self-similarity of a time series while studying on the optimum dam size for the Nile river in 1951. He developed a method for measuring the self-similarity which is known as rescale range analysis (RS) [6]. The result of RS analysis is expressed as a value between 0 and 1, which is called the Hurst exponent. There are many other methods that directly or indirectly compute the Hurst exponent for a time series. Now, we only name some of them and refer the interested readers to literature [2, 7]. As some instances, we briefly point out to the following methods, detrended fluctuation analysis (DFA) [8, 9, 3], power spectral analysis and its variants [10], wavelet method [11], methods based on the complex networks theory [12] (see also it's references).

In this work, we are intended to show that the multi-scale entropy (MSE) [13, 14, 15, 16, 17, 18, 19], can be also used for estimation of the Hurst exponent. For this end, we apply the MSE method to analyze the fractional Gaussian noise (FGN) time series with various Hurst exponents. Then we fit the obtained results by the decreasing q-exponential as a model function. We will find that the value of parameters are nicely related to the Hurst exponents.

We organize this paper as follows. In the next section we briefly review the fractional Gaussian noise and meaning of the Hurst exponent. The basic variant of multi-scale entropy is discussed in the third section. Fourth section is devoted to the results and their interpretation. We summarize our work at the final section.

2. Fractional Gaussian Noise

The fractional Brownian motion (FBM) is one of the most well known stochastic processes which has been widely studied analytically [20]. It is used in modelling various phenomena in science and engineering [21, 22]. Researchers are interested in studying and using FBM for its properties like as self-similarity [21, 22].

The one dimensional FBM which we denoted it as $B_H(t)$, is a non-stationary stochastic process which starts at zero, $B_H(0) = 0$. The process is known to have zero means, $\langle B_H(t)\rangle = 0$, and Gaussian distribution for its increments. The auto-covariance function of the process is,

$$\langle B_H(t+\tau)B_H(t)\rangle = \frac{1}{2}\left(|t+\tau|^{2H} + |t|^{2H} - |\tau|^{2H}\right). \qquad (1)$$

For simplicity, we assume that $\langle B_H^2(1)\rangle = 1$. The parameter H is the Hurst exponent, a real value between 0 and 1 which determines the persistence of process. For $H = 0.5$ we have ordinary Brownian motion or Wiener process.

Dependence of auto-covariance function on t is the reason for non-stationary feature of FBM. By using equation 1 in the case $\tau = 0$, we obtain, $< B_H^2(t) >= t^{2H}$ or $< B_H^2(at) >= a^{2H} < B_H^2(t) >$. By simple calculation it can be generalized to all other moments. This result indicates that the FBM process is self-similar in distribution, $B_H(at) \overset{d}{=} a^H B_H(t)$.

As we mentioned earlier, FBM time series has non-stationary character and is not suitable for modelling the stationary processes. The increments of FBM as denoted as, $\Delta B_H(t) = B_H(t+\Delta t) - B_H(t)$, define another stochastic process, named as fractional Gaussian noise (FGN). By using the auto-covariance function of FBM and some algebraic manipulation, we can prove that the auto-covariance of this new process is,

$$\langle \Delta B_H(t+\tau)\Delta B_H(t)\rangle = \frac{1}{2}\left(|\tau+\Delta t|^{2H} + |\tau-\Delta t|^{2H} - 2|\tau|^{2H}\right). \qquad (2)$$

In the case, $\tau \gg \Delta t$, we can approximate the auto-covariance function as,

$$\langle \Delta B_H(t+\tau)\Delta B_H(t)\rangle \sim H(2H-1)\tau^{2H-2}. \quad (3)$$

The case $H > 0.5$ demonstrates the existence of long range dependence in series. The FGN also inherits the property of self-similarity from its parent; by putting $\tau = 0$ in equation 2 we arrive at the desired result for second moment. The Gaussain distribution of $\Delta B_H(t)$ allows us to generalize the obtained result easily to all other moments.

It is important to note that the Hurst exponent is an indicator for both self-similarity and long range dependence in FGN time series.

3. Multi-scale Entropy

In statistical mechanics, entropy is a quantity for measuring disorder in system. A system may have many microscopic states and each state has certain probability of occurrence. The goal of statistical mechanics is to predict these probabilities. All the entropies like as Gibbs-Shannon , Renyi and Tsallis, are expressed in terms of the state probabilities.

When disorder is increased in the system, this means that most of the states are likely to occur then it is so hard to predict the state of system. Entropy also increases in this case. If disorder decreases, some states will be preferred and system becomes predictable, therefore entropy is decreased. The spatial and temporal patterns are indication of the regularity or order in a system. Any regularity makes the system predictable. Entropy can measure the amount of predictability or in other word the complexity of system.

