

Surface Wave Propagation using Ray Optics with Applications to Hamilton Harbour

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

Surface gravity waves are a commonly observed phenomenon in closed lakes. They are primarily generated by winds, but may also be generated by wakes of vessels in the water or tidal forces. These waves are known to undergo refraction and changes in energy (and therefore, amplitude and speed) as they propagate over variable depths in a closed lake.

In this thesis, we use the linearized ray optics equations to study the propagation of surface waves in closed lakes with variable bathymetry. We assume that the fluid is inviscid, irrotational and incompressible. We further assume that the wavelength of these waves is much smaller than the length scales over which bathymetry (depth) varies – in other words, we assume the waves propagate in a *slowly varying environment*.

We perform a series of numerical simulations of surface wave propagation with various initial conditions and bathymetries to validate the model, as well as gain insights on the effects of roughness of the bathymetry in diverting the expected trajectories of these waves.

Finally, we present a case study on Hamilton Harbour. This watershed has been under scrutiny for several years due to rising eutrophication levels affecting the dissolved oxygen levels in the basin. We apply the numerical model to the bathymetry of Hamilton Harbour to locate hotspots of wave accumulation in the lake, which may provide inference of regions in the lake where we expect sediment resuspension which is one of the primary factors of internal loading of phosphorus in lakes.

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Chapter 1

Introduction

Water waves have been of interest for a long period of time. Wave phenomena have been observed since a long time – particularly in the form of surface waves on water. These waves are typically generated by wind-induced shear at the air-water interface [1]. This thesis attempts to add to long-standing literature on modeling surface wave phenomena in oceans and lakes. In particular, we are interested in the problem of wave propagation and refraction. These topics have been explored using a mathematical lens since as early as the 1600s, with the 1800s representing important milestones in the development of contemporary mathematical theories [2, 3].

1.1 A Brief History of Water Wave Theory

The first notions of a mathematical theory of waves were presented in Isaac Newton’s *Principia* [2]. He argued correctly that the frequency of deep-water waves was directly proportional to the “square root of the breadth of the wave” (in other words, square root of the wavelength) [2]. Much later, Leonhard Euler derived the equations of hydrodynamics in his 1757 article [4]. Laplace and Lagrange both worked on reexamining wave motion and deriving linearized wave equations for small-amplitude waves, and deriving the solution for the equations in the shallow-water or long-wave approximation [5–8]. Lagrange [7] derived the expression for the phase speed of water waves in the shallow-water approximation, $c = \sqrt{gH}$, where g is the gravitational acceleration and H is the depth of water. The shallow-water approximation uses the “method of parallel sections,” where the liquid throughout its depth moves at the same velocity, but this velocity and depth is allowed to vary slowly, first applied by Daniel Bernoulli in *Hydrodynamica* [2, 9].

The 1800s marked an important century in the field of the development of fluid dynamics and in particular, the study of wave motion [2]. Gerstner [10] derived the first exact nonlinear solution for waves of finite amplitude in deep water [2], although, the solution was not for an irrotational wave which are not generated by conservative forces. A major milestone in wave theory was the analysis done somewhat independently by Poisson and Cauchy in the years 1818 and 1827 respectively [2, 11]. This work dealt with the general initial-value problem of linearized water waves with two-dimensional disturbances [2]. At the time, the usage of excessive mathematical rigor involving Fourier transforms (as they are called now) and asymptotic approximations was not appreciated by the academic community, but it provided a benchmark for many techniques used today in this field [2]. While theory was seeing progress, so were experiments. In his book, Brémontier [12] reports on practical observations of wave interference, reflection, and sediment transport [2]. In a book published in 1825, Weber and Weber [13] report on laboratory experiments in plane wavetrains in a channel, along with comments on theoretical work including work authored by Laplace (1776), Gerstner (1804) and Brémontier (1809) [2].

We witnessed one of the most important contributions to the mathematical theory of water waves in the form of a long article by George Biddell Airy in 1841 [2, 14]. Titled “Tides and Waves”, its primary attempt was to investigate tidal phenomena. However, Airy wrote a section on the “Theory of Waves in Canals” within the work [2, 14]. In his work, Airy uses potential theory to derive the standard linear theory of plane waves for a fluid that is inviscid, incompressible and irrotational; yielding the general dispersion relation, given by,

$$\left(\frac{n}{k}\right)^2 = \frac{g}{k} \frac{e^{kH} - e^{-kH}}{e^{kH} + e^{-kH}}, \quad (1.1)$$

where n is the frequency, k the wavenumber and H the depth [2, 14]. Among other results, in his article Airy also investigated long nonlinear finite-amplitude waves, small-amplitude waves in a canal with friction, and commented on Webers’ and Russell’s experimental results [2, 14].

An overview of wave theories is incomplete without mentioning solitary waves. Although not a major component of this thesis, it is important to point out that in 1845, Russell [15] reported a remarkable series of observations in seas, rivers and canals as well his own lab tank. In particular, this work is famous for discovery of the nonlinear solitary wave [2]. This was followed by several attempts at modeling solitary waves by Phillip Kelland (1844), Samuel Earnshaw (1847), Boussinesq (1871) and Rayleigh (1876) before the seminal paper by Korteweg and de Vries came out in 1895. [2, 16, 17].

1.2 Waves in Nature

The study in this thesis is largely motivated by natural wave phenomena. Surface waves in nature are governed by three major processes. First, wind imparts momentum to the water through a turbulent process, which may be parameterized in many different ways [18]. This process is slow and relatively inefficient. The second major process is that of weak nonlinear interactions within the wave and with other waves, resulting in exchange of energy and growth of the wave [18]. The third major process is that of wave dissipation. Waves have an upper limit on wave height that they can achieve, primarily due to wave breaking [18]. Wave energy is proportional to the square of the wave height. So, when waves undergo breaking, their wave height is reduced by a substantial factor, resulting in a large loss of energy in the process through turbulence and bubbles under the ocean surface and exchange of fluid particles with the air [18]. Fig. 1.1 shows surface waves in the ocean shoaling and eventually experiencing breaking as they reach the shore. Once they break, some of the waves end up being reflected in the opposite direction. We also notice in this figure that near-shore bathymetries are devoid of symmetries, and give room to many three-dimensional motions.



Figure 1.1: Great Ocean Road, Victoria, Australia. Surface waves shoaling and breaking onto the shore [19]. This is known as whitecapping.

In addition to wind and nonlinear processes mentioned above, the propagation of waves is also heavily influenced by the ambient current in the fluid and the varying water depth [3, 20, 21]. Fig. 1.2 shows waves refracting as they enter a channel with varying depth and possibly a background current within the channel. In this work, we focus on refraction of

waves due to depth variations. In these channel-like flows, since the waves have already broken, it is less likely for the waves to undergo major dissipation again. As a result, flows like this are good contenders for linear theory.



Figure 1.2: Waves entering a channel – an example of refraction due to variable depth [22].

1.3 Numerical Wave Models

Modelling wave phenomena presents one of the most challenging aspects of fluid dynamics. Even in deep-water conditions with no background current, the influence of many physical factors, the turbulent nature of waves, and the moving free-surface of the fluid makes it difficult to represent and analyze them scientifically [1]. Numerical wave models can be broadly categorized into two types – phase-resolving (or direct) models and phase-averaged (or spectral) models [23].

The former solve fluid equations of mass and momentum balance – typically variants of the Navier-Stokes equations under the Boussinesq approximation, the Hamiltonian formulation of fluid equations, or the nonlinear shallow water equations – to directly approximate the flow fields and sea surface elevation [24–26]. These are able to resolve specific details about wave diffraction, refraction, reflection, nonlinear shoaling as well as wave breaking and wave-wave interactions, albeit with need for parameterization of some of these processes [23, 24]. See Fig. 1.3 for a schematic of the several physical processes that require modeling. These models are used for small-scale flows like flows within a confined harbour

or nearshore circulation [24]. For larger scales, these models are not practical due to the limitations of computational resources [23].

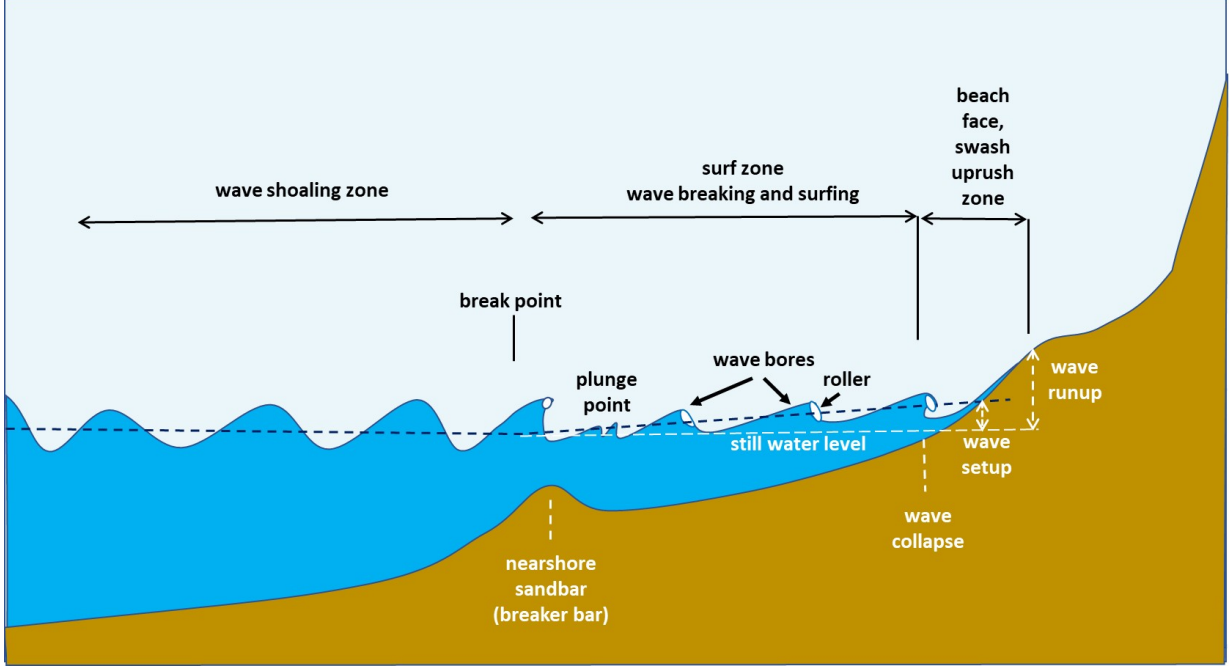


Figure 1.3: A schematic of important wave processes to be able to model in near-shore and oceanic waters [27].

Phase-averaged or spectral models, on the other hand, use the wave energy-balance or the wave action-balance equation to estimate the time evolution of the wave energy spectrum, i.e., the phase of individual waves is not tracked [23, 24]. This approach revolves around approximating the statistical properties of waves such as *significant wave height* or *peak period* and takes into account the stochastic processes associated with waves [3]. Spectral wave models have been through several generations of progress. First-generation models were developed throughout the 1960s and 1970s and employed simple parameterizations for wind-input along with a prescribed universal wave spectrum [24, 28]. Energy was artificially dissipated and there was no support for wave breaking and nonlinear wave-wave interactions [24]. The second-generation wave models from the early 1980s attempted to solve this by using simplified nonlinear parameterizations and variable wind fields [24, 28–30]. These models were still limited by the incomplete treatment of nonlinear energy transfer [24]. In the late 1980s and the 1990s, the third generation of spectral wave models arose which solved the wave-action balance equation explicitly with no a priori restrictions

on the shape of the spectrum. All physical processes mentioned in the paragraph above are parameterized, in addition to which, there is support for wave-wave interactions computed using the Discrete Interaction Approximation (DIA) [24, 28, 29, 31]. Examples of currently used third-generation wave models include WAM (Wave Model) [32], WAVEWATCH III [33], SWAN (Simulating Waves Nearshore) [28] and TOMAWAC [34].

While large-scale wave models have received much attention in the past, there have also been studies on smaller-scale phenomena such as wave-depth and wave-current interactions [3], particularly focused on refraction. Wave refraction is an important aspect of wave phenomena which is often parametrized in larger spectral wave models [20]. Due to recent advancements in computational resources, several studies on the effects of depth and currents on wave propagation have been undertaken in the last 50 years [20, 21, 23, 35–39]. Halsne et al. [20] developed an open-source package using Python which solve the ray optics equations in the presence of varying bathymetry and background currents. However, this package does not allow us to track wave amplitude as it propagates. While we must be wary of estimating wave amplitude in near-shore applications due to effects of wave-breaking taking over; in regions with finite depth where wave breaking and dissipation can be safely neglected, we can use the wave energy balance equation to calculate the wave energy and therefore, the wave amplitude for the waves being propagated. In the presence of a background current, the wave action generalized wave energy as the conserved quantity. Additionally, in the current work, we also study the sensitivity of wave refraction on rough bathymetries. We find that wave refraction is substantially sensitive to rough elevated surfaces in the seabed, and this must be taken into account while parameterizing these processes in spectral models.

