Some Sylvester-Gallai-Type Theorems for Higher-Dimensional Flats

by

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Author's Declaration

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of Contributions

The content of this thesis is based on three co-authored papers:

- A forthcoming paper with Jim Geelen and Peter Nelson on which Chapters 2 and 3 are based. This paper will supersede an earlier paper [25], which was joint work with Jim Geelen and has been published in *Combinatorica*. In this earlier paper, we proved the main result of Chapter 2 using a variation on the technique presented in this thesis.

- *Average plane-size in complex-representable matroids* with Rutger Campbell and Jim Geelen, which will soon appear in *Combinatorica*.
  Sections 6.1 and 6.2 are based on this paper.

- *Characterizing real-representable matroids with large average hyperplane-size* with Rutger Campbell and Ben Lund, which has been submitted for publication.
  Chapter 5 and Sections 6.3 and 6.4 are based on this paper.

Several passages in this thesis are taken verbatim from the above papers; any such passages were written by me.

Abstract

In response to an old question of Sylvester, Gallai proved that, for any finite set of points in the plane, not all contained in a single line, there is a line containing exactly two of them. This is known as the *Sylvester-Gallai Theorem*. In the language of matroid theory, which is used in this thesis, the Sylvester-Gallai Theorem says that every rank-3 real-representable matroid has a two-point line. Independently of Gallai and around the same time, Melchior proved the stronger result that, in a rank-3 real-representable matroid, the average number of points in a line is less than three. The purpose of this thesis is to prove theorems of this type for flats of higher dimension.

The Sylvester-Gallai Theorem was generalized to higher dimensions by Hansen, who proved that, for any positive integer k, a simple real-representable matroid of rank at least $2k-1$ has a rank-k independent flat. We will prove a theorem of this type for matroids representable over any given finite field by characterizing the "unavoidable flats" for matroids in this class. In particular, we show that any simple matroid of sufficiently high rank representable over a given finite field has a rank-k flat which is either independent or a projective or affine geometry. As a corollary, we will get the following Ramsey theorem: for any two-colouring of the points of a matroid of sufficiently high rank representable over a given finite field, there is a monochromatic rank-k flat.

The rest of the thesis concerns generalizing Melchior's theorem to higher dimensions. For planes, we prove that, in a real-representable matroid M of rank at least four, the average plane-size is at most an absolute constant, unless M is a direct sum of lines (in which case the average plane-size could be arbitrarily large). Generalizing this result to hyperplanes of arbitrary rank k, we prove that, if M is a simple rank-$(k+1)$ real-representable matroid whose average hyperplane-size is greater than some constant depending only on k, then all but a constant number b_k of the points of M are covered by a small collection of flats of relatively low rank. We also show that this constant b_k is best-possible.

Melchior's Theorem also implies that the average size of a flat (of any rank) in a rank-3 real-representable matroid is at most an absolute constant. We prove that, for any positive integer r, the average size of a flat in a rank-r real-representable matroid is at most some constant depending only on r.

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Over the years, I have benefitted greatly from being a student in the Department of Combinatorics and Optimization here at the University of Waterloo. I would like to express my gratitude for this superb learning environment. I thank all of the administrators and professors who contribute so much behind the scenes to keep this department humming along and maintain its standard of excellence. I would like to again thank Peter Nelson, not only for being a great collaborator in research, but also for being a gracious mentor and for always having his door open to talk about mathematics. I would also like to thank our graduate program coordinator, Melissa Quantrill, who is incredibly knowledgeable and has helped me so much over the years in navigating the various complexities of being a PhD student.

A significant portion of this thesis is based on joint work with Rutger Campbell and Ben Lund, both of whom work in the Discrete Mathematics Group (DIMAG) at the Institute for Basic Science in Daejeon, Korea. Much of this joint work took place while I was a visitor with DIMAG in January and July-August of 2024. I am very grateful to DIMAG, especially its Chief Investigator, Professor Sang-il Oum, for generously hosting me during these visits. I would also like to thank both Rutger and Ben for being fantastic collaborators and for being so generous to me while I was a visitor; I have learned so much from both of them. I would especially like to thank Rutger for inviting me to Daejeon in the first place, and for his kindness in showing me around, taking me on hikes, weekend trips and so on—in short, for going out of his way to make my first-ever trip to Korea special.

I would like to express my profound gratitude for my family and for their love and support, and to thank them for being the amazing people they are. I would especially like to thank my parents, who made me who I am today and sacrificed so much for me and for the sake of my education. Their support and unconditional love has been the greatest foundation in my life, which is why I dedicate this thesis to them.

Finally, I would like to express my heartfelt thanks to my supervisor, Jim Geelen. Jim is not only an incredibly talented mathematician, but a person of integrity, kindness and good humour; he has been a great role model and source of inspiration for me. My growth as a mathematician during these past four and a half years is thanks to his patience, generosity and dedication as a mentor. For this I am grateful beyond words.

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Dedication

to my parents

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Chapter 1

Introduction

Many classic theorems in discrete geometry concern a finite set of points in space and its spanned affine subspaces. A famous example is the following: for a finite set of non-collinear points in the plane, there is a line containing exactly two of them. A proof of this result was published by Gallai [24] in 1944 in response to an old question of Sylvester [52], although an equivalent theorem for line arrangements had been proved by Melchior [43] (who was apparently unaware of Sylvester's question) around the same time. For a nice introduction to the Sylvester-Gallai Theorem and surrounding literature, see Borwein and Moser's survey [8]. The main purpose of this thesis is to prove theorems of this type for spanned flats of higher dimension. We use the language of matroid theory, which seems especially suited for the presentation of our results and techniques. In this language, the Sylvester-Gallai Theorem is written as follows.

Sylvester-Gallai Theorem. *Every rank-*3 *real-representable matroid has a two-point line.*

The Sylvester-Gallai Theorem was generalized to higher dimensions by Hansen [32] who, confirming a conjecture of Motzkin [44], proved that for any integer $r \geqslant 3$, every rank-r real-representable matroid has a hyperplane comprising a point and a rank-$(r-2)$ flat. By a trivial inductive argument, we get the following corollary concerning the existence of flats which are also independent sets.

Hansen's Theorem. *For $k \geqslant 2$, every simple real-representable matroid with rank at least $2k-1$ has a rank-k independent flat.*

Hansen's rank bound of $2k-1$ is best-possible, as is demonstrated by the direct sum of $k-1$ lines, each containing at least three points.

The Sylvester-Gallai Theorem was famously extended to complex-representable matroids by Kelly [37], who proved that a complex-representable matroid always has a two-point line provided that its rank is at least four. This answered a question of Serre [50], who had asked whether complex-representable matroids with no two-point line, such as the ternary affine plane $\mathrm{AG}(2,3)$, necessarily have rank at most three. Hansen's Theorem has also been extended to the complex-representable matroids, but with weaker rank bounds [3, 18, 19, 26].

For matroids representable over any given finite field, the Sylvester-Gallai Theorem and Hansen's Theorem are patently false. For example, in any projective geometry representable over a finite field, every one of its flats forms a projective geometry over that field. In [47], Nelson and Nomoto gave a precise structure theorem for binary matroids with no three-point plane. For general matroids, Murty [46] showed that any rank-r matroid with fewer than $2^r - 1$ points has a two-point line. A stronger bound of this type for matroids representable over prime fields was proved by Bhattacharyya, Dvir, Shpilka and Saraf [5], who showed that, for a prime p and sufficiently large integer r, any rank-r $\mathrm{GF}(p)$-representable matroid with no two-point line has at least $p^{\Omega(r)}$ points.

In this thesis, we prove an analogue to Hansen's Theorem for matroids representable over a given finite field by characterizing the "unavoidable flats" for matroids in this class. In particular, we show that every simple matroid with sufficiently high rank representable over a given finite field contains a rank-k flat which is either independent, or is a projective or affine geometry.

We are also interested in the following relative of the Sylvester-Gallai Theorem, originally proved by Motzkin [45] and Rabin, concerning bicoloured configurations of points.

Motzkin-Rabin Theorem. *For any* 2*-colouring of the points of a rank-*3 *real-representable matroid, there is a monochromatic line.*

By combining our theorem on unavoidable flats with some classical Ramsey theory, we will prove a high-dimensional theorem of this type for matroids representable over a given finite field. Namely, we prove that for any such matroid of sufficiently high rank and any 2-colouring of its points, there is a monochromatic rank-k flat.

Around the same time that Gallai solved Sylvester's problem, Melchior [43] independently proved the following stronger theorem.

Melchior's Theorem. *In a rank-*3 *real-representable matroid, the average number of points in a line is less than three.*

Melchior's theorem, as originally formulated, concerns finite arrangements of lines in the projective plane and follows by a fairly straightforward application of Euler's Formula. (The formulation above then follows by projective duality.) In light of the simplicity of its proof, it is striking that the upper bound of three in Melchior's Theorem is sharp. To see this, consider a rank-3 matroid consisting of an arbitrarily large line and exactly one point outside of it.

In this thesis, we extend Melchior's Theorem to flats of higher dimension. In doing so, there is an important issue to navigate: namely, for $k \geqslant 3$, the average size of a rank-k flat in a real-representable matroid can be arbitrarily large compared to k. Consider, for instance, the direct sum of two (arbitrarily large) lines. In this matroid, every plane is forced to contain one of those lines, and so the average plane-size could be arbitrarily large. We will prove that this is the only thing that can go wrong for planes: that is, in a real-representable matroid of rank at least four which is not the direct sum of two lines, the average plane-size is at most an absolute constant. Together with Melchior's Theorem, this implies that the average size of a flat (of any rank) in a rank-4 real-representable matroid is at most an absolute constant. We will generalize both of these results to arbitrary rank.

When considering the average size of flats of a given rank in a real-representable matroid, we may always assume by truncation that they are hyperplanes. Our theorem on average plane-size cited above, phrased differently, says that every rank-4 real-representable matroid with large average plane-size is the direct sum of two lines. For hyperplanes of higher rank, things will not be so simple: there are real-representable matroids with arbitrarily large average hyperplane-size whose structure is considerably more complex than a direct sum of lines. We will generalize Melchior's Theorem to hyperplanes of arbitrary rank by characterizing these matroids. The theorem consists of two parts. First we show that, in a simple real-representable matroid M of given rank having large average hyperplane-size, there is a small collection of lines such that every hyperplane contains one of them. As with our characterization for planes, this ostensibly explains what causes the average to be large. The second part of the theorem further describes the structure of M. In particular, we prove that there is a partition (E_1, E_2) of the ground set where E_1 can be covered by a small collection of flats of relatively low rank and $|E_2|$ is bounded. We will in fact give a tight upper bound on $|E_2|$, based on a surprising connection to a classic problem in discrete geometry related to the Motzkin-Rabin Theorem. The following was first posed by Motzkin [44], and later by Grünbaum [30] and Erdős and Purdy [21] independently.

Motzkin's Problem. *Consider a rank-3 real-representable matroid M whose points are coloured red and blue, where there are k red points and the set of blue points is not collinear. If every line of M contains a red point, how many blue points can there be?*

In [44], Motzkin noted that there is no obvious generalization of this problem to higher dimensions. We will define such a generalization in this thesis, and we will use its solution to give a tight upper bound on the size of the set E_2 above.

For the remainder of this chapter, we provide detailed introductions to each of the main theorems of this thesis. It is worth pointing out explicitly that the classic theorems that we have stated (the Sylvester-Gallai Theorem, the Motzkin-Rabin Theorem and so on) were not originally formulated in the language of matroid theory, but rather as geometric theorems concerning points and lines in the plane. Because of the intrinsically geometric nature of this thesis, we include a section at the end of this chapter which provides a brief introduction to the language of matroid theory intended for geometers.

1.1 Unavoidable flats in matroids representable over a finite field

Our first main theorem, which we prove in Chapter 3, is a minimal list of the unavoidable flats in matroids representable over a given finite field. This was proved in joint work with Jim Geelen and Peter Nelson.

Unavoidable Flats Theorem. *For any prime-power q and positive integer k, there is a positive integer $N_q(k)$ such that, if M is a simple $\mathrm{GF}(q)$-representable matroid with $r(M) \geqslant N_q(k)$, then there is a rank-k flat F of M such that either F is independent, or $M|F$ is a projective or affine geometry over a subfield of $\mathrm{GF}(q)$.*

The proof of the Unavoidable Flats Theorem will follow in two steps, the first of which is to show that M contains either a large independent flat or a high-rank affine geometry as a restriction. We will prove this by adapting Kelly's technique [37] for showing that a rank-4 complex-representable matroid has a two-point line. This is explained in detail in Chapter 2. Once this first step is complete, we may assume that the matroid M contains an affine geometry as a spanning restriction. In fact, using a fairly straightforward contraction argument, we will be able to assume that M has a spanning *projective geometry* as a restriction. The second step of the proof will be to analyze this matroid, as is discussed further below. First we present some corollaries of the Unavoidable Flats Theorem.

By combining the Unavoidable Flats Theorem with some classical geometric Ramsey theory, we obtain a Ramsey theorem for matroids representable over a given finite field.

Corollary 1.1. *For any finite field $\mathbb{F}$ and positive integer k, if M is an $\mathbb{F}$-representable matroid with sufficiently high rank, then for any 2-colouring of the points of M, there is a monochromatic rank-k flat.*

To prove this result, we may assume by the Unavoidable Flats Theorem that M is either a free matroid (in which case we just consider a largest colour class) or a projective or affine geometry over a subfield of $\mathbb{F}$. In the latter case, the result will follow by known Ramsey theorems which we introduce in Chapter 2.

Because projective and affine geometries are full of low-rank dependencies, the Unavoidable Flats Theorem also has the surprising consequence that, by forbidding circuits of size at most four, independent flats become unavoidable. In fact, binary affine geometries are the only such matroids with no collinear triples, so for fields of odd characteristic, we can do slightly better. Recall that the *girth* of a matroid is the size of its smallest circuit.

Corollary 1.2. *For any finite field $\mathbb{F}$ and positive integer k, if M is a simple $\mathbb{F}$-representable matroid with sufficiently high rank and girth at least five, then M has a rank-k independent flat. Moreover, if $\mathbb{F}$ has odd characteristic, then girth at least four suffices.*

As mentioned above, the second half of the proof of the Unavoidable Flats Theorem consists of analyzing a $\mathrm{GF}(q)$-representable matroid M which contains a spanning projective geometry. The latter will be representable over the "base field" $\mathrm{GF}(p)$, where p is the characteristic of $\mathrm{GF}(q)$. The proof exploits a nice geometric relationship between the points of M and the flats of this spanning projective geometry.

Recall that, for p prime and a positive integer t, the additive group of $\mathrm{GF}(p^t)$ forms a t-dimensional vector space over $\mathrm{GF}(p)$. Because of this, if a $\mathrm{GF}(p^t)$-representable matroid M contains a projective geometry G_0 over the base field $\mathrm{GF}(p)$ as a spanning restriction, then any element e of M is spanned by a flat of G_0 of rank at most t. Moreover, there is a *unique* flat in G_0 of minimum rank which spans e, namely the intersection of all flats in G_0 which span e.

In our proof, we will view the matroid M as a spanning restriction of a projective geometry $G_1 \cong \mathrm{PG}(r-1, p^t)$ (where $p^t = q$ and p is prime), and assume that $G_0 \cong \mathrm{PG}(r-1, p)$ is a restriction of M. We know that every element of M is spanned by a flat of G_0 of rank at most t. Using some powerful tools from Ramsey theory which we introduce in Chapter 2, we will deduce (subject to the assumption that M has no rank-k affine or projective geometry over $\mathrm{GF}(q)$ as a flat) that every element of M is spanned by a flat of G_0 of rank *strictly less than* t. In this case, it is somewhat striking that M does indeed contain one of the unavoidable flats listed in the theorem. For instance, suppose that M

consists precisely of the elements of G_1 spanned by a rank-$(t-1)$ flat of G_0. In this case, M has no independent flat, and any projective or affine geometry contained in M must be over a *proper* subfield of $\mathrm{GF}(p^t)$. Moreover, if t happens to be prime, then the only such subfield is $\mathrm{GF}(p)$. In spite of this, we will show that this matroid does indeed contain a projective geometry—and a very high-rank one at that.

Theorem 1.3. *For p prime and $t \in \mathbb{N}$, let $G_1 \cong \mathrm{PG}(r-1, p^t)$ and let $G_0 \cong \mathrm{PG}(r-1, p)$ be a restriction of G_1. For $i \in \{1, \ldots, t-1\}$, let G be obtained from G_1 by deleting every element not spanned by a rank-i flat of G_0. If $r \geqslant \max\{ik, i(t+2)\}$, then G has a rank-k flat F such that $G|F$ is a projective geometry over a proper subfield of $\mathrm{GF}(p^t)$. In particular, if t is any prime, then $G|F \cong \mathrm{PG}(k-1, p)$.*

Theorem 1.3 is an immediate consequence of one of our main lemmas in the proof of the Unavoidable Flats Theorem (see Lemma 3.6).

1.2 Generalizing Melchior's Theorem

Along with the Unavoidable Flats Theorem, the other main contribution of this thesis is a series of generalizations of Melchior's Theorem to higher dimensions. We begin by considering average plane-size. Throughout this thesis, the *size* of a flat refers to its number of points (as opposed to its number of elements). As pointed out earlier, there are real-representable matroids with arbitrarily large average plane-size, particularly those which are the direct sum of two (arbitrarily large) lines. In joint work with Rutger Campbell and Jim Geelen [10], we showed that these are the only such matroids.

Average Plane-Size Theorem. *In a real-representable matroid of rank at least four which is not the direct sum of two lines, the average plane-size is at most an absolute constant.*

The Average Plane-Size Theorem is a special case of a more general theorem on average hyperplane-size which we formulate over the course of this and the following section. We will prove the Average Plane-Size Theorem on its own at the beginning of Chapter 6. Although its proof is subsumed by that of its generalization, we present it by itself because it is very intuitive and effectively demonstrates the broad strokes of the latter. In order to formulate our theorem for general hyperplanes, we must properly generalize the structure of two skew lines. As we will discuss, this turns out to be a rather complex problem.

The structure of two skew lines is generalized by the following notion of degeneracy introduced independently by Lovász [40] in the context of rigidity theory, and by Lund [41] and Do [15] in incidence geometry. For an integer $k \geqslant 2$, we call a matroid M *k-degenerate* if it either has rank at most one, or there is a collection $(F_1, \ldots, F_t)$ of flats of M, each having rank at least two, such that $E(M) \subseteq F_1 \cup \cdots \cup F_t$ and $\sum_{i=1}^{t}(r(F_i) - 1) \leqslant k - 1$. Given a k-degenerate matroid, we may always assume that the flats in $(F_1, \ldots, F_t)$ are pairwise-skew, as otherwise we can replace a non-skew pair with the span of their union. We call a matroid 1-*degenerate* if and only if it is the empty matroid. Observe that a matroid is 2-degenerate if and only if it has rank at most two, and it is 3-degenerate if and only if it is either a direct sum of two lines or has rank at most three. We call a subset $X \subseteq E(M)$ *k-degenerate* (in M) if $M|X$ is a k-degenerate matroid.

For a k-degenerate matroid M as defined above, it follows by the pigeonhole principle that every rank-k flat in M must contain one of the flats F_i, and so the average size of a rank-k flat could be arbitrarily large. This generalizes what occurs with planes in the direct sum of two lines, and so one optimistically might expect, in light of the Average Plane-Size Theorem, that forbidding k-degeneracy is sufficient for bounding the average size of a rank-k flat by an absolute constant (depending only on k). It turns out that this is *almost* true.

As shown by Lund [41] and Do [15], one can obtain powerful approximations on the number of rank-k flats in a real-representable matroid M by considering a largest k-degenerate set. For instance, Lund gave a best-possible approximation (within a multiplicative error) on the number of rank-k flats in M under the assumption that, for some constant c_k depending only on k, at least c_k points lie outside a largest k-degenerate set. We present the theorems of Lund and Do in Chapter 4, as they both play important roles in our proofs.

Broadly speaking, our approach for bounding the average size of a flat of given rank $k \geqslant 2$ will be to combine a good approximation on the number of rank-k flats in a matroid with a similar approximation on the *aggregate size* of the rank-k flats. For example, in Chapter 6 we will prove an upper bound, within a multiplicative error of Lund's approximation, on the aggregate size of the rank-k flats in a matroid. This means that the average size of rank-k flat in a real-representable matroid is absolutely bounded, provided that there are at least c_k points outside of a largest k-degenerate set. With a bit of extra effort, we will similarly use Lund's theorem to bound the average size of a flat of *any* rank. This result was also originally proved in [10].

Average Flat-Size Theorem. *For any integer $r \geqslant 1$, the average flat-size in a rank-r real-representable matroid is at most a constant depending only on r.*

For $r \leqslant 4$, the Average Flat-Size Theorem follows by Melchior's Theorem and (for $r = 4$) the Average Plane-Size Theorem. One appealing feature of this result is that, unlike the Average Plane-Size Theorem and its generalization, it does not depend on any considerations for the structure of the matroid. In contrast, for a given integer $k \geqslant 2$, there are real-representable matroids in which the average size of a rank-k flat is arbitrarily large. As explained above, we know from a theorem of Lund (which we have not yet stated explicitly) that there is a constant c_k such that all but c_k of the points in such a matroid are contained in a k-degenerate set. It turns out that this constant c_k is necessarily positive for all $k \geqslant 5$. To see this, consider a red-blue-coloured non-Fano matroid as in Fig. 1.1. Now, for some (arbitrarily large) positive integer m, let N be the matroid obtained from this non-Fano matroid by iteratively two-summing an m-point line onto each of the red points. This matroid N is rank-6 but is not 5-degenerate. Moreover, the average size of a hyperplane of N is at least m, because every hyperplane of N contains one of the three m-point lines. This is precisely because every line in Fig. 1.1 contains one of the three red points.

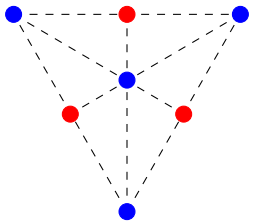


Figure 1.1: A red-blue colouring of F_7^- with no monochromatic blue line.

This construction demonstrates that k-degeneracy does not fully characterize the real-representable matroids where the average size of a rank-k flat is large. The rank-6 matroid N defined above does, however, share a common feature with 5-degenerate matroids: namely, there is a small collection of flats such that every hyperplane of N contains one of them, and this is ostensibly what causes the average hyperplane-size to be large. We will show that this is a characteristic of all real-representable matroids with large average hyperplane-size. The following was originally proved in joint work with Rutger Campbell and Ben Lund [11].

Average Hyperplane-Size Theorem, Part I. *For $k \geqslant 2$, if M is a rank-$(k+1)$ real-representable matroid, then either*

(i) the average size of a hyperplane of M is bounded by a constant depending only on k, or

(ii) there is a collection $L_1, \ldots, L_t$, where $t \leqslant k-1$, of lines of M such that every hyperplane of M contains one of them. Moreover, if M^+ is any matroid obtained from M by freely adding elements to these lines, then every hyperplane in M^+ still contains one of them.