The most challenging problem is the estimation of entropy for a given system from its time series data. Here we only focus on the multi-scale entropy analysis which is the most powerful method for investigating the complexity of time series. In following we describe the method and refer the interested readers for convincing statements to Refs. [13, 14].

Assume, we have a time series of length N which is denoted by series of values, $\{x_1, \ldots, x_N\}$.

The interaction between system and its environment may induce noises in the system time series. Short range correlations in noises can be accumulated and make a long range effect which is non-original. In the first step, it is necessary to reduce the effect of unwanted noises and short range correlations from signal by the coarse-graining procedure. In this step, we partitioned time series into non-overlapping windows with equal length τ, then the coarse grained time series is constructed by averaging on data in each window,

$$x_i^{(\tau)} = \frac{1}{\tau}\sum_{j=1}^{\tau} x_{(i-1)\tau+j} \tag{4}$$

and putting them in a new sequence, $\{x_1^{(\tau)}, \ldots, x_{N/\tau}^{(\tau)}\}$. In literature, τ is called the scale factor.

For the resulted time series we can define the $m-$dimensional vectors like as,

$$\mathbf{X}_m^{(\tau)}(i) = \{x_i^{(\tau)}, \ldots, x_{i+m-1}^{(\tau)}\}. \tag{5}$$

In the next step, we count the number of vector pairs that have distance less than r and denote it by $n_m(r,\tau)$. We repeat the same computation for $(m+1)-$dimensional vector pairs and obtain $n_{m+1}(r)$. The sample entropy is defined as,

$$S_E(m,r,\tau) = -\log(n_{m+1}(r,\tau)/n_m(r,\tau)). \tag{6}$$

It is clear that the sample entropy has zero or positive value, because $n_{m+1}(r,\tau)$ is less than $n_m(r,\tau)$.

In final step we plot the sample entropy against the scale factor. For the white noise we observe a decreasing behavior and it is proved analytically. For the short range correlated signals it is expected to find the same behavior as the white noise since coarse-graining procedure eliminates the short range correlations. The long range dependence in time series does not change after the coarse-graining operation then for $1/f$ noise the sample entropy does not vary with scale factor and remains constant. In the next section we analysis the FGN data series with different Hurst exponent by MSE method.

4. Results

First of all, we should generate the needed FGN series with different range of Hurst exponent for our analysis. The FGN samples are obtained through differencing of the fractional Brownian motion data. There are several ways for generating the Brownian motion time series with given Hurst exponent. Among them, we choose three algorithms; Hosking [23], random midpoint displacement [24] and Rambaldi-Pinazza [25]. These algorithms employ different approaches for doing this. The Hosking algorithm uses the

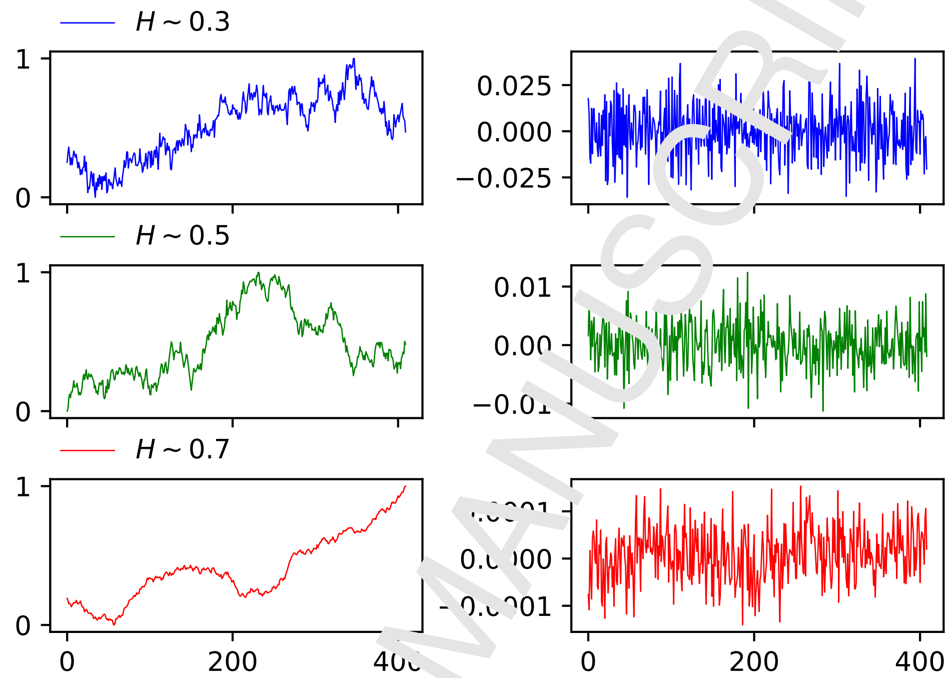

Figure 1: Part of the FBM (left) and FGN (right) time series for three values of the Hurst exponent, $\sim 0.3, \sim 0.5$ and ~ 0.7. For comparison purpose, the FBM time series are scaled to interval $[0, 1]$. Time series with small Hurst exponent is more denticulated.