1.4 Thesis Outline

The following material in this document is organized as follows. First, in Chapter 2 we go over the relevant theoretical background required for wave modeling – this includes the Navier-Stokes equations, and the linearized equations for surface gravity waves in a fluid with boundary conditions. We use the linearized wave equations to derive the general dispersion relation for surface waves and briefly discuss the deep-water and shallow-water approximations. We further derive the *ray optics* approximation for linear surface waves propagating in a *slowly* varying environment. This is followed by a discussion on contemporary wave models and their methods, and the section concludes with an introduction of a case study on the total phosphorus in Hamilton Harbour which will be a major part of the thesis.

Chapter 3 first validates the ray optics approximation. We then explore wave propagation over a radially symmetric bathymetry. We discuss refraction of waves in this setting. This is followed by investigating propagation over the same bathymetry with some sinusoidal perturbations added to it. We discuss the sensitivity of wave refraction to these perturbations. The final part of this chapter deals with applying the ray optics equations to the bathymetry of Hamilton Harbour as a real-world application.

Finally, chapter 4 draws primary conclusions regarding the utility of the ray optics equations in standard wave modeling, and also discusses the limitations of this method. There is an Appendix at the end of the thesis which lists the code used to perform these simulations.

Chapter 2

Theoretical Background

2.1 Fluid Equations

Fluids are complex physical systems. They are governed the Navier-Stokes equations – a system of equations representing the conservation of mass and momentum applied to fluid quantities [40].

There are some simplifying assumptions that can be made for water. We assume water to be an *incompressible* fluid, that is, the density does not vary with changes in pressure [40]. The conservation of mass equation, also known as the continuity equation for an incompressible fluid is,

$$\nabla \cdot \vec{u} = 0, \tag{2.1}$$

where $\vec{u}$ is the velocity field associated with the fluid. For incompressible fluids, the conservation of momentum equation is,

$$\frac{D\vec{u}}{Dt} = -\frac{1}{\rho}\nabla p + \vec{g} + \nu\nabla^2\vec{u}. \tag{2.2}$$

Here, ρ is the density, p is the pressure, $\vec{g}$ is the vectorized form of the acceleration due to gravity ($\vec{g} = -g\hat{k}$), and ν is the kinematic viscosity of the fluid [40].

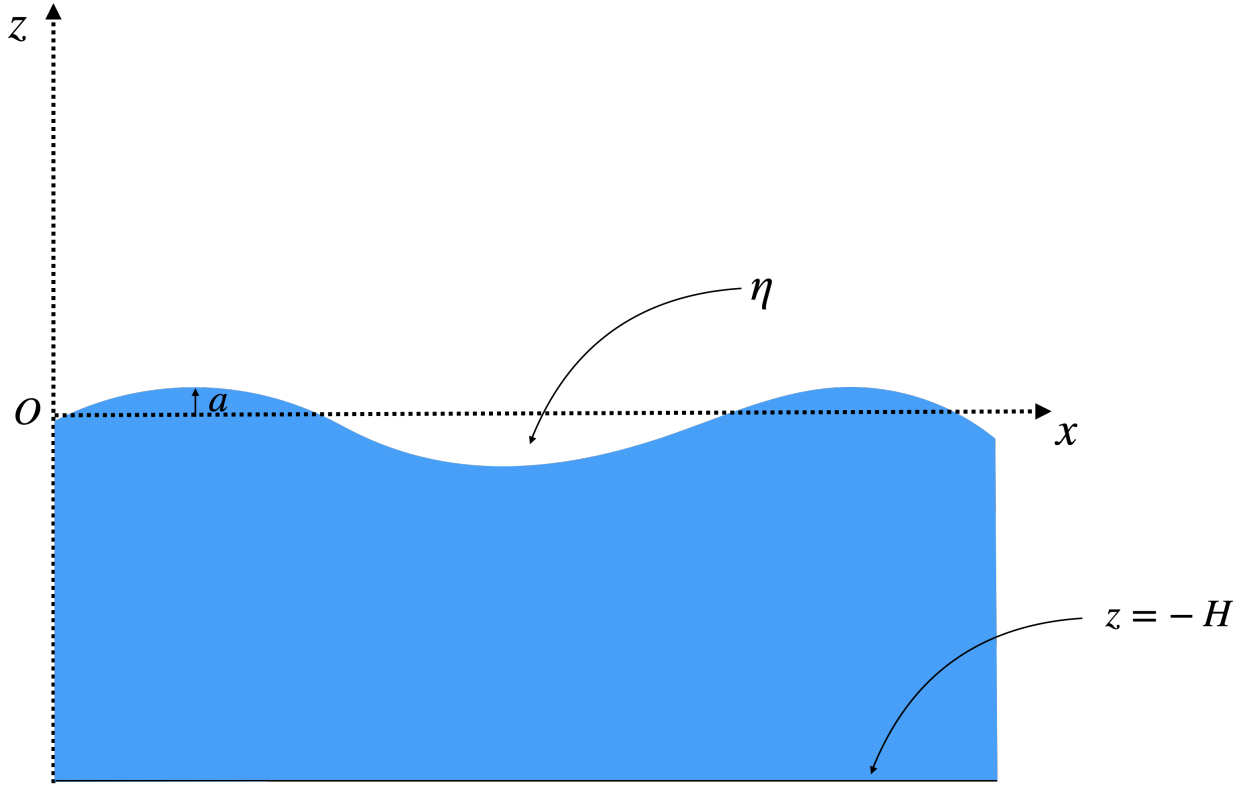


Figure 2.1: A simple schematic of a wave propagating in the x direction showing the construction of the free surface η . Adapted for the thesis from [40].

2.2 Surface Gravity Waves

In this section, we derive the equations of motion for gravity waves (waves with gravity as the restoring force) at the free surface of a liquid of uniform depth H . We further make the small-amplitude assumption, that is $a \ll \lambda$ and $a \ll H$ where a is the amplitude of the free-surface displacement in the vertical direction and λ is the wavelength.

2.2.1 Problem Formulation

Consider a wave propagating in the x and y (horizontal) directions with displacements in the z (vertical) direction. The motion is three-dimensional. The free surface displacement is taken to be $\eta(\vec{x}, t)$. A schematic showing a simplified version of the problem formulation

in two dimensions is shown in Fig. 2.1. We will limit our focus to *inviscid, incompressible and irrotational flow*, i.e.,

$$\nu = 0, \quad (2.3)$$

$$\nabla \cdot \vec{u} = 0, \quad (2.4)$$

$$\nabla \times \vec{u} = \vec{0}, \quad (2.5)$$

all of which are reasonable assumptions when considering surface gravity waves [40]. Following Kundu [40], this allows us to proceed with potential theory where the velocity can be written as the gradient of a *velocity potential* ϕ ,

$$\vec{u} = (u, v, w) = \nabla \phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right). \quad (2.6)$$

For the purposes of the derivation that follows, we will use the subscript h to refer to the part of the a vector corresponding to the horizontal directions. For example,

$$\vec{u}_h = (u, v), \quad (2.7)$$

$$\nabla_h = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right). \quad (2.8)$$

Substituting (2.6) in (2.4), we get Laplace's equation,

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0. \quad (2.9)$$

There are two boundary conditions at the free surface and one at the bottom boundary [40]. At the bottom boundary, the fluid has no normal velocity [40].

$$w = \frac{\partial \phi}{\partial z} = 0 \quad \text{at} \quad z = -H. \quad (2.10)$$

At the free surface the first boundary conditions is the *kinematic* boundary condition ([40]), which states that the fluid particles at the free surface remain at the surface,

$$\frac{D\eta}{Dt} = w \quad \text{at} \quad z = \eta, \quad (2.11)$$

$$\frac{\partial \eta}{\partial t} + (\vec{u}_h \cdot \nabla_h) \phi = \frac{\partial \phi}{\partial z} = w \quad \text{at} \quad z = \eta. \quad (2.12)$$

For small-amplitude waves, $\vec{u}_h$ and $\nabla \phi$ are both small, so the quadratic term $\vec{u}_h \cdot \nabla_h \phi$ in (2.12) is an order smaller than the other terms. Furthermore, since η is small, instead of evaluating the boundary condition at $z = \eta$, we can evaluate it at $z = 0$ (using a Taylor series approximation). Making these simplifications gives,

$$\frac{\partial \eta}{\partial t} = \frac{\partial \phi}{\partial z} \quad \text{at} \quad z = 0. \quad (2.13)$$

The second boundary condition at the free surface requires that the pressure, P , is given by the ambient pressure, $P_a = 0$, at the free surface $z = \eta$.

For irrotational motion, Bernoulli's principle holds [40],

$$\frac{\partial \phi}{\partial t} + \frac{1}{2} \vec{u} \cdot \vec{u} + \frac{p}{\rho} + gz = 0. \quad (2.14)$$

Neglecting the nonlinear term, $(1/2) \vec{u} \cdot \vec{u}$, the linearized form of the dynamic condition gives

$$\frac{\partial \phi}{\partial t} + g\eta = 0 \quad \text{at} \quad z = 0. \quad (2.15)$$

Recapitulating the work above, the problem to be solved is,

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0, \quad (2.16)$$

subject to the boundary conditions,

$$\frac{\partial \phi}{\partial z} = 0 \quad \text{at } z = -H, \quad (2.17)$$

$$\frac{\partial \phi}{\partial z} = \frac{\partial \eta}{\partial t} \quad \text{at } z = 0, \quad (2.18)$$

$$\frac{\partial \phi}{\partial t} = -g\eta \quad \text{at } z = 0. \quad (2.19)$$

To solve the equation, we assume that the free surface is in the form of a sinusoidal wave with a wavevector $\vec{k} = (k, l)$ and frequency ω ([40]),

$$\eta = A \cos(\vec{k} \cdot \vec{x} - \omega t). \quad (2.20)$$

Boundary conditions (2.18) and (2.19) require that ϕ be a sine function of $(\vec{k} \cdot \vec{x} - \omega t)$. Using $\phi = f(z) \sin(\vec{k} \cdot \vec{x} - \omega t)$, we can substitute this into (2.16) and the boundary conditions to find that,

$$\phi = \frac{A\omega}{\kappa} \frac{\cosh(\kappa(z + H))}{\sinh(\kappa H)} \sin(\vec{k} \cdot \vec{x} - \omega t), \quad (2.21)$$

where $\kappa = |\vec{k}| = \sqrt{k^2 + l^2}$. Differentiating ϕ gives us the velocity components,

$$u = \frac{A\omega k}{\kappa} \frac{\cosh(\kappa(z + H))}{\sinh(\kappa H)} \cos(\vec{k} \cdot \vec{x} - \omega t), \quad (2.22)$$

$$v = \frac{A\omega l}{\kappa} \frac{\cosh(\kappa(z + H))}{\sinh(\kappa H)} \cos(\vec{k} \cdot \vec{x} - \omega t), \quad (2.23)$$

$$w = \frac{A\omega}{\kappa} \frac{\sinh(\kappa(z + H))}{\sinh(\kappa H)} \sin(\vec{k} \cdot \vec{x} - \omega t) \quad (2.24)$$

Substituting (2.20) and (2.21) in (2.19) gives,

$$\omega = \sqrt{g\kappa \tanh(\kappa H)}. \quad (2.25)$$

This relation between the frequency and the wavenumber of the wave is known as the *dispersion* relation. The phase speed – the speed of constant-phase features such as wave crests or wave troughs – is defined as ([40]),

$$c = \frac{\omega}{\kappa} = \sqrt{\frac{g}{\kappa} \tanh(\kappa H)}. \quad (2.26)$$

Waves for which c depends on the wave vector are known as *dispersive* waves, akin to light waves of different wavelengths having different propagation speeds.

There are two common approximations that can be made in extreme cases. The first one assumes that $\kappa H \rightarrow 0$. This is known as the ‘shallow water’ or ‘long wave’ approximation. When this condition is satisfied, we can approximate $\tanh(\kappa H) \approx \kappa H$ using the Taylor expansion of $\tanh(x)$ at $x = 0$ [41]. This gives the following expression for the dispersion relation and phase speed of waves in shallow water,

$$\omega_{\text{SW}} = \kappa \sqrt{gH}, \quad (2.27)$$

$$c_{\text{SW}} = \frac{\omega_{\text{SW}}}{\kappa} = \sqrt{gH}. \quad (2.28)$$

Waves in shallow water are non-dispersive, i.e., their phase speed is independent of wave number. They only depend on the height of the fluid below. If we instead, assume that $\kappa H \rightarrow \infty$, we obtain the ‘deep water’ or ‘short wave’ approximation. In this case, $\tanh(\kappa H) \approx 1$, which gives,

$$\omega_{\text{DW}} = \sqrt{g\kappa}, \quad (2.29)$$

$$c_{\text{DW}} = \frac{\omega_{\text{DW}}}{\kappa} = \sqrt{\frac{g}{\kappa}}. \quad (2.30)$$

2.3 Ray Theory for Surface Wave Propagation

So far, we considered surface waves over constant bathymetry – that is, the bottom boundary is constant. However, in real water bodies, the bathymetry often varies. In particular, here we focus on the case where the length scale at which the bathymetry varies is much larger compared to the wavelength of the wave.