While this theorem provides a tangible geometric explanation as to why a given real-representable matroid has large average hyperplane-size, it is not helpful from a complexity point of view. In particular, verifying whether outcome (ii) of the theorem holds is at least as complex as calculating the average hyperplane-size by brute force. In contrast, the Average Plane-Size Theorem reduces the problem of bounding the average plane-size in a real-representable matroid to a very simple task: verifying whether the matroid is a direct sum of two lines. Similarly, for any integer $k \geqslant 2$, applying Lund's theorem as described above reduces the problem of bounding the average hyperplane-size in a rank-$(k+1)$ real-representable matroid to verifying whether all but at most c_k of its points are contained in a k-degenerate set, where c_k is some constant depending only on k. For the second part of the Average Hyperplane-Size Theorem, we will characterize the constant c_k.

1.3 Near-degeneracy of real-representable matroids with large average hyperplane-size

In the course of proving Part I of the Average Hyperplane-Size Theorem, we will show that, in a rank-$(k+1)$ real-representable matroid M with large average hyperplane-size, there is a k-degenerate set such that, when each of its component flats are "principally truncated" into points (formally defined in Chapter 6), every hyperplane in the resulting matroid contains one of them. Colouring these truncated points red and the remaining points blue, we obtain a bicoloured matroid, like the one in Fig. 1.1, where every monochromatic hyperplane is red. Consequently, if we can upper-bound the number of blue points in a bicoloured real-representable matroid with this property, then we will upper-bound the number of points outside a k-degenerate set in M. In rank three, this is equivalent to Motzkin's Problem.

Throughout this thesis, a 2-*colouring* of a simple matroid M refers to a partition (R, B) of its groundset into sets of 'red' and 'blue' elements. In response to a question of Graham (see [31]), Motzkin [45] and Rabin and independently Chakerian [12] proved that, for any 2-colouring (R, B) of a rank-3 real-representable matroid M, there is a monochromatic line. Intuitively, if every monochromatic line in M is red, then the number of blue points should not be too large compared to the number of red ones. Of course, for this to hold true, the

set of blue points cannot be collinear. In light of this, Motzkin [44], and independently Grünbaum [30] and Erdős and Purdy [21], asked: if the blue points in M are not collinear and every monochromatic line is red, how large can $|B|$ be compared to $|R|$? In his original paper [44], Motzkin proved that $|B| \leqslant (|R|-1)^2$. The best general upper bound currently is $|B| \leqslant \frac{11}{4}|R| + \frac{13}{4}$, as was proved by Huicochea, Leaños and Rivera [35] by building on work of Pinchasi [49]. On the other hand, Grünbaum [31] demonstrated that the best-possible upper bound is at least $|R| + 4$.

In the context of our proof, the bicoloured matroid we are dealing with may have rank greater than three. The question is: is the number of blue points bounded in this case? In rank three, the only necessary condition was for the set of blue points to be non-collinear. In higher dimensions, there is not such an obvious condition to be imposed. Indeed, consider a 2-colouring (R, B) of a simple rank-r real-representable matroid M, such that every monochromatic hyperplane is red. For $r \geqslant 4$, it is entirely possible that $|B|$ is arbitrarily large while B is spanning; consider a direct sum of two arbitrarily large lines, each containing one red point and whose remaining points are blue. Requiring that B is not $(r-1)$-degenerate also does not work, as can be demonstrated using the rank-6 matroid constructed from Fig. 1.1. We get around these issues by imposing an even stronger generalization of the non-collinearity condition: that for every line in M spanned by a pair of blue points, there is a monochromatic-blue rank-$(r-2)$ flat skew to it. As we will show, this condition gives rise to a well-defined generalization of Motzkin's Problem. It also turns out to work perfectly in the context of our proof.

For integers $k, t \geqslant 1$, we define $b_k(t)$ to be the largest number n for which there exists a simple rank-$(k+1)$ real-representable matroid M having a 2-colouring (R, B) with $|R| = t$, $|B| = n$ such that

(C1) $r_M(B) \geqslant 2$;

(C2) for every hyperplane H of M, $H \cap R \neq \varnothing$; and

(C3) for every line L of $M|B$, there is a rank-$(k-1)$ flat F of M skew to L such that $F \cap R = \varnothing$.

If, for given $k, t \geqslant 1$, there does not exist a number n for which the above is satisfied, then we set $b_k(t) = 0$. For $k = 1$, each point is a hyperplane, so (C2) implies that $b_1(t) = 0$ for all t. Also note that, for $k = 2$, (C3) is equivalent to the condition that the set of blue points is not collinear, and so $b_2(t)$ is precisely the solution to Motzkin's Problem.

In Chapter 5, we will prove some bounds on the numbers $b_k(t)$. Although we do not yet know its precise asymptotic dependence on k, we will prove that $b_k(t)$ is linear in t.

Theorem 1.4. *There is a constant $C > 0$ such that for all integers $k, t \geqslant 1$, we have $Ckt \leqslant b_k(t) \leqslant C_k t$ for some constant C_k depending only on k.*

Now we are ready to complete our theorem on average hyperplane-size. This and Theorem 1.4 were both originally proved in joint work with Rutger Campbell and Ben Lund [11].

Average Hyperplane-Size Theorem, Part II. *For each integer $k \geqslant 2$, there is a constant C_k such that, if M is a rank-$(k+1)$ real-representable matroid whose average hyperplane-size is greater than C_k, then for some $j \in \{1, \ldots, k-1\}$, M has a $(j+1)$-degenerate restriction containing all but at most $b_{k-j}(j)$ of its points.*

The lower bound given by this theorem on the size of a degenerate set is best-possible. To see this, we again consider the construction based on Fig. 1.1. It is easily verified that $b_2(3) = 4$, and so the rank-6 matroid N which we constructed from Fig. 1.1 has a 4-degenerate set containing all but precisely $b_2(3)$ of its points. More generally, for any $t \geqslant 3$, let (R, B) be a 2-colouring of a rank-3 real-representable matroid M such that B is not collinear, every monochromatic line of M is red, $|R| = t$, and $|B| = b_2(t)$. Now for an (arbitrarily large) positive integer m, let M' be the rank-$(t+3)$ matroid obtained from M by iteratively 2-summing an m-point line onto each red point. Then M' has average hyperplane-size at least m, and is not $(t+2)$-degenerate. But M' has a $(t+1)$-degenerate set (comprising the t large lines) containing all but exactly $b_2(t)$ of its points.

In Chapter 5 we will prove that $b_k(2) = 0$ for all k. Together with the fact that $b_2(3) = 4$, this yields a fairly precise characterization of the real-representable matroids of rank at most six with large average hyperplane-size.

Corollary 1.5. *Let M be a simple real-representable matroid. If $r(M) \leqslant 5$, then either M is $(r(M)-1)$-degenerate, or the average hyperplane-size in M is at most an absolute constant. If $r(M) = 6$ and M is not 5-degenerate, then either M has a 4-degenerate restriction of size at least $|E(M)| - 4$, or the average hyperplane-size in M is at most an absolute constant.*

1.4 Matroid terminology

All of the theorems in this thesis intrinsically concern a finite set of points in some affine or projective space and its spanned subspaces. The language of matroid theory allows us to handle such geometric information in a purely combinatorial manner and without referring

to any specific coordinatization. This section (which is essentially an appendix) is not meant to provide a comprehensive introduction to matroid theory, but rather to provide readers with the terminology and intuition necessary to understand the language of this thesis. For a proper introduction to the subject of matroid theory, we refer the reader to Oxley's textbook [48].

Let A be a matrix with entries in some field $\mathbb{F}$, let E be the set of column indices of A, and let $\mathcal{I}$ be the collection of subsets of E whose corresponding set of columns is linearly independent. Then the pair $M(A) = (E, \mathcal{I})$ is the *matroid represented by* A. The set E is called the *groundset* of $M(A)$, and the sets in $\mathcal{I}$ are called the *independent sets*. Any subset of E which is not independent in $M(A)$ is called *dependent*. Given an (abstract) matroid $M = (E, \mathcal{I})$ and a field $\mathbb{F}$, we say that M is $\mathbb{F}$-*representable* if there is a matrix A with entries in $\mathbb{F}$ such that $M = M(A)$; in this case, we say that A is an $\mathbb{F}$-*representation* of M. If $\mathbb{F}$ is the field of real numbers, then we say that M is *real-representable*. Note that we can also view an $\mathbb{F}$-representation of a matroid M simply as a map $R : E(M) \to \mathbb{F}^r$ (for some positive integer r). We will opt for this viewpoint at certain points in the proof of the Unavoidable Flats Theorem, where it will be helpful to view a representation as a subset of an additive group as opposed to the set of columns of a matrix. If two matroids M and N are isomorphic, then we write $M \cong N$.

In this thesis, we will only be concerned with representable matroids, since they correspond to finite sets of points in space. Indeed, consider a finite set E of points in the d-dimensional projective geometry over some field $\mathbb{F}$. For each point in E, we can fix a representative vector in $\mathbb{F}^{d+1}$ (based on its projective coordinates), and form a matrix B whose column set consists of these vectors (and is indexed by E). The matroid $M(B)$ then captures all of the linear-geometric information about the point-set E without recording its coordinatization.

In the matroid $M(A)$ described above, it is possible that an element $e \in E$ is represented by the zero vector in A. In this case, we say that e is a *loop* of M. For distinct non-loop elements $e, f \in E$, if the vectors representing e and f are scalar multiples of one another, then these elements are said to form a *parallel pair*. A matroid is called *simple* if it has no loops and no parallel pairs. The matroids corresponding to finite sets of points in space are simple, and so we are not interested in non-simple matroids in this thesis. However, non-simple matroids will arise in our proofs through the operation of *contraction* (defined below).

There are many different ways of thinking about the independent sets of a matroid $M = (E, \mathcal{I})$. An (inclusion-wise) maximal independent set of M is called a *basis* of M. For any subset $X \subseteq E$, every maximal independent subset of X has the same number of

elements; we may refer to such a set as a 'basis' of X (and only when the ambient matroid is obvious). It will also be convenient to consider minimal dependent sets of M; these sets are called the *circuits* of M. If the entire groundset of a rank-r matroid M forms a circuit, then we may write $M \cong U_{r,r+1}$ as a convenient shorthand. (This is because $U_{r,n}$ denotes the rank-r *uniform matroid* on n elements; see [48].) Given a basis B of M and any element $e \in E\backslash B$, the set $B \cup \{e\}$ contains a unique circuit of M, which we call the *fundamental circuit* of $B \cup \{e\}$.

The *rank* of a matroid $M = (E, \mathcal{I})$ refers to the size of a basis of M, and is denoted by $r(M)$. For a set $X \subseteq E$, we let $r_M(X)$ denote the size of a largest independent subset of X; we call this the *rank* of X in M. The map $r_M : 2^E \to \mathbb{Z}$ is called the *rank function* of M. (We may drop the subscript and denote the rank function simply by r when the corresponding matroid is clear.) The rank function has an important property called *submodularity*. That is, for any sets $X, Y \subseteq E$, we have

$$r(X) + r(Y) \geqslant r(X \cup Y) + r(X \cap Y).$$

Geometrically, rank can be thought of as "dimension plus one". If M is representable over some field $\mathbb{F}$, we can think of any representation of M as embedded in a projective geometry over $\mathbb{F}$. In this case, for any $X \subseteq E$, $r(X) - 1$ is the dimension of the projective subspace spanned by the representatives of X.

For $e \in E$ and $X \subseteq E$, we say that e is *spanned* by X (in M) if $r(X \cup \{e\}) = r(X)$. A *flat* of M is a subset of E which contains every element that it spans. For $X \subseteq E$, the *closure* of X, denoted $\mathrm{cl}_M(X)$, is the unique rank-$r(X)$ flat of M containing X. Viewing an $\mathbb{F}$-representation of M as embedded in a projective space over $\mathbb{F}$, the flats of M are the inclusion-wise maximal sets spanning some projective subspace. The rank-1, -2 and -3 flats of M are called *points*, *lines* and *planes* of M, respectively. A rank-$(r(M) - 1)$ flat of M is called a *hyperplane* of M. This terminology allows us to translate the classical theorems stated at the very beginning of this chapter from their usual geometric formulations into the language of matroid theory.

We finish this section by defining a couple of standard matroid operations which will be used many times throughout the thesis. For a matroid $M = (E, \mathcal{I})$ and a subset $X \subseteq E$, the *deletion* of X from M, denoted $M\backslash X$, is the matroid with groundset $E\backslash X$ where a set is independent if and only if it is independent in M. The *restriction* of M to X, denoted $M|X$, is the matroid obtained by deleting $E\backslash X$ from M. Finally, the *contraction* of X in M is the matroid $M/X = (E\backslash X, \mathcal{I}')$, where $I \in \mathcal{I}'$ if and only if $r(X \cup I) = r(X) + |I|$. Given an $\mathbb{F}$-representation of M, one obtains an $\mathbb{F}$-representation of M/X by "projecting out" or "quotienting on" the subspace spanned by X. As mentioned above, non-simple

matroids may arise in some of our proofs as a result of contraction. When this happens, we may pass to a "simplification" of the contracted matroid, whereby we delete all loops and all but one element from every parallel class. When convenient, we may use $\mathrm{si}(M)$ to denote an arbitrary simplification of a matroid M.

Chapter 2

Two-Point Lines in Matroids Representable over a Prime Field

The main purpose of this chapter is to introduce some preliminaries for the proof of the Unavoidable Flats Theorem, particularly a lifting argument of Kelly [37] (which we will adapt) as well as some classical geometric Ramsey theory. Along the way, we will prove a weaker version of the Unavoidable Flats Theorem concerning the existence of two-point lines in matroids representable over a given prime field. The proof of this weaker result will exhibit many of the key ideas in the proof of the Unavoidable Flats Theorem and demonstrate the roles of the preliminary tools and techniques that we introduce.

Theorem 2.1. *For any prime p and positive integer k, there exists a number $N_p(k)$ such that, if M is a simple* GF(p)*-representable matroid with $r(M) \geqslant N_p(k)$, then M contains either a two-point line or a rank-k affine or projective geometry as a flat.*

In [37], Kelly extended the Sylvester-Gallai Theorem to complex-representable matroids by showing that a rank-4 complex-representable matroid has a two-point line. We begin by explaining Kelly's proof technique and how we will adapt it in order to show that a matroid representable over a given finite field contains either a large independent set or affine geometry. After that, we introduce the Ramsey theorems which we will use to ultimately find the desired unavoidable flats. By the end of the chapter, we will have proved Theorem 2.1.

2.1 Kelly's lifting argument

The first step in the proof of the Unavoidable Flats Theorem is to show that a matroid with huge rank representable over a given finite field contains either a large independent set as a flat or a large affine geometry as a restriction. Our proof technique is an adaptation of Kelly's proof of the following.

Kelly's Theorem ([37]). *Every rank-4 complex-representable matroid has a two-point line.*

Recall that there are rank-3 complex-representable matroids, such as the ternary affine plane $\mathrm{AG}(2,3)$, with no two-point line, and so the rank bound of four in Kelly's Theorem is best-possible. There are currently three known proofs of Kelly's Theorem; see also [19, 20]. Kelly's original proof fundamentally relies on a powerful theorem of Hirzebruch [34], which guarantees the existence of a line with at most three points in any complex-representable matroid of rank at least three. In fact, Hirzebruch extended Melchior's Theorem to the setting of complex-representability.

Hirzebruch's Theorem. *In a complex-representable matroid of rank at least three, the average number of points in a line is less than four.*

Now, to prove Kelly's Theorem, consider a simple rank-4 complex-representable matroid M, and suppose to the contrary that M has no two-point line. Let $e \in E(M)$. By Hirzebruch's Theorem, M/e has a three-point line, and consequently there is a plane in M comprising three lines L_1, L_2 and L_3 containing e in their intersection. From here, the key to proving Kelly's Theorem is to show that a rank-3 complex-representable matroid comprising three co-punctual lines must have a two-point line. In his original proof, Kelly argues that, for any field $\mathbb{F}$, if an $\mathbb{F}$-representable matroid has this structure, then either there is a two-point line or $\mathbb{F}$ has positive characteristic. We essentially give the same argument below, with the slight difference that we translate the three-point lines of the matroid into additive dependencies in a group. This approach will be particularly helpful when we consider finite fields.

Consider the rank-3 matroid $N = M|L_1 \cup L_2 \cup L_3$. Then N has no two-point line, since it is a restriction of M to a flat. Now let $\{f_1, f_2, f_3\}$ be a line of N, where $f_i \in L_i \backslash \{e\}$ for each $i \in \{1,2,3\}$, and let $R : E(N) \to \mathbb{C}^3$ be a representation of N where $R(e) = (1,0,0)$, $R(f_2) = (0,1,0)$ and $R(f_3) = (0,0,1)$. By possibly performing some row- and column-scalings, we may assume that $R(f_1) = (0,1,1)$, and that the last non-zero entry of each vector in this representation is equal to one. In particular, this means that, for an element

$f \in L_i \backslash \{e\}$ and $i \in \{1,2,3\}$, the last two entries of $R(f)$ are given by $(1,1), (1,0)$ or $(0,1)$, respectively.

For each $f \in E(N)$, let $\rho(f) \in \mathbb{C}$ be the first entry of $R(f)$, and let $A_i = \{\rho(f) : f \in L_i \backslash \{e\}\}$ for each $i \in \{1,2,3\}$. The assumption that N has no two-point line then gives us the following.

Claim 2.2. $A_1 = A_2 + A_3$.

Proof. Let $\alpha \in A_1$. Then $R(f) = (\alpha, 1, 1)$ for some $f \in L_1 \backslash \{e\}$. Since N has no two-point line, there is an element $f' \in L_3 \backslash \{e\}$ such that $\{f, f_2, f'\}$ is a line. Since $R(f') = (\alpha', 0, 1)$ for some $\alpha' \in \mathbb{C}$, it follows that $\alpha = 0 + \alpha' \in A_2 + A_3$. Conversely, let $\alpha_2 \in A_2$ and $\alpha_3 \in A_3$, and let $e_2 \in L_2 \backslash \{e\}$ and $e_3 \in L_3 \backslash \{e\}$ such that $R(e_2) = (\alpha_2, 1, 0)$ and $R(e_3) = (\alpha_3, 0, 1)$. Since N has no two point line, there exists $e_1 \in (L_1 \backslash \{e\}) \cap \mathrm{cl}(\{e_2, e_3\})$. But $R(e_1) = (\alpha, 1, 1)$ for some $\alpha \in \mathbb{C}$, and $\alpha = \alpha_2 + \alpha_3$. □

Since $\rho(f_2) = \rho(f_3) = 0 \in A_2 \cap A_3$, we conclude from our next lemma that A_1 is an additive subgroup of $\mathbb{C}$. Since $|A_1| \geqslant 2$, this implies that $|A_1|$ is not finite, a contradiction. Note that we call a group *additive* if its operation is denoted by $+$ and it is abelian.

Proposition 2.3. *Let A, B and C be finite, non-empty subsets of an additive group Z, where $A = B + C$. If $0 \in B \cap C$ and $|A| \leqslant \min\{|B|, |C|\}$, then A is a subgroup of Z and $A = B = C$.*

Proof. Since $0 \in B \cap C$, we have that $0 \in A$ and that $B, C \subseteq A$. Since $|A| \leqslant |B|, |C|$, it follows that $A = B = C$. Therefore $A = A + A$, which means that A is a subgroup of Z. □

To summarize, the proof of Kelly's Theorem follows by lifting a three-point line to a plane consisting of three co-punctual lines, and then using complex-representability to force the existence of a two-point line in this matroid. Specifically, we showed that if such a matroid had no two-point line, then it would have a collinear set corresponding to a non-trivial additive subgroup of the complex numbers, contradicting the finiteness of the matroid. Applying this very same argument to a matroid representable over a finite field $\mathbb{F}$, we do not obtain the same contradiction, but we can exploit the correspondence to an additive subgroup in order to build up an affine geometry. In particular, by iteratively lifting a line of bounded length, we can find a set of elements in our matroid corresponding to a high-dimensional additive subgroup of a vector space over $\mathbb{F}$. These elements will then 'generate' all of the points of an affine geometry. (We had originally based this argument

on an application of Kneser's Theorem from additive combinatorics [38], before realizing that the more specialized Proposition 2.3 was sufficient.) The procedure is encapsulated in the next lemma, whose statement—which is slightly more technical than necessary for proving Theorem 2.1—is tailored for its later use in the inductive proof of the Unavoidable Flats Theorem.

Lemma 2.4. *Let p be prime and let t, k, ℓ be positive integers where $3 \leqslant \ell \leqslant p+1$. If M is a simple $\mathrm{GF}(p^t)$-representable matroid and F is a rank-k flat in M such that M/F has an ℓ-point line, then either:*

(i) there exists $f \in F$ and $e \in E(M)\backslash F$ such that $\{e, f\}$ is a two-point line in M;

(ii) there is a line in M which is skew to F and has at most $\ell - 1$ points; or

(iii) M has an $\mathrm{AG}(k, p)$-restriction contained in a point of M/F.

Proof. Assume that neither (i) nor (ii) holds. Let $I = \{e_1, \dots, e_k\}$ be a basis of F, and let P_1, P_2 and P_3 be points of M/F, where $|P_1| = \min\{|P_1|, |P_2|, |P_3|\}$. We will consider the matroid $N = M|F \cup P_1 \cup P_2 \cup P_3$. Because (ii) does not hold and N/F forms a three-point line, every line in N which is skew to F has exactly three points. Thus we can choose a line $\{f_1, f_2, f_3\}$ of N where $f_i \in P_i$ for each $i \in \{1, 2, 3\}$. Since (i) does not hold, we can choose elements $g_i \in \mathrm{cl}(\{f_1, e_i\})\backslash\{f_1, e_i\}$ for each $i \in \{1, \dots, k\}$.

Let $\mathbb{F} = \mathrm{GF}(p^t)$. We will now choose an $\mathbb{F}$-representation of N which translates the collinearity of triples in $P_1 \times P_2 \times P_3$ into additive dependencies in $\mathbb{F}^{k+2}$. For each $i \in \{1, \dots, k+2\}$, let v_i denote the ith standard basis vector in $\mathbb{F}^{k+2}$; for a vector $u \in \mathbb{F}^{k+2}$ and $i \in \{1, \dots, k+2\}$, let $u^i \in \mathbb{F}$ denote the ith coordinate of u.