FBM covariance property. In the random midpoint displacement method the self similarity and Gaussian distribution of increments are used and finally the Rambaldi Pinazza algorithm considers the integral representation of FBM. Thus, we construct 48 samples with Hurst exponent between 0.15 and 0.9 and size of 65535. In construction procedure all data are scaled between 0 and 1. This scaling does not alter the Hurst exponent. Figure 1 shows the plot for FBM and its associated FGN series for three values of the Hurst exponent, $\sim 0.3, \sim 0.5$ and ~ 0.7. We observe that by increasing the Hurst exponent the difference between adjacent values in FBM time series decreases. It should be noted that the procedure of generating FBM time series is not completely exact. Therefore it is necessary to measure the Hurst exponent again for all samples data. Here we use DFA for this purpose.

Before analyzing the prepared FGN data by MSE, we should set two parameters, m and r. The first one determines the dimension of vectors, see equation 5, and the latter defines a threshold for closeness of vectors. Here

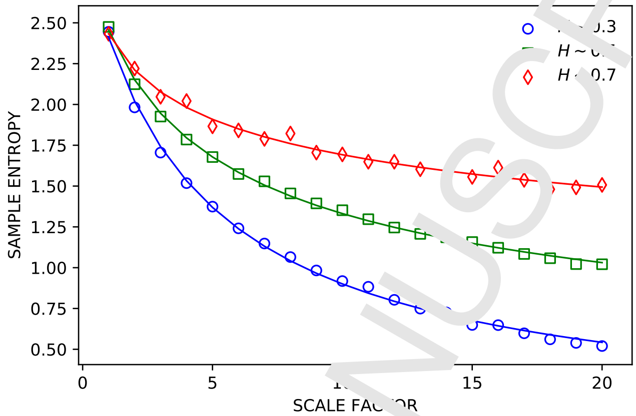

Figure 2: The MSE analysis for three sample time series with the Hurst exponent, $\sim 0.3, \sim 0.5$ and ~ 0.7. The plot for time series with small Hurst exponent decreases more rapidly than others. The results of fitting with the decreasing q-exponential are plotted by solid lines. The values of q parameter for these time series are $2.07, 3.42$ and 6.35 respectively.

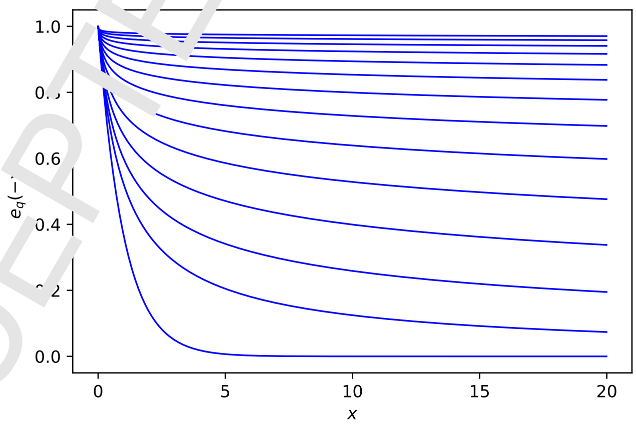

Figure 3: The decreasing q-exponential function for wide range of q from 1 to 400. The large value of q mimics the MSE result for $1/f$ noise.