We will derive and adapt the equations of wave propagation in inhomogeneous media to surface gravity waves using concepts from ‘geometrical optics’ or ‘ray optics’.

We consider a free-surface wave of the form,

$$\eta(x, y, t) = A(x, y) \exp(i(\vec{k} \cdot \vec{x} - \omega t)). \quad (2.31)$$

As is common in the study of waves, we refer to the quantity,

$$\theta = \vec{k} \cdot \vec{x} - \omega t, \quad (2.32)$$

as the *phase* [42].

Since we are dealing with inhomogeneous media, the wave vector and angular frequency depend explicitly on the position $\vec{x}$ and time t .

$$\vec{k} = \vec{k}(\vec{x}, t) \quad (2.33)$$

$$\omega = \omega(\vec{k}, \vec{x}, t), \quad (2.34)$$

where σ is the modified dispersion relation which depends on position due to the depth’s dependence on position. The local wave vector and angular frequency can be derived from the phase function,

$$\vec{k} = \nabla \theta|_t, \quad (2.35)$$

$$\omega = - \left. \frac{\partial \theta}{\partial t} \right|_{\vec{x}}. \quad (2.36)$$

Due to the symmetry of second derivatives, we know the following to be true,

$$\frac{\partial \vec{k}}{\partial t} + \nabla \omega = 0. \quad (2.37)$$

Using (2.34), we get,

$$\frac{\partial \omega}{\partial x} = \frac{\partial}{\partial x} \sigma(\vec{k}, \vec{x}, t) \quad (2.38)$$

$$= \nabla_{\vec{k}} \sigma \cdot \frac{\partial \vec{k}}{\partial x} + \frac{\partial \sigma}{\partial x}, \quad (2.39)$$

$$\frac{\partial \omega}{\partial y} = \frac{\partial}{\partial y} \sigma(\vec{k}, \vec{x}, t) \quad (2.40)$$

$$= \nabla_{\vec{k}} \sigma \cdot \frac{\partial \vec{k}}{\partial y} + \frac{\partial \sigma}{\partial y}. \quad (2.41)$$

$$(2.42)$$

Writing this in vectorized form, gives us,

$$\frac{\partial \vec{k}}{\partial t} + \vec{c}_g \cdot \nabla \vec{k} + \nabla \sigma = 0, \quad (2.43)$$

where $\vec{c}_g$ is the group velocity of the wave. Note that,

$$\frac{\partial \vec{k}}{\partial t} + \vec{c}_g \cdot \nabla \vec{k} = \frac{d\vec{k}}{dt} \quad \text{on the curve} \quad \frac{d\vec{x}}{dt} = \vec{c}_g. \quad (2.44)$$

We define a *ray* as a curve that satisfies the ray path equation, (2.45), since it propagates at the group velocity of the wave.

$$\frac{d\vec{x}}{dt} = \vec{c}_g. \quad (2.45)$$

We can now rewrite (2.43) as,

$$\frac{d\vec{k}}{dt} = -\nabla \sigma, \quad (2.46)$$

which we will refer to as the *wavevector* equation which describes how the wave vector changes as it moves along with the ray. Finally, we have,

$$\frac{\partial \omega}{\partial t} = \nabla_{\vec{k}} \sigma \cdot \frac{\partial \vec{k}}{\partial t} + \frac{\partial \sigma}{\partial t}, \quad (2.47)$$

$$\frac{\partial \omega}{\partial t} = -\vec{c}_g \cdot \nabla \sigma + \frac{\partial \sigma}{\partial t} \quad \text{from (2.37),} \quad (2.48)$$

$$\frac{d\omega}{dt} = \frac{\partial \sigma}{\partial t}, \quad (2.49)$$

which shows that the angular frequency of the wave is conserved along the ray except if the dispersion relation explicitly involves dependence on time.

These equations have been derived in the absence of a background current. With a background current $\vec{U}$, equations (2.45), (2.46) and (2.49) are modified as follows,

$$\frac{d\vec{x}}{dt} = \vec{c}_g + \vec{U}, \quad (2.50)$$

$$\frac{d\vec{k}}{dt} = -\nabla \sigma - \vec{k} \cdot \nabla \vec{U}, \quad (2.51)$$

$$\frac{d\omega}{dt} = \frac{\partial \sigma}{\partial t}. \quad (2.52)$$

Finally, we need one more equation to track how energy propagates along the ray. There are two approaches to this derivation. Johnson [41] uses a nondimensional formulation of surface waves, and uses the Wentzel-Kramers-Brillouin (WKB) approximation to construct an asymptotic approximation of wave amplitudes align with an application of Green's formula. He then calculates the phase-averaged energy density of the wave to be,

$$E = \frac{1}{2} \omega^2 A^2 \cosh^2(\kappa H). \quad (2.53)$$

and uses it to derive a conservation equation for the quantity E/ω' , where $\omega' = \omega - \vec{k} \cdot \vec{U}$ is the intrinsic frequency which changes along the ray. This quantity which is known as the *wave action*. Whitham [42] and Bretherton and Garrett [43], on the other hand, present a variational formulation that works for any general wave equation in slowly varying media. The full derivation involves substantial tedious algebra, and will be skipped. The variational formulation does not assume anything about the wave a priori. The Euler-Lagrange equations provide the relevant conservation equation once the expression for the Lagrangian has been derived,

$$\frac{\partial}{\partial t} \left(\frac{E}{\omega'} \right) + \nabla \cdot \left(\frac{E}{\omega'} (\vec{U} + \vec{c}_g) \right) = 0. \quad (2.54)$$

In the absence of a background current, which is the case in the current work, we get a much simpler conservation equation for energy density [42],

$$\frac{\partial E}{\partial t} + \nabla \cdot (E \vec{c}_g) = 0, \quad (2.55)$$

$$\frac{\partial E}{\partial t} + \vec{c}_g \cdot \nabla E = -E \nabla \cdot \vec{c}_g, \quad (2.56)$$

$$\frac{dE}{dt} = -E \nabla \cdot \vec{c}_g. \quad (2.57)$$

2.4 An Overview of Existing Surface Wave Models

Numerical modeling of sea surface waves has undergone significant transformation since its early development in the 1950s, evolving from basic linear and nonlinear wave theories toward the sophisticated third-generation spectral wind-wave models utilized today [44]. A foundational shift occurred in the 1980s when the WAMDIG group introduced a methodology to solve the wave energy balance equation, which led to the creation of current industry-standard models such as WAVEWATCH III (WW3) and Simulating Waves Nearshore (SWAN) [44]. These contemporary models are primarily based on the spectral action density balance equation, which tracks the evolution of wave energy through frequency and directional components rather than resolving individual wave phases [44]. While spectral models are efficient for large-scale applications, certain specialized high-resolution needs—such as modeling internal wave processes or complex vertical structures—require models that resolve vertical non-hydrostatic effects directly [44]. Examples of these direct resolution models include SUNTANS and ORCTM, which can accurately capture vertical velocity fields but are limited by extremely high computational costs compared to their spectral counterparts [44]. The two most prominent models currently in use, WW3 and SWAN, possess distinct capabilities and limitations depending on their intended environment. WAVEWATCH III generally utilizes structured rectangular grids and explicit numerical schemes, making it highly effective for simulations on global and basin-wide scales [44]. In contrast, SWAN is optimized for coastal and nearshore environments through its use of unstructured triangular meshes that better conform to complex

shorelines and bathymetry [44]. SWAN employs an implicit transport scheme that remains stable in shallow water, though it may suffer from higher numerical dissipation when applied to large-scale ocean scenarios [44]. A major limitation shared by standard third-generation models is their performance during extreme weather events like tropical cyclones; conventional drag coefficient parameterizations tend to overpredict surface drag at wind speeds exceeding 30 m/s, necessitating the development of revised schemes that account for sea-state and wave age [44]. Contemporary methodologies have recently expanded to include coupled modeling frameworks and deep learning surrogates. Systems such as Coastlines-LO utilize dynamically coupled Delft3D-SWAN models to account for wave-current interactions, where radiation stress from waves modifies circulation while currents alter wave refraction and relative wind speed [44, 45].

In this thesis, we will present an open-source module written in Python which requires fewer computational resources than the models discussed above. While it can work as a stand-alone tool to study wave propagation, tools such as this have also been used as a part of larger-scale models [28].

2.5 Case Study: Total Phosphorus in Hamilton Harbour

Tracking total phosphorus (TP) is a fundamental requirement for managing freshwater ecosystems because phosphorus serves as the primary limiting nutrient for algal growth and biomass [46]. The over-enrichment of this nutrient triggers cultural eutrophication, which causes toxic cyanobacterial blooms, significant reductions in water clarity, and seasonal hypoxia in deep-water habitats [46–48]. To address these impairments, international frameworks like the Great Lakes Water Quality Agreement (GLWQA) have established specific TP loading targets for different basins [47–49]. Effective stewardship relies on mass-balance models that link TP concentrations in the lake to external loading, water residence time (R_w), and the characteristic TP residence time (R_p), the latter of which determines the speed at which a system asymptotically approaches a new steady state following changes in nutrient inputs [49].

In Hamilton Harbour, a partially enclosed embayment of Lake Ontario designated as an Area of Concern (AOC), tracking TP is particularly vital due to a history of severe degradation from industrial effluents and urban runoff [37, 46]. While substantial external loading reductions have been achieved, the Harbour’s recovery is now complicated by significant internal loading, where legacy phosphorus is recycled from the sediment back into the

water column [37]. Research indicates that the reduction of industrial iron pollution—once a major source of contamination—has paradoxically decreased the sediment’s capacity to scavenge phosphorus, accelerating the release of reactive P into the hypolimnion [37]. This internal subsidy can delay water quality improvements, making it critical for managers to incorporate diagenetic sediment processes into recovery expectations and remediation plans for delisting the AOC [37, 46].

The accuracy of total phosphorus prediction is essential for forming effective remedial actions and evaluating management success [46]. Reliable predictions must account for the fact that the majority of annual TP loads—between 52% and 89%—are delivered during brief, high-flow events such as the spring freshet or intense rain storms [46]. Traditional monitoring often relies on infrequent grab samples that capture baseflow conditions, leading to an underestimation of true nutrient delivery [46].

Another source of phosphorus loading in lakes involve sediment resuspension which is influence by wind-generated surface waves. Chomicki et al. [50] and Bocaniov, Scavia, and Cappellen [49] pointed out that wind and wave-driven resuspension and subsequent dispersal of phosphorus from the sediment greatly affect the phosphorus concentrations in lakes, especially in shallower regions. Zhao et al. [51] investigated the heterogeneity of internal phosphorus loading due to wave-induced resuspension of sediment in shallower and non-stratified regions of the lake. Other recent studies in the literature which use wave models to investigate resuspension of sediment in small lakes of moderate depth [52, 53].

As such, we present a case study of applying the ray optics equations for wave propagation to the bathymetry of Hamilton Harbour. The purpose of this is twofold. First, we wish to identify patterns of wave propagation in the harbour in the absence of background current and wind effects – i.e., in baseflow conditions. Second, we attempt to gain insights on how these waves may affect the internal phosphorus loadings in the harbour based on what we know about surface wave effects on phosphorus cycles in lakes.

Chapter 3

Results

In what follows, we first validate the model using two test cases – Snell’s law for surface waves as mentioned in [3], and comparing outputs from the model to the one-dimensional WKB approximation for waves traveling in a *slowly* varying environment [41]. Next, we investigate the behaviour of rays passing over Gaussian bathymetry. We note that flat depths with a Gaussian shoal or hole act akin to convex and concave lenses for light rays respectively.

We further investigate the sensitivity of wave refraction to perturbations in the bathymetry in both cases, observing that small perturbations can significantly alter the expected path of a ray due to compounded effects of local depth variation within the Gaussian artifacts. Finally, we end with a section on the application of the model to Hamilton Harbour. We study how the propagation of surface waves are affected by the initial positions and velocities.

3.1 Model Validation

We verify that the ray optics equations are accurate through a series of test cases. Holthuijsen [3] states that Snell’s law is applicable in bathymetries consisting of parallel depth contours. Snell’s law for surface waves is given by,

$$\frac{\sin \theta_2}{\sin \theta_1} = \frac{c_2}{c_1}, \tag{3.1}$$

where the subscripts refer to regions of different depth separated by an interface. θ_1 and θ_2 are the angles that the ray makes with the normal to the interface in the region. c_1 and c_2 are the magnitudes of the group velocity in the respective regions. Fig. 3.1 shows a ray propagating across a depth interface and undergoing refraction. Table 3.1 shows the values output by the ray optics solver of the wave vector, energy and amplitude of the wave in a depth of 25 m given the initial conditions set for $H = 50$ m.

H (m)	k	l	E	A
50	0.001000	0.0005	0.000100	0.571137
25	0.001503	0.0005	0.000141	0.677882

Table 3.1: Values of the wave vector, energy and amplitude of the wave in both depths.