Claim 2.4.1. *There is an $\mathbb{F}$-representation $R : E(N) \to \mathbb{F}^{k+2}$ of N such that*

1. *for all $e \in E(N)$, $R(e)^m = 1$, where m is largest element in the support of $R(e)$;*
2. *for each $i \in \{1, \dots, k\}$, $R(e_i) = v_i$;*
3. *$R(f_2) = v_{k+1}$, $R(f_3) = v_{k+2}$, and $R(f_1) = v_{k+1} + v_{k+2}$; and*
4. *for each $i \in \{1, \dots, k\}$, $R(g_i) = R(e_i) + R(f_1)$.*

Proof of claim. Since $I \cup \{f_2, f_3\}$ is a basis of N, there is an $\mathbb{F}$-representation R of N such that $R(e_i) = v_i$ for each $i \in \{1, \dots, k\}$, $R(f_2) = v_{k+1}$, and $R(f_3) = v_{k+2}$. We consider a matrix A whose set of columns is $\{R(e) : e \in E(N)\}$. By scaling a row or column of

A, we obtain another $\mathbb{F}$-representation of N. In particular, by column-scaling, we may always assume that (1) holds. Since $f_1 \in \mathrm{cl}_M(\{f_2, f_3\})$, $\mathrm{supp}(R(f_1)) = \{k+1, k+2\}$. By scaling, we may assume that $R(f_1) = v_{k+1} + v_{k+2}$. Now consider $i \in \{1, \ldots, k\}$. By (1), $R(g_i) = R(f_1) + \alpha v_i$ for some $\alpha \in \mathbb{F}$. By possibly scaling the ith row of A and the column $R(e_i)$, we may assume that $\alpha = 1$. □

Let R be an $\mathbb{F}$-representation of N as given by the claim. For each $e \in E(N)$, let $\rho(e)$ be the vector in $\mathbb{F}^k$ defined by the first k entries of $R(e)$, and let $A_i = \{\rho(e) : e \in P_i\}$ for each $i \in \{1, 2, 3\}$. Note that, by condition (1) of the claim, the map $\rho : P_i \to \mathbb{F}^k$ is injective for each $i \in \{1, 2, 3\}$. In particular, this means that $|A_1| \leqslant |A_2|, |A_3|$. Because every line in N which is skew to F has exactly three points, it follows by the very same argument as in the proof of Claim 2.2, that $A_1 = A_2 + A_3$. Since $\rho(f_2) = \rho(f_3) = 0 \in A_2 \cap A_3$, it follows by Proposition 2.3 that A_1 is an additive subgroup of $\mathbb{F}^k$. Since A_1 contains the k standard basis vectors $\rho(g_1), \ldots, \rho(g_k)$, we have that $\mathrm{GF}(p)^k \subseteq A_1$. It follows that $M|P_1$ has an $\mathrm{AG}(k, p)$-restriction. □

In the proof of Theorem 2.1, we consider a high-rank $\mathrm{GF}(p)$-representable matroid with no $\mathrm{AG}(k, p)$-restriction, and apply Lemma 2.4 iteratively so as to find smaller and smaller lines. In this way, we ultimately end up with the desired two-point line. In Chapter 3, we will generalize Lemma 2.4 in order to similarly find independent flats of any fixed rank. This argument as described, however, does have a glaring issue: namely, we apply the induction to a minor M/F, where F is a rank-k flat of our matroid; so what do we do if M/F has an $\mathrm{AG}(k, p)$-restriction? As we will see in the next section, we can get around this issue using Ramsey theory.

2.2 Ramsey theory

The proof of the Unavoidable Flats Theorem relies heavily on Ramsey theorems for finite affine and projective geometries. For affine geometries, we have the following classic and well-known result [29, 51].

Affine Ramsey Theorem. *For any prime-power q and positive integers k, t, ℓ, there is a number $N = \mathrm{AR}_q(k, t, \ell)$ such that, for any ℓ-colouring of the rank-t flats of $G \cong \mathrm{AG}(N, q)$, there is a monochromatic rank-k flat.*

Formally, the conclusion of this theorem says that there is a rank-k flat F of G such that every rank-t flat of $G|F$ has the same colour. In the proof of the Unavoidable Flats

Theorem, we will use the Affine Ramsey Theorem to deduce that a bounded lift of an affine geometry of huge rank contains an affine geometry of large rank, thereby answering the question at the end of Section 2.1. We state this formally in our next lemma, which was already proved in a stronger form by Geelen and Nelson [27, Lemma 5.2]. Note that, for a representation A of a matroid M and a subset $X \subseteq E(M)$, we denote by $A|X$ the submatrix of A comprising the columns of A corresponding to the elements of X.

Lemma 2.5. *For any prime-power q and positive integers m and k, if J is an m-element independent set in a $\mathrm{GF}(q)$-representable matroid M and M/J has an $\mathrm{AG}(N, q')$-restriction, where $\mathrm{GF}(q')$ is a subfield of $\mathrm{GF}(q)$ and $N \geqslant \mathrm{AR}_{q'}(k, 1, q^m)$, then M has an $\mathrm{AG}(k-1, q')$-restriction.*

Proof. We may assume that $M/J \cong \mathrm{AG}(N, q')$. Thus M has a $\mathrm{GF}(q)$-representation of the form

$$\begin{pmatrix} I & B \\ 0 & A \end{pmatrix},$$

where I denotes the $(m \times m)$-identity matrix, and A is a representation of $\mathrm{AG}(N, q')$ whose first row is the all-ones vector. Consider the q^m-colouring of the points of M/J where each point is coloured by its associated column in B. By the Affine Ramsey Theorem, M/J has a monochromatic rank-k flat F. Because the first row of A is the all-ones vector, and the columns of $B|F$ are all the same, we see that $A|F$ is a representation of $M|F$. Thus $M|F \cong \mathrm{AG}(k-1, q')$. □

Lemmas 2.4 and 2.5 together imply that a matroid with huge rank representable over a given prime field either has a two-point line or an affine geometry as a restriction.

Lemma 2.6. *For any prime p and positive integers k, t, where $2 \leqslant t \leqslant p+1$, there is a number $N_p(k, t)$ such that, if M is a $\mathrm{GF}(p)$-representable matroid with $r(M) \geqslant N_p(k, t)$, then M has either an $\mathrm{AG}(k, p)$-restriction or a line with at most t points.*

Proof. For all k, set $N_p(k, p+1) = 2$. Now let $2 \leqslant t \leqslant p$, and assume that $N_p(k, t+1)$ exists for all k. Let M be a $\mathrm{GF}(p)$-representable matroid with

$$r(M) \geqslant k + N_p(\mathrm{AR}_p(k, 1, p^k), t+1).$$

Let F be a rank-k flat in M. By induction, M/F has either a $(t+1)$-point line or an $\mathrm{AG}(\mathrm{AR}_p(k, 1, p^k), p)$-restriction. In the former case, the result follows by Lemma 2.4. In the latter case, the result follows by Lemma 2.5. □

Lemma 2.6 gets us very close to proving Theorem 2.1. In particular, it allows us to assume that our $\mathrm{GF}(p)$-representable matroid M contains a spanning affine geometry. In order to find the unavoidable flats in this case, we use a Ramsey theorem for finite *projective* geometries. The following was originally proved by Graham, Leeb and Rothschild [28].

Projective Ramsey Theorem. *For any prime-power q and positive integers k, t, ℓ, there is a number $N = \mathrm{PR}_q(k, t, \ell)$ such that, for any ℓ-colouring of the rank-t flats of $G \cong \mathrm{PG}(N, q)$, there is a monochromatic rank-k flat.*

By taking $r(M) \geqslant N_p(\mathrm{PR}_p(k, 1, 2), 2)$, the proof of Theorem 2.1 is completed by Lemma 2.6 and the following.

Lemma 2.7. *For any prime-power q and positive integer k, if M is a simple $\mathrm{GF}(q)$-representable matroid with an $\mathrm{AG}(N, q)$-restriction, where $N \geqslant \mathrm{PR}_q(k, 1, 2)$, then M has a rank-k flat F such that either $M|F \cong \mathrm{AG}(k-1, q)$ or $M|F \cong \mathrm{PG}(k-1, q)$.*

Proof. We may assume that $r(M) = N+1$, that is, M has a spanning $\mathrm{AG}(N, q)$-restriction. Let $X \subseteq E(M)$ such that $M|X \cong \mathrm{AG}(N, q)$, and choose $G \cong \mathrm{PG}(N, q)$ such that M is a restriction of G. Then $E(G)\backslash X$ is a hyperplane in G, and so $G\backslash X \cong \mathrm{PG}(N-1, q)$. Consider the 2-colouring of the points of $G\backslash X$ based on membership in M: by the Projective Ramsey Theorem, there is a rank-k flat F of $G\backslash X$ such that either $F \subseteq E(M)$ or $F \subseteq E(G)\backslash E(M)$. Now fix an element $e \in X$, and consider that flat $H = \mathrm{cl}_G(F \cup \{e\}) \cap E(M)$ of M. In the former case, $M|H \cong \mathrm{PG}(k, q)$, and in the latter, $M|H \cong \mathrm{AG}(k, q)$, as desired. □

Although we have not yet proved the Unavoidable Flats Theorem, we note as an aside that, by combining it with the Projective and Affine Ramsey Theorems, we can easily derive Corollary 1.1, our Ramsey theorem for matroids representable over a given finite field. In fact, we can prove a stronger theorem on colouring the flats of any given rank using any given number of colours.

Theorem 2.8. *For any finite field $\mathbb{F}$ and positive integers k, t, ℓ, if M is an $\mathbb{F}$-representable matroid of sufficiently high rank, then for any ℓ-colouring of the rank-t flats of M, there is a monochromatic rank-k flat.*

Proof. By the Unavoidable Flats Theorem, we may assume that M is either a free matroid or a projective or affine geometry over a subfield of $\mathbb{F}$. In the former case, the result follows by Ramsey's theorem for hypergraphs. Otherwise, the result follows by the Projective or Affine Ramsey Theorem. □

When dealing with matroids representable over a non-prime finite field, we will require another powerful tool from geometric Ramsey theory, which essentially says that, if a matroid representable over a given finite field has huge rank and is dense, then it contains a large affine geometry. This was originally proved by Furstenberg and Katznelson [22], generalizing a similar theorem of Brown and Buhler [9] concerning the existence of collinear triples. It also follows as a natural consequence of the Density Hales-Jewett Theorem, which was also first proved by Furstenberg and Katznelson [23].

Geometric Density Hales-Jewett Theorem. *For any prime-power q, $\epsilon > 0$ and positive integer k, there is a number $\mathrm{DHJ}_q(\epsilon, k)$ such that, if M is a simple $\mathrm{GF}(q)$-representable matroid with $r(M) \geqslant \mathrm{DHJ}_q(\epsilon, k)$ and $|E(M)| \geqslant \epsilon q^{r(M)}$, then M has an $\mathrm{AG}(k, q)$-restriction.*

As explained in Chapter 1, the proof of the Unavoidable Flats Theorem follows in two steps, the first of which involves showing that the matroid M contains either a large independent flat or affine geometry, thereby reducing the proof to the case that M has a spanning affine geometry (over the base field). This will follow by an argument similar to the proof of Theorem 2.1; the key difference is that, if M itself is not representable over the base field, then we cannot directly apply Lemma 2.7 to find the unavoidable flats, as we did above. Instead, we will exploit the structure of the spanning affine geometry and apply the Ramsey theorems introduced in this section in order to impose more and more structure on M, until we can find the unavoidable flats. This (very broadly) describes the second step of the proof. In this context, the Density Hales-Jewett Theorem together with Lemma 2.7 will allow us to make the very important assumption that M is sparse.

Chapter 3

The Unavoidable Flats Theorem

In this chapter, we prove the Unavoidable Flats Theorem. In the first section, we generalize the proof technique for Theorem 2.1 to show that, in a matroid of huge rank representable over a given finite field, there is a high-rank affine geometry or independent flat. This reduces the proof of the Unavoidable Flats Theorem to the case that the matroid has a spanning projective geometry. This case is handled in Section 3.2. In Section 3.3, we combine the results to ultimately prove the theorem.

3.1 Matroids without an independent flat

The first step in the proof of the Unavoidable Flats Theorem is to show the following, our main result for this section.

Theorem 3.1. *For a prime p and integers $k, m, t \in \mathbb{N}$, there is a number $N_1 = N_1(p^t, k, m)$ such that, if M is a simple $\mathrm{GF}(p^t)$-representable matroid with $r(M) \geqslant N_1$, then M contains either a rank-m independent flat or an $\mathrm{AG}(k-1, p)$-restriction.*

The proof of Theorem 3.1 is similar to the proof of Lemma 2.6, where we showed that, for any prime p, a $\mathrm{GF}(p)$-representable matroid with sufficiently high rank and no $\mathrm{AG}(k, p)$-restriction has a two-point line. Using Lemma 2.4, we proved this by iteratively lifting lines of bounded size in order to find smaller ones. The first step in the proof of Theorem 3.1 is to generalize Lemma 2.4, so that we can do a similar lifting argument with flats of any given rank. In order to do this, however, it turns out that there is some additional information that we will need to account for, namely the connectivity of the

flat which we are lifting. (A matroid is *connected* if, for any pair of distinct elements of its groundset, there is a circuit containing both of them.) This was not an issue in the proof of Lemma 2.6, since every line with at least three points is connected. For this reason, we start by adapting Lemma 2.4 in a way that specifically handles the lifting of *circuits*.

Lemma 3.2. *For p prime and positive integers t, k, m where $m \geqslant 2$, let M be a simple $\mathrm{GF}(p^t)$-representable matroid with a rank-k flat F such that $\mathrm{si}(M/F) \cong U_{m,m+1}$. Then either*

(i) there exists $f \in F$ and $e \in E(M)\backslash F$ such that $\{e, f\}$ is a two-point line in M;

(ii) there exists a rank-m independent flat in M skew to F; or

(iii) M has an $\mathrm{AG}(k, p)$-restriction contained in a point of M/F.

Proof. We proceed by induction on $m \geqslant 2$. The base case $m = 2$ follows by Lemma 2.4. Now suppose that $m \geqslant 3$, and assume that neither (i) nor (ii) holds. Let $F_1, \ldots, F_{m+1}$ be the points of M/F. Since (ii) does not hold, there is a circuit $\{f_1, \ldots, f_{m+1}\}$ of M where $f_i \in F_i$ for each $i \in \{1, \ldots, m+1\}$. Let $M' = M/f_{m+1}\backslash F_{m+1}$. Then $\mathrm{si}(M'/F) \cong U_{m-1,m}$. If there exists $f \in F$ and $e \in E(M')\backslash F$ such that $\{e, f\}$ is a line in M', then $\{e, f\}$ is a line in M. If there is a rank-$(m-1)$ independent flat J of M' which is skew to F in M', then $J \cup \{f_{m+1}\}$ is a rank-m independent flat in M which is skew to F. Hence, by induction, M' has an $\mathrm{AG}(k, p)$-restriction contained in a point of M'/F. We may assume that this restriction is contained in F_1; that is, there exists $A \subseteq F_1$ such that $M'|A \cong \mathrm{AG}(k, p)$. Since $M|F_1 = (M/f_{m+1})|F_1$, we have $M|A \cong \mathrm{AG}(k, p)$, as desired. □

Now we can generalize the lifting argument to flats of any rank.

Lemma 3.3. *For p prime and positive integers t, k, m, n, where $n > m \geqslant 2$, if M is a $\mathrm{GF}(p^t)$-representable matroid of rank at least $m + k$ such that every rank-m flat of M has at least n points, then either*

(1) M has an $\mathrm{AG}(k, p)$-restriction,

(2) there is a rank-k flat F in M such that every rank-m flat in M/F has at least $n + 1$ points, or

(3) M has a rank-m flat H such that $M|H$ is not connected.

Proof. Let M' be a $\mathrm{GF}(p^t)$-representable matroid with $r(M') \geqslant m + k$ such that every rank-m flat in M' has at least n points. We may assume that M' is simple. Let F be a rank-k flat in M'; we may assume that there is a rank-m flat G in M'/F with at most n points. By hypothesis, $(M'/F)|G$ has exactly n points. Moreover, we may assume that $(M'/F)|G$ is connected.

Let M be the restriction of M' to the flat $F \cup G$. So M is a rank-$(m + k)$ matroid where every rank-m flat has at least n points, and M/F is a connected rank-m matroid with exactly n points. We will show that M has an $\mathrm{AG}(k, p)$-restriction.

Claim 3.3.1. *For every $f \in F$ and $e \in E(M)\backslash F$, $|\operatorname{cl}_M(\{e, f\})| \geqslant 3$.*

Proof of claim. Let $f \in F$ and let $e \in E(M)\backslash F$. Suppose to the contrary that $\{e, f\}$ is a line in M. Let H be a rank-m flat in M which is skew to F, such that $e \in H$. Because $|H| \geqslant n$, $M|H \cong \mathrm{si}(M/F)$ (and so $M|H$ has exactly n points). Let $N = M|\operatorname{cl}_M(\{f\} \cup H)$, and note that $\mathrm{si}(N/f) \cong M|H$.

First we show that there exists an element $h \in H$ such that such that $|\operatorname{cl}_M(\{f, h\})| \geqslant 3$. Suppose otherwise. Then, because $\mathrm{si}(N/f) \cong M|H$, f is a coloop in N. Since $N|H$ is connected, there exists a rank-$(m - 1)$ flat H' in $N|H$ such that $|H'| \leqslant |H| - 2$. But then $|\operatorname{cl}_M(\{f\} \cup H')| \leqslant n - 1$, a contradiction. Thus there exists a line L_2 in N containing f and at least two other elements g_1 and g_2.

Now let G be a hyperplane of N with $e, g_1 \in G$ and $f \notin G$. Then $N|G \cong \mathrm{si}(N/f)$, and so there is a circuit C of $N|G$ containing e and g_1 (since this matroid is connected). Let $h \in C\backslash\{e, g_1\}$, and let $X = C\backslash\{e, g_1, h\}$. Then $X \cup \{h, g_2, f\}$ is independent in N, and so we can choose a hyperplane G' of N with $f \notin G'$ and $X \cup \{h, g_2\} \subseteq G'$. Since $|G'| = n$, G' must intersect the line $\{e, f\}$, that is, $e \in G'$. But $X \cup \{e, h, g_2\}$ is independent in N and a circuit in M/f. So f is spanned by $X \cup \{e, h, g_2\}$ in N, a contradiction. □

Let $C = \{f_1, \ldots, f_{\ell+1}\}$, where $\ell \geqslant 2$, be a circuit in M/F. For each $i \in \{1, \ldots, \ell+1\}$, let F_i be the point of M/F containing f_i, and let N be the restriction of M to $F \cup F_1 \cup \cdots \cup F_{\ell+1}$. So $\mathrm{si}(N/F) \cong U_{\ell,\ell+1}$.

Claim 3.3.2. *For any rank-ℓ flat J of N, if J is skew to F, then J is a circuit.*

Proof of claim. Suppose to the contrary that J is an independent flat of N. Since J is skew to F, $|J \cap F_i| \leqslant 1$ for each $i \in \{1, \ldots, \ell + 1\}$. Without loss of generality, we may assume that $J = \{e_1, \ldots, e_\ell\}$, where $e_i \in F_i$ for each $i \in \{1, \ldots, \ell\}$. So $\operatorname{cl}_M(J) \cap F_{\ell+1} = \varnothing$. Let I be a basis of M/F containing J, and let $H = \operatorname{cl}_M(I)$. Since $|H| \geqslant n$ and H is skew to

F in M, there exists an element $e_{\ell+1} \in H \cap F_{\ell+1}$. Since $M|H = (M/F)|H$, $e_{\ell+1} \in \mathrm{cl}_M(J)$, a contradiction. □

By Lemma 3.2, N has an $\mathrm{AG}(k,p)$-restriction. □

Lemma 3.3 translates to the following extremal theorem, from which we will easily derive Theorem 3.1.

Lemma 3.4. *Let $k,t \geqslant 1$ and $n \geqslant m \geqslant 1$ be integers and let p be prime. There is a number $N_2 = N_2(p^t,k,n,m)$ such that, if M is a simple $\mathrm{GF}(p^t)$-representable matroid with $r(M) \geqslant N_2$, then M has either an $\mathrm{AG}(k-1,p)$-restriction or a rank-m flat which is either disconnected or has at most n points.*

Proof. Let $q = p^t$. Let m be any positive integer, and observe that, if $n \geqslant \frac{q^m-1}{q-1}$, then the number N_2 exists for all $k \geqslant 1$. (Just take $N_2 = m$.) Now let $n < \frac{q^m-1}{q-1}$, and assume that $N_2(q,k,n+1,m)$ exists for all $k \geqslant 1$. Let M be a simple $\mathrm{GF}(q)$-representable matroid with

$$r(M) \geqslant k + m + N_2(q, \mathrm{AR}_q(k,1,q^k) + 1, n+1, m).$$

We may assume that every rank-m flat in M is connected and has at least $n+1$ points. Furthermore, by Lemma 3.3, we may assume that there is a rank-k flat F in M such that every rank-m flat in M/F has at least $n+2$ points. Since every rank-m flat in M/F is connected, it follows by induction that M/F has an $\mathrm{AG}(\mathrm{AR}_q(k,1,q^k),p)$-restriction. Thus, by Lemma 2.5, M has an $\mathrm{AG}(k-1,p)$-restriction. □

Now we finish the proof.

Proof of Theorem 3.1. For $k \in \mathbb{N}$, we recursively define

$$N_1(p^t,k,m) := N_2(p^t,k,2N_1(p^t,k,m-1),2N_1(p^t,k,m-1)),$$

where $N_1(p^t,k,1) = 1$. Now let $m \geqslant 1$ and assume $r(M) \geqslant N_1(p^t,k,m)$. The result follows if $m = 1$, because every point in a simple matroid forms a rank-1 independent flat. For $m \geqslant 2$, we may assume by Lemma 3.4 that M has a rank-$2N_1(p^t,k,m-1)$ flat F such that $M|F$ is not connected. Let N be a smallest component of $M|F$. By induction, we may assume that $M|F\backslash E(N)$ has a rank-$(m-1)$ independent flat I. But then $I \cup \{e\}$ is a rank-m independent flat for any $e \in E(N)$. □

3.2 Matroids with a spanning projective geometry

With Theorem 3.1, we have reduced the proof of the Unavoidable Flats Theorem to the case that our matroid contains a spanning affine geometry over the base field. Our main theorem for this section allows us to handle a matroid with this structure inductively. In particular, we show that a $\mathrm{GF}(q)$-representable matroid of huge rank with a spanning affine geometry contains either one of the desired unavoidable flats or a high-rank flat which is representable over a proper subfield of $\mathrm{GF}(q)$.

As we will show in Section 3.3 using a contraction argument, it suffices to consider a matroid containing a spanning projective (rather than affine) geometry. Therefore we prove the following.