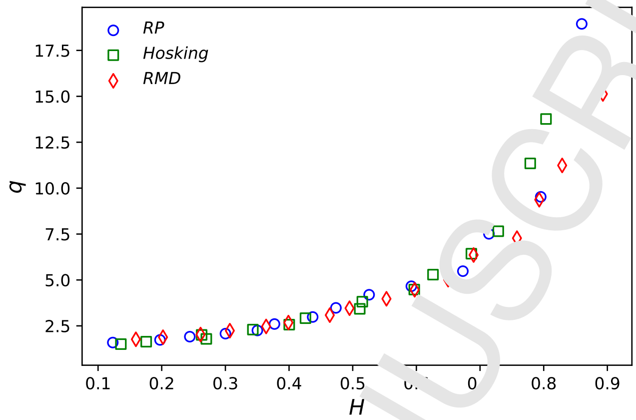

Figure 4: The parameter q against the Hurst exponent. The value of q rapidly grows by increasing the Hurst exponent, which ensures us for simulating the behavior of $1/f$ noise.

we put $m = 2$ and $r = 0.15\sigma$ where σ is the standard deviation of time series. In figure 2 we plot the result for three time series with the Hurst exponent equal to, $\sim 0.3, \sim 0.5$ and ~ 0.7. As is seen in this figure, all plots show decreasing behavior but for large value of the Hurst exponent, this behavior is more gentle.

This behavior could be justified by closer look at the different stages of the MSE calculation. In the coarse graining procedure, we elimiate the short range correlated (high frequency) noises in time series, hence the variance of the resulting coarse grained time series decreases. Since r is constant, decreasing in variance causes that $n_{m+1}(r,\tau)$ tends to $n_m(r,\tau)$ and consequently the sample entropy decreases too [18]. It is expected that the time series with lesser Hurst exponent shows more decrease in the variance and also in the sample entropy because the high frequency noises are dominated for lower values of H.

It is important to mention that the FGN series with $H = 0.5$ is the same as the white noise and as we expected, its result completely overlaps with the white noise result.

In order to find the relation between the MSE analysis and the Hurst exponent, it is required to model the result of analysis with a model function. This function should be well fitted to all the FGN results only by changing some parameters. The number of these parameters must be at minimum. Here we choose, the following function which is defined in terms

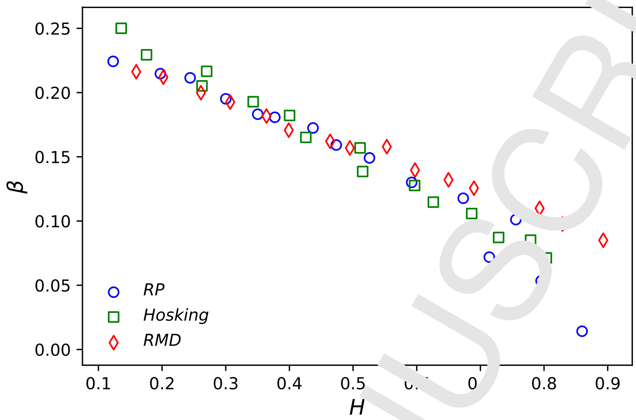

Figure 5: The parameter β against the Hurst exponent. The value of β decreases by increasing the Hurst exponent. It seems that for large value of the Hurst exponent, this decreasing becomes more rapidly.

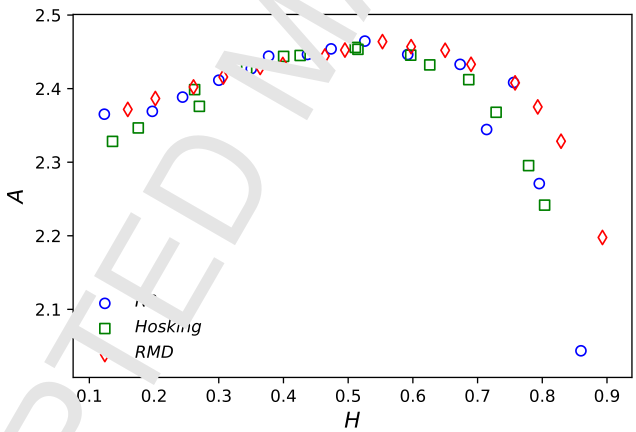

Figure 6: The parameter A against the Hurst exponent. The behavior of A is different for $H < 0.5$ and $H > 0.5$. First it increases slowly then rapid decreasing is observed.

of q-exponential function,

$$f(x; A, q, \beta) = Ae_q(-\beta x) = A[1 - \beta(q-1)x]^{\frac{1}{1-q}} \tag{7}$$

A, q and β are three real positive numbers. q must be one or greater than one to have decreasing behavior for the model function. In figure 2 we fit the results of analysis for the above mentioned three sample time series by this model function. As is seen, the analysis results are well fitted to the