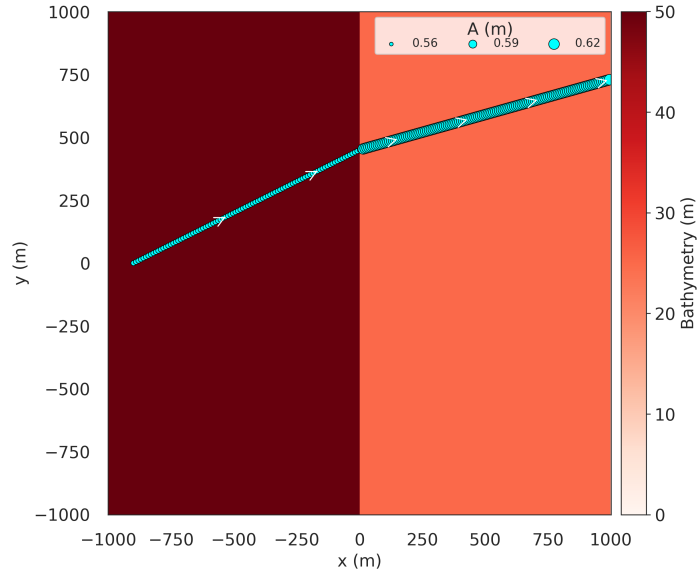


Figure 3.1: A ray propagating across an interface between two parallel depth contours. The ray undergoes refraction as it crosses the interface.

The values of $\sin \theta/c$ for the ray in both depths are given below,

$$\frac{\sin \theta_1}{c_1} = 0.02019 \quad (3.2)$$

$$\frac{\sin \theta_2}{c_2} = 0.02015, \quad (3.3)$$

which is in accordance with Snell's law.

We also consider the one-dimensional case where based on the WKB approximation, the amplitude A and the depth H are known to have a relationship given by [41],

$$A \sim O(H^{-1/4}). \quad (3.4)$$

Figure 3.2 show a log-log plot of amplitude A against depth H confirming this relationship.

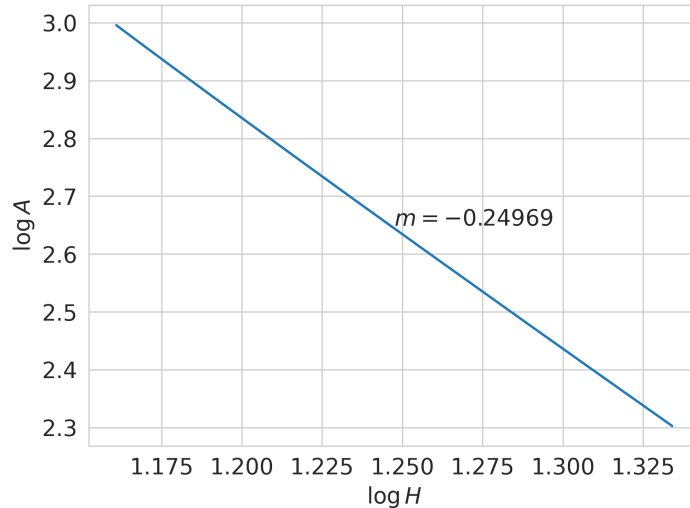


Figure 3.2: A one-dimensional application of the ray optics model. The plot shows the natural logarithms of amplitude A and depth H plotted against each other. The slope of the line has been calculated and shown to be in agreement with the WKB approximation.

In the event of waves approaching the a parallel shoreline, the depth solely depends on x – that is, the height is written as $H = H(x)$, and the components of the wave vector are $k = k(x)$, $l = l(x)$. Using 2.27, and 2.49, we have,

$$\omega = \sigma(\vec{k}, \vec{x}) = \sqrt{gH(x)(k(x)^2 + l(x)^2)}, \text{ and} \quad (3.5)$$

$$\frac{d\omega}{dt} = 0, \quad (3.6)$$

which implies,

$$\sqrt{gH_0(k_0^2 + l_0^2)} = \sqrt{gH(x)(k(x)^2 + l(x)^2)}, \quad (3.7)$$

where H_0 is the depth at the initial position, k_0 and l_0 are the initial components of the wave vector. Using 2.46, we have,

$$\frac{dl}{dt} = \frac{\partial \omega}{\partial y} = 0, \quad (3.8)$$

$$\implies l(x) = l_0. \quad (3.9)$$

As a result, we finally obtain,

$$k(x)^2 = \frac{H_0}{H(x)}(k_0^2 + l_0^2) - l_0^2. \quad (3.10)$$

As $H(x) \rightarrow 0$, $k(x) \rightarrow \infty$, which implies that in this limit, the wave approaches the shore normal to the shoreline. Fig. 3.3 shows such an example of multiple wave approaching the shoreline from different angles, but they all end up incident on the shoreline along the normal.

3.2 Gaussian Bathymetry

Below, we will describe some examples of wave path solutions generated by solving the ray optics equations. The figures show the paths of several wavefronts over two kinds of varying bathymetry – namely, Gaussian ridges and Gaussian ‘holes’. Both these functions

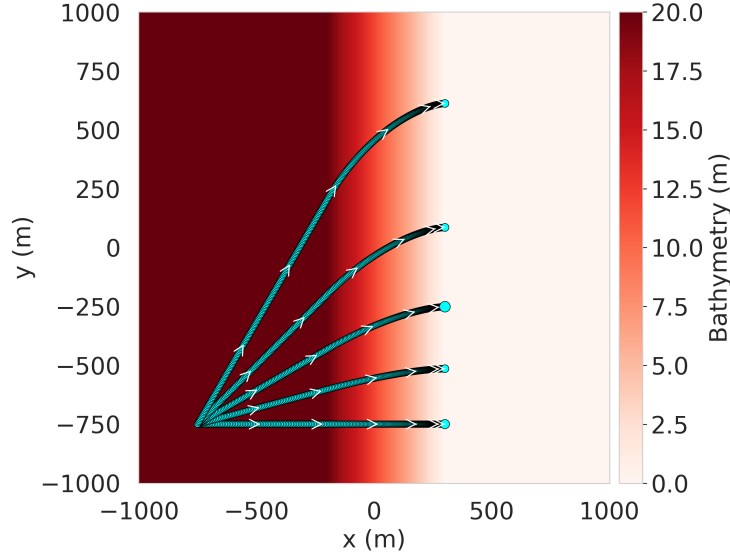


Figure 3.3: Waves propagating towards the shore at different angles. Notice that all of the waves end up incident normal to the shoreline.

consist of a mean depth, along with a Gaussian function that increases or decreases depth around a given center. Ridges are elevated terrain while holes are depressed terrain.

The depth field is given by,

$$d = H_0 - H_R \exp \left(-\frac{(x - \mu_x)^2}{2\sigma_x^2} - \frac{(y - \mu_y)^2}{2\sigma_y^2} \right), \quad (3.11)$$

where μ_x and μ_y are the coordinates around which the Gaussian function is centered and σ_x and σ_y are the standard deviations in the corresponding horizontal directions. H_0 is the undisturbed depth and H_R is the displacement of the bathymetry away from H_0 . H_R is positive for a ridge and negative for a hole.

Figure 3.4 (a) shows a series of wavefronts starting at various y locations and initially only travelling in the x direction going over a Gaussian ridge centered at $(0, 0)$ with standard deviations $\sigma_x = \sigma_y = 200$ m and getting refracted. The size of the points plotted changes according to the amplitude. The larger size of the waves with larger amplitude is exaggerated to be visibly different. The size s of the points plotted relates to the amplitude a according to the following,

$$s = s_{\min} + \left(\frac{A - A_{\min}}{A_{\max} - A_{\min}} \right) (s_{\max} - s_{\min}), \quad (3.12)$$

where $A_{\min(\max)}$ is the minimum (maximum) amplitude among all the plotted points, and $s_{\min(\max)}$ is the minimum (maximum) size of the point in units of pts^2 , where $1 \text{ pt} = 1/72$ inches.

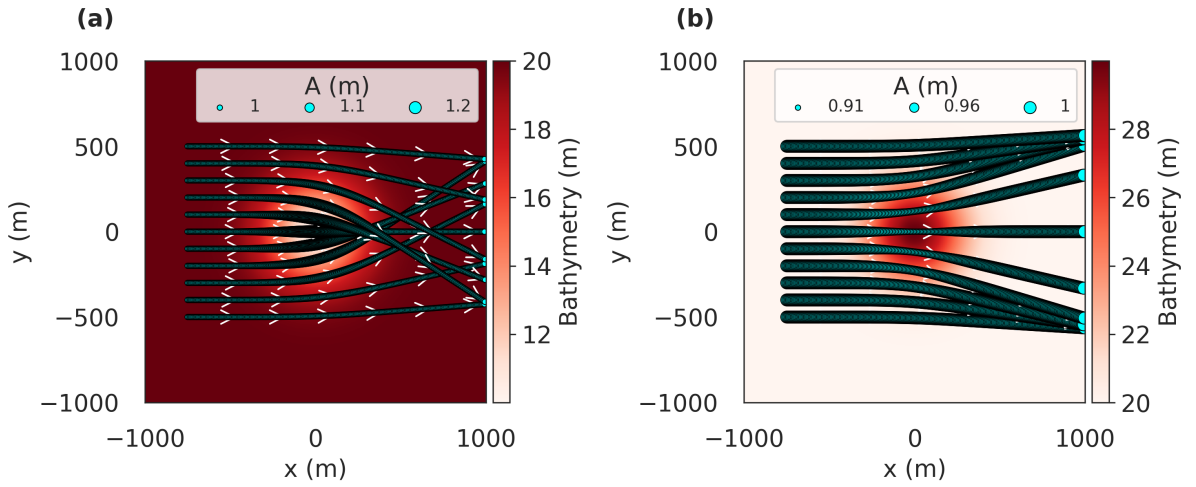


Figure 3.4: A series of plane waves traveling over (a) a Gaussian ridge and (b) a Gaussian hole showcasing the refraction they undergo. The white arrows show the direction of wave propagation. The wave path has been plot as a scatterplot depicting wave locations for each timestep. The size of the points plotted corresponds directly with the wave amplitude at that location. Since the ridge is circular, the refraction observed is symmetric about $y = 0$ in both cases.

Conversely, for a Gaussian hole, parallel rays traveling over the center diverge instead of converging, as shown in Fig. 3.4. Note that in both cases, the waves refract symmetrically about the center of the ridge due to radial symmetry. This has been discussed substantially in [41].

3.3 Perturbed Gaussian Bathymetry

Bathymetry in real lakes is often not smooth. So, some focus needs to be given to roughness in the bathymetry and how that manifests in surface wave propagation. We investigate the effects of small perturbations in the bathymetry in the way waves propagate in a closed system using sinusoidal perturbations defined by,

$$d = H_0 - H_R \exp \left(-\frac{(x - \mu_x)^2}{2\sigma_x^2} - \frac{(y - \mu_y)^2}{2\sigma_y^2} + a(\sin(12\pi(x - \mu_x)) + \sin(12\pi(y - \mu_y))) \right), \quad (3.13)$$

where a is the perturbation amplitude – a metric representing how much the bathymetry deviates from the idealized Gaussian case.

Figure 3.5 shows wavefronts at the same initial locations and with the same wavenumbers as Fig. 3.4 traveling over a Gaussian ridge with a sinusoidal perturbation. Panel(a) has a perturbation amplitude $a = 0.2$, which results in some disturbance of the wave path compared to the unperturbed case, but the wave paths still remain symmetric about $y = 0$. Panel (b), on the other hand, shows a perturbation with a higher amplitude, and therefore a larger deviation of the ray paths from the unperturbed case. The symmetry is now broken. Figure 3.6 shows the same for a Gaussian hole. We draw the reader’s attention to two observations. First, as expected the deviations from the original paths seen in 3.4 is more pronounced in panels (b) of Figs. 3.5 and 3.6, as are the differences in amplitudes attained as the waves pass over the Gaussian bathymetry. Second, the deviations are more severe in Gaussian ridges. While the rays in Fig. 3.6 do show some deviation, the overall nature of diverging rays remains as is. However, in Fig. 3.5, some rays – for example, the fifth and seventh ones from the top – show significant deviation in the final positions along the y -axis compared to those observed in Fig. 3.4. This seems to suggest that perturbations in an elevated region tend to disturb the path of a ray more than perturbations in a depressed region.