Theorem 3.5. *For p prime and $k, t \in \mathbb{N}$, there is a number $N_2 = N_2(p^t, k)$ such that, if M is a simple $\mathrm{GF}(p^t)$-representable matroid with a $\mathrm{PG}(r(M)-1, p)$-restriction and $r(M) \geqslant N_2$, then M has a rank-k flat F such that either $M|F \cong \mathrm{AG}(k-1, p^t)$, $M|F \cong \mathrm{PG}(k-1, p^t)$, or $M|F$ is representable over a proper subfield of $\mathrm{GF}(p^t)$.*

The proof of this result relies heavily on the Ramsey theorems presented in Section 2.2. The idea is as follows. Let G_0 denote the spanning projective geometry in M, and recall that every element in M is spanned by a flat of G_0 of rank at most t. We start by colouring each flat in G_0 of rank at most t according to the number of elements in M for which it is the minimal spanning flat in G_0. By repeated application of the Projective Ramsey Theorem, we may assume that, for each $i \in \{1, \dots, t\}$, the rank-i flats of G_0 have the same colour. If each rank-t flat of G_0 is the minimal spanning flat for an element of M, then M is dense, in which case the result follows by the Density Hales-Jewett Theorem and Lemma 2.7. Therefore we may assume that every element of M is spanned by a flat of G_0 with rank strictly less than t. In this case, we will show that M has a rank-k flat which is representable over a proper subfield of $\mathrm{GF}(p^t)$.

The idea will be to choose $i \in \{1, \dots, t-1\}$ as large as possible such that every rank-i flat of G_0 is the minimal spanning flat of some element of M, and then choose k skew rank-i flats $F_1, \dots, F_k$ of G_0 and elements $e_1, \dots, e_k$ of M minimally spanned by each of them. Then we show that the flat $F = \mathrm{cl}_M(\{e_1, \dots, e_k\})$ is representable over a proper subfield of $\mathrm{GF}(p^t)$. Because M is $\mathrm{GF}(p^t)$-representable, we may view M as a restriction of a projective geometry $G_1 \cong \mathrm{PG}(r(M)-1, p^t)$. Having chosen the flat F, we may assume that M contains every element of G_1 spanned by a rank-i flat of G_0. The following lemma will then complete the proof.

Lemma 3.6. *For p prime and $i, t, k \in \mathbb{N}$, where $i < t \leqslant k-2$, let $G_1 \cong \mathrm{PG}(ik-1, p^t)$, let $G_0 \cong \mathrm{PG}(ik-1, p)$ be a restriction of G_1, and let M be the matroid obtained from G_1 by deleting every element not spanned by a rank-i flat of G_0. Let $F_1, \ldots, F_k$ be a collection of rank-i flats of G_0 with $r(F_1 \cup \cdots \cup F_k) = ik$. For each $j \in \{1, \ldots, k\}$, let e_j be an element of M whose minimal spanning flat in G_0 is F_j. If $F = \mathrm{cl}_M(\{e_1, \ldots, e_k\})$, then $M|F$ is a projective geometry over a proper subfield of $\mathrm{GF}(p^t)$.*

To prove Lemma 3.6, we will show that $M|F$ is *modular*, that is, for all flats H_1, H_2 of $M|F$, $r(H_1) + r(H_2) = r(H_1 \cup H_2) + r(H_1 \cap H_2)$. Then we apply the standard fact that a connected modular matroid is a projective geometry [48, Proposition 6.9.1], as well as the fact that any finite projective geometry of rank $k \geqslant 4$ is isomorphic to $\mathrm{PG}(k-1, q)$ for some prime-power q [48, Theorem 6.1.2]. Using an easy counting argument, we then show that $M|F \ncong \mathrm{PG}(k-1, p^t)$, which will complete the proof.

The following proposition will be helpful in finding a modular flat. Recall, from Oxley [48], that $\sqcap(A, B) = r(A) + r(B) - r(A \cup B)$.

Proposition 3.7. *If F_1 and F_2 are flats in a simple matroid M with $r(F_1 \cap F_2) < \sqcap(F_1, F_2)$, then there is a flat $H \subseteq F_1$, disjoint from F_2, with $\sqcap(H, F_2) = 1$.*

Proof. Choose a flat $K \subseteq F_1$ of M which is skew to F_2 and such that $r(K) + r(F_2) = r(F_1 \cup F_2)$. Then $F_1 \backslash K$ is spanned by F_2 in M/K, and $r_{M/K}(F_1 \backslash K) = \sqcap_M(F_1, F_2) > r_M(F_1 \cap F_2)$. Thus we can choose an element $e \in F_1 \backslash K$ which is not spanned by $F_1 \cap F_2$ in M/K. The result follows for $H = \mathrm{cl}_M(K \cup \{e\})$. □

Now we are ready to prove the lemma.

Proof of Lemma 3.6. Let $r = ik$. If $i = 1$, then $M = G_0$ and the result follows; so let $2 \leqslant i \leqslant t-1$. By modularity of G_0, for all flats F_0 and H_0 of G_0, $\mathrm{cl}_M(F_0 \cap H_0) = \mathrm{cl}_M(F_0) \cap \mathrm{cl}_M(H_0)$, and similarly $\mathrm{cl}_{G_1}(F_0 \cap H_0) = \mathrm{cl}_{G_1}(F_0) \cap \mathrm{cl}_{G_1}(H_0)$. We use these facts implicitly throughout the proof.

Claim 3.7.1. *Each flat H of $M|F$ is not spanned by any flat of G_0 of rank less than $ir(H)$.*

Proof of claim. First we show that F is not spanned by any flat of G_0 of rank less than r. Indeed, suppose to the contrary that there is a flat F_0 of G_0 such that $F \subseteq \mathrm{cl}_M(F_0)$ and F_0 does not span G_0. Then for some $j \in \{1, \ldots, k\}$, $F_j \nsubseteq F_0$. But then, since F_j is the minimal spanning flat for e_j in G_0, we have that $e_j \notin \mathrm{cl}_M(F_j \cap F_0)$. But $e_j \in \mathrm{cl}_M(F_j) \cap \mathrm{cl}_M(F_0)$, a contradiction.

Now let H be a flat of $M|F$, and suppose to the contrary that there is a flat H_0 of G_0 such that H_0 spans H in M and $r(H_0) < ir(H)$. Let I be a basis of H, and let B be a basis of F containing I. Since each element of $B\backslash I$ is spanned by a set in G_0 with rank at most i, F is spanned by a set in G_0 of rank at most

$$i|B\backslash I| + r(H_0) = i(r(F) - r(H)) + r(H_0) < ir(F) = r,$$

a contradiction. □

By Claim 3.7.1 and hypothesis, for every element $e \in F$, the minimal spanning flat in G_0 of e has rank exactly i; fix a basis B_e of this flat. A consequence of Claim 3.7.1 is that G_0 is related to every flat of $M|F$ in this manner.

Claim 3.7.2. *Each flat H of $M|F$ is spanned by a unique rank-$ir(H)$ flat of G_0.*

Proof of claim. Since each element of a basis of H is spanned by a rank-i flat of G_0, there exists a flat H_0 of G_0 with $H \subseteq \mathrm{cl}_M(H_0)$ and $r(H_0) \leqslant ir(H)$. By Claim 3.7.1, $r(H_0) = ir(H)$. Suppose to the contrary that there is another rank-$ir(H)$ flat H_0' of G_0 which spans H. Then $H \subseteq \mathrm{cl}_M(H_0 \cap H_0')$ and $r(H_0 \cap H_0') < ir(H)$, contrary to Claim 3.7.1. □

The next step of the proof is to use the correspondence established in Claim 3.7.2, together with the modularity of the projective geometry G_0, to show that $M|F$ is modular. In order to do this, we must show that minimal spanning flats are well-behaved under taking intersections.

Claim 3.7.3. *For flats H_1, H_2 of $M|F$, if H_i' is the minimum spanning flat of H_i in G_0 for $i \in \{1, 2\}$, then*

$$r(H_1' \cap H_2') = i(r(H_1) + r(H_2) - r(H_1 \cup H_2)).$$

Proof of claim. Let H' denote the minimal spanning flat of $H_1 \cup H_2$ in G_0. We first show that $H' = \mathrm{cl}_{G_0}(H_1' \cup H_2')$. Indeed, let B_1 be a basis of H_1, and let B be a basis of $H_1 \cup H_2$ containing B_1. By Claim 3.7.1, $\bigcup_{e \in B} B_e$ is a basis of H', and $\bigcup_{e \in B_1} B_e$ is a basis for H_1'. Thus $H_1' \subseteq H'$; and similarly $H_2' \subseteq H'$. Therefore $\mathrm{cl}_{G_0}(H_1' \cup H_2') \subseteq H'$. By Claims 3.7.1 and 3.7.2,

$$ir(H_1 \cup H_2) \leqslant r(H_1' \cup H_2') \leqslant r(H') = ir(H_1 \cup H_2),$$

and so $\mathrm{cl}_{G_0}(H_1' \cup H_2') = H'$. Now by modularity of G_0, we have

$$i(r(H_1) + r(H_2)) = r(H_1') + r(H_2') = ir(H_1 \cup H_2) + r(H_1' \cap H_2'),$$

as desired. □

The next two claims establish the modularity of $M|F$.

Claim 3.7.4. *For flats H_1, H_2 of $M|F$, if $\sqcap(H_1, H_2) = 1$, then $H_1 \cap H_2 \neq \varnothing$.*

Proof of claim. Let H_1', H_2' respectively be the minimum spanning flats of H_1, H_2 in G_0. By Claim 3.7.3, we have that $r(H_1' \cap H_2') = i$. Also, since $\mathrm{cl}_{G_1}(H_1' \cap H_2') = \mathrm{cl}_{G_1}(H_1') \cap \mathrm{cl}_{G_1}(H_2')$, we have that $\mathrm{cl}_{G_1}(H_1) \cap \mathrm{cl}_{G_1}(H_2) \subseteq \mathrm{cl}_{G_1}(H_1' \cap H_2') \cap \mathrm{cl}_{G_1}(F)$. By hypothesis, $\mathrm{cl}_{G_1}(H_1) \cap \mathrm{cl}_{G_1}(H_2)$ is a point in G_1, say $\{g\}$. Thus $g \in \mathrm{cl}_{G_1}(H_1' \cap H_2')$. In particular, this means that g is spanned by a rank-i flat of G_0, and so $g \in E(M)$. Thus $g \in H_1 \cap H_2$. □

Claim 3.7.5. *For flats H_1, H_2 of $M|F$, $\sqcap(H_1, H_2) = r(H_1 \cap H_2)$.*

Proof of claim. Suppose that $r(H_1 \cap H_2) < \sqcap(H_1, H_2)$. By Proposition 3.7, there are flats $E_1 \subseteq H_1$ and $E_2 \subseteq H_2$ such that $\sqcap(E_1, E_2) = 1$ and $r(E_1 \cap E_2) = 0$. But this contradicts Claim 3.7.4. □

By Claim 3.7.5, the matroid $M|F$ is modular. Recall that each connected component of a modular matroid is either a point or a projective geometry. We claim that $M|F$ is connected. Indeed, let $e, f \in F$, and let H_e and H_f denote their minimal spanning flats in G_0. By Claim 3.7.1, H_e and H_f are skew, and $H = \mathrm{cl}_{G_0}(H_e \cup H_f)$ is the minimal spanning flat in G_0 for $\mathrm{cl}_M(\{e, f\})$. Since $G_0|H \cong \mathrm{PG}(2i-1, p)$, there is a rank-$i$ flat H_g in $G_0|H$ such that H_e, H_f and H_g are pairwise-skew, and H_g is the minimal spanning flat in G_0 for an element $g \in \mathrm{cl}_{G_1}(\{e, f\}) \backslash \{e, f\}$. By assumption, $g \in F$. It follows that $M|F$ is connected, and in particular, $M|F$ is a rank-k projective geometry. Since $k \geqslant t + 2 \geqslant 4$, $M \cong \mathrm{PG}(k-1, q)$ for some prime-power q. But since $M|F$ is $\mathrm{GF}(p^t)$-representable, $\mathrm{GF}(q)$ is a subfield of $\mathrm{GF}(p^t)$. It remains to show that $M|F \not\cong \mathrm{PG}(k-1, p^t)$.

Recall that, for $e, f \in H$ distinct, B_e and B_f are skew. (Otherwise $\mathrm{cl}(\{e, f\})$ is spanned by a flat of G_0 of rank less than $2i$, contrary to Claim 3.7.1.) It follows that $|E(G_0)| > |F|$. Now if $M|F \cong \mathrm{PG}(k-1, p^t)$, then we have

$$\frac{(p^t)^k - 1}{p^t - 1} = |F| < |E(G_0)| = \frac{p^{ik} - 1}{p - 1},$$

which in turn implies that $p^k < p^t$, contrary to our choice of k. Thus $M|F$ is a projective geometry over a proper subfield of $\mathrm{GF}(p^t)$. □

We are now ready to prove Theorem 3.5.

Proof of Theorem 3.5. We may assume that $k \geqslant t+2$ (which we will need to invoke Lemma 3.6). Let $q = p^t$, let $N = \mathrm{PR}_q(k, 1, 2)$, and let

$$m = \max\{\mathrm{DHJ}_q((2^t p^{2t^2})^{-1}, N), tk\}.$$

Finally, let $M_2 = \mathrm{PR}_p(m, 2, q^t)$, and recursively define $M_i = \mathrm{PR}_p(M_{i-1}, i, q^t)$ for each $i \in \{3, \dots, t\}$. We will show that $N_2 = M_t$ works.

Let M be a simple $\mathrm{GF}(p^t)$-representable matroid with a restriction $G_0 \cong \mathrm{PG}(r(M) - 1, p)$ and rank $r(M) \geqslant M_t$. Recall that every element e of M is spanned by a flat of G_0 of rank at most t, and that there is a unique such flat of minimum rank, which we call the minimum spanning flat of e. For each $i \in \{2, \dots, t\}$ and each rank-i flat F of G_0, let $\phi_i(F)$ be the number of elements in M for which F is the minimal spanning flat. Then, for each $i \in \{2, \dots, t\}$, ϕ_i is a q^t-colouring of the rank-i flats of G_0. By repeated application of the Projective Ramsey Theorem, there is a rank-m flat F_0 of G_0 such that, for each $i \in \{2, \dots, t\}$, every rank-i flat of $G_0|F_0$ has the same colour under ϕ_i.

Let $G_0' = G_0|F_0$ and $M' = M|\operatorname{cl}_M(F_0)$. We may assume that $M' \neq G_0'$. Choose $i \in \{2, \dots, t\}$ as large as possible such that $\phi_i \not\equiv 0$.

Case 1. $i < t$.

By choice of i, every element of M' is spanned by a rank-i flat of G_0', and every rank-i flat of G_0' is the minimal spanning flat of some element of M'. Since $r(G_0') \geqslant tk$, there is a collection $F_1, \dots, F_k$ of rank-i flats of G_0' such that $r(F_1 \cup \cdots \cup F_k) = ik$. For each $j \in \{1, \dots, k\}$, choose an element e_j of M' for which F_j is the minimal spanning flat in G_0, and let $F = \operatorname{cl}_M(\{e_1, \dots, e_k\})$. By Lemma 3.6, $M|F$ is representable over a proper subfield of $\mathrm{GF}(p^t)$.

Case 2. $i = t$.

In this case, the number of elements of M' is at least the number of rank-t flats of G_0', which equals the Gaussian coefficient $\binom{m}{t}_p$ (see [48, p. 162]). Therefore

$$|E(M')| \geqslant \binom{m}{t}_p > \frac{1}{2^t p^{2t^2}} p^{mt} = \frac{1}{2^t p^{2t^2}} q^{r(M')}.$$

Since $r(M') \geqslant \mathrm{DHJ}_q((2^t p^{2t^2})^{-1}, N)$, it follows by the Density Hales-Jewett Theorem that M' has an $\mathrm{AG}(N, q)$-restriction. The result now follows by Lemma 2.7. □

3.3 Putting the pieces together

We now finish the proof of the Unavoidable Flats Theorem by combining the main results from the previous two sections, both of which we restate here for ease of reference.

Theorem (Theorem 3.1)**.** *For a prime p and integers $k, m, t \in \mathbb{N}$, there is a number $N_1 = N_1(p^t, k, m)$ such that, if M is a simple $\mathrm{GF}(p^t)$-representable matroid with $r(M) \geqslant N_1$, then M contains either a rank-m independent flat or an $\mathrm{AG}(k-1,p)$-restriction.*

Theorem (Theorem 3.5)**.** *For p prime and $k, t \in \mathbb{N}$, there is a number $N_2 = N_2(p^t, k)$ such that, if M is a simple $\mathrm{GF}(p^t)$-representable matroid with a $\mathrm{PG}(r(M)-1,p)$-restriction and $r(M) \geqslant N_2$, then M has a rank-k flat F such that either $M|F \cong \mathrm{AG}(k-1,p^t)$, $M|F \cong \mathrm{PG}(k-1,p^t)$, or $M|F$ is representable over a proper subfield of $\mathrm{GF}(p^t)$.*

The Unavoidable Flats Theorem follows by applying Theorem 3.1 to pass to a high-rank matroid with a spanning affine geometry restriction, and then applying Theorem 3.5 and induction. The only problem is that the latter requires that the matroid has a spanning *projective* geometry as a restriction. We get around this issue by contracting an element in the affine restriction and then lifting after applying induction.

Unavoidable Flats Theorem. *For every prime-power q and positive integer k, there is a number $N_q(k)$ such that, if M is a simple $\mathrm{GF}(q)$-representable matroid with rank at least $N_q(k)$, then M has a rank-k flat F such that either F is independent in M, or $M|F$ is isomorphic to a projective or affine geometry over a subfield of $\mathrm{GF}(q)$.*

Proof. We proceed by induction on $t \geqslant 1$, where $q = p^t$ for some prime p. For the base case $t = 1$, let M be a simple $\mathrm{GF}(p)$-representable matroid with $r(M) \geqslant N_1(p, N+1, k)$, where $N = \mathrm{PR}_p(k,1,2)$. By Theorem 3.1, we may assume that M has an $\mathrm{AG}(N,p)$-restriction. By Lemma 2.7, M has a rank-k flat F such that either $M|F \cong \mathrm{AG}(k-1,p)$ or $M|F \cong \mathrm{PG}(k-1,p)$, as desired.

Now let $t \geqslant 2$, and assume that the number $N_{p^i}(\kappa)$ exists for all $i \in \{1,\dots,t\}$ and $\kappa \in \mathbb{N}$. Let k be any positive integer, and define $m = \mathrm{PR}_q(k,1,2)$,

$$\ell = \max\{\mathrm{DHJ}_q(q^{-2}, m), N_{p^{t-1}}(k)\},$$

and $n = N_{p^{t-1}}(\ell)$. Let M be a simple $\mathrm{GF}(q)$-representable matroid of rank at least $N_1(q, N_2(q,n)+1, k)$. By Theorem 3.1, we may assume that, for some $S \subseteq E(M)$, $M|S \cong \mathrm{AG}(N_2(q,n),p)$. Let $M' = M|\operatorname{cl}(S)$ and let $e \in S$. Then $\operatorname{si}((M'|S)/e) \cong \mathrm{PG}(N_2(q,n)-1,p)$. Therefore, by Theorem 3.5 and induction, M'/e has a rank-ℓ flat

F such that $\mathrm{si}((M'/e)|F)$ is either a free matroid or a projective or affine geometry over a subfield of $\mathrm{GF}(q)$. In the former case, we can lift to obtain a rank-ℓ independent flat in M', and so we may assume that $\mathrm{si}((M'/e)|F)$ is a projective or affine geometry.

Case 1. $\mathrm{si}((M'/e)|F)$ is a projective or affine geometry over a proper subfield of $\mathrm{GF}(q)$.

Let $N = M|\operatorname{cl}_M(F \cup \{e\})$, and let H be a hyperplane of N which does not contain e. Then $(N/e)|H = N|H$, and so $N|H$ is representable over a proper subfield of $\mathrm{GF}(q)$. Since $r(H) = \ell$, it follows by induction that $N|H$ contains a rank-k flat H' such that either H' is independent, or $N|H' = M|H'$ is an affine or projective geometry over a proper subfield of $\mathrm{GF}(q)$.

Case 2. $\mathrm{si}((M'/e)|F)$ is isomorphic to either $\mathrm{PG}(\ell-1,q)$ or $\mathrm{AG}(\ell-1,q)$.

Again, let $N = M|\operatorname{cl}_M(F \cup \{e\})$, and observe that $|E(N)| \geqslant q^{\ell-1} = q^{-2}q^{r(N)}$. By the Density Hales-Jewett Theorem, N has an $\mathrm{AG}(m,q)$-restriction. By Lemma 2.7, N has a rank-k flat F such that either $N|F \cong \mathrm{AG}(k-1,q)$ or $N|F \cong \mathrm{PG}(k-1,q)$. This completes the proof. □

Chapter 4

Incidence Geometry and Counting Hyperplanes

In this chapter, we provide an introduction to the theory of counting hyperplanes in real-representable matroids. This will play a fundamental role in the proofs of the Average Hyperplane-Size Theorem, the Average Plane-Size Theorem and the Average Flat-Size Theorem. We begin in low dimensions, with a classic theorem of József Beck on approximating the number of lines. As a simple demonstration of our general proof technique, we show how to derive Melchior's Theorem (with a weaker bound on the average) from Beck's Theorem. We also provide a nice proof of Beck's Theorem due to Szekély [53], with the aim of providing the reader with an introduction to the surrounding theory of *incidence geometry*. For further reading on this subject, we refer the reader to an excellent survey by Dvir [17], which was a helpful resource for the writing of this section. In the second section, we present two generalizations of Beck's theorem, due to Lund and Do, to higher dimensions, both of which will play important roles in our proofs.

Throughout this chapter (and the remainder of the thesis), we may find it handy to use the standard asymptotic notation $O(\cdot)$ and $\Omega(\cdot)$ to denote upper and lower bounds up to a (hidden) positive multiplicative constant. If the latter depends on some parameter, say k, then we will use a subscripted $O_k(\cdot)$ or $\Omega_k(\cdot)$.

4.1 Beck's Theorem

The first step in proving the Average Hyperplane-Size Theorem (as well as the Average Plane-Size Theorem) is to obtain a good approximation on the number of hyperplanes in

a real-representable matroid. We do this by generalizing a classic theorem in incidence geometry, Beck's theorem on counting spanned lines.

Theorem 4.1 (Beck's Theorem, [4])**.** *If M is a simple rank-3 real-representable matroid on n elements, and L is a largest line of M, then the number of lines in M is at least $\Omega(n(n-|L|))$.*

It is easily verified that, in the matroid M above, the number of lines is at most $n(n-|L|)$, and so the bound in Beck's Theorem is best-possible up to a multiplicative constant.

Beck's Theorem was generalized to higher dimensions by Lund [41], who gave a similar approximation on the number of flats of any given rank. In Chapter 6, we prove a strengthening Lund's theorem; the details are discussed in the next section. For now, we illustrate the role such approximations play in the proof of the Average Hyperplane-Size Theorem by showing how to derive its "base case" from Beck's Theorem. In other words, we use Beck's Theorem to give a new proof of Melchior's Theorem (with a weaker bound on the average). The reasoning is very simple: because Beck's Theorem provides a good approximation on the number of lines, it suffices to show that we can similarly approximate the *aggregate size* of the lines.