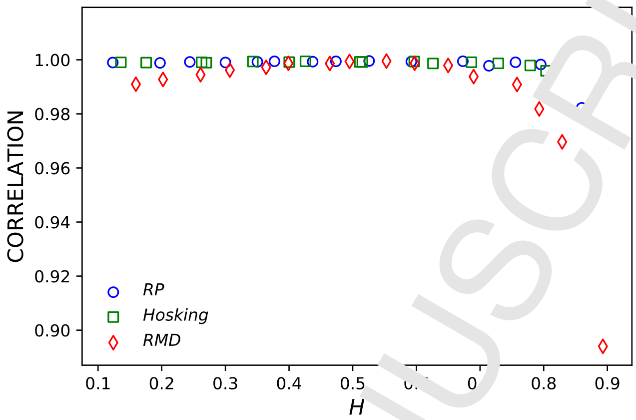

Figure 7: The correlation measure of fitting procedure. The MSE analysis results for time series that are generated by the Hosking algorithm and the Rambaldi-Pinazza method are well fitted by our suggested function.

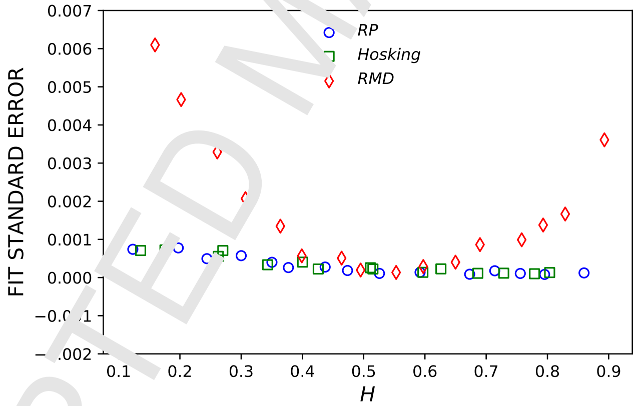

Figure 8: The fit standard error measure of fitting procedure. The MSE analysis results for time series that are generated by the Hosking algorithm and the Rambaldi-Pinazza method are well fitted by our suggested function.

suggested function. The values of q parameter are $2.07, 3.42$ and 6.35 for time series with the Hurst exponent $\sim 0.3, \sim 0.5$ and ~ 0.7 respectively. The larger Hurst exponent leads to the greater value of q parameter and the gentle decrease of the model function.

One advantage of using such function, is mimicking behavior of the MSE results from white noise to $1/f$ noise by only one function. This statement is illustrated graphically in figure 3. By increasing the value of q, the function becomes close to the horizontal straight line which is the behavior of the obtained result for the $1/f$ noise.

We fit all the results that obtained from the MSE analysis of the FGN time series by the mentioned function. Figures 4 and 5 show the fit parameters, q and β against the Hurst exponent. We observe increasing behavior for q and decreasing behavior for β. The parameter q varies between 1.5 for $H = 0.136$ to 18.9 for $H = 0.86$. Simultaneously, β changes in the range of 0.25 to 0.01. The parameter A softly increases from 2.3 to 2.45 for $H < 0.5$ then rapid decreasing can be observed, see the figure 6. We know that the FGN process, when $H \to 1$ simulates the $1/f$ noise. It can be understood through the large value of q and small value of β for time series with large Hurst exponent, because the resulting model function becomes very close to the horizontal straight line.

The fit standard error and correlation are two important measures which determine the goodness of fitting. It is worth to note that both of them indicate that all the curve fittings are well done. The correlation coefficient for nearly all data is more than 0.99 (see figure 7) and the fit standard error for most of the fitting procedures are less than 0.001 (see figure 8). The results for time series which were generated by the Hosking algorithm and the Rambaldi-Pinazza method are better fitted to the model function than others.

Finally, we mention that for the MSE analysis we use the program written by M. Costa *etal.*, which can be downloaded from physionet.org [26]. In fitting procedure we benefit from the curve fitting function in SciPy which is the Python library for the scientific and technical computing.

5. Conclusion

The MSE analysis is favorable way for the abnormally detection in physiological time series. It has been widely used for analysis of the heart beats and the electroencephalogram time series. Many diseases were distinguished by such analysis without any need to use the invasive methods. This way of the analysis of complexity has been applied to technical time series too. In this study, we were interested to answer a fundamental question, how MSE can be used for detection of the long range dependence in time series. In other word, we were looking to explore the relation between the MSE analysis result and the Hurst exponent of a time series. We chose the FGN time series with different Hurst exponent as our data samples for analysis. It was found that by the function, $Ae_q(-\beta x)$, all the results for the FGN time series can be modeled. We explored the nice relation between parameters of model function; q, β and A, with the Hurst exponent. We also reported the curve fitting measures which demonstrated that the well fitting were done. In future, we would interested to find the effect of trends on the result of the MSE analysis and apply it on natural time series, particularly the daily temperature data and finding a way for the climate classification.

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