We quantify this difference in final positions in Fig. 3.8 which shows the error between the final distances traveled by waves with the same initial position, traveling over a Gaussian ridge with and without a sinusoidal perturbation applied to it. See Fig. 3.7 for a diagrammatic representation of how the ‘distance error’ is calculated. The blue points show the trajectory the wave would follow for an unperturbed bathymetry, while the orange points show the trajectory in the bathymetry plotted. There are six cases in total: an

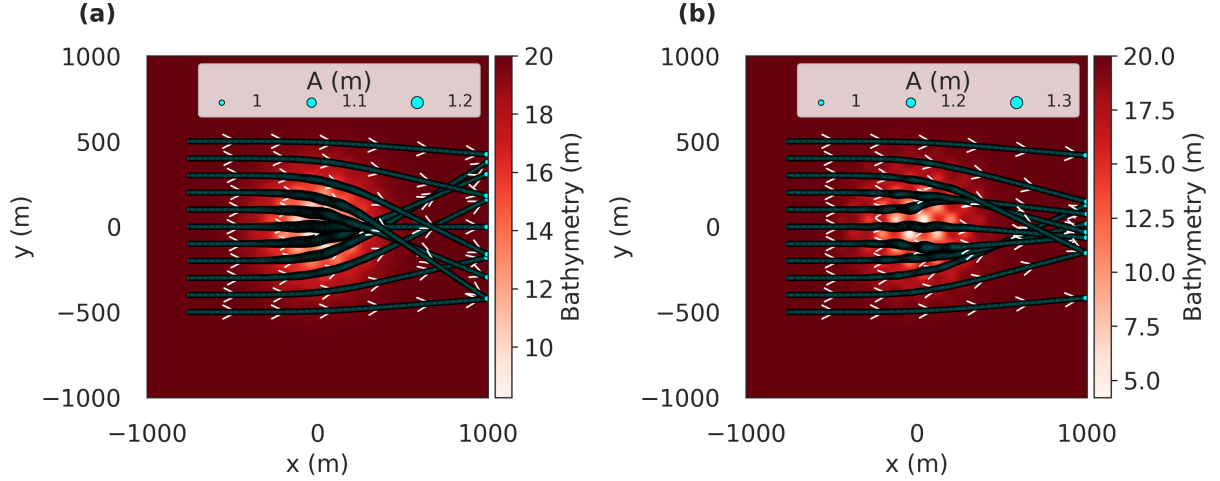


Figure 3.5: A series of wavefronts going over a perturbed Gaussian ridges with amplitudes (a) $a = 0.2$ and (b) $a = 0.5$. The initial locations for the wavefronts are the same as the ones employed in Fig. 3.4.

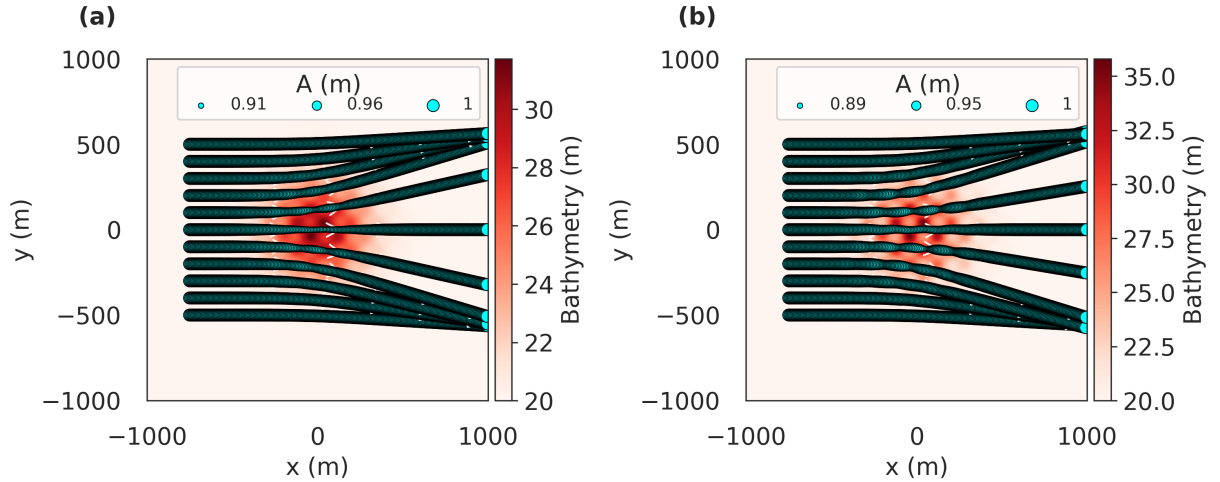


Figure 3.6: A series of wavefronts going over a perturbed Gaussian holes with amplitudes (a) $a = 0.2$ and (b) $a = 0.5$. The initial locations for the wavefronts are the same as the ones employed in Fig. 3.4.

unperturbed ridge, a perturbed ridge with amplitude $a = 0.2$, a perturbed ridge with amplitude $a = 0.5$, and the same for a hole. Note that the errors are much smaller for the hole than the ridge. This further emphasizes the larger sensitivity of ray paths while passing over an elevated shoal with perturbations compared to a perturbed depressed region.

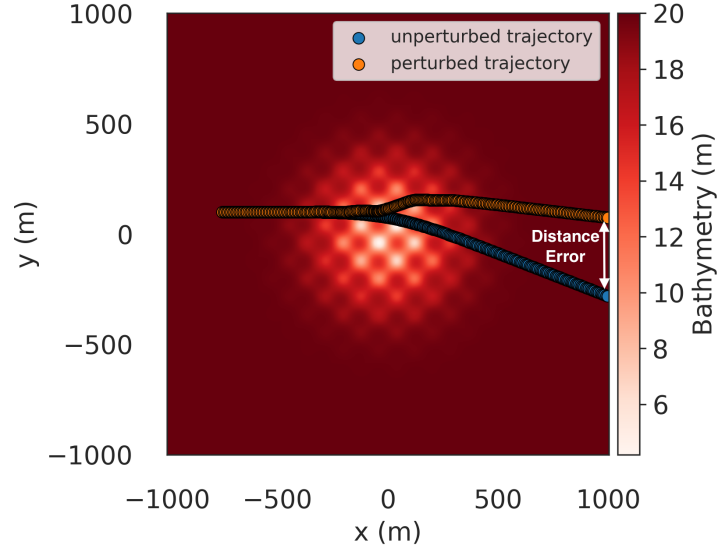


Figure 3.7: A schematic representation of the how the ‘distance error’ plotted in Fig. 3.8 is calculated.

Fig. 3.9 shows a wave propagation diagram for a subset of waves from Fig. 3.8 panel (b) near $y = 0$. This figure shows how perturbations on a Gaussian ridge can introduce a large deviation from the expected path. The cumulative effect of the wave moving away from depressed regions and towards shallower regions sharply varies the path as is shown in panel (b) – a zoomed version close to the point $(0, 0)$.

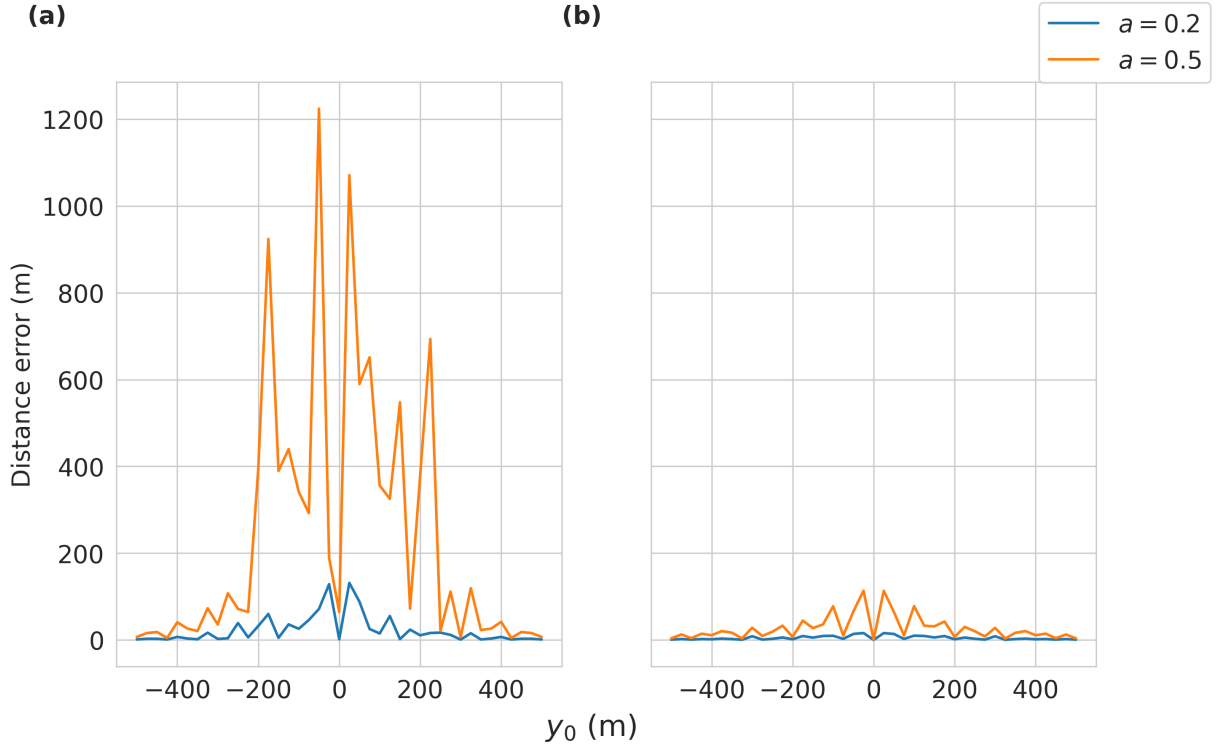


Figure 3.8: The norm of the difference in the final positions of wavefronts with horizontal initial velocity traveling over perturbed and unperturbed bathymetry for (a) Gaussian ‘ridge’ and (b) Gaussian ‘hole’ centered at the origin. There are two perturbation amplitudes applied: $a = 0.2$ and $a = 0.5$. The x axis shows the initial vertical position of the wavefront y_0 .

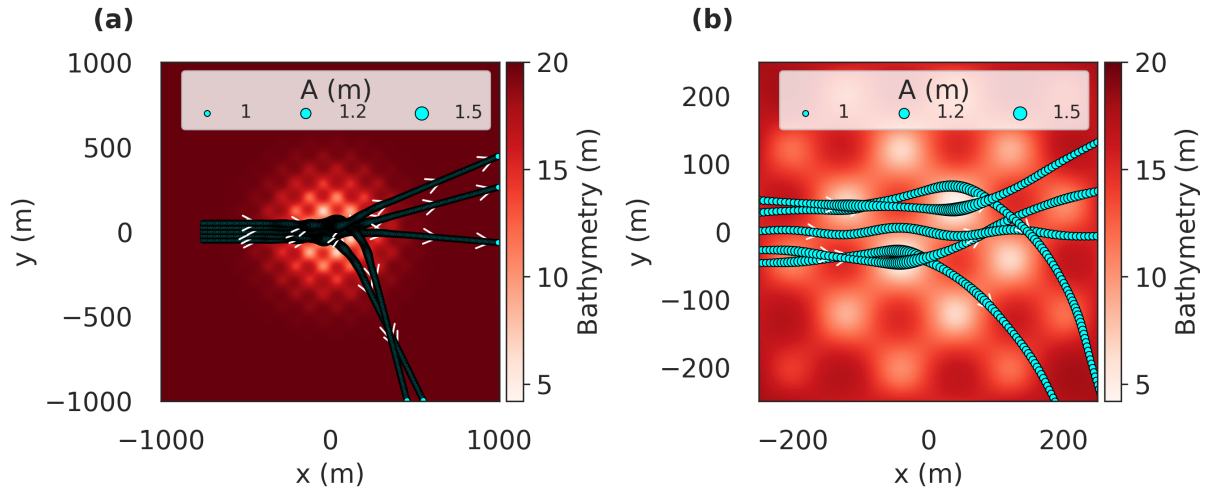


Figure 3.9: A wave propagation diagram to get a closer look at the rays passing over a perturbed ridge. The rays shown are correspond to the ones for which the distance error was plotted in Fig. 3.8 (b). Panel (a) shows the entire domain while panel (b) shows a limited domain to focus on the local variation in depth affecting the refraction of waves within the ridge.