Proposition 4.2. *If M is a simple rank-3 matroid on n elements, and L is a largest line of M, then*

$$\sum_{H \in \mathcal{L}(M)} |H| \leqslant 3n(n-|L|),$$

where $\mathcal{L}(M)$ denotes the collection of lines of M.

Proof. Let $\mathcal{L}' = \mathcal{L}(M) \backslash \{L\}$, and consider a line $H \in \mathcal{L}'$. Since H intersects L in at most one element, there are at least $|H|-1$ pairs $(e,f) \in H \times (H \backslash L)$ where $e \neq f$. Because any pair of distinct elements spans a unique line in M, it follows that

$$\frac{1}{2} \sum_{H \in \mathcal{L}'} |H| \leqslant \sum_{H \in \mathcal{L}'} (|H|-1) \leqslant n(n-|L|).$$

The result now follows because $|L| \leqslant n$ and $n - |L| \geqslant 1$. □

It follows immediately from Beck's Theorem and Proposition 4.2 that, in any real-representable matroid of rank at least three, the average size of a line is at most an absolute

constant. The proof of the Average Hyperplane-Size Theorem will follow in the very same manner, namely by combining a suitable generalization of Beck's Theorem with a similar generalization of Proposition 4.2. Note that the latter does not require M to be real-representable.

The remainder of this section comprises a proof of Beck's Theorem. This will have no technical bearing on anything else in the thesis, and is included only for enrichment and context, in light of the fundamental role Beck's Theorem plays in our proofs.

Beck's original proof of his theorem appeared around the same time as a paper of Szemerédi and Trotter [54] which proved an even stronger theorem on point-line incidences. We provide a short and elegant proof, due to Szekély [53], of Szemerédi and Trotter's theorem, and then derive Beck's Theorem from the latter using the standard argument from Beck's original paper. (Beck had in fact proved a slightly weaker version of Szemerédi and Trotter's result in [4].) The Szemerédi-Trotter Theorem has two well-known formulations, one of which is a sharp upper bound on the number of incidences between finitely many points and lines in the plane. We will use the other formulation, a bound on the number of lines having large intersection with a given finite set of points in the plane.

Szemerédi-Trotter Theorem. *For a set P of n points in the real plane and any integer $k \geqslant 2$, we have*

$$|\mathcal{L}_k| \leqslant O\Big(\frac{n^2}{k^3} + \frac{n}{k}\Big),$$

where $\mathcal{L}_k$ denotes the collection of lines containing at least k points from P.

Most proofs of the Szemerédi-Trotter Theorem, including the original, are based on a geometric technique known as cell partitioning, in which the plane is cut up into finitely many open sets, and then point-line incidences are counted in each of these sets. Szekély's proof is entirely different, and is based on the theory of crossing numbers. Recall that the *crossing number* $\mathrm{cr}(G)$ of a graph G is the smallest number κ for which there exists a drawing of G in the plane with at most κ crossings. (In particular, a graph is planar if and only if its crossing number is zero.) For a drawing D of a graph, we will denote its number of crossings by $\kappa(D)$. Szekély's proof of the Szemerédi-Trotter Theorem is based on a well-known lower bound on the crossing number originally proved independently by Ajtai, Chvátal, Newborn and Szemerédi [2] and by Leighton [39].

The Crossing Number Inequality says that, for a sufficiently dense graph $G = (V, E)$, $\mathrm{cr}(G) \geqslant \Omega(|E|^3/|V|^2)$. To prove this, we start with the fact that if G is any graph, then $\mathrm{cr}(G) \geqslant |E| - 3|V|$. This is an immediate consequence of Euler's Formula, and not very

helpful on its own. To derive the Crossing Number Inequality, we apply this weak bound to a random induced subgraph of G, and then tease out a stronger bound using the first moment method; this proof is originally due to Chazelle, Sharir and Welzl (see [1]).

Crossing Number Inequality ([2], [39])**.** *For any graph $G = (V, E)$, if $|E| \geqslant 4|V|$, then* $\mathrm{cr}(G) \geqslant |E|^3/64|V|^2$.

Proof. Let $p = 4|V|/|E|$ (so $p \in [0, 1]$), and choose each vertex $v \in V$ to be in the set V' independently with probability p. Let $G' = (V', E')$ denote the resulting random induced subgraph of G. Let X be the random variable on this probability space defined by $X(G') = |E'| - 3|V'|$, and similarly define Y by $Y(G') = \kappa(G')$. By Euler's Formula, $E(Y) \geqslant E(X)$.

Claim 4.2.1. *$E(X) = p^2|E| - 3p|V|$, and $E(Y) \leqslant p^4\mathrm{cr}(G)$.*

Proof of claim. For each $e \in E$ and $v \in V$, let X_e and X_v be the natural indicator random variables for e and v on the above probability space. Then $E(X_e) = p^2$ and $E(X_v) = p$. The first assertion now follows by linearity of expectation.

To prove the second assertion, fix a drawing D of G with $\kappa(D) = \mathrm{cr}(G)$, and define the same probability space on the induced subgraphs of D. Let Z be the random variable on this space defined by $Z(D') = \kappa(D')$, and observe that $E(Z) \geqslant E(Y)$. Now for each crossing in D, the probability that it exists in the random induced subgraph D' of D is p^4. Hence by linearity of expectation, $E(Z) = p^4\mathrm{cr}(G)$. □

Now by the claim, we have that $p^4\mathrm{cr}(G) \geqslant p^2|E| - 3p|V|$. The result follows. □

We can now elegantly derive the Szemerédi-Trotter Theorem; this proof is due to Székely [53].

Proof of the Szemerédi-Trotter Theorem. Define a pair of distinct points in P to be adjacent if and only if they lie on a common line in $\mathcal{L}_k$ and no other point of P is in the line segment adjoining them; let $G = (P, E)$ be the resulting graph. Then $|E| \geqslant (k-1)|\mathcal{L}_k|$. Thus, if $|E| < 4n$, then $|\mathcal{L}_k| \leqslant O(\frac{n}{k})$. So we may assume that $|E| \geqslant 4n$. Hence, by the Crossing Number Inequality, we have that

$$\mathrm{cr}(G) \geqslant \frac{|E|^3}{64n^2} \geqslant \frac{(k-1)^3|\mathcal{L}_k|^3}{64n^2}.$$

On the other hand, because any two distinct lines in $\mathcal{L}_k$ intersect in at most one point, $\mathrm{cr}(G) \leqslant |\mathcal{L}_k|^2$. The result follows. □

Now we derive Beck's Theorem from the Szemerédi-Trotter Theorem. To do this, we will prove an intermediate result, which is also referred to as "Beck's Theorem" in the literature. The two theorems are equivalent up to the specification of multiplicative constants.

Theorem 4.3 (Beck, [4]). *There exist $\epsilon, \gamma \in (0,1)$ such that the following holds. For any rank-3 real-representable matroid M on n points, either*

(i) there is a line in M of size at least ϵn, or

(ii) M has at least γn^2 lines.

Proof. We may assume that M is simple. The idea is to show that, for some constant $C > 0$ (which we will choose later), if M has maximum line-size at most $\frac{n}{C}$, then there are $\Omega(n^2)$ *pairs of points* spanning a line of size at most C. We then conclude that there are $\Omega_C(n^2)$ lines in M.

Let P be a set of n points in the real plane which forms a representation of M, and let $\mathcal{L}$ be the collection of lines in the plane spanned by pairs of points in P. For each integer $j \geqslant 1$, let $\mathcal{L}_j = \{L \in \mathcal{L} : 2^j \leqslant |L \cap P| < 2^{j+1}\}$, and let $\mathcal{P}_j$ be the collection of pairs of points in P which span a line in $\mathcal{L}_j$. Then $|\mathcal{P}_j| \leqslant O(|\mathcal{L}_j| 2^{2j})$. Hence by the Szemerédi-Trotter Theorem,

$$|\mathcal{P}_j| \leqslant O(2^{-j} n^2 + 2^j n) \tag{4.1}$$

for each $j \geqslant 1$.

For some $C > 0$ to be chosen later, let $\mathcal{P}_s$ be the collection of pairs of points in P which span a line containing fewer than C points of P, and let $\mathcal{P}_m$ be the collection of pairs which span a line containing at least C and fewer than n/C points of P. By (4.1), we have that

$$|\mathcal{P}_m| \leqslant n^2 O\Big(\sum_j 2^{-j} + \frac{2^j}{n} \Big),$$

where j is taken over all integers $\log(C) \leqslant j \leqslant \log(n/C)$. The latter coefficient tends to 0 as $C \to \infty$. Hence, by taking $C > 0$ sufficiently large, we have that $|\mathcal{P}_m| \leqslant n^2/100$. Consequently, if every line in M has size at most n/C, then $|\mathcal{P}_s| \geqslant n^2/8$, in which case the number of lines in M is at least $n^2/8C^2$. □

Beck's Theorem is an easy consequence of Theorem 4.3.

Proof of Beck's Theorem. In this proof, let ϵ and γ be as in Theorem 4.3. By Theorem 4.3, we may assume that $|L| \geqslant \epsilon n$. (We may also assume that n is large compared to $\frac{1}{\epsilon}$.) Now choose a subset $Z \subseteq E(M) \backslash L$ of size $\lceil \epsilon(n - |L|) \rceil$. By counting the lines of M spanned by a point in L and a point in Z, we see that

$$\begin{aligned} |\mathcal{L}(M)| &\geqslant |L||Z| - \binom{|Z|}{2} \\ &\geqslant \epsilon(n - |L|)\Big(\epsilon n - \frac{1}{2}\lceil \epsilon(n - |L|) \rceil\Big) \\ &\geqslant \Omega_\epsilon(n(n - |L|)), \end{aligned}$$

as desired. □

Before moving on to the next section, we make note of a strengthening of Theorem 4.3, which is also readily seen to be equivalent to Beck's Theorem.

Theorem 4.4 (Beck, [4])**.** *For all $\epsilon > 0$, there exists $\gamma_2(\epsilon) > 0$ such that, if M is a rank-3 real-representable matroid on n points, then either*

(i) M has a line of size at least $(1 - \epsilon)n$, or

(ii) M has at least $\gamma_2(\epsilon)n^2$ lines.

In the next section, we will see a generalization, due to Do [15], of Theorem 4.4 to higher dimensions, which will play an important role in our proofs. We use the subscripted "γ_2" in Theorem 4.4 so that its notation coincides with that of Do's theorem.

4.2 Counting hyperplanes

In this section, we present some generalizations of Beck's Theorem to higher dimensions. The original theorem of this kind is a generalization of Theorem 4.3, also proved by Beck in [4].

Theorem 4.5 (Beck's Theorem for Hyperplanes)**.** *For each positive integer $k \geqslant 2$, there exist $\alpha_k, \beta_k > 0$ such that, if M is a rank-$(k+1)$ real-representable matroid on n points, then either*

(i) M has a hyperplane of size at least $\alpha_k n$, or

(ii) M has at least $\beta_k n^k$ hyperplanes.

Beck's Theorem for Hyperplanes is best-possible in the sense that Theorem 4.4 does not generalize in the same way. To this end, Lund proved in [42] that, for any $k \geqslant 2$ and $\epsilon \in (0, \frac{1}{2})$, a rank-$(k+1)$ real-representable matroid on n points either has $\Omega_\epsilon(n^k)$ hyperplanes or a hyperplane of size at least ϵn. However, this bound on ϵ is best-possible, as can be seen by considering the direct sum of two lines of size $\frac{n}{2}$ for an arbitrary even number n. In light of this we see that, in order to get a good approximation on the number of hyperplanes in a real-representable matroid of rank at least four, we need to take more into consideration than just the size of a largest hyperplane.

As was discovered independently by Lund [41] and Do [15], it turns out that the solution is to instead consider largest *degenerate sets*. Recall, from Section 1.2, that a set X in a matroid M is called k-degenerate if there is a collection $(F_1, \ldots, F_t)$ of flats of M, each of rank at least two, such that $X \subseteq F_1 \cup \cdots \cup F_t$ and $\sum_{i=1}^{t}(r_M(F_i) - 1) \leqslant k - 1$. Using this notion, Do generalized Theorem 4.4 as follows.

Theorem 4.6 (Do's Theorem, [15])**.** *For every $\epsilon > 0$ and integer $k \geqslant 2$, there exists $\gamma_k(\epsilon) > 0$ such that, if M is a real-representable matroid on n points, then either*

(i) M has a k-degenerate set of size at least $(1-\epsilon)n$, or

(ii) the number of rank-k flats in M is at least $\gamma_k(\epsilon)n^k$.

Do's Theorem plays a crucial role in the proofs of the Average Hyperplane-Size Theorem and the Average Plane-Size Theorem. It will allow us to assume, in various instances, that the matroid under consideration has a huge degenerate set (otherwise we get a good approximation on the number of hyperplanes). Following an idea of Lund [41], our approach will be to keep track of a largest i-degenerate set for every possible i. For an integer $k \geqslant 2$ and a matroid M, a *k-stratification* of M is a sequence $(X_2, \ldots, X_k)$ of subsets of its ground set such that, for each $i \in \{2, \ldots, k\}$, X_i is i-degenerate and $X_{i-1} \subseteq X_i$. (In this context, X_1 denotes the empty set, and X_{k+1} denotes the ground set of M.) If, for each $i \in \{2, \ldots, k\}$, X_i is a largest i-degenerate subset of X_{i+1}, then the k-stratification is called *optimal*. Lund proved the following generalization of Beck's Theorem.

Theorem 4.7 (Lund's Theorem, [41])**.** *For each integer $k \geqslant 2$, there exists an integer $c_k \geqslant 0$ such that, if M is a simple rank-$(k+1)$ real-representable matroid on n elements, $(X_2, \ldots, X_k)$ is an optimal k-stratification of M, and $n - |X_k| > c_k$, then*

$$|\mathcal{H}(M)| \geqslant \Omega_k\Big(n\prod_{i=2}^{k}(n - |X_i|)\Big),$$

where $\mathcal{H}(M)$ is the collection of hyperplanes of M.

We note that the above is a slight (and equivalent, up to the multiplicative constant) variation of Lund's original formulation of this result, see [10, 41] or Theorem 6.4.

Lund's Theorem gives a good approximation on the number of flats of any given rank. Indeed, we will prove in Chapter 6 that, for any simple matroid M on n elements and integer $k \geqslant 2$, if M is not k-degenerate and $(X_2, \ldots, X_k)$ is an optimal k-stratification of M, then

$$\sum_{H \in \mathcal{H}(M)} (|H| - k) \leqslant O_k\Big(n \prod_{i=2}^{k} (n - |X_i|)\Big). \tag{4.2}$$

Just like Proposition 4.2, (4.2) will hold regardless of the representability of M. Combining (4.2) with the bound in Lund's Theorem bounds the average size of a rank-k flat by a constant depending only on k. Of course, in order to apply Lund's Theorem for a given $k \geqslant 2$, the matroid M must have more than c_k points outside of a largest k-degenerate set. As we saw in Section 1.2, for $k \geqslant 5$, there are rank-$(k+1)$ real-representable matroids which are not k-degenerate but have arbitrarily large average hyperplane-size. In light of (4.2), this means that the constant c_k is necessarily positive for $k \geqslant 5$. In proving the Average Plane-Size Theorem, we will show that $c_3 = 0$. More generally, in proving the Average Hyperplane-Size Theorem, we will show that $c_k \leqslant b_k(k-2)$ for all $k \geqslant 2$.

The theorems of Beck, Do and Lund given in this section, as well as the generalizations of Lund's Theorem which we prove in Chapter 6, follow by inductive arguments which are, after the invokation of Beck's Theorem in the base case, entirely matroidal. Because the Szemerédi-Trotter Theorem holds for the classes of complex-representable matroids [55, 56] and orientable matroids [53], these higher-dimensional results also extend to these settings, as do the Average Plane-, Hyperplane- and Flat-Size Theorems. This is explained in slightly more detail in [11].

Chapter 5

Motzkin's Problem in Higher Dimensions

In order to formulate Part II of the Average Hyperplane-Size Theorem, we defined the following generalization of Motzkin's Problem to higher dimensions. For a positive integer k, given a simple rank-$(k+1)$ real-representable matroid M and a 2-colouring (R, B) of M satisfying

(C1) $r_M(B) \geqslant 2$;

(C2) for every hyperplane H of M, $H \cap R \neq \varnothing$; and

(C3) for every line L of $M|B$, there is a rank-$(k-1)$ flat F of M skew to L such that $F \cap R = \varnothing$,

what is the maximum possible size of B in terms of $|R|$ and k? We denote the solution to this problem by $b_k(t)$, where t is the number of red elements. (So $b_2(t)$ is the solution to Motzkin's Problem.) Recall that, if for given $k, t \geqslant 1$, there does not exist such (M, R, B) satisfying (C1), (C2) and (C3), then we set $b_k(t) = 0$.

Using the numbers $b_k(t)$, the Average Hyperplane-Size Theorem gives a best-possible lower bound on the size of a k-degenerate set in a rank-$(k+1)$ real-representable matroid with large average hyperplane-size. The purpose of this brief chapter is to show that this problem is well-defined and to give some preliminary results on the asymptotics of $b_k(t)$, culminating in the proof of Theorem 1.4, which says that $b_k(t) \geqslant \Omega(kt)$ and $b_k(t) \leqslant O_k(t)$. It remains open to determine the precise asymptotic dependence of $b_k(t)$ on k.

Before proving Theorem 1.4, we show that our general lower bound is not best-possible by demonstrating that $b_k(3) \geqslant \Omega(k^2)$.

Proposition 5.1. *For each integer $k \geqslant 2$, we have $b_k(3) \geqslant \binom{k+2}{2} - 2$.*

Proof. We start by defining the standard representation of the rank-$(k+1)$ complete graphic matroid. Let G be the complete graph on $k+2$ vertices with vertex set $\{1, 2, \ldots, k+2\}$. For each edge $ij \in E(G)$ with $i < j$, let e_{ij} denote the vector $x \in \mathbb{R}^{k+1}$ such that, if $j < k+2$, then $x_i = 1, x_j = -1$ and $x_k = 0$ for all $k \neq i, j$; and if $j = k+2$, then $x_i = 1$ and $x_k = 0$ for all $k \neq i$. Note that the vectors e_{ij} form a representation of the graphic matroid M_G of G. Now let $e = (1, 1, -1, -1, 0, \ldots, 0) \in \mathbb{R}^{k+1}$, and let M be the matroid represented by the set of vectors $\{e_{ij} : i, j \in E(G)\} \cup \{e\}$. We consider the 2-colouring (R, B) of M where $R = \{e_{12}, e_{34}, e\}$.

The fact that $r_M(B) \geqslant 2$ is immediate. To verify that (R, B) satisfies properties (C2) and (C3), we use the fact that, for $\ell \geqslant 0$, any rank-$(k+1-\ell)$ flat of M_G is naturally identified with a unique partition of $V(G)$ into $\ell + 1$ (non-empty) parts, where the elements of the flat are precisely the edges of G both of whose endpoints belong to the same part of the partition. Indeed, for (C2), consider a hyperplane H of M with $e_{12}, e_{34} \notin H$. Then there is a partition (V_1, V_2) of $V(G)$ neither side of which contains the edges 12 or 34. Without loss of generality, we may assume that $k+2 \in V_2$, and so the span of H in $\mathbb{R}^{k+1}$ is defined by the equation $\sum_{i \in V_1} x_i = 0$. Hence $e \in H$.

For (C3), consider a line L of $M|B$. Note that, for some $i \in \{1, 2\}$ and $j \in \{3, 4\}$, $e_{ij} \notin L$; we may assume that $e_{13} \notin L$. Now in the graph G, L is represented either by a pair of non-incident blue edges, a blue triangle, or a pair of incident blue edges which together form a triangle with one of the red edges 12 or 34. In any case, there is a partition $\mathcal{P} = (V_1, V_2, V_3)$ of $V(G)$ such that $1, 3 \in V_1$, $2 \in V_2$ and $4 \in V_3$, and such that no edge of L has both its endpoints in one part of the partition. Then the rank-$(k-1)$ flat F of M defined by $\mathcal{P}$ is skew to L. Moreover, for some $j \in \{2, 3\}$ (depending on which part of $\mathcal{P}$ contains $k+2$) every vector $x \in \mathbb{R}^{k+1}$ spanned by F satisfies $\sum_{i \in V_j} x_i = 0$, and so $e \notin F$. □

Our next construction will prove the general lower bound $b_k(t) \geqslant \Omega(kt)$. The construction is based on the fact, as shown by Jamison and Hill [36], that for any $t \geqslant 3$, there is a set S of at most $t+1$ points in the plane whose spanned lines define exactly t directions. The idea is to take k "blue" copies of such an arrangement S (in projective space), all of which share a common line at infinity. For each direction determined by S, we place a corresponding red point on the line at infinity. This will define a 2-colouring of a simple real-representable matroid which satisfies the desired conditions.

Proposition 5.2. *For integers $k, t \geqslant 3$, we have $b_k(t) \geqslant (k-1)t$.*

Proof. Let S be a set of at most $t+1$ points in $\mathbb{R}^2$ which determines exactly t directions. We use projective coordinates to define a bicoloured configuration in k-dimensional real projective space consisting of $k-1$ blue copies of S together with a red set of collinear points at infinity corresponding to the directions defined by S. Specifically, for $i \in \{1, \ldots, k-1\}$, let

$$B_i = \{(1, a, b, x_2, \ldots, x_{k-1}) : (a, b) \in S\},$$

where $x_j = 1$ if $j = i$ and $x_j = 0$ otherwise. Let $B = \bigcup_{i=1}^{k-1} B_i$ and let

$$R = \{(0, a-a', b-b', 0, \ldots, 0) : (a, b), (a', b') \in S, (a, b) \neq (a', b')\}.$$

Now we show that the 2-colouring (R, B) of the matroid M represented by the points $R \cup B$ satisfies (C1), (C2) and (C3).

For (C2), consider a hyperplane H of M, and suppose to the contrary that $H \cap R = \varnothing$. By the pigeonhole principle, $r_M(H \cap B_i) \geqslant 2$ for some $i \in \{1, \ldots, k-1\}$. But then, by definition of R and S, we have $H \cap R \neq \varnothing$, a contradiction. For (C3), consider a line L of $M|B$. Let L' be the image of the projection of L onto the first three coordinates. Since S is not collinear, there is a point $(a, b) \in S \backslash L'$. Now let F be the set of points in B which, under the above coordinatization, have a and b as their second and third coordinates respectively. By construction, F is skew to both L and R. □

It is easy to see that $b_2(2) = 0$, since at least three red points are needed to hit the three lines spanned by three non-collinear blue points. Now we show that $b_k(2) = 0$ for all k. The proof of this fact does not depend in any way on the representability of M.