3.4 Hamilton Harbour

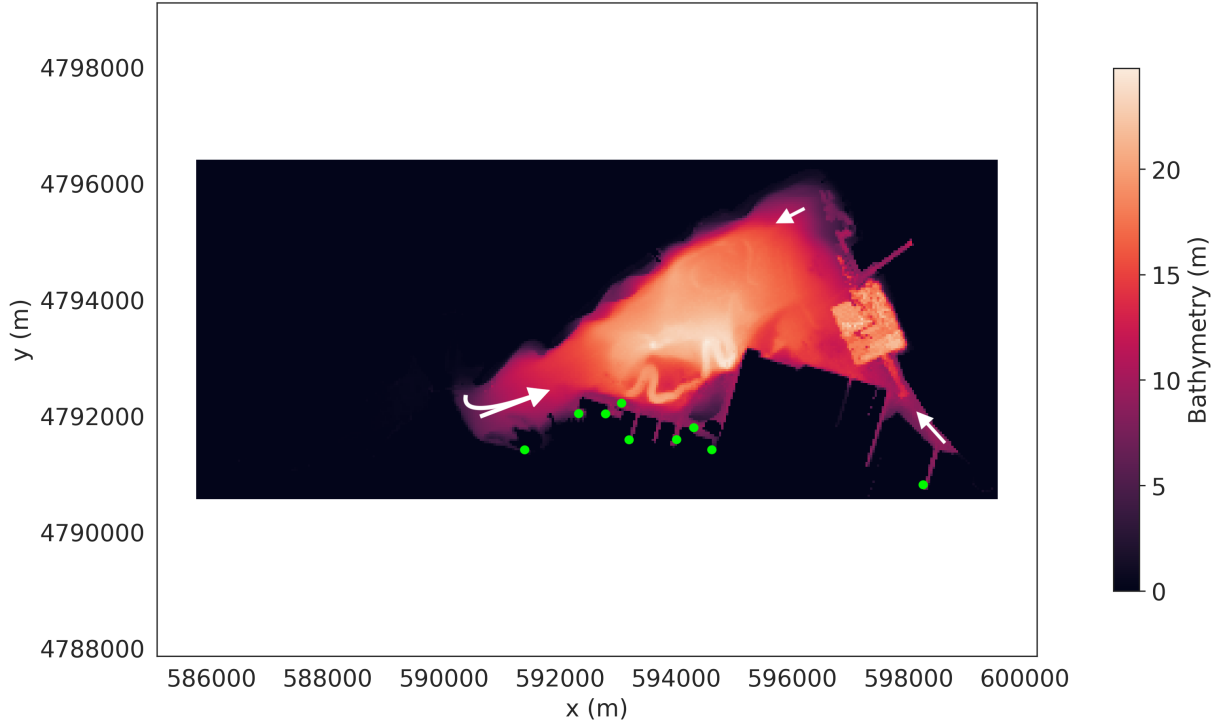


Figure 3.10: The bathymetry field of Hamilton Harbour. The white arrows show the rough locations of treated wastewater inputs and tributary inflows. The green dots show locations of combined sewer outflows located near the harbour that release into it [54].

We now apply the ray optics to real bathymetry data for Hamilton Harbour provided by Environment and Climate Change Canada (ECCC). This makes an interesting area of study for wave propagation due to the reasons mentioned in 2.5. The harbour has a roughly triangular shape with a length of 8 km along its main axis and width of 6 km, with a maximum depth of 23 m in the central basin [55]. It receives inputs from three major streams: Grindstone Creek at the north west end, Red Hill Creek at the south east end and Cootes Paradise at the west end. Other sources of input are wastewater treatment plans and combined sewer outflows [55]. Finally, it is connected to Lake Ontario through a narrow manmade canal called the Burlington Ship Channel [56].

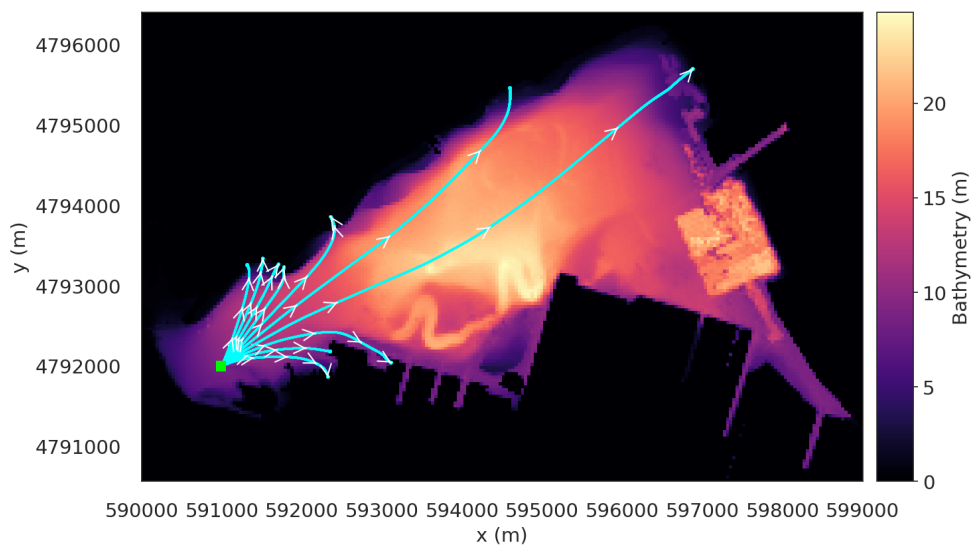
Figure 3.10 shows the bathymetry field of Hamilton Harbour. The bathymetry map shows an interesting meandering feature just below the central basin. Generally interpreted

as a partially infilled river valley [57], the feature is of much interest to wave propagation. As will be noted later, waves passing over this submerged channel are sensitive to being refracted. The white arrows show rough locations of tributary inflows – Cootes’ Paradise and Grindstone Creek on the west end – and inflows from wastewater treatment plants (WWTP) – Skyway WWTP at the north eastern corner and Woodward WWTP at the south eastern corner [58]. The green dots show locations of combined sewer outflows that release into the harbour [54].

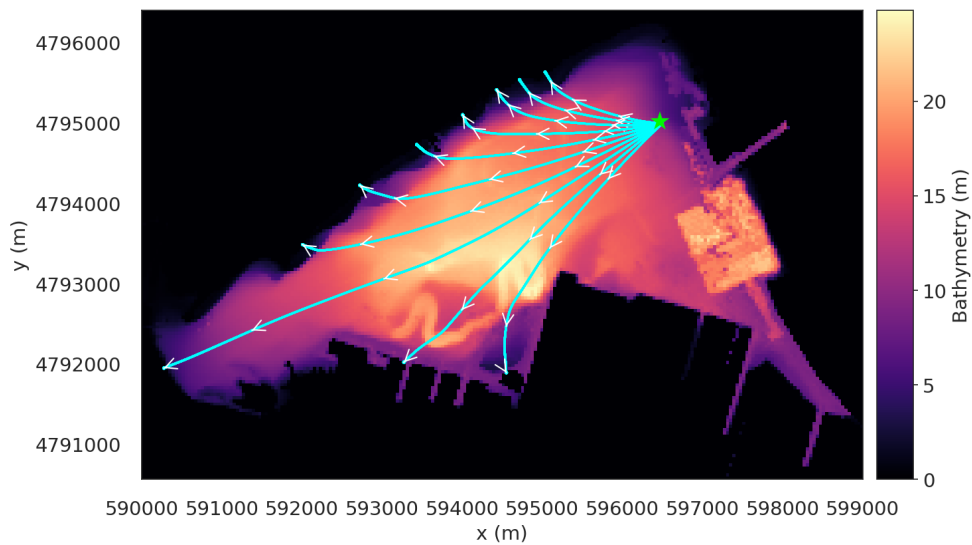
As proof of concept, we present Fig. 3.11 showing three wave propagation diagrams over the Hamilton Harbour bathymetry with different initial positions (marked by a green square, circle and a star) corresponding to the three primary tributary outflows. Notice that in Fig. 3.11a, most waves turn towards shallower regions. We see that the second and third waves from the bottom experience the most refraction due to a large gradient in depth in the region they travel over, while waves travelling over deeper regions with less of a gradient in depth around them are able to move much farther without having their direction of propagation significantly altered until they hit a shore. Figs. 3.11b and 3.11c shows encounters of two waves with the river valley during which quick changes in their direction are observed as they propagate over it. While diagrams like these can be helpful to visualize directions of propagating waves or a proof of concept of the ray optics model, they provide a limited amount of information in terms of practical usage. As a result, we present a technique that can put this tool to better use in the rest of this section.

Figs. 3.13, 3.14, and 3.15 all show heatmaps of the total distance travelled by a wavefront when the initial position is changed over a small region in the domain. The three regions considered are within the vicinity of the three primary corners of the harbour, as has already been done in 3.11. Figure 3.12 shows locations of these regions on the bathymetry. In each of the figures, the initial wavenumber vector is $2\pi/5(\cos(\theta_0 + \phi_0), \sin(\theta_0 + \phi_0))$, where θ_0 is constant for each figure ($0, 3\pi/2, \pi/2$ respectively) and ϕ_0 changes over the panels - $\phi_0 =$ (a) $\pi/6$, (b) $\pi/4$ and (c) $\pi/3$. The angles chosen are somewhat arbitrary. These heatmaps provide some statistics on how much of a region is susceptible to large (or small) displacement of water waves, given some initial direction of wave propagation and the bathymetry of the harbour.

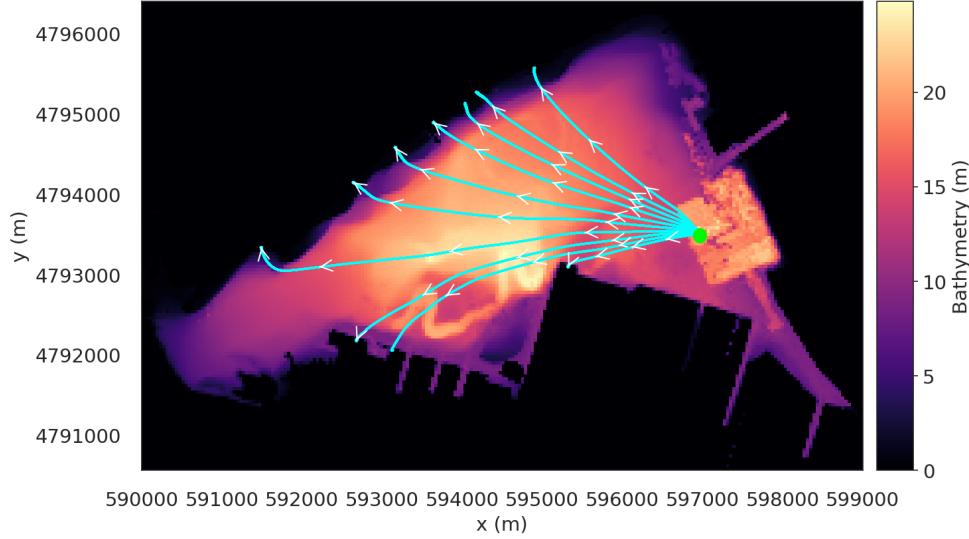
In Fig. 3.13 (corresponding to region 1 in Fig. 3.12) panel (a), we notice a dark band of locations from which the waves are able to propagate far, approximately across the entire length of the harbour. This band sits in the top-half of the region, indicating that the bathymetry across the top half of the harbour is smooth and devoid of any large perturbation. In the lower-half of this region, the waves are not able to propagate as far and begin to hit the shore before crossing the harbour. The distance that the waves are able to travel reduces gradually on either side of the dark band. However, there are some



(a) Waves beginning at the south-west corner.



(b) Waves beginning at the north-east corner.



(c) Waves beginning at the south-east corner.

Figure 3.11: Wave propagation from three initial positions - (a) the green square at the south-west corner, (b) the green circle at the north-east corner, and (c) the green star at the south-east corner is shown. The angles have been varied at each position to show possible directions waves can take.

anomalies shown as dark spots amid a region of lighter shades which possibly point to some encounter with the meandering river valley which introduces substantial refraction of these waves. Panel (b) of the same figure shows a similar pattern with the dark band now shifted to the lowest quarter of the region. This is likely due to a combination of two factors – the initial direction of propagation at an angle of 45° with the respect to the horizontal and the lack of severe refraction in the upper half. Finally, panel (c) shows the waves initially propagating at an angle of 60° with the horizontal. As expected, most of them end up hitting the northern shore without experiencing much refraction, with a few at the very bottom of the region being able to travel further.

Fig. 3.14 corresponds to region 2 in Fig. 3.12. The initial wavenumber vector is, in general, directed towards the south-west direction which is the third quadrant in Cartesian coordinate system. A dark triangular region at the bottom-right is seen in panel(a) suggesting that the waves originating inside this triangle are able to travel to the other end of the harbour, and this distance gradually reduces as the initial location of the wave moves away from the triangle. Panel (b) has an initial direction of propagation that is more southward than in panel (a), which we see manifest in the regions from which waves

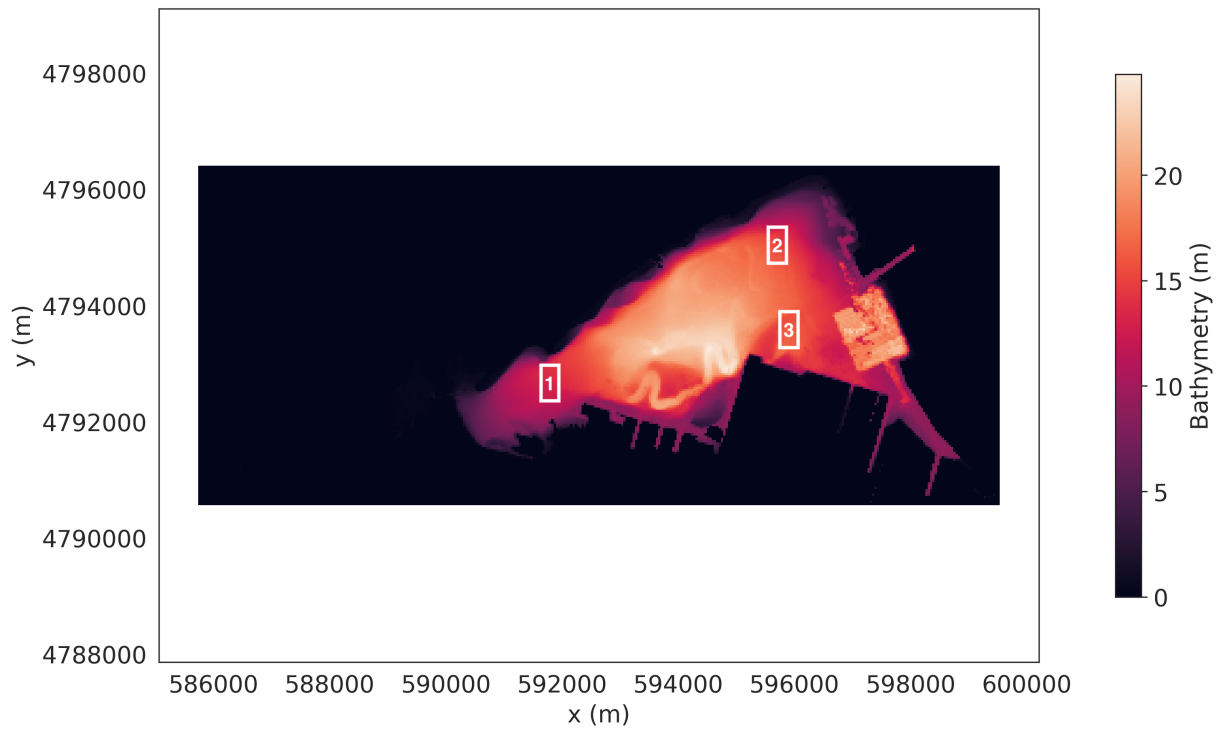


Figure 3.12: Locations of the regions considered in Figs. 3.13 (region 1), 3.14 (region 2), and 3.15 (region 3).

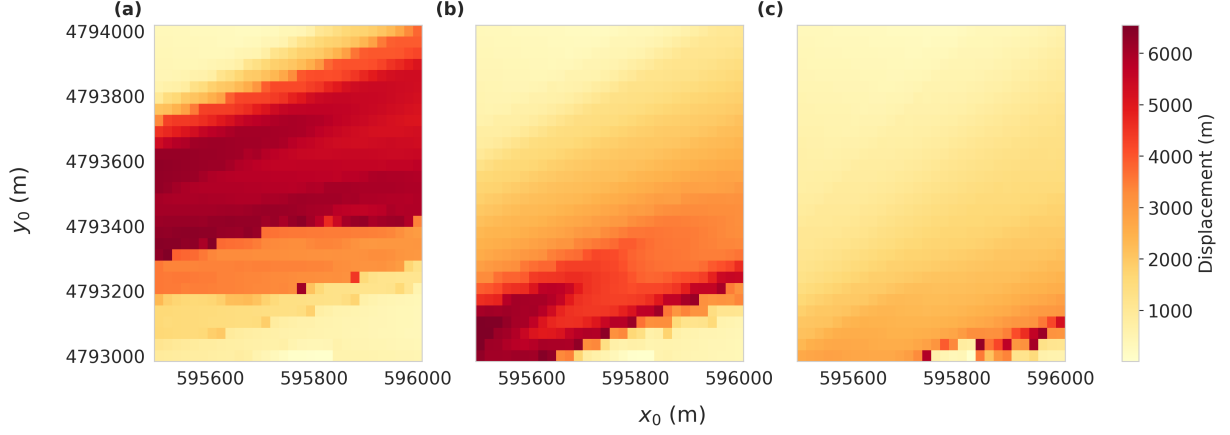


Figure 3.13: Heatmaps of distance travelled by waves originating at initial points with x_0 between 591,500 m and 592,000 m, and y_0 between 4.792×10^6 m and 4.793×10^6 m (region 1 in Fig. 3.12). The initial wavenumber is $\vec{k}_0 = 2\pi/5(\cos \phi, \sin \phi)$, where $\phi =$ (a), $\pi/6$, (b), $\pi/4$ and (c) $\pi/3$. The color bar has been normalized to represent the maximum and minimum across distances recorded in all three cases.

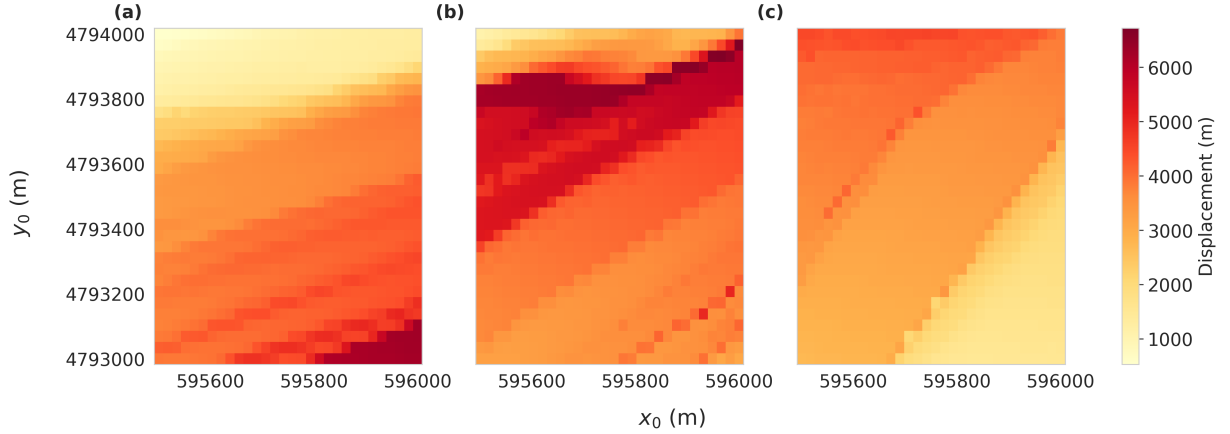


Figure 3.14: A similar figure as Fig. 3.13, with x_0 between 595,500 m and 596,000 m, and y_0 between 4.7945×10^6 m and 4.7955×10^6 m (region 2 in Fig. 3.12). The values of ϕ in this case are $3\pi/2$ added to the ones in Fig. 3.13.

travel farthest moving northward. In this panel, towards the bottom right, we also notice a few anomalous dark spots in an otherwise light region – anomalous initial points from which waves are able to travel farther than if they began at neighbouring points, indicating some refraction being encountered along the wave path. Panel (c) does not show any regions where waves travel farther as the ones seen in panels (a) and (b), while still showing gradual decrease in displacement as the initial location varies from the top-left to the bottom-right corner of the region.

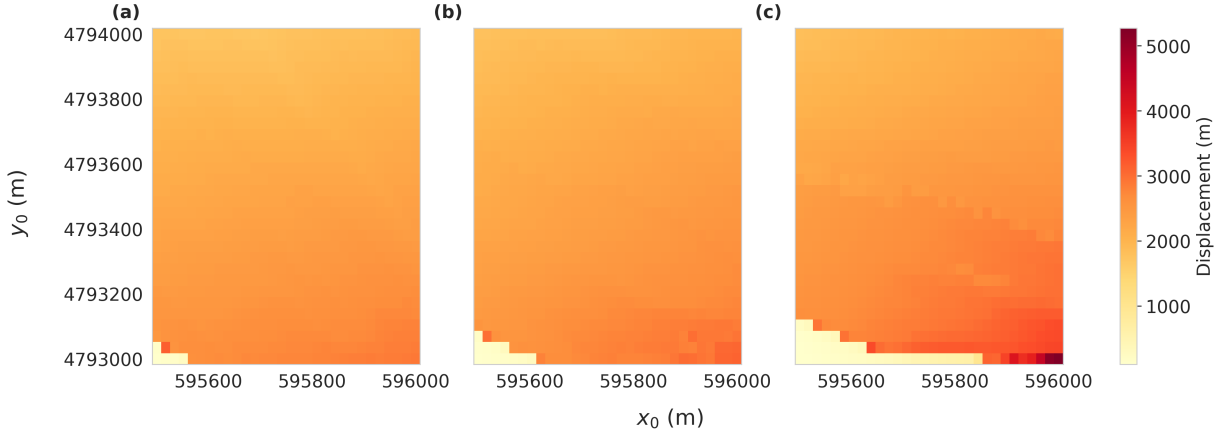


Figure 3.15: A similar figure as Fig. 3.13, with x_0 between 595,500 m and 596,000 m, and y_0 between 4.793×10^6 m and 4.794×10^6 m (region 3 in Fig. 3.12). The values of ϕ in this case are $\pi/2$ added to the ones in Fig. 3.13.

Fig. 3.15 corresponds to region 3 in Fig. 3.12. Since this is located towards the south-east corner, waves originating from here are not able to cross to the other side of the harbour with the initial directions chosen. In all of the panels, most of the distances covered lie between 2000 and 3000 m. In the bottom-left corner of all three panels, we notice a light region likely due to the wave path encountering a shore. Further investigation is required to make definite conclusions in this case.

Chapter 4

Discussion and Conclusions

In the previous chapter, we have reported on the development of an open-source Python module that solves the ray optics equations for surface wave propagation over slowly varying bathymetry. While Halsne et al. [20] discusses a similar tool, in this work, we add the feature of tracking wave energies and amplitudes along the ray. We showed that the model passes standard validation tests including Snell’s Law, the amplitude-height relationship in one dimension, and ensuring that wavefronts approaching the shore do so normal to the shore [3, 41, 59].

Smooth elevations and depressions are idealized bathymetry variations that have been studied in the context of ray theory [41]. In the current work, we investigate multiple waves propagating over radially symmetric elevated and depressed regions using a surface defined by a two-dimensional Gaussian function with. Using these, we are able to draw parallels between ray optics and surface waves. We noticed that elevated ‘ridges’ drive the rays to converge towards each other, akin to a convex lens; while depressed ‘holes’ result in the diverging of rays away from each other, similar to a concave lens.

Lake bathymetries are often rough and not as smooth as a Gaussian function [60]. Adding sinusoidal perturbations to the Gaussian ridges and holes give us an insight on the sensitivity of wave propagation on bathymetry roughness. We investigated this by comparing the distances travelled by waves with and without the perturbations. We found that propagating waves over elevated regions tend to be more sensitive to bathymetry roughness than over their depressed counterparts. We found that with perturbations in an elevated region, the degree to which waves converge can change drastically, with the distance between the final positions of a wave traveling over an unperturbed and a perturbed ridge going as high as 1.2 km. This is due to a combined effect of locally shallow and deep

regions refracting the waves, along with ridges increasing the density of rays travelling over it, while holes resulting in diverging rays which reduces both the number and the extent to which the ray paths affected. This results in some waves converging sharply as seen in Fig. 3.9 panel (a), and other waves not converging at all as seen in Fig. 3.7. The same is not observed in the case of holes.

We further showed the application of this module to real bathymetry data for Hamilton Harbour obtained from Environment and Climate Change Canada (ECCC) [61]. Surface waves and currents are known transporters of sediment and floating particles in lakes and other enclosed bodies of water [62, 63]. Studying wave propagation over this bathymetry can provide insight into how contaminants are transported. Waves originating from the primary tributary outflows were the focus in the current work since these are sources of contaminants. Locations that can be further considered include combined sewer outflows (CSO), which dispense treated industrial and municipal wastewater into the harbour [54]. While CSOs generally dispense fluid away from the surface, surface waves of larger wavelength can continue to affect transport even in these deeper regions [40, 63]. We demonstrated a method to study multiple waves in aggregate using heatmaps such as Figs. 3.14, 3.15 and 3.13. These serve the purpose of showing general trends in which waves may propagate across the harbour given an initial location and direction, but also provide insight on some regions which are anomalous. In particular, for Hamilton Harbour, the meandering river valley located in the southern part of the harbour [57] acts as a region of sharp variation in depth, resulting in substantial refraction as surface waves propagate above it. The initial directions of propagation were chosen arbitrarily in the current work. In practice, they can be chosen based on wind conditions in the region. There are two weather stations near the harbour – one located at the eastern end close to the Burlington Ship Channel and another located close to the western end, north of Cootes’ paradise [64].

There are clear limitations in the current version of the module. The absence of important wave processes such as wave breaking and dissipation due to it acts as a barrier to prevent accurate amplitude estimation close to the shoreline. In theory, the amplitudes can become infinitely large as waves approach the shore, but this is not true for surface waves observed in real life. Waves are primarily wind-driven and may experience dissipation while propagating due to various factors such obstacles or vegetation in the fluid. Support for energy sink and source terms is not available in the module. Wave-wave interactions are also not considered. Many enclosed water bodies, including Hamilton Harbour, have a wall as part of their boundary instead of a gentle incline to shore [56]. Walls can absorb and reflect energy, and by extension, waves can be reflected off of them [59]. This process of reflection is not straightforward to model and is not being considered in this thesis. Finally, waves can also be refracted by background currents in the fluid [59]. This was briefly discussed

in Chapter 2 while deriving the ray optics equations. The combination of the effects of currents and bathymetry on wave propagation has been studied extensively in oceans [20, 35, 36, 65]. The current module does not support propagation of waves in the presence of a background current. Allowing a constant background current is straightforward. However, for enclosed bodies such as a harbour or a lake, background currents vary in time. With time-stepping in ray optics, the presence of a time-dependent background current introduces complications and requires considerable work to be dealt with. Therefore, we do not consider it to be within the scope of this thesis. Finally, in near-shore or coastal waters, nonlinear processes begin to dominate [3]. As wave amplitudes vary spatially and when obstacles are present, diffraction also becomes evident [3]. These processes result in behaviour not well-predicted by the ray optics equations. See Fig. 4.1 in which waves entering from the right do not completely turn towards the shore before breaking, showing an example where ray theory fails to model wave refraction near the shore accurately.