Proposition 5.3. *For $k \geqslant 2$, let M be a simple rank-$(k+1)$ matroid. If (R, B) is a 2-colouring of M satisfying (C1), (C2) and (C3), then $|R| \geqslant 3$.*

Proof. Suppose to the contrary that $R = \{r_1, r_2\}$. Assume first that r_1 and r_2 are in different connected components of M, and let (X_1, X_2) be a separation of M with $r_1 \in X_1$ and $r_2 \in X_2$. By (C1) and (C3), $r(B) = r(M)$, which means that $|X_1|, |X_2| \geqslant 2$. We claim that $|X_1|, |X_2| \geqslant 3$. Indeed, suppose that $X_1 = \{r_1, e\}$ for some $e \in B$. Choose $f \in B \cap X_2$, and let F be a monochromatic-blue rank-$(k-1)$ flat in M skew to $\{e, f\}$. Since $r(F \cap X_2) \leqslant r(X_2) - 1 = k-2$, we have $r_1 \in F$, a contradiction. Thus $|X_1|, |X_2| \geqslant 3$. Now let $e, f \in X_1 \cap B$, and let F be a monochromatic-blue rank-$(k-1)$ flat in M skew to

$\{e, f\}$. Then r_2 is in neither of the hyperplanes $\mathrm{cl}(F \cup \{e\})$ and $\mathrm{cl}(F \cup \{f\})$, and r_1 is in at most one of them. This contradicts (C2).

Now assume that r_1 and r_2 are in the same connected component of M, and let C be a circuit containing both of them. Let I be a basis of $M|B$ containing $C \backslash \{r_1, r_2\}$, and let C_1 and C_2 be the fundamental circuits of $I \cup \{r_1\}$ and $I \cup \{r_2\}$ respectively. By choice of I, there exists $e \in C_1 \cap C_2$. But then $r_1, r_2 \notin \mathrm{cl}(I \backslash \{e\})$, contrary to (C2). □

We finish this chapter by showing that $b_k(t) \leqslant O_k(t)$, thereby completing the proof of Theorem 1.4. Our proof relies on the following operation: given a line L in a matroid M, the *principal truncation* of L in M, which we denote by $M \div L$, is the matroid obtained by freely adding a new element to M inside L and then contracting this new element. For our purposes, the key property of $M \div L$ is that its hyperplanes containing (the point) L are precisely the hyperplanes of M containing L, and its hyperplanes avoiding L are precisely the rank-$(r(M)-2)$ flats of M which are skew to L. In Chapter 6, we will define principal truncation for higher-rank flats, which will play a fundamental role in the proof of the Average Hyperplane-Size Theorem.

We will also use two classic theorems from discrete geometry, one of which is Beck's Theorem for Hyperplanes. The other is a weak version of a well-known conjecture of Dirac [14], which was first proved by Beck [4] and Szemerédi and Trotter [54].

Weak Dirac Theorem. *There is a number $c_d > 0$ such that, if M is a simple real-representable matroid on n elements and $r(M) \geqslant 3$, then there is an element of M incident to at least $c_d n$ lines.*

Our proof that $b_k(t) \leqslant O_k(t)$ proceeds as follows. We consider a 2-colouring (R, B) of a rank-$(k+1)$ real-representable matroid M satisfying (C1), (C2) and (C3), and show that $|B| \leqslant O_k(|R|)$ by induction on $k \geqslant 2$. Using the Weak Dirac Theorem, we start by contracting a blue element incident to $\Omega(|B|)$ lines in M. Then we colour the resulting matroid in the natural way, where the red points correspond to lines containing a red element. This colouring satisfies (C1) and (C2), so if it satisfies (C3), then we win by induction. If it does not, then we can principally truncate a line to obtain a matroid which (with the natural resulting colouring) again satisfies (C1) and (C2). Using part (ii) of the following lemma, we can repeat this process until we obtain a bicoloured truncated matroid satisfying (C3).

Lemma 5.4. *For $k \geqslant 2$, let M be a simple rank-$(k+1)$ matroid and let L be a line of M. Let (R, B) be a 2-colouring of M such that every hyperplane of M contains a red element.*

(i) *If there is a monochromatic blue rank-$(k-1)$ flat skew to L, then $|L| \leqslant |R|$.*

(ii) *If every rank-$(k-1)$ flat skew to L contains a red element, then every hyperplane in $M \div L$ contains a red element.*

Proof. For (1), observe that if there is a rank-$(k-1)$ flat F skew to L with $F \cap R = \varnothing$, then for all $e \in L$, $\mathrm{cl}_M(F \cup \{e\}) \cap R \neq \varnothing$, and so $|L| \leqslant |R|$. For (2), observe that the set of hyperplanes of $M \div L$ consists precisely of the hyperplanes of M containing L and the rank-$(k-1)$ flats of M skew to L. □

There is one problem with the argument described above. Namely, in order to derive the desired bound on $|B|$ after applying induction, we need the number of blue points in the truncated matroid to be $\Omega(|B|)$. We ensure this by initially bounding the size of each flat in M by $O(|B|)$. This will bound the number of blue points in each truncated line as needed.

Theorem 5.5. *For any positive integer k, we have $b_k(t) \leqslant O_k(t)$.*

Proof. We proceed by induction on $k \geqslant 1$. By definition, $b_1(t) = 0$ for all $t \geqslant 1$. Now let $k \geqslant 2$ and assume that, for all $i \in \{1, \dots, k-1\}$, there is a constant $C_i \geqslant 0$ such that $b_i(t) \leqslant C_i t$ for all $t \geqslant 1$. Let M be a simple rank-$(k+1)$ real-representable matroid, and let (R, B) be a 2-colouring of M satisfying (C1), (C2) and (C3). Let $t = |R|$.

Using the constants c_d from the Weak Dirac Theorem and α_i and β_i from Theorem 4.5, we define $\delta_k = (1/4k)c_d$, and for each $i \in \{1, \dots, k-1\}$, we recursively define $\delta_{i-1} = (1/2)\alpha_{i-1}\delta_i$. We may assume that $|B| \geqslant \max\{\frac{2t}{c_d \beta_{i-1} \delta_i^{i-1}} : 2 \leqslant i \leqslant k\}$, where we set $\gamma_1 = 1$.

Claim 5.5.1. *For each $i \in \{2, \dots, k\}$, every rank-i flat of $M|B$ has size at most $\delta_i|B|$.*

Proof of claim. Suppose otherwise, and choose $i \in \{2, \dots, k\}$ as small as possible such that there exists a rank-i flat F of $M|B$ with $|F| \geqslant \delta_i|B|$. By Lemma 5.4, every line of $M|B$ has size at most $t < \delta_2|B|$, and so $i \geqslant 3$. By choice of i, every hyperplane of $M|F$ has size at most $\delta_{i-1}|B| < \alpha_{i-1}|F|$. Hence, by Theorem 4.5, there are at least $\beta_{i-1}\delta_i^{i-1}|B|^{i-1}$ hyperplanes in $M|F$.

By (C3), there is a monochromatic-blue rank-$(k+1-i)$ flat P in M which is skew to F. To see this, take a line L of $M|F$ and a monochromatic-blue rank-$(k-1)$ flat F' skew to L. Then let $P = \mathrm{cl}_M(I)$, where I is a maximal independent subset of $F' \backslash F$ in M/F. Now $(M/P)|F = M|F$, and every hyperplane of M/P contains a red element. Since each

red element is spanned by at most $|B|^{i-2}$ hyperplanes of $M|F$ in M/P, we have that there are at least

$$\beta_{i-1}\delta_i^{i-1}|B|^{i-1} - t|B|^{i-2} = \Big(\beta_{i-1}\delta_i^{i-1} - \frac{t}{|B|}\Big)|B|^{i-1} > 0$$

hyperplanes in M/P containing no red element, a contradiction. □

By the Weak Dirac Theorem, there exists $e \in B$ incident to at least $c_d|B|$ lines in $M|B$. Let M' be a simplification of M/e, and let (R', B') be a 2-colouring of M' where $f \in R'$ if and only if $\mathrm{cl}_M(\{e, f\}) \cap R \neq \varnothing$. Then (R', B') satisfies (C1) and (C2). Also $|R'| \leqslant t$ and $|B'| \geqslant c_d|B| - t \geqslant \frac{1}{2}c_d|B|$. Henceforth, we refer to the elements of R' as *red* and the elements of B' as *blue*.

We have the following as an immediate consequence of Claim 5.5.1.

Claim 5.5.2. *For $1 \leqslant i \leqslant k-1$, if F is a rank-i flat of $M'|B'$, then $|F| \leqslant \frac{2\delta_i}{c_d}|B'|$.*

By repeated application of Lemma 5.4, there is a matroid M'', obtained from M' by principally truncating $\ell \leqslant k-2$ lines in turn, such that every hyperplane of M'' contains a red element, and, for every line of M'' containing two monochromatic-blue points, there is a monochromatic-blue rank-$(r(M'')-2)$ flat of M'' skew to it. Let B'' be the set of all blue elements which are not contained in a non-trivial parallel class of M''. (Equivalently, B'' is the set of blue elements not contained in one of the lines which were truncated to obtain M''.) For each line which was principally truncated to obtain M'', its set of blue elements is a flat of $M'|B'$ of rank at most $k-1$. Therefore, by Claim 5.5.2,

$$|B''| \geqslant |B'| - \frac{2\ell\delta_k}{c_d}|B'| \geqslant \frac{1}{2}|B'| \geqslant \frac{c_d}{4}|B|.$$

By induction, $|B''| \leqslant C_{k-\ell}t$. The result follows. □

Chapter 6

Average Hyperplane- and Flat-Size

In this chapter we prove the Average Plane-Size Theorem, the Average Flat-Size Theorem and the Average Hyperplane-Size Theorem. Broadly speaking, the proof of each of these theorems follows the same steps as the proof of Melchior's Theorem from Section 4.1. Namely, we start with a good approximation (within a multiplicative error) on the number of hyperplanes (or flats of a given rank) given by some generalization of Beck's Theorem, and then derive a similar approximation on the aggregate hyperplane-size.

We start by proving the Average Plane-Size Theorem, which says that the average plane-size in a real-representable matroid of rank at least four is at most an absolute constant, provided that it is not a direct sum of lines. Although this result is a corollary of the Average Hyperplane-Size Theorem, its proof serves as a good warm-up for the proof of the latter, which is similar but considerably more technical. After that, in Section 2, we derive a general upper bound on the aggregate hyperplane-size in a matroid. Combining this with Lund's Theorem, we prove the Average Flat-Size Theorem. In Section 3, we introduce the operation of principal truncation in full generality, and prove some relevant lemmas which will play important roles in the proof of the Average Hyperplane-Size Theorem. Finally, in Section 4, we complete the proof of the Average Hyperplane-Size Theorem.

Throughout this section, note that $\mathcal{L}(M)$, $\mathcal{P}(M)$ and $\mathcal{H}(M)$ will denote the collections of lines, planes and hyperplanes, respectively, of a matroid M. For a positive integer k, $\mathcal{F}_k(M)$ will denote the collection of rank-k flats of M.

6.1 Proof of the Average Plane-Size Theorem

We begin by proving a generalization of Beck's Theorem (Theorem 4.1) to planes. As discussed in Section 4.2, our approximation will be based on the consideration of a largest 3-degenerate set in our matroid. Recall that, in a matroid M, a set $X \subseteq E(M)$ is 3-*degenerate* if it has rank at most three or is contained in the union of two lines; if X_3 is a largest 3-degenerate set in M, and X_2 is a largest line in $M|X_3$, then (X_2, X_3) is called an *optimal* 3-*stratification* of M. Lund's Theorem (Theorem 4.7) says that there exists a constant $c_3 \geqslant 0$ such that, if M is a simple real-representable matroid on n elements, and (X_2, X_3) is an optimal 3-stratification of M, then $|\mathcal{P}(M)| \geqslant \Omega(n(n-|X_2|)(n-|X_3|))$, provided that $n - |X_3| > c_3$. We will strengthen this by proving that $c_3 = 0$. This theorem was originally proved in joint work with Rutger Campbell and Jim Geelen [10]; the proof we give here is a variation on the proof from [10]. It is essentially a simple case of the proof of the Average Hyperplane-Size Theorem.

Our proof employs a classic theorem of De Bruijn and Erdős [13], which says that, in any matroid of rank at least three, the number of lines is at least the number of points.

Theorem 6.1. *There exists $\rho_3 > 0$ such that, if M is a simple real-representable matroid on n elements, and (X_2, X_3) is an optimal 3-stratification of M, then*

$$|\mathcal{P}(M)| \geqslant \rho_3 n(n-|X_2|)(n-|X_3|).$$

Proof. We may assume that M is not 3-degenerate. Also, if $|X_3| < \frac{1}{2}n$, then by Theorem 4.6, $|\mathcal{P}(M)| \geqslant \gamma_3(\frac{1}{2})n^3$. So we may further assume that $|X_3| \geqslant \frac{1}{2}n$.

Case 1. $|X_2| < \frac{1}{4}n$.

In this case, X_3 is a plane; otherwise it is a union of skew lines, one of which is larger than $\frac{1}{4}n$, contrary to our choice of X_2. By Theorem 4.4,

$$|\mathcal{L}(M|X_3)| \geqslant \gamma_2(1/2)|X_3|^2 \geqslant \frac{1}{4}\gamma_2(1/2)n^2.$$

Now for $e \in E(M)\backslash X_3$, let $\mathcal{P}_e$ denote the collection of planes spanned by e and a line of $M|X_3$. For each $f \in E(M)\backslash(X_3 \cup \{e\})$, f is contained in at most n planes in $\mathcal{P}_e$. Hence

$$\sum_{P \in \mathcal{P}_e} |P\backslash X_3| \leqslant |\mathcal{P}_e| + n(n-|X_3|) \leqslant 2n^2.$$

Since $|\mathcal{P}_e| \geqslant \frac{1}{4}\gamma_2(\frac{1}{2})n^2$, it follows by Markov's Inequality that, for at least half of the planes $P \in \mathcal{P}_e$, $|P\backslash X_3| \leqslant 16/\gamma_2(\frac{1}{2})$. Letting $\mathcal{P}'_e$ denote this subcollection, we have that

$$\frac{1}{8}\gamma_2(1/2)n^2(n-|X_3|) \leqslant \sum_{e\in E(M)\backslash X_3}\sum_{P\in\mathcal{P}'_e} 1 \leqslant \frac{16}{\gamma_2(\frac{1}{2})}|\mathcal{P}(M)|,$$

as desired.

Case 2. $|X_2| \geqslant \frac{1}{4}n$.

Let $N = (M \div X_2)\backslash X_2$. Because M is not 3-degenerate, $r(N) \geqslant 3$. By choice of X_3, $X_3\backslash X_2$ is a largest line in N. Thus by Theorem 4.1, there are at least $\beta_2(n-|X_2|)(n-|X_3|)$ lines in N, where $\beta_2 > 0$ is some absolute constant. Since at most $\frac{1}{2}|E(N)|$ lines of N span the point X_2 in $M \div X_2$, it follows by the De Bruijn-Erdős Theorem that at least half of the lines of N are skew to the point X_2 in $M \div X_2$. Let $\mathcal{L}_\div$ denote this collection of lines, and note that every line in $\mathcal{L}_\div$ is a line in M which is skew to X_2. By Proposition 4.2, the average size of a line in $\mathcal{L}_\div$ is at most $\frac{6}{\beta_2}$. By Markov's Inequality, there is a subcollection $\mathcal{L}'_\div$ of $\mathcal{L}_\div$ such that $|\mathcal{L}'_\div| \geqslant \frac{1}{2}|\mathcal{L}_\div|$, and every line in $\mathcal{L}'_\div$ has at most $\frac{12}{\beta_2}$ points.

For $L \in \mathcal{L}'_\div$, let $\mathcal{P}_L$ be the collection of planes in M spanned by L and an element of X_2, and observe that

$$\sum_{P\in\mathcal{P}_L}|P| \leqslant |X_2||L| + n \leqslant \Big(\frac{12}{\beta_2}+1\Big)n.$$

Since $|\mathcal{P}_L| = |X_2| \geqslant \frac{1}{4}n$, it follows by Markov's Inequality that there is a subcollection $\mathcal{P}'_L$ of $\mathcal{P}_L$ such that $|\mathcal{P}'_L| \geqslant \frac{1}{2}|\mathcal{P}_L|$ and $|P| \leqslant 8(\frac{12}{\beta_2}+1)$ for all $P \in \mathcal{P}'_L$. Now each plane in $\mathcal{P}'_L$ contains at most $64(\frac{12}{\beta_2}+1)^2$ lines from $\mathcal{L}'_\div$. Hence

$$\frac{1}{32}\beta_2 n(n-|X_2|)(n-|X_3|) \leqslant \frac{1}{8}n|\mathcal{L}'_\div| \leqslant \sum_{L\in\mathcal{L}'_\div}\sum_{P\in\mathcal{P}'_L} 1 \leqslant 64\Big(\frac{12}{\beta_2}+1\Big)^2|\mathcal{P}(M)|,$$

as desired. □

The proof of the Average Plane-Size Theorem is now completed by the following.

Theorem 6.2. *Let M be a simple matroid on n elements, and let (X_2, X_3) be an optimal 3-stratification of M. If M is not 3-degenerate, then*

$$\sum_{P\in\mathcal{P}(M)}|P| \leqslant 5n(n-|X_2|)(n-|X_3|).$$

Proof. We partition $\mathcal{P}(M)$ into three collections based on how each plane intersects X_3, and show that the desired bound holds for each collection. Let $\mathcal{P}_1$ be the collection of planes P of M such that $r(P \cap X_2) \leqslant 1$ and $r(P \cap X_3) \leqslant 2$. For $P \in \mathcal{P}_1$, every maximal independent subset of P has an ordering (e, f, g) such that $f \notin X_2$ and $g \notin X_3$. By basis exchange, P has at least $|P| - 2 \geqslant \frac{1}{3}|P|$ maximal independent subsets. Because any rank-3 independent set in M spans a unique plane, it follows that

$$\frac{1}{3} \sum_{P \in \mathcal{P}_1} |P| \leqslant \sum_{P \in \mathcal{P}_1} (|P| - 2) \leqslant n(n - |X_2|)(n - |X_3|). \tag{6.1}$$

Now let $\mathcal{P}_2$ be the collection of planes P of M such that $r(P \cap X_2) = 2$. Because every element of $E(M) \backslash X_2$ is contained in exactly one plane of $\mathcal{P}_2$, we have that

$$\sum_{P \in \mathcal{P}_2} |P| \leqslant |X_2|(n - |X_2|) + n \leqslant 2n(n - |X_2|)(n - |X_3|). \tag{6.2}$$

Finally, let $\mathcal{P}_3$ be the collection of planes P of M such that $r(P \cap X_2) \leqslant 1$ and $r(P \cap X_3) = 3$. If X_3 is a plane, then $\mathcal{P}_3 = \{X_3\}$, in which case $\sum_{P \in \mathcal{P}_3} |P| \leqslant n$. Now assume that X_3 is the union of two skew lines, X_2 and $X_3 \backslash X_2$. Then for $P \in \mathcal{P}_3$, it must be the case that $r(P \cap X_2) = 1$, and so $\mathcal{P}_3$ consists precisely of the planes of M spanned by the line $X_3 \backslash X_2$ and a point in X_2. Consequently,

$$\sum_{P \in \mathcal{P}_3} |P| \leqslant |X_2||X_3 \backslash X_2| + n \leqslant 2n(n - |X_2|)(n - |X_3|). \tag{6.3}$$

The result now follows by combining (6.1), (6.2) and (6.3). □

In the next section, we generalize Theorem 6.2 to hyperplanes of any rank. In Section 6.4, we will generalize Theorem 6.1, which will complete the proof of the Average Hyperplane-Size Theorem. The proofs of both of these more general results are generalizations of the proofs of Theorems 6.1 and 6.2.

6.2 Proof of the Average Flat-Size Theorem

The main purpose of this section is to generalize Theorem 6.2 to flats of any given rank. This will essentially get us 'half way' through the proof of the Average Hyperplane-Size Theorem. We also prove the Average Flat-Size Theorem in this section, which will turn out to be a fairly easy consequence of our bound and Lund's Theorem.

Recall from Section 4.2 that, for an integer $k \geqslant 2$ and a matroid M, a *k-stratification* is a sequence $(X_2, \dots, X_k)$, where, for each $i \in \{2, \dots, k\}$, $X_i \subseteq E(M)$ is i-degenerate in M (see Section 1.2) and $X_{i-1} \subseteq X_i$. Here X_1 denotes the empty set and $X_{k+1} = E(M)$. We call a k-stratification $(X_2, \dots, X_k)$ *optimal* if, for each $i \in \{2, \dots, k\}$, X_i is a largest i-degenerate subset of X_{i+1}.

Theorem 6.3. *For an integer $k \geqslant 2$, let M be a simple matroid on $n \geqslant 1$ elements, and let $(X_2, \dots, X_k)$ be an optimal k-stratification of M. If $d \in \{1, \dots, k\}$ is as large as possible such that M is not d-degenerate, then*

$$\sum_{F \in \mathcal{F}_k} (|F| - k) \leqslant 2^{k(k-1)} \prod_{i=1}^{d} (n - |X_i|).$$

Proof. Let $\mathcal{H}_0$ denote the collection of rank-k flats F of M such that $r(F \cap X_i) < i$ for each $i \in \{2, \dots, k\}$. It is relatively easy to show that the desired bound holds for flats in $\mathcal{H}_0$.

Claim 6.3.1. $\sum_{F \in \mathcal{H}_0} (|F| - k) \leqslant \prod_{i=1}^{d} (n - |X_i|)$.

Proof of claim. If $d < k$, then $\mathcal{H}_0 = \varnothing$, so we may assume that $d = k$. Now for $F \in \mathcal{H}_0$ and for every maximal independent subset I of F, there is an ordering $(x_1, \dots, x_k)$ of the elements of I such that $x_i \notin X_i$ for all $i \in \{1, \dots, k\}$. Moreover, by basis exchange, there are at least $|F| - k$ maximal independent subsets of F. The claim now follows because each rank-k independent set in M spans a unique flat. □

Now we partition $\mathcal{F}_k \backslash \mathcal{H}_0$ into $2^{k(k-1)} - 1$ subcollections, and show that the aggregate size of the flats in each subscollection can be bounded just as in Claim 6.3.1. This will complete the proof.

For $i \in \{2, \dots, k\}$, let $\mathcal{P}_i$ be a collection of flats, each of rank at least two, such that $X_i = \bigcup_{P \in \mathcal{P}_i} P$, and $\sum_{P \in \mathcal{P}_i} (r(P_i) - 1) \leqslant i - 1$. Recall that, by the pigeonhole principle, if F is a rank-k flat with $r(F \cap X_i) \geqslant i$, then $P \subseteq F$ for some $P \in \mathcal{P}_i$. Therefore, letting $\mathcal{W}$ denote the collection of sets formed by taking unions of non-empty subcollections of $\mathcal{P}_2 \cup \cdots \cup \mathcal{P}_k$, we have that every flat in $\mathcal{F}_k \backslash \mathcal{H}_0$ contains a set in $\mathcal{W}$. For each $W \in \mathcal{W}$, let $\mathcal{H}_W$ denote the collection of flats $F \in \mathcal{F}_k$ such that $W \subseteq F$ and F does not contain any larger set in $\mathcal{W}$. Note that, because $|\mathcal{P}_i| \leqslant k - 1$ for each $i \in \{2, \dots, k\}$, we have that $|\mathcal{W}| \leqslant 2^{k(k-1)} - 1$.

For each $i \in \{1, \dots, k\}$, let $\mathcal{I}_i$ denote the collection of i-element independent sets I such that $r(I \cap X_j) < j$ for every $j \in \{2, \dots, k\}$.

Claim 6.3.2. *Let $W \in \mathcal{W}$ and $F \in \mathcal{H}_W$. Then for every maximal independent subset I of $F\backslash W$ in M/W and every $e \in \mathrm{cl}_M(W)$, we have $I \cup \{e\} \in \mathcal{I}_{k-r(W)+1}$.*

Proof of claim. Since I and $\mathrm{cl}_M(W)$ are skew in M, $I \cup \{e\}$ is independent in M. Now suppose to the contrary that $r_M((I \cup \{e\}) \cap X_j) \geqslant j$ for some $j \in \{2, \ldots, k\}$. Then $P \subseteq \mathrm{cl}_M(I \cup \{e\})$ for some $P \in \mathcal{P}_j$. However, since $r_M(P) \geqslant 2$ and I is skew to W, $P \nsubseteq W$. Hence $W \cup P$ is contained in F but properly contains W, a contradiction. □

Claim 6.3.3. *For each $W \in \mathcal{W}$, $\sum_{F \in \mathcal{H}_W}(|F| - k) \leqslant \prod_{i=1}^{d}(n - |X_i|)$.*

Proof of claim. If $r(W) \geqslant 1$, then $|\mathcal{H}_W| \leqslant 1$, and so $\sum_{F \in \mathcal{H}_W}(|F| - k) \leqslant n$. So we may assume that $r(W) < k$. For each $F \in \mathcal{H}_W$, let $\mathcal{I}_F$ denote the collection of independent sets of the form $I \cup \{e\}$, where $e \in \mathrm{cl}_M(W)$ and I is a maximal independent subset of $F\backslash W$ in M/W. By basis exchange, we have that

$$|\mathcal{I}_F| \geqslant |\mathrm{cl}_M(W)|(|F| - |\mathrm{cl}_M(W)| - (r_M(F) - r_M(W))) \geqslant |F| - k.$$

Now for distinct $F, F' \in \mathcal{H}_W$, $\mathcal{I}_F \cap \mathcal{I}_{F'} = \varnothing$ and, by Claim 6.3.2, $\mathcal{I}_F \subseteq \mathcal{I}_{k-r(W)+1}$. Since $k - r(W) + 1 \leqslant d$, we now have that

$$\sum_{F \in \mathcal{H}_W}(|F| - k) \leqslant \sum_{F \in \mathcal{H}_W} |\mathcal{I}_F| \leqslant |\mathcal{I}_d| \leqslant \prod_{i=1}^{d}(n - |X_i|),$$

as desired. □

The result now follows by Claims 6.3.1 and 6.3.3. □

The formulation of Lund's Theorem given in Section 4.2 is actually a slight variation on Lund's original theorem from [41], which considers a largest i-degenerate set *in* M for each i rather than a k-stratification (where there is an assumption of containment for the degenerate sets). As shown below, the two statements are equivalent up to the specification of a multiplicative constant. The formulation from Section 4.2 is given for the sake of straightforward comparison with our results. However, for the proof of the Average Flat-Size Theorem, we require Lund's original formulation.

Theorem 6.4 (Lund's Theorem, [41])**.** *For each integer $k \geqslant 2$, there exists $\rho_k > 0$ and an integer $c_k \geqslant 0$ such that, if M is a simple real-representable matroid on n elements with $n - g_k > c_k$, then*

$$|\mathcal{F}_k(M)| \geqslant \rho_k n \prod_{i=2}^{k}(n - g_i),$$

where, for each $i \in \{2, \ldots, k\}$, g_i is the size of a largest i-degenerate set in M.

The following, which was originally proved in [10], shows that the two versions of Lund's Theorem are equivalent.

Proposition 6.5. *Let M be a simple matroid on n elements, and let $(X_2, \ldots, X_k)$ be an optimal k-stratification of M. Then for each $i \in \{2, \ldots, k\}$, $n - |X_i| \leqslant (k+1-i)(n-g_i)$.*

Proof. For $i = k$, the result follows immediately. Now let $i \in \{2, \ldots, k-1\}$ and assume that $n - |X_{i+1}| \leqslant (k-i)(n-g_{i+1})$. Let Z be a largest i-degenerate set in M. Then

$$|X_i| \geqslant |Z \cap X_{i+1}| \geqslant |Z| - (n - |X_{i+1}|) \geqslant g_i - (k-i)(n-g_{i+1}).$$

Hence

$$n - |X_i| \leqslant n - g_i + (k-i)(n-g_{i+1}) \leqslant (k-i+1)(n-g_i),$$

as desired. □

Now we are ready to prove the Average Flat-Size Theorem, which we re-state here for convenience.

Average Flat-Size Theorem. *For any integer $r \geqslant 1$, the average flat-size in a rank-r real-representable matroid is at most a constant depending only on r.*

Proof. Let M be a rank-r real-representable matroid on n elements; we may assume that $r \geqslant 3$ and that M is simple. Choose $t \in \{2, \ldots, r-1\}$ as large as possible such that $n - g_t > c_t$. (Recall that $c_2 = 0$ by Beck's Theorem, and so this choice is well-defined.) By Theorem 6.4, we have

$$|\mathcal{F}(M)| \geqslant |\mathcal{F}_t(M)| \geqslant \rho_t n \prod_{i=2}^{t}(n - g_i). \tag{6.4}$$

Now let $d \in \{t, \ldots, r-1\}$ be as large as possible such that M is not d-degenerate. Since $n - g_i \leqslant c_i$ for each $i \in \{t+1, \ldots, d\}$, it follows from (6.4) that

$$|\mathcal{F}(M)| \geqslant \frac{\rho_t}{c_{t+1} \ldots c_d} n \prod_{i=2}^{d}(n - g_i). \tag{6.5}$$

On the other hand, it follows by Theorem 6.3 and Proposition 6.5 that, for each $k \in \{2, \ldots, r-1\}$,

$$\sum_{F \in \mathcal{F}_k} |F| \leqslant b_k n \prod_{i=2}^{d}(n - g_i) + k|\mathcal{F}_k|,$$

for some constant $b_k > 0$ depending only on k. Hence

$$\sum_{F \in \mathcal{F}(M)} |F| \leqslant \Big(2 + \sum_{j=2}^{r-1} b_j\Big) n \prod_{i=2}^{d} (n - g_i) + r^2 |\mathcal{F}(M)|. \tag{6.6}$$

The result now follows from (6.5) and (6.6). □

6.3 Principal truncation

One of the important steps in the proof of the Average Plane-Size Theorem was to reduce to a rank-3 matroid by principally truncating a line. In the proof of the Average Hyperplane-Size Theorem, we will principally truncate a flat of rank at least two; this is a standard operation in matroid theory which we define below. Given an optimal 3-stratification (X_2, X_3) of a matroid M of rank at least four, it is relatively easy to see that $X_3 \backslash X_2$ is a largest line in $(M \div X_2) \backslash X_2$. (This was an important observation in the proof of the Average Plane-Size Theorem.) In this section, we will prove several helpful lemmas on principal truncation, the most important of which tells us an optimal k-stratification in a principally truncated matroid of rank $k+1$.

Let M be a matroid and let F be a set in M of rank at least two. The *principal truncation of F in M* is the matroid with N ground set $E(N) = E(M)$ and rank function

$$r_N(X) = \begin{cases} r_M(X) - 1 & \text{if } F \subseteq \mathrm{cl}_M(X) \\ r_M(X) & \text{otherwise.} \end{cases}$$

Equivalently, N is the matroid obtained from M by freely extending in $\mathrm{cl}_M(F)$ by a new element and then contracting this element. Note that if M is a simple matroid, then so is N, unless $r_M(F) = 2$, in which case $\mathrm{cl}_M(F)$ is a point in N, and $N \backslash \mathrm{cl}_M(F)$ is simple. For any $1 \leqslant k \leqslant r_M(F) - 1$, this operation may be reiterated k times; we refer to this as the *k-fold principal truncation* of F in M. The $(r_M(F) - 1)$-fold principal truncation of F in M is called the *complete principal truncation* of F in M, which we denote by $M \div F$.

We begin by laying out some standard properties of principal truncation which will be particularly helpful to us. We will prove these properties in full, although they are well-known consequences of the fact that principal truncation is a quotient operation on a matroid, see [48, Section 7.3].

Proposition 6.6. *Let M be a matroid, and let $F \subseteq E(M)$ with $2 \leqslant r_M(F) \leqslant r(M) - 2$. Let N be the 1-fold principal truncation of F in M.*

(1) For $X \subseteq E(M)$, if $F \nsubseteq \mathrm{cl}_N(X)$, then $F \nsubseteq \mathrm{cl}_M(X)$. Consequently, if $F \nsubseteq \mathrm{cl}_{M \div F}(X)$, then $F \nsubseteq \mathrm{cl}_M(X)$.

(2) If H is a flat in N, then H is a flat in M. Consequently, if H is a flat in $M \div F$, then H is a flat in M.

(3) If H is a flat in $M \div F$ which does not contain F, then H is a rank-$r_{M \div F}(H)$ flat in M which is skew to F.

(4) If H is a hyperplane in M which does not contain F, then there exists $X \subseteq H$ such that X is a hyperplane in $M \div F$ (which is disjoint from F).

Proof. For (1), suppose that $F \nsubseteq \mathrm{cl}_N(X)$. Then

$$r_M(X \cup F) - 1 = r_N(X \cup F) \geqslant r_N(X) + 1 \geqslant r_M(X),$$

as desired.

For (2), suppose to the contrary that there exists $e \in \mathrm{cl}_M(H) \backslash H$. If $F \subseteq \mathrm{cl}_M(H)$, then

$$r_N(H \cup \{e\}) = r_M(H \cup \{e\}) - 1 = r_M(H) - 1 = r_N(H),$$

a contradiction. On the other hand, if $F \nsubseteq \mathrm{cl}_M(H)$, then

$$r_N(H \cup \{e\}) - 1 = r_N(H) = r_M(H) = r_M(H \cup \{e\}),$$

again a contradiction.

For (3), it suffices to prove that if H is a flat in N which is skew to F, then H is a flat in M which is skew to F. (The result then follows immediately by induction on $r_M(F)$.) Indeed, let H be a flat in N which is skew to F. By (2), H is a flat in M. By (1), $F \nsubseteq \mathrm{cl}_M(H)$, and so $r_M(H) = r_N(H)$. Hence

$$r_M(X \cup F) - 1 = r_N(X \cup F) = r_N(X) + r_N(F) = r_M(X) + r_M(F) - 1,$$

and so H and F are skew in M.

For (4), take a flat $X \subseteq H$ in M which is skew to F and has rank $r(M) - r_M(F)$. □

The reason principal truncation is the right operation to use in our proof is that, subject to mild assumptions, it behaves very nicely with respect to an optimal stratification. The main lemma of this section will allow us to obtain an optimal stratification in our truncated matroid that preserves much of the relevant data from the original. We first prove a slightly weaker result. For an optimal k-stratification $(X_2, \ldots, X_k)$ of a matroid M and $i \in \{2, \ldots, k\}$, we call $(X_i, \ldots, X_k)$ an optimal *partial* k-stratification of M.

Proposition 6.7. *For $k \geqslant 4$, let M be a simple rank-$(k+1)$ matroid, and let $(X_2, \ldots, X_k)$ be an optimal k-stratification of M. For $i \in \{3, \ldots, k-1\}$, if X_i is a rank-i flat in M, and $|X_{i-1}| < \frac{1}{k}|X_i|$, then $(X_i, \ldots, X_k)$ is an optimal partial $(k-1)$-stratification of the rank-1 principal truncation of X_i in M.*

Proof. Let N denote the 1-fold principal truncation of X_i in M. Clearly X_i is a rank-$(i-1)$ flat in N, and so it is $(i-1)$-degenerate in N. Now let $\ell \in \{i+1, \ldots, k\}$, and let $(F_1, \ldots, F_t)$ be a collection of flats of M, each having rank at least two, such that $X_\ell = F_1 \cup \cdots \cup F_t$ and $\sum_{j=1}^{t}(r_M(F_j) - 1) \leqslant \ell - 1$. Choose $j \in \{1, \ldots, t\}$ such that $|X_i \cap F_j|$ is as large as possible. Since $t \leqslant k$ and $X_i \subseteq X_\ell$, $|X_i \cap F_j| \geqslant \frac{1}{k}|X_i|$. If $X_i \nsubseteq F_j$, then $X_i \cap F_j$ is an $(i-1)$-degenerate set, contrary to the choice of X_{i-1}. Thus $X_i \subseteq F_j$. It follows that X_ℓ is $(\ell-1)$-degenerate in N.

It remains to show that, for each $\ell \in \{i, \ldots, k\}$, X_ℓ is a largest $(\ell-1)$-degenerate subset of $X_{\ell+1}$ in N. (Here $X_{k+1} = E(N)$.) Indeed, for $\ell \in \{i, \ldots, k\}$, suppose to the contrary that there exists $Y \subseteq X_{\ell+1}$ such that $|Y| > |X_\ell|$ and Y is $(\ell-1)$-degenerate in N. Let $(F_1, \ldots, F_t)$ be a collection of flats of N, each with rank at least two, such that $Y = F_1 \cup \cdots \cup F_t$ and $\sum_{j=1}^{t}(r_N(F_j) - 1) \leqslant \ell - 2$. Recall that we may assume that the flats $F_1, \ldots, F_t$ are pairwise-skew in N, and so at most one of them contains X_i. If $X_i \subseteq F_j$ for some $j \in \{1, \ldots, t\}$, then, by Proposition 6.6(1), Y is ℓ-degenerate in M; otherwise Y is $(\ell-1)$-degenerate in M, in either case a contradiction to the choice of X_ℓ. □

Now we are ready to prove our main lemma on principal truncation.

Lemma 6.8. *For $k \geqslant 2$, let M be a simple rank-$(k+1)$ matroid, and let $(X_2, \ldots, X_k)$ be an optimal k-stratification of M. For $i \in \{2, \ldots, k-1\}$, if X_i is a rank-i flat in M, and $|X_{i-1}| + 1 < \frac{1}{k}|X_i|$, then $(X_{i+1}\backslash X_i, \ldots, X_k\backslash X_i)$ is an optimal $(k+1-i)$-stratification of $(M \div X_i)\backslash X_i$.*

Proof. We first prove the result for $i = 2$, and then proceed by induction. Let $\ell \in \{3, \ldots, k\}$, and let $(F_1, \ldots, F_t)$ be a collection of flats of M, each having rank at least two, such that $X_\ell = F_1 \cup \cdots \cup F_t$ and $\sum_{j=1}^{t}(r_M(F_j) - 1) \leqslant \ell - 1$. If $X_2 \nsubseteq F_j$ for all $j \in \{1, \ldots, t\}$, then $X_2 \leqslant t < k$, a contradiction. It follows that, for all $\ell \in \{3, \ldots, k\}$, X_ℓ is an $(\ell-1)$-degenerate set in $M \div X_2$. Thus $(X_3\backslash X_2, \ldots, X_k\backslash X_2)$ is a $(k-1)$-stratification of $(M \div X_2)\backslash X_2$; it remains to show that it is optimal. Indeed, let $\ell \in \{3, \ldots, k\}$, and suppose to the contrary that there exists $Y \subseteq X_{\ell+1}\backslash X_2$ such that $|Y| > |X_\ell\backslash X_2|$ and Y is $(\ell-1)$-degenerate in $M \div X_2$. Then, by Proposition 6.6(1), $Y \cup X_2$ is ℓ-degenerate in M, a contradiction. This completes the proof for $i = 2$.

Now suppose that $i \in \{3, \dots, k-1\}$, and let N be the 1-fold principal truncation of X_i in M. Then X_i is a rank-$(i-1)$ flat in N, and, by Proposition 6.7, $(X_i, \dots, X_k)$ is an optimal partial $(k-1)$-stratification of N. Let Z be a largest $(i-2)$-degenerate set in N contained in X_i. Since X_i is a rank-i flat in M and $Z \subseteq X_i$, Z is an $(i-2)$-degenerate set in M, and so $|X_i| > k(|Z|+1)$. By induction, $(X_{i+1}\backslash X_i, \dots, X_k\backslash X_i)$ is an optimal $(k+1-i)$-stratification of $(N \div X_i)\backslash X_i = (M \div X_i)\backslash X_i$. □

6.4 Bounding average hyperplane-size

In this section, we finish the proof of the Average Hyperplane-Size Theorem, which we re-state here in full for convenience.

Average Hyperplane-Size Theorem. *For $k \geqslant 2$, if M is a rank-$(k+1)$ real-representable matroid, then either*

(i) the average size of a hyperplane of M is at most a constant C_k depending only on k, or

(ii) there is a collection $L_1, \dots, L_t$, where $t \leqslant k-1$, of lines of M such that each hyperplane of M contains one of them. Moreover, by freely adding new elements to M inside any of these lines, every hyperplane in the resulting matroid still contains one of them.

Additionally, if the average hyperplane-size in M is greater than C_k, then for some $j \in \{1, \dots, k-1\}$, M has a $(j+1)$-degenerate restriction containing all but at most $b_{k-j}(j)$ of its points.

The main technical lemma of this section is an approximation on the number of hyperplanes in a real-representable matroid which, when combined with Theorem 6.3, gives the Average Hyperplane-Size Theorem. This approximation is in fact a strengthening of Lund's Theorem (Theorem 4.7), but it has a rather technical formulation to facilitate its inductive proof. To state the result, we require a special notion of degeneracy for bicoloured matroids. For a sequence of sets $(F_1, \dots, F_t)$ in a matroid M, we let $M \div F_1 \div \cdots \div F_t$ denote the matroid obtained by completely principally truncating F_1 in M, then completely principally truncating F_2 in $M \div F_1$, and so on. For $k \geqslant 2$ and $\ell \geqslant 0$, let M be a simple rank-$(k+1)$ matroid, and let (R, B) be a 2-colouring of M with $|R| = \ell$. We say that (M, R, B) is (k, ℓ)-*degenerate* if either B is k-degenerate, or there exists a sequence $(F_1, \dots, F_t)$ of disjoint subsets of B such that

(D1) $t \leqslant k-2$;

(D2) $2 \leqslant r_M(F_1) \leqslant k-1$, and, for each $j \in \{2,\dots,t\}$, $2 \leqslant r_{M_j}(F_j) \leqslant r(M_j)-2$, where $M_j = M \div F_1 \div \cdots \div F_{j-1}$; and

(D3) every hyperplane in $M \div F_1 \div \cdots \div F_t$ has non-empty intersection with $R \cup F_1 \cup \cdots \cup F_t$.

Our technical lemma is as follows.

Theorem 6.9. *For integers $k \geqslant 2$ and $\ell \geqslant 0$, there exist $n_k(\ell) \geqslant 0$ and $\rho_k(\ell) > 0$ such that the following holds. Let M be a simple rank-$(k+1)$ real-representable matroid, let (R,B) be a 2-colouring of M where $|R| = \ell$, and let $(X_2,\dots,X_k)$ be an optimal k-stratification of $M|B$. If $|B| \geqslant n_k(\ell)$ and (M,R,B) is not (k,ℓ)-degenerate, then there are at least $\rho_k(\ell)|B|\prod_{i=2}^{k}(|B|-|X_i|)$ monochromatic-blue hyperplanes in M.*

Before proving Theorem 6.9, we show that it indeed implies the Average Hyperplane-Size Theorem. To prove the second part of the theorem, we will need the following lemma.

Lemma 6.10. *For $k \geqslant 2$, let M be a simple rank-$(k+1)$ real-representable matroid, and let (R,B) be a 2-colouring of M. If every hyperplane of M contains a red element, then there exists $j \in \{0,1,\dots,k-1\}$ such that $M|B$ has a $(j+1)$-degenerate restriction with at least $|B| - b_{k-j}(|R|)$ elements.*

Proof. We proceed by induction on $k \geqslant 2$. For $k=2$, the result follows with $j=1$ if B is collinear; otherwise, the result follows with $j=0$. Now let $k \geqslant 3$. Suppose first that for every line L of $M|B$, there is a rank-$(k-1)$ flat F skew to L that contains no red points. In this case, we have $|B| \leqslant b_k(|R|)$ by definition, and so the result follows with $j=0$. Thus we may assume that there is a line L of $M|B$ such that every rank-$(k-1)$ flat F skew to L contains a red point. Let N be a simplification of the principal truncation of L in M such that the element corresponding to L is coloured red if and only if $L \cap R \neq \varnothing$. Let (R',B') be the resulting 2-colouring of N. By Lemma 5.4, every hyperplane of N contains a red element. By induction, there exists $j \in \{0,1,\dots,k-2\}$ such that $X \subseteq B'$ such that $N|X$ is $(j+1)$-degenerate and $|X| \geqslant |B'| - b_{k-1-j}(|R'|)$. Now $X \cup L$ is $(j+2)$-degenerate in M. The result now follows because $|B'| + |L \backslash X| \geqslant |B|$. □

We apply Lemma 6.10 to a matroid obtained from M by a series of principal truncations, where we colour the truncated points red. The truncated points together with the blue degenerate set will then form the desired degenerate set in M. In order to prove this, we need the following rank formula, whose easy inductive proof is omitted.

Proposition 6.11. *For $k \geqslant 2$, let M be a rank-$(k+1)$ matroid, and let $(F_1, \ldots, F_t)$ be a sequence of subsets of M satisfying (D2). Let $M_1 = M$, and let $M_j = M \div F_1 \div \cdots \div F_{j-1}$ for each $j \in \{2, \ldots, t+1\}$. For any flat H of M_{t+1},*

$$r_M(H) = r_{M_{t+1}}(H) + \sum_{i \in \mathcal{I}} (r_{M_i}(F_i) - 1),$$

where $\mathcal{I} = \{i \in \{1, \ldots, t\} : F_i \subseteq H\}$.

Now we are ready to derive the Average Hyperplane-Size Theorem from Theorem 6.9.

Proof of the Average Hyperplane-Size Theorem. Let $C_k = \max\{2^{k(k-1)}/\rho_k(0) + k, n_k(0)\}$, where n_k and ρ_k are as in Theorem 6.9. We may assume that M is simple and not k-degenerate, and that $|E(M)| \geqslant n_k(0)$. Now let $R = \varnothing$ and $B = E(M)$. By Theorems 6.3 and 6.9, we may assume that (M, R, B) is not $(k, 0)$-degenerate, for otherwise the average hyperplane-size is at most C_k. So there is a sequence $(F_1, \ldots, F_t)$ of disjoint subsets of $E(M)$ satisfying (D1), (D2) and (D3). For each $i \in \{1, \ldots, t\}$, take $S_i \subseteq F_i$ with $r_M(S_i) = 2$ and define $L_i = \mathrm{cl}_M(S_i)$. Let $M_{t+1} = M \div F_1 \div \cdots \div F_t$. Then every hyperplane of M_{t+1} contains F_i for some $i \in \{1, \ldots, t\}$. By Proposition 6.6(4), every hyperplane of M contains one of the sets $F_1, \ldots, F_t$. This proves the first part of the theorem.