In terms of future work, there are a few directions. A natural next step is modifying the source code to support the presence of sources and sinks of energy. Since waves are primarily wind-driven, having the ability to use publicly available meteorological data to drive the formation and initial propagation direction of waves would substantially improve the applications of this module. Vegetation at the bottom of lakes is also known to dissipate wave energy and can be parameterized as a sink term in the wave energy (or wave action) equation [67–69]. Additional wave processes such as wave reflection due to walls, diffraction and scattering, all have extensive theoretical background within ray optics [59], and it would be worthwhile to have support for these processes in the module. As has been discussed before, wave refraction and energy are affected by background currents. Modifying the ray optics equations to include the background current terms and using the wave-action equation in place of the wave-energy equation would require some effort but is worthwhile to pursue to refine the module. Finally, in terms of code structure, a lot can be done in terms of optimization. The low-hanging fruit in this case is parallelizing the numerical integration of different wave paths since they are all computed independently of each other.



Figure 4.1: Shoaling waves observed in Cape Byron in Byron Bay, Australia. The waves enter from the right but do not turn completely perpendicular to the shore upon entry. This is a counter-example to ray theory [\[66\]](#).

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APPENDICES

Appendix A

Code Used to Numerically Solve the Ray Optics Equations

A.1 An Overview of the Code Structure

We used Python 3.11 to numerically integrate the ray optics equations. Inspiration for how to structure the code was drawn from [20], but all of the code written was our own. We used common open-source libraries designed for Python such as pandas, scipy and numpy to manipulate arrays and store the results in a tabulated format, matplotlib and seaborn for plotting data and creating figures [70–75].

Each wave is stored in an object called `CartWaveSol` which is initialized with the initial wave parameters including position, wavenumber, energy, the bathymetry within the domain, the total time of the simulation along with the timestep, and the choice of the dispersion relation (between the general dispersion relation and the shallow water approximation) and the numerical scheme to be used.

A.2 `CartWaveSol`

Below is the code that has been used to numerically integrate the ray optics equation. `CartWaveSol` stores user-defined parameters such as initial position, wavenumbers, energy and the bathymetry of the domain, along with numerical parameters such as spatial and temporal step sizes. Also defined within the class are functions used to interpolate the

bathymetry and calculate the frequency and group velocity at any given point. There are three ‘step’ methods defined within the class, each one using one of the three numerical schemes supported – Euler, symplectic Euler and Adams-Bashforth. Finally the ‘solve’ function numerically integrates the ray optics equations and stores the results in a pandas dataframe which is an accessible tabular format that has read-write support in Python.

```

1  % importing libraries
2  import numpy as np
3  import matplotlib.pyplot as plt
4  import pandas as pd
5  from scipy.interpolate import RegularGridInterpolator as RGI
6  from warnings import warn
7  % class definition and implementation
8
9  class CartWaveSol:
10
11     def __init__(self,
12                 init_k,
13                 x,
14                 y,
15                 bathymetry,
16                 x0,
17                 y0,
18                 E0,
19                 T,
20                 dt,
21                 dispersion=None,
22                 method='euler'
23                 ):
24
25         '''
26         Initializes the grid and bathymetry along with initial wave conditions.
27         Assumes that the wavefront arrives from the left.
28
29         Parameters
30         -----
31         init_k:
32             initial wave vector
33         x:
34             x grid (m), uniformly spaced
35         y:
36             y grid (m), uniformly spaced
37         bathymetry:

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38         2d array of lake bathymetry (m)
39     vegetation:
40         2d binary array of where vegetation is present
41     x0:
42         initial x position of the ray
43     y0:
44         initial y position of the ray
45     E0:
46         initial amplitude of the wave
47     T:
48         simulation time (seconds)
49     dt:
50         time step size (seconds)
51
52     Returns
53     -----
54     None
55     '''
56
57     # storing class variables to be used for each wave
58
59     self.g = 9.81 # m/s^2
60     self.init_k = init_k
61     self.x0 = x0
62     self.y0 = y0
63     self.E0 = E0
64     self.bathymetry = bathymetry
65     self.x, self.y = np.meshgrid(x, y)
66     self.x_min, self.x_max = np.min(x), np.max(x)
67     self.y_min, self.y_max = np.min(y), np.max(y)
68     self.T = T
69     self.dt = dt
70
71     # choice of dispersion relation
72     if dispersion is None or dispersion == 'shallow':
73         self.dispersion = self.sigma_sw
74     elif dispersion == 'full':
75         self.dispersion = self.sigma_full
76     else:
77         print("Dispersion must be 'shallow' or 'full'.")
78
79     # choosing the numerical scheme to work with
80     if method == 'euler':
81         self.step = self.euler_step
82     elif method == 'symplectic_euler':

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83         self.step = self.symplectic_euler_step
84     elif method == 'adams-bashforth':
85         self.step = self.adams_bashforth_step
86     else:
87         raise UserWarning("The method must be one of 'euler',
88                             'symplectic_euler', or 'adams-bashforth'.")
89
90     # derived quantities
91     self.ts = np.arange(0, T, dt)
92     self.dx = x[1] - x[0]
93     self.dy = y[1] - y[0]
94
95     # interpolate the bathymetry to a function f(x, y)
96     self.bathymetry_interpolate = RGI((x, y), self.bathymetry.T, method='linear')
97     self.H0 = self.bathymetry_interpolate((x0, y0)) # depth at the initial point
98
99     # initial amplitude
100    d_init = self.bathymetry_interpolate((self.x0, self.y0))
101    self.a0 = np.sqrt(2 * self.E0) / self.dispersion(self.init_k[0], self.init_k[1], d_init)
102
103    # empty arrays to store solutions
104    self.ks = np.zeros(len(self.ts))
105    self.ls = np.zeros(len(self.ts))
106    self.ray_xs = np.zeros(len(self.ts))
107    self.ray_ys = np.zeros(len(self.ts))
108    self.energies = np.zeros(len(self.ts))
109    self.energies_1 = np.zeros(len(self.ts))
110    self.amps = np.zeros(len(self.ts))
111    self.depths = np.zeros(len(self.ts))
112    self.df = pd.DataFrame(columns=['time', 'x', 'y', 'k', 'l', 'E', 'A', 'H'])
113
114
115    if self.bathymetry.shape != self.x.shape:
116        warn("The shape of the bathymetry must be the same as the
117              gridsize you wish to solve for. Please try again")
118        return None
119
120    return None
121
122    def sigma_sw(self, k, l, d):
123        '''
124        Dispersion relation for shallow water
125        '''
126        return np.sqrt(self.g * d * (k ** 2 + l ** 2))
127

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128     def sigma_full(self, k, l, d):
129         '''
130         Full dispersion relation for surface waves
131         '''
132         k_mag = np.sqrt(k**2 + l**2)
133         return np.sqrt(self.g * k_mag * np.tanh(k_mag * d))
134
135     def solve(self):
136         '''
137         Solves the wave energy equation and ray tracing equations
138         '''
139
140         self.ks[0] = self.init_k[0]
141         self.ls[0] = self.init_k[1]
142         self.ray_xs[0] = self.x0
143         self.ray_ys[0] = self.y0
144         self.energies[0] = self.E0
145         self.energies[1] = self.E0
146         self.amps[0] = self.a0
147         self.depths[0] = self.H0
148
149         for i in range(1, len(self.ks)):
150
151             # Check if wavefront is out of or close to the domain
152             x = self.ray_xs[i-1]
153             y = self.ray_ys[i-1]
154             if self.bounds_check(x, y):
155                 break
156
157             # calculate bathymetry at current position
158             # if depth too small, the model won't proceed and stop
159             d = self.bathymetry_interpolate((x, y))
160             if d < 1e-10:
161                 print("The depth is approximately zero.")
162                 break
163
164             # perform the integration step
165             # if step does not work, break out of loop
166             # typically happens when the wavefront runs out of the domain
167             try:
168                 xnow, ynow, know, lnow, Enow = self.step(i)
169             except:
170                 print("wavefront out of the domain")
171                 break
172

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173         # record current step output
174         self.ray_xs[i] = xnow
175         self.ray_ys[i] = ynow
176         self.ks[i] = know
177         self.ls[i] = lnow
178         self.energies[i] = np.round(Enow, 8)
179         amp_now = np.sqrt(2 * Enow) / (self.dispersion(know, lnow, d))
180         self.amps[i] = np.round(amp_now, 8)
181         self.depths[i] = np.round(d, 8)
182
183         # collect everything in a dataframe at the end
184         # truncate dataframe to include points until breaks
185         self.df = pd.DataFrame({
186             'time': self.ts,
187             'x': self.ray_xs,
188             'y': self.ray_ys,
189             'k': self.ks,
190             'l': self.ls,
191             'E': self.energies,
192             'A': self.amps,
193             'H': self.depths
194         }).iloc[:i-1]
195
196         return None
197
198     def euler_step(self, i):
199         '''
200         Performs one step of Euler's integration method
201         '''
202         x = self.ray_xs[i-1]
203         y = self.ray_ys[i-1]
204         k = self.ks[i-1]
205         l = self.ls[i-1]
206         E = self.energies[i-1]
207
208         # ray equation
209         # dx/dt = cg
210         d = self.bathymetry_interpolate((x, y))
211         xnew = x + self.dt * self.cg(k, l, d)[0]
212         ynew = y + self.dt * self.cg(k, l, d)[1]
213
214         # wavenumber equation
215         # dk/dt = -grad(sigma)
216         sigma_x = (
217             self.dispersion(k, l, self.bathymetry_interpolate((x + self.dx, y))) -

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218     self.dispersion(k, l, self.bathymetry_interpolate((x - self.dx, y)))
219     ) * \
220     (1 / (2 * self.dx))
221     sigma_y = (
222     self.dispersion(k, l, self.bathymetry_interpolate((x, y + self.dy))) -
223     self.dispersion(k, l, self.bathymetry_interpolate((x, y - self.dy)))
224     ) * \
225     (1 / (2 * self.dy))
226     knew = k - sigma_x * self.dt
227     lnew = l - sigma_y * self.dt
228
229     # energy equation
230     # dE/dt = -E div(cg)
231     diff_x_cgx = (1 / (2 * self.dx)) * (
232         self.cg(k, l, self.bathymetry_interpolate((x + self.dx, y)))[0] -
233         self.cg(k, l, self.bathymetry_interpolate((x - self.dx, y)))[0]
234     )
235     diff_y_cgy = (1 / (2 * self.dy)) * (
236         self.cg(k, l, self.bathymetry_interpolate((x, y + self.dy)))[1] -
237         self.cg(k, l, self.bathymetry_interpolate((x, y - self.dy)))[1]
238     )
239     Enew = E - self.dt * E * (diff_x_cgx + diff_y_cgy)
240
241     return xnew, ynew, knew, lnew, Enew
242
243     def cg(self, k, l, d):
244         '''
245         Computes the group velocity using the specified dispersion relation
246         '''
247         dk = 1e-6 * k if k != 0 else 1e-6
248         dl = 1e-6 * l if l != 0 else 1e-6
249         sigma_k = (self.dispersion(k + dk, l, d) - self.dispersion(k - dk, l, d)) / (2 * dk)
250         sigma_l = (self.dispersion(k, l + dl, d) - self.dispersion(k, l - dl, d)) / (2 * dl)
251         return np.array([sigma_k, sigma_l])
252
253     def bounds_check(self, x, y):
254         '''
255         Checks whether the current location or the location
256         of the wavefront after a step of dx or dy is out of the domain.
257
258         Parameters
259         -----
260         x: int
261         y: int
262

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263     Returns
264     -----
265     None
266     '''
267     if x < self.x_min or x > self.x_max or \
268     y < self.y_min or y > self.y_max:
269         warn("The wavefront has moved out of the domain.")
270         self.df = pd.DataFrame({
271             'time': self.ts,
272             'x': self.ray_xs,
273             'y': self.ray_ys,
274             'k': self.ks,
275             'l': self.l_s,
276             'E': self.energies,
277             'A': self.amps
278         })
279         return True
280     elif x - self.dx < self.x_min or \
281     x + self.dx > self.x_max or \
282     y - self.dy < self.y_min or
283     y + self.dy > self.y_max:
284         warn("The wavefront is close to the edge of the domain; numerical errors may occur.")
285         self.df = pd.DataFrame({
286             'time': self.ts,
287             'x': self.ray_xs,
288             'y': self.ray_ys,
289             'k': self.ks,
290             'l': self.l_s,
291             'E': self.energies,
292             'A': self.amps
293         })
294         return True
295     return False

```