Let $\mathcal{P} = \{\mathrm{cl}_{M_{t+1}}(F_i) : i \in \{1, \ldots, t\}\}$ be the collection of truncated points in M_{t+1}. (Note that, for $i \neq j$, it is possible that $\mathrm{cl}_{M_{t+1}}(F_i) = \mathrm{cl}_{M_{t+1}}(F_j)$.) Let N be a simplification of M_{t+1}, and let R be the set of elements of N contained in a point in $\mathcal{P}$. Let $B = E(N) \backslash R$. (Note also that $B = E(M) \backslash R$.) Then (R, B) is a 2-colouring of N such that every hyperplane contains a red element. By Lemma 6.10, there exists $h \in \{0, 1, \ldots, r(N) - 2\}$ such that $N|B$ has an $(h+1)$-degenerate set X such that $|X| \geqslant |B| - b_{r(N)-1-h}(|R|)$. Let $(G_1, \ldots, G_s)$ be a collection of pairwise-skew flats of $N|B$, each having rank at least two, such that $X \subseteq G_1 \cup \cdots \cup G_s$ and $\sum_{i=1}^{s}(r_N(G_i) - 1) \leqslant h$. Now let $j = r(M) + h - r(N)$, and observe that $b_{k-j}(|R|) \leqslant b_{k-j}(j)$, since $|R| \leqslant r(M) - r(N) \leqslant j$. Therefore the set $X \cup (\bigcup_{P \in \mathcal{P}} P)$ has size at least $|E(M)| - b_{k-j}(j)$. It remains to show that this set is $(j+1)$-degenerate in M.

Because the sets $G_1, \ldots, G_s$ are pairwise-skew in M_{t+1}, any point of $\mathcal{P}$ is spanned by at most one of them in M_{t+1}. Let $\mathcal{P}'$ be the collection of points in $\mathcal{P}$ not spanned by any of $G_1, \ldots, G_s$ in M_{t+1}. Now by Proposition 6.11,

$$\sum_{P \in \mathcal{P}'} (r_M(P) - 1) + \sum_{i=1}^{s} (r_M(\mathrm{cl}_{M_{t+1}}(G_i)) - 1) = \sum_{i=1}^{t} (r_{M_i}(F_i) - 1) + \sum_{j=1}^{s} (r_N(G_j) - 1) \leqslant j,$$

as required. This proves the second part of the theorem. □

Now we finish the proof.

Proof of Theorem 6.9. We proceed by induction on $k \geqslant 2$. For the base case, assume that $|B| \geqslant \max\{4\ell, \ell\gamma_2(\frac{1}{4})^{-1}\}$. Suppose first that $|X_2| < \frac{3}{4}|B|$. In this case, there are at least $\gamma_2(\frac{1}{4})|B|^2$ lines in $M|B$. Since each red element is spanned by at most $\frac{1}{2}|B|$ lines of $M|B$, there are at most $\frac{1}{2}\ell|B|$ lines of $M|B$ which span a red element in M. Hence the number of monochromatic-blue lines in M is at least $(\gamma_2(\frac{1}{4}) - \frac{\ell}{2|B|})|B|^2 \geqslant \frac{1}{2}\gamma_2(\frac{1}{4})|B|^2$. Thus we may assume that $|X_2| \geqslant \frac{3}{4}|B|$. For each $e \in B\backslash X_2$, there are at least $|X_2| - (|B| - |X_2| + \ell)$ (monochromatic-blue) lines in M which contain only e and an element of X_2. Hence the number of monochromatic-blue lines in M is at least

$$\begin{aligned}(2|X_2| - |B| - \ell)(|B| - |X_2|) &\geqslant (\frac{1}{2}|B| - \ell)(|B| - |X_2|)\\ &\geqslant \frac{1}{4}|B|(|B| - |X_2|),\end{aligned}$$

as required.

Now let $k \geqslant 3$ and $\ell \geqslant 0$. Let $\mathcal{H}^b$ be the collection of monochromatic-blue hyperplanes of M. Suppose first that $|X_k| < \frac{1}{2}|B|$. In this case, $M|B$ has at least $\gamma_k(\frac{1}{2})|B|^k$ hyperplanes by Theorem 4.6. Since each element of R is spanned by at most $|B|^{k-1}$ hyperplanes of $M|B$, we have that

$$|\mathcal{H}^b| \geqslant \Big(\gamma_k(1/2) - \frac{\ell}{|B|}\Big)|B|^k \geqslant \frac{1}{2}\gamma_k(1/2)|B|^k,$$

provided that $|B|$ is sufficiently large as a function of k and ℓ.

Now assume that $|X_k| \geqslant \frac{1}{2}|B|$. Choose $i \in \{2, \dots, k\}$ as small as possible such that $|X_i| \geqslant |B|(2k)^{-(1+2(k-i))}$ and $|X_{i-1}| < |B|(2k)^{-(1+2(k-i+1))}$. (Recall that X_1 denotes the empty set.)

Claim 6.11.1. *X_i is a rank-i flat in $M|B$.*

Proof of claim. Suppose otherwise. Since X_i is i-degenerate, there is a collection $(F_1, \dots, F_t)$, where $t \geqslant 2$, of pairwise-skew flats in M, each with rank at least two, such that $X_i = F_1 \cup \cdots \cup F_t$, and $\sum_{j=1}^{t}(r_M(F_j) - 1) \leqslant i - 1$. Then there exists $j \in \{1, \dots, t\}$ such that $|F_j| \geqslant \frac{|X_i|}{t} \geqslant \frac{|X_i|}{k} > |X_{i-1}|$. But, since $t \geqslant 2$, $r_M(F_j) \leqslant i - 1$, that is, F_j is an $(i-1)$-degenerate set, contrary to the choice of X_{i-1}. □

By Claim 6.11.1 and Theorem 4.6, we have that

$$|\mathcal{F}_{i-1}(M|X_i)| \geqslant \gamma_{i-1}(1-(2k)^{-2})|X_i|^{i-1} \geqslant C_1|B|^{i-1} \tag{6.7}$$

for some constant $C_1 > 0$ depending only on k.

Claim 6.11.2. *If $i = k$, then $|\mathcal{H}^b| \geqslant \frac{C_1^2}{128}|B|^{k-1}(|B|-|X_k|)$.*

Proof of claim. Let $e \in B\backslash X_k$, and let $\mathcal{F}_e$ denote the collection of hyperplanes in $\mathcal{H}^b$ spanned by e and a hyperplane of $M|X_k$. Since each element of R is spanned by at most $|B|^{k-2}$ hyperplanes of $M|X_k$ in M/e, we have

$$|\mathcal{F}_e| \geqslant C_1|B|^{k-1} - \ell|B|^{k-2} \geqslant \frac{C_1}{2}|B|^{k-1}, \tag{6.8}$$

provided that $|B|$ is sufficiently large as a function of k and ℓ. Similarly, since each element of $B\backslash X_k$ is spanned by at most $|B|^{k-2}$ hyperplanes of $M|X_k$ in M/e, we have that

$$\sum_{F\in\mathcal{F}_e} |F\backslash X_k| \leqslant |B|^{k-2}(|B|-|X_k|) + |B|^{k-1} \leqslant 2|B|^{k-1}. \tag{6.9}$$

Let $\mathcal{F}_e'$ be the collection of flats $F \in \mathcal{F}_e$ such that $|F\backslash X_k| \leqslant \frac{8}{C_1}$. Combining (6.8) and (6.9) and applying Markov's Inequality, we have that $|\mathcal{F}_e'| \geqslant \frac{C_1}{4}|B|^{k-1}$. Hence

$$\frac{C_1}{4}|B|^{k-1}(|B|-|X_k|) \leqslant \sum_{e\in E(M)\backslash X_k}\sum_{F\in\mathcal{F}_e'} 1 \leqslant \frac{8}{C_1}|\mathcal{H}^b|,$$

as desired. □

By Claim 6.11.2, we may assume that $i < k$. Let M' be a simplification of $M \div X_i$, and let x_i denote the element of $E(M') \cap X_i$. Let (R', B') be the 2-colouring of M' given by $R' = (R\backslash X_i) \cup \{x_i\}$ and $B' = B\backslash X_i$, and let $\ell' = |R'|$. Then $\ell' \leqslant \ell + 1$, and (M', R', B') is not $(k-i+1, \ell')$-degenerate. Furthermore, by Lemma 6.8, $(X_{i+1}\backslash X_i, \ldots, X_k\backslash X_i)$ is an optimal $(k+1-i)$-stratification of $M'|B'$.

It remains to consider two cases.

Case 1. $|B| - |X_i| < n_{k-i+1}(\ell')$.

Since (M, R, B) is not (k, ℓ)-degenerate, there is a hyperplane H of M' disjoint from R'. By Proposition 6.6(3), H is a monochromatic-blue rank-$(k-i+1)$ flat in M which is

skew to X_i. Since $M|X_i = (M/H)|X_i$, it follows by (6.7) that M/H has at least $C_1|B|^{i-1}$ hyperplanes. Now, because each element of R is spanned by at most $|B|^{i-2}$ hyperplanes of $M|X_i$ in M/H, we have that

$$|\mathcal{H}^b| \geqslant C_1|B|^{i-1} - \ell|B|^{i-2} \geqslant \frac{C_1}{2}|B|^{i-1},$$

provided that $|B|$ is sufficiently large as a function of k and ℓ. The result now follows because $|B| - |X_j| < n_{k+1-i}(\ell')$ for each $j \in \{i, \dots, k\}$.

Case 2. $|B| - |X_i| \geqslant n_{k-i+1}(\ell')$.

By induction, M' has $\rho_{k-i+1}(\ell')\prod_{j=i}^{k}(|B|-|X_j|)$ hyperplanes which are disjoint from R'. Let $\mathcal{H}_{\div}$ denote this collection of hyperplanes. By Proposition 6.6(3), each hyperplane in $\mathcal{H}_{\div}$ is a monochromatic-blue rank-$(k-i+1)$ flat in M which is skew to X_i. By Theorem 6.3, the average size of a hyperplane in $\mathcal{H}_{\div}$ at most C_2, where $C_2 > 0$ is a constant depending only on k. Therefore, letting $\mathcal{H}'_{\div} = \{H \in \mathcal{H}_{\div} : |H| \leqslant 2C_2\}$, we have by Markov's Inequality that $|\mathcal{H}'_{\div}| \geqslant \frac{1}{2}|\mathcal{H}_{\div}|$.

For $H \in \mathcal{H}'_{\div}$, let $\mathcal{F}_H$ denote the collection of monochromatic-blue hyperplanes of M spanned by H and a hyperplane of $M|X_i$. Now $M|X_i = (M/H)|X_i$, and each element of R is spanned by at most $|B|^{i-2}$ hyperplanes of $M|X_i$ in M/H. Hence, by (6.7),

$$|\mathcal{F}_H| \geqslant C_1|B|^{i-1} - \ell|B|^{i-2} \geqslant \frac{C_1}{2}|B|^{i-1}, \tag{6.10}$$

provided that $|B|$ is sufficiently large as a function of k and ℓ. Now observe that

$$\frac{1}{|\mathcal{F}_H|}\sum_{F\in\mathcal{F}_H}|F\backslash X_i| \leqslant \frac{1}{|\mathcal{F}_H|}(|B|^{i-1} + |\mathcal{F}_H||H|) \leqslant \frac{2}{C_1} + 2C_2.$$

Let $\mathcal{F}'_H = \{F \in \mathcal{F}_H : |F\backslash X_i| \leqslant 4/C_1 + 4C_2\}$. By Markov's Inequality and (6.10), $|\mathcal{F}'_H| \geqslant \frac{C_1}{4}|B|^{i-1}$. Now, because every flat of $\mathcal{H}_{\div}$ is skew to X_i in M, each hyperplane in $\mathcal{F}'_H$ contains at most $(4/C_1 + 4C_2)^{k+1-i}$ flats in $\mathcal{H}_{\div}$. Therefore

$$\frac{C_1}{8}|B|^{i-1}|\mathcal{H}_{\div}| \leqslant \frac{C_1}{4}|B|^{i-1}|\mathcal{H}'_{\div}| \leqslant \sum_{H\in\mathcal{H}'_{\div}}\sum_{F\in\mathcal{F}'_H} 1 \leqslant \Big(\frac{4}{C_1} + 4C_2\Big)^{k+1-i}|\mathcal{H}^b|.$$

The result now follows, because $|\mathcal{H}_{\div}| \geqslant \rho_{k-i+1}(\ell')\prod_{j=i}^{k}(|B| - |X_j|)$. □

Chapter 7

Directions for Future Research

We close the thesis with a short chapter highlighting some open problems and conjectures, all of which were originally posed either by Jim Geelen or jointly with him. We begin with some natural and 'easy-to-state' problems related to the literature referenced in Chapter 1, particularly the Sylvester-Gallai Theorem and Hansen's Theorem. After that, we advertise some conjectures from [10] on average hyperplane- and flat-size in real- and complex-representable matroids. We then finish with some speculation on how to generalize the results of this thesis.

Hansen's Theorem is the prototypical Sylvester-Gallai-type theorem for higher-rank flats. As noted in Chapter 1, Hansen's original theorem (which is stronger than the one we formulated) says that every real-representable matroid M has an *ordinary* hyperplane, that is, a hyperplane comprising a point and a rank-$(r(M)-2)$ flat. While this is arguably the correct way to generalize the Sylvester-Gallai Theorem to higher dimensions, it is more natural to think about *independent* flats. Hansen's Theorem, as stated in Chapter 1, gives a tight upper bound on the rank of a real-representable matroid with no rank-k independent flat. We believe that it should be possible to strengthen this into a more precise characterization. At this point, we do not have a conjecture for such a characterization, so we will just pose it as an open problem to come up with one. We will only discuss the problem for planes, since this is the natural starting point.

Hansen's Theorem tells us that a real-representable matroid with no three-point plane has rank at most four. It was first observed by Motzkin [44] that there exist rank-4 real representable matroids with no three-point plane, namely $M(K_5)$ and the direct sum of two lines with at least three points each. Further examples, including some infinite families, were given by Bonnice and Kelly [7] and Hansen [33]. Our first open problem is to come

up with some kind of characterization of these rank-4 matroids.

Problem 7.1. *Characterize the real-representable matroids with no three-point plane.*

The same general question extends to the complex-representable matroids, where even less seems to be known. Kelly's Theorem says that every complex-representable matroid with no two-point line has rank at most three, but there seems to be no characterization of these matroids beyond this.

Problem 7.2. *Characterize the complex-representable matroids with no two-point line.*

Another natural problem is to extend Hansen's Theorem beyond real-representability. It is already conjectured (see [6, Sect. 6.5]) that every orientable matroid has an ordinary hyperplane. In [26], we showed that, for $k \geqslant 2$, every rank-$3(k-1)$ orientable matroid has an ordinary rank-k flat. In the case of complex-representability, Dvir, Saraf and Wigderson [19] proved that, if a complex-representable matroid has rank at least $\Omega(k)$, then it has an ordinary rank-k flat. In [26], we proved this with a multiplicative constant of four, and also conjectured the following.

Conjecture 7.3. *For $k \geqslant 2$, if M is a rank-$(k+2)$ complex-representable matroid, then M has a rank-k ordinary flat.*

Turning our attention to the results of this thesis, perhaps the most natural question that arises is: what are the *best-possible* bounds on the average hyperplane-size or the average flat-size in a real-representable matroid of a given rank? Melchior's Theorem gives a sharp upper bound of three on the average line-size. We generalized Melchior's Theorem to real-representable matroids of arbitrary rank $r \geqslant 3$ by proving the existence of constants $f_r, h_r > 0$ (depending only on r) such that the average flat-size is at most f_r and, subject to some necessary structural considerations, the average hyperplane-size is at most h_r. However, the bounds arising from our proofs are probably far from best-possible.

Although we only discussed real-representable matroids in this thesis, the Average Flat-Size and Average Hyperplane-Size theorems generalize to complex-representable matroids [10, 11]. This is essentially because the Szemerédi-Trotter Theorem holds for this class [55, 56]. We now highlight some conjectures of Geelen (also given in [10]) on the bounds on these averages. Broadly speaking, Geelen's conjectures say that, for average flat- and hyperplane-size in complex-representable matroids, the worst-case behaviour is exhibited by *Dowling Geometries*, which are a special class of matroids defined by finite abelian groups (see [16]). In particular, for any integer $r \geqslant 3$, the class of rank-r Dowling Geometries defined

by the finite cyclic groups $\mathbb{Z}_t$ gives a family of rank-r complex-representable matroids whose average hyperplane- and flat-sizes tend to $\binom{r}{2}$ as t goes to infinity. The proof of this, as well as the definition of Dowling Geometries, is available in the appendix of [10].

Before stating the conjectures for matroids of arbitrary rank, we discuss the special cases of lines and planes. Recall that Hirzebruch's Theorem (see Section 2.1) says that the average line-size in a complex-representable matroid is less than four. Peter Nelson (personal communication) asked whether this bound is in fact best-possible. The best construction we know of in this regard is the dual configuration of Wiman's line arrangement in $\mathrm{PG}(2, \mathbb{C})$. This matroid has 45 points and average line-size roughly 3.58. Like what we witness with lines, we believe that the average plane-size in real- and complex-representable matroids should also be very small. Note that, in the following conjectures, $o_n(1)$ denotes a function which tends to zero as n tends to infinity.

Conjecture 7.4 (Geelen)**.** *Let M be a simple complex-representable matroid on n elements.*

1. *If $r(M) \geqslant 3$, then the average line-size in M is at most $3 + o_n(1)$.*
2. *If $r(M) \geqslant 4$ and M is not the direct sum of two lines, then the average plane-size of M is at most $6 + o_n(1)$.*

In light of the discussion above, we also have the following general conjecture on average flat-size. When dealing with matroids of arbitrary (but fixed) rank r, the function $o_n(1)$ again denotes some function which tends to zero as n tends to infinity, although it may depend on r.

Conjecture 7.5 (Geelen)**.** *For any integer $r \geqslant 3$, if M is a simple rank-r complex-representable matroid on n elements, then the average flat-size in M is at most $\binom{r}{2} + o_n(1)$.*

In light of the Average Hyperplane-Size Theorem (which appeared about a year after [10]), we will append to Geelen's conjectures the same bound for hyperplanes.

Conjecture 7.6. *For an integer $r \geqslant 3$, let M be a simple rank-r complex-representable matroid on n elements, and let X_{r-1} be a largest $(r-1)$-degenerate set in M. If $n - |X_{r-1}|$ is at least some constant depending only on r, then the average hyperplane-size in M is at most $\binom{r}{2} + o_n(1)$.*

There is one further conjecture from [10] of a different nature, concerning the average size of flats of a given rank in a general matroid. Essentially, the question is whether a

given bound on the average line-size implies a bound on the average size of flats of higher rank. We will focus our discussion on lines and planes, since this is the simplest case and we do not know the answer.

Conjecture 7.7. *For every number $C_2 > 0$, there exists $C_3 > 0$ such that the following holds. If M is a matroid with average line-size at most C_2 and $r(M) \geqslant 6$, then the average size of a plane in M is at most C_3.*

The rank bound of six in this conjecture is necessary; a looser version would only require that the rank of M is sufficiently high. Refuting an earlier version of the conjecture, Ben Lund (personal communication) demonstrated that the bound of six is best-possible by constructing rank-5 matroids which are the direct sum of a projective plane of order p and an n-point line. By taking n sufficiently large in this construction (relative to p), the average plane-size is at least p but the average line-size is less than three.

In light of Conjecture 7.7, we believe that a converse must surely hold, and also ought to have a relatively simple proof. So far, we have not found it.

Conjecture 7.8 ([10])**.** *For every number $C_3 > 0$, there exists $C_2 > 0$ such that the following holds. If M is a matroid of rank at least four with average plane-size at most C_3, then M has average line-size at most C_2.*

The remaining conjectures that we pose are generalizations or extensions of the results of this thesis. The Unavoidable Flats Theorem as well as our theorems on average flat- and hyperplane-size do not generalize to all matroids. The following conjectures are based on the belief that the unavoidability of independent flats and the absolute-boundedness on the average size of flats of a given rank are only obstructed by the existence of low-rank dependencies. Indeed, for matroids representable over a given finite field, this is demonstrated by the Unavoidable Flats Theorem, as noted explicitly in Corollary 1.2. We conjecture that this generalizes to all matroids.

Conjecture 7.9 (Geelen)**.** *For any positive integer k, there exists a number C_k such that, if M is a simple matroid with girth at least five and $r(M) \geqslant C_k$, then M has a rank-k independent flat.*

Note that Conjecture 7.9, if true, would imply a Ramsey theorem for matroids of girth at least five. It is not possible to generalize our Ramsey theorem (Corollary 1.1) for matroids representable over a given finite field to matroids in general, because the Ramsey numbers for finite projective geometries are known to grow with the size of the underlying field. However, Geelen conjectured the following general Ramsey theorem which is stronger than the one implied by Conjecture 7.9.

Conjecture 7.10 (Geelen)**.** *For any positive integers k, ℓ, there is a number $N(k, \ell)$ such that, if M is a matroid with maximum line-size at most ℓ and $r(M) \geqslant N(k, \ell)$, then for any two-colouring of the points of M, there is a monochromatic rank-k flat.*

We also pose a similar conjecture for the average size of flats of given rank.

Conjecture 7.11. *For any positive integer k, there exists a number C_k such that, if M is a matroid with girth at least five and $r(M) \geqslant C_k$, then the average size of a rank-k flat in M is at most a constant depending only on k.*

Our proof techniques for the Unavoidable Flats Theorem and our average theorems have no bearing on Conjectures 7.9, 7.10 and 7.11, and we have no idea how to prove these conjectures at this point in time. Given that all of our results so far deal with matroids representable over finite fields and fields of characteristic zero, considering matroids representable over an infinite field of non-zero characteristic seems like a sensible next step. In this context, we believe that the unavoidability of independent flats is obstructed only by the presence of large affine geometries.

Conjecture 7.12 (Geelen)**.** *For p prime and any positive integer k, if M is a simple matroid of sufficiently high rank representable over a field of characteristic p, then M has either a rank-k independent flat or an $\mathrm{AG}(k, p)$-restriction.*

We also believe that something similar should be true for averages.

Conjecture 7.13 (Geelen)**.** *For p prime and any positive integer k, there is a constant $B_{k,p} > 0$ such that, if M is a matroid of sufficiently high rank representable over a field of characteristic p, then either the average size of a rank-k flat in M is at most $B_{k,p}$, or M has an $\mathrm{AG}(k, p)$-restriction.*

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